Appendix F

Self-dual Connection Formulation and Ashtekar's new Variables

F.1 Self-dual Connection Formulation

F.1.1 Self-dual Curvature

We will need the totally antisymmetry tensor or Levi-Civita symbol, ϵ_{IJKL}. Recall that this is equal to either +1 or -1 depending on whether $IJKL$ is either an even or odd permutation of 0123, respectively, and zero if any two indices take the same value. The internal indices of ϵ_{IJKL} are raised with the Minkowski metric η^{IJ}.

Given any anti-symmetric tensor T^{IJ}, we define its dual as

$$*T^{IJ} = \frac{1}{2}\epsilon_{KL}{}^{IJ}T^{\ KL} \tag{F.1}$$

The self-dual part of any tensor T^{IJ} is defined as

$${}^{+}T^{IJ} := \frac{1}{2}\Big(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}\Big) \tag{F.2}$$

with the anti-self-dual part defined as

$${}^{-}T^{IJ} := \frac{1}{2}\Big(T^{IJ} + \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}\Big) \tag{F.3}$$

(the appearance of the imaginary unit i is related to the Minkowski signature as we will see below).

Tensor decomposition

Now given any anti-symmetric tensor T^{IJ}, we can decompose it as

$$T^{IJ} = \frac{1}{2}(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}) + \frac{1}{2}(T^{IJ} + \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}) = {}^{+}T^{IJ} + {}^{-}T^{IJ} \tag{F.4}$$

where ${}^{+}T^{IJ}$ and ${}^{-}T^{IJ}$ are the self-dual and anti-self-dual parts of T^{IJ} respectively. Define the projector onto (anti-)self-dual part of any tensor as

$$P^{(\pm)} = \frac{1}{2}(1 \mp i*). \tag{F.5}$$

The meaning of these projectors can be made explicit. Let us concentrate of P^{+},

$$(P^{+}T)^{IJ} = (\frac{1}{2}(1 - i*)T)^{IJ} = \frac{1}{2}(\delta^{I}{}_{K}\delta^{J}{}_{L} - i\frac{1}{2}\epsilon_{KL}{}^{IJ})T^{KL} = \frac{1}{2}(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}) = {}^{+}T^{IJ}. \tag{F.6}$$

Then

${}^{\pm}T^{IJ} = (P^{(\pm)}T)^{IJ}$.

The Lie bracket

An important object is the Lie bracket defined by

$$[F, G]^{IJ} := F^{IK}G_{K}^{\ J} - G^{IK}F_{K}^{\ J}, \tag{F.7}$$

it appears in the curvature tensor (see the last two terms of ()), it also defines the algebraic structure. We have the results (proved below):

$$P^{(\pm)}[F, G]^{IJ} = [P^{(\pm)}F, G]^{IJ} = [F, P^{(\pm)}G]^{IJ} = [P^{(\pm)}F, P^{(\pm)}G]^{IJ} \tag{F.8}$$

and

$$[F, G] = [P^{+}F, P^{+}G] + [P^{-}F, P^{-}G]. \tag{F.9}$$

That is the Lie bracket, which defines an algebra, decomposes into two separate independent parts.

Identities for the totally anti-symmetric tensor

The internal indices of ϵ_{IJKL} are raised with the Minkowski metric η^{IJ}. Since η_{IJ} has signature $(-,+,+,+)$, it follows that

$$\epsilon^{IJKL} = -\epsilon_{IJKL}.$$

to see this consider,

$$\begin{aligned} \epsilon^{0123} &= \eta^{0I}\eta^{1J}\eta^{2K}\eta^{3L}\epsilon_{IJKL} \\ &= (-1)(+1)(+1)(+1)\epsilon_{0123} = -\epsilon_{0123}. \end{aligned}$$

We then have the identities,

$$\begin{aligned} \epsilon^{IJKO}\epsilon_{LMNO} &= -6\delta^I_{[L}\delta^J_M\delta^K_{N]} && \text{(F.10)} \\ \epsilon^{IJMN}\epsilon_{KLMN} &= -4\delta^I_{[K}\delta^J_{L]} \\ &= -2(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K). && \text{(F.11)} \end{aligned}$$

Definition of self-dual tensor

Therefore the square of the duality operator is minus the identity,

$$**T^{IJ} = \frac{1}{4}\epsilon_{KL}{}^{IJ}\epsilon_{MN}{}^{KL}T^{\ MN} = -T^{IJ} \tag{F.12}$$

The minus sign here is due to the minus sign in (F.11), which is in turn due to the Minkowski signature. Had we used Euclidean signature, i.e. $(+,+,+,+)$, instead there would have been a positive sign. We define S^{IJ} to be self-dual if and only if

$$*S^{IJ} = iS^{IJ}. \tag{F.13}$$

(with Euclidean signature the self-duality condition would have been $*S^{IJ} = S^{IJ}$). Say S^{IJ} is self-dual, write it as a real and imaginary part,

$$S^{IJ} = \frac{1}{2}U^{IJ} + i\frac{1}{2}V^{IJ}.$$

Write the self-dual condition in terms of U and V,

$$*(U^{IJ}+iV^{IJ})=\frac{1}{2}\epsilon_{KL}{}^{IJ}(U^{KL}+iV^{KL})=i(U^{IJ}+iV^{IJ}).$$

Equating real parts we read off

$$V^{IJ}=-\frac{1}{2}\epsilon_{KL}{}^{IJ}U^{KL}$$

and so

$$S^{IJ}=\frac{1}{2}(U^{IJ}-\frac{i}{2}\epsilon_{KL}{}^{IJ}U^{KL}) \tag{F.14}$$

where U^{IJ} is the real part of $2T^{IJ}$. Now given any tensor T^{IJ}, we can decompose it as

$$\begin{aligned} T^{IJ} &= \frac{1}{2}(T^{IJ}-\frac{i}{2}\epsilon_{KL}{}^{IJ}T^{IJ})+\frac{1}{2}(T^{IJ}+\frac{i}{2}\epsilon_{KL}{}^{IJ}T^{IJ}) \\ &= {}^{+}T^{IJ}+{}^{-}T^{IJ} \end{aligned} \tag{F.15}$$

since $*({}^{+}T^{IJ})=i\,{}^{+}T^{IJ}$ and $*({}^{-}T^{IJ})=-i\,{}^{-}T^{IJ}$, ${}^{+}T^{IJ}$ and ${}^{-}T^{IJ}$ are the self-dual and anti-self-dual parts of T^{IJ}.

Important lengthy calculation

The following lengthy calculation is important as all the other important formula can easily be derived from it. From the definition of the Lie bracket and with the use of (F.10) we have

$$
\begin{aligned}
*[F,*G]^{IJ} &= \frac{1}{2}\epsilon_{MN}{}^{IJ}(F^{MK}(*G)_K{}^N - (*G)^{MK}F_K{}^N) \\
&= \frac{1}{2}\epsilon_{MN}{}^{IJ}(F^{MK}\frac{1}{2}\epsilon_{OPK}{}^{N}G^{OP} - \frac{1}{2}\epsilon_{OP}{}^{MK}G^{OP}F_K{}^N) \\
&= \frac{1}{4}(\epsilon_{MN}{}^{IJ}\epsilon_{OP}{}^{KN} + \epsilon_{NM}{}^{IJ}\epsilon_{OP}{}^{NK})F^M{}_K G^{OP} \\
&= \frac{1}{2}\epsilon_{MN}{}^{IJ}\epsilon_{OP}{}^{KN}F^M{}_K G^{OP} \\
&= \frac{1}{2}\epsilon^{MIJN}\epsilon_{OPKN}F_M{}^K G^{OP} \\
&= -\frac{1}{2}\epsilon^{KIJN}\epsilon_{OPMN}F^M{}_K G^{OP} \\
&= \frac{1}{2}(\delta^K_O\delta^I_P\delta^J_M + \delta^K_M\delta^I_O\delta^J_P + \delta^K_P\delta^I_M\delta^J_O \\
&\quad - \delta^K_P\delta^I_O\delta^J_M - \delta^K_M\delta^I_P\delta^J_O - \delta^K_O\delta^I_M\delta^J_P)F^M{}_K G^{OP} \\
&= \frac{1}{2}(F^J{}_K G^{KI} + F^K{}_K G^{IJ} + F^I{}_K G^{JK} \\
&\quad - F^J{}_K G^{IK} - F^K{}_K G^{JI} - F^I{}_K G^{KJ}) \\
&= -F^{IK}G_K{}^J + G^{IK}F_K{}^J \\
&= -[F,G]^{IJ}
\end{aligned}
\tag{F.16}
$$

That gives the formula

$$
*[F,*G]^{IJ} = -[F,G]^{IJ}. \tag{F.17}
$$

from which everything else is then easy to derive.

Derivation of important results

First consider

$$
\begin{aligned}
*[*F,G]^{IJ} &= -*[G,*F]^{IJ} \\
&= [G,F]^{IJ} = -[F,G]^{IJ}.
\end{aligned}
$$

where in the first step we have used the anti-symmetry of the Lie bracket to swap $*F$ and G, in the second step we used (F.17) and in the last step we used the anti-symmetry of the Lie bracket again. Now using this we obtain

$$
\begin{aligned}
*[F,G]^{IJ} &= *(-*[*F,G]^{IJ}) \\
&= -**[*F,G]^{IJ} \\
&= [*F,G]^{IJ}.
\end{aligned}
$$

where we used $**=-1$ in the third step. Similarly we have

$$*[F,G]^{IJ}=[F,*G]^{IJ}$$

Now if we took $*[F,G]^{IJ}=[*F,G]^{IJ}$ and simply replaced G with $*G$ we would get $*[F,*G]^{IJ}=[*F,*G]^{IJ}$. Combining $-[F,G]^{IJ}=*[F,*G]^{IJ}$ (F.17) and $*[F,*G]^{IJ}=[*F,*G]^{IJ}$ we obtain

$$-[F,G]^{IJ}=[*F,*G]^{IJ}.$$

Summarising, we have

$$\begin{aligned} *[F,*G]^{IJ} &= -[F,G]^{IJ}=*[*F,G]^{IJ} && \text{(F.18)} \\ *[F,G]^{IJ} &= [*F,G]^{IJ}=[F,*G]^{IJ} && \text{(F.19)} \\ [*F,*G]^{IJ} &= -[F,G]^{IJ} && \text{(F.20)} \end{aligned}$$

Then

$$\begin{aligned} (P^{(\pm)}[F,G])^{IJ} &= \frac{1}{2}([F,G]^{IJ}\mp i*[F,G]^{IJ}) \\ &= \frac{1}{2}([F,G]^{IJ}+[\mp i*F,G]^{IJ}) \\ &= [P^{(\pm)}F,G]^{IJ} \qquad \text{(F.21)} \end{aligned}$$

Similarly we have $(P^{(\pm)}[F,G])^{IJ}=[F,P^{(\pm)}G]^{IJ}$. Now consider $[P^+F,P^-G]^{IJ}$,

$$\begin{aligned} [P^+F,P^-G]^{IJ} &= \frac{1}{4}[(1-i*)F,(1+i*)G]^{IJ} \\ &= \frac{1}{4}[F,G]^{IJ}-\frac{1}{4}i[*F,G]^{IJ}+\frac{1}{4}i[F,*G]^{IJ}+\frac{1}{4}[*F,*G]^{IJ} \\ &= \frac{1}{4}[F,G]^{IJ}-\frac{1}{4}i[*F,G]^{IJ}+\frac{1}{4}i[*F,G]^{IJ}-\frac{1}{4}[F,G]^{IJ} \\ &= 0. \qquad \text{(F.22)} \end{aligned}$$

Similarly $[P^-F,P^+G]^{IJ}=0$. This implies

$$[P^{(\pm)}F,G]^{IJ}=[P^{(\pm)}F,P^{(\pm)}G+P^{(\mp)}G]^{IJ}=[P^{(\pm)}F,P^{(\pm)}G]^{IJ}.$$

Summary of main results

Altogether we have,

$$P^{(\pm)}[F,G]^{IJ} = [P^{(\pm)}F,G]^{IJ} = [F,P^{(\pm)}G]^{IJ} = [P^{(\pm)}F,P^{(\pm)}G]^{IJ}. \tag{F.23}$$

We then have that any bracket splits as

$$\begin{aligned}
[F,G]^{IJ} &= [P^+F + P^-F, P^+G + P^-F]^{IJ} \\
&= [P^+F, P^+G]^{IJ} + [P^-F, P^-G]^{IJ}.
\end{aligned} \tag{F.24}$$

into a part that depends only on self-dual Lorentzian tensors and is itself the self-dual part of $[F,G]^{IJ}$ by (F.23), and a part that depndends only on anti-self-dual Lorentzian tensors and is the anit-self-dual part of $[F,G]^{IJ}$ again by (F.23).

We can write

$$so(1,3)_{\mathbb{C}} = so(1,3)^+_{\mathbb{C}} + so(1,3)^-_{\mathbb{C}} \tag{F.25}$$

where $so(1,3)^{\pm}_{\mathbb{C}}$ contains only the self-dual (anti-self-dual) elements of $so(1,3)_{\mathbb{C}}$.

Self-dual curvature

Instead of considering the connection $\omega_a{}^{IJ}$ we will consider its self-dual part, ${}^+A_a{}^{IJ}$, with respect to the internal indices, that is,

$$i\,{}^+A_a{}^{IJ} = \frac{1}{2}\epsilon_{MN}{}^{IJ}\,{}^+A_a{}^{MN}$$

${}^+A_a{}^{IJ}$ is related to $\omega_a{}^{IJ}$ by

$${}^+A_a{}^{IJ} = \frac{1}{2}\omega_a{}^{IJ} - \frac{i}{4}\epsilon_{MN}{}^{IJ}\omega_a{}^{MN} \tag{F.26}$$

Define $F_{ab}{}^{JK}$ as the curvature of the self-dual connection,

$$F_{ab}{}^{IJ} = \nabla_a\,{}^+A_b{}^{IJ} - \nabla_b\,{}^+A_a{}^{IJ} + {}^+A_a{}^{IK}\,{}^+A_{bK}{}^{J} - {}^+A_b{}^{IK}\,{}^+A_{aK}{}^{J} \tag{F.27}$$

We use the above results to show this corresponds to the self-dual part of the curvature of the usual connection:

$$
\begin{aligned}
F_{ab}{}^{IJ} &= \nabla_a\, {}^+A_b{}^{IJ} - \nabla_b\, {}^+A_a{}^{IJ} + \, {}^+A_a{}^{IK}\, {}^+A_{bK}{}^J - \, {}^+A_b{}^{IK}\, {}^+A_{aK}{}^J \\
&= (\nabla_a (P^+\omega_b)^{IJ} - \nabla_b (P^+\omega_a)^{IJ}) + [P^+\omega_a, P^+\omega_b]^{IJ} \\
&= (P^+ 2\nabla_{[a}\omega_{b]})^{IJ} + (P^+[\omega_a, \omega_b])^{IJ} \\
&= (P^+\Omega_{ab})^{IJ} \qquad \text{(F.28)}
\end{aligned}
$$

Which was the desired result. Thus we have,

$$
\begin{aligned}
F_{ab}{}^{IJ} &= \frac{1}{2}\Big(\Omega_{ab}{}^{IJ} - \frac{i}{2}\epsilon_{MN}{}^{IJ}\Omega_{ab}{}^{MN}\Big) \\
&= \nabla_{[a}\omega_{b]}{}^{IJ} + \omega_{[a}{}^{IK}\omega_{b]K}{}^{J} - \frac{i}{2}\epsilon_{MN}{}^{IJ}(\nabla_{[a}\omega_{b]}{}^{MN} + \omega_{[a}{}^{MK}\omega_{b]K}{}^{N}). \qquad \text{(F.29)}
\end{aligned}
$$

Using

$$
\begin{aligned}
2\nabla_{[a}\omega_{b]}{}^{IJ} + [\omega_a, \omega_b]^{IJ} &= (P^+ 2\nabla_{[a}\omega_{b]})^{IJ} + (P^- 2\nabla_{[a}\omega_{b]})^{IJ} + [P^+\omega_a + P^-\omega_a, P^+\omega_b + P^-\omega_b]^{IJ} \\
&= \Big((P^+ 2\nabla_{[a}\omega_{b]})^{IJ} + [P^+\omega_a, P^+\omega_b]^{IJ}\Big) + \Big((P^- 2\nabla_{[a}\omega_{b]})^{IJ} + [P^-\omega_a, P^-\omega_b]^{IJ}\Big). \qquad \text{(F.30)}
\end{aligned}
$$

we see that the Palatini curvature decomposes as,

$$
\Omega_{ab}{}^{IJ}[\omega] = \Omega_{ab}{}^{IJ}[{}^+A] + \Omega_{ab}{}^{IJ}[{}^-A] \qquad \text{(F.31)}
$$

and therefore the Palatini action can be written in terms of a self-dual and anti-self-dual part which depend respectively only on the self-dual and anti-self-dual connections.

F.1.2 Self-dual Action

In the self-dual formulation of general relativity, the variables are a self-dual Lorentz connection.

The other variable is a complex tetrad. The action in the self-dual formulation is built using the curvature of the self-dual Lorentz connection.

As in the Palatini formulism, one can use the tetrad to define a metric g on M by

$$
g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta
$$

However, because the tetrad is complex, the metric is now complex.

The self-dual action can be written

$$S_{EH} = \int d^4x \; e \; e_I^a e_J^b F_{ab}{}^{IJ} \tag{F.32}$$

We introduce a connection compatible with the tetrad via $\nabla_a e_I^b = 0$. We will like to proceed analogously as with the Palatini case and show that one of the equations of motion implies that ${}^+\mathcal{D}_a$ is the self-dual part of the unique, torsion-free covariant derivative ∇_a compatible with e_a^I.

If $\Gamma_{aI}{}^J$ denotes the Chrristoffel symbol of ∇_a, we define the self-dual part ${}^+\nabla_a$ of ∇_a by

$${}^+\nabla_a v_I := \partial_a v_I + {}^+\Gamma_{aI}{}^J v_J, \tag{F.33}$$

where ${}^+\Gamma_{aI}{}^J$ is the self-dual part of $\Gamma_{aI}{}^J$. The difference between these two connections when applied to a tensor with purely internal indices is

$${}^+C_{aI}{}^J V_J = ({}^+\mathcal{D}_a - {}^+\nabla_a) V_I. \tag{F.34}$$

Note that ${}^+C_{aI}{}^J = A_{aI}{}^J - {}^+\Gamma_{aI}{}^J$ is indeed the self-dual part of $C_{aI}{}^J$.

$$F_{ab}{}^{IJ} - {}^+R_{ab}{}^{IJ} = 2\nabla_{[a} {}^+C_{b]}{}^{IJ} + 2\, {}^+C_{[a}{}^{IK}\, {}^+C_{b]K}{}^J \tag{F.35}$$

The self-dual action is written as

$$S_{EH} = \int d^4x \; e \; e_I^a e_J^b ({}^+R_{ab}{}^{IJ} + 2\nabla_{[a} {}^+C_{b]}{}^{IJ} + 2\, {}^+C_{[a}{}^{IK}\, {}^+C_{b]K}{}^J) \tag{F.36}$$

Variation with respect to ${}^+C_{b]}{}^{IJ}$ produces

$${}^+C_{bI}{}^K e_K^{[a} e_J^{b]} + {}^+C_{bJ}{}^K e_I^{[a} e_K^{b]} = 0. \tag{F.37}$$

where we have used ${}^+C_{bKI} = - {}^+C_{bIK}$. This can be writtem more compactly as

$$e_M^{[a} e_N^{b]} \delta_{[I}^M \delta_{J]}^K \, {}^+C_{bK}{}^N = 0. \tag{F.38}$$

The variation with respect to the tetrad goes along the similarly except $\Omega_{ab}{}^{IJ}$ replaced everywhere by $F_{ab}{}^{IJ}$.

F.2 Ashtekar's Canonical Formalism

GR can be expressed in terms of a complex field $A_a^i(x)$ and a 3d real field $\tilde{E}_i^a(x)$, defined on a three-dimensional space Σ without boundaries, satisfying the reality condition

$$A_a^i - \bar{A}_a^i = \Sigma_a^i[E] \tag{F.39}$$

where Σ is defined in appendix A. The theory is defined by the Hamiltonian system

$$\tilde{E}_i^a(x) = \frac{\delta S}{\delta A_a^i(x)} \tag{F.40}$$

This indicates that the quantity $\tilde{E}_i^a(x)$ is the momentum conjugate to $A_a^i(x)$. In Maxwell and Yang-Mills theories, the momentum conjugate to the three dimensional connection A is called the electric field.

The Lagrainian and Ashtekar Lagrangian give the same equations of motion from the varaiation of the spin connection, and once this equation is solved and substituted into, the two Lagrangians differ only by a term that vanishes due to the Bianchi identity.

F.3 Generators of Symmetry Transformations

$$\{E_i^a(x), G(y)\} = \epsilon^{ijk}\Lambda_j \tilde{E}_k^a \tag{F.41}$$

This Poisson bracket generates an infintesmal roatation on the internal space index of E_i^a. $R(\delta\theta)^{ij} = \delta^{ij} + \epsilon^{ijk}\Lambda_j$.

$$\{A_a^i(x), G(y)\} = \mathcal{D}_a\Lambda_j, \tag{F.42}$$

The theory contains the "vector" constraint ()

$$V(\vec{N}) = \int_\Sigma d^3 N^a E^{bi} F_{ab}^i \tag{F.43}$$

which when combined with the Gauss constraint, gives the statial diffeomorphism constraint

$$\begin{aligned} D(\vec{N}) = V(\vec{N}) - G(A_a^i N^a) = \int_\Sigma d^3x[N^a E^{bi} F_{ab}^i = \\ = \int_\Sigma d^3x[E^{bi}\partial_a A_b^i - \partial_b(E^{bi}A_a^i)] \end{aligned} \tag{F.44}$$

The diffeomorphism constraint generates the in intesimal change in the fields along the diffeomorphism flow. When applied to a Wilson loop we dont need to include the Gauss gauge term $G(A^i_a N^a)$ and there is no difference between the constraints $V(\vec{N})$ and $D(\vec{N})$.

$$\begin{aligned}\{A^i_a, D(\vec{N})\} &= \mathcal{L}_{\vec{N}} A^i_a,\\ \{E^{ai}, D(\vec{N})\} &= \mathcal{L}_{\vec{N}} E^{ai}\end{aligned} \tag{F.45}$$

The Hamiltonian constraint generates

$$\{A^i_a, H(N)\} = 2\beta N \epsilon^{ijk} E^{bi} F^k_{ab}, \tag{F.46}$$

$$\{E^{ai}, H(N)\} = -2\beta N D_b(N E^{aj} E^{bk}). \tag{F.47}$$

F.3.1 The Gauss-law Constraint Generates Gauge Transformations

$$\widehat{E_i(x)}\Psi = -i\frac{\delta\Psi}{\delta A^i(x)}. \tag{F.48}$$

$$|\psi\rangle \to |\psi\rangle + i\epsilon\left(\int d^3x \Lambda \nabla^i \widehat{E_i(x)}\right)|\psi\rangle \tag{F.49}$$

$$\Psi[A] \to \Psi[A] + \epsilon \int d^3x \Lambda \nabla^i \frac{\delta\Psi}{\delta A^i(x)}. \tag{F.50}$$

Integrating by parts, this gives us

$$\Psi[A] \to \Psi[A] + \epsilon \int d^3x (\nabla^i \Lambda) \frac{\delta\Psi}{\delta A^i(x)}. \tag{F.51}$$

But this is just the first terms of an infinite-dimensional Taylor expansion, and we have

$$\Psi[A] \to \Psi[A + \epsilon\nabla^i\Lambda] \tag{F.52}$$

which is a gauge transformation. The constraint operators are the generators of the gauge group. It follows that any physical state, i.e. any state satisfying

$$\nabla^i \widehat{E_i(x)}\,|\psi\rangle = 0, \tag{F.53}$$

must be gauge invariant.

F.3.2 Incorperating Matter in the Quantum Theory

fermions at open ends of loops - natural gauge invariant objects.

Lattice Gauge Theories

F.4 Toy Model: Free Particle described using Half-Complex Coordinates.

F.4.1 Complex Variables and Reality Conditions

Canonical transformation

$$P\dot{Q} - \tilde{\mathcal{H}}(Q,P) = p\dot{q} - \mathcal{H}(q,p) - \frac{\partial F}{\partial t}. \tag{F.54}$$

$$\begin{aligned} \frac{\partial \tilde{\mathcal{H}}}{\partial P} &= \dot{Q} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial Q} &= -\dot{P} \end{aligned} \tag{F.55}$$

$$P = -\frac{\partial F_1}{\partial Q}, \quad Q = q, \tag{F.56}$$

$$F_1(q,Q) = -qQ + ipQ \tag{F.57}$$

$Q = q$ $P = z = q - ip$

$$z = q - ip. \tag{F.58}$$

In terms of these variables the Hamiltonian reads

$$\mathcal{H}_0(x,z) = -\frac{1}{2m}(x-z)^2. \tag{F.59}$$

Consider z as a configuration variable

To find the solutions corresponding to real x and p, we have to impose the condition

$$z + \bar{z} = 2x \tag{F.60}$$

which x is the real part of z

Equation (F.60) is called the *reality condition.*

F.4.2 Quantization in Compex Coordinates.

A complex cononical transformation $x \to x, \;\; p \to x - ip$

If we know what x and p are then we know what x and z are and conversely if we know what x and z are we know what x and p are; so they are equivalent descriptions.

a free particle described in the coordinates

The Shrödinger equation in complex coordinates is

$$i\hbar\frac{\partial\psi(z,t)}{\partial t} = \mathcal{H}_0\left(\hbar\frac{\partial}{\partial z}, z\right)\psi(z,t) = -\frac{1}{2m}\left(\hbar\frac{\partial}{\partial z} - z\right)^2\psi(z,t). \tag{F.61}$$

$$z + z^\dagger = 2\hbar\frac{\partial}{\partial z}. \tag{F.62}$$

Now, equation (F.62) only makes sense after we have specified a the scalar product, because the adjoint of an operator is defined in terms of the the scalar product.

$$(\psi, \phi) = \int d\bar{z}dz f(z,\bar{z})\overline{\psi(z)}\phi(z). \tag{F.63}$$

We will know what the inner product if we can specify f.

$$(z + \bar{z})f(z,\bar{z}) = -2\hbar\frac{\partial}{\partial z}f(z,\bar{z}) \tag{F.64}$$

This gives

$$f(z,\bar{z}) = e^{-\frac{(z+\bar{z})^2}{4\hbar}} \tag{F.65}$$