Chapter 3

Isolated and Dynamical Horizons - Generalizations of Stationary Black Holes

A spacetime that looks Minkowskian at infinity is refered to as asymptotically flat.

An event horizon is a surface beyond which timelike curves cannot escape to infintity. Recall that the causal past $J^-(S)$ of some region S is the set of all points one can reach from that region by moving along past-directed timelike or null curves. The causal past of **future null inifinity** $J^-(\zeta^+)$ lie outside of the event horizon of the black hole, see Fig 3.18. The event horizon is formed by the set of points on the verge of lying in $J^-(\zeta^+)$ but not quite ie. they are the boundary separating $J^-(\zeta^+)$ from the inside of the black hole, $\dot{J}^-(\zeta^+)$. This definition has draw-back in that one needs to know the entire future history of the spacetime to identify the event horizon.

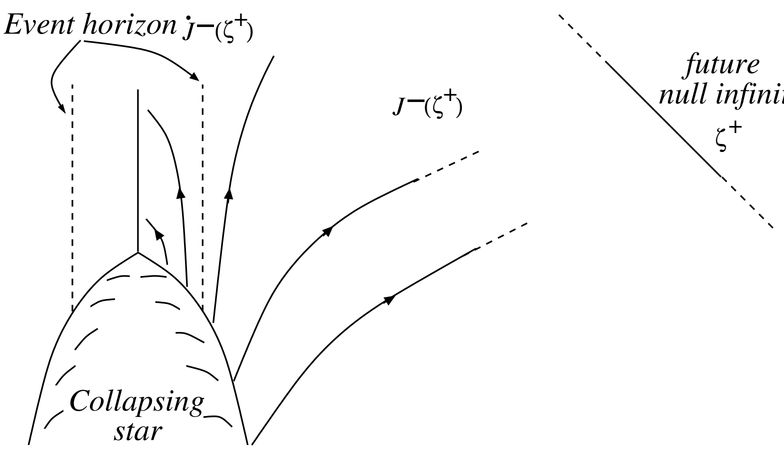


Figure 3.1: eventhorizon.

the gravitational field at its surface is so strong that the light cones will be bent inward.

Standard BH thermodynamics is based on globally stationary space-times with event horizons.

Standard treatments of BH mechanics are elegent from the space-time geometry point of view but are not well suited for quantization. They do not lead to a well defined action principle and Hamiltonian framework.

In the usual formulation of the black hole thermodynamics the energy and anular momentum are defined by ADM at infinity. ??To define a temperature of a black hole one must make refference to a Killing vector field at infinity??

The presentation will become progressively more involved.

???Surface area of a black hole - is it invariant under active diffeomorphisms? A surface defined as a loci of points of a coordinate system is not invariant under active diffeomorphisms, just as the proper-time between two spacetime points is not invariant under active diffeomorphisms. The horizon is defined in such a way that under an active diffeomorphism, the horizon is dragged across together with the metric field - just as with the proper time between particle coincident points are dragged across with the metric.???

3.1 Review of Stationary Black Holes

3.1.1 Mass and Angular Momentum of Bodies in Newtonian Gravity and Special Realtivity

Mass of Body

$$T^{ab} = \rho v^a v^b \tag{3.1}$$

$$E = \int_V \rho \, dV \tag{3.2}$$

Angular Momentum

$$\mathbf{J} = \mathbf{P} \times \mathbf{r} \tag{3.3}$$

$$J_d = -m\epsilon_{abcd} v^a x^b w^c = -m\epsilon_{abcd} v^a x^b p^c \tag{3.4}$$

$$p^a = E v^a + P^a \tag{3.5}$$

$$J_d = -m\epsilon_{abcd} v^a x^b P^c \tag{3.6}$$

Mass and angular momentum in GR

phase space of GR for the case when no internal boundary are present configuration space is the space of all metrics q_{ab} which satisfy the fall-off conditions at infinity

$$q_{ab} = \left(1 + \frac{M(\theta, \phi)}{r}\right) f_{ab} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{3.7}$$

The momentum conjugate to q_{ab} is given by

$$p^{ab} = \sqrt{q}\left(K^{ab} - KQ^{ab}\right). \tag{3.8}$$

Mass of Body

$$E = ke_\xi = -8\pi \int_V (T_{ab} - \frac{1}{2} T g_{ab}) \xi^a v^a \, d^3x. \tag{3.9}$$

$$E = 2 \int_V (T_{ab} - \frac{1}{2} T g_{ab}) v^a v^b \, d^3x = \int_V d^3x \rho. \tag{3.10}$$

Angular Momentum

$$\chi_a = \epsilon_{abcd} v^a x^b e^d \tag{3.11}$$

$$\chi_a v^a = \chi_a e^a = 0 \tag{3.12}$$

$$J = k' e_\chi = -8\pi k' \int_V (T_{ab} - \frac{1}{2} T g_{ab}) \chi^a v^b \, dV. \tag{3.13}$$

$$J = \frac{1}{8\pi} \int_S \nabla_a \chi_b v^a n^b \, dS. \tag{3.14}$$

The net flow through the faces

$$l^2 c(T^{01}(x) - T^{01}(x+l)] = -l^3 c \frac{\partial T^{01}}{\partial x} \tag{3.15}$$

$$l^3 \frac{\partial T^{00}}{\partial t} = -l^3 c \left(\frac{\partial T^{01}}{\partial x} + \frac{\partial T^{02}}{\partial y} + \frac{\partial T^{03}}{\partial z} \right) \tag{3.16}$$

Notion of energy and angular momentum have played a key role in analyzing behaviour of physical theories. For theories of fields on a fixed, background spacetime, a locally conserved stress-energy tensor, T_{ab}, normally can be defined. If the background spaceime has a Killing field, k^a, then $J^a = T^a{}_b k^b$ is a locally conserved current. If Σ is a Cauchy surface, then $q = \int_\Sigma J^a d\Sigma_a$ defines a conserved quantity associated with k^a; if Σ is a timelike or null surface, then $\int_\Sigma J^a d\Sigma_a$ has the interpretation of the flux of this quantity through Σ.

However, in diffeomorphism covariant theories such as general relativity, there is no notion of the local stress-energy tensor of the gravitational field, so conserved quantities () cannot and their fluxes cannot be defined by the above procedures, even when Killing fields are present. Nevertheless, in general relativity, for asymptotic symmetries have been defined at spacial infinity.

A definition of mass-energy are radiated energy at null infinity, $\mathcal{I}$, was first given by Trautman and Bondi et al. This definition was arrived at via a detailed study of asymptotic behaviour of the metric, and the main justification advanced for this definition has been in agreement with other notions of mass in some simply cases as well as the fact that the radiated energy is always positive

Ashtekar illustrates this *"Thus for example, while an event horizon may well be developing in the room in which you are now sitting in anticipation of a future gravitational collapse."*

3.2 The Schwarzschild Metric

The most general spherically symmetric metric is

$$d\tau^2 = A(t,r)dt^2 - B(t,r)dr^2 - 2C(t,r)dtdr - D(t,r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.17}$$

We introduce a new radial coordinate

$$r' = D^{1/2}(t,r)$$

The line element then becomes

$$d\tau^2 = A'(t,r')dt^2 - B'(t,r')dr'^2 - 2C'(t,r')dtdr - r'^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.18}$$

Consider the differential

$$A'(t,r')dt - B'(t,r')dr'$$

The theory of ordinary differential equations tells us that there exists a function (so called integration factor) $f(t,r')$ such that

$$\frac{\partial t'}{\partial t} = A'(t,r')f(t,r'), \qquad \frac{\partial t'}{\partial r'} = -B'(t,r')f(t,r')$$

for some $t' = t'(t,r')$, making t' a perfect differential:

$$\begin{aligned} dt' &= \frac{\partial t'}{\partial t}dt + \frac{\partial t'}{\partial r'}dr' \\ &= \big(A'(t,r')dt - B'(t,r')dr'\big)f(t,r') \end{aligned} \tag{3.19}$$

We need to know something of integrating factors and perfect differentials.

Proof:

Consider the ordinary differential equation of first order in two variables:

$$(\frac{dy}{dx} - \frac{Y(x,y)}{X(x,y)}) = g(x,y). \tag{3.20}$$

Observe that

$$\frac{d}{dx}\left(ye^{-\int^x Y(x',y)/X(x',y)dx'}\right) = \left(\frac{dy}{dx}e^{-\int^x Y(x',y)/X(x',y)dx'} - \frac{Y(x,y)}{X(x,y)}e^{-\int^x Y(x')/X(x')dx'}\right)$$

Thus we multiply a given differential equation by the factor

$$e^{-\int^x Y(x',y)/X(x',y)dx'}$$

which turns the LHS of the equation into a perfect differential

$$\frac{d}{dx}\left(ye^{-\int^x Y(x',y)/X(x',y)dx'}\right) = g(x,y)e^{-\int^x Y(x',y)/X(x',y)dx'}$$

or

$$
\begin{aligned}
d\left(y(x,y)e^{-\int^x Y(x',y)/X(x',y)dx'}\right) &= (dy - \frac{Y(x,y)}{X(x,y)}dx)e^{-\int^x Y(x')/X(x')dx'} \\
&= (X(x,y)dy - Y(x,y)dx)\big(e^{-\int^x Y(x',y)/X(x',y)dx'}/X(x,y)\big)
\end{aligned}
\tag{3.21}
$$

Put

$$
y'(x,y) = y(x,y)e^{-\int^x Y(x',y)/X(x',y)dx'}
$$

and

$$
f(x,y) = e^{-\int^x Y(x',y)/X(x',y)dx'}/X(x,y)
$$

then we have

$$
dy' = (X(x,y)dy - Y(x,y)dx)f(x,y)
$$

□

We use this result to define a new time coordinate t' for which

$$
dt' = \big(A'(t,r')dt - B'(t,r')dr'\big)f(t,r') \tag{3.22}
$$

Squaring, we obtain

$$
dt'^2 = \big(A'^2dt^2 - 2A'B'dtdr' + B'^2dr'^2\big)f^2(t,r')
$$

or

$$
A'dt^2 - 2B'dtdr' = A'^{-1}f^{-2}dt'^2 - A'^{-1}B'^2dr'^2
$$

the line element then becomes

Dropping the primes, the line element then becomes

$$
d\tau^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.23}
$$

Dropping the primes, the line element then becomes

$$d\tau^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.24}$$

where

$$\nu = \nu(t, r), \qquad \lambda = \lambda(t, r) \tag{3.25}$$

The metric components are

$$g_{00} = A, \quad g_{11} = -B, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \tag{3.26}$$

$$g_{00} = e^{\nu}, \quad g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \tag{3.27}$$

The metric is diagonal so that $g^{00} = 1/g_{00}$. The coordinates are labelled

$$x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi.$$

By definition

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}).$$

There are $4^3/2 = 32$ independent Γ^a_{bc}s. However many will be zero. For example, it is obvious that if all components are distinct the connection vanishes.

Let us do some examples.

$$\begin{aligned}
\Gamma^0_{01} &= \frac{1}{2}g^{0d}(g_{0d,1} + g_{1d,0} - g_{01,d}) \\
&= \frac{1}{2}g^{00}(g_{00,1} + g_{10,0} - g_{01,0}) \\
&= \frac{\nu'}{2}
\end{aligned} \tag{3.28}$$

where we have used that the metric is diagonal and the prime denotes differentiation with respect to r.

Consider Γ^1_{00}

$$
\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}g^{11}(g_{01,0} + g_{01,0} - g_{00,1}) \\
&= -\frac{1}{2}g^{11}g_{00,1} \\
&= \frac{e^{-\lambda}}{2}\nu' e^{\nu} \\
&= \frac{e^{\nu-\lambda}}{2}\nu' && (3.29)
\end{aligned}
$$

Consider Γ^1_{11}

$$
\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) \\
&= \frac{1}{2}g^{11}g_{11,1} \\
&= \frac{e^{-\lambda}}{2}\lambda' e^{\lambda} \\
&= \frac{\lambda'}{2} && (3.30)
\end{aligned}
$$

Consider Γ^1_{22}

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}(g_{21,2} + g_{21,b} - g_{22,1}) \\
&= -\frac{1}{2}g^{11}g_{22,1} \\
&= \frac{e^{-\lambda}}{2}(-2r) \\
&= -re^{-\lambda} && (3.31)
\end{aligned}
$$

Consider Γ^2_{12}

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2}g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) \\
&= \frac{1}{2}g^{22}g_{22,1} \\
&= \frac{1}{2r^2}2r \\
&= 1/r && (3.32)
\end{aligned}
$$

Consider Γ^3_{13}

$$\begin{aligned}
\Gamma^3_{13} &= \frac{1}{2}g^{33}(g_{13,3} + g_{33,1} - g_{13,3}) \\
&= \frac{1}{2}g^{33}g_{33,1} \\
&= -\frac{1}{2r^2\sin^2\theta}(-2r\sin^2\theta) \\
&= 1/r.
\end{aligned} \tag{3.33}$$

Consider Γ^1_{33}

$$\begin{aligned}
\Gamma^1_{33} &= \frac{1}{2}g^{1d}(g_{3d,3} + g_{3d,3} - g_{33,d}) \\
&= \frac{1}{2}g^{11}(g_{31,3} + g_{31,3} - g_{33,1}) \\
&= -g^{11}g_{33,1} \\
&= -r\sin^2\theta/B.
\end{aligned} \tag{3.34}$$

Consider Γ^3_{32}

$$\begin{aligned}
\Gamma^3_{32} &= \frac{1}{2}g^{33}(g_{33,2} + g_{23,3} - g_{32,3}) \\
&= \frac{1}{2}g^{33}g_{33,2} \\
&= -\frac{1}{2r^2\sin^2\theta}\frac{\partial}{\partial\theta}(-r^2\sin^2\theta) \\
&= \frac{\cos\theta}{\sin\theta} = \cot\theta.
\end{aligned} \tag{3.35}$$

Consider Γ^3_{33}

$$\begin{aligned}
\Gamma^3_{33} &= \frac{1}{2}g^{33}(g_{33,3} + g_{33,3} - g_{33,3}) \\
&= \frac{1}{2}g^{33}g_{33,3} = 0
\end{aligned} \tag{3.36}$$

As none of the metric components depend on ϕ. There are many other examples that turn out to be zero.

Consider Γ^0_{00}

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}g^{00}(g_{00,0} + g_{00,0} - g_{00,0}) \\
&= \frac{1}{2}g^{00}g_{00,0} \\
&= \frac{e^{-\nu}}{2}\dot{\nu}e^{\nu} \\
&= \frac{\dot{\nu}}{2}
\end{aligned} \tag{3.37}$$

Consider Γ^0_{11}

$$\begin{aligned}
\Gamma^0_{01} &= \frac{1}{2}g^{00}(g_{00,1} + g_{10,0} - g_{01,0}) \\
&= \frac{1}{2}g^{00}g_{00,1} \\
&= \frac{e^{-\nu}}{2}\nu' e^{\nu} \\
&= \frac{\nu'}{2}
\end{aligned} \tag{3.38}$$

Consider Γ^0_{11}

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2}g^{00}(g_{10,1} + g_{10,1} - g_{11,0}) \\
&= -\frac{1}{2}g^{00}g_{11,0} \\
&= \frac{e^{\lambda-\nu}}{2}\dot{\lambda}
\end{aligned} \tag{3.39}$$

Consider Γ^1_{00}

$$\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}g^{11}(g_{01,0} + g_{01,0} - g_{00,1}) \\
&= -\frac{1}{2}g^{11}g_{00,1} \\
&= -\frac{e^{\nu-\lambda}}{2}\nu'
\end{aligned} \tag{3.40}$$

Consider Γ^1_{01}

$$\begin{aligned}
\Gamma^1_{01} &= \frac{1}{2}g^{11}(g_{01,1} + g_{11,0} - g_{01,1}) \\
&= \frac{1}{2}g^{11}g_{11,0} \\
&= \frac{e^{-\lambda}}{2}\dot{\lambda}e^{\lambda} \\
&= \frac{\dot{\lambda}}{2}
\end{aligned} \tag{3.41}$$

We find altogether:

$$\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}e^{\nu-\lambda}\nu' \\
\Gamma^1_{11} &= \frac{1}{2}\lambda' \\
\Gamma^1_{22} &= -re^{-\lambda}, \\
\Gamma^1_{33} &= -re^{-\lambda}\sin^2\theta, \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = 1/r \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \\
\Gamma^3_{32} &= \Gamma^3_{23} = \cot\theta \\
\Gamma^0_{00} &= \frac{\dot{\nu}}{2}, \\
\Gamma^0_{01} &= \frac{\nu'}{2} \\
\Gamma^0_{11} &= \frac{e^{\lambda-\nu}}{2}\dot{\lambda} \\
\Gamma^1_{00} &= -\frac{e^{\nu-\lambda}}{2}\nu' \\
\Gamma^1_{01} &= \frac{1}{2}\dot{\lambda}
\end{aligned} \tag{3.42}$$

All others being zero.

We can calculate the Riemann tensor using

$$R^a{}_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \tag{3.43}$$

$$R^a{}_{bcd} = -R^a{}_{bdc} \tag{3.44}$$

The non-vanishing components of the Riemann tensor are

$$
\begin{aligned}
R^0{}_{101} &= e^{\lambda-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\nu'\dot{\lambda}}{4}\right)+\left(\frac{\nu'\lambda'}{4}-\frac{\nu''}{2}-\frac{\nu'^2}{4}\right)\\
R^0{}_{202} &= -re^{-\lambda}\nu'/2\\
R^0{}_{303} &= -re^{-\lambda}\sin^2\theta\nu'/2\\
R^0{}_{212} &= -re^{-\nu}\dot{\lambda}/2\\
R^0{}_{313} &= -re^{-\nu}\sin^2\theta\dot{\lambda}/2\\
R^1{}_{212} &= re^{-\lambda}\lambda'/2\\
R^1{}_{313} &= re^{-\lambda}\sin^2\theta\lambda'/2\\
R^2{}_{323} &= (1-e^{-\lambda})\sin^2\theta
\end{aligned}
\tag{3.45}
$$

We use these to calculate the components of the Ricci tensor.

$$
R_{ab}=\partial_b\Gamma^c_{ac}-\partial_c\Gamma^c_{ab}+\Gamma^d_{ac}\Gamma^c_{bd}-\Gamma^d_{ab}\Gamma^c_{dc}
\tag{3.46}
$$

The non-vanishing components of the Ricci tensor are

$$
R_{00} = \left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\nu}\dot{\lambda}}{4}\right)+e^{\lambda-\nu}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{1}{r}\nu'\right)
\tag{3.47}
$$

$$
R_{11} = -\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}-\frac{1}{r}\lambda'\right)+e^{\lambda-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\nu}\dot{\lambda}}{4}\right)
\tag{3.48}
$$

$$
R_{01} = \frac{1}{r}\dot{\lambda}
\tag{3.49}
$$

$$
R_{22} = e^{-\lambda}\left(\frac{r}{2}(\lambda'-\nu')-1\right)+1
\tag{3.50}
$$

$$
R_{33} = R_{22}\sin^2\theta
\tag{3.51}
$$

(3.49) vanishing implies

$$
\lambda=\lambda(r).
$$

From R_{00} and R_{11} vanishing independently

$$
0=e^{\lambda-\nu}R_{00}+R_{11}=\frac{1}{r}(\lambda'+\nu')
\tag{3.52}
$$

implying

$$\lambda + \nu = h(t) \tag{3.53}$$

where $h(t)$ is an arbitrary function of integration. As λ is purely a function of r, (3.50) is an ordinary differential equation, which we write

$$e^{-\lambda} - re^{-\lambda}\lambda' = 1,$$

or

$$(re^{-\lambda})' = 1.$$

Integrating, we get

$$re^{-\lambda} = r + Const.$$

so

$$e^{\lambda} = (1 + Const./r)^{-1}. \tag{3.54}$$

Note

$$e^{\nu} = e^{h(t)-\lambda}$$

The line element becomes

$$ds^2 = e^{h(t)}\left(1 + \frac{Const.}{r}\right)dt^2 - \frac{dr^2}{1 + Const./r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.55}$$

Write

$$t' = \int^t e^{h(u)/2} du \tag{3.56}$$

then

$$dt'^2 = e^{h(t)}dt^2$$

Dropping primes the most general spherically symmetric solution of the vacuum field equations is

$$ds^2 = \left(1 + \frac{Const.}{r}\right) dt^2 - \frac{dr^2}{1 + Const./r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.57}$$

All of the metric components are independent of a time coordinate. We have therefore proven that any spherically symmetric vacuum metric is stationary.

Now consider the Newtonian limit. A point mass M situated at the origin in Newtonian theory gives rise to a potential

$$\phi = -GM/r$$

Using this in the weak field limit gives

$$g_{00} \simeq 1 + 2\phi/c^2 = 1 - 2GM/c^2 r.$$

We see that the constant turns out to be $-2GM/c^2$

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{dr^2}{1 - 2GM/c^2 r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.58}$$

3.3 Schwarzschild Black Hole

3.3.1 Eddington-Finkelstein Coordinates

In the Eddington-Finkelstein coordinates radial null geodesics become straight lines

$$\bar{t} = t + 2m \ln(r - 2m) \tag{3.59}$$

$$dt = d\bar{t} - \frac{2m}{r - 2m} dr \tag{3.60}$$

Squaring gives

$$\begin{aligned}\left(1-\frac{2M}{r}\right)dt^2 &= \left(1-\frac{2M}{r}\right)\left[dt-\frac{2m}{r}\left(1-\frac{2m}{r}\right)^{-1}dr\right]^2 \\ &= \left(1-\frac{2M}{r}\right)d\bar{t}^2-2\frac{2m}{r}d\bar{t}dr+\frac{4m^2}{r^2}\left(1-\frac{2m}{r}\right)^{-1}dr^2 \qquad (3.61)\end{aligned}$$

Now we substitute it into the Schwarzschild line element.

$$\begin{aligned}ds^2 &= \left(1-\frac{2M}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2) \\ &= \left(1-\frac{2M}{r}\right)d\bar{t}^2-\frac{4m}{r}d\bar{t}dr-\left(1-\frac{4m^2}{r^2}\right)\left(1-\frac{2m}{r}\right)^{-1}dr^2 \\ &\quad -r^2(d\theta^2+\sin^2\theta d\phi^2) \\ &= \left(1-\frac{2M}{r}\right)d\bar{t}^2-\frac{4m}{r}d\bar{t}dr-\left(1-\frac{2M}{r}\right)dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (3.62)\end{aligned}$$

and obtain the line element

$$ds^2=\left(1-\frac{2M}{r}\right)d\bar{t}^2-\frac{4m}{r}d\bar{t}dr-\left(1-\frac{2M}{r}\right)dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (3.63)$$

Notice that in this new coordinate system we no longer have a coordinate signularity at $R=2M$. In fact it is regular for the whole range $0<r<2m$.

Advanced time parameter

As an external observer we cannot chart the course of a particle upon entering the event horizon. However, a particle entering would pass through the event horizon unaware of anything strange. The singularity at $R=2M$ of the Schwarzschild metric is unphysical, it is simply a coordinate singularity.

To show it isn't a physical singularity, let us make a change of coordinates and derive a new metric. We will keep r,θ but replace t with

$$t=v-r-2M\ \ln\left|\frac{r}{2M}-1\right| \qquad (3.64)$$

Let us find dt and substitute it back into the Schwarzschild line element. We begin by taking the derivative of t,

$$\begin{aligned}
dt &= dv - dr - \frac{dr}{r/2M-1} \\
&= dv - \left(\frac{r/2M-1+1}{r/2M-1}\right) dr \\
&= dv - \frac{rdr}{r-2M} \\
&= dv - \left(1-\frac{2M}{r}\right)^{-1} dr
\end{aligned} \tag{3.65}$$

Squaring this we get

$$dt^2 = dv^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dvdr + \left(1-\frac{2M}{r}\right)^{-2} dr^2. \tag{3.66}$$

Now we substitute it into the Schwarzschild line element.

$$\begin{aligned}
ds^2 &= \left(1-\frac{2M}{r}\right) dt^2 - \frac{dr^2}{1-2M/r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&= \left(1-\frac{2M}{r}\right) \left[dv^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dvdr + \left(1-\frac{2M}{r}\right)^{-2} dr^2\right] \\
&\quad - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&= \left(1-\frac{2M}{r}\right) dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned} \tag{3.67}$$

We have the line element in the Eddington-Finkelstein coordinates

$$ds^2 = \left(1-\frac{2M}{r}\right) dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.68}$$

Notice that in this new coordinate system we no longer have a coordinate singularity at $R = 2M$. We are now able to explore what happens inside the Schwarzschild black hole. Notice however we still have a singularity at $r = 0$. This is a real physical singularity, and not due to the coordinates used.

3.4 Internal Schwarzschild Solution

Perfect Fluid Schwarzschild Solution

The metric has the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.69}$$

We need to solve the full Einstein equation,

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}. \tag{3.70}$$

The Ricci scalar is giveb by

$$R = e^{-\lambda}\left(\nu'' + \frac{1}{2}\nu'2 - \frac{1}{2}\nu'\lambda' + \frac{2}{r}(\nu' - \lambda') + \frac{2}{r^2}\right) - \frac{2}{r^2}. \tag{3.71}$$

The nonvanishing components of the Einstein tensor G_{ab} are given by

$$\begin{aligned}
G_{00} &= \frac{1}{r^2}e^{\nu-\lambda}\left(r\lambda' - 1 + e^{\lambda}\right)\\
G_{11} &= \frac{1}{r^2}\left(r\nu' + 1 - e^{\lambda}\right)\\
G_{22} &= r^2 e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{1}{2r}(\nu' - \lambda')\right)\\
G_{33} &= \sin^2\theta G_{22}
\end{aligned} \tag{3.72}$$

The star is modelled by a perfect fluid

$$T_{ab} = (\rho + p)U_a U_b - p g_{ab}. \tag{3.73}$$

where ρ and p are the energy density and uniform pressure as measured in the rest frame of the fluid, and U is the fluid four velocity - assumed to be

$$U_a = (e^{\nu/2}, 0, 0, 0) \tag{3.74}$$

$U^a U_a = 1$

$$T_{ab} = \begin{pmatrix} e^{\nu}\rho & 0 & 0 & 0 \\ 0 & e^{\lambda}p & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta\, p \end{pmatrix} \tag{3.75}$$

Einstein's equations read

$$\frac{1}{r^2} e^{-\lambda}\left(r\lambda' - 1 + e^{\lambda}\right) = 8\pi G\rho, \tag{3.76}$$

$$\frac{1}{r^2} e^{-\lambda}\left(r\nu' + 1 - e^{\lambda}\right) = 8\pi G p, \tag{3.77}$$

$$e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{1}{2r}(\nu' - \lambda')\right) = 8\pi G p \tag{3.78}$$

Note that the first of these equations involves only $\lambda(r)$ and the density $\rho(r)$. Write

$$m(r) = \frac{1}{2G}(r - re^{-\lambda}) \tag{3.79}$$

then the first equation becomes

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{3.80}$$

which that can be integrated

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'. \tag{3.81}$$

Here, ρ is the density and so we interpret $m(r)$ as the total mass of the star enclosed by a sphere of radius r. Say or star extends to a radius R, after which spacetime is described by the Schwarzschild solution. We must have

$$M = m(R) = 4\pi \int_0^R \rho(r') r^2 dr. \tag{3.82}$$

where M is the Schwarzschild mass.

$$e^{\lambda} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1} \tag{3.83}$$

so that the line element becomes

$$ds^2 = e^{\nu(r)}dt^2 - \left(1 - \frac{2Gm(r)}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.84}$$

Equation (3.77) becomes

$$\frac{1}{r^2}\left(1 - \frac{2Gm(r)}{r}\right)(r\nu' + 1) - \frac{1}{r^2} = 8\pi Gp,$$

or

$$(r - 2Gm(r))(r\nu' + 1) = r + 8\pi Gpr^3,$$

or

$$(r\nu' + 1) = \frac{r + 8\pi Gpr^3}{(r - 2Gm(r))},$$

and so

$$\frac{d\nu}{dr} = \frac{2Gm(r) + 8\pi Gpr^3}{r\,(r - 2Gm(r))} \tag{3.85}$$

From the energy-momentun $\nabla_a T^{ab} = 0$, the component $b = 1$ gives

$$(\rho + p)\frac{d\nu}{dr} = -\frac{dp}{dr} \tag{3.86}$$

we use this to eliminate ν from (3.85):

$$\frac{dp}{dr} = -\frac{(\rho + p)[2Gm(r) + 8\pi Gpr^3]}{r\,(r - 2Gm(r))} \tag{3.87}$$

This is the equation of hydrostatic equilibrium, describing the balance between compression due to gravity and the pressure gradiant force in the opposite direction.

3.5 Penrose-Carter diagrams

Let $\mathcal{M}$ denote physical space-time with metric g_{ab}. The idea is to construct another "unphysical" manifold $\tilde{\mathcal{M}}$ with boundary $\mathcal{I}$ and metric $\tilde{g}_{ab}$, such that $\mathcal{M}$ is conformal to the interior of $\tilde{\mathcal{M}}$ with

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{3.88}$$

(where Ω is the conformal factor) and so that the "infinity" of $\mathcal{M}$ is represented by the finite hypersurface $\mathcal{I}$. We realise the whole physical spacetime $\mathcal{M}$ as a subset of the unphysical spacetime $\tilde{\mathcal{M}}$.

Asymptotic properties of $\mathcal{M}$ and of fields in $\mathcal{M}$ can be investigated by studying $\mathcal{I}$, and the local behaviour of the fields at $\mathcal{I}$ provided the relavent information is conformally invariant.

We will show that the null geodesics of conformally related metrics are the same, and hence have the same causal structure.

In chapter 7 we use such diagrams when investigating Hawking radiation.

Geodesics under conformal transformations

We denote the inverse metric to $\tilde{g}_{ab}$ by $\tilde{g}^{ab}$. Obviously,

$$\tilde{g}^{ab} = \Omega^{-2} g^{ab}.$$

Let $\tilde{\nabla}_a$ denote the derivative operator associated with $\tilde{g}_{ab}$.

$$\tilde{\nabla}_a \tilde{g}_{ab} = 0 \tag{3.89}$$

implies

$$\tilde{\Gamma}^a_{bc} = \frac{1}{2} \tilde{g}^{ad} (\nabla_b \tilde{g}_{cd} + \nabla_c \tilde{g}_{bd} - \nabla_d \tilde{g}_{bc}) \tag{3.90}$$

But since $\nabla_a g_{bc} = 0$, we have

$$\nabla_b \tilde{g}_{cd} = \nabla_b (\Omega^2 g_{cd}) = 2\Omega g_{cd} \nabla_b \Omega \tag{3.91}$$

$$
\begin{aligned}
\tilde{\Gamma}^a_{bc} &= \Omega^{-1} g^{ad}(g_{cd}\nabla_b\Omega + g_{bd}\nabla_c\Omega - g_{bc}\nabla_d\Omega) \\
&= 2\delta^a_{(b}\nabla_{c)} \ln\Omega - g_{bc}\nabla_d \ln\Omega
\end{aligned}
\tag{3.92}
$$

The tangent, v^a, to an affinely parameterised geodesic γ with respect to ∇_b satifies

$$
v^b\nabla_b v^a = 0 \tag{3.93}
$$

Hence

$$
\begin{aligned}
v^b\tilde{\nabla}_b v^a &= v^b\nabla_b v^a + v^b\tilde{\Gamma}^a_{bc} v^c \\
&= v^a(2v^c\nabla_c \ln\Omega) - (g_{bc}v^b v^c) g^{ad}\nabla_d \ln\Omega
\end{aligned}
\tag{3.94}
$$

Thus, γ fails to be a geodesic with respect to $\tilde{\nabla}_b$ unless $g_{bc}v^b v^c = 0$, in which case it is the non-affinely parameterised geodesic equation - the RHS is proportional to v^a. Hence we have proved the result.

3.5.1 Penrose-Carter Diagram for Minkowski Spacetime.

As a first example to illustrate the idea of a Penrose diagram we consider the procedure for Minkowski spacetime.

We introduce coordinates

$$
v = t + r, \tag{3.95}
$$

$$
w = t - r, \tag{3.96}
$$

It is obvious that

$$
-\infty < v < \infty, \quad -\infty < w < \infty \tag{3.97}
$$

$$
v \geq w, \tag{3.98}
$$

the Minkowski spacetime becomes

$$
ds^2 = dv dw - \frac{1}{4}(v-w)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.99}
$$

We define new coordinates p and q by

$$p = \tan^{-1} v, \tag{3.100}$$
$$q = \tan^{-1} w, \tag{3.101}$$

with coordinate ranges

$$-\frac{1}{2}\pi < p < \frac{1}{2}\pi, \quad -\frac{1}{2}\pi < q < \frac{1}{2}\pi \tag{3.102}$$
$$p \geq q. \tag{3.103}$$

$$ds^2 = \frac{1}{4}\sec^2 p\ \sec^2 q[4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.104}$$

and the line element of the unphysical metric is

$$d\tilde{s}^2 = 4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.105}$$

$$\Omega = \frac{1}{4}\sec^2 p\ \sec^2 q \tag{3.106}$$

We introduce coordinates

$$t' = p + q, \tag{3.107}$$
$$r' = p - q, \tag{3.108}$$

The unphysical metric is now

$$d\tilde{s}^2 = dt'^2 - dr'^2 - \sin^2 r'(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.109}$$

subject to the coordinate range

$$-\pi < t' + r' < \pi, \tag{3.110}$$
$$-\pi < t' - r' < \pi, \tag{3.111}$$
$$r' \geq 0. \tag{3.112}$$

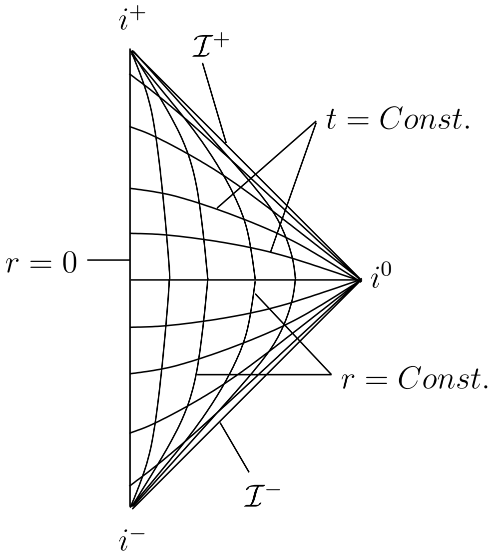


Figure 3.2: Penrose diagram of Minkowski spacetime

The whole of Minkowski spacetime has been shrunk into a finite or compact region. The process is called conformal compactification.

$\mathcal{I}^+$ is future null infinity,

$\mathcal{I}^-$ is past null infinity,

i^+ is future timelike infinity,

i^- is past timelike infinity,

i^0 is past spacelike infinity.

3.5.2 Maximal Extensions

3.5.3 The Kruskal Solution

The coordinate choice most useful for path integral investigation into black hole radiation.

3.6 Charged Balck Holes

We now obtain the Reissener-Nordstrom solution that describes a charged non-rotating black hole. We look for a static, asymptotically flat, shrerically symmetric solution of the Einstein-Maxwell field equations. The Einstein-Maxwell equations are

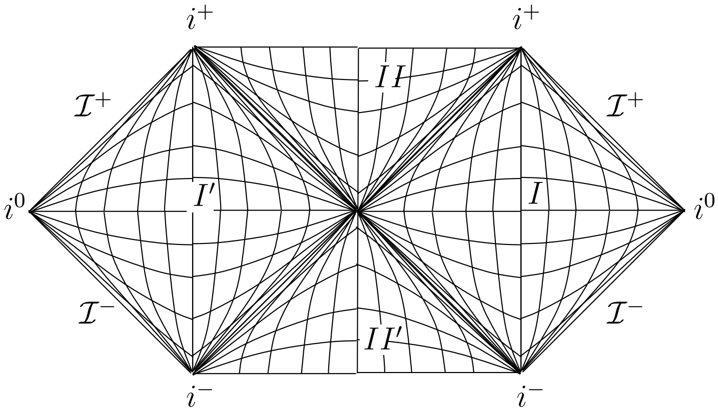


Figure 3.3: Penrose diagram of the Kruskal solution

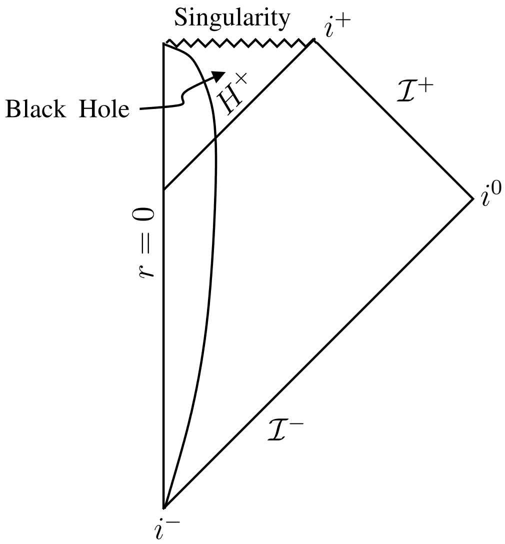


Figure 3.4: Penrose diagram of a black hole.

$$G_{ab} = 8\pi T_{ab} \tag{3.113}$$

where T_{ab} is the energy-momentum tensor of electromagnetism

$$T_{ab} = \frac{1}{4\pi}(-g^{cd}F_{ac}F_{bd} + \frac{1}{4}g_{ab}F_{cd}F^{cd}) \tag{3.114}$$

and where F_{ab} is the field strength tensor. Note that T_{ab} has zero trace,

$$T = g^{ab}T_{ab} = \frac{1}{4\pi}(-g^{ab}g^{cd}F_{ac}F_{bd} + \frac{1}{4}g^{ab}g_{ab}F_{cd}F^{cd}) = 0. \tag{3.115}$$

This implies the vanishing of the Ricci scalar as the trace of the Einstein tensor must vanish

$$R - \frac{1}{2}4R = T^a{}_a = 0 \quad \Rightarrow R = 0$$

We can then instead use

$$R_{ab} = 8\pi T_{ab} \tag{3.116}$$

In source-free regions the tensor F_{ab} must satisfy Maxwell's equations

$$\begin{aligned} \nabla_b F^{ab} &= 0, & (3.117)\\ \partial_{[a} F_{bc]} &= 0. & (3.118) \end{aligned}$$

We can assume there are coordinates (t, r, θ, ϕ) so thay the metric reduces to the form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.119}$$

If we impose the condition that the solution is static, then this requires ν and λ are functions of r only,

$$\nu = \nu(r), \qquad \lambda = \lambda(r). \tag{3.120}$$

We also assume the solution to be asymptotically flat.

We read off from (3.119) that

$$g_{ab} = \begin{pmatrix} e^\nu & 0 & 0 & 0 \\ 0 & -e^\lambda & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{3.121}$$

and so

$$g^{ab} = \begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \tag{3.122}$$

and

$$\sqrt{|g|} = \sqrt{-\det(g_{ab})} = e^{\frac{1}{2}(\nu+\lambda)} r^2 \sin\theta. \tag{3.123}$$

The non-zero components of the Ricci tensor R_{ab} are then

$$\begin{aligned}
R_{00} &= e^{\lambda-\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{1}{r}\nu'\right) \\
R_{11} &= -\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} - \frac{1}{r}\lambda'\right) \\
R_{22} &= e^{-\lambda}\left(\frac{r}{2}(\lambda' - \nu') - 1\right) + 1 \\
R_{33} &= R_{22}\sin^2\theta
\end{aligned} \tag{3.124}$$

which follow from (3.47)-(3.51) and the assumption that $\nu = \nu(r)$ and $\lambda = \lambda(r)$.

Solving the Maxwell equations and the electric charge

In spherical polar coordinates the Maxwell tensor takes the form

$$F_{ab} = E(r)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.125}$$

To confirm this ansatz we will take the definition of the Maxwell tensor in Minkoski coordinates (t, x, y, z):

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{3.126}$$

and find its components in spherical polar coordinates (t, r, θ, ϕ)

$$\begin{aligned}
t &= t \\
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned} \tag{3.127}$$

via the tansformation formula

$$F'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}F_{\sigma\rho}, \tag{3.128}$$

where we easily find that,

$$\frac{\partial x}{\partial x'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{3.129}$$

while inserting the assumption that the elecric field is radial:

$$\begin{aligned} E_x &= E(r)\sin\theta\cos\phi \\ E_y &= E(r)\sin\theta\sin\phi \\ E_z &= E(r)\cos\theta. \end{aligned} \tag{3.130}$$

We obtain from (3.128)

$$
\begin{aligned}
&\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ 0 & r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta\\ 0 & -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0\end{pmatrix}\\
&\begin{pmatrix}0 & -E(r)\sin\theta\cos\phi & -E(r)\sin\theta\sin\phi & -E(r)\cos\theta\\ E(r)\sin\theta\cos\phi & 0 & 0 & 0\\ E(r)\sin\theta\sin\phi & 0 & 0 & 0\\ E(r)\cos\theta & 0 & 0 & 0\end{pmatrix}\\
&\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ 0 & \cos\theta & -r\sin\theta & 0\end{pmatrix}\\
= &\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ 0 & r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta\\ 0 & -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0\end{pmatrix}\\
&\begin{pmatrix}0 & -E(r) & 0 & 0\\ E(r)\sin\theta\cos\phi & 0 & 0 & 0\\ E(r)\sin\theta\sin\phi & 0 & 0 & 0\\ E(r)\cos\theta & 0 & 0 & 0\end{pmatrix}\\
= &\begin{pmatrix}0 & -E(r) & 0 & 0\\ E(r) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}
\end{aligned}
\tag{3.131}
$$

confirming the ansatz.

Raising the indices using the inverse metric, we obtain

$$
\begin{aligned}
F^{ab} &= E(r)\begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\qquad\qquad\qquad\qquad \begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \\
&= E(r)\begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} 0 & e^{-\lambda} & 0 & 0 \\ e^{-\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= e^{-\nu-\lambda}E(r)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (3.132)
\end{aligned}
$$

We verify that (3.118) is automatically satisfied using (3.125). We can use $F_{ab} = -F_{ba}$ to write

$$
\partial_{[a}F_{bc]} = \frac{2}{3!}(\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca}) \qquad (3.133)
$$

We need only check for

$$
\begin{aligned}
& a = 0, b = 1, c = 2 \\
& a = 0, b = 1, c = 3 \\
& a = 0, b = 2, c = 3 \\
& a = 1, b = 2, c = 3. \qquad (3.134)
\end{aligned}
$$

First

$$
\begin{aligned}
\partial_{[0}F_{12]} &= \frac{2}{3!}(\partial_0 F_{12} + \partial_2 F_{01} + \partial_1 F_{20}) \\
&= 0 \qquad (3.135)
\end{aligned}
$$

then

$$\begin{aligned}\partial_{[0}F_{13]} &= \frac{2}{3!}(\partial_0 F_{13} + \partial_3 F_{01} + \partial_1 F_{30}) \\ &= 0 \end{aligned} \tag{3.136}$$

then

$$\begin{aligned}\partial_{[0}F_{23]} &= \frac{2}{3!}(\partial_0 F_{23} + \partial_3 F_{02} + \partial_2 F_{30}) \\ &= 0 \end{aligned} \tag{3.137}$$

and lastly

$$\begin{aligned}\partial_{[0}F_{23]} &= \frac{2}{3!}(\partial_0 F_{23} + \partial_3 F_{02} + \partial_2 F_{30}) \\ &= 0. \end{aligned} \tag{3.138}$$

To write out (3.117) we will use the following expression that holds for any antisymmetric rank two tensor $X^{ab} = -X^{ba}$

$$\nabla_a X^{ab} = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}X^{ab}). \tag{3.139}$$

To prove this result we need

$$\Gamma^a_{ab} = \frac{1}{\sqrt{|g|}}\partial_b\sqrt{|g|}. \tag{3.140}$$

Observe for a rank two tensor X^{ab}

$$\begin{aligned}\nabla_a X^{ab} &= \partial_a X^{ab} + \Gamma^a_{ac}X^{cb} + \Gamma^b_{ac}X^{ac} \\ &= \partial_a X^{ab} + \frac{1}{\sqrt{|g|}}(\partial_c\sqrt{|g|})X^{cb} + \Gamma^b_{ac}X^{ac} \\ &= \frac{1}{\sqrt{|g|}}\partial_c(\sqrt{|g|}X^{cb}) + \Gamma^b_{ac}X^{ac} \end{aligned} \tag{3.141}$$

Since we require the tensor X^{ab} to be antisymmetric the last term vanishes and we have established the result.

We now apply it to the tensor F^{ab}:

$$\begin{aligned}\nabla_a F^{ab} &= \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}F^{ab}) \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_a(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{ab}) \end{aligned} \tag{3.142}$$

From the t component of Maxwell's equation (3.117) and using (3.132) we must have

$$\begin{aligned} 0 &= \nabla_a F^{a0} \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_a(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{a0}) \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2}\partial_1(e^{\frac{1}{2}(\nu+\mu)}r^2 F^{10}) \\ &= -\frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2}\partial_1(e^{-\frac{1}{2}(\nu+\mu)}r^2 E). \end{aligned} \tag{3.143}$$

Integrating this gives

$$E = e^{\frac{1}{2}(\nu+\lambda)}\epsilon/r^2 \tag{3.144}$$

where ϵ is the constant of integration. Assuming the solution is asymptotically flat requires

$$\nu, \lambda \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \tag{3.145}$$

and so asymptotically

$$E \sim \epsilon/r^2$$

We therefore interpret ϵ as the charge of the blackhole.

We confirm the other Maxwell's eqations. From the r component

$$\begin{aligned}\nabla_a F^{a1} &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_a(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{a1}) \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_0(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{01}) \\ &= 0 \end{aligned} \tag{3.146}$$

as there is no time-dependence. The other two are zero by spherical symmetry assumption. In particular as $F^{a2} = 0$

$$
\begin{aligned}
\nabla_a F^{a2} &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2 \sin\theta} \partial_a (e^{\frac{1}{2}(\nu+\mu)} r^2 \sin\theta F^{a2}) \\
&= 0. \qquad (3.147)
\end{aligned}
$$

Similarly for $\nabla_a F^{a3} = 0$.

Calculation of the energy-momentum tensor

We employ the ansatz (3.125) together with (3.121) and (3.122) to compute the Maxwell energy momentum tensor:

$$
T_{ab} = \frac{1}{4\pi}(-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}). \qquad (3.148)
$$

First we compute

$$
\begin{aligned}
F_{cd} F^{cd} &= g^{ce} g^{df} F_{cd} F_{ef} \\
&= g^{0e} g^{1f} F_{01} F_{ef} + g^{1e} g^{0f} F_{10} F_{ef} \\
&= 2 g^{0e} g^{1f} F_{01} F_{ef} \\
&= 2 g^{00} g^{11} F_{01} F_{01} \\
&= -2 e^{-\nu-\lambda} E^2. \qquad (3.149)
\end{aligned}
$$

We find the componets of T_{ab}: First T_{00}

$$
\begin{aligned}
T_{00} &= \frac{1}{4\pi}(-g^{cd} F_{0c} F_{0d} + \frac{1}{4} g_{00} F_{cd} F^{cd}) \\
&= \frac{1}{4\pi}(-g^{11} F_{01} F_{01} + \frac{1}{4} g_{00} F_{cd} F^{cd}) \\
&= \frac{1}{4\pi}(e^{-\lambda} E^2 - \frac{1}{2} e^{\nu} e^{-\nu-\lambda} E^2) \\
&= \frac{1}{8\pi} e^{-\lambda} E^2 \qquad (3.150)
\end{aligned}
$$

Next T_{11}

$$
\begin{aligned}
T_{11} &= \frac{1}{4\pi}(-g^{cd}F_{1c}F_{1d} + \frac{1}{4}g_{11}F_{cd}F^{cd}) \\
&= \frac{1}{4\pi}(-g^{00}F_{10}F_{10} + \frac{1}{4}g_{11}F_{cd}F^{cd}) \\
&= -\frac{1}{4\pi}(e^{-\nu}E^2 - \frac{1}{2}e^{\lambda}e^{-\nu-\lambda}E^2) \\
&= -\frac{1}{8\pi}e^{-\nu}E^2 \qquad (3.151)
\end{aligned}
$$

Then T_{22}

$$
\begin{aligned}
T_{22} &= \frac{1}{4\pi}(-g^{cd}F_{2c}F_{2d} + \frac{1}{4}g_{22}F_{cd}F^{cd}) \\
&= \frac{1}{16\pi}g_{22}F_{cd}F^{cd} \\
&= \frac{1}{8\pi}r^2e^{-\nu-\lambda}E^2. \qquad (3.152)
\end{aligned}
$$

It is easy to see that $T_{33} = T_{22}\sin^2\theta$. We compute T_{01}:

$$
\begin{aligned}
T_{01} &= \frac{1}{4\pi}(-g^{cd}F_{0c}F_{1d} + \frac{1}{4}g_{01}F_{cd}F^{cd}) \\
&= -\frac{1}{4\pi}g^{10}F_{01}F_{10} \\
&= 0. \qquad (3.153)
\end{aligned}
$$

In acordance with (3.124), (note in the derivation of the Schwarzschild solution that $R_{01} = 8\pi T_{01} = 0$ implied $\lambda(t,r) = \lambda(r)$ by (3.49)).

The other components of T_{ab} are easily seen to be zero, which are in accordance with (3.124).

Solving the field equations

The equations $R_{00} = 8\pi T_{00}$ and $R_{11} = 8\pi T_{11}$ can be combined

$$e^{\lambda-\nu}R_{00} + R_{11} = 8\pi[e^{\lambda-\nu}T_{00} + T_{11}] \qquad (3.154)$$

By (3.124), (3.150) and (3.151), from the 00 and 11 equations we get

$$\lambda' + \nu' = 0 \tag{3.155}$$

which by (3.145) means that

$$\lambda = -\nu. \tag{3.156}$$

The 22 equation

$$R_{22} = 8\pi T_{22} = r^2 e^{-\nu-\lambda} E^2$$

becomes on using (3.124), (3.144) and (3.156)

$$e^{\nu}(-r\nu' - 1) + 1 = r^2 E^2 = \epsilon^2/r^2 \tag{3.157}$$

or

$$(re^{\nu})' = 1 - \epsilon^2/r^2 \tag{3.158}$$

integrating

$$e^{\nu} = 1 + Const./r + \epsilon^2/r^2 \tag{3.159}$$

(note that because $R_{33} = R_{22}\sin^2\theta$ and $T_{33} = T_{22}\sin^2\theta$ the field equation $R_{33} = 8\pi T_{33}$ gives the same equation as $R_{22} = 8\pi T_{22}$). We obtain the line element

$$ds^2 = \left(1 + \frac{Const.}{r} + \frac{\epsilon^2}{r^2}\right) dt^2 - \left(1 + \frac{Const.}{r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.160}$$

When $\epsilon = 0$, this reduces to the Schwarzschild line element and so

$$Const. = -2MG/c^2.$$

We obtain the Reissner-Nordstrom solution

$$ds^2 = \left(1 - \frac{2MG}{c^2 r} + \frac{\epsilon^2}{r^2}\right) dt^2 - \left(1 - \frac{2MG}{c^2 r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.161}$$

3.6.1 Event Horizons

Recall that at the event horizon was the point after which the coefficients of dr and dt change signs.

The horizon function $H(r)$ is given by

$$H(r) = \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) \tag{3.162}$$

is quadratic with two distinct roots. Write

$$r^2 H(r) = r^2 - 2mr + \epsilon^2 = (r-m)^2 + \epsilon^2 - M^2. \tag{3.163}$$

Notice that we must have $\epsilon^2 \leq m^2$ otherwise the roots would be complex.

The two roots are given by

$$\begin{aligned} r_+ &= m + (m^2 - \epsilon^2)^{1/2} \\ r_- &= m - (m^2 - \epsilon^2)^{1/2} \end{aligned} \tag{3.164}$$

These two roots correspond to two different event horizons, one at r_+ and the other at r_-.

The line element has three regular regions

$$\begin{aligned} I. &\quad r_+ < r < \infty, \\ II. &\quad r_- < r < r_+, \\ III. &\quad 0 < r < r_-. \end{aligned} \tag{3.165}$$

The outer horizon at r_+ is much like the event horizon of $2m$ for a Schwarzschild black hole.

Space and time change roles.

At the inner horizon, the Cauchy horizon, space and time change roles again.

Inside the Cauchy horizon the sigularity is space-like.

As the more charge is entered into the black hole, the inner event horizon gets larger, while the outer event horizon starts to shrink. When we reach the maximum possible charge, i.e. $\epsilon^2 = m^2$, the two horizons merge and only the regions I and III exist.

3.6.2 Analogue of Eddington-Finkelstein Coordinates

$\epsilon^2 < m^2$

$$t = \bar{t} - \frac{r_+^2}{r_+ - r_-} \ln(r - r_+) + \frac{r_-^2}{r_+ - r_-} \ln(r - r_-) \tag{3.166}$$

$$r_+ - r_- = 2(m^2 - \epsilon^2)^{1/2}$$

$$\begin{aligned} r_+ - r_- &= 2(m^2 - \epsilon^2)^{1/2} \\ r_+ + r_- &= 2m && (3.167) \\ r_+ r_- &= \epsilon^2 && (3.168) \end{aligned}$$

$$\begin{aligned} dt &= d\bar{t} - \frac{r_+^2}{r_+ - r_-}\frac{dr}{r - r_+} + \frac{r_-^2}{r_+ - r_-}\frac{dr}{r - r_-} \\ &= d\bar{t} - \frac{dr}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} + \frac{r_-^2}{r - r_-}\right) && (3.169) \end{aligned}$$

$$dt^2 = d\bar{t}^2 - 2\frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right) d\bar{t}dr + \frac{dr^2}{(r_+ - r_-)^2}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)^2 \tag{3.170}$$

Notice that the horizon function is

$$r^2 H(r) = (r - r_+)(r - r_-)$$

$$\begin{aligned} ds^2 &= \frac{(r - r_+)(r - r_-)}{r^2} dt^2 - \frac{r^2}{(r - r_+)(r - r_-)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= \frac{(r - r_+)(r - r_-)}{r^2} d\bar{t}^2 - 2\frac{(r - r_+)(r - r_-)}{r^2}\frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right) d\bar{t}dr \\ &\quad + \frac{(r - r_+)(r - r_-)}{r^2}\frac{dr^2}{(r_+ - r_-)^2}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)^2 \\ &\quad - \frac{r^2}{(r - r_+)(r - r_-)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) && (3.171) \end{aligned}$$

Consider the coefficient of $-2d\bar{t}dr$:

$$\begin{aligned}
& \frac{(r-r_+)(r-r_-)}{r^2}\frac{1}{r_+-r_-}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right) \\
= \quad & \frac{1}{r^2(r_+-r_-)}\left(r_+^2(r-r_-)-r_-^2(r-r_+)\right) \\
= \quad & \frac{1}{r^2(r_+-r_-)}\left(r(r_+^2-r_-^2)-(r_+^2r_--r_-^2r_+)\right) \\
= \quad & \frac{r_++r_-}{r}-\frac{r_+r_-}{r^2} \\
= \quad & \frac{2m}{r}-\frac{\epsilon^2}{r^2}
\end{aligned} \tag{3.172}$$

Consider the coefficient of $-dr^2$:

$$\begin{aligned}
& -\frac{(r-r_+)(r-r_-)}{r^2}\frac{1}{(r_+-r_-)^2}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right)^2+\frac{r^2}{(r-r_+)(r-r_-)} \\
= \quad & -\frac{r^2}{(r-r_+)(r-r_-)}\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2+\frac{r^2}{(r-r_+)(r-r_-)} \\
= \quad & \frac{r^2}{(r-r_+)(r-r_-)}\left[1-\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2\right] \\
= \quad & \frac{1}{1-2m/r-\epsilon^2/r^2}\left[1-\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2\right] \\
= \quad & 1+\frac{2m}{r}-\frac{\epsilon^2}{r^2}
\end{aligned} \tag{3.173}$$

Introduce the function f by

$$f = 1-g_{00} = \frac{2m}{r}-\frac{\epsilon^2}{r^2} \tag{3.174}$$

Then the line element can becomes

$$ds^2 = (1-f)d\bar{t}^2 - 2fd\bar{t}dr - (1+f)dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{3.175}$$

Notice we no longer have any coordinate singularities.

3.6.3 Penrose Diagram

3.6.4 Double null coordinates

Introduce double null coordinates

$$v = \bar{t} + r, \qquad w = 2t - v, \tag{3.176}$$

this implies

$$v = t + r^*, \qquad w = t - r^*, \tag{3.177}$$

where

$$r^* = r + \frac{r_+^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2}{r_+ - r_-} \ln(r - r_-) \tag{3.178}$$

Obviously

$$\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) dvdw = \frac{(r - r_+)(r - r_-)}{r^2}(dt^2 - dr^{*2}) \tag{3.179}$$

The coefficient of dt^2 can seen immediately to be correct. Now

$$dr^* = \left[1 + \frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)\right] dr \tag{3.180}$$

Squaring

$$dr^{*2} = \left[1 + 2\frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right) + \frac{1}{(r_+ - r_-)^2}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)^2\right] dr^2 \tag{3.181}$$

$$
\begin{aligned}
&\frac{(r-r_+)(r-r_-)}{r^2}dr^{*2} \\
= \;& \Big[\frac{(r-r_+)(r-r_-)}{r^2}dr^{*2} \\
&+2\frac{(r-r_+)(r-r_-)}{r^2}\frac{1}{r_+-r_-}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right) \\
&+\frac{(r-r_+)(r-r_-)}{r^2}\frac{1}{(r_+-r_-)^2}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right)^2\Big]dr^2 \\
= \;& \frac{r^2}{(r-r_+)(r-r_-)}\Big[\left(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)^2+ \\
&+2\left(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)+\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2\Big]dr^2 \\
= \;& \frac{r^2}{(r-r_+)(r-r_-)}\Big[1-2\left(\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)+\left(\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)^2 \\
&+2\left(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)+\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2\Big]dr^2 \\
= \;& \frac{r^2}{(r-r_+)(r-r_-)}dr^2. \qquad (3.182)
\end{aligned}
$$

Therefore

$$\left(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)dvdw=\left(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)dt^2-\left(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)^{-1}dr^2$$

Thus the line element can be written in double null coordinates as

$$\left(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}\right)dvdw-r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (3.183)$$

3.6.5 Maximal extension

$\epsilon^2<m^2$, we define new coordinates

$$v''=\tan^{-1}\left(\exp\frac{r_+-r_-}{4r_+^2}v\right), \qquad w''=\tan^{-1}\left(-\exp\frac{r_+-r_-}{4r_+^2}w\right) \qquad (3.184)$$

Inverting we see

$$v = \frac{4r_+^2}{r_+ - r_-} \ln\ \tan v'', \qquad w = -\frac{4r_+^2}{r_+ - r_-} \ln\ \tan w'' \tag{3.185}$$

As

$$d \ln \tan x = \frac{d \tan x}{\tan x} = \frac{1}{\sin x \cos x} dx = \frac{2}{\sin 2x} dx = 2\text{cosec} 2x\ dx$$

we get

$$dvdw = -4^2 \times 2^2 \left(\frac{4r_+^2}{r_+ - r_-} \right)^2 \text{cosec} 2v'' \text{cosec} 2w''\ dv'' dw''$$

Consider the product $\tan v'' \tan w''$

$$\begin{aligned}
& \tan v'' \tan w'' \\
= & -\exp\left(\frac{r_+ - r_-}{4r_+^2} v \right) \exp\left(\frac{r_- - r_+}{4r_+^2} w \right) \\
= & -\exp\left(\frac{r_+ - r_-}{4r_+^2} (v - w) \right) \\
= & -\exp\left(\frac{r_+ - r_-}{2r_+^2} r^* \right) \\
= & -\exp\left(\frac{r_+ - r_-}{2r_+^2} r \right) \\
\times & \exp\left(\frac{r_+ - r_-}{2r_+^2} \left[\frac{r_+^2}{r_+ - r_-} \ln(r - r_+) \right] \right) \exp\left(\frac{r_+ - r_-}{2r_+^2} \left[\frac{r_-^2}{r_+ - r_-} \ln(r - r_-) \right] \right) \\
= & -\exp\left(\frac{r_+ - r_-}{2r_+^2} r \right) (r - r_+)^{1/2} (r - r_-)^{r_-^2 1/2r_+^2}.
\end{aligned} \tag{3.186}$$

Therefore the line element has the form

$$\begin{aligned}
ds^2 &= -64 \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2} \right) \frac{4r_+^4}{(r_+ - r_-)^2} \text{cosec} 2v'' \text{cosec} 2w''\ dv'' dw'' \\
&= -r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\end{aligned} \tag{3.187}$$

where r is defined implicitly by

$$\tan v'' \tan w'' = -\exp\left(\frac{r_+ - r_-}{2r_+^2} r\right)(r - r_+)^{1/2}(r - r_-)^{r_-^2/2r_+^2}. \tag{3.188}$$

This line element is the analogue of the Kruskel solution and represents the maximal analytic extension of the Reissner-Nordstrm solution for $\epsilon^2 < m^2$.

The Penrose diagram for this maximal extension is shown in fig (3.6.5)

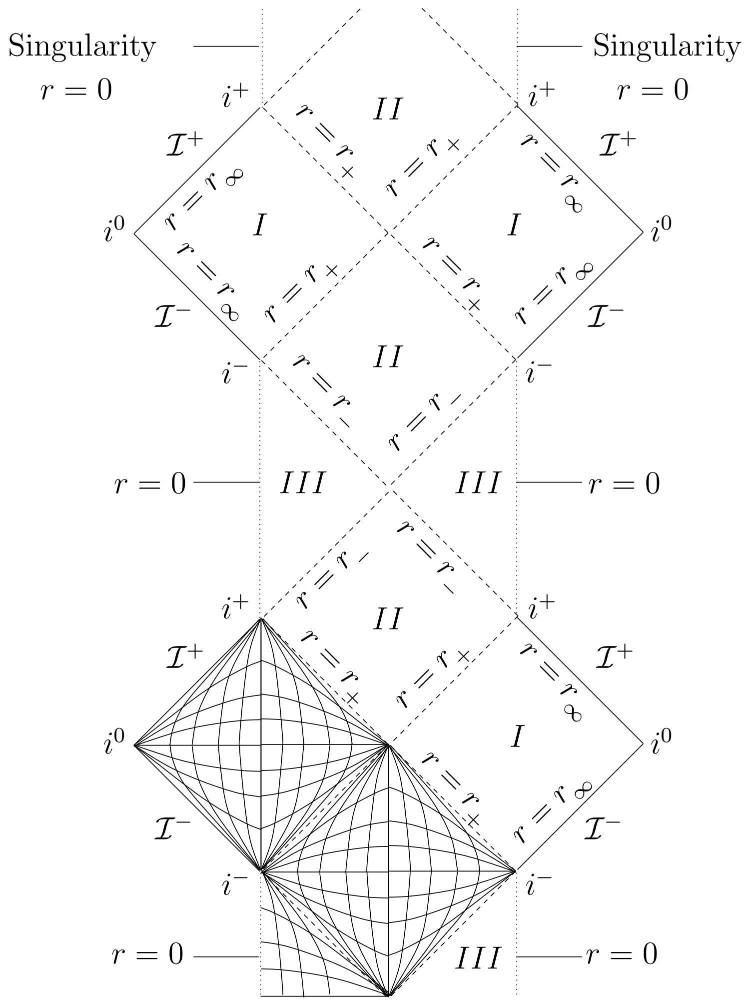


Figure 3.5: Penrose diagram of a black hole.

3.7 Rotating Black Holes

3.7.1 Field Equations

Degenerate metric

A metric of the form

$$g_{ab} = \eta_{ab} - 2ml_a l_b, \quad l_a l_b \eta^{ab} = 0, \quad m \text{ is arbitrary.} \tag{3.189}$$

is called a degenerate metric. The Schwarzschild metric is an example as can be seen from the Eddington-Finkelstein form (3.63)

$$\begin{aligned}
ds^2 &= \left(1 - \frac{2M}{r}\right) d\bar{t}^2 - \frac{4m}{r} d\bar{t}dr - \left(1 - \frac{2M}{r}\right) dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&= d\bar{t}^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{2m}{r}(d\bar{t} + dr)^2 \\
&= d\bar{t}^2 - (dx^2 + dy^2 + dz^2) - \frac{2m}{r}\left(d\bar{t} + \frac{xdx + ydy + zdz}{r}\right)^2
\end{aligned} \tag{3.190}$$

Therefore the Schwarzschild metric can be written in degenerate form with l_a given by

$$l_a = \frac{1}{\sqrt{r}}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{3.191}$$

It turns out that the solution for a rotating black-hole is also of the form of a degenerate metric. Let us proceed. Define

$$l^a := \eta^{ab} l_b. \tag{3.192}$$

The inverse matrix of g_{ab} is

$$g^{ab} = \eta^{ab} + 2ml^a l^b \tag{3.193}$$

as is easily seen:

$$\begin{aligned}
g_{ac}g^{cb} &= (\eta_{ac} - 2ml_a l_c)(\eta^{cb} + 2ml^c l^b) \\
&= \delta_a{}^b + 2ml_a l^b - 2ml^b l_a + 4m^2 l_a l^b l_c l^c \\
&= \delta_a{}^b
\end{aligned}$$

It follows from (3.193) that the contravariant four vector corresponding to l_a is the same as $\eta^{ab}l_b$:

$$l^a := g^{ac}l_c = \eta^{ac}l_c. \tag{3.194}$$

Therefore its indices han be raised and lowered by either the true metric or the Lorentz metric. Since l_a is null it has the property

$$0 = \frac{1}{2}\partial_d(\eta^{ab}l_a l_b) = l^c\partial_d l_c = l_c\partial_d l^c. \tag{3.195}$$

We can consider, g, the determinat of the metric. At any point l_a is a flat-space null vector: $l_a l_b \eta^{ab} = 0$. We can perform a proper rotation of coordinates in three-space that leaves η^{ab} invariant and brings l_a into the form

$$\begin{pmatrix} a \\ a \\ 0 \\ 0 \end{pmatrix}, \tag{3.196}$$

see

$$(a, a, 0, 0)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ a \\ 0 \\ 0 \end{pmatrix} = 0. \tag{3.197}$$

In this system we have

$$\begin{aligned} g &= \begin{vmatrix} 1-2ma^2 & -2ma^2 & 0 & 0 \\ -2ma^2 & -1-2ma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 1-2ma^2 & -2ma^2 \\ -2ma^2 & -1-2ma^2 \end{vmatrix} \\ &= -(1-2ma^2)(1+2ma^2) - 4m^2a^4 \\ &= -1 \end{aligned} \tag{3.198}$$

Since a three-dimensional rotation has a unit Jacobian, the metric transforms as a scalar under this transformation. Thus, $g = -1$ for any degenerate metric. It follows that

$$\Gamma^a_{ca} = \frac{\partial}{\partial x^c} \ln \sqrt{-g} = 0 \tag{3.199}$$

This simplifies the field equations $R_{ab} = 0$,

$$R_{ab} = -\partial_c \Gamma^c_{ab} + \Gamma^d_{ac}\Gamma^c_{db} = 0 \tag{3.200}$$

Now R_{ab} involves different powers of m and since m is arbitrary each order must vanish separately. If we note that the Christoffel symbol of the first kind, $\Gamma_{ab,d}$, is linear in m, power counting is easy and we isolate terms corresponding to different powers of m,

$$\begin{aligned}
R_{ab} &= -\partial_c \Gamma^c_{ab} + \Gamma^d_{ac}\Gamma^c_{db} \\
&= -\partial_c (g^{cd}\Gamma_{ab,d}) + g^{cd}\Gamma_{ea,d}\, g^{ef}\Gamma_{cb,f} \\
&= -\partial_c \big((\eta^{cd} + 2ml^c l^d)\Gamma_{ab,d}\big) + (\eta^{cd} + 2ml^c l^d)(\eta^{ef} + 2ml^e l^f)\, \Gamma_{ea,d}\Gamma_{cb,f} \\
&= -\eta^{cd}\partial_c \Gamma_{ab,d} \\
&\quad -2m\partial_c \big(l^c l^d \Gamma_{ab,d}\big) + \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} \\
&\quad +2m(l^c l^d \eta^{ef} + l^e l^f \eta^{cd})\Gamma_{ea,d}\Gamma_{cb,f} \\
&\quad +4m^2 l^c l^d l^e l^f \Gamma_{ea,d}\Gamma_{cb,f}
\end{aligned} \tag{3.201}$$

We just rewrite one term by swapping the dummy variables d and f:

$$l^c l^d \eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} = l^c l^f \eta^{ed}\Gamma_{ea,f}\Gamma_{cb,d}$$

This gives four sets of ten equations:

$$\begin{aligned}
&-\eta^{cd}\partial_c \Gamma_{ab,d} = 0 && \mathcal{O}(m) \\
&2m\partial_c \big(l^c l^d \Gamma_{ab,d}\big) - \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} = 0 && \mathcal{O}(m^2) \\
&l^e l^f \eta^{cd}\Gamma_{ea,d}\Gamma_{cb,f} + l^c l^f \eta^{ed}\Gamma_{ea,f}\Gamma_{cb,d} = 0 && \mathcal{O}(m^3) \\
&4m^2 l^c l^d l^e l^f \Gamma_{ea,d}\Gamma_{cb,f} = 0 && \mathcal{O}(m^4)
\end{aligned} \tag{3.202}$$

Order m^4 equations:

We need to satisfy

$$l^c l^d l^e l^f \Gamma_{ea,d}\Gamma_{cb,f} = 0$$

From the form of the degenerate metric (3.189) we have

$$\partial_a g_{bc} = -2m\partial_a(l_b l_c). \tag{3.203}$$

We will use this a number of times in what follows. Consider the part $l^d l^e \Gamma_{ea,d}$

$$\begin{aligned}
l^d l^e \Gamma_{ea,d} &= l^d l^e \frac{1}{2} l^e(\partial_e g_{ad} + \partial_a g_{ed} - \partial_d g_{ea}) \\
&= -ml^d l^e \big(\partial_e(l_d l_a) + \partial_a(l_d l_e) - \partial_d(l_e l_a)\big) \\
&= -m\big[l^e l_a(l^d \partial_e l_d) + (l^d l_d) l^e \partial_e l_a + \\
&\quad + (l^e l_e) l^d \partial_a l_d + (l^d l_d) l^e \partial_a l_e \\
&\quad - l^d l_a(l^e \partial_d l_e) - (l^e l_e) l^d \partial_d l_a \,\big] \\
&= 0
\end{aligned} \tag{3.204}$$

Therefore the m^4 equations are automatically stisfied by degenerate metrics.

Order m^3 equations:

$$l^e l^f \eta^{cd} \Gamma_{ea,d} \Gamma_{cb,f} + l^c l^f \eta^{ed} \Gamma_{ea,f} \Gamma_{cb,d} = 0 \tag{3.205}$$

First note that

$$l^c l^f \eta^{ed} \Gamma_{ea,f} \Gamma_{cb,d} = l^e l^f \eta^{cd} \Gamma_{eb,d} \Gamma_{ca,f} \tag{3.206}$$

shows the second term in on the LHS (3.205) is the same as the first term with a and b exchanged. Thus the field equations are

$$l^e l^f \eta^{cd} \Gamma_{ea,d} \Gamma_{cb,f} + a \leftrightarrow b = 0 \tag{3.207}$$

Consider

$$\begin{aligned}
l^e \Gamma_{ea,d} &= \frac{1}{2} l^e(\partial_e g_{ad} + \partial_a g_{ed} - \partial_d g_{ea}) \\
&= -ml^e[\partial_e(l_a l_d) + \partial_a(l_e l_d) - \partial_d(l_e l_a)] \\
&= -ml^e \partial_e(l_a l_d),
\end{aligned} \tag{3.208}$$

where again we have used (3.203), and also consider

$$
\begin{aligned}
l^f \Gamma_{cb,f} &= \frac{1}{2} l^f (\partial_c g_{bf} + \partial_b g_{cf} - \partial_f g_{cb}) \\
&= m l^f \partial_f (l_c l_b).
\end{aligned} \tag{3.209}
$$

Combining them we have

$$
\begin{aligned}
l^e l^f \eta^{cd} \Gamma_{ea,d} \Gamma_{cb,f} &= -m^2 l^e l^f \eta^{cd} \partial_e (l_a l_d) \partial_f (l_c l_b) \\
&= -m^2 l^e l^f \eta^{cd} [l_d \partial_e l_a + l_a \partial_e l_d][l_b \partial_f l_c + l_c \partial_f l_b] \\
&= -m^2 l^e l^f \eta^{cd} [l_d l_b (\partial_e l_a)(\partial_f l_c) + l_a l_b (\partial_e l_d)(\partial_f l_c) + l_a l_d \partial_e l_c \partial_f l_b].
\end{aligned} \tag{3.210}
$$

Let us consider the terms separately, first,

$$
\begin{aligned}
-m^2 l^e l^f \eta^{cd} l_d l_b (\partial_e l_a)(\partial_f l_c) &= ...\eta^{cd} l_d \partial_f l_c \\
&= ...l^c \partial_f l_c \\
&= 0.
\end{aligned} \tag{3.211}
$$

where we have used (3.195). Next

$$
-m^2 l^e l^f \eta^{cd} l_a l_b (\partial_e l_d)(\partial_f l_c) = -m^2 l_a l_b (l^e \partial_e l_d)(l^f \partial_f l^d) \tag{3.212}
$$

by a simple rearangement of terms. Next

$$
\begin{aligned}
-m^2 l^e l^f \eta^{cd} l_a l_d \partial_e l_c \partial_f l_b &= ...\eta^{cd} l_d \partial_e l_c \\
&= ...l^c \partial_e l_c \\
&= 0.
\end{aligned} \tag{3.213}
$$

where we have used (3.195) again.

Therefore

$$
l^e l^f \eta^{cd} \Gamma_{ea,d} \Gamma_{cb,f} = -m^2 l_a l_b (l^e \partial_e l_d)(l^f \partial_f l^d) \tag{3.214}
$$

Using this in (3.207), the m^3 equations lead to

$$-m^2 l_a l_b (v^c v_c) = 0 \tag{3.215}$$

where

$$v^c := l^d \partial_d l^c. \tag{3.216}$$

Therefore v^c is a null vector. It is also orthogonal to the null vector l^c, as easily seen

$$v^b l_b = (l^c \partial_c l^b) l_b = l^c (l_b \partial_c l^b) = 0 \tag{3.217}$$

where we have used (3.195). Notice that

$$\begin{aligned} v_a &= g_{ab} v^b \\ &= (\eta_{ab} - 2m l_a l_b) v^b \\ &= \eta_{ab} v^b \end{aligned} \tag{3.218}$$

so that the indices of v^c can be lowered and raised with the Lorentz metric, as with l^c. At any chosen point l^a and v^a may be written

$$l^b = (|\vec{l}|, \vec{l}), \qquad v^b = (|\vec{v}|, \vec{v}) \tag{3.219}$$

where $\vec{l}$ an $\vec{v}$ are ordinary three-vectors in Eucldean space. If θ is the angle between $\vec{l}$ an $\vec{v}$ then

$$\cos\theta = \frac{\vec{l} \cdot \vec{v}}{|\vec{l}|\ |\vec{v}|} \tag{3.220}$$

Now because l^b and v^b are orthogonal (from (3.217)),

$$\begin{aligned} l^b v_b &= l^a v^b \eta_{ab} = l^0 v^0 - \vec{l} \cdot \vec{v} \\ &= |\vec{l}||\vec{v}|(1 - \cos\theta) = 0 \end{aligned} \tag{3.221}$$

Thus $\cos\theta = 1$, and so $\vec{v}$ is parallel to $\vec{l}$ at any given point. We may therefore write

$$v^b = l^c \partial_c l^b = -A(x^a) l^b \tag{3.222}$$

where A is a scalar field.

We will next consider the linear order in m equations and return to the order m^2 equations later, where we will show that they are identically satisified.

Equation from linear in m equation

Using (3.203) in the linear in m equation

$$-\eta^{cd}\partial_c\Gamma_{ab,d} = 0 \tag{3.223}$$

gives

$$\begin{aligned}
-\eta^{cd}\partial_c\Gamma_{ab,d} &= -\eta^{cd}\partial_c[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}] \\
&= 2m\eta^{cd}[-\partial_c\partial_d(l_a l_b) + \partial_c\partial_a(l_b l_d) + \partial_c\partial_b(l_a l_d)] \\
&= 2m[-\Box^2(l_a l_b) + \eta^{cd}\partial_c\partial_a(l_b l_d) + \eta^{cd}\partial_c\partial_b(l_a l_d)] \\
&= 0
\end{aligned} \tag{3.224}$$

where we have introduced the D'Alembertian opertator

$$\eta^{cd}\partial_c\partial_d = \Box^2 = \frac{\partial^2}{\partial x^{02}} - \nabla^2.$$

Define

$$L := -\partial_a l^a. \tag{3.225}$$

Expanding $\eta^{cd}\partial_c(l_b l_d)$ gives

$$\begin{aligned}
\eta^{cd}\partial_c(l_b l_d) &= \eta^{cd} l_d \partial_c l_b + \eta^{cd} l_b \partial_c l_d \\
&= -((-l^c\partial_c l_b) - (l_b\partial_c l^c)) \\
&= -(L + A) l_b
\end{aligned} \tag{3.226}$$

implying

$$\eta^{cd}\partial_c\partial_a(l_b l_d) = \partial_a\big(\eta^{cd}\partial_c(l_b l_d)\big) = -\partial_a[(L + A) l_b]$$

Substituting this into equation (3.224) we obtain the equation for l_a

$$-\Box^2(l_a l_b) = \partial_b[(L+A)l_a] + \partial_a[(L+A)l_b] \tag{3.227}$$

We use this equation in the following to prove the order m^2 equations are satisfied, proving the order m^2 equations are automatically satisfied by any solution to (3.227). The entire content of the field equations are thus embodied in (3.227). After dealing with the m^2 equations we will specialise (3.227) to the stationary case where all x^0 derivatives vanish.

Order m^2 equations

Recall the order m^2 equations

$$2m\partial_c\big(l^c l^d \Gamma_{ab,d}\big) - \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} = 0. \tag{3.228}$$

Consider the first term, again we use (3.203), we have

$$\begin{aligned}
2m\partial_c\big(l^c l^d \Gamma_{ab,d}\big) &= \frac{2m}{2}\partial_c[l^c l^d(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})] \\
&= -2m^2\partial_c[l^c l^d\big(\partial_a(l_b l_d) + \partial_b(l_a l_d) - \partial_d(l_a l_b)\big)]
\end{aligned} \tag{3.229}$$

Looking at the first term in the brackets

$$l^d\partial_a(l_b l_d) = l_b(l^d\partial_a l_d) + (l^d l_d)\partial_a l_b = 0$$

and similarly the second

$$l^d\partial_b(l_a l_d) = 0.$$

So

$$
\begin{aligned}
2m\partial_c\big(l^c l^d \Gamma_{ab,d}\big) &= 2m^2\partial_c[l^c l^d(l_b\partial_d l_a + l_a\partial_d l_b)] \\
&= 2m^2\partial_c[l^c l^d l_b\partial_d l_a + l^c l^d l_a\partial_d l_b)] \\
&= 2m^2\partial_c[(l^d\partial_d l_a)l^c l_b + (l^d\partial_d l_b)l^c l_a] \\
&= 2m^2\partial_c[(-Al_a)l^c l_b + (-Al_b)l^c l_a] \\
&= -4m^2\partial_c[(Al^c)(l_a l_b)] \\
&= -4m^2[l_a l_b\partial_c(Al^c) + Al^c\partial_c(l_a l_b)] \\
&= -4m^2[l_a l_b\partial_c(Al^c) + Al^c l_b\partial_c l_a + Al^c l_a\partial_c l_b] \\
&= -4m^2[l_a l_b\partial_c(Al^c) + Al_b(l^c\partial_c l_a) + Al_a(l^c\partial_c l_b)] \\
&= -4m^2[l_a l_b\partial_c(Al^c) - 2A^2 l_a l_b] \\
&= -4m^2 l_a l_b[\partial_c(Al^c) - 2A^2]
\end{aligned}
\tag{3.230}
$$

So that

$$
2m\partial_c\big(l^c l^d\Gamma_{ab,d}\big) = -2m^2 l_a l_b[2\partial_c(Al^c) - 4A^2] \tag{3.231}
$$

Now lets us look at the second term in (3.228)

$$
-\eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f}.
$$

We have

$$
\begin{aligned}
-\eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} &= -\frac{\eta^{cd}\eta^{ef}4m^2}{4}[\partial_e(l_a l_d) + \partial_a(l_e l_d) - \partial_d(l_e l_a)]\times \\
&\quad \times[\partial_c(l_b l_f) + \partial_b(l_c l_f) - \partial_f(l_c l_b)] \\
&= -m^2\eta^{cd}\eta^{ef}[\partial_e(l_a l_d)\partial_c(l_b l_f) + \partial_e(l_a l_d)\partial_b(l_c l_f) - \partial_e(l_a l_d)\partial_f(l_c l_b) \\
&\qquad \partial_a(l_e l_d)\partial_c(l_b l_f) + \partial_a(l_e l_d)\partial_b(l_c l_f) - \partial_a(l_e l_d)\partial_f(l_c l_b) \\
&\qquad -\partial_d(l_e l_a)\partial_c(l_b l_f) - \partial_d(l_e l_a)\partial_b(l_c l_f) + \partial_d(l_e l_a)\partial_f(l_c l_b)].
\end{aligned}
\tag{3.232}
$$

Tedious but straightforward calculations give

$$
\begin{array}{ll}
1. & \eta^{cd}\eta^{ef}\partial_e(l_a l_d)\partial_c(l_b l_f) = l_a l_b[3A^2 + \partial_d l^c \partial_c l^d] \\
2. & \eta^{cd}\eta^{ef}\partial_e(l_a l_d)\partial_b(l_c l_f) = 0 \\
3. & -\eta^{cd}\eta^{ef}\partial_e(l_a l_d)\partial_f(l_c l_b) = -l_a l_b(\partial_d l^c \partial^d l_c) \\
4. & \eta^{cd}\eta^{ef}\partial_a(l_e l_d)\partial_c(l_b l_f) = 0 \\
5. & \eta^{cd}\eta^{ef}\partial_a(l_e l_d)\partial_b(l_c l_f) = 0 \\
6. & -\eta^{cd}\eta^{ef}\partial_a(l_e l_d)\partial_f(l_c l_b) = 0 \\
7. & -\eta^{cd}\eta^{ef}\partial_d(l_e l_a)\partial_c(l_b l_f) = -l_a l_b(\partial_d l^c \partial^d l_c) \\
8. & -\eta^{cd}\eta^{ef}\partial_d(l_e l_a)\partial_b(l_c l_f) = 0 \\
9. & \eta^{cd}\eta^{ef}\partial_d(l_e l_a)\partial_f(l_c l_b) = l_a l_b[3A^2 + \partial_d l^c \partial_c l^d]
\end{array} \tag{3.233}
$$

so that

$$
\begin{aligned}
-\eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} &= -m^2[1. + \cdots + 9.] \\
&= -2m^2 l_a l_b[3A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c]
\end{aligned} \tag{3.234}
$$

Putting together (3.231) and (3.234)

$$
\begin{aligned}
& 2m\partial_c\big(l^c l^d \Gamma_{ab,d}\big) - \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} \\
= \; & -2m^2 l_a l_b[2\partial_c(l^c A) - 4A^2] \\
& - 2m^2 l_a l_b[3A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c] \\
= \; & -2m^2 l_a l_b[2\partial_c(l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c]
\end{aligned} \tag{3.235}
$$

we see the m^2 order equations vanish when

$$
l_a l_b[2\partial_c(l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c] = 0. \tag{3.236}
$$

which implies the scalar equation

$$
2\partial_c(l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c = 0. \tag{3.237}
$$

Let us consider the third term, using $l^d \partial_d l^c = -A l^c$ and $L = -\partial_c l^c$

$$\begin{aligned}\partial_d l^c \partial_c l^d &= \partial_c(l^d \partial_d l^c) - l^d \partial_c \partial_d l^c \\ &= \partial_c(-A l^c) + l^d \partial_d L \\ &= \partial_c[(L-A) l^c] + L^2 \end{aligned} \tag{3.238}$$

Similarly we find for the fourth term using $l^c \partial^d l_c = 0$

$$\begin{aligned}\partial_d l^c \partial^d l_c &= \partial_d(l^c \partial^d l_c) - l^c \partial_d \partial^d l_c \\ &= -l^c \partial_d \partial^d l_c \end{aligned} \tag{3.239}$$

Expanding (3.227) gives

$$\begin{aligned}&-l_a \partial_d \partial^d l_b - l_b \partial_d \partial^d l_a - 2(\partial^d l_a)(\partial_d l_b) \\ &\quad = l_b \partial_a (L+A) + l_a \partial_b (L+A) + (L+A)(\partial_b l_a + \partial_a l_b). \end{aligned} \tag{3.240}$$

Contracting with l^a gives

$$\begin{aligned}-l_b l^a \partial_d \partial^d l_a &= l^a l_b \partial_a (L+A) + (L+A) l^a \partial_a l_b \\ &= l_b l^a \partial_a (L+A) - (L+A) A l_b \end{aligned} \tag{3.241}$$

(where we have used $l^a l_a = 0$ as well as $l^a \partial_b l_a = 0$ and $l^a \partial_a l_b = -A l_b$). As l_b is a common factor we get

$$\begin{aligned}-l^a \partial_d \partial^d l_a &= l^a \partial_a (L+A) - (L+A) A \\ &= \partial_a[(L+A) l^a] - \partial_a l^a (L+A) - (L+A) A \\ &= \partial_a[(L+A) l^a] + L^2 - A^2 \end{aligned} \tag{3.242}$$

(where we have used the definition $L := -\partial_a l^a$). By using (3.239) the left hand side of this equation can be replaced by $\partial_d l^c \partial^d l_c$. Thus the fourth term of (3.237) can be written

$$\partial_d l^c \partial^d l_c = L^2 - A^2 + \partial_a[(L+A) l^a]. \tag{3.243}$$

We now sustitute (3.238) and (3.243) into the LHS of the scalar equation (3.237), to obtain

$$
\begin{aligned}
&2\partial_c(l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c = \\
&\quad 2\partial_c(l^c A) - A^2 + \left(\partial_a[(L-A)l^a] + L^2\right) - \left(\partial_a[(L+A)l^a] + L^2 - A^2\right) \\
&\quad 2\partial_c(l^c A) - 2\partial_c(l^c A) \\
&\quad 0
\end{aligned} \tag{3.244}
$$

Thus any solution of (3.227) makes the m^2 equations vanish identically.

Stationary field equations from the m order equation

We have found that the full content of the field equations is embodied in

$$
-\Box^2(l_a l_b) = \partial_b[(L+A)l_a] + \partial_a[(L+A)l_b]. \tag{3.245}
$$

We now consider the stationary, or time-independent, case. We will find a simlpification to the algebraic manipulations will occur with the introduction of the three-vector λ_j via

$$
l_a = l_0(1, \lambda_1, \lambda_2, \lambda_3) \tag{3.246}
$$

Since l_a is a flat-space null vector ($l_a l_b \eta^{ab} = 0$), λ_j is a flat-space unit vector,

$$
\vec{\lambda}^2 = 1.
$$

For $a = b = 0$, in the time-independent case we are now considering, (3.245) reduces to

$$
\nabla^2(l_0^2) = 0 \tag{3.247}
$$

For $a = 0$, $b = j \neq 0$ (3.245) reduces to

$$
\nabla^2(l_0^2 \lambda_j) = \partial_j[(L+A)l_0] \tag{3.248}
$$

For $a = i \neq 0$, $b = j \neq 0$ (3.245) reduces to

$$
\nabla^2(l_0^2 \lambda_i \lambda_j) = \partial_j[(L+A)l_0\lambda_i] + \partial_i[(L+A)l_0\lambda_j] \tag{3.249}
$$

We can take (3.249) and simplify it to a first-order diferential equation by using (3.248) and (3.247). First expand the RHS of (3.249) and then use (3.248),

$$\begin{aligned}\nabla^2(l_0^2\lambda_i\lambda_j) &= \partial_j[(L+A)l_0\lambda_i] + \partial_i[(L+A)l_0\lambda_j] \\ &= \lambda_i\partial_j[(L+A)l_0] + \lambda_j\partial_i[(L+A)l_0] + (L+A)l_0(\partial_j\lambda_i + \partial_i\lambda_j) \\ &= \lambda_i\nabla^2(l_0^2\lambda_j) + \lambda_j\nabla^2(l_0^2\lambda_i) + (L+A)l_0(\partial_j\lambda_i + \partial_i\lambda_j) \qquad (3.250)\end{aligned}$$

Rearanging gives

$$\partial_j\lambda_i + \partial_i\lambda_j = \frac{1}{(L+A)l_0}[\nabla^2(l_0^2\lambda_i\lambda_j) - \lambda_i\nabla^2(l_0^2\lambda_j) - \lambda_j\nabla^2(l_0^2\lambda_i)] \qquad (3.251)$$

Let us expand the derivatives in $\nabla^2(l_0^2\lambda_i\lambda_j)$:

$$\begin{aligned}\nabla^2(l_0^2\lambda_i\lambda_j) &= \partial_k\partial_k(l_0^2\lambda_i\lambda_j) \\ &= \partial_k[\lambda_i\lambda_j\partial_k(l_0^2) + l_0^2\lambda_j\partial_k\lambda_i + l_0^2\lambda_i\partial_k\lambda_j] \\ &= \lambda_i\lambda_j\nabla^2(l_0^2) + \lambda_j\partial_k(l_0^2)(\partial_k\lambda_i) + \partial_k(l_0^2)\lambda_i\partial_k\lambda_j \\ &+ \partial_k(l_0^2)(\partial_k\lambda_i)\lambda_j + l_0^2(\partial_k\partial_k\lambda_i)\lambda_j + l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \\ &+ \partial_k(l_0^2)\lambda_i\partial_k\lambda_j + l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) + l_0^2\lambda_i\partial_k\partial_k\lambda_j \\ \\ &= \lambda_i\lambda_j\nabla^2(l_0^2) + 2(\partial_k(l_0^2))(\lambda_j\partial_k\lambda_i + \lambda_i\partial_k\lambda_j) \\ &+ l_0^2(\lambda_j\nabla^2\lambda_i + \lambda_i\nabla^2\lambda_j) + 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j). \qquad (3.252)\end{aligned}$$

Let us expand the derivatives in $\lambda_i\nabla^2(l_0^2\lambda_j)$:

$$\begin{aligned}\lambda_i\nabla^2(l_0^2\lambda_j) &= \lambda_i\partial_k\partial_k(l_0^2\lambda_j) \\ &= \lambda_i\partial_k\big((\partial_k l_0^2)\lambda_j + l_0^2\partial_k\lambda_j\big) \\ &= \lambda_i\lambda_j\nabla^2(l_0^2) + 2(\partial_k(l_0^2))(\lambda_i\partial_k\lambda_j) + l_0^2\,\lambda_i\nabla^2\lambda_j. \qquad (3.253)\end{aligned}$$

With these results, let us expand the content of the square brackets on the RHS of (3.251)

$$\begin{aligned}&\nabla^2(l_0^2\lambda_i\lambda_j) - \lambda_i\nabla^2(l_0^2\lambda_j) - \lambda_j\nabla^2(l_0^2\lambda_i) \\ =\ & (\nabla^2(l_0^2))\lambda_i\lambda_j + 2\partial_k(l_0^2)(\lambda_j\partial_k\lambda_i + \lambda_i\partial_k\lambda_j) \\ +\ & l_0^2(\lambda_j\nabla^2\lambda_i + \lambda_i\nabla^2\lambda_j) + 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \\ -\ & \lambda_i\lambda_j\nabla^2(l_0^2) - 2(\partial_k l_0^2)\lambda_i\partial_k\lambda_j - l_0^2\lambda_i\nabla^2\lambda_j \\ -\ & \lambda_j\lambda_i\nabla^2(l_0^2) - 2(\partial_k l_0^2)\lambda_j\partial_k\lambda_i - l_0^2\lambda_j\nabla^2\lambda_i \\ =\ & -(\nabla^2(l_0^2))\lambda_i\lambda_j + 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \\ =\ & 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \qquad (3.254)\end{aligned}$$

where we used (3.247) in the last line. We get

$$\partial_j \lambda_i + \partial_i \lambda_j = \frac{2l_0}{L+A}(\partial_k \lambda_i)(\partial_k \lambda_j) \tag{3.255}$$

Define

$$p := \frac{L+A}{2l_0} \tag{3.256}$$

then

$$\partial_j \lambda_i + \partial_i \lambda_j = \frac{1}{p}(\partial_k \lambda_i)(\partial_k \lambda_j) \tag{3.257}$$

The gravitational field is now described by (3.247), (3.248), and (3.257).

Let

$$M_{ik} := \partial_k \lambda_i \tag{3.258}$$

Then (3.257) becomes

$$M_{ij} + M_{ji} = M_{ik}(M^T)_{kj} \tag{3.259}$$

or

$$M + M^T = \frac{1}{p} M M^T \tag{3.260}$$

The constant length of λ_j, implies

$$\frac{1}{2}\partial_k(\lambda_i \lambda_i) = \lambda_i \partial_k \lambda_i = 0 \tag{3.261}$$

or

$$M^T \lambda = 0. \tag{3.262}$$

That is $\vec{\lambda}$ is in the null space of M^T. Moreover, $l^c \partial_c l_b = -A l_b$ with $b = 0$ gives

$$\lambda_j \partial_j l_0 = A \tag{3.263}$$

and with $b = k \neq 0$

$$\lambda_j \partial_j (l_0 l_k) = A\lambda_k \tag{3.264}$$

where we used the definition of λ given in (3.246), but

$$\begin{aligned} \lambda_j \partial_j (l_0 \lambda_k) &= l_0(\lambda_j \partial_j \lambda_k) + \lambda_k \lambda_j \partial_j l_0 \\ &= l_0(\lambda_j \partial_j \lambda_k) + A\lambda_k \end{aligned} \tag{3.265}$$

where we used (3.263). By comparing (3.264) and (3.265) we see that

$$\lambda_j \partial_j \lambda_k = 0 \tag{3.266}$$

or

$$M\lambda = 0. \tag{3.267}$$

We will now be able to solve (3.260), (3.262), and (3.267) for M as a function of $\vec{\lambda}$. Consider a rotation such that

$$R\lambda = \lambda' \tag{3.268}$$

where

$$\lambda' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.269}$$

If $\vec{\lambda}$ is in the null space of M and M^T, then $\vec{\lambda}'$ is in the null space of M' and M'^T, where

$$M' = RMR^T, \qquad M'^T = RM^T R^T. \tag{3.270}$$

Now a rotation matrix satisfies $RR^T = I$. Let us write this out in component form,

$$RR^T = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Writing out some of the terms

$$\begin{aligned} R_{11}R_{11} + R_{12}R_{12} + R_{13}R_{13} &= 1 \\ R_{21}R_{11} + R_{22}R_{12} + R_{23}R_{13} &= 0 \\ \vdots \quad + \quad \vdots \quad + \quad \vdots &= \vdots \end{aligned}$$

Generally:

$$R_{i1}R_{k1} + R_{i2}R_{k2} + R_{i3}R_{k3} = \delta_{ik} \tag{3.271}$$

In other words the three rows constitue three orthonormal vectors in 3 spacial dimensions. Define

$$\begin{aligned} \vec{R}_1 &= (R_{11}, R_{12}, R_{13}) \\ \vec{R}_2 &= (R_{21}, R_{22}, R_{23}) \\ \vec{R}_3 &= (R_{31}, R_{32}, R_{33}) \end{aligned}$$

We are considering rotation such that

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.272}$$

Notice that you could exchange the second and third rows of R without changing (3.272). Let us choose the ordering such that:

$$\vec{R}_2 \times \vec{R}_3 = \vec{R}_1. \tag{3.273}$$

We will use this later. From that

$$\lambda' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and that $M'\lambda' = 0$ and $M'^T\lambda' = 0$ we see that M' must have the form

$$M' = \begin{pmatrix} 0 & a & b \\ c & N'_{11} & N'_{12} \\ d & N'_{21} & N'_{22} \end{pmatrix} \tag{3.274}$$

The values a, b, c, d can be determined from the property of M',

$$M' + M'^T = \frac{1}{p} M' M'^T \tag{3.275}$$

We have first

$$M' + M'^T = \begin{pmatrix} 0 & a+c & b+d \\ a+c & \cdot & \cdot \\ b+d & \cdot & \cdot \end{pmatrix} \tag{3.276}$$

but then

$$\begin{aligned} \frac{1}{p} M' M'^T &= \frac{1}{p} \begin{pmatrix} 0 & a & b \\ c & \cdot & \cdot \\ d & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 & c & d \\ a & \cdot & \cdot \\ b & \cdot & \cdot \end{pmatrix} \\ &= \frac{1}{p} \begin{pmatrix} a^2+b^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{aligned} \tag{3.277}$$

which implies $a^2 + b^2 = 0$ or $a = b = 0$. Now consider $\frac{1}{p} M' M'^T$ again:

$$\begin{aligned} \frac{1}{p} M' M'^T &= \frac{1}{p} \begin{pmatrix} 0 & 0 & 0 \\ c & \cdot & \cdot \\ d & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 & c & d \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix} \\ &= \frac{1}{p} \begin{pmatrix} 0 & \cdot & \cdot \\ 0 & \cdot & \\ 0 & & \end{pmatrix} \end{aligned} \tag{3.278}$$

implying $c = d = 0$. So M' must be of the form

$$M' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & N'_{11} & N'_{21} \\ 0 & N'_{12} & N'_{12} \end{pmatrix} \tag{3.279}$$

The matrix N' satisfies the same equation, (3.275), as M':

$$N' + N'^T = \frac{1}{p} N' N'^T \tag{3.280}$$

Write

$$U = I - \frac{1}{p} N'$$

then the relation (3.280) becomes the following relation for U,

$$\begin{aligned} UU^T &= \left(I - \frac{1}{p}N'\right)\left(I - \frac{1}{p}N'^T\right) \\ &= I - \frac{1}{p}\left(N' + N'^T - \frac{1}{p}N'N'^T\right) \\ &= I \end{aligned} \tag{3.281}$$

Put

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.282}$$

Then $UU^T = I$ reads

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} &= \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{3.283}$$

$$\begin{aligned} a^2+b^2 &= 1 \\ c^2+d^2 &= 1 \\ ac+bd &= 0 \end{aligned} \tag{3.284}$$

Now we consider what the result is of taking product U^TU:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} \tag{3.285}$$

From

$$det\, U det\, U^T = 1$$

we have

$$det\, U = \mp 1$$

or

$$ad - bc = \mp 1$$

multiplying this by c

$$acd - bc^2 = \mp c$$

and using $ac = -bd$ gives

$$-bd^2 - bc^2 = \mp c$$

or

$$-b(c^2 + d^2) = \mp c$$

and as $c^2 + d^2 = 1$ we therefore have

$$b = \pm c. \tag{3.286}$$

Using this in (3.285) we have

$$\begin{aligned} a^2 + c^2 &= a^2 + b^2 = 1 \\ b^2 + d^2 &= c^2 + d^2 = 1 \\ ab + cd &= a(\pm c) + (\pm b)d \\ &= \pm(ac + bd) = 0 \end{aligned} \tag{3.287}$$

therefore

$$U^T U = I.$$

Now $UU^T = U^T U = I$ is the well known condition for unitary matrices. Summarising

$$U = I - \frac{1}{p}N', \quad UU^T = U^T U = I \tag{3.288}$$

Since N' is a 2×2 real matrix, it is therefore either a proper rotation or a an improper rotation, that is, a rotation plus inversion. Thus it may be written

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.289}$$

The first case corresponds to $detU = +1$ (a proper rotation) and the second $detU = -1$ (an improper rotation). The first case leads to interesting results. For N' and M' we have

$$N' = p \begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix} \tag{3.290}$$

$$M' = p \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\cos\theta & \sin\theta \\ 0 & -\sin\theta & 1-\cos\theta \end{pmatrix} \tag{3.291}$$

We now need to rotate back to the originl coordinates to get

$$M = R^T M' R.$$

The simple form of M' allows us to write

$$\begin{aligned} M_{ik} &= R_{li}R_{jk}M'_{lj} \\ &= R_{2i}R_{2k}M'_{22} + R_{3i}R_{3k}M'_{33} \\ &+ R_{2i}R_{3k}M'_{23} + R_{3i}R_{2k}M'_{32} \\ &= p(1-\cos\theta)(R_{2i}R_{2k} + R_{3i}R_{3k}) + p\sin\theta(R_{2i}R_{3k} - R_{3i}R_{2k}) \end{aligned} \tag{3.292}$$

Let us look at properties of the rotation matrix

$$
R^T R = \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Writing out some terms

$$
\begin{array}{rcl}
R_{11}R_{11} + R_{21}R_{21} + R_{31}R_{31} & = & 1 \\
R_{12}R_{11} + R_{22}R_{21} + R_{32}R_{31} & = & 0 \\
\vdots \quad + \quad \vdots \quad + \quad \vdots & = & \vdots
\end{array}
$$

Or generaly:

$$
R_{1i}R_{1k} + R_{2i}R_{2k} + R_{3i}R_{3k} = \delta_{ik}. \tag{3.293}
$$

Now take (3.273), which can be written in component form

$$
\epsilon_{lmn} R_{2m} R_{3n} = R_{1l},
$$

contracting with ϵ_{ijl} gives

$$
\begin{aligned}
\epsilon_{ijl}\epsilon_{lmn} R_{2m} R_{3n} &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) R_{2m} R_{3n} \\
&= R_{2i}R_{3j} - R_{3i}R_{2j}
\end{aligned} \tag{3.294}
$$

meaning

$$
R_{2i}R_{3j} - R_{3i}R_{2j} = \epsilon_{ijl} R_{1l} \tag{3.295}
$$

Let us write $R_{1i} = R_i$. Using the above results in (3.292) gives

$$
M_{ik} = p(1 - \cos\theta)(\delta_{ik} - R_i R_k) + p \sin\theta \epsilon_{ikl} R_l. \tag{3.296}
$$

Now we had

$$
\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.297}
$$

implying

$$R_{11}\lambda_1 + R_{12}\lambda_2 + R_{13}\lambda_3 = 1$$

or

$$R_{1i}\lambda_i = 1$$

Using the notation $R_{1i} = R_i$, this becomes

$$\vec{R} \cdot \vec{\lambda} = 1, \tag{3.298}$$

where $\vec{R}$ denotes the vector with components R_i. As both $\vec{R}$ and $\vec{\lambda}$ are unit vectors,

$$\vec{R} \cdot \vec{\lambda} = \cos\varphi = 1. \tag{3.299}$$

implies $\varphi = 0$ and hence

$$\vec{R} = \vec{\lambda} \tag{3.300}$$

or $R_i = \lambda_i$. Then we have our end result

$$M_{ik} = p(1 - \cos\theta)(\delta_{ik} - \lambda_i\lambda_k) + p\sin\theta\epsilon_{ikl}\lambda_l. \tag{3.301}$$

We have now replaced the non-linear implicit relation (3.257) by the above simple explicit expression for $\partial_k\lambda_i$.

Laplace and eikonal equations

We rewrite (3.301) in terms of new parameters α and β

$$\partial_k\lambda_i = \alpha(\delta_{ik} - \lambda_i\lambda_k) + \beta\epsilon_{ikl}\lambda_l. \tag{3.302}$$

It will turn out that α and β determine the metric. A number of important three-vector realtion follow directly from (3.302). Set $i = k$ and sum

$$\begin{aligned}\partial_i \lambda_i &= \alpha(\delta_{ii} - \lambda_i\lambda_i) + \beta\epsilon_{iil}\lambda_l \\ &= \alpha(3-1) + 0\end{aligned}$$

giving

$$\nabla \cdot \lambda = 2\alpha. \tag{3.303}$$

Multiplying (3.302) by ϵ_{jki} and summing over i and k gives

$$\begin{aligned}\epsilon_{jki}\partial_k \lambda_i &= \alpha(\delta_{ik} - \lambda_i\lambda_k)\epsilon_{jki} + \beta\epsilon_{ikl}\epsilon_{jki}\lambda_l \\ &= \beta\epsilon_{ikl}\epsilon_{jki}\lambda_l \\ &= -2\beta\lambda_j\end{aligned}$$

or

$$\nabla \times \lambda = -2\beta\lambda. \tag{3.304}$$

The Laplacian of λ can be obtaind in two ways. First differentiating by (3.302) with respect to x^k gives

$$\begin{aligned}\nabla^2\lambda_i &= \partial_k[\alpha(\delta_{ik} - \lambda_i\lambda_k) + \beta\epsilon_{ikl}\lambda_l] \\ &= \partial_i\alpha - \lambda_i(\lambda_k\partial_k\alpha) - \alpha\lambda_k\partial_k\lambda_i - \alpha\lambda_i\partial_k\lambda_k + \epsilon_{ikl}\lambda_l\partial_k\beta + \beta\epsilon_{ikl}\partial_k\lambda_l \\ &= \partial_i\alpha - \lambda_i(\nabla\alpha \cdot \lambda) - \alpha\lambda_i(\nabla \cdot \lambda) + \beta(\nabla \times \lambda)_i + (\nabla\beta \times \lambda)_i \\ &= \partial_i\alpha - \lambda_i(\nabla\alpha \cdot \lambda) - 2(\alpha^2 + \beta^2)\lambda_i + (\nabla\beta \times \lambda)_i\end{aligned}$$

where we used (3.303), (3.304), and

$$\begin{aligned}\lambda_k\partial_k\lambda_i &= \alpha(\lambda_i - \lambda_i\lambda \cdot \lambda) + \beta\epsilon_{ikl}\lambda_k\lambda_l \\ &= 0\end{aligned} \tag{3.305}$$

(which is just $M\lambda = 0$). We have obtained our first equation for $\nabla^2\lambda$:

$$\nabla^2 \lambda = \nabla\alpha - \lambda(\nabla\alpha \cdot \lambda) - 2(\alpha^2 + \beta^2)\lambda + \nabla\beta \times \lambda \tag{3.306}$$

To obtain the alternative equation we first derive the vector identity

$$\nabla \times (\nabla \times \lambda) = \nabla(\nabla \cdot \lambda) - \nabla^2 \lambda$$

from

$$\begin{aligned}
[\nabla \times (\nabla \times \lambda)]_m &= \epsilon_{mni}\partial_n(\epsilon_{ikl}\partial_k \lambda_l) \\
&= \epsilon_{mni}\epsilon_{ikl}\partial_n\partial_k\lambda_l \\
&= (\delta_{mk}\delta_{nl} - \delta_{ml}\delta_{nk})\partial_n\partial_k\lambda_l \\
&= \partial_m(\partial_n\lambda_n) - \partial_n\partial_n\lambda_m.
\end{aligned}$$

We then have

$$\begin{aligned}
\nabla^2\lambda &= \nabla(\nabla\cdot\lambda) - \nabla \times (\nabla \times \lambda) \\
&= \nabla(2\alpha) - \nabla \times (-2\beta\lambda) \\
&= 2\nabla\alpha + 2(\nabla\beta \times \lambda) + 2\beta\nabla \times \lambda \\
&= 2\nabla\alpha + 2(\nabla\beta \times \lambda) - 4\beta^2\lambda
\end{aligned} \tag{3.307}$$

where again we used (3.303) and (3.304). We equate the expressions (3.306) and (3.307)

$$\begin{aligned}
&\nabla\alpha - \lambda(\nabla\alpha\cdot\lambda) - 2(\alpha^2+\beta^2)\lambda + \nabla\beta\times\lambda \\
&= 2\nabla\alpha + 2(\nabla\beta\times\lambda) - 4\beta^2\lambda
\end{aligned}$$

from which we obtain

$$\nabla\alpha = -\nabla\beta \times \lambda - \lambda(\nabla\alpha\cdot\lambda) - 2(\alpha^2 - \beta^2)\lambda. \tag{3.308}$$

Dotting this with λ gives

$$\nabla\alpha \cdot \lambda = -(\nabla\alpha\cdot\lambda) - 2(\alpha^2 - \beta^2)$$

and rearanging we have

$$\nabla\alpha \cdot \lambda = \beta^2 - \alpha^2 \tag{3.309}$$

and substituting this into (3.308) gives

$$\nabla\alpha = (\beta^2 - \alpha^2)\lambda - \nabla\beta \times \lambda \tag{3.310}$$

Equations (3.309) and (3.310) are very important.

Equations analogous to (3.309) and (3.310) with β replacing α on the left hand side can be obtained. The divergence of (3.304) is zero, implying the divergence of $\beta\lambda$ is zero,

$$\nabla \cdot (\beta\lambda) = \beta\nabla \cdot \lambda + \nabla\beta \cdot \lambda = 0 \tag{3.311}$$

using (3.303) in this gives

$$\nabla\beta \cdot \lambda = -2\alpha\beta \tag{3.312}$$

Now cross (3.310) with λ and use $(\nabla\beta \times \lambda) \times \lambda = -\lambda \cdot \lambda\nabla\beta + \lambda(\lambda \cdot \nabla\beta)$. We obtain

$$\nabla\beta = \lambda(\lambda \cdot \nabla\beta) + (\nabla\alpha \times \lambda) \tag{3.313}$$

which usong (3.312) becomes

$$\nabla\beta = -2\alpha\beta\lambda + (\nabla\alpha \times \lambda). \tag{3.314}$$

Introduce the complex function γ

Eqations (3.309), (3.310), (3.312), and (3.314) are very important and can be expressed in a more concise way by the introduction of the complex function $\gamma = \alpha + i\beta$:

$$\begin{aligned} \nabla\gamma \cdot \lambda &= \nabla\alpha \cdot \lambda + i\nabla\beta \cdot \lambda \\ &= [\beta^2 - \alpha^2] + i[-2\alpha\beta] \\ &= -(\alpha + i\beta)^2 \\ &= -\gamma^2 \end{aligned} \tag{3.315}$$

$$\begin{aligned}
\nabla\gamma &= [(\beta^2-\alpha^2)\lambda - \nabla\beta\times\lambda] + i[-2\alpha\beta\lambda + (\nabla\alpha\times\lambda)] \\
&= [(\beta^2-\alpha^2) + i(-2\alpha\beta)]\lambda + i[\nabla\alpha + i\nabla\beta]\times\lambda \\
&= -\gamma^2\lambda + i(\nabla\gamma\times\lambda). \qquad (3.316)
\end{aligned}$$

We will obtain a pair of simple differential equations that determine γ and show in that γ in turn determines the metric via l_0 and λ_j.

The first differential equation is found by forming the Laplacian for γ from (3.316) and using (3.303), (3.315), and (3.304) we obtain

$$\begin{aligned}
\nabla^2\gamma &= \nabla\cdot\nabla\gamma \\
&= \nabla\cdot[-\gamma^2\lambda + i(\nabla\gamma\times\lambda)] \\
&= -\gamma^2\nabla\cdot\lambda - 2\gamma\nabla\gamma\cdot\lambda + i\nabla\cdot(\nabla\gamma\times\lambda) \\
&= -2\alpha\gamma^2 + 2\gamma^3 + i\partial_i[\epsilon_{ijk}(\partial_j\gamma)\lambda_k] \\
&= -2\alpha\gamma^2 + 2\gamma^3 - i\partial_i\gamma(\epsilon_{ijk}\partial_j\lambda_k) \\
&= -2\alpha\gamma^2 + 2\gamma^3 - i\nabla\gamma\cdot(\nabla\times\lambda) \\
&= -2\alpha\gamma^2 + 2\gamma^3 - i\nabla\gamma\cdot(-2\beta\lambda) \\
&= -2\alpha\gamma^2 + 2\gamma^3 - 2i\beta\gamma^2 \\
&= -2\gamma^2(\alpha + i\beta - \gamma) \\
&= 0 \qquad (3.317)
\end{aligned}$$

Thus γ is a complex harmonic function. The second differential equation is obtained by squaring (3.316) and using (3.315)

$$\begin{aligned}
(\nabla\gamma)^2 &= [-\gamma^2\lambda + i(\nabla\gamma\times\lambda)]^2 \\
&= \gamma^4 - (\nabla\gamma\times\lambda)\cdot(\nabla\gamma\times\lambda) \\
&= \gamma^4 + i(\nabla\gamma\times\lambda)\cdot(\nabla\gamma + \gamma^2\lambda) \\
&= \gamma^4 \qquad (3.318)
\end{aligned}$$

we get

$$\nabla^2\gamma = 0, \quad (\nabla\gamma)^2 = \gamma^4 \qquad (3.319)$$

Let us introduce $\omega \equiv 1/\gamma$, then

$$\nabla\omega = \nabla\left(\frac{1}{\gamma}\right) = -\frac{1}{\gamma^2}\nabla\gamma \tag{3.320}$$

Then the equation $(\nabla\gamma)^2 = \gamma^4$ becomes

$$(\nabla\omega)^2 = \frac{1}{\gamma^4}(\nabla\gamma)^2 = 1.$$

where we used (3.318). Repeating (3.317), altogether we have

$$\nabla^2\gamma = 0, \quad (\nabla\omega)^2 = 1, \quad \omega \equiv \frac{1}{\gamma} \tag{3.321}$$

These equations determine the function γ completely, dependent on consistent boundary conditions.

As we shall now show, these equations completely replace the field equations since the metric functions l_0 and λ are determined by γ.

Expression for λ

It is more convenient to express (3.315) and (3.316) in terms of ω. For equation (3.315) we have

$$\nabla\left(\frac{1}{\omega}\right)\cdot\lambda = -\frac{\lambda\cdot\nabla\omega}{\omega^2} = -\frac{1}{\omega^2} \tag{3.322}$$

where we have used (3.320) and (3.315). For equation (3.316) we have

$$\nabla\left(\frac{1}{\omega}\right) = -\frac{1}{\omega^2}\lambda + i(\nabla\left(\frac{1}{\omega}\right)\times\lambda) \tag{3.323}$$

or

$$-\frac{1}{\omega^2}\nabla\omega = -\frac{1}{\omega^2}\lambda - \frac{i}{\omega^2}(\nabla\omega\times\lambda)$$

Substituting (3.323) into (3.322) gives

$$\lambda\cdot\nabla\omega = 1.$$

Altogether we have

$$\lambda \cdot \nabla\omega = \lambda \cdot \nabla\omega^* = 1, \qquad \nabla\omega = \lambda + i(\nabla\omega \times \lambda) \tag{3.324}$$

Thus

$$\begin{aligned}
\nabla\omega \times \nabla\omega^* &= [\lambda + i(\nabla\omega \times \lambda)] \times [\lambda - i(\nabla\omega^* \times \lambda)] \\
&= -i\lambda \times (\nabla\omega^* \times \lambda) + i(\nabla\omega \times \lambda) \times \lambda + (\nabla\omega \times \lambda) \times (\nabla\omega^* \times \lambda) \\
&= -i[\nabla\omega^* - \lambda(\lambda \cdot \nabla\omega^*)] - i[\nabla\omega - \lambda(\lambda \cdot \nabla\omega)] + (\nabla\omega \times \lambda) \times (\nabla\omega^* \times \lambda) \\
&= -i(\nabla\omega^* - \lambda) - i(\nabla\omega - \lambda) + (\nabla\omega \times \lambda) \times (\nabla\omega^* \times \lambda) \\
&= -i[\nabla\omega^* + \nabla\omega] + 2i\lambda + (\nabla\omega \times \lambda) \times (\nabla\omega^* \times \lambda)
\end{aligned} \tag{3.325}$$

where we used the vector identity $\lambda \times (\nabla f \times \lambda) = \lambda \cdot \lambda \nabla f - \lambda(\lambda \cdot \nabla f)$ and that $\lambda \cdot \nabla\omega = \lambda \cdot \nabla\omega^* = 1$.

Consider the last term,

$$\begin{aligned}
[(\nabla\omega \times \lambda) \times (\nabla\omega^* \times \lambda)]_p &= \epsilon_{pil}(\nabla\omega \times \lambda)_i(\nabla\omega^* \times \lambda)_l \\
&= \epsilon_{pil}(\epsilon_{ijk}\lambda_k\partial_j\omega)(\epsilon_{lmn}\lambda_n\partial_m\omega^*) \\
&= \epsilon_{pil}\epsilon_{lmn}(\epsilon_{ijk}\lambda_k\lambda_n\partial_j\omega\partial_m\omega^*) \\
&= (\delta_{pm}\delta_{in} - \delta_{pn}\delta_{im})(\epsilon_{ijk}\lambda_k\lambda_n\partial_j\omega\partial_m\omega^*) \\
&= \epsilon_{ijk}\lambda_k\lambda_i\partial_j\omega\partial_p\omega^* - \lambda_p\epsilon_{ijk}\lambda_k\partial_j\omega\partial_i\omega^* \\
&= \lambda_p\epsilon_{ijk}\lambda_i\partial_j\omega\partial_k\omega^* \\
&= \lambda_p[\lambda \cdot (\nabla\omega \times \nabla\omega^*)]
\end{aligned} \tag{3.326}$$

So

$$\begin{aligned}
\nabla\omega \times \nabla\omega^* &= -i[\nabla\omega^* + \nabla\omega] + 2i\lambda + \lambda[\lambda \cdot (\nabla\omega \times \nabla\omega^*)] \\
&= -i[\nabla\omega^* + \nabla\omega] + B\lambda
\end{aligned} \tag{3.327}$$

Now as

$$(\partial_i\omega)\varepsilon_{ijk}\partial_j\omega\partial_k\omega^* = 0$$

or

$$\nabla\omega \cdot (\nabla\omega \times \nabla\omega^*) = 0$$

dotting (3.327) with $\nabla\omega$ and using $(\nabla\omega)^2 = 1$ and $\lambda \cdot \nabla\omega = 1$ we obtain for B

$$B = i[1 + \nabla\omega \cdot \nabla\omega^*] \tag{3.328}$$

So we get for λ

$$\lambda = \frac{\nabla\omega + \nabla\omega^* - i(\nabla\omega \times \nabla\omega^*)}{1 + \nabla\omega \cdot \nabla\omega^*} \tag{3.329}$$

and we have our expression for λ_j in terms of ω.

Expression for l_0

We prove that the choice

$$l_0^2 = Re\ (\gamma) = \alpha. \tag{3.330}$$

leads to a solution of the first two field equations (3.247) and (3.248) as we will now show. We have first from the first equation in (3.321) that

$$0 = \nabla^2\alpha = \nabla^2(l_0^2). \tag{3.331}$$

which solves (3.247). Next consider

$$\nabla^2(\alpha\lambda_j) = \alpha\nabla^2\lambda_j + 2(\partial_k\alpha)(\partial_k\lambda_j) \tag{3.332}$$

By using (3.307), (3.302), and (3.309) we get

$$\begin{aligned}
\nabla^2(\alpha\lambda_j) &= \alpha\nabla^2\lambda_j + 2(\partial_k\alpha)(\partial_k\lambda_j) \\
&= \alpha[2\partial_j\alpha + 2(\nabla\beta \times \lambda)_j - 4\beta^2\lambda_j] \\
&\quad + 2(\partial_k\alpha)[\alpha(\delta_{jk} - \lambda_j\lambda_k) + \beta\epsilon_{jkl}\lambda_l] \\
&= 4\alpha\partial_j\alpha - 4\alpha\beta^2\lambda_j - 2\lambda_j\alpha(\lambda \cdot \nabla\alpha) + 2\alpha(\nabla\beta \times \lambda)_j + 2\beta(\nabla\alpha \times \lambda)_j \\
&= 4\alpha\partial_j\alpha - 4\alpha\beta^2\lambda_j - 2\lambda_j\alpha(\beta^2 - \alpha^2) + 2\alpha(\nabla\beta \times \lambda)_j + 2\beta(\nabla\alpha \times \lambda)_j \\
&= 4\alpha\partial_j\alpha + 2\alpha(\alpha^2 - 3\beta^2)\lambda_j + 2\alpha(\nabla\beta \times \lambda)_j + 2\beta(\nabla\alpha \times \lambda)_j
\end{aligned} \tag{3.333}$$

We simplify further using (3.310) and (3.314),

$$
\begin{aligned}
\nabla^2(\alpha\lambda_j) &= 4\alpha\partial_j\alpha + 2\alpha(\alpha^2 - 3\beta^2)\lambda_j + 2\alpha[(\beta^2 - \alpha^2)\lambda_j - \partial_j\alpha] \\
&\quad + 2\beta[2\alpha\beta\lambda_j + \partial_j\beta] \\
&= 2\alpha\partial_j\alpha + 2\beta\partial_j\beta \\
&= \partial_j(\alpha^2 + \beta^2). \qquad (3.334)
\end{aligned}
$$

Or

$$\nabla^2(\alpha\lambda) = \nabla(\alpha^2 + \beta^2). \qquad (3.335)$$

We calculate the RHS of (3.335). From $\alpha = p(1 - \cos\theta)$ and $\beta = p\sin\theta$ we have

$$\alpha^2 + \beta^2 = 2p^2(1 - \cos\theta) = 2\alpha p \qquad (3.336)$$

From the definition of p,

$$p = \frac{L + A}{2l_0}. \qquad (3.337)$$

We then have

$$L + A = \frac{l_0}{\alpha}(\alpha^2 + \beta^2). \qquad (3.338)$$

Thus with $l_0^2 = \alpha$ the RHS of (3.335) is

$$\partial_j[(L + A)l_0] = \partial_j(\alpha^2 + \beta^2). \qquad (3.339)$$

We see that $l_0^2 = \alpha$ is indeed a solution.

3.7.2 The Kerr Solution

A solution to the field equations

Consider the function

$$\gamma = [(x-a)^2 + (y-b)^2 + (z-c)^2]^{-1/2} \tag{3.340}$$

We will calculate $\nabla^2\gamma$ and $(\nabla\omega)^2$, $(\omega = 1/\gamma)$. First

$$\frac{\partial}{\partial x}[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-1/2} = -(x-a)[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-3/2}$$

and

$$\begin{aligned}
& \frac{\partial^2}{\partial x^2}[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-1/2} \\
= \; & -[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-3/2} \\
& + 3(x-a)^2[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-5/2}
\end{aligned} \tag{3.341}$$

so that

$$\begin{aligned}
\nabla^2\gamma &= \nabla^2[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-1/2} \\
&= -3[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-3/2} \\
&\quad + 3[(x-a)^2 + (y-b)^2 + (z-c)^2][(x-a)^2 + (y-b)^2 + (z-c)^2]^{-5/2} \\
&= 0.
\end{aligned} \tag{3.342}$$

Now consider $\nabla\omega$

$$\begin{aligned}
\nabla\omega &= \nabla[(x-a)^2 + (y-b)^2 + (z-c)^2]^{1/2} \\
&= \frac{[(x-a)\hat{i} + (y-b)\hat{j} + (z-c)\hat{k}]}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{1/2}}
\end{aligned} \tag{3.343}$$

so obviously we have

$$(\nabla\omega)^2 = 1. \tag{3.344}$$

Therefore (3.340) is a solution of the field equations (3.321).

The Schwarzschild solution

Consider the choice

$$\gamma = (x^2 + y^2 + z^2)^{-1/2} \tag{3.345}$$

We obviously have

$$l_0^2 = \frac{1}{r}. \tag{3.346}$$

Next as $\omega = 1/\gamma$, $\omega = r$ and

$$\begin{aligned} \nabla\omega &= \nabla(x^2 + y^2 + z^2)^{1/2} \\ &= \frac{x\hat{i} + y\hat{j} + z\hat{i}}{(x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{i} \end{aligned} \tag{3.347}$$

Obviously

$$\nabla\omega \times \nabla\omega^* = \nabla\omega \times \nabla\omega = 0 \tag{3.348}$$

Next

$$\nabla\omega \cdot \nabla\omega^* = \nabla\omega \cdot \nabla\omega = 1 \tag{3.349}$$

Recall λ is given by

$$\begin{aligned} \lambda &= \frac{\nabla\omega + \nabla\omega^* - i(\nabla\omega \times \nabla\omega^*)}{1 + \nabla\omega \cdot \nabla\omega^*} \\ &= \nabla\omega \\ &= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{i} + \frac{z}{r}\hat{i} \end{aligned} \tag{3.350}$$

Reading off the components of λ we get

$$\lambda_1 = \frac{x}{r}, \quad \lambda_2 = \frac{y}{r}, \quad \lambda_3 = \frac{z}{r}. \tag{3.351}$$

Combining this with (3.346) gives for l_a

$$l_a = l_0(1, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{\sqrt{r}}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{3.352}$$

in agreement with (3.191).

The Kerr solution

Consider the complex valued choice,

$$\gamma = (x^2 + y^2 + (z - ia)^2)^{-1/2} \tag{3.353}$$

Let us calculate the functions l_0 and λ. We first split ω into real and imaginary parts

$$\omega = 1/\gamma = \rho + i\sigma = (x^2 + y^2 + (z - ia)^2)^{1/2} = (r^2 - a^2 - 2iaz)^{1/2} \tag{3.354}$$

Squaring and equating real and imaginary parts gives

$$\rho^2 - \sigma^2 = r^2 - a^2, \qquad \sigma = -\frac{az}{\rho} \tag{3.355}$$

substituting for σ in the real part gives

$$\rho^2 - \frac{a^2 z^2}{\rho^2} = r^2 - a^2$$

resulting in the quadratic equation in ρ^2:

$$\rho^4 - \rho^2(r^2 - a^2) - a^2 z^2 = 0. \tag{3.356}$$

Completeing the square we obatain

$$\rho^2 = \frac{r^2 - a^2}{2} + \left[\frac{(r^2 - a^2)^2}{4} + a^2 z^2\right]^{1/2} \tag{3.357}$$

where we have choosen the plus sign so that for $r \gg a$, $\rho \sim r$. Now $l_0^2 = \mathrm{Re}\,(\gamma)$. From

$$\gamma = \frac{1}{\rho + i\sigma} = \frac{\rho}{\rho^2 + \sigma^2} - \frac{i\sigma}{\rho^2 + \sigma^2}$$

we have

$$l_0^2 = \frac{\rho}{\rho^2 + \sigma^2} = \frac{\rho^3}{\rho^4 + a^2 z^2}. \tag{3.358}$$

Now let us move onto the calculation of λ. First

$$\begin{aligned}
\nabla\omega &= \nabla\left(\frac{1}{\gamma}\right) \\
&= \nabla(r^2 - 2iaz - a^2)^{1/2} \\
&= \frac{1}{2\omega}(2x\hat{i} + 2y\hat{j} + 2z\hat{k} - 2ia\hat{k}) \\
&= \frac{\hat{r} - ia\hat{k}}{\omega}
\end{aligned} \tag{3.359}$$

then

$$\begin{aligned}
\nabla\omega + \nabla\omega^* &= \frac{\hat{r} - ia\hat{k}}{\omega} + \frac{\hat{r} + ia\hat{k}}{\omega^*} \\
&= \frac{\omega^*(\hat{r} - ia\hat{k}) + \omega(\hat{r} + ia\hat{k})}{|\omega|^2} \\
&= \frac{(\rho - i\sigma)(\hat{r} - ia\hat{k}) + (\rho + i\sigma)(\hat{r} + ia\hat{k})}{|\omega|^2} \\
&= 2\frac{\rho\hat{r} - a\sigma\hat{k}}{|\omega|^2}
\end{aligned} \tag{3.360}$$

and

$$\nabla\omega \cdot \nabla\omega^* = \frac{\hat{r} - ia\hat{k}}{\omega} \cdot \frac{\hat{r} + ia\hat{k}}{\omega^*} = \frac{r^2 + a^2}{|\omega|^2}$$

and

$$\nabla\omega \times \nabla\omega^* = \frac{\hat{r} - ia\hat{k}}{\omega} \times \frac{\hat{r} + ia\hat{k}}{\omega^*} = 2ia\frac{\mathbf{r} \times \mathbf{k}}{|\omega|^2}$$

so λ is

$$\begin{aligned} \lambda &= \frac{\nabla\omega + \nabla\omega^* - i(\nabla\omega \times \nabla\omega^*)}{1 + \nabla\omega \cdot \nabla\omega^*} \\ &= \frac{2[\rho\hat{r} - a\sigma\hat{k} + a(\mathbf{r} \times \mathbf{k})]}{|\omega|^2 + r^2 + a^2}. \end{aligned} \tag{3.361}$$

We can simplify this by noting

$$\begin{aligned} |\omega|^2 &= [(r^2 - a^2)^2 + 4a^2 z^2]^{1/2} \\ &= 2[\frac{(r^2 - a^2)^2}{4} + a^2 z^2]^{1/2} \\ &= 2\rho^2 - (r^2 - a^2) \end{aligned}$$

because then

$$|\omega|^2 + r^2 + a^2 = 2(\rho^2 + a^2).$$

Substituting this into (3.361) we then have for λ

$$\lambda = \frac{[\rho\hat{r} - a\sigma\hat{k} + a(\mathbf{r} \times \mathbf{k})]}{\rho^2 + a^2} \tag{3.362}$$

As

$$\mathbf{r} \times \mathbf{k} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \times \hat{\mathbf{k}} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$$

and as $\sigma = -az/\rho$, λ is

$$\begin{aligned} \lambda &= \frac{1}{\rho^2 + a^2}\left[\rho\hat{r} + \frac{a^2 z}{\rho}\hat{k} + a(y\hat{\mathbf{i}} - x\hat{\mathbf{j}})\right] \\ &= \frac{1}{\rho^2 + a^2}\left[(\rho x + ay)\hat{i} + (\rho y - ax)\hat{j} + \frac{(\rho^2 + a^2)z}{\rho}\hat{k}\right]. \end{aligned} \tag{3.363}$$

We can read off the components

$$\lambda_1 = \frac{\rho x + ay}{a^2+\rho^2},$$

$$\lambda_2 = \frac{\rho y - ax}{a^2+\rho^2},$$

and

$$\lambda_3 = \frac{z}{\rho}.$$

Then

$$\begin{aligned} l_\mu dx^\mu &= l_0 \left[dx^0 + \frac{\rho x + ay}{a^2+\rho^2}dx + \frac{\rho y - ax}{a^2+\rho^2}dy + \frac{z}{\rho}dz\right] \\ &= l_0 \left[dx^0 + \frac{\rho}{a^2+\rho^2}(xdx+ydy) + \frac{a}{a^2+\rho^2}(ydx - xdy) + \frac{z}{\rho}dz\right] \end{aligned}$$

Finally, as

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu - 2ml_\mu l_\nu dx^\mu dx^\nu$$

we have

$$\begin{aligned} ds^2 &= (dx^0)^2 - (d\mathbf{x})^2 - \frac{2m\rho^3}{\rho^4+a^2z^2}\left[dx^0 + \frac{\rho x}{a^2+\rho^2}(xdx+ydy)\right. \\ &\qquad \left. + \frac{a}{a^2+\rho^2}(ydx-xdy) + \frac{z}{\rho}dz\right]^2 \end{aligned} \tag{3.364}$$

This is the form obtained by Kerr in 1963.

3.7.3 Independence of Metric on Angular Variable φ

We now make axially symmetry of this solution manifest. Consider

$$
\begin{aligned}
u &= x^0 + u \\
x &= \rho\sin\theta\cos\varphi - a\rho\sin\theta\sin\varphi \\
y &= \rho\sin\theta\cos\varphi + a\rho\sin\theta\sin\varphi \\
z &= \rho\cos\theta.
\end{aligned} \tag{3.365}
$$

The middle two transformations can be written as a complex expression

$$(\rho + ia)e^{i\varphi}\sin\theta = x + iy \tag{3.366}$$

Now let us find the differential dz^2

$$dz^2 = (\cos\theta d\rho - \rho\sin\theta d\theta)^2 \tag{3.367}$$

Now let us find the differential $dx^2 + dy^2$

$$
\begin{aligned}
dx^2 + dy^2 &= |d(x + iy)|^2 \\
&= |d[(\rho + ia)e^{i\varphi}\sin\theta]|^2 \\
&= |e^{i\varphi}\sin\theta d\rho + i(\rho + ia)e^{i\varphi}\sin\theta d\varphi + (\rho + ia)e^{i\varphi}\cos\theta d\theta|^2 \\
&= |\sin\theta d\rho + i(\rho + ia)\sin\theta d\varphi + (\rho + ia)\cos\theta d\theta|^2 \\
&= |\sin\theta d\rho - a\sin\theta d\varphi + \rho\cos\theta d\theta + i[\rho\sin\theta d\varphi + a\cos\theta d\theta]|^2 \\
&= (\sin\theta d\rho - a\sin\theta d\varphi + \rho\cos\theta d\theta)^2 + (\rho\sin\theta d\varphi + a\cos\theta d\theta)^2
\end{aligned} \tag{3.368}
$$

Now let us find the differential $xdx + ydy$

$$
\begin{aligned}
xdx + ydy &= \frac{1}{2}d|x + iy|^2 \\
&= \frac{1}{2}d|(\rho + ia)e^{i\varphi}\sin\theta|^2 \\
&= \frac{1}{2}d|(\rho + ia)\sin\theta|^2 \\
&= \frac{1}{2}d[(\rho^2 + a^2)\sin^2\theta] \\
&= (\rho^2 + a^2)\sin\theta\cos\theta d\theta + \rho\sin^2\theta d\rho
\end{aligned} \tag{3.369}
$$

Now let us find the differential $xdy - ydx$

$$
\begin{aligned}
xdy - ydx &= -Im[(x+iy)d(x-iy)] \\
&= -Im\{[(\rho+ia)e^{i\varphi}\sin\theta]d[(\rho-ia)e^{-i\varphi}\sin\theta]\} \\
&= -Im\{[(\rho+ia)e^{i\varphi}\sin\theta][e^{-i\varphi}\sin\theta d\rho - i(\rho-ia)e^{-i\varphi}\sin\theta d\varphi + \\
&\quad + (\rho-ia)e^{i\varphi}\cos\theta d\theta]\} \\
&= -Im\{(\rho+ia)\sin^2\theta d\rho - i(\rho^2+a^2)\sin^2\theta d\varphi + (\rho^2+a^2)\sin\theta\cos\theta d\theta\} \\
&= (\rho^2+a^2)\sin\theta^2 d\varphi - a\sin^2\theta d\rho
\end{aligned} \tag{3.370}
$$

Now let us find the differential zdz

$$
zdz = \rho\cos^2\theta d\rho - \rho^2\sin\theta\cos\theta d\theta \tag{3.371}
$$

$$
\frac{2m\rho^3}{\rho^4+a^2z^2} = \frac{2m\rho}{\rho^2+a^2\cos^2\theta} \tag{3.372}
$$

Substituting these results into (3.364)

$$
\begin{aligned}
ds^2 &= (du-d\rho)^2 - (\sin\theta d\rho - a\sin\theta d\varphi + \rho\cos\theta d\theta)^2 - (\rho\sin\theta d\varphi + a\cos\theta d\theta)^2 \\
&\quad -(\cos\theta d\rho - \rho\sin\theta d\theta)^2 \\
&\quad - \frac{2m\rho}{\rho^2+a^2\cos^2\theta}\Big[du - d\rho + \frac{\rho}{a^2+\rho^2}\left\{(\rho^2+a^2)\sin\theta\cos\theta d\theta + \rho\sin^2\theta d\rho\right\} \\
&\quad - \frac{a}{a^2+\rho^2}\left\{(\rho^2+a^2)\sin^2\theta d\varphi - a\sin^2\theta d\rho\right\} + \cos^2\theta d\rho - \rho\sin\theta\cos\theta d\theta\Big]^2
\end{aligned} \tag{3.373}
$$

We multiply out the first and second line and get

$$
\begin{aligned}
& du^2 + (1-\sin^2\theta-\cos^2)\theta d\rho^2 - [(\rho^2+a^2)\cos^2\theta + \rho^2\sin^2\theta]d\theta^2 - (\rho^2+a^2)\sin^2\theta d\varphi^2 \\
& - 2dud\rho - (2\rho\sin\theta\cos\theta - 2\rho\sin\theta\cos\theta)d\rho d\theta + 2a\sin^2\theta d\rho d\varphi \\
& + (a\rho\sin\theta\cos\theta - \rho a\sin\theta\cos\theta)d\theta d\varphi \\
= \;& du^2 - (\rho^2+a^2\cos^2\theta)d\theta^2 - (\rho^2+a^2)\sin^2\theta d\varphi^2 - 2dud\rho + 2a\sin^2\theta d\rho d\varphi
\end{aligned} \tag{3.374}
$$

The last two lines can be simplified,

$$
\begin{aligned}
& - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du + \left(-1 + \frac{\rho^2}{a^2+\rho^2}\sin^2\theta + \frac{a^2}{a^2+\rho^2}\sin^2\theta + \cos^2\theta\right)d\rho \\
& + \left(\rho\sin\theta\cos\theta - \rho\sin\theta\cos\theta\right)d\theta + \left(-a\sin^2\theta\right)d\varphi\Big]^2 \\
= & - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du + \left(\frac{\rho^2}{a^2+\rho^2} + \frac{a^2}{a^2+\rho^2} - 1\right)\sin^2\theta d\rho - a\sin^2\theta d\varphi\Big]^2 \\
= & - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du - a\sin^2\theta d\varphi\Big]^2
\end{aligned} \tag{3.375}
$$

Putting together (3.374) and (3.375) we get

$$
\begin{aligned}
ds^2 = & \; du^2 - (\rho^2 + a^2\cos^2\theta)d\theta^2 - (\rho^2+a^2)\sin^2\theta d\varphi^2 - 2dud\rho + 2a\sin^2\theta d\rho d\varphi \\
& - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du - a\sin^2\theta d\varphi\Big]^2 .
\end{aligned} \tag{3.376}
$$

Simplifying, we finally obtain the line element

$$
\begin{aligned}
ds^2 = & \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right)du^2 - 2dud\rho + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(2a\sin^2\theta)dud\varphi \\
& + 2a\sin^2\theta d\rho d\varphi - (\rho^2 + a^2\cos^2\theta)d\theta^2 - \left((\rho^2+a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right)d\varphi^2 .
\end{aligned} \tag{3.377}
$$

There is no dependence of the line element on the angular coordinate φ, so that the solution (3.377) is manifestly axially symmetric. This is the advanced Eddington-Finkelstein form of Kerr's solution.

3.7.4 Boyer-Lindquist Coordinates

We wish to make a change of coordinates that puts the line element in a form suh that the only cross-term is $d\varphi dt$.

write the element (3.377) as

$$
ds^2 = g_{00}du^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 + 2g_{03}dud\varphi + 2g_{01}dud\rho + 2g_{13}d\rho d\varphi \tag{3.378}
$$

We guess the form of the desired transformation

$$
\begin{aligned}
t &= u - A(\rho) \qquad & du &= dt + A' d\rho \\
\tilde{\varphi} &= \varphi - B(\rho) \qquad & d\varphi &= d\tilde{\varphi} + B' d\rho
\end{aligned}
\tag{3.379}
$$

(3.378) is then

$$
\begin{aligned}
ds^2 &= g_{00}(dt + A'd\rho)^2 + g_{22}d\theta^2 + g_{33}(d\tilde{\varphi} + B'd\rho)^2 + 2g_{03}(dt + A'd\rho)(d\tilde{\varphi} + B'd\rho) \\
&\quad + 2g_{01}(dt + A'd\rho)d\rho + 2g_{13}d\rho(d\tilde{\varphi} + B'd\rho) \\
&= g_{00}dt^2 + (g_{00}A'^2 + g_{33}B'^2 + 2g_{01}A' + 2g_{13}B' + 2g_{03}A'B')d\rho^2 \\
&\quad + g_{22}d\theta^2 + g_{33}d\tilde{\varphi}^2 + 2g_{03}dtd\tilde{\varphi} + 2(g_{03}A' + g_{33}B' + g_{13})d\tilde{\varphi}d\rho \\
&\quad + 2(g_{00}A' + g_{03}B' + g_{01})dtd\rho
\end{aligned}
\tag{3.380}
$$

We now demand that the coefficients of $d\tilde{\varphi}d\rho$ and $dtd\rho$ zanish, we must have

$$
g_{03}A' + g_{33}B' + g_{13} = 0 \tag{3.381}
$$

and

$$
g_{00}A' + g_{03}B' + g_{01} = 0 \tag{3.382}
$$

respectively. Multiplying (3.381) by g_{03} and (3.382) by g_{33} and subtracting implies

$$
A' = \frac{g_{33}g_{01} - g_{03}g_{13}}{g_{03}^2 - g_{00}g_{03}}. \tag{3.383}
$$

Multiplying (3.381) by g_{00} and (3.382) by g_{03} and subtracting implies

$$
B' = \frac{g_{00}g_{13} - g_{03}g_{01}}{g_{03}^2 - g_{00}g_{03}}. \tag{3.384}
$$

$$
\begin{aligned}
g_{33}g_{01} - g_{03}g_{13} &= -\left((\rho^2 + a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right)(-1) \\
&\quad - \left(\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right)a\sin^2\theta \\
&= (\rho^2 + a^2)\sin^2\theta
\end{aligned}
\tag{3.385}
$$

$$
\begin{aligned}
g_{00}g_{13} - g_{03}g_{01} &= \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right) a\sin^2\theta - \left(\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right)(-1) \\
&= a\sin^2\theta \qquad (3.386)
\end{aligned}
$$

$$
\begin{aligned}
g_{03}^2 - g_{00}g_{33} &= \left(\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right)^2 - \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right) \times \\
&\times - \left((\rho^2 + a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right) \\
&= \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta}[\frac{4m^2\rho^2a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta} + \\
&+ (\rho^2 + a^2\cos^2\theta - 2m\rho)[(\rho^2 + a^2) + \frac{2m\rho a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}] \\
&= \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta}[(\rho^2 + a^2\cos^2\theta)(\rho^2 + a^2) + 2m\rho a^2\sin^2\theta - 2m\rho(\rho^2 + a^2)] \\
&= \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta}[(\rho^2 + a^2\cos^2\theta)(\rho^2 + a^2) - 2m\rho(\rho^2 + a^2\cos^2\theta)] \\
&= (\rho^2 + a^2 - 2m\rho)\sin^2\theta \qquad (3.387)
\end{aligned}
$$

Therefore

$$
A' = \frac{\rho^2 + a^2}{\rho^2 + a^2 - 2m\rho} \qquad (3.388)
$$

and

$$
B' = \frac{a}{\rho^2 + a^2 - 2m\rho} \qquad (3.389)
$$

The line element is now given by (3.380) with the last two terms vanishing by construction. The calculation of the coefficient of $d\rho^2$ can be simplified

$$
\begin{aligned}
& g_{00}A'^2 + g_{33}B'^2 + 2g_{01}A' + 2g_{13}B' + 2g_{03}A'B' \\
&= A'(A'g_{00} + B'g_{03} + g_{01}) + B'(A'g_{03} + B'g_{33} + g_{13}) + g_{01}A' + g_{13}B' \\
&= g_{01}A' + g_{13}B' \\
&= (-1)\frac{\rho^2 + a^2}{\rho^2 + a^2 - 2m\rho} + a\sin^2\theta\frac{a}{\rho^2 + a^2 - 2m\rho} \\
&= -\frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho} \qquad (3.390)
\end{aligned}
$$

The line element is then

$$ds^2 = g_{00}dt^2 + (g_{01}A' + g_{13}B')d\rho^2 + g_{22}d\theta^2 + g_{33}d\tilde{\varphi}^2 + 2g_{03}dtd\tilde{\varphi} \tag{3.391}$$

$$\begin{aligned} ds^2 &= \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right) dt^2 - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho} d\rho^2 - (\rho^2 + a^2\cos^2\theta)d\theta^2 \\ &- \left((\rho^2 + a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right) d\tilde{\varphi}^2 + \frac{4ma\rho\sin^2\theta}{\rho^2 + a^2\cos^2\theta} dtd\tilde{\varphi} \end{aligned} \tag{3.392}$$

This is the Kerr solution in Boyer-Lindquist coordinates and is analogous to the Schwarzschild coordinates for a non-rotating black hole.

3.7.5 Interptetation as Rotating Body

$$\begin{aligned} ds^2 &= \left(1 - \frac{2mr}{\rho^2}\right) dt^2 + \frac{2mar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) \\ &- \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2 \end{aligned} \tag{3.393}$$

where

$$\Delta(r) = r^2 - 2mr + a^2 \tag{3.394}$$

and

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{3.395}$$

$$a = J/M \tag{3.396}$$

where J is the Komar anagular momentum.

3.7.6 Basic Propertire of the Kerr Solution

Boyer-Lindquist cordinates

If we let $a \to 0$ we obtain

$$\begin{aligned} ds^2 &= \left(1 - \frac{2mr}{r^2}\right) dt^2 - \frac{r^2}{r^2 - 2GMr} dr^2 \\ &\quad - r^2 d\theta^2 - \frac{\sin^2\theta}{r^2} r^4 d\phi^2 \end{aligned} \tag{3.397}$$

i.e. as $a \to 0$ Boyer-Lindquist cordinates reduce to Schwarschild coordinates.

The Boyer-Lindquist form is the most useful one for investigating the properties of the Kerr solution.

• If we set $a = 0$, the regain the Schwarschild solution in Schwarschild coordinates and so m is identified with the geometric mass;

• The metric coefficients are independent of t and hence the solution is stationary;

• The metric coefficients are independent of ϕ and hence the solution is axially symmetric, i.e. there is an axis such that the solution is invariant under rotation about this axis.

• As for discrete symmetries, the solution is invariant under the symultaneous inversion of t and ϕ, that is under the transformation

$$t \to -t, \qquad \phi \to -\phi. \tag{3.398}$$

This suggests that the Kerr solution may correspond to a spinning source, since running time backwards with negative spin direction is equivalent to unning time forward with positive spin directin.

Cartesian Coordinates

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta \cos\phi \\ y &= \sqrt{r^2 + a^2} \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{3.399}$$

3.7.7 Singularities and Event Horizons

Event horizons are null surfaces beyond which it is impossible to return to a certain region of space.

The stataionary limit surface is timelike exery where except where it s tangent to the event horizons at the poles. It represents the place past which it is impossible to remail stationay.

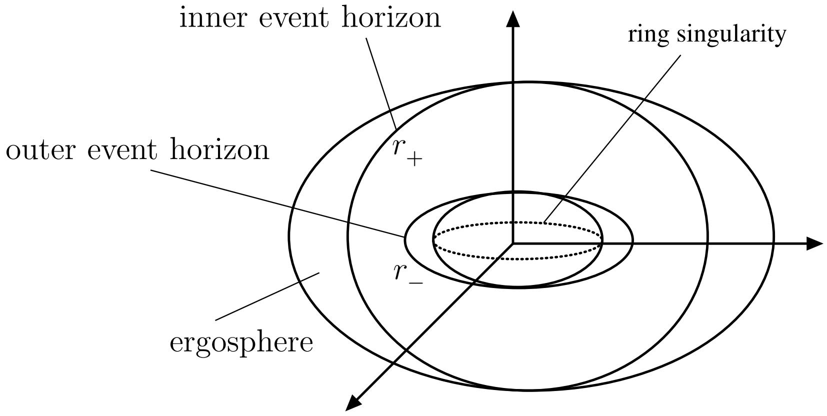


Figure 3.6: Rotating blackhole

$$\Delta = r^2 - 2mr + a^2 = 0, \tag{3.400}$$

two null event horizons

$$r_{\pm} = m \pm (m^2 - a^2)^{1/2}. \tag{3.401}$$

Then in a similarly way in which the Reissner-Nordstrom solution is regular in three regions:

I. $r_+ < r < \infty$

II. $r_- < r < r_+$

III. $0 < r < r_-$

In the limit $a \to 0$

3.8 Raychaudhuri equations

3.8.1 Null geodesic congruences

These equations are not very different (in structure as well as consequences thereof) from the equations for timelike congruences.

3.9 Properties of Null Surfaces

tangent vector is also normal

-

- Null surfaces

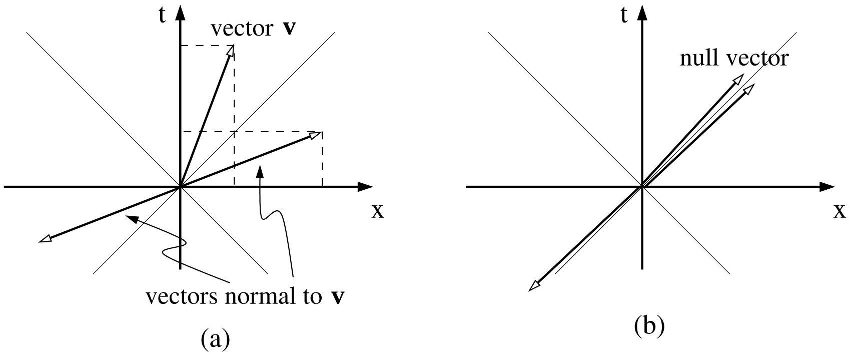
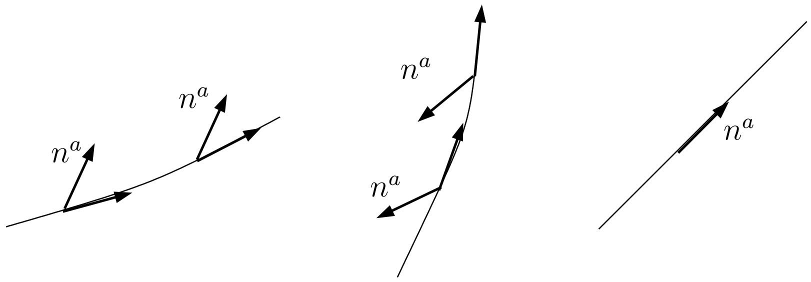


Figure 3.7: Normal and tangent vector to a tangent



(a) timelike surface (b) spacelike surface (c) null surface

Figure 3.8: (a) Normal to a timelike surface, (b) Normal to a spacelike surface, (c) Normal to a null surface.

$$n^a \nabla_a u = 0 \tag{3.402}$$

n^a is tangent to the level surfaces of u **null generators** of u.

$$n^b\nabla_b n^a = (\nabla^b u)\nabla_b\nabla_a u = (\nabla^b u)\nabla_a\nabla_b u = \frac{1}{2}\nabla_a(\nabla^b u\nabla_b u) = 0 \tag{3.403}$$

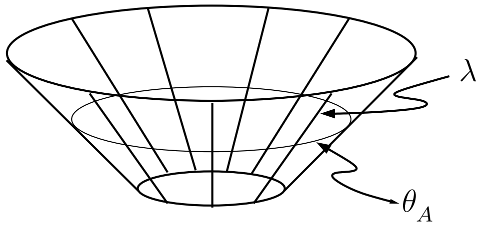


Figure 3.9: Coordinatizing a null surface in Minkowsian spacetime - λ, θ_A.

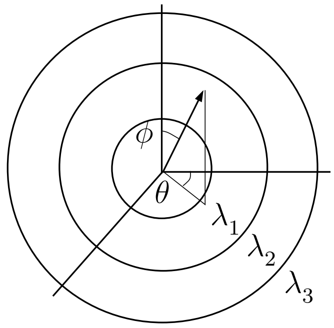


Figure 3.10: Each two sphere. Coordinates λ, θ, ϕ.

If we follow the null geodesics forward in time, instant by instant, the outward rays take us along a sequence of shperes increasing in radius and area, while the inward rays take us along a sequence of spheres whose radius and area decreases.

unlike Killing horizons, isolated horizons do not require there to be a Killing vector field in their neighbourhood.

Additionary properties of black hole horizons can be inferred from the Raychaudhuri's equation.

A trapped surface $\theta_{(\ell)} < 0$ and $\theta_{(n)} < 0$

A marginally trapped surface is one where the above requirements are weakened so as to: $\theta_{(\ell)}, \theta_{(n)} \leq 0$

An outer marginally trapped surface $\theta_{(\ell)} \leq 0.$ is the boundary of a three-dimension volume whose expansion of the outgoing family of null geodesics orthogonal to S is everywhere non-postive, $\theta_{(\ell)} \leq 0$.

An apparent horizon is a closed, spacelike, two-surface is an apparent horizon if it is the outermost marginally traped surface.

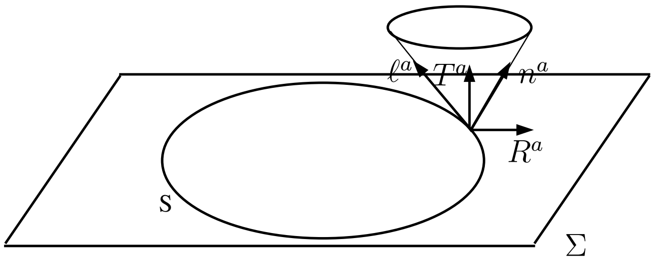


Figure 3.11: A spacial two-sphere S embedded in the spacial slice Σ (which in turn is embedded in spacetime $\mathcal{M}$), with two sets of orthogonal null vector fields. The vector field n^a is the unit timelike normal to Σ, R^a is the unit spacial normal to S, and n^a and ℓ^a are, respevtively, the outgoing and ingoing null vectors orthogonal to S.

For null congruences the condition $n \cdot l$ is not sufficient to eliminate the ambiguity in the choice of η because

$$
\begin{aligned}
\eta' \cdot l &= (\eta + al) \cdot l = \eta \cdot l + al \cdot l \\
&= \eta \cdot l
\end{aligned}
\tag{3.404}
$$

so that $\eta' \cdot l = 0$ whenever $\eta \cdot l = 0$.

$$
\boxed{n^a n_a = 0, \quad n^a l_a = -1} \tag{3.405}
$$

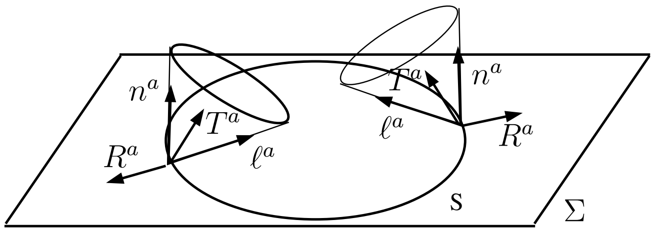


Figure 3.12: A spacial two-sphere S .

Since the metric q_{ab} induced on the hupersurface is degenerate ($q_{ab}\ell^b = 0$)

$$
\mathcal{D}_a q_{bc} \triangleq \partial_a q_{bc} + C^d_{ab} q_{cd} + C^d_{ac} q_{bd} = 0 \tag{3.406}
$$

$$
C^d_{ab} q_{dc} \triangleq \frac{1}{2}(\partial_c q_{ab} - \partial_a q_{bc} - \partial_b q_{ac}). \tag{3.407}
$$

This we see that q_{ab} does not uniquely determine $C^d_{ab} q_{dc}$ - if C^d_{ab} is a solution of (3.407), then so is $C^d_{ab} + \alpha_{ab}\ell^d$ for any symemtric tensor α_{ab}.

If l is tangent to an out going radial null geodesic, then n is tangent to an out going one.

The vector η now span a two-dimensional subspace, $T_\perp$, of the tangent space, that is orthogonal to both l and n, i.e. $P\eta$, where

$$P^a{}_b = \delta^a{}_b + n^a l_b + l^a n_b \tag{3.408}$$

projects onto $T_\perp$.

We decompose B as we did for timelike congruences:

$$\hat{B}_{ab} = \frac{1}{2}\theta P_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab} \tag{3.409}$$

where

$$\begin{aligned} \theta &= \hat{B}^a{}_a && \text{expansion} \\ \hat{\sigma}_{ab} &= \hat{B}_{(ab)} - \frac{1}{2}P_{ab}\theta && \text{shear} \\ \hat{\omega}_{ab} &= \hat{B}_{[ab]} && \text{twist} \end{aligned} \tag{3.410}$$

Lemma $l_{[a}\hat{B}_{bc]} = l_{[a}B_{bc]}$

Proof:

bf Proposition The tangents l are normal to a family of null hypersurfaces if and only if $\hat{\omega} = 0$.

Proof:

$$\begin{aligned} 0 &= l_{[a}\hat{\omega}_{bc]} \equiv l_{[a}\hat{B}_{bc]} \\ &= l_{[a}B_{bc]} \\ &= l_{[a}D_c l_{b]} \end{aligned} \tag{3.411}$$

Expansion and Shear

$$\mathcal{A} = \epsilon^{abcd} l_a n_b \eta_c^{(1)} \eta_d^{(2)} \tag{3.412}$$

Since $l^a \nabla_a l^b = 0$ and $l^a \nabla_a n^b = 0$, we have

$$\begin{aligned}
\frac{\partial \mathcal{A}}{\partial \lambda} = l^a \partial_a \mathcal{A} &= l^a \nabla_a \mathcal{A} = \epsilon^{abcd} l_a n_b (\nabla_l \eta_c^{(1)} \eta_d^{(2)})(\eta_c^{(1)} \nabla_l \eta_d^{(2)}) \\
&= \epsilon^{abcd} l_a n_b \left[\hat{B}_c{}^e \eta_e^{(1)} \eta_d^{(2)} + \eta_c^{(1)} \hat{B}_d{}^e \eta_e^{(2)} \right] \\
&= \epsilon^{abcd} l_a n_b \hat{B}_c{}^e \eta_{[e}^{(1)} \eta_{d]}^{(2)} \\
&= \theta \mathcal{A} \qquad (3.413)
\end{aligned}$$

i.e. θ mearsures the rate of increase of the magnitude of the area element. If $\theta > 0$ neighboring geodesics are ***diverging***, if $\theta < 0$ they are ***converging***.

3.9.1 geodesics: Expansion, Rotation, and Shear

weak energy condition

$$\text{If } v^\alpha \text{ non} - \text{spacelike } T^{\alpha\beta} v_\alpha v_\beta \geq 0. \qquad (3.414)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \qquad (3.415)$$

we contract with $v^\mu v^\nu$

$$R_{\mu\nu} v^\mu v^\nu - \frac{1}{2} g_{\mu\nu} v^\mu v^\nu R = \kappa T_{\mu\nu} v^\mu v^\nu \qquad (3.416)$$

$$T_{\mu\nu} v^\mu v^\nu \qquad (3.417)$$

$$\boxed{\frac{d^2 \mathcal{A}^{1/2}}{d\lambda^2} = -(|\sigma|^2 + \frac{1}{2} R_{\alpha\beta} k^\alpha k^\beta) \mathcal{A}^{1/2} \qquad (3.418)}$$

$$T_{\alpha\beta} W^\alpha W^\beta \stackrel{*}{=} T_{00} \geq 0 \qquad (3.419)$$

by continuity

$$T_{\alpha\beta} k^\alpha k^\beta \geq 0 \quad k^2 = 0. \qquad (3.420)$$

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \sim T_{\alpha\beta} \qquad (3.421)$$

implies

$$R_{\alpha\beta}k^\alpha k^\beta \geq 0 \tag{3.422}$$

implies

$$\boxed{\frac{d\mathcal{A}^{1/2}}{d\lambda^2} \leq 0. \qquad (3.423)}$$

Raycherdhuri's equation for null geodesic congruences.

These equations are not very different (in structure as well as consequences thereof) from the equations for timelike congruences.

3.9.2 Frobenius' Theorem

$$t_a = \lambda f_{,a}, \tag{3.424}$$

$$t_a \partial_b t_c = \lambda f_{,a} \lambda_b f,_c + \lambda^2 f_{,a} f_{,c} \tag{3.425}$$

$$t_{[a} \partial_b t_{c]} = 0 \tag{3.426}$$

because of the symmetry of the lower two indices of the connection, i.e, $\Gamma^c_{ab} = \Gamma^c_{bb}$, we can replace the partial derivative in the above expression by the covariant derivative,

$$t_{[a} \nabla_b t_{c]} = 0 \tag{3.427}$$

Timelike Case

We are assuming that $t^2 \neq 0$.

$$t_{[a} \nabla_b t_{c]} = 0 \tag{3.428}$$

We wish to show that if this condition is met then $t_a = \lambda f_{,a}$. $t_a = t^2 f_{,a}$ $\lambda = t^2$. That is, there is a function f such that $f_{,a} = t_a / t^2$ - this would be equivalent to showing

$$\partial_a \left(\frac{t_b}{t^2} \right) = \partial_b \left(\frac{t_a}{t^2} \right), \tag{3.429}$$

This indeed what we shall do. which would be a problem if we didn't have $t^2 \neq 0$. For the purposes of comparision we use the quotient diferentiation rule then multiply by $(t^2)^2$ the condition (3.429) becomes

$$t^2\partial_a t_b - t_b\partial_a t^2 = t^2\partial_b t_a - t_a\partial_b t^2 \tag{3.430}$$

This proof works if $t^2 \neq 0$; the null case $t^2 = 0$ will be considered separately.

$\nabla_a t_b = -\nabla_b t_a$

$$t_a\nabla_b t_c + t_c\nabla_a t_b + t_b\nabla_c t_a = 0, \tag{3.431}$$

contracting with t^c we get

$$t_a t^c\nabla_b t_c + t^2\nabla_a t_b + t_c t^c\nabla_c t_a = 0 \tag{3.432}$$

using Killing's equation again,

$$t_a t^c\nabla_b t_c - t^2\nabla_a t_b - t_c t^c\nabla_c t_a = 0 \tag{3.433}$$

using $t^c\nabla_a t_c = t_c\nabla_a t^c$

3.9.3 Null Case

$$\begin{aligned}\tilde{B}_{ab} &= (g_a{}^c + k_aN^c + N_ak^c)(g_b{}^d + k_bN^d + N_bk^d)B_{cd}\\ &= (g_a{}^c + k_aN^c + N_ak^c)(B_{cb} + k_bB_{cd}N^d)\\ &= B_{ab} + k_aN^cB_{cb} + k_bN^cB_{ac} + k_ak_bB_{cd}N^cN^d\end{aligned} \tag{3.434}$$

$$\nabla_{[a}k_bk_{c]} = 0. \tag{3.435}$$

implies

$$B_{[ab]}k_c + B_{[ca]}k_b + B_{[bc]}k_a = 0 \tag{3.436}$$

$$
\begin{aligned}
B_{[ab]} &= B_{[ca]}k_b N^c + B_{[bc]}k_a N^c \\
&= \frac{1}{2}(B_{ca}k_b - B_{ac}k_b + B_{bc}k_a - B_{cb}k_a)N^c) \\
&= B_{c[a}k_{b]}N^c + k_{[a}B_{b]c}N^c
\end{aligned}
\tag{3.437}
$$

but from eq (4.3.4)

$$
\tilde{B}_{ab} = B_{[ab]} - B_{c[a}k_{b]}N^c - k_{[a}B_{b]c}N^c \tag{3.438}
$$

3.9.4 Killing Horizons

Defintion: A null hypersurface $\mathcal{N}$ is a killing horizon of a Killing vector field ξ if, on $\mathcal{N}$, ξ is normal to $\mathcal{N}$.

Formula for surface gravity

$$
\xi_{[a}\nabla_b\xi_{c]}|_{\mathcal{N}} = 0 \tag{3.439}
$$

$$
\kappa = -\frac{1}{2}(\nabla^a\xi^b)(\nabla_a\xi_b)|_{\mathcal{N}} \tag{3.440}
$$

Surface gravity is the acceleration of a static particle near the horizon as a measured at spacial infinity.

Killing horizon:

surface gravity

$$
\nabla_{[a}t_b t_{c]} \tag{3.441}
$$

Using the Kiling's equation $\nabla_a t_b = -\nabla_b t_a$

$$
t_a\nabla_b t_c + t_c\nabla_a t_b + t_b\nabla_c t_a = 0, \tag{3.442}
$$

contracting with $\nabla^a t^b$

$$
\begin{aligned}
(\nabla^a t^b)(\nabla_a t_b)t_c &= -(\nabla_a t_c)(\nabla_b t^a)t^b + (\nabla_c t_b)(\nabla_a t^b)t^a \\
&= -\kappa\nabla_a t_c)t^a + \kappa\nabla_c t_b t^b \\
&= -2\kappa^2 t_c.
\end{aligned}
\tag{3.443}
$$

$$
\kappa^2 = -\frac{1}{2}\nabla^b t^a \nabla_b t_a \tag{3.444}
$$

3.10 Laws of (Stationary) Black Hole Mechanics

3.10.1 Zeroth law

3.10.2 First law

Some matter falls into a black hole. The resulting black hole will oscilate, emmitting gravitational radiation, but it quickly settles down to another Schwarzschild black hole. Could the energy carried away by gravitational radiation be greater than the matter that fell in, leaving a lower mass Schwarzschild black hole than the initial one? This cannot happen because the area of Schwarzschild black holes are related to their mass by $A = 4\pi(2M)^2$.

$$
\begin{aligned}
M_H - 2\Omega_H J_H &= -\frac{1}{8\pi}\int_{\partial V} \nabla^a(t^a + \Omega\phi^b)dS_{ab} \\
&= -\frac{1}{8\pi}\int_{\partial V} \nabla^a \xi^b ds_{ab} \\
&= -\frac{1}{4\pi}\int_{\partial V} \xi_a n_b \nabla^a \xi^b dS \\
&= -\frac{1}{4\pi}\int_{\partial V} \kappa\xi^b n_b dS \\
&= -\frac{1}{4\pi}\int_{\partial V} dS.
\end{aligned}
\tag{3.445}
$$

$$
\delta M = -\int T^a{}_b t^b d\Sigma_a, \qquad \delta J = \int T^a{}_b \phi^b d\Sigma_a \tag{3.446}
$$

$$
dS_{ab} = 2\xi_{[a} n_{b]} dS \tag{3.447}
$$

$$
\begin{aligned}
\delta M - \Omega_H \delta J &= \int T_{ab}(t^a + \Omega_H \phi^b)\xi^a dS dv \\
&= \int dv \int T_{ab}\xi^a \xi^b dS.
\end{aligned} \tag{3.448}
$$

$$
\frac{d\theta}{dv} = \kappa\theta - 8\pi T_{ab}\xi^a \xi^b. \tag{3.449}
$$

$$
\begin{aligned}
\delta M - \Omega_H \delta J &= \int (\frac{d\theta}{dv} - \kappa\theta) dS \\
&= v|_{-\infty}^{\infty} - \kappa \int (\frac{1}{dS}\frac{dA}{dv}) dS \\
&= \delta A.
\end{aligned} \tag{3.450}
$$

3.10.3 Second law

The second law of black hole mechanics states that the area of the event horizon cannot decrease.

Irreversible tendency for geodesics to converge toward each other.

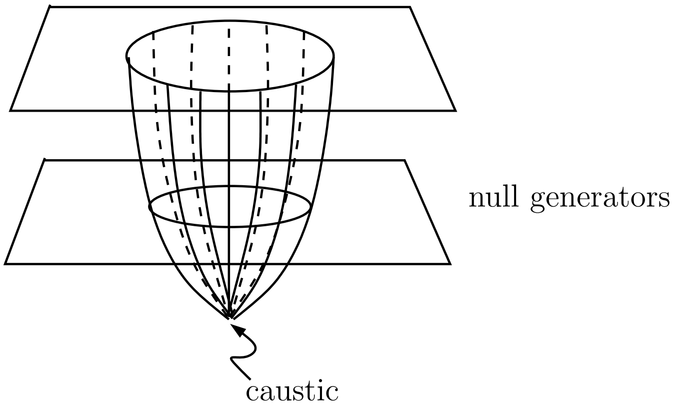


Figure 3.13: In Schwarzschild black hole the horizon is generated by the radial light rays, which meet at the center.

Glossing over technical details... If there were points on the horizon that can be joined by a timelike curve, it is then possible to show that there is a small deformation of this curve that results in a future-directed timelike curve starting inside the black hole and ending outside the event horizon - contradicting the very defination of an event horizon.

The chronological future $I^+(S)$ of a set S is the set of points of the spacetime $\mathcal{M}$ can be reached from p by a future-directed timelike curve.

Open sets - A defining property of open sets is that there is always 'room for variation'. An open interval of the real line is the set of points between a and b excluding a and b. As the end points are removed, any point in the set (a, b) is always a non-zero distance from the edge of the set, there is always room to move from side to side.

Locally the geometry is Minkowski spacetime (even at the event horizon!) so we know locally any non-spacelike separated points on the horizon are joined by a null geodesic. The same is true globally. Suppose that p and q were two points of the event horizon which are timlike separated. As chronological futures and pasts are open sets, there is always room for the timelike curve between p and q to be deformed to a nearby timelike curve between new points p' and q' with p' inside the event horizon but q' outside. We have a contradiction, the timelike curve between p and q cannot exist.

We know that in Minkowsian spacetime that the chronogolical future (past) of a set is open. The reason why the chronoligical future (past) of a set is open in general follows from the fact that if a region is sufficiently small, its geometry is indistinguishable from that of flat spacetime. Such regions are refered to as simple regions or normal neightbourhoods.

There is a future-directed timelike γ joing p to q. Consider a point z on γ in the past of q which is sufficiently close to q so that a normal neighbourhood $U(z)$ exists which contains q.

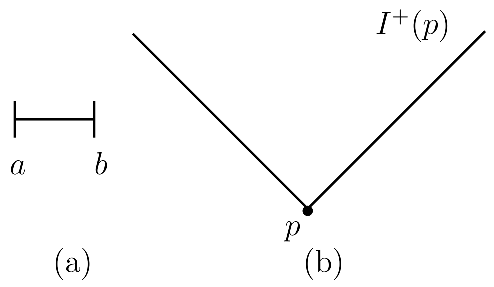


Figure 3.14: (a) An open interval of the real line is the set of points between a and b excluding a and b. (b) .

A neighbourhood of the point q that is sufficiently small lies entirely within the chronological future of p. Similarly, a neighbourhood of the point p, that is sufficiently small, lies entirely within the chronological past of the point q. From this it follows that any point in $U(p)$ can be joined to any point in $V(q)$ by a timelike curve (see fig(3.17)). A slight deformation of the curve which keeps it timelike will make it link an arbitrary event p in some sufficiently small neighbourhood $U(a)$ to an arbritary event in some small neighbourhood $V(a)$.

As a lies on the event horizon, the neighbourhood $U(a)$ overlaps with both the interior and exterior of the black hole, and similarly for $V(b)$. Take any point in the neighbourhood

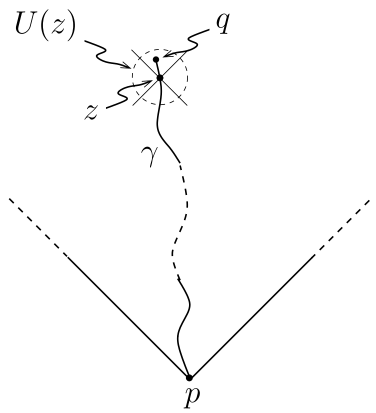


Figure 3.15: q lies in the chronological future of z.

$U(a)$, which lies within the event horizon, label it p and any point in the neighbourhood $V(b)$ which lies outside the event horizon, label it q. These two points can be joined a timelike curve, γ' (see (3.17)).

Conjugate points of neighbouring geodesics...

From Raychaudhuri's equation, if two null geodesics start to converge, they will intersect within affine parameter time. This is the instability of the atractive influence of gravity.

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}\ell^a\ell^b, \tag{3.451}$$

where the vorticity is zero because hypersurface orthogonal. We have the inequality

The Raychaudhuir equation implies the following inequality for the expansion $\theta(\lambda)$

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \tag{3.452}$$

Integrating this

$$\int \frac{d\theta}{\theta^2} \leq -\frac{1}{2}\int_0^\lambda d\lambda \tag{3.453}$$

gives us

$$\theta^{-1}(\lambda) \geq \theta^{-1}(0) + \frac{\lambda}{4} \tag{3.454}$$

From this inequality we see that if the congruence is initially converging ($\theta(0) < 0$), then $\theta^{-1}(\lambda) \to -\infty$ within an affine parameter $\lambda \leq 2/|\theta(0)|$.

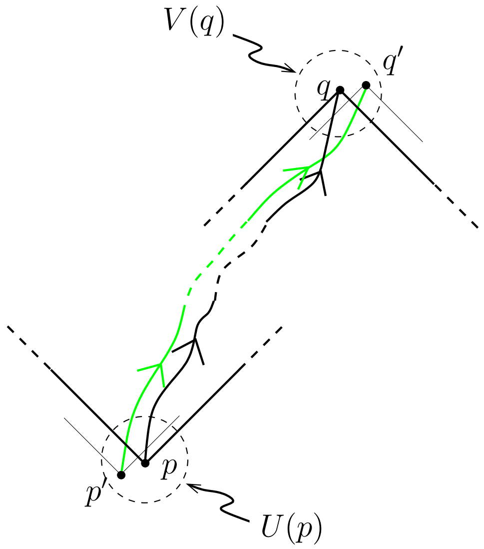


Figure 3.16: The chronological future $I^+(p)$ of p is an open set; given any point $q \in I^+(p)$, there exists a sufficiently small neighbourhood $V(q)$ contained in $I^+(p)$. Similar statements hold for the p in the chronological past $I^-(q)$ of q.

It is obvious in Minkowskian spacetime that if one has path that is partly null and partly timelike, then there is a timelike curve between the two points (fig(5.12)). To prove the same is true in general on curved spacetimes, we will again employ the fact that if a region is sufficiently small, its geometry is indistinguishable from that of flat spacetime.

We are making use of the fact that it is true in Minkowski spacetime fig. (5.12). Cover the null geodesic with a finite number of normal neighbourhoods.fig. (5.13). Pick a point between x_1 and call it z_1.

That is, if one of the geodesics encounters a singularity before it has a chance to cross the other geodesic.

If one of these geodesics in fig.(5.13) encountered a singularity befor it crossed the other geodesic. We required that the spacetime manifold to be smooth, and also the metric. The singular point at the centre of the Schwartzchild solution where the curvature diverges, must be missing. The radial geodesics thus come to a full stop. Before the second conjugate point is reached, a singularity by a geodesic comming to a full stop if the null geodesic in fig (3.18). The geodesic cannot be extended within the manifold, but nevertheless ends at a finite value of the affine parameter. We require that any null generator of the horizon does not meet a singularity. the *cosmic cencership conjecture* holds. Hawking relaxed this condition that is less strict and so emcoposess more spacetimes, called the *weak cosmic censorship condition.*

In this last part prove the statements in the Schwartzchild solution.

$\dot{J}^-(\mathcal{I}^+)$ is generated by null geodesics that never leave the horizon. The generator followed into the past lies in the surface of the event horizon or is in $J^-(\mathcal{I}^+)$.

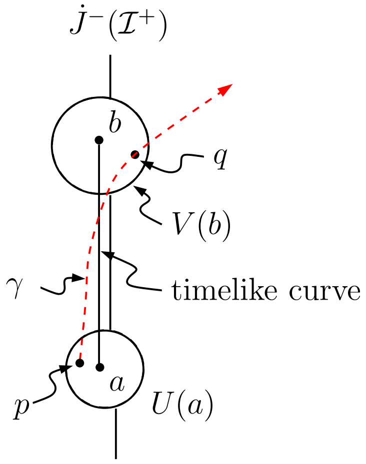


Figure 3.17: If two points on the event horizon are timelike separated, we can produce a timelike curve starting inside the black hole which joins to a point outside the event horizon.

There is a timelike curve, D, from p to b. Because $b \gg p$ there exists a neighbourhood, $V(b)$, of b which is entirely contained inside $I^+(p)$. As $U(a)$ was an arbitrarily small neighbourhood a cannot be inside the event horizon, rather it is in either $\dot{J}^-(\mathcal{I}^+)$ or $J^-(\mathcal{I}^+)$.

So any that comes to be a null generator must come from outside the black hole.

3.10.4 Third law

The third law of black hole mechanics states that the surface gravity of a black hole cannot be reduced to zero within a finite advanced time.

3.10.5 Quasi-Local Generalizations

The notion of an isolated horizon is arrived at by extracting from the definition of a Killing hroizon the minimum conditions necessary for black hole thermodynamics.

3.10.6 Black Hole Thermodynamics

from theomodynamics we se the temperature and entropy arises from underlying statistical mechanics. What microstaes are responsible for black hole thermodynamics?

a classical, stationary black hole is determined completely by its mass, charge, and angular momentum, with no room for additional microscopic states to account for thermal

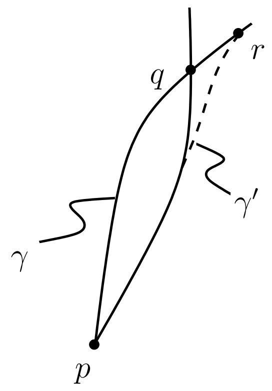


Figure 3.18: When neighbouring null geodesics have conjugate points there exists a timlike curve joining the two conjugate points. The dashed line represents a timelike curve joining to the null geodesic. The points q and q' are timelike separated - rounding off the corner. We make this argument more rigouress in the appendix M.

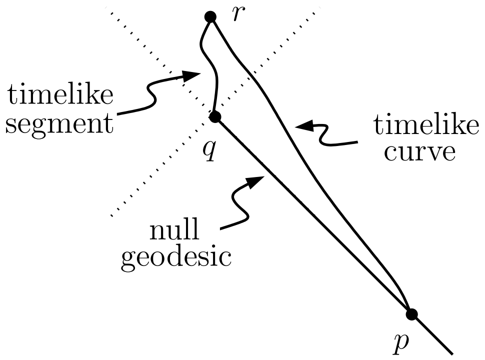


Figure 3.19: In flat spacetime, when a null geodesic curve joins onto a timlike curve, there exists a timelike curve between p and q.

behaviour.

If black hole thermodynamics has a statistical mechanical origin then the relavent states must therefore be non-classical.

3.11 Definitions

A trapped surface $\theta_{(\ell)} < 0$ and $\theta_{(n)} < 0$

A marginally trapped surface $\theta_{(\ell)}, \theta_{(n)} \leq 0$

An outer marginally trapped surface $\theta_{(\ell)} \leq 0.$ is the boundary of a three-dimension volume whose expansion of the outgoing family of null geodesics orthogonal to S is everywhere

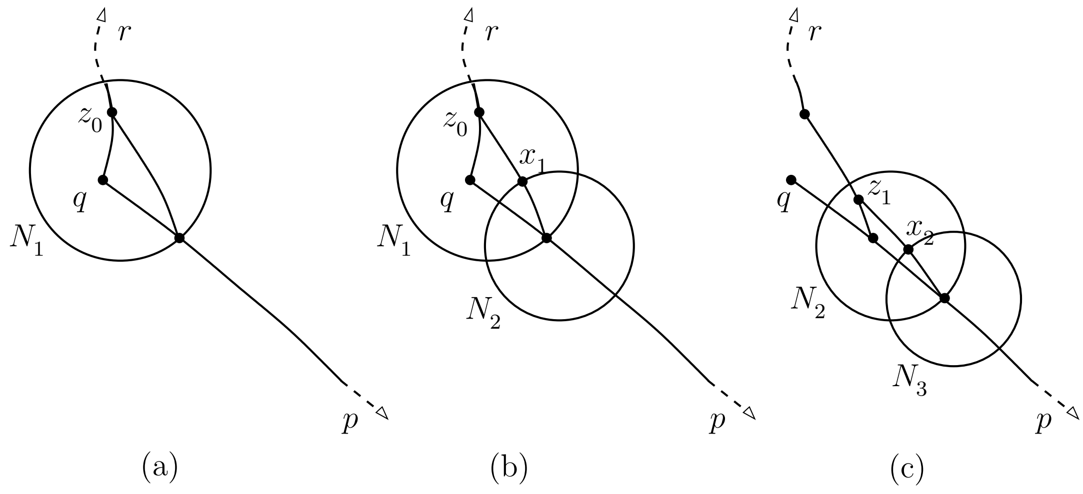


Figure 3.20: There exists a timlike curve joining the two conjugate points. A timelike curve joining to the null geodesic. (b) Contiuing in this way, we "peel" away a timelike curve that joins r and p.

non-postive, $\theta_{(\ell)} \leq 0$.

An apparent horizon

A closed, spacelike, two-surface is an apparent horizon if it is the outermost marginally traped surface.

3.12 Non-Expanding Horizons (NEH)

The very acsessible [64], *A 3+1 perspective on null hypersurfaces and isolated horizons*

Any null normal 'is expansion free and by defnition. It is also automatically twist free because it is normal to a smooth surface. Additional properties use the Raychaudhuri equation

$$0 = -\mathcal{L}_\ell \theta_\ell \triangleq |\sigma_{(\ell)}|^2 + R_{ab}\ell^a \ell^b \tag{3.455}$$

From the energy condition it follows that $R_{ab}\ell^a \ell^b \triangleq 8\pi G T_{ab}\ell^a \ell^b$

Since $|\sigma_{(\ell)}|^2$ is also positive, it follows from (2.3) that

$$|\sigma_{(\ell)}|^2 = 0, \qquad R_{ab}\ell^a \ell^b = 0. \tag{3.456}$$

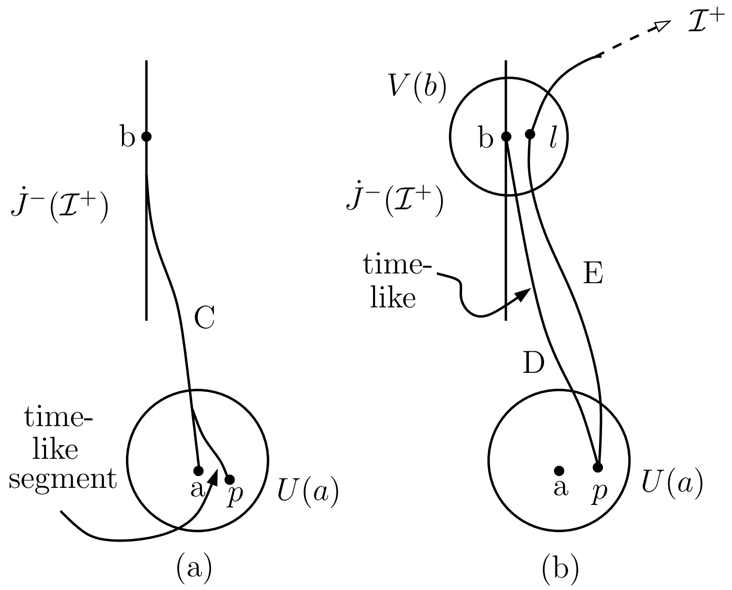


Figure 3.21: There exists a timlike curve joining the two is way, we a timelike curve that joins r and p.

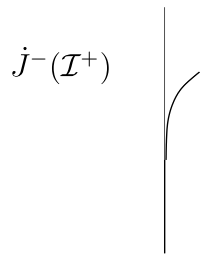


Figure 3.22: There exists a timlike curve joining the two is way, we a timelike curve that joins r and p.

Since ℓ is expansion, shear and twist free, it follows that there must exist a one-form $\omega_a^{(\ell)}$ associated with ℓ such that

$$\nabla_a \ell^b \triangleq \omega_a^{(\ell)} \ell^b \tag{3.457}$$

$$\ell^a \nabla_a \ell^b \triangleq \ell^a \omega_a^{(\ell)} \ell^b = \kappa_{(\ell)} \ell^b \tag{3.458}$$

where $\kappa_{(\ell)} := \ell^a \omega_a^{(\ell)}$ the surface gravity of Δ.

Under the rescalings $\ell \mapsto \ell' = f\ell$, of the null normal ℓ, it transforms via:

$$\omega_a \mapsto \tilde{\omega}_a = \omega_a + \nabla_{\underleftarrow{a}} \ln f \tag{3.459}$$

and the surface gravity transforms as

$$\kappa_{(\ell')} := \ell'^a \omega_a^{(\ell')} = f\kappa_{(\ell)} + \ell^a \partial_a f \tag{3.460}$$

From (4.3.4) we also get

$$\mathcal{L}_\ell q_{ab} \triangleq \mathcal{L}_\ell g_{ab} \triangleq 2\nabla_a \ell_b \triangleq 0. \tag{3.461}$$

Thus, every null normal ℓ is a 'Killing field' of the degenerate metric on Δ. Although ℓ is a 'Killing field' of the intrinsic horizon geometry, the space-time metric g_{ab} need not admit a Killing field in any neighborhood of Δ. Robinson-Trautman metrics [??] provide explicit examples of this type.

3.13 Weakly Isolated Horizons (WIH) and Generalization of the Laws of Black Hole Mechanics

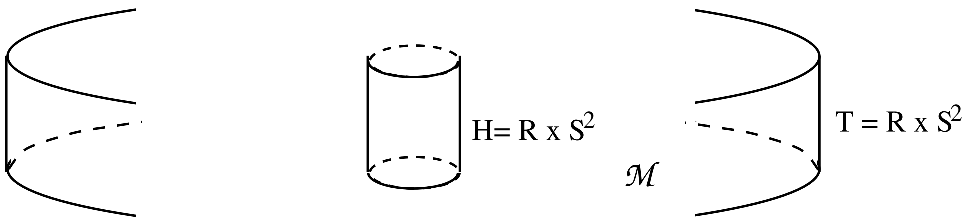


Figure 3.23: Classical bourndary conditions for weakly isolated horizons.

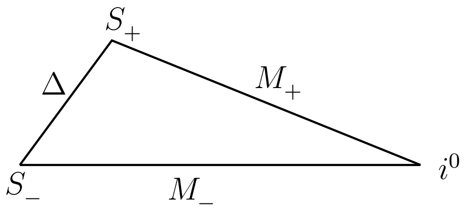


Figure 3.24: Conformal spacetime diagram of a WHI.

The lapse function N and shift N^a will define the flow of time relative to this surface in the usual way. That is, Ndt units of proper time pass for every dt units of coordinate time, and an observer swept along by the flow of time will move with velocity $\sqrt{N^a N_a}/N$.

Keep in mind that depending on the spacetime, such "motion" may or may not correspond to "real" movement. For example, a particle which is static on the Schwarzchild horizon

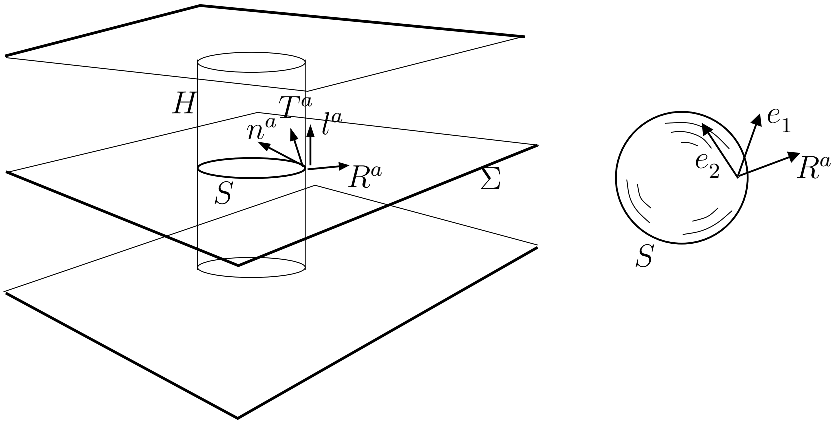


Figure 3.25: Construction of a null tetrad for a NEH. Any spatial two surface S determines uniquely, up to rescaling, two null vectors orthogonal to S.

will have a velocity equal to the speed of light, but doesn't actually move by any normal definition of movement.

3.13.1 Zeroth law

black hole mechanics says that the surface gravity of a weakly isolated horizon is constant. This result hold even when the horizon is highly distorted so long as it is in equilibrium.

$$\kappa_{(\ell)} := \ell^a \omega_a \tag{3.462}$$

$$\mathcal{L}_\ell \omega_a = 0. \tag{3.463}$$

the curl of ω_a is related to the imaginary part of Ψ_2:

$$\mathcal{D}_{[a}\omega_{b]} = (Im\Psi_2)\epsilon_{ab} \tag{3.464}$$

or, as is usually written, in terms of the covariant exterior derivative

$$d\omega = 2(Im\Psi_2)\epsilon \tag{3.465}$$

where ϵ is the area 2-form on Δ ($\mathcal{L}_\ell \epsilon$ and $\ell \cdot \epsilon$). Hence we conclude $\ell \cdot d\omega = 0$ which implies that $\kappa_{(\ell)}$ is constant on the horizon:

$$0 = \mathcal{L}_\ell \omega = d(\ell \cdot \omega) = d\kappa_{(\ell)} \tag{3.466}$$

3.13.2 First law

Hamiltonian theory well defined equivalent to the first law of black hole mechanics.

Two Hamiltonian frameworks are available. The first uses a covariant phase space which consists of the solutions to field equations which satisfy the required boundary conditions [26, 15]. Here the calculations are simplest if one uses a first order formalism for gravity, so that the basic gravitational variables are orthonormal tetrads and Lorentz connections. The second uses a canonical phase space consisting of initial data on a Cauchy slice M ofinner boundary

In this report, we use a Hamiltonian framework to calculate conserved quantities such as angular momentum and mass. We therefore begin by describing the phase space we are interested in. Let $\mathcal{M}$ be the region of spacetime that we are interested in. The boundary of M consists of four components: the timelike cylinder 1 at spatial infinity, two spacelike surfaces M which are the future and past boundaries of $\mathcal{M}$ and and inner boundary which is a weakly isolated horizon with a preferred class of null normals [‘] (see figure). The two spheres S are the intersections of M with .

shall use a first order formalism in which the fundamental fields are the gravitational connection A aI J and a tetrad e I a which satisfy appropriate boundary conditions (see [] for a formulation in terms of metrics and extrinsic curvature). The lowercase latin indices refer to the spacetime wnd the uppercase letters refer to a fixed internal four dimensional Lorentzian vector space with internal metric η_{IJ} with signature $(-;+;+;+)$. A Lorentz connection A_J denes a derivative operator acting on internal indices

$$D_a k_I := \partial_a k_I + A_{aI}{}^J k_J \tag{3.467}$$

where @ a is an arbitrary at fiducial derivative operator. Fix a preferred internal null tetrad $(‘I; nI; mI; mI)$ at . The allowed field config- urations $(AaIJ; eIa)$ are those which satisfy the appropriate fall-of conditions at infnity

to ensure asymptotic flatness; and at are such that (i) $a = ‘IeaI$ is a member of the preferred equivalence class $[\ell]$ fixed at and (ii) (; ‘) is a WIH. It turns out (see [3] for details) that due to the zeroth law, the standard gravitational action is a viable action even in the presence of the internal boundary Δ:

$$S(e, A) = -\frac{1}{16\pi G}\int_{\mathcal{M}} \Sigma^{IJ} \wedge F_{IJ} + \frac{1}{16\pi G}\int_{\tau_\infty} \Sigma^{IJ} \wedge A_{IJ} \tag{3.468}$$

where

$$\Sigma^{IJ} := \frac{1}{2}\epsilon_{IJKL} e^K \wedge e^L \tag{3.469}$$

and

$$F_I{}^J = dA_I{}^J + A_I{}^K \wedge A_K^J \tag{3.470}$$

is the curvature of the derivative operator D a . Our phase space , known as the covariant phase space, is the set of solutions to the eld equations satisfying the boundary conditions specied above. The symplectic structure is obtained by second variations of the action and leads to the following expression

3.13.3 Second law

3.14 Isolated Horizons

Boundary conditions at T : asymptotic flatness.

Boundary conditions at H : $\partial_\alpha v = n_\alpha$, $n_\alpha l^\alpha = 1$.

(i) H is null, $l^\alpha l_\alpha = 0$, l^α is tangent.

(ii) Gauge fixing - σ has:

$$l^\alpha = \sigma^\alpha_{AA'} l^A \bar{l}^{A'} \quad n^\alpha = \sigma^\alpha_{AA'} o^A \bar{o}^{A'} \tag{3.471}$$

(iii) Field equations hold at H.

(iv) Main conditions:

$$\begin{aligned} o^A D_\alpha o_A &= 0 \\ l^A D_\alpha l_A &= -if(v)\sigma^{AA'}_\alpha l_A \bar{o}^{A'} \quad f > 0 \end{aligned} \tag{3.472}$$

on H.

3.14.1 Boundary Conditions for Isolated Horizons

Boundary conditions that define an isolated horizon:

Consequences of the boundary conditions:

- The 2-spheres (v = const) are marginally trapped (expansion of l vanishes).

• The Horizon H is non-rotating (l and n are shear- and twist-free; expansion of n is spherically).

shear-free:

$$\sigma = m^a m^b \nabla_a l_b = 0. \tag{3.473}$$

twist-free:

$$l_{[a}\partial_b l_{c]} = 0. \tag{3.474}$$

(Let us assume that l^a is orthogonal to a family of hypersurfaces, the ***wave fronts***, i.e., there exist scalar fields v, w such that

$$l_a = v\nabla_a w. \tag{3.475}$$

Then it is easy to show that

$$l_{[a}\partial_b l_{c]} = 0. \tag{3.476}$$

In fact the converse holds: (3.476) implies (3.475) as a consequence of Frobenius' theorem.)

• H is a future horizon (expansion of n is negative).

• Metric on 2-sphere ($v =$ const) is independent of v *area is constant*, $\mathcal{A}$

• Gauge-fixing reduces A_α^{AB} to a (complex) $U(1)$ connection on H; only real $U(1)$ is dynamical variable.

• If $\Sigma^{AB} = \sigma^{AA'} \wedge \sigma_{A'}^{B}$ then

$$F_{\alpha\beta}^{AB} = -\frac{2\pi}{\mathcal{A}}\Sigma_{\alpha\beta}^{AB} \tag{3.477}$$

curvature of A pulled back to the sphere ($v =$ const).

3.15 Rotating Isolated Horizons

There are two distinct notions of multipole moments - source multipoles which encode the distribution of mass (or charge-current), and field multipoles which arise as coefficients in the asymptotic expansions of fields.

IH geometries may be distorted because of matter rings, stars, black holes in the exterior region. As we will see, a diffeomorphism invariant characterization can be provided by a set of mass and angular momentum multipole moments M_n, J_n.

3.15.1 Basic Review of Multipoles

Gravitainal (Electrostatic) Multipoles

Earth's gravitational field comming from the pair shaped deformation of the Earth.

The potential $\Phi(\vec{r})$ of the gravitational fieldb at $\vec{r}$ produced by a mass m at $\vec{r}'$ is

$$\Phi(\vec{r}) = \frac{m}{|\vec{r} - \vec{r}'|} \tag{3.478}$$

as in Fig.(3.15.1).

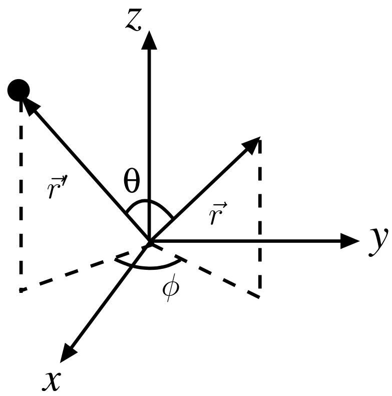


Figure 3.26: multipole is the position of the mass density source and $\vec{r}$ is the requested position for the potential $\Phi(\vec{r})$.

First, say we wish to calculate the gravitation field at a point r produced by matter located at $\vec{r}_s$ where is r_s a distance d away form the origin in the $z-$direction, see (Fig.3.15.1). The mass will be given as the density $\rho(\vec{r}_s)$ times a small volume element ΔV.

$$\Phi(r - r_s) = \frac{\rho(r_s)\Delta V}{|r - r_s|} = \left[1 + \left(-d\frac{\partial}{\partial z}\right) + \frac{1}{2!}\left(-d\frac{\partial}{\partial z}\right)^2 + \ldots\right]\Phi(r) \tag{3.479}$$

The terms in this expansion are easily evaluated using the following identity

$$\frac{\partial}{\partial z}\frac{1}{r^m} = 2z\frac{\partial}{\partial r^2}\frac{1}{(r^2)^{m/2}} = -\frac{mz}{r^{m+2}} = -\frac{m\cos\theta}{r^{m+1}}. \tag{3.480}$$

where $\cos\theta = \vec{r}\cdot\vec{r}_s/rr' = z/r$. We find for the first few terms

$$\Phi(r - r_s) = \rho(r_s)\Delta V \left(\frac{1}{r} + \frac{d\cos\theta}{r^2} + \frac{d^2}{2}\frac{3\cos^2\theta - 1}{r^3} + \ldots\right) \tag{3.481}$$

The first 3 terms are the monopole, dipole and quadrapole respectively. $P_n(\cos\theta)$. The first few polynomials are

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{3x^2-1}{2}, \quad P_3 = \frac{5x^3-3x}{2} \tag{3.482}$$

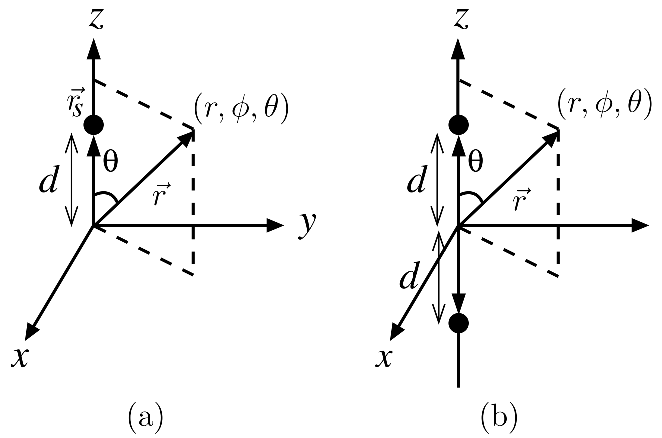


Figure 3.27: dipolemass (a) monpole. (b) dipole (c) quadrapole

We make the replacement $\cos\theta \to \vec{r}\cdot\vec{r}'/rr'$ and get:

$$\begin{aligned}\Phi(\vec{r}) &= \frac{1}{r}\int d^3x'\rho(\vec{r}') + \frac{\vec{r}}{r^3}\cdot\int d^3x'\rho(\vec{r}')\vec{r}' + \\ &+ \frac{1}{2r^5}\cdot\int d^3x'\rho(\vec{r}')\frac{(3\vec{r}\cdot\vec{r}' - r'^2r^2)}{2} + \ldots\end{aligned} \tag{3.483}$$

where

$$\begin{aligned} E_1 &= \int d^3x'\rho(\vec{r}') && \text{Monopole Moment} \\ E_2 &= \int d^3x'\rho(\vec{r}')\vec{r}' && \text{Dipole Moment} \\ E_3 &= \int d^3x'\rho(\vec{r}')\frac{(3\vec{r}\cdot\vec{r}' - r'^2r^2)}{2} && \text{Quadrupole Moment}\end{aligned} \tag{3.484}$$

The result of this integration, in the axisymmetric case, will be of the form:

$$
\begin{aligned}
\Phi(\vec{r}) &= \int r^n \left(\frac{1}{r} + \frac{d\cos\theta}{r^2} + \frac{d^2}{2}\frac{3\cos^2\theta - 1}{r^3} + \dots\right)\rho(\vec{r}')d^3x' \\
&= \frac{1}{r} - \sum_{n=1}^{\infty} E_n \frac{1}{r^{n+1}} \qquad (3.485)
\end{aligned}
$$

the coeffients E_n are defined as the *moments*

$$
E_n := \int r^n P_n(\cos\theta)\rho d^3x \qquad (3.486)
$$

Note that these terms only exist because of our choise of coordinates - take care with your origin (in section ?? we will see that the framework for the type II isolated horizon automatically places you in the 'centre of mass frame'). Consider the dipole (Fig. 3.15.1 a). They have the same mass and both are a distance d from the origin.

$$
\begin{aligned}
\Phi(r,\theta) &= \rho(r_s)\Delta V \left(\frac{1}{|r - r_s|} + \frac{1}{|r + r_s|}\right) \\
&= 2\rho(r_s)\Delta V \left(\frac{1}{r} + \frac{d^2}{2}\frac{3\cos^2\theta - 1}{r^3} + \dots\right) \qquad (3.487)
\end{aligned}
$$

Generally, if the matter distribution axisymmetric, and if the term $P_1(\cos\theta)$ is missing it is because we are sitting in the centre of mass frame.

The coefficients of the Legendre series match a known solution at a boundary. bounadry value problem.

$$
\Phi(r,\theta) = \frac{GM}{R}\left[\frac{R}{r} - \sum_{n=2}^{\infty} a_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta)\right] \qquad (3.488)
$$

P_1 is omitted since it would represent a displacement and not deformation. For a spherical case all the moments zanish apart from the zeroth $P_0 = 1$ and Φ is a function of r only. The scalar curvature of the surface of the Earth is constant over the surface and is just $1/R$.

When the mass is concentrated on a sphere S defined as $r = R$, the expression simplifies to

$$
E_n = R^n \oint_S P_n(\cos\theta)\tilde{\rho} d^2\tilde{V} \qquad (3.489)
$$

Say we have constant mass density over the surface of an elipsoid, defined by

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = R^2. \tag{3.490}$$

The mass density $\rho(x, y, z) = C$.

We can choose new coordinates to describe where the elipsoid is a sphere, if so the density, in the new coordinates will vary of the surface of this sphere. We introduce the coordinates

$$ax' = x, \quad ay' = y, \quad bz' = z. \tag{3.491}$$

$$x'^2 + y'^2 + z'^2 = R^2. \tag{3.492}$$

and the density is

$$\rho'(x', y', z') = \tag{3.493}$$

Angular Momentum (Magnetic) Multipoles

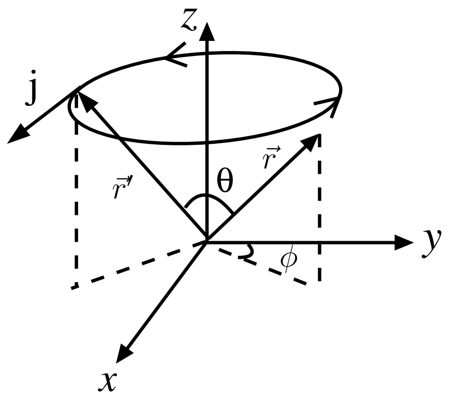


Figure 3.28: magmulty is the position of the current density and $\vec{r}$ is the requested position for the magnetic potential $\Phi_m(\vec{r})$.

$$dB = I\frac{dl \times \vec{r}_0}{r^2} \tag{3.494}$$

$$B = \int I\frac{dl \times \vec{r}_0}{r^2} \tag{3.495}$$

The vector potential $\mathbf{A}$ of a magnetic dipole, dipole moment $\mathbf{m}$, is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}(\mathbf{m} \times \frac{\mathbf{r}_0}{r^3}) \tag{3.496}$$

using the identity $\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B - B(\nabla \cdot A) + A(\nabla \cdot B)$, the magnetic field is given by

$$\mathbf{B} = \frac{3\mathbf{r}_0(\mathbf{r}_0 \cdot \mathbf{m}) - \mathbf{m}}{r^3} \tag{3.497}$$

where $\cos\theta = \mathbf{r}_0 \cdot \mathbf{m}$

$$\phi_m = \frac{\vec{\nabla} \cdot (\vec{r} \times \vec{j})}{|\vec{r} - \vec{r}'|} \tag{3.498}$$

$$\begin{aligned}\Phi_m(\vec{r}) &= \frac{1}{r}\int d^3x'\vec{\nabla} \cdot (\vec{r}' \times \vec{j}) + \frac{\vec{r}}{r^3} \cdot \int d^3x'\vec{r}'\vec{\nabla} \cdot (\vec{r}' \times \vec{j}) + \\ &+ \frac{1}{2r^5} \cdot \int d^3x' \frac{(3\vec{r} \cdot \vec{r}' - r'^2 r^2)}{2}\vec{\nabla} \cdot (\vec{r}' \times \vec{j}) + \ldots\end{aligned} \tag{3.499}$$

where

$$\begin{aligned} M_1 &= \int d^3x'\vec{\nabla} \cdot (\vec{r}' \times \vec{j}) && \text{Monopole Moment} \\ M_2 &= \int d^3x'\vec{r}'\vec{\nabla} \cdot (\vec{r}' \times \vec{j}) && \text{Dipole Moment} \\ M_3 &= \int d^3x' \frac{(3\vec{r} \cdot \vec{r}' - r'^2 r^2)}{2}\vec{\nabla} \cdot (\vec{r}' \times \vec{j}) && \text{Quadrupole Moment} \end{aligned} \tag{3.500}$$

the coeffients M_n are defined as the *moments*

$$M_n := \int r^n P_n(\cos\theta)\vec{\nabla} \cdot (\vec{r} \times \vec{j})d^3x \tag{3.501}$$

$$M_n = -R^{n+1} \oint_S (\tilde{\epsilon}^{ab}\tilde{D}_b P_n(\cos\theta))\tilde{j}_a d^2\tilde{V} \tag{3.502}$$

3.15.2 Spherical Harmonics

For functions that live on the unit sphere there is an expansion by a in *spherical harmonics*, analogous to Forurier expansion of functions that live on the real line. The reader probably have come accross them before as they solve the Schrödinger equation for the Hydrogen atom. These form a complete orthonormal set on the unit sphere. The orthonormality is expreesed as

$$\int Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)d\Omega = \delta_{ll'}\delta_{mm'} \tag{3.503}$$

Complete because for any square integrable function Δ there is a ??unique?? expansion

$$\Delta(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm},(\theta,\phi) \tag{3.504}$$

where

$$a_{lm} = \int Y^*_{lm}(\theta,\phi)\Delta(\theta,\phi)d\Omega. \tag{3.505}$$

in solid angle notation

$$d\Omega = \sin\theta d\theta d\phi. \tag{3.506}$$

unit normal vector to the sphere vector

$$\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{3.507}$$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, \tag{3.508}$$

where P_l^m are the associated Legendre functions. For our purposes we all the geometries under consideration are axisymmetric, we only need the Y_{lm} with $m = 0$. The functions we need are the P_l^m with $m = 0$, these are prescisely the functions that appeared in the moment expansion in the previous section. These are called the Legendre polynomials.

the generating function for Legendre polynomials:

$$g(t,x) = (1-2tx+t^2)^{-1} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |t|<1. \tag{3.509}$$

3.15.3 Invariant Coordinates on the Horizon

Label points on the submanifold in a way that is invariant under spacetime coordinate transformations $x^a \mapsto x^{a\prime}$. They are 'coordinates' in that they specify individual points on the sphere but they transform as scalars under spacetime coordinate transformations.

The thesis [78]

In Euclidean 3-space,

Let S be the unit 2-sphere, equipped with metric $\tilde{q}_{ab}$.

$$\epsilon_{ab}\phi^b = \partial_a \zeta \tag{3.510}$$

$\epsilon_{11} = \epsilon_{00} = 0$, $\epsilon_{10} = -\epsilon_{01} = 1$. $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ so that $\tilde{q}_{ab} = (1, \sin^2\theta)$. In polar coordinates $\tilde{\epsilon}_{ab} = \sqrt{\tilde{q}}[ab] = \sin\theta[ab]$. The Killing's vector field is in polar coordinates $\mathbf{\Phi} = r\hat{\phi} + 0\hat{r} + 0\hat{\theta}$. restricted to the unit sphere $\mathbf{\Phi} = \sin\hat{\phi} + 0\hat{r} + 0\hat{\theta}$

$\partial_\phi \zeta = \epsilon_{\phi r}\Phi^r + \epsilon_{\phi\theta}\Phi^\theta = 0;$

$\partial_\theta \zeta = \epsilon_{\theta r}\Phi^r + \epsilon_{\theta\phi}\Phi^\phi = \epsilon_{\theta\phi}\sin\theta$

$\partial_\theta \zeta = -\sin\theta$

So that

$\zeta = \cos\theta + C$

The condition $\oint_S \zeta\tilde{\epsilon} \equiv \oint_S \zeta d^2S$ to eliminate the freedom to add to ζ a constant.

$$\oint_S \zeta d^2S = \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} (\cos\theta + C)d\phi \sin\theta d\theta = 0 + 4\pi C \tag{3.511}$$

so we should take $C = 0$. On the unit 2-sphere in Euclidean 3-space, this function is

$$\zeta = \cos\theta. \tag{3.512}$$

vector field ζ^a on $S' = (S-$ poles) via:

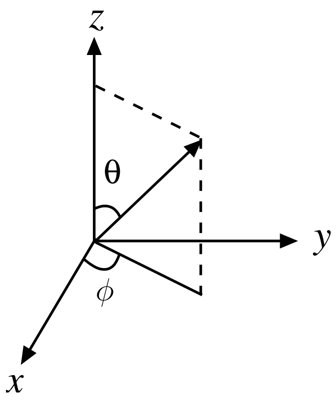


Figure 3.29: polarcoo

$$\tilde{q}_{ab}\zeta^a\phi^b = 0 \tag{3.513}$$

fixing the norm of the vector field ζ^a by the condition

$$\zeta^\theta \partial_\theta \cos\theta = 1 \tag{3.514}$$

$$\zeta^\theta = -\frac{1}{\sin\theta} \tag{3.515}$$

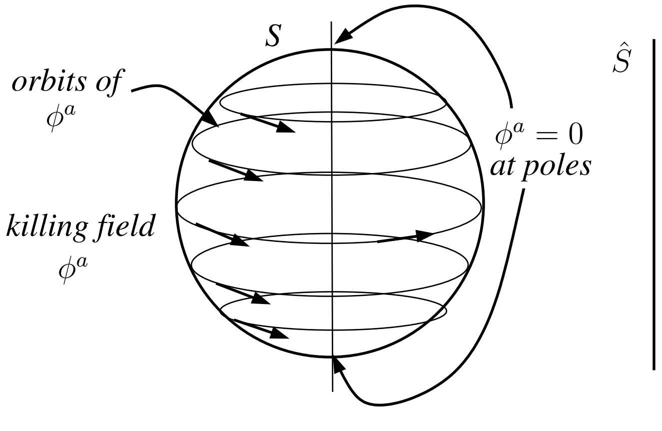


Figure 3.30: axicoords. In addaptive coordinates

$$\tilde{q}_{ab} = R^2(f^{-1}\partial_a\zeta\partial_b\zeta + f\partial_a\phi\partial_b\phi) \quad \text{and} \quad \tilde{q}^{ab} = \frac{1}{R^2}(f\zeta^a\zeta^b + f^{-1}\phi^a\phi^b) \tag{3.516}$$

$$\begin{aligned} \tilde{q}^{ab}\tilde{q}_{bc} &= (f\zeta^a\zeta^b + f^{-1}\phi^a\phi^b)(f^{-1}\partial_b\zeta\partial_c\zeta + f\partial_b\phi\partial_c\phi) \\ &= \zeta^a\zeta^b\partial_b\zeta\partial_c\zeta + f^2\zeta^a\zeta^b\partial_b\phi\partial_c\phi \end{aligned} \tag{3.517}$$

Invariant Coordinates

Let S be a manifold with the topology of a 2-sphere, equipped with metric $\tilde{q}_{ab}$. a the area of S and by R its radius, defined by $a = 4\pi R^2$.

we can define on S a function ζ via

The definition of ζ (3.510) and (3.511) can be put into a covariant form as

$$\tilde{D}_a\zeta = \tilde{\epsilon}_{ab}\phi^b \quad \text{and} \quad \int \zeta\sqrt{\tilde{q}}d^2S \tag{3.518}$$

Obviously $\partial_\phi\zeta = 0$. This fact can be put into a coordinate invariant form as $\mathcal{L}_\phi\zeta = 0$

Such metric manifolds carry invariantly defined coordinates.

Consider the orbits of the axial Killing field ϕ^a of q_{ab} on S.

introduce a vector field ζ^a on $S' = (S-$ poles) via:

$$\tilde{q}_{ab}\zeta^a\phi^b = 0 \quad \text{and} \quad \zeta^a\tilde{D}_a\zeta = 1. \tag{3.519}$$

Then it follows

$$\zeta^a = \frac{R^4}{\mu^2}\tilde{q}^{ab}\tilde{D}_b\zeta \tag{3.520}$$

where $\mu^2 = \tilde{q}_{ab}\phi^a\phi^b$ is the squared norm of ϕ^a.

$$\tilde{q}_{ab} = R^2(f^{-1}\tilde{D}_a\zeta\tilde{D}_b\zeta + f\tilde{D}_a\phi\tilde{D}_b\phi) \quad \text{and} \quad \tilde{q}^{ab} = \frac{1}{R^2}(f\zeta^a\zeta^b + f^{-1}\phi^a\phi^b) \tag{3.521}$$

where $f = \mu^2/R^2$.

$$\tilde{\mathcal{R}}(\zeta,\phi) = -\frac{1}{R^2}f''(\zeta). \tag{3.522}$$

one can reconstruct the function f from the scalar curvature:

$$f = -R^2\left[\int_{-1}^{\zeta} d\zeta_1 \int_{-1}^{\zeta_1} d\zeta_2\tilde{\mathcal{R}}(\zeta_2)\right] + 2(\zeta+1) \tag{3.523}$$

canonical round, 2-sphere metric $\tilde{q}^0_{ab}$ on S:

$$\tilde{q}^0_{ab} = R^2(f_0^{-1}\tilde{D}_a\zeta\tilde{D}_b\zeta + f_0\tilde{D}_a\phi\tilde{D}_b\phi) \tag{3.524}$$

where $f_0 = 1 - \zeta^2$. The availability of $\tilde{q}^0_{ab}$ enables us to perform a natural harmonic decomposition on S. One uses spherical harmonics defined by $\tilde{q}^0_{ab}$.

3.15.4 Definition of Geometric Multipoles

We showed that ζ is a good coordinate on the manifold of orbits of and all geometrical fields such as Ψ_2. In Euclidean space, smooth functions on the unit 2-sphere which are Lie dragged by ϕ^a, can be decomposed uniquely in terms of Legendre polynomials. We follow the same idea to define multipoles of Ψ_2. We set

$$I_n - iJ_n := \frac{1}{8\pi G}\oint_S \zeta^n\Psi_2\epsilon \tag{3.525}$$

Magnetostatics in Flat Space

$$M_n = \int r^n P_n(\cos\theta)\vec{\nabla}\cdot(\vec{x}\times\vec{j})d^3x, \tag{3.526}$$

Angular Momentum Moments

$$\begin{aligned} J_n &:= -\frac{R_\Delta^{n+1}}{8\pi G}\oint_S(\tilde{\epsilon}^{ab}\tilde{D}_bP_n(\zeta))\tilde{\omega}_a d^2\tilde{V} \\ &= -\sqrt{\frac{4\pi}{2n+1}}\frac{R_\Delta^{n+1}}{4\pi G}\oint_S Y_n^0(\zeta)\mathrm{Im}\Psi_2 d^2\tilde{V} \\ &= -\sqrt{\frac{4\pi}{2n+1}}\frac{R_\Delta^{n+1}}{4\pi G}L_n \end{aligned} \tag{3.527}$$

Electrostatics in Flat Space

$$E_n = \int r^n P_n(\cos\theta)\rho d^3x, \tag{3.528}$$

Moments

$$
\begin{aligned}
M_n &:= -\sqrt{\frac{4\pi}{2n+1}}\frac{M_\Delta R^n_\Delta}{4\pi G}\oint_S Y^0_n(\zeta)\mathrm{Re}\Psi_2 d^2\tilde{V} \\
&= -\sqrt{\frac{4\pi}{2n+1}}\frac{M_\Delta R^n_\Delta}{4\pi G} I_n \qquad (3.529)
\end{aligned}
$$

Structure of IHs

However, the connection is now complex:

$$\underline{A}^i_a r_i = \underline{\Gamma}^i_a r_i - i\omega_a \qquad (3.530)$$

where $\underline{\Gamma}^i_a$ is a real $U(1)$ connection and ω_a is globally defined and real valued on Δ (which carries the angular momentum information). The scalar curvature of the 2-metric on the horizon need not be constant; the horizon may be arbitrarily distored (subject to axisymmetry).

The curvature is given by:

$$F_{ab} = 2dV = -2\Psi_2 \underline{\Sigma}^i_{ab} r_i \qquad (3.531)$$

Properties of moments:

ζ and $\mathrm{Im}\Psi_2$ are completely determined, in a covariant manner, by the metric $\tilde{q}_{ab}$, the rotation one-form $\tilde{\omega}_a$, and a rotational killing field ϕ^a. Therefore, it is immediate that if (Δ, q_{ab}, D) and (Δ', q'_{ab}, D') are related by an active diffeomorphism, we have $I_n = I'_n$ and $J_n - J'_n$.

i) I_n, J_n are diffeomorphism invariant. That is, (S, ϕ^a, V) and its image $(S, \phi^{a\prime}, V')$ under any diffeomorphism define exactly the same set of numbers I_n, J_n; and

ii) The multipole moments I_n, J_n of any Ψ_2 enable one to reconstruct that Ψ_2 up to diffeomorphism.

3.16 Dynamical Horizons

In this chapter study generalizations of the isolated horizon framework by allowing matter fields to fall into the black hole. The method is based on the fact that the world tube of apparent horizons is a spacelike surface in the dynamical regime and therefore the usual Hamiltonian and momentum constraints must hold on this surface.

We point out the similarities of this method with the analysis of isolated horizon mechanics described in the earlier chapters and we also describe some subtleties regarding the ADM angular momentum.

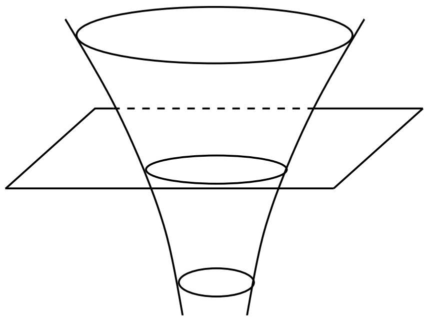


Figure 3.31: .

Defnition: A smooth three dimensional sub-manifold H of a spacetime $(\mathcal{M}, g_{ab})$ will be said to be a dynamical horizon if

(i) H is topologically $S^2 \times \mathcal{I}$ and spacelike where $\mathcal{I}$ is an interval on the real line.

(ii) There is a preferred foliation of H by two-spheres and each leaf of this foliation is an outer-marginally-trapped-surface; i.e. the expansion $\theta_{(\ell)}$ of any outgoing null normal ℓ^a is identically zero and the expansion $\theta_{(n)}$ of any ingoing null normal n a is negative.

(iii) All equations of motion hold at H and all matter fields satisfy the condition that $-T^a_b X^b$ is future-directed and causal for any future-directed null vector X_a .

$$\int_V (16\pi N t_{ab} \tilde{T}^a \tilde{T}^b) d^3x = \int_V N(\mathcal{R} + \mathcal{K}^2 - \mathcal{K}_{ab}\mathcal{K}^{ab}) d^3x \tag{3.532}$$

where N is an arbitrary ositive function (the lapse) and V is the region on H bounded by S_1 and S_2

$$\mathcal{G}_{ab} \tilde{R}^a \tilde{R}^b = -\frac{1}{2}\left(\tilde{R} - \mathcal{K}^2 + \mathcal{K}_{ab}\mathcal{K}^{ab}\right) \tag{3.533}$$

$$\mathcal{R} = 2(\mathcal{R}_{ab} - \mathcal{G}_{ab})\tilde{R}^a \tilde{R}^b \tilde{\mathcal{R}} + (\mathcal{K})^2 - \mathcal{K}_{ab}\mathcal{K}^{ab} + 2\mathcal{D}_a \alpha^a. \tag{3.534}$$

Substituiting this result in (4.3.4)

$$\int_V (16\pi N T_{ab}\tilde{T}^a\tilde{T}^b) d^3x = \int_V N\left(\tilde{\mathcal{R}} + (\mathcal{K})^2 - \mathcal{K}_{ab}\mathcal{K}^{ab} + 2\mathcal{D}_a\alpha^a\right) d^3x. \tag{3.535}$$

This equation relates the flux of energy along the normal to $H((\tilde{T}^a)$ with quantities intrinsic to H.

$$\begin{aligned}\int_V (16\pi N T_{ab}\tilde{T}^a\tilde{T}^b) d^3x &= \int_V \left(2N\tilde{R}^b\mathcal{D}^a(\mathcal{K}_{ab} - \mathcal{K}q_{ab})\right) d^3x \\ &= \int_V 2N\left(\mathcal{D}_a\beta^a - \mathcal{P}^{ab}\mathcal{D}_a\tilde{R}_b\right) d^3x \end{aligned} \tag{3.536}$$

where $\beta^a := \mathcal{K}^{ab}\tilde{R}_b - \mathcal{K}\tilde{R}^a$ and, as before, $\mathcal{P}_{ab} = \mathcal{K}_{ab} - \mathcal{K}q_{ab}$.

$$\zeta^a := \tilde{W}^a + \tilde{D}^a \ln N_R = \tilde{q}^{ab}R^c\nabla_c\ell_b \tag{3.537}$$

$$\mathcal{F}^{(R)}_{\Delta H} = \int_{\Delta H} N_R\tilde{\mathcal{R}} d^3V = 16\pi G\int_{\Delta H} T_{ab}T^a\xi^a_{(R)} d^3V + \int_{\Delta H} N_R\{|\sigma|^2 + 2|\zeta|^2\} d^3V \tag{3.538}$$

3.16.1 Gravitational Energy Flux

With these decompositions, the following results are very easily verifed

$$\begin{aligned} {}^2\mathcal{K}_{ab}\,{}^2\mathcal{K}^{ab} &= \frac{1}{2}({}^2\mathcal{K})^2 + {}^2\sigma_{ab}\,{}^2\sigma^{ab} \\ \mathcal{K} &= 2A + B \\ \mathcal{K}_{ab}\mathcal{K}^{ab} &= \frac{1}{2}({}^2\mathcal{K})^2 + \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} + 2W_aW^a + B^2 \\ {}^2\mathcal{K} &= -2A \end{aligned} \tag{3.539}$$

Vanishing in Spherical symmetry: By Birkhoff's theorem

Vaidya solution

any metric $g_{ab}(x)$ soves Einsteins field equations so long as we take the energy momentum-tensor to be given by $T_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$, where we have calculated R_{ab} using $g_{ab}(x)$. What distinguishes a *physical* solution from a *non-physical* solution is whether the corresponding energy-momentum tensor satisfies physically reasonable conditions, such as the various energy conditions. Even it satifies these conditions, one still has to a physical interpretation of the source of this energy-momentum tensor.

We want m to depend on u, $m = m(u)$

$$f = 1 - \frac{2m(u)}{r} \tag{3.540}$$

The only non-vanishing component of the Einstein tensor is $G_{uu} = -(2/r^2)/(dm/du)$ (exercise). Then the energy-momentum tensor is of the form

$$T_{ab} = -\frac{dm/du}{4\pi r^2}\ell_a \ell_b, \tag{3.541}$$

where $\ell_a = -\partial u$ is tangent to radial, outgoing null geodesics.

3.16.2 Rotating Dynamical Horizons

3.17 Null Tetrads and Spinor Analysis

3.17.1 Null Tetrads

We start with four linerly independent vector fields e_i^a, where i serves to label the vectors.

$$\sum_i \alpha_i e_i^a = 0 \quad \textit{implies } \alpha_i = 0.$$

At a particular point, we define a matrix of scalars g_{ij}, called the frame metric, by

$$g_{ij} = g_{ab} e_i^a e_j^b. \tag{3.542}$$

Since e_i^a are linearly independent and that g_{ab} is non-singular, it follows that the matrix g_{ij} is non-singular and hence invertible. To see this first consider the eigenvector equation

$$\sum_j g_{ij} v_j = \lambda v_i$$

as g_{ij} is real and symmetric it has eigenvales. To prove g_{ij} is non-singular we assume $\lambda = 0$. Then

$$\sum_j g_{ij} v_j = e_i^a [\sum_j g_{ab} e_j^b v_j] = 0.$$

As the e_i^a are linearly independent, this implies

$$\sum_j g_{ab} e_j^b v_j = 0.$$

Using that g_{ab} is invertible, this implies

$$\sum_j e_j^b v_j = 0$$

and again by linearly independence of e_j^b we have $v_j = 0$. Hence there is no eigenvector with eigenvalue zero and so g_{ij} is invertible.

We denote the inverse as g^{ij},

$$g_{ij} g^{jk} = \delta_i^k. \tag{3.543}$$

we then use the frame metric to raise and lower frame indices. Then we can write

$$\delta_j^i = g^{ik} g_{kj} = g^{ik} g_{ab} e_k^a e_j^b = e_a^i e_j^a. \tag{3.544}$$

Using this it is easy to verify the inverse relationship to (3.542) is

$$g_{ab} = g_{ij} e_a^i e_b^j \tag{3.545}$$

as

$$\begin{aligned} g_{ab} e_i^a e_j^b &= (g_{kl} e_a^k e_b^l) e_i^a e_j^b \\ &= g_{kl} (e_a^k e_i^a)(e_b^l e_j^b) \\ &= g_{ij} \end{aligned}$$

Suppose for a given spacetime we have defined an orthonormal tetrad as follows:

$$\begin{array}{ll} v^a & \text{timelikevector} \\ i^a, j^a, k^a & \text{spacelikevectors.} \end{array} \tag{3.546}$$

We can construct a null teterad via

$$e_{\mathbf{0}}^{\ a} = l^a := \frac{1}{\sqrt{2}}(v^a + i^a), \tag{3.547}$$

$$e_{\mathbf{1}}^{\ a} = n^a := \frac{1}{\sqrt{2}}(v^a - i^a) \tag{3.548}$$

in whcih case l^a and n^a are *null* vectors, that is

$$l^a l_a = n^a n_a = 0 \tag{3.549}$$

and satisfy the normalization condition

$$l^a n_a = 1. \tag{3.550}$$

next we introduce a *complex* null vector defined by

$$m^a := \frac{1}{\sqrt{2}}(j^a + ik^a) \tag{3.551}$$

together with its complex conjugate

$$m^a := \frac{1}{\sqrt{2}}(j^a - ik^a) \tag{3.552}$$

It easily follows that the vector are null,

$$m^a m_a = \overline{m}^a \overline{m}_a = 0, \tag{3.553}$$

and satisfy the normalization condition

$$m^a \overline{m}_a = -1. \tag{3.554}$$

Including

$$e^a_{\mathbf{2}} = m^a, \quad e^a_{\mathbf{3}} = \overline{m}^a, \tag{3.555}$$

then the defines the frame metric via (3.542) is then

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.556}$$

and the inverse frame metric is

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0. \end{pmatrix} \tag{3.557}$$

$$\begin{aligned} g^{ab} &= g_{ij}e^i_a e^j_b \\ &= (l_a, n_a, m_a \overline{m}_a) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} l^b \\ n^b \\ m^b \\ \overline{m}^b \end{pmatrix} \\ &= l_a n_b + n_a l_b - m_a \overline{m}_b - m_b \overline{m}_a. \end{aligned}$$

The metric g_{ab} is decomposed into products the null tetrads according to

$$g_{ab} = l_a n_a + l_b n_a - m_a \overline{m}_b - m_b \overline{m}_a. \tag{3.558}$$

$$\begin{aligned} g^{ab} &= g^{ij}e^a_i e^b_j \\ &= (l^a, n^a, m^a \overline{m}^a) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} l^b \\ n^b \\ m^b \\ \overline{m}^b \end{pmatrix} \\ &= l^a n^b + n^a l^b - m^a \overline{m}^b - m^b \overline{m}^a. \end{aligned} \tag{3.559}$$

$$\begin{aligned} &n \cdot l = -1 \quad n \cdot m = 0 \quad n \cdot \overline{m} = 0 \\ &l \cdot m = 0 \quad l \cdot \overline{m} = 0 \quad m \cdot \overline{m} = 1. \end{aligned} \tag{3.560}$$

Example

Conbsider the flat Minkowski metric, written in spherical polar coordinates:

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

$$\begin{aligned} l_\mu &= \frac{1}{\sqrt{2}}(1,1,0,0), \qquad n_\mu = \frac{1}{\sqrt{2}}(1,-1,0,0) \\ m_\mu &= \frac{1}{\sqrt{2}}(0,0,r,ir\sin\theta), \qquad \overline{m}_\mu = \frac{1}{\sqrt{2}}(0,0,r,-ir\sin\theta) \end{aligned} \tag{3.561}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \tag{3.562}$$

$$\begin{aligned} l\cdot l &= g^{\mu\nu} l_\mu l_\nu \\ &= \frac{1}{2}(1,1,0,0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{3.563}$$

$$\begin{aligned} n\cdot n &= g^{\mu\nu} n_\mu n_\nu \\ &= \frac{1}{2}(1,-1,0,0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{3.564}$$

$$\begin{aligned} l\cdot n &= g^{\mu\nu} l_\mu n_\nu \\ &= \frac{1}{2}(1,1,0,0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= 1. \end{aligned} \tag{3.565}$$

$$
\begin{aligned}
m \cdot m &= g^{\mu\nu} m_\mu m_\nu \\
&= \frac{1}{2}(0,0,r,ir\sin\theta)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ r \\ ir\sin\theta \end{pmatrix} \\
&= \frac{1}{2}(0,0,r,ir\sin\theta)\begin{pmatrix} 0 \\ 0 \\ -1/r \\ -i/r\sin\theta \end{pmatrix} \\
&= 0.
\end{aligned} \tag{3.566}
$$

Obviously $\overline{m} \cdot \overline{m} = 0$.

$$
\begin{aligned}
m \cdot \overline{m} &= g^{\mu\nu} m_\mu m_\nu \\
&= \frac{1}{2}(0,0,r,ir\sin\theta)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ r \\ -ir\sin\theta \end{pmatrix} \\
&= \frac{1}{2}(0,0,r,ir\sin\theta)\begin{pmatrix} 0 \\ 0 \\ -1/r \\ i/r\sin\theta \end{pmatrix} \\
&= -1.
\end{aligned} \tag{3.567}
$$

Obviously $l \cdot m = l \cdot \overline{m} = 0$ and $n \cdot m = n \cdot \overline{m} = 0$

3.17.2 Newman-Penrose Formulism

The Weyl tensor in terms of the curvature tensor is

$$
\begin{aligned}
C_{abcd} &= R_{abcd} + \frac{1}{2}\left(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}\right) \\
&\quad + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R.
\end{aligned} \tag{3.568}
$$

Define the scalars

$$
\begin{aligned}
\Psi_0 &= -C_{abdc} l^a m^b l^c m^d, \\
\Psi_1 &= i C_{abdc} l^a m^b l^c n^d, \\
\Psi_2 &= C_{abdc} l^a m^b \overline{m}^c n^d, \\
\Psi_3 &= -i C_{abdc} l^a n^b \overline{m}^c n^d, \\
\Psi_4 &= -C_{abdc} \overline{m}^a n^b \overline{m}^c n^d.
\end{aligned} \tag{3.569}
$$

The tetrad components of the Ricci tensor are given by:

$$
\begin{aligned}
\Phi_{00} &= \frac{1}{2} R_{ab} l^a l^b, & \Phi_{01} &= i\frac{1}{2} R_{ab} l^a m^b, \\
\Phi_{02} &= -\frac{1}{2} R_{ab} m^a m^b, & \Phi_{10} &= -i\frac{1}{2} R_{ab} l^a \overline{m}^b, \\
\Phi_{11} &= \frac{1}{4} R_{ab} (l^a n^b + m^a \overline{m}^b), & \Phi_{12} &= i\frac{1}{2} R_{ab} n^a m^b, \\
\Phi_{20} &= -\frac{1}{2} R_{ab} \overline{m}^a \overline{m}^b, & \Phi_{21} &= -i\frac{1}{2} R_{ab} n^a \overline{m}^b, \\
\Phi_{22} &= \frac{1}{2} R_{ab} n^a n^b.
\end{aligned} \tag{3.570}
$$

$$
\begin{aligned}
\kappa &= -m^a l^b \nabla_b l_a & \epsilon &= \frac{1}{2}(\overline{m}^a l^b \nabla_b m_a - n^a l^b \nabla_b l_a) & \pi &= \overline{m}^a l^b \nabla_b n_a \\
\sigma &= -m^a m^b \nabla_b l_a & \beta &= \frac{1}{2}(\overline{m}^a m^b \nabla_b m_a - n^a m^b \nabla_b l_a) & \mu &= \overline{m}^a m^b \nabla_b n_a \\
\rho &= -m^a \overline{m}^b \nabla_b l_a & \alpha &= \frac{1}{2}(\overline{m}^a \overline{m}^b \nabla_b m_a - n^a \overline{m}^b \nabla_b l_a) & \lambda &= \overline{m}^a \overline{m}^b \nabla_b n_a \\
\tau &= -m^a n^b \nabla_b l_a & \gamma &= \frac{1}{2}(\overline{m}^a n^b \nabla_b m_a - n^a n^b \nabla_b l_a) & \nu &= \overline{m}^a n^b \nabla_b n_a
\end{aligned} \tag{3.571}
$$

3.17.3 Spinor Analysis in GR

Spinors and Vectors

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.572}
$$

We form the matrix

$$
u^{AA'} = u^0 \sigma_0^{AA'} + u^1 \sigma_1^{AA'} + u^2 \sigma_2^{AA'} + u^3 \sigma_3^{AA'} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{3.573}
$$

We demand that this matrix by Hermitian

$$u^\dagger = u \tag{3.574}$$

as then and only then are the coordinates (t, x, y, z) guareanteed to be real. One can represent the effect of the Lorenz transformation by matrix multiplication

$$u' = LuL^\dagger \tag{3.575}$$

Here u' is also Hermitian

$$\begin{aligned}(u')^\dagger &= (LuL^\dagger)^\dagger \\ &= (L^\dagger)^\dagger u^\dagger L^\dagger \\ &= LuL^\dagger \\ &= u'.\end{aligned} \tag{3.576}$$

and the new coordinates (t', x', y', z') are guareanteed to be real too. Explicitly

$$\begin{pmatrix} t'+z' & x'+iy' \\ x'-iy' & t'-z' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \tag{3.577}$$

A Lorentx transformation is defined as a linear operation that leaves the interval invariant:

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2 \tag{3.578}$$

Note that the determinant of u is

$$det\ u = t^2 - x^2 - y^2 - z^2 \tag{3.579}$$

so that the condition for the preservation of the interval is

$$det\ u' = det\ u \tag{3.580}$$

or

$$(det\ L)(det\ u')(det\ L^\dagger) = det\ u \tag{3.581}$$

This requirement is fulfilled by

$$det\ L = 1. \tag{3.582}$$

We take the transformation rule for a two component quantitiy ξ^A to be

$$\xi'^A = L^A{}_B \xi^B. \tag{3.583}$$

This is the definition of a two-component spinor. To recover the formula $u' = LuL^\dagger$, namely,

$$u'^{AB'} = L^A{}_C \overline{L}^{B'}{}_{D'} u^{CD'} \tag{3.584}$$

we introduce another spinor $\eta^{B'}$ which transforms according to the conjugate of the Lorentz transformation

$$\eta'^{A'} = \overline{L}^{A'}{}_{B'} \eta^{B'} \tag{3.585}$$

because then the transformation law for a second rank spinor $\xi^A \eta^{B'}$,

$$\xi'^A \eta'^{B'} = L^A{}_C \overline{L}^{B'}{}_{D'} \xi^C \eta^{D'}, \tag{3.586}$$

is the same transformation law for $u^{AA'}$.

We see in a sense a spinor is the "square-root of a vector".

Dual spinor

In components with respect to this basis we have

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.587}$$

We define ϵ^{AB} as

$$\epsilon^{AB} = -(\epsilon^{-1})^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.588}$$

We can use ϵ_{AB} lower indices of spinors,

$$\epsilon_{AB}\xi^A\eta^B = (\epsilon_{AB}\xi^A)\eta^B. \tag{3.589}$$

The quantity in the brackets is the dual of ξ^B, and so we have

$$\xi_B = \epsilon_{AB}\xi^A = \xi^A\epsilon_{AB} = -\epsilon_{BA}\xi^A \tag{3.590}$$

Using the inverse of ϵ_{AB} we have

$$(\epsilon^{-1})^{BC}\xi_B = \xi^A\epsilon_{AB}(\epsilon^{-1})^{BC} = \xi^A\delta_A{}^C, \tag{3.591}$$

where $\delta_A{}^C$ is the spinor Kronecker delta. Note th eposition of the indices

$$-\epsilon^{BC}\xi_B = \xi^A\delta_A{}^C = \xi^C. \tag{3.592}$$

These can be used for the raising or lowering of spinor indices, in a way analogous to g^{ab} and g_{ab} for tensors, but here one has to be careful with the up-down rule.

$$\xi_B = \xi^A\epsilon_{AB}, \qquad \xi^C = \epsilon^{CB}\xi_B. \tag{3.593}$$

For multiple-component spinors, for example we have

$$\xi^A{}_D{}^C = \xi^{ABC}\epsilon_{BD} \tag{3.594}$$

Null vectors.

A space time vector corresponding to a spinor of the form

$$X^{AA'} = \alpha^A\beta^{A'} \tag{3.595}$$

is null as its determinate is zero,

$$\epsilon_{AB}\epsilon_{A'B'}(\alpha^A\beta^{A'})(\alpha^B\beta^{B'}) = (\alpha^A\alpha_A)(\beta^{A'}\beta_{A'}) = 0. \tag{3.596}$$

In fact any null vector has this spinorial form. A four-vector is null if and only if $\det(X^{AA'}) = 0$. This means rows/columbs must be linearly dependent.

A real null vector satisfies the condition $\beta = \overline{\alpha}$ whenever X is.

Converting spinors into vectors

$$u^{AA'} = x^\mu \sigma_\mu{}^{AA'} \tag{3.597}$$

As

$$u_{AA'} = u^{BB'} \epsilon_{BA} \epsilon_{B'A'}$$

we define the matrix $\sigma^\mu{}_{AA'}$ via

$$u_{AA'} = x_\mu \sigma^\mu{}_{AA'} \tag{3.598}$$

we have

$$\begin{aligned} u_{AA'} &= x^\mu \sigma_\mu{}^{BB'} \epsilon_{BA} \epsilon_{B'A'} \\ &= x_\mu (\eta^{\mu\nu} \sigma_\nu{}^{BB'} \epsilon_{BA} \epsilon_{B'A'}) \\ &= x_\mu \sigma^\mu{}_{AA'} \end{aligned} \tag{3.599}$$

implying

$$\sigma^\mu{}_{AA'} = \eta^{\mu\nu} \sigma_\nu{}^{BB'} \epsilon_{BA} \epsilon_{B'A'}. \tag{3.600}$$

Let us calculate this from

$$\sigma_0{}^{AA'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1{}^{AA'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2{}^{AA'} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3{}^{AA'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.601}$$

Firstly

$$\begin{aligned} \sigma^0{}_{AA'} &= \eta^{0\mu} \sigma_\mu{}^{BB'} \epsilon_{BA} \epsilon_{B'A'} \\ &= -(\epsilon^T)_{AB} \sigma_0{}^{BB'} \epsilon_{B'A'} \\ &= -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{3.602}$$

then

$$
\begin{aligned}
\sigma^{1}{}_{AA'} &= \eta^{1\mu}\sigma_{\mu}{}^{BB'}\epsilon_{BA}\epsilon_{B'A'} \\
&= (\epsilon^{T})_{AB}\sigma_{1}{}^{BB'}\epsilon_{B'A'} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\end{aligned} \tag{3.603}
$$

then

$$
\begin{aligned}
\sigma^{2}{}_{AA'} &= \eta^{2\mu}\sigma_{\mu}{}^{BB'}\epsilon_{BA}\epsilon_{B'A'} \\
&= (\epsilon^{T})_{AB}\sigma_{2}{}^{BB'}\epsilon_{B'A'} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{aligned} \tag{3.604}
$$

and

$$
\begin{aligned}
\sigma^{3}{}_{AA'} &= \eta^{3\mu}\sigma_{\mu}{}^{BB'}\epsilon_{BA}\epsilon_{B'A'} \\
&= (\epsilon^{T})_{AB}\sigma_{3}{}^{BB'}\epsilon_{B'A'} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{3.605}
$$

Altogether we have

$$
\begin{aligned}
&\sigma^{0}{}_{AA'} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{1}{}_{AA'} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{2}{}_{AA'} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
&\sigma^{3}{}_{AA'} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{3.606}
$$

We have the following orthogonality and normalisation relations

$$\sigma_{\mu}{}^{AA'}\sigma^{\mu}{}_{BB'} = -2\delta^{A}_{B}\delta^{A'}_{B'} \tag{3.607}$$

and

$$\sigma_{\mu}{}^{AA'}\sigma^{\nu}{}_{AA'} = -2\delta^{\nu}_{\mu} \tag{3.608}$$

Proof:

(i) Proof of (3.607)

We check by direct calculation. There arer 16 possibilities

$A = A'$ and $B \neq B'$

$$\begin{aligned}
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{00'}\sigma^{\mu}{}_{01'} &= 0\\
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{00'}\sigma^{\mu}{}_{10'} &= 0\\
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{11'}\sigma^{\mu}{}_{01'} &= 0\\
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{11'}\sigma^{\mu}{}_{10'} &= 0
\end{aligned} \tag{3.609}$$

$B = B'$ and $A \neq A'$

$$\begin{aligned}
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{01'}\sigma^{\mu}{}_{00'} &= 0\\
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{10'}\sigma^{\mu}{}_{00'} &= 0\\
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{01'}\sigma^{\mu}{}_{11'} &= 0\\
\sum_{\nu=0}^{3}\sigma_{\mu}{}^{10'}\sigma^{\mu}{}_{11'} &= 0
\end{aligned} \tag{3.610}$$

$A = A'$ and $B = B'$

$$\begin{aligned}
\sum_{\nu=0}^{3} {\sigma_\mu}^{00'} {\sigma^\mu}_{00'} &= -2 \\
\sum_{\nu=0}^{3} {\sigma_\mu}^{00'} {\sigma^\mu}_{11'} &= 0 \\
\sum_{\nu=0}^{3} {\sigma_\mu}^{11'} {\sigma^\mu}_{00'} &= 0 \\
\sum_{\nu=0}^{3} {\sigma_\mu}^{11'} {\sigma^\mu}_{11'} &= -2 \qquad (3.611)
\end{aligned}$$

$A \neq A'$ and $B \neq B'$

$$\begin{aligned}
\sum_{\nu=0}^{3} {\sigma_\mu}^{01'} {\sigma^\mu}_{01'} &= -2 \\
\sum_{\nu=0}^{3} {\sigma_\mu}^{01'} {\sigma^\mu}_{10'} &= 0 \\
\sum_{\nu=0}^{3} {\sigma_\mu}^{10'} {\sigma^\mu}_{01'} &= 0 \\
\sum_{\nu=0}^{3} {\sigma_\mu}^{10'} {\sigma^\mu}_{10'} &= -2 \qquad (3.612)
\end{aligned}$$

confirming (3.607).

(ii) Proof of (3.608).

Calculate ${\sigma_\mu}^{AA'} {\sigma^\nu}_{AA'}$. First say $\mu \neq \nu$

$${\sigma_\mu}^{AA'} {\sigma^\nu}_{AA'} = Tr[-\sigma_\mu^T \sigma_\nu] = 0 \qquad (3.613)$$

as what appears in the bracket is the positive or negative of one of the sigma matrices. Now say $\mu = \nu$, first $\mu = 0$

$${\sigma_0}^{AA'} {\sigma^0}_{AA'} = Tr[-\sigma_0^T \sigma_0] = -2.$$

Now say $\mu = 1$

$$\sigma_1{}^{AA'}\sigma^1{}_{AA'} = Tr[-\sigma_1^T\sigma_1] = -2.$$

Similarly for σ_3. Now say $\mu = 2$

$$\sigma_2{}^{AA'}\sigma^2{}_{AA'} = Tr[\sigma_2^T\sigma_2] = -2.$$

Now say $\mu = 3$

$$\sigma_3{}^{AA'}\sigma^3{}_{AA'} = Tr[\sigma_3^T\sigma_3] = -2.$$

Altogether

$$\sigma_\mu{}^{AA'}\sigma^\nu{}_{AA'} = -2\delta_\mu^\nu \tag{3.614}$$

□

Now

$$\begin{aligned}
\sigma^\nu{}_{AA'}u^{AA'} &= \sigma^\nu{}_{AA'}(x^\mu\sigma_\mu{}^{AA'}) \\
&= x^\mu\sigma^\nu{}_{AA'}\,\sigma_\mu{}^{AA'} \\
&= -2x^\mu\delta_\mu^\nu \\
&= -2x^\nu
\end{aligned}$$

therefore the contravariant components of the vector are

$$x^\mu = -\frac{1}{2}\sigma^\mu{}_{AA'}u^{AA'} \tag{3.615}$$

Now

$$\begin{aligned}
\sigma_\nu{}^{AA'}u_{AA'} &= \sigma_\nu{}^{AA'}(x_\mu\sigma^\mu{}_{AA'}) \\
&= x_\mu\sigma^\mu{}_{AA'}\,\sigma_\nu{}^{AA'} \\
&= -2x_\nu
\end{aligned}$$

and the covariant components of the vector are

$$x_\mu = -\frac{1}{2}\sigma_\mu{}^{AA'}u_{AA'}. \tag{3.616}$$

A tensor T can be expressed in spinor language, for example for a mixed tensor

$$T_{AA'}{}^{BB'CC'} = \sigma^\mu{}_{AA'}\sigma_\nu{}^{BB'}\sigma_\sigma{}^{CC'}T_\mu{}^{\nu\sigma} \tag{3.617}$$

and the inverting this via

$$T_\mu{}^{\nu\sigma} = \left(-\frac{1}{2}\right)^3 \sigma_\mu{}^{AA'}\sigma^\nu{}_{BB'}\sigma^\sigma{}_{CC'}T_{AA'}{}^{BB'CC'} \tag{3.618}$$

$$\epsilon_{AA'BB'CC'DD'} = \epsilon_{\mu\nu\sigma\rho}\sigma^\mu{}_{AA'}\sigma^\nu{}_{BB'}\sigma^\sigma{}_{CC'}\sigma^\rho{}_{DD'} \tag{3.619}$$

Spin basis

It is easy to find if two spinors μ_A and λ_A are linearly independent $\mu_A = Const.\lambda_A$; there scalar product $\lambda_A\mu^A = 0$. Therefore a nonvanishing scalar product

$$\lambda_A\mu^A \neq 0 \tag{3.620}$$

is a necessary and sufficient condition for the linear independence of two spinors.

A general spinor can be written as a linear combination of two basis spinors:

$$\xi = \xi^0 o + \xi^1 \imath. \tag{3.621}$$

$$o^A = (1,0), \qquad \imath^A = (0,1) \tag{3.622}$$

The conition that $(o,\imath)$ is a spin basis are

$$\epsilon_{AB}o^Ao^B = \epsilon_{AB}\imath^A\imath^B = 0, \quad \epsilon_{AB}o^A\imath^B = 1. \tag{3.623}$$

$$\epsilon_{AB} = o_A\imath_B - \imath_A o_A \tag{3.624}$$

since both sides give the same result when applied to o^A or $\imath^A$:

$$
\begin{aligned}
\epsilon_{AB}o^A &= o_A o^A \iota_B - \iota_A o^A o_B = o_B \\
\epsilon_{AB}\iota^A &= o_A \iota^A \iota_B - \iota_A \iota^A o_B = \iota_B
\end{aligned} \tag{3.625}
$$

Jacobi identity

We have the Jacobi identity

$$\epsilon_{A[B}\epsilon_{CD]} = 0 = \epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC} \tag{3.626}$$

Lemma 3.17.1 *Let $\tau_{...CD...}$ be a multivalent spinor. Then*

$$\tau_{...AB...} = \tau_{...(AB)...} + \frac{1}{2}\epsilon_{AB}\tau_{...C}{}^{C}{}_{...} \tag{3.627}$$

Proof:

It is sufficient to consider the case where τ has valence two. So multiply (3.626) with the CD indices raised with τ_{CD},

$$(\epsilon_{AB}\epsilon^{CD} + \epsilon_A{}^C\epsilon^D{}_B + \epsilon_A{}^D\epsilon_B{}^C)\tau_{CD} = \epsilon_{AB}\tau_C{}^C - \tau_{AB} + \tau_{BA} = 0, \tag{3.628}$$

or

$$\tau_{[AB]} = \frac{1}{2}\epsilon_{AB}\tau_C{}^C. \tag{3.629}$$

So that

$$\tau_{AB} = \tau_{(AB)} + \tau_{[AB]} = \tau_{(AB)} + \frac{1}{2}\epsilon_{AB}\tau_C{}^C. \tag{3.630}$$

□

The alternating tensor. The alternating, as defined before, by $\epsilon_{abcd} = \epsilon_{[abcd]}$, with $\epsilon_{0123} = 1$. We prove that

$$\epsilon_{abcd} = i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}. \tag{3.631}$$

First note this corresponds to a real tensor since complex conjugation interchanges the two terms with i replaced with $-i$. We check that it is anti-symmetric under interchanging a and b:

$$\epsilon_{abcd} = i\epsilon_{BC}\epsilon_{AD}\epsilon_{B'D'}\epsilon_{A'C'} - i\epsilon_{BD}\epsilon_{AC}\epsilon_{B'C'}\epsilon_{A'D'} = -\epsilon_{bacd}. \tag{3.632}$$

Similarly it can be shown to be anti-symmetric under the interchange of c and d. Finally we consider the interchange between b and c. We have using the Jacobi identity (3.626),

$$\begin{aligned}
\epsilon_{abcd} + \epsilon_{acbd} &= i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'} \\
&+ i\epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{C'B'} - i\epsilon_{AD}\epsilon_{CB}\epsilon_{A'B'}\epsilon_{C'D'} \\
&= i(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AB}\epsilon_{CD})\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}(\epsilon_{A'C'}\epsilon_{D'B'} + \epsilon_{A'B'}\epsilon_{C'D'}) \\
&= i(-\epsilon_{AD}\epsilon_{BC})\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}(-\epsilon_{A'D'}\epsilon_{B'C'}) \\
&= 0.
\end{aligned} \tag{3.633}$$

From anti-symmetry in the pairs ab, bc and cd, it follows that we have total anti-symmetry:

$$\epsilon_{abcd} = \epsilon_{[abcd]}. \tag{3.634}$$

An equivalent expression for ϵ_{abcd} is:

$$i\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}. \tag{3.635}$$

We use the Jacobi identity (3.626) to show they are equivalent:

$$\begin{aligned}
& i\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} \\
&= -i(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC})\epsilon_{A'C'}\epsilon_{B'D'} + i\epsilon_{AC}\epsilon_{BD}(\epsilon_{A'C'}\epsilon_{D'B'} + \epsilon_{A'D'}\epsilon_{B'C'}) \\
&= -i\epsilon_{AC}\epsilon_{DB}\epsilon_{A'C'}\epsilon_{B'D'} \underbrace{- i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}} \\
& i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{D'B'} \underbrace{+ i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'}} \\
&= i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}.
\end{aligned} \tag{3.636}$$

Null tetrads and spinors

We can contract the $\sigma^{\hat{a}}{}_{AA'}$ with a tetrad $e_{\hat{a}}{}^{a}$

$$u^a = u^{\hat{a}} e_{\hat{a}}{}^a = u^{AA'} \sigma^{\hat{a}}{}_{AA'} e_{\hat{a}}{}^a, \tag{3.637}$$

so that

$$u^{AA'} = \sigma_{\hat{a}}{}^{AA'} u^{\hat{a}}. \tag{3.638}$$

$$k^a = \pm \kappa^A \overline{\kappa}^{A'} \tag{3.639}$$

$$s^a = 2^{-1/2}(\kappa^A \overline{\mu}^{A'} + \mu^A \overline{\kappa}^{A'}), \tag{3.640}$$

is a unit spacetime vector orthogonal to k^a, and is unique up to an additive multiple of k^a.

Another real unit spacelike vector orthogonal to k^a is

$$t^a = 2^{-1/2}(\kappa^A \overline{\mu}^{A'} - \mu^A \overline{\kappa}^{A'}) \tag{3.641}$$

$$\tau^{AA'} = \xi o^A \overline{o}^{A'} + \eta \imath^A \overline{\imath}^{A'} + \zeta o^A \overline{\imath}^{A'} + \sigma \imath^A \overline{o}^{A'} \tag{3.642}$$

$$l^a = o^A \overline{o}^{A'}, \;\; n^a = \imath^A \overline{\imath}^{A'}, \;\; m^a = o^A \overline{\imath}^{A'}, \overline{m}^a = \imath^A \overline{o}^{A'}, \tag{3.643}$$

$$l_a = o_A \overline{o}_{A'}, \;\; n_a = \imath_A \overline{\imath}_{A'}, \;\; m_a = o_A \overline{\imath}_{A'}, \overline{m}_a = \imath_A \overline{o}_{A'}. \tag{3.644}$$

$$l^a l_a = o_A \overline{o}_{A'} o^A \overline{o}^{A'} = 0 \tag{3.645}$$

Theorem: Suppose $\tau_{AB\ldots C}$ is totally symmetric. Then there exists univalent spinors $\alpha_A, \beta_B, \ldots, \gamma_C$ such that

$$\tau_{AB\ldots C} = \alpha_{(A} \beta_B \ldots \gamma_{C)}. \tag{3.646}$$

The $\alpha, \beta, \ldots \gamma$ are called *principal spinors of* τ. The corresponding *null directions of* τ.

Proof:

First we let $\xi^A = (x, y)$ and define

$$\tau(\xi) = \tau_{AB\ldots C}\xi^A\xi^B\ldots\xi^C. \tag{3.647}$$

For simplicity, let us consider the simple case of a valent 2 spinor, τ_{AB}. Now

$$\tau_{AB} = \tau_{AB}\xi^A\xi^B$$

is obviously a polynomial of degree 2 in (complex) x and y:

$$\begin{aligned}
\tau_{00}x^2 - \tau_{01}xy - \tau_{10}yx + \tau_{11}y^2 &= \tau_{00}x^2 - 2!\tau_{01}xy + \tau_{11}y^2 \\
&= y^2\left[\tau_{00}\left(\frac{x}{y}\right)^2 - 2!\tau_{01}\left(\frac{x}{y}\right) + \tau_{11}\right] \\
&= y^2 p_2\left(\frac{x}{y}\right).
\end{aligned} \tag{3.648}$$

The polynomial in (x/y) can be factorised

$$\tau_{00}(x/y - a_1)(x/y - a_2) \tag{3.649}$$

where a_1 and a_2 are roots of the equation $p_2(x/y) = 0$. So

$$\begin{aligned}
\tau(\xi) &= \tau_{AB}\xi^A\xi^B \\
&= y^2 p_2\left(\frac{x}{y}\right) \\
&= y^2(\alpha_0\frac{x}{y} - \alpha_1)(\beta_0\frac{x}{y} - \beta_1) \\
&= (\alpha_0 x - \alpha_1 y)(\beta_0 x - \beta_1 y) \\
&= \alpha_A\beta_B\xi^A\xi^B
\end{aligned} \tag{3.650}$$

Therefore

$$\tau_{AB}\xi^A\xi^B = \alpha_A\beta_B\xi^A\xi^B \tag{3.651}$$

$$\begin{aligned}
\tau_{00}x^2 - 2!\tau_{01}xy + \tau_{11}y^2 &= \tau_{00}x^2 - 2!\tau_{10}xy + \tau_{11} \\
&= \alpha_0\beta_0x^2 - \alpha_0\beta_1xy - \alpha_1\beta_0yx + \alpha_1\beta_1y^2
\end{aligned} \tag{3.652}$$

Differentiating $\partial^2/\partial x^p \partial y^q$ for $p+q=2$ we get

$$\begin{aligned}\tau_{00} &= \alpha_{(0}\beta_{0)} \\ \tau_{01} &= \alpha_{(0}\beta_{1)} \\ \tau_{11} &= \alpha_{(1}\beta_{1)}\end{aligned} \tag{3.653}$$

or

$$\tau_{AB} = \alpha_{(A}\beta_{B)} \tag{3.654}$$

For τ with valence n

$$\begin{aligned}\tau(\xi) &= \tau_{AB\ldots C}\xi^A\xi^B\ldots\xi^C \\ &= y^n p_n\left(\frac{x}{y}\right) \\ &= y^2(\alpha_0\frac{x}{y}-\alpha_1)(\beta_0\frac{x}{y}-\beta_1)\ldots(\gamma_0-\gamma_1\frac{x}{y}) \\ &= (\alpha_0 x-\alpha_1 y)(\beta_0 x-\beta_1 y)\ldots(\gamma_0 x-\gamma_1 y) \\ &= \alpha_A\beta_B\ldots\gamma_C\xi^A\xi^B\ldots\xi^C\end{aligned} \tag{3.655}$$

Differentiating as $\partial^2/\partial x^p \partial y^q$ for $p, q = 0, 1, \ldots, n$, such that $p+q=n$ we obtain

$$\tau_{AB\ldots C} = \alpha_{(A}\beta_B\ldots\gamma_{C)}. \tag{3.656}$$

□

The spinor equivalent of $T_{ab} = -T_{ba}$ is $T_{ABA'B'} = -T_{BAB'A'}$. Define

$$\phi_{AB} = \frac{1}{2}T_{ABC'}{}^{C'}$$

By

$$T_{ABC'}{}^{C'} = T_{ABA'B'}\epsilon^{A'B'} = -T_{BAB'A'}\epsilon^{A'B'} = T_{BAA'B'}\epsilon^{A'B'} = T_{BAC'}{}^{C'}$$

we have

$$\phi_{AB} = \phi_{BA}.$$

$$\begin{aligned} T_{ABA'B'} &= T_{AB(A'B')} + T_{AB[A'B']} \\ &= T_{AB(A'B')} + \phi_{AB}\epsilon_{A'B'} \end{aligned} \tag{3.657}$$

where $\phi_{AB} = \frac{1}{2}T_{ABC'}{}^{C'}$. Aplying this again

$$T_{ABA'B'} = T_{(AB)(A'B')} + \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\overline{\phi}_{A'B'} \tag{3.658}$$

Note that

$$\begin{aligned} T_{(AB)(A'B')} &= \frac{1}{2!}\frac{1}{2!}(T_{ABA'B'} + T_{BAB'A'} + T_{BAA'B'} + T_{ABB'A'}) \\ &= \frac{1}{2!}\frac{1}{2!}(T_{ABA'B'} - T_{ABA'B'} + T_{BAA'B'} - T_{BAA'B'}) \\ &= 0. \end{aligned} \tag{3.659}$$

So that we have

$$T_{ABA'B'} = \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\overline{\phi}_{A'B'}. \tag{3.660}$$

The dual of T is defined by $T^*_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}T_{cd}$. Let us calculate $T^*_{ABA'B'}$,

$$\begin{aligned} T^*_{ABA'B'} &= \frac{1}{2}i(\epsilon_{AB}\epsilon^{CD}\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'} - \epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'B'}\epsilon^{C'D'})T_{CDC'D'} \\ &= \frac{i}{2}(\epsilon_{AB}\epsilon^{CD}\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'} - \epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'B'}\epsilon^{C'D'})(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\overline{\phi}_{C'D'}) \\ &= \frac{i}{2}[\epsilon_{AB}(\epsilon^{CD}\phi_{CD})(\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'}\epsilon_{C'D'}) + \epsilon_{AB}(\epsilon^{CD}\epsilon_{CD})(\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'}\overline{\phi}_{C'D'}) \\ &\quad - (\epsilon_A{}^C\epsilon_B{}^D\phi_{CD})\epsilon_{A'B'}(\epsilon^{C'D'}\epsilon_{C'D'}) - (\epsilon_A{}^C\epsilon_B{}^D\epsilon_{CD})\epsilon_{A'B'}(\epsilon^{C'D'}\overline{\phi}_{C'D'})] \\ &= i(\epsilon_{AB}\overline{\phi}_{A'B'} - \phi_{AB}\epsilon_{A'B'}) \end{aligned} \tag{3.661}$$

where we used $\epsilon^{CD}\phi_{CD} = \epsilon^{C'D'}\phi_{C'D'} = 0$ and $\epsilon^{CD}\epsilon_{CD} = \epsilon^{C'D'}\epsilon_{C'D'} = 2$. From this and (3.660) we have

$$T_{ab} + iT^*_{ab} = 2\phi_{AB}\epsilon_{A'B'} \tag{3.662}$$

which proves that ϕ_{AB} and T_{ab} are fully equivalent as

$$T_{ab} = Re\,(2\phi_{AB}\epsilon_{A'B'}).$$

We can decompose ϕ_{AB} as it is symmetric,

$$\phi_{AB} = \alpha_{(A}\beta_{B)} \tag{3.663}$$

If α and β are proportional then α is called a repeated principal spinor of ϕ, and ϕ is called algebraically special.

3.17.4 Curvature Spinors

$$R_{abcd} = R_{AA'BB'CC'DD'} \tag{3.664}$$

From $R_{abcd} = -R_{bacd}$

$$R_{abcd} = \frac{1}{2}R_{AX'B}{}^{X'}{}_{CC'DD'}\epsilon_{A'B'} + \frac{1}{2}R_{XA'}{}^{X}{}_{B'CC'DD'}\epsilon_{AB} \tag{3.665}$$

We have the symmetries

$$\begin{aligned} R_{AX'B}{}^{X'}{}_{CC'DD'} &= R_{(A|X'|B)}{}^{X'}{}_{CC'DD'} \\ R_{XA'}{}^{X}{}_{B'CC'DD'} &= R_{X(A'}{}^{X}{}_{B')CC'DD'} \end{aligned} \tag{3.666}$$

From $R_{abcd} = -R_{abdc}$ we then have

$$\begin{aligned} R_{abcd} &= X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\ &\quad + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \end{aligned} \tag{3.667}$$

where

$$X_{ABCD} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'}, \quad \Phi_{ABC'D'} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{YC'}{}^{Y}{}_{D'} \tag{3.668}$$

The complex conucates appear to make R_{abcd} real. We have the symmetries

$$
\begin{aligned}
R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'} &= R_{(A|X'|B)}{}^{X'}{}_{CY'D}{}^{Y'} = R_{(A|X'|B)}{}^{X'}{}_{(C|Y'|D)}{}^{Y'} \\
R_{XA'}{}^{X}{}_{B'YC'}{}^{Y}{}_{D'} &= R_{X(A'}{}^{X}{}_{B')YC'}{}^{Y}{}_{D'} = R_{(A|X'|B)}{}^{X'}{}_{Y(C'}{}^{Y}{}_{D')}
\end{aligned} \tag{3.669}
$$

or

$$
\begin{aligned}
X_{ABCD} &= X_{(AB)CD} = X_{(AB)(CD)} = X_{AB(CD)} \\
\Phi_{ABC'D'} &= \Phi_{(AB)C'D'} = \Phi_{(AB)(C'D')} = \Phi_{AB(C'D')}
\end{aligned} \tag{3.670}
$$

The interchange symmetry

$$
\begin{aligned}
& X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
= \ & X_{CDAB}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Phi}_{C'D'AB}\epsilon_{A'B'}\epsilon_{CD} + \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} + \overline{X}_{C'D'A'B'}\epsilon_{AB}\epsilon_{CD}
\end{aligned} \tag{3.671}
$$

Contracting both sides with $\epsilon^{A'B'}\epsilon^{C'D'}$ gives

$$
X_{ABCD} = X_{CDAB} \tag{3.672}
$$

Contracting both sides with $\epsilon^{A'B'}\epsilon^{CD}$ gives

$$
\Phi_{ABC'D'} = \overline{\Phi}_{C'D'AB} \tag{3.673}
$$

Therefore the interchange symmetry is equivalent to

$$
X_{ABCD} = X_{CDAB}, \qquad \overline{\Phi}_{ABC'D'} = \Phi_{ABC'D'} \tag{3.674}
$$

The second of these equations implies that $\Phi_{ABA'B'}$ corresponds to a real tensor Φ_{ab} while $\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} = \Phi_{BAB'A'}$ implies $\Phi_{ab} = \Phi_{ba}$ and $\Phi_{C}{}^{C}{}_{C'}{}^{C'} = \Phi_{ABA'B'}\epsilon^{AB}\epsilon^{A'B'} = \Phi_{(AB)(A'B')}\epsilon^{AB}\epsilon^{A'B'} = 0$ implies $\Phi_{a}{}^{a} = 0$, altogether

$$
\Phi_{ABA'B'} = \Phi_{ab} = \Phi_{ba} = \overline{\Phi}_{ab}, \quad \Phi_{a}{}^{a} = 0 \tag{3.675}
$$

that is Φ_{ab} is real, symmetric and trace-free. Note also $X_{ABCD} = X_{(AB)(CD)}$ and $X_{ABCD} = X_{CDAB}$ implies

$$
\begin{aligned}
X_{A(BC)D}\epsilon^{AD} &= \frac{1}{2}(X_{ABCD} + X_{ACBD})\epsilon^{AD} \\
&= \frac{1}{2}(X_{BADC} + X_{BDAC})\epsilon^{AD} \\
&= X_{B(AD)C}\epsilon^{AD} \\
&= 0
\end{aligned} \tag{3.676}
$$

that is

$$
X_{A(BC)}{}^{A} = 0. \tag{3.677}
$$

The first type of dual of $R^*_{abcd} = \frac{1}{2}\epsilon_{cd}{}^{ef}R_{abef}$

$$
\begin{aligned}
R^*_{ABCDA'B'C'D'} &= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'})R_{ABEFA'B'E'F'} \\
&= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'}) \\
&\quad (X_{ABEF}\epsilon_{A'B'}\epsilon_{E'F'} + \Phi_{ABE'F'}\epsilon_{A'B'}\epsilon_{EF} \\
&\quad + \overline{\Phi}_{A'B'EF}\epsilon_{AB}\epsilon_{E'F'} + \overline{X}_{A'B'E'F'}\epsilon_{AB}\epsilon_{EF}) \\
&= \frac{1}{2}i(2\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + 2\overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
&\quad - 2X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - 2\overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}) \\
&= iR_{ABCDA'B'D'C'}
\end{aligned} \tag{3.678}
$$

Similarly we have

$$
{}^*R_{abcd} = {}^*R_{ABCDA'B'C'D'} = iR_{ABCDB'A'C'D'} \tag{3.679}
$$

Also

$$
\begin{aligned}
{}^*R^*{}_{abcd} &= \frac{1}{4}\epsilon_{ab}{}^{ef}\epsilon_{cd}{}^{gh}R_{efgh} \\
&= {}^*R^*_{ABCDA'B'C'D'} \\
&= i\,{}^*R_{ABCDA'B'D'C'} \\
&= -R_{ABCDB'A'D'C'}
\end{aligned} \tag{3.680}
$$

Altogether

$$
\begin{aligned}
R^*{}_{abcd} &= iR_{ABCDA'B'D'C'} \\
{}^*R_{abcd} &= iR_{ABCDB'A'C'D'} \\
{}^*R^*{}_{abcd} &= -R_{ABCDB'A'D'C'}
\end{aligned}
\tag{3.681}
$$

Clearly all three duals satisfy share the anti-symmetry of R_{abcd} ($R_{abcd} = R_{[ab][cd]}$). In addition as

$$
\epsilon_{ab}{}^{ef}\epsilon_{cd}{}^{gh}R_{efgh} = \epsilon_{ab}{}^{ef}\epsilon_{cd}{}^{gh}R_{ghef} = \epsilon_{cd}{}^{ef}\epsilon_{ab}{}^{gh}R_{efgh}
$$

i.e.

$$
{}^*R^*{}_{abcd} = {}^*R^*{}_{cdab} \tag{3.682}
$$

and

$$
{}^*R^*{}_{a[bcd]} = \frac{1}{4}\epsilon_{a[b}{}^{ef}\epsilon_{cd]}{}^{gh}R_{efgh} = \frac{1}{4}\epsilon_{a[b}{}^{ef}\epsilon_{c]d}{}^{gh}R_{efgh} = 0
$$

i.e.

$$
{}^*R^*{}_{a[bcd]} = 0. \tag{3.683}
$$

$$
\begin{aligned}
R_{abcd} &= X + \Phi + \overline{\Phi} + \overline{X} \\
R^*{}_{abcd} &= -iX + i\Phi - i\overline{\Phi} + i\overline{X} \\
{}^*R_{abcd} &= -iX - i\Phi + i\overline{\Phi} + i\overline{X} \\
{}^*R^*{}_{abcd} &= -X + \Phi + \overline{\Phi} - \overline{X}
\end{aligned}
\tag{3.684}
$$

Because of $X_{ABCD} = X_{CDAB}$

$$
X_{CB}{}^{C}{}_{D} = \epsilon^{AC}X_{ABCD} = \epsilon^{AC}X_{CDAB} = -\epsilon^{AC}X_{ADCB} = -X_{CD}{}^{C}{}_{B} \tag{3.685}
$$

and so

$$
X_{CB}{}^{C}{}_{D} = 3\Upsilon\epsilon_{BD}. \tag{3.686}
$$

We translate the symmetry $R_{[abc]d} = 0$ or equivalently $R_{a[bcd]} = 0$ into spinors. To simplify the calculation we establish an equivalence, first note

$$R^*{}_{ab}{}^{bc} = \frac{1}{2}\epsilon^{bcef}R_{abef} = -\frac{1}{2}\epsilon^{cbef}R_{a[bef]} \tag{3.687}$$

therefore $R_{a[bcd]} = 0$ implies $R^*{}_{ab}{}^{bc} = 0$. Next

$$\begin{aligned} R_{a[bcd]} &= \frac{1}{3!}\delta^{efg}_{bcd}R_{aefg} \\ &= -\frac{2}{3!}\epsilon_{hbcd}\frac{1}{2}\epsilon^{ehfg}R_{aefg} \\ &= -\frac{2}{3!}\epsilon_{hbcd}R^*{}_{ae}{}^{eh} \end{aligned} \tag{3.688}$$

therefore $R^*{}_{ab}{}^{bc} = 0$ implies $R_{a[bcd]} = 0$. Thus $R_{a[bcd]} = 0$ is equivalent to

$$R^*{}_{ab}{}^{bc} = 0. \tag{3.689}$$

To obtain the cyclic identity

$$\begin{aligned} R^*{}_{ab}{}^{b}{}_{c} &= -iX_{AB}{}^{B}{}_{C}\epsilon_{A'B'}\epsilon^{B'}{}_{C'} + i\Phi_{AB}{}^{B'}{}_{C'}\epsilon_{A'B'}\epsilon^{B}{}_{C} \\ &\quad - i\overline{\Phi}_{A'B'}{}^{B}{}_{C}\epsilon_{AB}\epsilon^{B'}{}_{C'} + i\overline{X}_{A'B'}{}^{B'}{}_{C'}\epsilon_{AB}\epsilon^{B}{}_{C} \\ &= -iX_{AB}{}^{B}{}_{C}\epsilon_{A'B'}\epsilon^{B'}{}_{C'} + i\overline{X}_{A'B'}{}^{B'}{}_{C'}\epsilon_{AB}\epsilon^{B}{}_{C} \\ &\quad + i(\Phi_{ACA'C'} - \Phi_{ACA'C'}) \\ &= 0 \end{aligned} \tag{3.690}$$

implies

$$X_{AB}{}^{B}{}_{C}\epsilon_{A'C'} = \overline{X}_{A'B'}{}^{B'}{}_{C'}\epsilon_{AC} \tag{3.691}$$

or

$$\Upsilon\epsilon_{AC}\epsilon_{A'C'} = \overline{\Upsilon}\epsilon_{A'C'}\epsilon_{AC} \tag{3.692}$$

or on contracting with $\epsilon^{AC}\epsilon^{A'C'}$ gives

$$\Upsilon = \overline{\Upsilon} \tag{3.693}$$

Let us define Ψ_{ABCD} by

$$\Psi_{ABCD} = X_{ABCD} - \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) \tag{3.694}$$

where

$$\Upsilon = \frac{1}{6}\epsilon^{AC}\epsilon^{BD}X_{ABCD}. \tag{3.695}$$

The symmetries $X_{ABCD} = X_{(AB)(CD)}$

$$\begin{aligned}
\Psi_{(AB)(CD)} - \Psi_{ABCD} &= X_{(AB)(CD)} - X_{ABCD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) \\
&\quad - \frac{\Upsilon}{(2!)^2}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD} + \epsilon_{BC}\epsilon_{AD} + \epsilon_{AC}\epsilon_{BD} \\
&\quad + \epsilon_{AD}\epsilon_{BC} + \epsilon_{BD}\epsilon_{AC} + \epsilon_{BD}\epsilon_{AC} + \epsilon_{AD}\epsilon_{BC}) \\
&= 0.
\end{aligned} \tag{3.696}$$

or

$$\Psi_{ABCD} = \Psi_{(AB)(CD)} \tag{3.697}$$

and $X_{ABCD} = X_{CDAB}$ imply

$$\begin{aligned}
\Psi_{ABCD} - \Psi_{CDAB} &= X_{ABCD} - X_{CDAB} \\
&\quad - \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) + \Upsilon(\epsilon_{CA}\epsilon_{DB} + \epsilon_{DA}\epsilon_{CB}) \\
&= 0.
\end{aligned} \tag{3.698}$$

or

$$\Psi_{ABCD} = \Psi_{CDAB}. \tag{3.699}$$

By construction

$$
\begin{aligned}
\epsilon^{AC}\Psi_{ABCD} &= \epsilon^{AC}X_{ABCD} - \epsilon^{AC}\Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) \\
&= 3\Upsilon\epsilon_{BD} - \Upsilon(2\epsilon_{BD} + \epsilon_{BD}) \\
&= 0. \qquad (3.700)
\end{aligned}
$$

Therefore Ψ_{ABCD} is totally symmetric

$$
\Psi_{ABCD} = \Psi_{(ABCD)}. \qquad (3.701)
$$

We have now found the spinor equivalents of all the symmetries of R_{abcd}. Next we compute the Ricci tensor $R_{ac} = R_{abc}{}^{b}$. From (3.667) we get

$$
\begin{aligned}
R_{ac} &= R_{abc}{}^{b} \\
&= X_{ABC}{}^{B}\epsilon_{A'B'}\epsilon_{C'}{}^{B'} + \Phi_{ABC'}{}^{B'}\epsilon_{A'B'}\epsilon_{C}{}^{B} \\
&\quad + \overline{\Phi}_{A'B'C}{}^{B}\epsilon_{AB}\epsilon_{C'}{}^{B'} + \overline{X}_{A'B'C'}{}^{B'}\epsilon_{AB}\epsilon_{C}{}^{B} \\
&= 6\Upsilon\epsilon_{AC}\epsilon_{A'C'} - 2\Phi_{ACA'C'} \qquad (3.702)
\end{aligned}
$$

or

$$
R_{ab} = 6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}. \qquad (3.703)
$$

which may written

$$
R_{ab} = 6\Upsilon g_{ab} - 2\Phi_{ab}. \qquad (3.704)
$$

Hence, for the scalar curvature $R = R_{a}{}^{a}$ we find, using $\Phi_{a}{}^{a} = 0$

$$
R = 24\Upsilon. \qquad (3.705)
$$

Thus

$$
\Phi_{ab} = -\frac{1}{2}(R_{ab} - \frac{1}{4}Rg_{ab}) \qquad (3.706)
$$

$$
\begin{aligned}
R_{ABCDA'B'C'D'} &= (\Psi_{ABCD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}))\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\
&\quad + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + (\overline{\Psi}_{A'B'C'D'} + \Upsilon(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{B'C'}\epsilon_{A'D'}))\epsilon_{AB}\epsilon_{CD} \\
&= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\epsilon_{A'B'}\epsilon_{C'D'} \\
&\quad + c.c. \qquad (3.707)
\end{aligned}
$$

Weyl tensor

Define the real tensor

$$C_{ABCDA'B'C'D'} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \tag{3.708}$$

Rearanging (3.707) gives

$$\begin{aligned} C_{ABCDA'B'C'D'} &= R_{ABCDA'B'C'D'} \\ &\quad -\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} - \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\epsilon_{A'B'}\epsilon_{C'D'} \\ &\quad -\overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} - \Upsilon(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{B'C'}\epsilon_{A'D'})\epsilon_{AB}\epsilon_{CD} \end{aligned} \tag{3.709}$$

We use the Jacobi identity

$$\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{A'C'}\epsilon_{B'D'} = 0$$

on the terms proportional to Υ,

$$\begin{aligned} & -\Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\epsilon_{A'B'}\epsilon_{C'D'} + c.c. \\ = & -\Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})(\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{A'D'}\epsilon_{B'C'}) + c.c. \\ = & -\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'} - \epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'} \\ & - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} + \epsilon_{BC}\epsilon_{AD}\epsilon_{A'C'}\epsilon_{B'D'}) + c.c \\ = & -2\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'} - \epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'}) \end{aligned} \tag{3.710}$$

Substituting this into (3.709)

$$\begin{aligned} C_{ABCDA'B'C'D'} &= R_{ABCDA'B'C'D'} - \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} - \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} \\ &\quad -2\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'} - \epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'}) \end{aligned} \tag{3.711}$$

Comparing

$$-\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} - \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} \tag{3.712}$$

with

$$\Phi_{ACA'C'}\epsilon_{B'D'}\epsilon_{BD} - \Phi_{ADA'D'}\epsilon_{B'C'}\epsilon_{BC} - \Phi_{BCB'C'}\epsilon_{A'D'}\epsilon_{AD} + \Phi_{BDB'D'}\epsilon_{A'C'}\epsilon_{AC} \tag{3.713}$$

for the distinct indices

$A = 0, B = 0, C = 0, D = 0;$
$A = 1, B = 1, C = 1, D = 1;$
$A = 1, B = 1, C = 1, D = 0;$
$A = 0, B = 1, C = 1, D = 1;$
$A = 0, B = 1, C = 0, D = 1;$
$A = 0, B = 0, C = 0, D = 1;$
$A = 0, B = 1, C = 0, D = 0;$

proves they are equivalent. Substituting (3.713) into (3.711) gives

$$\begin{aligned} C_{ABCDA'B'C'D'} &= R_{ABCDA'B'C'D'} + \Phi_{ACA'C'}\epsilon_{B'D'}\epsilon_{BD} - \Phi_{ADA'D'}\epsilon_{B'C'}\epsilon_{BC} \\ &\quad - \Phi_{BCB'C'}\epsilon_{A'D'}\epsilon_{AD} + \Phi_{BDB'D'}\epsilon_{A'C'}\epsilon_{AC} \\ &\quad - 2\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'} - \epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'}) \end{aligned} \tag{3.714}$$

Converting into tensors gives

$$\begin{aligned} C_{ABCDA'B'C'D'} &= R_{abcd} + (\Phi_{ac}g_{db} - \Phi_{ad}g_{bc} - \Phi_{bc}g_{ad} + \Phi_{bd}g_{ac}) - 2\Upsilon(g_{ac}g_{bd} - g_{bc}g_{ad}) \\ &= R_{abcd} - \frac{1}{2}(R_{ac}g_{db} - R_{ad}g_{bc} - R_{bc}g_{ad} + R_{bd}g_{ac}) \\ &\quad + \frac{1}{8}R(g_{ac}g_{db} - g_{ad}g_{bc} - g_{bc}g_{ad} + g_{bd}g_{ac}) - \frac{R}{12}(g_{ac}g_{bd} - g_{bc}g_{ad}) \\ &= R_{abcd} - \frac{1}{2}(R_{ac}g_{db} - R_{ad}g_{bc} - R_{bc}g_{ad} + R_{bd}g_{ac}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R \end{aligned} \tag{3.715}$$

This agrees with the definition of the Weyl tensor

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R \tag{3.716}$$

so we can identify

$$C_{ABCDA'B'C'D'} = C_{abcd} \tag{3.717}$$

Worked exercise:

Use that C_{abcd} only differes from R_{abcd} by terms involving the Ricci tensor and the Ricci scalar, that C_{abcd} have the same symmetries as R_{abcd}:

$$R_{abcd} = -R_{abdc} = -R_{bacd} \tag{3.718}$$

and that

$$C_{acb}{}^{c} = 0$$

to determine C_{abcd}.

Solution:

The most general expression for C_{abcd} involving R_{abcd}, R_{ab}, R and g_{ab} is

$$\begin{aligned} C_{abcd} &= R_{abcd} + [C_1 R_{ab} g_{cd} + C_2 R_{ac} g_{bd} + C_3 R_{ad} g_{bc}] + [C_4 R_{bc} g_{ad} + C_5 R_{bd} g_{ac}] + C_6 R_{cd} g_{ab} \\ &\quad + R(C_7 g_{ab} g_{cd} + C_8 g_{ac} g_{bd} + C_9 g_{ad} g_{bc}). \end{aligned} \tag{3.719}$$

Using

$$\begin{aligned} C_{abcd} &= -C_{bacd} \\ &= R_{abcd} - [C_1 R_{ab} g_{cd} + C_2 R_{bc} g_{ad} + C_3 R_{bd} g_{ac}] - [C_4 R_{ac} g_{bd} + C_5 R_{ad} g_{bc}] - C_6 R_{cd} g_{ab} \\ &\quad - R(C_7 g_{ab} g_{cd} + C_8 g_{bc} g_{ad} + C_9 g_{bd} g_{ac}) \end{aligned} \tag{3.720}$$

which implies $C_1 = 0$ and $C_2 = -C_4$ and $C_3 = -C_5$ and $C_6 = 0$ and $C_7 = 0$ and $C_8 = -C_9$. So that

$$\begin{aligned} C_{abcd} &= R_{abcd} + C_2(R_{ac} g_{bd} - R_{bc} g_{ad}) + C_3(R_{ad} g_{bc} - R_{bd} g_{ac}) \\ &\quad + C_8 R(g_{ac} g_{bd} - g_{ad} g_{bc}) \end{aligned} \tag{3.721}$$

Now using

$$\begin{aligned} C_{abcd} &= -C_{abdc} \\ &= R_{abcd} - C_2(R_{ad} g_{bc} - R_{bd} g_{ac}) - C_3(R_{ac} g_{bd} - R_{bc} g_{ad}) \\ &\quad - C_8 R(g_{ad} g_{bc} - g_{ac} g_{bd}) \end{aligned} \tag{3.722}$$

So that $C_2 = -C_3$ and so

$$\begin{aligned} C_{abcd} &= R_{abcd} + C_2(R_{ac}g_{bd} - R_{bc}g_{ad} - R_{ad}g_{bc} + R_{bd}g_{ac}) \\ &\quad + C_8 R(g_{ac}g_{bd} - g_{ad}g_{bc}). \end{aligned} \tag{3.723}$$

Now using $C_{adc}{}^{d} = 0$ we obtain

$$0 = R_{ac} + C_2(4R_{ac} - R_{ac} - R_{ac} + Rg_{ac}) + C_8 R(4g_{ac} - g_{ac})$$

or

$$0 = (1 + 2C_2)R_{ac} + (C_2 + 3C_8)Rg_{ac}.$$

So that

$$C_{abcd} = R_{abcd} - \frac{1}{2}(R_{ac}g_{bd} - R_{bc}g_{ad} - R_{ad}g_{bc} + R_{bd}g_{ac}) + \frac{1}{6}R(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{3.724}$$

□

The dual of the Weyl tensor is

$$C^*{}_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef}C_{abef}$$

In spinor notation

$$\begin{aligned} C^*_{ABCDA'B'C'D'} &= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'})C_{ABEFA'B'E'F'} \\ &= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'}) \\ &\quad \Psi_{ABEF}\epsilon_{A'B'}\epsilon_{E'F'} + \overline{\Psi}_{A'BE'F'}\epsilon_{AB}\epsilon_{EF} \\ &= i(\overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} - \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}) \end{aligned} \tag{3.725}$$

Therfore

$$C_{\ abcd} + iC^*{}_{abcd} = 2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} \tag{3.726}$$

so that $C_{\ abcd}$ and Ψ_{ABCD} are equivalent,

$$C_{\ abcd} = Re\ (2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}). \tag{3.727}$$

As Ψ_{ABCD} is totally symetric we can decompose it as

Petrov classification

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}. \tag{3.728}$$

There are six distict cases which constitute the so-called Petrov classification

Type I or $\{1,1,1,1\}$. None of the four pricipal null directions coincide.

Type II or $\{2,1,1\}$. Two directions coincide.

Type D or $\{2,2\}$. Two are two different pairs of repeated null directions.

Type III or $\{3,1\}$. Three pricipal null directions coincide.

Type N or $\{4\}$. All four pricipal null directions coincide.

Type O The Weyl tensor vanishes and spacetime is conformally flat.

Spinor Covariant Derivative

$$(\nabla_{AA'}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_t + \partial_z & \partial_x - i\partial_y \\ \partial_x i + \partial_y & \partial_t - \partial_z \end{pmatrix} \tag{3.729}$$

where the ∂_a are derivatives with respect to inertial coordinates.

3.17.5 Curvature in spinors

$$u^a R_{abcd} = 2\nabla_{[c}\nabla_{d]}u_b \tag{3.730}$$

It is easily seen that $\nabla_{[c}\nabla_{d]}$ annihiltes all scalar fields:

$$\begin{aligned} \nabla_{[c}\nabla_{d]}\phi &= \nabla_c\partial_d\phi - \nabla_d\partial_c\phi \\ &= \partial_c\partial_d\phi - \partial_d\partial_c\phi + (\Gamma^e_{cd} - \Gamma^e_{dc})\partial_e\phi = 0. \end{aligned} \tag{3.731}$$

$$
\begin{aligned}
u^a R_{abcd} &= 2\nabla_{[c}\nabla_{d]}(u_{\hat{b}}e^{\hat{b}}{}_b) \\
&= 2u_{\hat{b}}\nabla_{[c}\nabla_{d]}e^{\hat{b}}{}_b \quad (\text{since } \nabla_{[c}\nabla_{d]}u_{\hat{b}} = 0) \\
&= 2(u^a e_{\hat{b}a})\nabla_{[c}\nabla_{d]}e^{\hat{b}}{}_b
\end{aligned} \tag{3.732}
$$

It follows that

$$
R_{abcd} = 2e_{\hat{b}a}\nabla_{[c}\nabla_{d]}e^{\hat{b}}{}_b. \tag{3.733}
$$

$$
2\nabla_{[c}\nabla_{d]} = \nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'} \tag{3.734}
$$

The spinor equivalent is

$$
\begin{aligned}
R_{ABCDA'B'C'D'} &= 2\epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}\nabla_{[c}\nabla_{d]}\left(\epsilon^{\hat{B}}{}_B\epsilon^{\hat{B}'}{}_{B'}\right) \\
&= \epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}(\nabla_c\nabla_d - \nabla_d\nabla_c)\left(\epsilon^{\hat{B}}{}_B\epsilon^{\hat{B}'}{}_{B'}\right) \\
&= \epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}[\nabla_c(\epsilon^{\hat{B}'}{}_{B'}\nabla_d\epsilon^{\hat{B}}{}_B + \epsilon^{\hat{B}}{}_B\nabla_d\epsilon^{\hat{B}'}{}_{B'}) \\
&\quad - \nabla_d(\epsilon^{\hat{B}'}{}_{B'}\nabla_c\epsilon^{\hat{B}}{}_B + \epsilon^{\hat{B}}{}_B\nabla_c\epsilon^{\hat{B}'}{}_{B'}) \\
&= \epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}\epsilon^{\hat{B}'}{}_{B'}(\nabla_c\nabla_d - \nabla_d\nabla_c)\epsilon^{\hat{B}}{}_B + c.c. \\
&= 2\epsilon_{\hat{B}A}\epsilon_{B'A'}\nabla_{[c}\nabla_{d]}\epsilon^{\hat{B}}{}_B + c.c.
\end{aligned} \tag{3.735}
$$

We define

$$
\begin{aligned}
\Box_{CD} &= \epsilon^{C'D'}\nabla_{[CC'}\nabla_{DD']} \\
&= \frac{1}{2}\epsilon^{C'D'}(\nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'}) \\
&= \frac{1}{2}(\nabla_{CC'}\nabla_D{}^{C'} + \nabla_{DD'}\nabla_C{}^{D'}) \\
&= \nabla_{C'(C}\nabla_{D)}{}^{C'}.
\end{aligned} \tag{3.736}
$$

$$
\begin{aligned}
\nabla_{CC'}\nabla_{DD'} &= \nabla_{C'(C}\nabla_{D)D'} + \frac{1}{2}\epsilon_{CD}\nabla_{C'E}\nabla^E{}_{D'} \\
&= \nabla_{(C'(C}\nabla_{D)D')} + \frac{1}{2}\epsilon_{C'D'}\nabla_{E'(C}\nabla_{D)}{}^{E'} \\
&\quad + \frac{1}{2}\epsilon_{CD}\nabla_{E(C'}\nabla_{D')}{}^E + \frac{1}{4}\epsilon_{CD}\epsilon_{C'D'}\nabla_{EE'}\nabla^{EE'}
\end{aligned} \tag{3.737}
$$

$$
\begin{aligned}
& \nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'} \\
= \; & \nabla_{(C'(C}\nabla_{D)D')} + \frac{1}{2}\epsilon_{C'D'}\nabla_{E'(C}\nabla_{D)}{}^{E'} \\
& + \frac{1}{2}\epsilon_{CD}\nabla_{E(C'}\nabla_{D')}^{E} - \frac{1}{4}\epsilon_{CD}\epsilon_{C'D'}\nabla_{EE'}\nabla^{EE'} \\
& - \nabla_{(D'(D}\nabla_{C)C')} - \frac{1}{2}\epsilon_{D'C'}\nabla_{E'(D}\nabla_{C)}{}^{E'} \\
& - \frac{1}{2}\epsilon_{DC}\nabla_{E(D'}\nabla_{C')}^{E} - \frac{1}{4}\epsilon_{DC}\epsilon_{D'C'}\nabla_{EE'}\nabla^{EE'}
\end{aligned} \tag{3.738}
$$

Or

$$
\begin{aligned}
\nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'} &= \epsilon_{C'D'}\nabla_{E'(C}\nabla_{D)}{}^{E'} + \epsilon_{CD}\nabla_{E(D'}\nabla_{C')}^{E} \\
&= \epsilon_{C'D'}\Box_{CD} + \epsilon_{CD}\Box_{C'D'}.
\end{aligned} \tag{3.739}
$$

Consider the term

$$
\epsilon_{\hat{B}A}\Box_{CD}\epsilon^{\hat{B}}{}_{B} \tag{3.740}
$$

This is obviously symmetric in CD and from

$$
\begin{aligned}
0 &= \Box_{CD}(\epsilon_{\hat{B}A}\epsilon^{\hat{B}}{}_{B}) \\
&= \epsilon_{\hat{B}A}\Box_{CD}\epsilon^{\hat{B}}{}_{B} + \epsilon^{\hat{B}}{}_{B}\Box_{CD}\epsilon_{\hat{B}A} \\
&= \epsilon_{\hat{B}A}\Box_{CD}\epsilon^{\hat{B}}{}_{B} - \epsilon_{\hat{B}B}\Box_{CD}\epsilon^{\hat{B}}{}_{A}
\end{aligned} \tag{3.741}
$$

we see it is symmeric in AB. We decompose it into symmetric spinors

$$
\begin{aligned}
\epsilon_{\hat{B}A}\Box_{CD}\epsilon^{\hat{B}}{}_{B} &= \frac{1}{3}\epsilon_{\hat{B}A}(\Box_{CD}\epsilon^{\hat{B}}{}_{B} + \Box_{DB}\epsilon^{\hat{B}}{}_{C} + \Box_{BC}\epsilon^{\hat{B}}{}_{D}) \\
&+ \frac{1}{3}\epsilon_{\hat{B}A}(\Box_{CD}\epsilon^{\hat{B}}{}_{B} - \Box_{BD}\epsilon^{\hat{B}}{}_{C}) \\
&+ \frac{1}{3}\epsilon_{\hat{B}A}(\Box_{CD}\epsilon^{\hat{B}}{}_{B} - \Box_{CB}\epsilon^{\hat{B}}{}_{D}) \\
&= \epsilon_{\hat{B}A}\Box_{(CD}\epsilon^{\hat{B}}{}_{B)} - \frac{1}{3}\epsilon_{\hat{B}A}\epsilon_{CB}\Box_{D}{}^{F}\epsilon^{\hat{B}}{}_{F} - \frac{1}{3}\epsilon_{\hat{B}A}\epsilon_{DB}\Box_{C}{}^{F}\epsilon^{\hat{B}}{}_{F} \\
&= \epsilon_{\hat{B}A}\Box_{(CD}\epsilon^{\hat{B}}{}_{B)} - \frac{1}{3}\epsilon_{CB}\epsilon_{\hat{B}(A}\Box_{D)}{}^{F}\epsilon^{\hat{B}}{}_{F} - \frac{1}{6}\epsilon_{AD}\epsilon_{CB}\epsilon_{\hat{B}E}\Box^{EF}\epsilon^{\hat{B}}{}_{F} \\
&- \frac{1}{3}\epsilon_{DB}\epsilon_{\hat{B}(A}\Box_{C)}{}^{F}\epsilon^{\hat{B}}{}_{F} - \frac{1}{6}\epsilon_{AC}\epsilon_{DB}\epsilon_{\hat{B}E}\Box^{EF}\epsilon^{\hat{B}}{}_{F} \\
&= \epsilon_{\hat{B}A}\Box_{(CD}\epsilon^{\hat{B}}{}_{B)} - (\frac{1}{6}\epsilon_{\hat{B}E}\Box^{EF}\epsilon^{\hat{B}}{}_{F})(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{CB}) \\
&- \frac{1}{3}[\epsilon_{CB}\epsilon_{\hat{B}(A}\Box_{D)}{}^{F}\epsilon^{\hat{B}}{}_{F} + \epsilon_{DB}\epsilon_{\hat{B}(A}\Box_{C)}{}^{F}\epsilon^{\hat{B}}{}_{F}]
\end{aligned} \tag{3.742}
$$

Using the symmetry in AB and CD,

$$
\begin{aligned}
\epsilon_{\hat{B}A}\Box_{CD}\epsilon^{\hat{B}}{}_{B} &= \epsilon_{\hat{B}(A}\Box_{|(CD)|}\epsilon^{\hat{B}}{}_{B)} \\
&= \epsilon_{\hat{B}A}\Box_{(CD}\epsilon^{\hat{B}}{}_{B)} - (\frac{1}{6}\epsilon_{\hat{B}E}\Box^{EF}\epsilon^{\hat{B}}{}_{F})2\epsilon_{(A(C}\epsilon_{D)B)} \\
&= \Psi_{ABCD} - 2\Upsilon\epsilon_{(A(C}\epsilon_{D)B)}
\end{aligned} \tag{3.743}
$$

where

$$
\Psi_{ABCD} = \epsilon_{\hat{B}A}\Box_{(CD}\epsilon^{\hat{B}}{}_{B)} = \Psi_{(ABCD)} \tag{3.744}
$$

and

$$
\Upsilon = \frac{1}{6}\epsilon_{\hat{B}E}\Box^{EF}\epsilon^{\hat{B}}{}_{F}. \tag{3.745}
$$

Also we write

$$
\epsilon_{\hat{B}A}\Box_{C'D'}\epsilon^{\hat{B}}{}_{B} = \Phi_{ABC'D'} \tag{3.746}
$$

which is symmetric in AB and $C'D'$. Combining all of this into (3.735)

$$
R_{ABCDA'B'C'D'} = \epsilon_{A'B'}\epsilon_{C'D'}\left[\Psi_{ABCD} - 2\Upsilon\epsilon_{(A(C}\epsilon_{D)B)}\right] + \epsilon_{A'B'}\epsilon_{CD}\Phi_{ABC'D'} + c.c. \tag{3.747}
$$

3.17.6 Spinor Form of the Ricci Identies

Recall the covariant derivative satisfies the product rule

$$\nabla_a(T_A S_B) = T_A \nabla_a S_B + S_B \nabla_a T_A. \tag{3.748}$$

Then

$$\begin{aligned}
(\nabla_a\nabla_b - \nabla_b\nabla_a)(T_A S_B) &= \nabla_a(T_A\nabla_b S_B + S_B\nabla_b T_A) \\
&\quad -\nabla_b(T_A\nabla_a S_B + S_B\nabla_a T_A) \\
&= T_A\nabla_a\nabla_b S_B + S_B\nabla_a\nabla_b T_A \\
&\quad -T_A\nabla_b\nabla_a S_B + S_B\nabla_b\nabla_a T_A \\
&= 2T_A\nabla_{[a}\nabla_{b]}S_B + 2S_B\nabla_{[a}\nabla_{b]}T_A
\end{aligned} \tag{3.749}$$

$$2\nabla_{[a}\nabla_{b]} = \epsilon_{A'B'}\Box_{AB} + \epsilon_{AB}\Box_{A'B'}$$

$$\Box_{AB}(\phi_{\mathcal{C}}\chi_{\mathcal{D}}) = \phi_{\mathcal{C}}\Box_{AB}\chi_{\mathcal{D}} + \chi_{\mathcal{D}}\Box_{AB}\phi_{\mathcal{C}} \tag{3.750}$$

We consider the self-dual null bivector

$$T^{ab} = \xi^A\xi^B\xi^{A'B'} \tag{3.751}$$

The Ricci identity says

$$2\nabla_{[a}\nabla_{b]}T^{cd} = R_{abe}{}^{c}T^{ed} + R_{abe}{}^{d}T^{ce} \tag{3.752}$$

Using (3.749)

$$2\xi^C\epsilon^{C'D'}\nabla_{[a}\nabla_{b]}\xi^D + 2\xi^D\epsilon^{C'D'}\nabla_{[a}\nabla_{b]}\xi^C = R_{abEE'}{}^{CC'}\xi^E\xi^D\epsilon^{E'D'} + R_{abEE'}{}^{DD'}\xi^C\xi^E\epsilon^{C'E'} \tag{3.753}$$

or

$$4\epsilon^{C'D'}\xi^{(C}\nabla_{[a}\nabla_{b]}\xi^{D)} = R_{abEE'}{}^{CC'}\xi^E\xi^D\epsilon^{E'D'} + R_{abEE'}{}^{DD'}\xi^C\xi^E\epsilon^{C'E'} \tag{3.754}$$

Consider

$$\begin{aligned}
\epsilon_{C'D'} R_{AA'BB'EE'}{}^{CC'} \epsilon^{E'D'} &= \delta_{C'}^{E'} (X_{ABE}{}^{C} \epsilon_{A'B'} \epsilon_{E'}{}^{C'} + \Phi_{ABE'}{}^{C'} \epsilon_{A'B'} \epsilon_{E}{}^{C} \\
&\quad + \Phi_{A'B'E}{}^{C} \epsilon_{AB} \epsilon_{E'}{}^{C'} + \overline{X}_{A'B'E'}{}^{C'} \epsilon_{AB} \epsilon_{E}{}^{C}) \\
&= 2X_{ABE}{}^{C} \epsilon_{A'B'} + \Phi_{ABC'}{}^{C'} \epsilon_{A'B'} \epsilon_{E}{}^{C} \\
&\quad + 2\Phi_{A'B'E}{}^{C} \epsilon_{AB} + \overline{X}_{A'B'C'}{}^{C'} \epsilon_{AB} \epsilon_{E}{}^{C} \\
&= 2X_{ABE}{}^{C} \epsilon_{A'B'} + 2\Phi_{A'B'E}{}^{C} \epsilon_{AB} \qquad (3.755)
\end{aligned}$$

where we used $\Phi_{ABC'}{}^{C'} = \epsilon^{C'D'} \Phi_{ABC'D'} = 0$, and $\overline{X}_{A'B'C'}{}^{C'} = \epsilon^{C'D'} \overline{X}_{A'B'C'D'} = 0$. We also have

$$\begin{aligned}
\epsilon_{C'D'} R_{AA'BB'EE'}{}^{DD'} \epsilon^{C'E'} &= \epsilon_{D'C'} R_{AA'BB'EE'}{}^{DD'} \epsilon^{E'C'} \\
&= \epsilon_{C'D'} R_{AA'BB'EE'}{}^{DC'} \epsilon^{E'D'} \\
&= 2X_{ABE}{}^{D} \epsilon_{A'B'} + 2\Phi_{A'B'E}{}^{D} \epsilon_{AB} \qquad (3.756)
\end{aligned}$$

where in the first step we made the replacements $\epsilon_{C'D'} = -\epsilon_{D'C'}$ $\epsilon^{C'E'} = -\epsilon^{E'C'}$, and in the second we swapped the dummy variables C' and D', then comparison with the LHS of (3.755) gives the last line. Substitution of these results into the contraction of (3.754) with $\epsilon_{C'D'}$ gives

$$2\xi^{(D} \nabla_{[a} \nabla_{b]} \xi^{C)} = \xi^{(D} (\epsilon_{A'B'} X_{ABE}{}^{C)} + \epsilon_{AB} \Phi_{A'B'E}{}^{C)}) \xi^{E} \qquad (3.757)$$

If we have

$$\phi^{(A} \chi^{B)} = 0$$

then either $\phi^A = 0$ or $\chi^B = 0$

We obtain

$$2\nabla_{[a} \nabla_{b]} \xi^{C} = (\epsilon_{A'B'} X_{ABD}{}^{C} + \epsilon_{AB} \Phi_{A'B'D}{}^{C}) \xi^{D} \qquad (3.758)$$

or

$$\epsilon_{A'B'} \Box_{AB} \xi^{C} + \epsilon_{AB} \Box_{A'B'} \xi^{C} = (\epsilon_{A'B'} X_{ABE}{}^{C} + \epsilon_{AB} \Phi_{A'B'E}{}^{C}) \xi^{E} \qquad (3.759)$$

contracting with $\epsilon^{A'B'}$ gives

$$\Box_{AB}\xi_C = X_{ABCD}\xi^D \tag{3.760}$$

contracting with ϵ^{AB} gives

$$\Box_{A'B'}\xi_C = \Phi_{A'B'CD}\xi^D \tag{3.761}$$

If we substitute $X_{ABCD} = \Psi_{ABCD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})$ in (3.760) we obtain

$$\begin{aligned} \Box_{AB}\xi_C &= \Psi_{ABCD}\xi^D + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\xi^D \\ &= \Psi_{ABCD}\xi^D - \Upsilon(\xi_B\epsilon_{AC} + \xi_A\epsilon_{BC}) \end{aligned} \tag{3.762}$$

Symmetrising over ABC gives

$$\Box_{(AB}\xi_{C)} = \Psi_{ABCD}\xi^D \tag{3.763}$$

Contracting (3.760) with ϵ^{BC} and using $\epsilon^{BC}X_{ABCD} = 3\Upsilon\epsilon_{AD}$ we obtain

$$\Box_{AB}\xi^B = -3\Upsilon\xi_A \tag{3.764}$$

We collect these formula together:

$$\begin{aligned} &\Box_{AB}\xi_C = \Psi_{ABCD}\xi^D - 2\Upsilon\xi_{(A}\epsilon_{B)C} \\ &\Box_{(AB}\xi_{C)} = \Psi_{ABCD}\xi^D \\ &\Box_{AB}\xi^B = -3\Upsilon\xi_A \\ &\Box_{A'B'}\xi_C = \xi^D\Phi_{CDA'B'}. \end{aligned} \tag{3.765}$$

These are the spinor forms for the Ricci identities.

3.17.7 Einstein's equations

The vacuum field equations are

$$R_{ab} = 0. \tag{3.766}$$

or in terms of spinors

$$6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'} = 0. \tag{3.767}$$

Symmetrising (3.767) over AB implies

$$\Phi_{ab} = \Phi_{ABA'B'} = 0 \quad \text{then} \quad \Upsilon = 0. \tag{3.768}$$

Obviously (3.772) implies (3.767), thus they are equivalent.

If a cosmological constant is included in the field equations are

$$R_{ab} - \frac{1}{2}g_{ab}R - \Lambda g_{ab} = 0. \tag{3.769}$$

Contracting gives $R = -4\Lambda$ which upon substitution back into the field equations gives

$$R_{ab} = -\Lambda g_{ab} \tag{3.770}$$

which in spinor form becomes

$$6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'} = -\Lambda\epsilon_{AB}\epsilon_{A'B'} \tag{3.771}$$

and is equivalent to

$$\Phi_{ab} = \Phi_{ABA'B'} = 0, \qquad \Upsilon = -\frac{1}{6}\Lambda. \tag{3.772}$$

In the general case, where sources are present, the fielf equations with cosmological term are

$$G_{ab} - \Lambda g_{ab} = 8\pi G T_{ab} \tag{3.773}$$

using $R = 24\Upsilon$, this can be written

$$\Phi_{ab} + (3\Upsilon + \frac{1}{2}\Lambda)g_{ab} = -4\pi G T_{ab} \tag{3.774}$$

which we rewrite as

$$\Phi_{ab} + (3\Upsilon + \frac{1}{2}\Lambda)g_{ab} = -4\pi G[(T_{ab} - \frac{1}{4}T^c_c g_{ab}) + \frac{1}{4}T^c_c g_{ab}] \tag{3.775}$$

As Φ_{ab} represents the trace-free part of the RHS we have

$$\Phi_{ab} = -4\pi G(T_{ab} - \frac{1}{4}T^c_c g_{ab}) \tag{3.776}$$

and

$$\Upsilon = -\frac{1}{3}\pi G T^c_c - \frac{1}{6}\Lambda \tag{3.777}$$

Recall that

$$X_{AB}{}^B{}_C = 3\Upsilon\epsilon_{AC}$$

thus the vanishing of Υ which occurs for vacuum field equations implies that X_{ABCD} is symmetric in BC and since it is symetric in AB and CD, it is symmetric in all its indices. The curvature tensor $R_{ABCDA'B'C'D'}$ is given by the Weyl tensor (in accordance with) $C_{ABCDA'B'C'D'}$. In a vacuum the curvature can be fully characterised by a totally symmetric four-index spinor.

3.17.8 Spinor form of the Bianchi identity

Recall the Bianchi identity

$$\nabla_{[a}R_{bc]de} = 0. \tag{3.778}$$

Consider

$$\begin{aligned} \nabla^a \, {}^*R_{abcd} &= \frac{1}{2}\epsilon_{ab}{}^{ef}\nabla^a R_{efcd} \\ &= \frac{1}{2}\epsilon^a{}_b{}^{ef}\nabla_{[a}R_{ef]cd} \end{aligned} \tag{3.779}$$

This proves $\nabla_{[a}R_{bc]de} = 0$ implies $\nabla^a \, {}^*R_{abcd} = 0$. Now consider

$$
\begin{aligned}
\nabla_{[a} R_{bc]de} &= \frac{1}{3!}\delta^{efg}_{abc}\nabla_f R_{ghde} \\
&= \frac{1}{3!}\epsilon_{iabc}\epsilon^{ifgh}\nabla_f R_{ghde} \\
&= -\frac{2}{3!}\epsilon^{i}{}_{abc}\nabla^f\left(\frac{1}{2}\epsilon_{fi}{}^{gh}R_{ghde}\right) \\
&= -\frac{2}{3!}\epsilon^{i}{}_{abc}\nabla^f\ {}^*R_{fide} \qquad (3.780)
\end{aligned}
$$

This proves $\nabla^a\ {}^*R_{abcd} = 0$ implies $\nabla_{[a}R_{bc]de} = 0$, therefore the Bianchi identity is equivalent to

$$
\nabla^a\ {}^*R_{abcd} = 0 \qquad (3.781)
$$

From (4.3.4) this in spinor form is

$$
\begin{aligned}
\nabla^a\ {}^*R_{abcd} &= -i\nabla^{AA'}[X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\
&\quad - \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} - \overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}] \\
&= 0. \qquad (3.782)
\end{aligned}
$$

Or

$$
\epsilon_{C'D'}\nabla^{A}{}_{B'}X_{ABCD} + \epsilon_{CD}\nabla^{A}{}_{B'}\Phi_{ABC'D} - \epsilon_{C'D'}\nabla_{B}{}^{A'}\Phi_{CDA'B'} - \epsilon_{CD}\nabla_{B}{}^{A'}\overline{X}_{A'B'C'D'} = 0 \qquad (3.783)
$$

Contracting with $\epsilon^{C'D'}$ gives

$$
\nabla^{A}{}_{B'}X_{ABCD} = \nabla_{B}{}^{A'}\Phi_{CDA'B'}. \qquad (3.784)
$$

Contracting with ϵ^{CD} gives its complex conjugate. Thus (3.784) is the spinor form of the Bianchi identity.

3.17.9 Newman-Penrose Formalism in Spinor Form

Newman-Penrose scalars in terms of spinors

Now

$$
\begin{aligned}
m^a\nabla l_a &= o^A\bar{\imath}^{A'}\nabla(o_A\bar{o}_{A'}) \\
&= o^A\bar{\imath}^{A'}(o_A\nabla\bar{o}_{A'} + \bar{o}_{A'}\nabla o_A) \\
&= o^A\nabla o_A. \qquad (3.785)
\end{aligned}
$$

Note $\nabla(o^A\imath_A) = 0$ implies

$$
o^A\nabla\imath_A = \imath^A\nabla o_A \qquad (3.786)
$$

Now consider

$$
\begin{aligned}
\frac{1}{2}(n^a\nabla l_a - \overline{m}^a\nabla m_a) &= \frac{1}{2}(\imath^A\bar{\imath}^{A'}\nabla(o_A\bar{o}_{A'}) - \imath^A\bar{o}^{A'}\nabla(o_A\bar{\imath}_{A'})) \\
&= \frac{1}{2}[\imath^A\bar{\imath}^{A'}(o_A\nabla\bar{o}_{A'} + \bar{o}_{A'}\nabla o_A) - \imath^A\bar{o}^{A'}(o_A\nabla\bar{\imath}_{A'} + \bar{\imath}_{A'}\nabla o_A)] \\
&= \frac{1}{2}(\bar{\imath}^{A'}\nabla\bar{o}_{A'} + \imath^A\nabla o_A - \bar{o}^{A'}\nabla\bar{\imath}_{A'} + \imath^A\nabla o_A) \\
&= \imath^A\nabla o_A = o^A\nabla\imath_A. \qquad (3.787)
\end{aligned}
$$

$$
\begin{aligned}
-\overline{m}^a\nabla n_a &= -\imath^A\bar{o}^{A'}\nabla(\imath_A\bar{\imath}_{A'}) \\
&= -\imath^A\bar{o}^{A'}(\imath_A\nabla\bar{\imath}_{A'} + \bar{\imath}_{A'}\nabla\imath_A) \\
&= \imath^A\nabla\imath_A. \qquad (3.788)
\end{aligned}
$$

Altogether we have

$$
\begin{aligned}
m^a\nabla l_a &= o^A\nabla o_A \\
\frac{1}{2}(n^a\nabla l_a - \overline{m}^a\nabla m_a) &= o^A\nabla\imath_A = \imath^A\nabla o_A \\
-\overline{m}^a\nabla n_a &= \imath^A\nabla\imath_A \qquad (3.789)
\end{aligned}
$$

Recall $D = l^a\nabla_a$, $\Delta = n^a\nabla_a$, $\delta = m^a\nabla_a$, and $\overline{\delta} = \overline{m}^a\nabla_a$. The Newman-Penrose scalars can then be written:

$$
\begin{array}{lll}
\kappa = o^A D o_A & \epsilon = o^A D\imath_A & \pi = \imath^A D\imath_A \\
\sigma = o^A\delta o_A & \beta = o^A\delta\imath_A & \mu = \imath^A\delta\imath_A \\
\rho = o^A\overline{\delta} o_A & \alpha = o^A\overline{\delta}\imath_A & \lambda = \imath^A\overline{\delta}\imath_A \\
\tau = o^A\Delta o_A & \gamma = o^A\Delta\imath_A & \nu = \imath^A\Delta\imath_A
\end{array} \qquad (3.790)
$$

Consider Do_A, we can write

$$Do_A = ao_A + b\,\imath_A \tag{3.791}$$

we can determine a and b by contracting with $\imath^A$ and o^A respectively,

$$a = \imath^A Do_A, \qquad b = -o^A Do_A, \tag{3.792}$$

so that, using (3.790),

$$Do_A = \epsilon \nabla o_A - \kappa \imath_A. \tag{3.793}$$

We can derive the following in a similar manner:

$$\begin{aligned}
Do_A &= \epsilon o_A - \kappa \imath_A, \quad D\imath_A = \pi o_A - \epsilon \imath_A, \\
\Delta o_A &= \gamma o_A - \tau \imath_A, \quad \Delta \imath_A = \nu o_A - \gamma \imath_A, \\
\delta o_A &= \beta o_A - \sigma \imath_A, \quad \delta \imath_A = \mu o_A - \beta \imath_A, \\
\overline{\delta} o_A &= \alpha o_A - \rho \imath_A, \quad \overline{\delta} \imath_A = \lambda o_A - \alpha \imath_A.
\end{aligned} \tag{3.794}$$

Weyl tensor written in terms of spinors

We move to the Newman-Penrose components of the Weyl tensor. Consider $\Psi_0 = C_{abcd} l^a m^b l^c m^d$

$$\begin{aligned}
C_{abcd} l^a m^b l^c m^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD}) o^A \overline{o}^{A'} o^B \overline{\imath}^{B'} o^C \overline{o}^{C'} o^D \overline{\imath}^{D'} \\
&= \Psi_{ABCD} o^A o^B o^C o^D (\epsilon_{A'B'} \overline{o}^{A'} \overline{\imath}^{B'})(\epsilon_{C'D'} \overline{o}^{C'} \overline{\imath}^{D'}) + \ldots \epsilon_{AB} o^A o^B \ldots \\
&= \Psi_{ABCD} o^A o^B o^C o^D.
\end{aligned} \tag{3.795}$$

Consider $\Psi_1 = C_{abcd} l^a m^b l^c n^d$

$$\begin{aligned}
C_{abcd} l^a m^b l^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD}) o^A \overline{o}^{A'} o^B \overline{\imath}^{B'} o^C \overline{o}^{C'} \imath^D \overline{\imath}^{D'} \\
&= \Psi_{ABCD} o^A o^B o^C \imath^D (\epsilon_{A'B'} \overline{o}^{A'} \overline{\imath}^{B'})(\epsilon_{C'D'} \overline{o}^{C'} \overline{\imath}^{D'}) \\
&= \Psi_{ABCD} o^A o^B o^C \imath^D.
\end{aligned} \tag{3.796}$$

Consider $\Psi_2 = C_{abcd} l^a m^b \overline{m}^c n^d$

$$
\begin{aligned}
C_{abcd}l^a m^b \overline{m}^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})o^A\overline{o}^{A'}o^B\overline{\imath}^{B'}\imath^C\overline{o}^{C'}\imath^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}o^A o^B \imath^C \imath^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) \\
&= \Psi_{ABCD}o^A o^B \imath^C \imath^D. \qquad (3.797)
\end{aligned}
$$

Consider $\Psi_3 = C_{abcd}l^a n^b \overline{m}^c n^d$

$$
\begin{aligned}
C_{abcd}l^a n^b \overline{m}^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})o^A\overline{o}^{A'}\imath^B\overline{\imath}^{B'}\imath^C\overline{o}^{C'}\imath^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}o^A \imath^B \imath^C \imath^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) + \ldots \epsilon_{CD}\imath^C\imath^D \\
&= \Psi_{ABCD}o^A \imath^B \imath^C \imath^D. \qquad (3.798)
\end{aligned}
$$

Lastly, consider $\Psi_4 = C_{abcd}\overline{m}^a n^b \overline{m}^c n^d$

$$
\begin{aligned}
C_{abcd}\overline{m}^a n^b \overline{m}^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})\imath^A\overline{o}^{A'}\imath^B\overline{\imath}^{B'}\imath^C\overline{o}^{C'}\imath^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}\imath^A \imath^B \imath^C \imath^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) \\
&= \Psi_{ABCD}\imath^A \imath^B \imath^C \imath^D. \qquad (3.799)
\end{aligned}
$$

The Newman-Penrose components of the Weyl tensor written in terms of spinors are given by:

$$
\begin{aligned}
\Psi_0 &= \Psi_{ABCD}o^A o^B o^C o^D, \\
\Psi_1 &= \Psi_{ABCD}o^A o^B o^C \imath^D, \\
\Psi_2 &= \Psi_{ABCD}o^A o^B \imath^C \imath^D, \\
\Psi_3 &= \Psi_{ABCD}o^A \imath^B \imath^C \imath^D, \\
\Psi_4 &= \Psi_{ABCD}\imath^A \imath^B \imath^C \imath^D. \qquad (3.800)
\end{aligned}
$$

Ricci tensor written in terms of spinors

We move to the tetrad components of the Ricci tensor written in terms of spinors. Consider $\Phi_{00} = -\frac{1}{2}R_{ab}l^a l^b$

$$
\begin{aligned}
-\frac{1}{2}R_{ab}l^a l^b &= -\frac{1}{2}(6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'})o^A\overline{o}^{A'}o^B\overline{o}^{B'} \\
&= \Phi_{ABA'B'}o^A o^B \overline{o}^{A'}\overline{o}^{B'}. \qquad (3.801)
\end{aligned}
$$

Consider $\Phi_{01} = -\frac{1}{2}R_{ab}l^a m^b$

$$
\begin{aligned}
-\frac{1}{2}R_{ab}l^a m^b &= -\frac{1}{2}(6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'})o^A\overline{o}^{A'}o^B\overline{\imath}^{B'} \\
&= \Phi_{ABA'B'}o^A o^B\overline{o}^{A'}\overline{\imath}^{B'}. \qquad (3.802)
\end{aligned}
$$

Consider $\Phi_{02} = -\frac{1}{2}R_{ab}m^a m^b$

$$
\begin{aligned}
-\frac{1}{2}R_{ab}m^a m^b &= -\frac{1}{2}(6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'})o^A\overline{\imath}^{A'}o^B\overline{\imath}^{B'} \\
&= \Phi_{ABA'B'}o^A o^B\overline{\imath}^{A'}\overline{\imath}^{B'}. \qquad (3.803)
\end{aligned}
$$

Consider $\Phi_{10} = -\frac{1}{2}R_{ab}l^a\overline{m}^b = \overline{\Phi_{01}}$, hence

$$
\Phi_{10} = \Phi_{ABA'B'}o^A\imath^B\overline{o}^{A'}\overline{o}^{B'}. \qquad (3.804)
$$

Consider $\Phi_{20} = -\frac{1}{2}R_{ab}\overline{m}^a\overline{m}^b = \overline{\Phi_{02}}$, hence

$$
\Phi_{20} = \Phi_{ABA'B'}i^A\imath^B\overline{o}^{A'}\overline{o}^{B'}. \qquad (3.805)
$$

Consider $\Phi_{11} = -\frac{1}{4}R_{ab}(l^a n^b + m^a\overline{m}^b)$

$$
\begin{aligned}
-\frac{1}{4}R_{ab}(l^a n^b + m^a\overline{m}^b) &= -\frac{1}{4}(6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'})(o^A\overline{o}^{A'}\imath^B\overline{\imath}^{B'} + o^A\overline{\imath}^{A'}\imath^B\overline{o}^{B'}) \\
&= \frac{1}{2}\Phi_{ABA'B'}(o^A\imath^B\overline{o}^{A'}\overline{\imath}^{B'} + o^A\imath^B\overline{\imath}^{A'}\overline{o}^{B'}) \\
&\quad - \frac{3\Upsilon}{2}\epsilon_{AB}\epsilon_{A'B'}(o^A\imath^B\overline{o}^{A'}\overline{\imath}^{B'} + o^A\imath^B\overline{\imath}^{A'}\overline{o}^{B'}) \\
&= \Phi_{ABA'B'}o^A\imath^B\overline{o}^{A'}\overline{\imath}^{B'}. \qquad (3.806)
\end{aligned}
$$

where we used the symmetry in $A'B'$ of $\Phi_{ABA'B'}$. Consider $\Phi_{12} = -\frac{1}{2}R_{ab}n^a m^b$

$$
\begin{aligned}
-\frac{1}{2}R_{ab}n^a m^b &= -\frac{1}{2}(6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'})\imath^A\overline{\imath}^{A'}o^B\overline{\imath}^{B'} \\
&= \Phi_{ABA'B'}\imath^A\overline{\imath}^{A'}o^B\overline{\imath}^{B'} \\
&= \Phi_{ABA'B'}o^A\imath^B\overline{\imath}^{A'}\overline{\imath}^{B'}. \qquad (3.807)
\end{aligned}
$$

where we used the symmetry in AB of $\Phi_{ABA'B'}$. Consider $\Phi_{21} = -\frac{1}{2}R_{ab}n^a\overline{m}^b = \overline{\Phi_{12}}$, hence

$$\Phi_{21} = \Phi_{ABA'B'}\imath^A\imath^B\overline{o}^{A'}\overline{\imath}^{B'}. \tag{3.808}$$

Consider $\Phi_{22} = -\frac{1}{2}R_{ab}n^an^b$

$$\begin{aligned}
-\frac{1}{2}R_{ab}n^an^b &= -\frac{1}{2}(6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'})\imath^A\overline{\imath}^{A'}\imath^B\overline{\imath}^{B'} \\
&= \Phi_{ABA'B'}\imath^A\overline{\imath}^{A'}\imath^B\overline{\imath}^{B'} \\
&= \Phi_{ABA'B'}\imath^A\imath^B\overline{\imath}^{A'}\overline{\imath}^{B'}.
\end{aligned} \tag{3.809}$$

Collecting these results together the tetrad components of the Ricci tensor written in terms of spinors are given by:

$$\begin{aligned}
\Phi_{00} &= \Phi_{ABA'B'}o^Ao^B\overline{o}^{A'}\overline{o}^{B'}, \\
\Phi_{01} &= \Phi_{ABA'B'}o^Ao^B\overline{o}^{A'}\overline{\imath}^{B'}, \\
\Phi_{02} &= \Phi_{ABA'B'}o^Ao^B\overline{\imath}^{A'}\overline{\imath}^{B'}, \\
\Phi_{10} &= \Phi_{ABA'B'}o^A\imath^B\overline{o}^{A'}\overline{o}^{B'}, \\
\Phi_{11} &= \Phi_{ABA'B'}o^A\imath^B\overline{o}^{A'}\overline{\imath}^{B'}, \\
\Phi_{12} &= \Phi_{ABA'B'}o^A\imath^B\overline{\imath}^{A'}\overline{\imath}^{B'}, \\
\Phi_{20} &= \Phi_{ABA'B'}\imath^A\imath^B\overline{o}^{A'}\overline{o}^{B'}, \\
\Phi_{21} &= \Phi_{ABA'B'}\imath^A\imath^B\overline{o}^{A'}\overline{\imath}^{B'}, \\
\Phi_{22} &= \Phi_{ABA'B'}\imath^A\imath^B\overline{\imath}^{A'}\overline{\imath}^{B'}.
\end{aligned} \tag{3.810}$$

Lorentz transformations

Class I transformation:

They correspond to

$$(\hat{o}, \hat{\imath}) = (o, \imath + ao) \tag{3.811}$$

we have

$$\begin{aligned}
\hat{l} &= \hat{o}^A\overline{\hat{o}}^{A'} = o^A\overline{o}^{A'} \\
\hat{m} &= \hat{o}^A\overline{\hat{\imath}}^{A'} = o^A\overline{\imath}^{A'} + \overline{a}o^A\overline{o}^{A'} \\
\hat{n} &= \hat{\imath}^A\overline{\hat{\imath}}^{A'} = \imath^A\overline{\imath}^{A'} + ao^A\overline{\imath}^{A'} + \overline{a}\imath^A\overline{o}^{A'} + a\overline{a}o^A\overline{o}^{A'}
\end{aligned} \tag{3.812}$$

or

$$\hat{l} = l, \quad \hat{m} = m + \overline{a}l, \quad \hat{n} = n + am + \overline{am} + a\overline{a}l. \tag{3.813}$$

Class II transformation:

They correspond to

$$(\hat{o}, \hat{\imath}) = (o + b\imath, \imath) \tag{3.814}$$

we have

$$\begin{aligned} \hat{n} &= \hat{\imath}^A \overline{\hat{\imath}}^{A'} = \imath^A \overline{\imath}^{A'} \\ \hat{m} &= \hat{o}^A \overline{\hat{\imath}}^{A'} = o^A \overline{\imath}^{A'} + b\imath^A \overline{\imath}^{A'} \\ \hat{l} &= \hat{o}^A \overline{\hat{o}}^{A'} = o^A \overline{o}^{A'} + \overline{b} o^A \overline{\imath}^{A'} + b\imath^A \overline{o}^{A'} + b\overline{b}\imath^A \overline{\imath}^{A'} \end{aligned} \tag{3.815}$$

or

$$\hat{n} = n, \quad \hat{m} = m + bn, \quad \hat{l} = l + \overline{b}m + b\overline{m} + b\overline{b}l. \tag{3.816}$$

Class III transformation:

They correspond to

$$(\hat{o}, \hat{\imath}) = (\lambda o, \lambda^{-1}\imath), \qquad \lambda = c\exp(i\theta), \tag{3.817}$$

we have

$$\hat{l} = c^2 l, \quad \hat{n} = c^{-2} n, \quad , \quad \hat{m} = e^{2i\theta} m. \tag{3.818}$$

Transformation of Weyl scalars

The Weyl scalars transform under class I transformations as

$$
\begin{aligned}
\hat{\Psi}_0 &= \Psi_0, \\
\hat{\Psi}_1 &= \Psi_1 + a\Psi_0, \\
\hat{\Psi}_2 &= \Psi_2 + 2a\Psi_1 + a^2\Psi_0, \\
\hat{\Psi}_3 &= \Psi_3 + 3a\Psi_2 + 3a^2\Psi_1 + a^3\Psi_0, \\
\hat{\Psi}_4 &= \Psi_4 + 4a\Psi_3 + 6a^2\Psi_2 + 4a^3\Psi_1 + a^4\Psi_0.
\end{aligned}
\tag{3.819}
$$

The Weyl scalars transform under class II transformations as

$$
\begin{aligned}
\hat{\Psi}_0 &= \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4, \\
\hat{\Psi}_1 &= \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4, \\
\hat{\Psi}_2 &= \Psi_2 + 2b\Psi_3 + b^2\Psi_4, \\
\hat{\Psi}_3 &= \Psi_3 + b\Psi_4, \\
\hat{\Psi}_4 &= \Psi_4.
\end{aligned}
\tag{3.820}
$$

The Weyl scalars transform under class III transformations as

$$
\begin{aligned}
\tilde{\Psi}_0 &= c^4 e^{4i\theta}\Psi_0, \\
\tilde{\Psi}_1 &= c^2 e^{2i\theta}\Psi_1, \\
\tilde{\Psi}_2 &= \Psi_2, \\
\tilde{\Psi}_3 &= c^{-2} e^{-2i\theta}\Psi_3, \\
\tilde{\Psi}_4 &= c^{-4} e^{-4i\theta}\Psi_4.
\end{aligned}
\tag{3.821}
$$

3.17.10 Petrov Classification

Recall

$$
\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}. \tag{3.822}
$$

There are six distict cases which constitute the so-called Petrov classification

Type I or $\{1,1,1,1\}$. None of the four pricipal null directions coincide.

Type II or $\{2,1,1\}$. Two directions coincide.

Type D or $\{2,2\}$. Two are two different pairs of repeated null directions.

Type III or $\{3, 1\}$. Three pricipal null directions coincide.

Type N or $\{4\}$. All four pricipal null directions coincide.

Type O The Weyl tensor vanishes and spacetime is conformally flat.

Petrov classification via scalars

The condition for $e_{\hat{0}}{}^{a}$ to be a principal null vector is

Type I is when $\Psi_0 = \Psi_4 = 0$ and $\Psi_1, \Psi_2, \Psi_3 \neq 0$

Condition for double:

Type II is when $\Psi_0 = \Psi_1 = \Psi_4 = 0$ and $\Psi_2, \Psi_3 \neq 0$

Condition for a pair of doubles:

Type D is when $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and $\Psi_2 \neq 0$

Condition for triple:

Type III is when $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$ and $\Psi_3 \neq 0$

Condition for quadruple:

Type N is when $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ and $\Psi_4 \neq 0$

Type O is when $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$.

Proof:

Assume spacetime is not conformally flat and not all Weyl scalars vanish. Let $\Psi_4 \neq 0$. If it happens to be zero in the chosen frame, we can make it non-zero by a rotation of class I. By a class II transformation, $\hat{\Psi}_0$ can be made to vanish if b is a root of the equation

$$b^4\Psi_4 + 4b^3\Psi_3 + 6b^2\Psi_2 + 4b\Psi_1 + \Psi_0 = 0 \tag{3.823}$$

This always has four roots and the correspondingnew directions of l,

$$l + \bar{b}m + b\overline{m} + b\bar{b}n, \tag{3.824}$$

are the principal null directions of the Weyl tensor.

We can easily derive from (3.820) that

$$
\begin{aligned}
\frac{1}{4}\frac{d}{db}\hat{\Psi}_0(b) &= b^3\Psi_4 + 3b^2\Psi_3 + 3b\Psi_2 + \Psi_1 = \hat{\Psi}_1(b) \\
\frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b) &= b^2\Psi_4 + 2b\Psi_3 + \Psi_2 = \hat{\Psi}_2(b) \\
\frac{1}{2}\frac{d}{db}\hat{\Psi}_2(b) &= b\Psi_4 + \Psi_3 = \hat{\Psi}_3(b) \\
\frac{d}{db}\hat{\Psi}_3(b) &= \Psi_4 = \hat{\Psi}_4(b),
\end{aligned} \tag{3.825}
$$

and that

$$
\hat{\Psi}_0(b) = \Psi_4(b-b_1)(b-b_2)(b-b_3)(b-b_4). \tag{3.826}
$$

We have

$$
\begin{aligned}
\hat{\Psi}_1(b) &= \frac{1}{4}\frac{d}{db}\hat{\Psi}_0(b) \\
&= \frac{\Psi_4}{4}\frac{d}{db}(b-b_1)(b-b_2)(b-b_3)(b-b_4) \\
&= \frac{\Psi_4}{4}[(b-b_2)(b-b_3)(b-b_4) + (b-b_1)(b-b_3)(b-b_4) \\
&\quad +(b-b_1)(b-b_2)(b-b_4) + (b-b_2)(b-b_3)(b-b_4)]
\end{aligned} \tag{3.827}
$$

and

$$
\begin{aligned}
\hat{\Psi}_2(b) &= \frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b) \\
&= \frac{\Psi_4}{6}[(b-b_1)(b-b_2) + (b-b_1)(b-b_3) + (b-b_1)(b-b_4) \\
&\quad +(b-b_2)(b-b_3) + (b-b_2)(b-b_4) + (b-b_3)(b-b_4)
\end{aligned} \tag{3.828}
$$

and

$$
\begin{aligned}
\hat{\Psi}_3(b) &= \frac{1}{2}\frac{d}{db}\hat{\Psi}_2(b) \\
&= \frac{\Psi_4}{4}(4b - b_1 - b_2 - b_3 - b_4)
\end{aligned} \tag{3.829}
$$

and finally

$$\hat{\Psi}_4(b) = \frac{d}{db}\hat{\Psi}_3(b) = \Psi_4. \tag{3.830}$$

a) **Petrov type I**. All four roots are distinct. Then a rotation of class II with parameter $b = b_1$ (say) we can make

$$\hat{\Psi}_0 = b_1^4\Psi_4 + 4b_1^3\Psi_3 + 6b_1^2\Psi_2 + 4b_1\Psi_1 + \Psi_0 = 0.$$

Taking $\hat{\Psi}_0$ as a fucntion of b, we have

For (3.827) the parameter $b = b_1$

$$\hat{\Psi}_1 = \hat{\Psi}_1(b = b_1) = \frac{\Psi_4}{2}(b_1 - b_2)(b_1 - b_3)(b_1 - b_4) \neq 0. \tag{3.831}$$

For (3.828)

$$\hat{\Psi}_2(b) = \frac{\Psi_4}{6}[(b - b_1)(3b - b_2 - b_3 - b_4) + (b - b_2)(2b - b_3 - b_4) + (b - b_3)(b - b_4)] \tag{3.832}$$

$$\hat{\Psi}_2 = \hat{\Psi}_2(b = b_1) = \frac{\Psi_4}{6}[(b_1 - b_2)(b_1 - b_3) + (b_1 - b_2)(b_1 - b_4) + (b_1 - b_3)(b_1 - b_4)]. \tag{3.833}$$

As we have already seen

$$\hat{\Psi}_3(b) = \frac{\Psi_4}{4}(4b - b_1 - b_2 - b_3 - b_4) \tag{3.834}$$

so that

$$\hat{\Psi}_3 = \hat{\Psi}_3(b = b_1) = \frac{\Psi_4}{4}[(b_1 - b_2) + (b_1 - b_3) + (b_1 - b_4)]. \tag{3.835}$$

We have that $\hat{\Psi}_1$ is guaranteed to be non-zero. If we have $Re(b_1) > Re(b_2), Re(b_3), Re(b_4)$ or $Im(b_1) > Im(b_2), Im(b_3), Im(b_4)$ then $\hat{\Psi}_2, \hat{\Psi}_3 > 0$.

A rotation of class I (which does not effect Ψ_0) we can make Ψ_4 vanish with appropriate value of parameter a

$$\hat{\hat{\Psi}}_4 = \Psi_4 + 4a\hat{\Psi}_3 + 6a^2\hat{\Psi}_2 + 4a^3\hat{\Psi}_1 = 0. \tag{3.836}$$

We also have that $\hat{\Psi}_1$ is invariant under this rotation and so

$$\hat{\hat{\Psi}}_1 = \hat{\Psi}_1 \neq 0. \tag{3.837}$$

Only if the above polynomial (3.836) has three distinct roots can $\hat{\hat{\Psi}}_2$ and $\hat{\hat{\Psi}}_3$ be non-zero.

b) **Petrov type II**. Two roots coincide $b_1 = b_2$ the other two different and distinct $b_1 \neq b_3 \neq b_4$ and $b_3 \neq b_4$. We have

$$\hat{\Psi}_0(b) = \Psi_4(b - b_1)^2(b - b_3)(b - b_4) \tag{3.838}$$

and under a transformation of class II we have (using (3.827) with $b_1 = b_2$)

$$\hat{\Psi}_1(b) = \Psi_4(b - b_1)[2(b - b_3)(b - b_4) + (b - b_1)(b - b_3) + (b - b_1)(b - b_4)] \tag{3.839}$$

Setting $b = b_1$ we make Ψ_0, and Ψ_1 vanish simultaneously.

$$\hat{\Psi}_2 = \frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b)\Big|_{b=b_1} = \frac{2}{3}\Psi_4(b_1 - b_3)(b_1 - b_4) \neq 0. \tag{3.840}$$

and from (3.829)

$$\hat{\Psi}_3 = \hat{\Psi}_3(b = b_1) = \frac{\Psi_4}{4}(2b_1 - b_3 - b_4) \tag{3.841}$$

Under a transformation of class I with parameter a we uneffect the vanishing of Ψ_0 and Ψ_1 while make Ψ_4 vanish if we chose a to be a root of

$$\hat{\hat{\Psi}}_4 = \Psi_4 + 4a\hat{\Psi}_3 + 6a^2\hat{\Psi}_2 = 0. \tag{3.842}$$

Under such a transformation we have

$$\hat{\hat{\Psi}}_2 = \hat{\Psi}_2 \neq 0 \tag{3.843}$$

and

$$\hat{\hat{\Psi}}_3 = \hat{\Psi}_3 + 3a\hat{\Psi}_2$$

We have

$$\frac{1}{4}\frac{d}{da}\hat{\hat{\Psi}}_4(a) = \hat{\Psi}_3 + 3a\hat{\Psi}_2 = \hat{\hat{\Psi}}_3$$

So $\hat{\hat{\Psi}}_3$ will vanish if (3.842) has repeated roots. The condition for the qudratic equation (3.842) to have repeated roots reduces to

$$2\hat{\Psi}_3^2 - 3\hat{\Psi}_2\Psi_4 = 0.$$

From (3.841) we have

$$\begin{aligned} 2\hat{\Psi}_3^2 &= 2\frac{\Psi_4^2}{16}[(b_1 - b_3) + (b_1 - b_4)]^2 \\ &= \frac{\Psi_4^2}{8}[(b_1 - b_3)^2 + 2(b_1 - b_3)(b_1 - b_4) + (b_1 - b_4)^2], \end{aligned}$$

and from (3.840) we have

$$3\hat{\Psi}_2\Psi_4 = 2\Psi_4^2(b_1 - b_3)(b_1 - b_4).$$

We write

$$2\hat{\Psi}_3^2 - 3\hat{\Psi}_2\Psi_4 = \frac{1}{8}[(b_1 - b_3)^2 - 14(b_1 - b_3)(b_1 - b_4) + (b_1 - b_4)^2]$$

Put $\rho = b_1 - b_4$ then look for roots for the quadratic equation in γ,

$$\gamma^2 - 14\gamma\rho + \rho^2 = 0.$$

Say α is a root, if it happens that $b_3 = b_1 - \alpha$ then $2\hat{\Psi}_3^2 - 3\hat{\Psi}_2\Psi_4 = 0$ and (3.842) has repeated roots.

c) **Petrov type D**. We have two distinct double roots b_1 and b_2. And so putting $b_3 = b_1$ and $b_4 = b_2$ we have

$$\hat{\Psi}_0(b) = \Psi_4(b - b_1)^2(b - b_2)^2 \tag{3.844}$$

$$\hat{\Psi}_1(b) = \frac{1}{2}\Psi_4(b-b_1)(b-b_2)(2b-b_1-b_2). \tag{3.845}$$

$$\hat{\Psi}_2(b) = \frac{1}{3}\Psi_4[(b-b_2)(b-b_2) + \frac{1}{2}(2b-b_1-b_2)^2]. \tag{3.846}$$

$$\hat{\Psi}_3(b) = \frac{1}{2}\Psi_4(2b-b_1-b_2). \tag{3.847}$$

$$\hat{\Psi}_4 = \Psi_4. \tag{3.848}$$

With the choice, $b = b_1$

$$\begin{aligned} &\hat{\Psi}_0 = \hat{\Psi}_1 = 0 \qquad \hat{\Psi}_2 = \frac{1}{6}\Psi_4(b_1-b_2)^2 \\ &\hat{\Psi}_3 = \frac{1}{2}\Psi_4(b_1-b_2), \quad \text{and} \quad \hat{\Psi}_4 = \Psi_4 \end{aligned} \tag{3.849}$$

We now subject the frame to a class I transformation with parameter a. First we also have

$$\hat{\hat{\Psi}}_0 = \hat{\hat{\Psi}}_1 = 0 \tag{3.850}$$

which follows from

$$\begin{aligned} \hat{\hat{\Psi}}_0 &= \hat{\Psi}_0 = 0 \\ \hat{\hat{\Psi}}_1 &= \hat{\Psi}_1 + a\hat{\Psi}_0 = 0. \end{aligned} \tag{3.851}$$

Then

$$\begin{aligned} \hat{\hat{\Psi}}_2 &= \hat{\Psi}_2 + 2a\hat{\Psi}_1 + \hat{\Psi}_0 \\ &= \hat{\Psi}_2 \\ &= \frac{1}{6}\Psi_4(b_1-b_2)^2 \neq 0 \end{aligned} \tag{3.852}$$

Consider

$$\begin{aligned}\hat{\hat{\Psi}}_3 &= \hat{\Psi}_3 + 3a\hat{\Psi}_2 \\ &= \frac{1}{2}\Psi_4(b_1 - b_2) + 3a\frac{1}{6}\Psi_4(b_1 - b_2)^2 \\ &= \frac{1}{2}\Psi_4(b_1 - b_2)[1 + a(b_1 - b_2)] \end{aligned} \tag{3.853}$$

and

$$\begin{aligned}\hat{\hat{\Psi}}_4 &= \Psi_4 + 4a\hat{\Psi}_3 + 6a^2\hat{\Psi}_2 \\ &= \Psi_4 + 4a\frac{1}{2}\Psi_4(b_1 - b_2) + 6a^2\frac{1}{6}\Psi_4(b_1 - b_2)^2 \\ &= \Psi_4[1 + a(b_1 - b_2)]^2. \end{aligned} \tag{3.854}$$

With the choice

$$a = -(b_1 - b_2)^{-1} \tag{3.855}$$

we have

$$\hat{\hat{\Psi}}_3 = \hat{\hat{\Psi}}_4 = 0. \tag{3.856}$$

Thus Ψ_0, Ψ_1, Ψ_3, and Ψ_4 have all been reduced to zero with Ψ_2 the only nonvanishing scalar.

d) **Petrov type III**. Three roots coincide $b_1 = b_2 = b_3 \neq b_4$. With a class II transformation with parameter b_1 then we can make Ψ_0, Ψ_1, and Ψ_2 vanish. We have

$$\begin{aligned}\hat{\Psi}_3(b) &= \frac{1}{4}\Psi_4(4b - b_4 - 3b_1) \\ \hat{\Psi}_4(b) &= \Psi_4 \end{aligned} \tag{3.857}$$

Putting $b = b_1$

$$\begin{aligned}\hat{\Psi}_3 &= \frac{1}{4}(b_1 - b_4) \neq 0. \\ \hat{\Psi}_4 &= \Psi_4 \end{aligned} \tag{3.858}$$

Then by a subsequent transformation of class I with parameter a we have

$$\begin{aligned}
\hat{\hat{\Psi}}_3 &= \hat{\Psi}_3 \neq 0 \\
\hat{\hat{\Psi}}_4 &= \hat{\Psi}_4 + 4a\hat{\Psi}_3 \\
&= \Psi_4 + \Psi_4 a(b_1 - b_4) \\
&= \Psi_4(1 + a(b_1 - b_4))
\end{aligned} \tag{3.859}$$

With the choice $a = -(b_1 - b_4)^{-1}$ we can make Ψ_4 vanish. And Ψ_3 is the only non-zero scalar.

e) **Petrov type N**. All four roots coincide with on distinct root b_1. Then a transformation of class II with parameter b_1 we can make Ψ_0, Ψ_1, Ψ_2, and Ψ_3 vanish simultaneously and Ψ_4 will be the only non-vanishing scalar, as is easily seen from:

$$\begin{aligned}
\hat{\Psi}_1(b) &= \frac{1}{4}\frac{d}{db}\hat{\Psi}_0(b) = \Psi_4 \frac{1}{4}\frac{d}{db}(b - b_1)^4 = \Psi_4(b - b_1)^3 \\
\hat{\Psi}_2(b) &= \frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b) = \Psi_4(b - b_1)^2 \\
\hat{\Psi}_3(b) &= \frac{1}{2}\frac{d}{db}\hat{\Psi}_2(b) = \Psi_4(b - b_1) \\
\hat{\Psi}_4(b) &= \frac{d}{db}\hat{\Psi}_3(b) = \Psi_4.
\end{aligned} \tag{3.860}$$

□

3.17.11 Equivalence of Petrov Classification Schemes

We are interested in the roots of $\hat{\Psi}_4(b) = 0$ which is quartic in b and so can be written,

$\hat{\Psi}_0(b) = \Psi_4(b - b_1)(b - b_2)(b - b_3)(b - b_4)$.

Case (a) First we consider the case where the four roots b_1, b_2, b_3, b_4 are distinct. Write

$$\begin{aligned}
\rho_1^A &:= o^A + b_1 i^A \\
\rho_2^A &:= o^A + b_2 i^A \\
\rho_3^A &:= a^A + b_3 i^A \\
\rho_4^A &:= a^A + b_4 i^A
\end{aligned} \tag{3.861}$$

Then $\hat{\Psi}_0 = 0$ implies the 4 equations for $\alpha, \beta, \gamma, \delta$:

$$\begin{aligned}
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C\rho_1^D &= 0 \\
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_2^A\rho_2^B\rho_2^C\rho_2^D &= 0 \\
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_3^A\rho_3^B\rho_3^C\rho_3^D &= 0 \\
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_4^A\rho_4^B\rho_4^C\rho_4^D &= 0
\end{aligned} \tag{3.862}$$

which reduce to the 4 equations:

$$(\alpha_A\rho_1^A)(\beta_B\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0 \tag{3.863}$$
$$(\alpha_A\rho_2^A)(\beta_B\rho_2^B)(\gamma_C\rho_2^C)(\delta_D\rho_2^D) = 0 \tag{3.864}$$
$$(\alpha_A\rho_3^A)(\beta_B\rho_3^B)(\gamma_C\rho_3^C)(\delta_D\rho_3^D) = 0 \tag{3.865}$$
$$(\alpha_A\rho_4^A)(\beta_B\rho_4^B)(\gamma_C\rho_4^C)(\delta_D\rho_4^D) = 0 \tag{3.866}$$

Now we use that for spinors, $\alpha_A\rho^A = 0$ if and only if α is proportional to ρ (we write $\alpha_A = \lambda\rho_A$).

We are considering the case where all the roots b_1, b_2, b_3, b_4 are all different and as such that the spinors $\rho_1, \rho_2, \rho_3, \rho_4$ are not proportional to each other. Then (3.863) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (3.864) cant then vanish because ρ_1 is not proportional to ρ_2, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_2\rho_{2A} = \lambda_2(o_A + b_2 i_A)$. The first two brackets of (3.865) cant vanish, so at least one of the other two vanish, say it is the 3rd bracket then $\gamma_A = \lambda_3\rho_{3A} = \lambda_3(o_A + b_3 i_A)$. The first 3 brackets of (3.866) can't vanish and so it must be the last bracket that vanishes, and so $\delta_A = \lambda_4\rho_{4A} = \lambda_4(o_A + b_4 i_A)$. And So $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ are all distinct and each representing a principal null direction.

Case (b) We consider the case where just two roots coincide, say $b_1 = b_2$. As $\rho_1 = \rho_2$ we have three independent equations from $\hat{\Psi}_0 = 0$:

$$(\alpha_A\rho_1^A)(\beta_B\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0 \tag{3.867}$$
$$(\alpha_A\rho_3^A)(\beta_B\rho_3^B)(\gamma_C\rho_3^C)(\delta_D\rho_3^D) = 0 \tag{3.868}$$
$$(\alpha_A\rho_4^A)(\beta_B\rho_4^B)(\gamma_C\rho_4^C)(\delta_D\rho_4^D) = 0 \tag{3.869}$$

Then (3.867) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (3.868) cant then vanish because ρ_1 is not proportional to ρ_3, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_3 \rho_{3A} = \lambda_3(o_A + b_3 i_A)$. The first two brackets of (3.869) cant vanish, so at least one of the other two vanish, say it is the 3rd bracket then $\gamma_A = \lambda_4 \rho_{4A} = \lambda_4(o_A + b_4 i_A)$.

It can easily be shown that with parameter $b = b_1 (= b_2)$ $\hat{\Psi}_0$ and $\hat{\Psi}_1$ will vanish. So we also have the equation

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A \rho_1^B \rho_1^C i^D = 0$$

or

$$\rho_{1(A}\rho_{3B}\rho_{4C}\delta_{D)}\ \rho_1^A \rho_1^B \rho_1^C i^D = 0$$

which reduce to

$$(\rho_{1A} i^A)(\rho_{3B}\rho_1^B)(\rho_{4C}\rho_1^C)(\delta_D \rho_1^D) = 0$$

implying $\delta_A = \rho_{1A} = \lambda_1(o_A + b_1 i_a)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with two directions coinciding.

Case (c) Two distinct double roots b_1 and b_2. As $\rho_1 = \rho_3$ and $\rho_2 = \rho_4$ we have two independent equations from $\hat{\Psi}_0 = 0$:

$$(\alpha_A \rho_1^A)(\beta_B \rho_1^B)(\gamma_C \rho_1^C)(\delta_D \rho_1^D) = 0 \tag{3.870}$$
$$(\alpha_A \rho_2^A)(\beta_B \rho_2^B)(\gamma_C \rho_2^C)(\delta_D \rho_2^D) = 0 \tag{3.871}$$

Then (3.870) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1 \rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (3.871) cant then vanish because ρ_1 is not proportional to ρ_4, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_2 \rho_{2A} = \lambda_2(o_A + b_2 i_A)$.

It is easily shown that with parameter $b = b_1$ we have $\hat{\Psi}_1 = 0$

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A \rho_1^B \rho_1^C i^D = 0$$

or

$$\rho_{1(A}\rho_{2B}\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0$$

which reduces to

$$(\rho_{1A}i^A)(\rho_{2B}\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0$$

So that at least one of the last two brackets vanish. Say the third bracket vanishes, then $\gamma_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$.

It is easily shown that with parameter $b = b_2$ we have $\Psi_1 = 0$

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_2^A\rho_2^B\rho_2^C i^D = 0$$

or

$$\rho_{1(A}\rho_{2B}\rho_{1C}\delta_{D)}\ \rho_2^A\rho_2^B\rho_2^C i^D = 0$$

which reduces to

$$(\rho_{1A}\rho_2^A)(\rho_{2B}i^B)(\rho_{1C}\rho_2^C)(\delta_D\rho_2^D) = 0$$

So that at least one of the last two brackets vanish. Say the third bracket vanishes, then $\delta_A = \lambda_2\rho_{2A} = \lambda_2(o_A + b_2 i_A)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with two different pairs repeated.

Case (d) Three roots coincide and $b = b_1 (= b_2 = b_3)$. As $\rho_1 = \rho_2 = \rho_3$ we have two independent equations from $\hat{\Psi}_0 = 0$:

$$(\alpha_A\rho_1^A)(\beta_B\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0 \tag{3.872}$$

$$(\alpha_A\rho_4^A)(\beta_B\rho_4^B)(\gamma_C\rho_4^C)(\delta_D\rho_4^D) = 0 \tag{3.873}$$

Then (3.872) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (3.873) cant then vanish because ρ_1 is not proportional to ρ_4, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_4\rho_{4A} = \lambda_4(o_A + b_4 i_A)$.

It is easily shown that with parameter $b = b_1 (= b_2 = b_3)$ $\hat{\Psi}_0$, $\hat{\Psi}_1$ and $\hat{\Psi}_2$ will vanish simultaneously. So we also have the equations

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0 \tag{3.874}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B i^C i^D = 0 \tag{3.875}$$

(3.874) is

$$\rho_{1(A}\rho_{4B}\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0$$

which reduces to

$$(\rho_{1A}i^A)(\rho_{4B}\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0$$

This implies that $\gamma_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$. (3.875) then reads

$$\rho_{1(A}\rho_{4B}\rho_{1C}\delta_{D)}\ \rho_1^A\rho_1^B i^C i^D = 0$$

which reduces to

$$(\rho_{1A}i^A)(\rho_{4B}\rho_1^B)(\rho_{1C}i^C)(\delta_D\rho_1^D) = 0$$

which means that $\delta_D = \rho_{1D} = \lambda_1(o_a + b_1 i_A)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with three directions coinciding.

Case (e) All roots coincide and we have for $b = b_1$ that $\hat{\Psi}_0 = \hat{\Psi}_1 = \hat{\Psi}_2 = \hat{\Psi}_3 = 0$. We have the equations:

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C\rho_1^D = 0 \tag{3.876}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0 \tag{3.877}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B i^C i^D = 0 \tag{3.878}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A i^B i^C i^D = 0 \tag{3.879}$$

(3.876) reduces to

$$(\alpha_A\rho_1^A)(\beta_B\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0$$

At least one of the brackets vanish, say the first. So that $\alpha_A = \lambda_1 \rho_{1A} = \lambda_1(o_A + b_1 i_A)$. (3.877) reduces to

$$(\rho_{1A} i^A)(\beta_B \rho_1^B)(\gamma_C \rho_1^C)(\delta_D \rho_1^D) = 0.$$

At least one the last three brackets vanish, say the second bracket vanishes. Then $\beta_A = \lambda_1 \rho_{1A} = \lambda_1(o_a + b_1 i_A)$. (3.878) reduces to

$$(\rho_{1A} i^A)(\rho_{1B} i^B)(\gamma_C \rho_1^C)(\delta_D \rho_1^D) = 0.$$

At least one the last two brackets vanish, say the third bracket vanishes. Then $\gamma_A = \lambda_1 \rho_{1A} = \lambda_1(o_a + b_1 i_A)$. (3.879) reduces to

$$(\rho_{1A} i^A)(\rho_{1B} i^B)(\rho_{1C} i^C)(\delta_D \rho_1^D) = 0.$$

The last bracket must vanish, therefore $\delta_A = \lambda_1 \rho_{1A} = \lambda_1(o_a + b_1 i_A)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with all four directions coinciding.

Proof of the converse:

We now prove the converse.

Case (a) Assume

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$$

where

$$\begin{aligned} \alpha &= \alpha_0 o_A + \alpha_1 i_A, \\ \beta &= \beta_0 o_A + \beta_1 i_A, \\ \gamma &= \gamma_0 o_A + \gamma_1 i_A, \\ \delta &= \delta_0 o_A + \delta_1 i_A \end{aligned} \tag{3.880}$$

are disinct (not proportional to each other). We then wish to show that all roots of

$$\Psi_0 \to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^2\Psi_3 + b^4\Psi_4 = 0$$

are distinct.

$$
\begin{aligned}
\Psi_0 &\to \Psi_{ABCD}(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)\\
&= \alpha_{(A}\beta_B\gamma_C\delta_{D)}(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)\\
&= \alpha_A\beta_B\gamma_C\delta_D(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)\\
&= (\alpha_A o^A + b\,\alpha_A i^A)(\beta_B o^B + b\,\beta_B i^B)(\gamma_C o^C + b\,\gamma_C i^C)(\delta_D o^D + b\delta_D i^D)\\
&= (-\alpha_1 + b\alpha_0)(-\beta_0 + b\beta_0)(-\gamma_0 + b\gamma_1)(-\delta_0 + b\delta_1)\\
&= 0. \qquad (3.881)
\end{aligned}
$$

We see that it has 4 distinct roots.

Case (b) Assume

$$
\Psi_{ABCD} = \alpha_{(A}\alpha_B\gamma_C\delta_{D)}
$$

where α, β, γ are distinct, then

$$
\begin{aligned}
\Psi_0 &\to \Psi_{ABCD}(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)\\
&= \alpha_A\alpha_B\gamma_C\delta_D(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)\\
&= (\alpha_A o^A + b\,\alpha_A i^A)(\alpha_B o^B + b\,\alpha_B i^B)(\gamma_C o^C + b\,\gamma_C i^C)(\delta_D o^D + b\delta_D i^D)\\
&= (-\alpha_1 + b\alpha_0)^2(-\gamma_0 + b\gamma_1)(-\delta_0 + b\delta_1)\\
&= 0. \qquad (3.882)
\end{aligned}
$$

We see that two roots coincide.

The other cases work through the same way.

3.17.12 Petrov classification via Eigenbivectors of the Weyl Tensor

Eigenbivectors of the Weyl Spinor

Given any spin-frame (o^A, i^A) we can construct a corresponding orthonormal basis $\overset{1}{\delta}_{AB}$, $\overset{2}{\delta}_{AB}$, and $\overset{3}{\delta}_{AB}$ ($\overset{\alpha}{\delta}_{AB}\overset{\beta}{\delta}{}^{AB} = \delta_{\alpha\beta}$) for $\mathcal{C}_{(AB)}$

$$
\overset{1}{\delta}_{AB} = -\frac{i}{\sqrt{2}}(o_A o_B - i_A i_B), \quad \overset{2}{\delta}_{AB} = \frac{1}{\sqrt{2}}(o_A o_B + i_A i_B), \quad \overset{3}{\delta}_{AB} = i\sqrt{2}\, o_{(A} i_{B)}. \qquad (3.883)
$$

We have

$$
\begin{aligned}
\overset{\alpha}{\delta}_{AB}\overset{\alpha}{\delta}{}^{CD} &= -\frac{i}{\sqrt{2}}(o_Ao_B - i_Ai_B)\cdot -\frac{i}{\sqrt{2}}(o^Co^D - i^Ci^D) \\
&\quad +\frac{1}{\sqrt{2}}(o_Ao_B + i_Ai_B)\cdot \frac{1}{\sqrt{2}}(o^Co^D + i^Ci^D) \\
&\quad +\frac{i}{\sqrt{2}}(o_Ai_B + o_Bi_A)\cdot \frac{i}{\sqrt{2}}(o^Ci^D + o^Di^C) \\
&= \frac{1}{2}\big(-o_Ao_Bo^Co^D + o_Ao_Bi^Ci^D + i_Ai_Bo^Co^D - i_Ai_Bi^Ci^D\big) \\
&\quad +\frac{1}{2}\big(o_Ao_Bo^Co^D + o_Ao_Bi^Ci^D + i_Ai_Bo^Co^D + i_Ai_Bi^Ci^D\big) \\
&\quad +\frac{1}{2}\big(-o_Ai_Bo^Ci^D - o_Ai_Bo^Di^C - o_Bi_Ao^Ci^D - o_Bi_Ao^Di^C\big) \\
&= \frac{1}{2}\big(o_Ao_Bi^Ci^D - o_Ai_Bi^Co^D - i_Ao_Bo^Ci^D + i_Ai_Bo^Co^D \\
&\quad +o_Bo_Ai^Ci^D - o_Bi_Ai^Co^D - i_Bo_Ao^Ci^D + i_Bi_Ao^Co^D\big) \\
&= \frac{1}{2}(o_Ai^C - i_Ao^C)(o_Bi^D - i_Bo^D) + \frac{1}{2}(o_Bi^C - i_Bo^C)(o_Ai^D - i_Ao^D) \\
&= \frac{1}{2}(\delta_A{}^C\delta_B{}^D + \delta_B{}^C\delta_A{}^D)
\end{aligned}
\tag{3.884}
$$

The components of ϕ_{AB} with respect to the basis (3.883) are

$$
\overset{1}{\phi} = \frac{-i}{\sqrt{2}}(\phi_{00} - \phi_{11}), \quad \overset{2}{\phi} = \frac{1}{\sqrt{2}}(\phi_{00} + \phi_{11}), \quad \overset{3}{\phi} = i\sqrt{2}\phi_{01}. \tag{3.885}
$$

The components of $\Psi_{AB}{}^{CD}$ with respect to the basis (3.883) are

$$
\mathbf{\Psi} = \begin{pmatrix} \frac{1}{2}(-\Psi_0 + 2\Psi_1 - \Psi_4) & \frac{-i}{2}(\Psi_0 - \Psi_4) & (\Psi_1 - \Psi_3) \\ \frac{-i}{2}(\Psi_0 - \Psi_4) & \frac{1}{2}(\Psi_0 + 2\Psi_1 + \Psi_4) & i(\Psi_1 + \Psi_3) \\ (\Psi_1 - \Psi_3) & i(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix} \tag{3.886}
$$

From (3.884)

$$
\begin{aligned}
\overset{\alpha\alpha}{\Psi} &= \Psi_{AB}{}^{CD}\,\overset{\alpha}{\delta}{}^{AB}\overset{\alpha}{\delta}_{CD} \\
&= \Psi_{AB}{}^{CD}\frac{1}{2}(\delta^A_C\delta^B_D + \delta^A_D\delta^B_C) \\
&= \Psi_{AB}{}^{AB} \\
&= \Psi_{ABCD}\epsilon^{CA}\epsilon^{DB} \\
&= 0.
\end{aligned}
\tag{3.887}
$$

From (3.884) we have that the eigen-equation

$$\overset{\alpha\beta}{\Psi}\,\phi_\beta = \lambda\phi_\alpha \tag{3.888}$$

can be written:

$$\begin{aligned}
\overset{\alpha\beta}{\Psi}\,\phi_\beta &= \Psi_{AB}{}^{CD}\,\overset{\alpha\,AB}{\delta}\,\overset{\beta}{\delta}_{CD}\,\phi_{EF}\,\overset{\beta\,EF}{\delta} \\
&= \Psi_{AB}{}^{CD}\,\overset{\alpha\,AB}{\delta}\,\frac{1}{2}(\delta^E_C\delta^F_D + \delta^E_D\delta^F_C)\phi_{EF} \\
&= \Psi_{AB}{}^{CD}\phi_{CD}\,\overset{\alpha\,AB}{\delta} \\
&= \lambda\phi_{AB}\,\overset{\alpha AB}{\delta}\;\;.
\end{aligned} \tag{3.889}$$

Therefore the expressing the eigen-equation

$$\Psi_{AB}{}^{CD}\phi_{CD} = \lambda\phi_{AB} \tag{3.890}$$

in components according to the basis (3.883), we see that λ is also an eigenvalue in the normal sense of the matrix $\boldsymbol{\Psi}$. If $\lambda_1, \lambda_2, \lambda_3$ are the three eigenvalues of $\boldsymbol{\Psi}$ we have

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= \Psi_{AB}{}^{AB} = 0 \\
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= \Psi_{AB}{}^{CD}\Psi_{CD}{}^{AB} =: I \\
\lambda_1^3 + \lambda_2^3 + \lambda_3^3 &= \Psi_{AB}{}^{CD}\Psi_{CD}{}^{EF}\Psi_{EF}{}^{AB} =: J.
\end{aligned} \tag{3.891}$$

From

$$\begin{aligned}
0 &= (\lambda_1 + \lambda_2 + \lambda_3)^3 - 3(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \\
&= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + 3(\lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_2^2\lambda_3 + \lambda_2^2\lambda_1 + \lambda_3^2\lambda_1 + \lambda_3^2\lambda_2) + 6\lambda_1\lambda_2\lambda_3 \\
&\quad - 3(\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + (\lambda_2 + \lambda_3)\lambda_1^2 + (\lambda_1 + \lambda_3)\lambda_2^2 + (\lambda_1 + \lambda_2)\lambda_3^2) \\
&= -2(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + 6\lambda_1\lambda_2\lambda_3
\end{aligned}$$

we have

$$J = 3\lambda_1\lambda_2\lambda_3. \tag{3.892}$$

Note that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are roots of

$$6\lambda^3 - 3I\lambda - 2J = 0$$

as is easily seen from

$$6(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 6\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\lambda - 6\lambda_1\lambda_2\lambda_3$$

Using

$$\begin{aligned} \Psi_{ABCD} &= \Psi_0 i_A i_B i_C i_D - 4\Psi_1 o_{(A} i_B i_C i_{D)} + 6\Psi_2 o_{(A} o_B i_C i_{D)} \\ &\quad - 4\Psi_1 o_{(A} o_B o_C i_{D)} + \Psi_4 o_A o_B o_C o_D \end{aligned} \tag{3.893}$$

which follows as it gives (3.800) upon contraction with appropriate combination of o's and i's. For example

$$\begin{aligned} \Psi_{ABCD} o^A o^B i^C i^D &= (\Psi_0 i_A i_B i_C i_D - 4\Psi_1 o_{(A} i_B i_C i_{D)} + 6\Psi_2 o_{(A} o_B i_C i_{D)} \\ &\quad - 4\Psi_1 o_{(A} o_B o_C i_{D)} + \Psi_4 o_A o_B o_C o_D) o^A o^B i^C i^D \\ &= 6\Psi_2 o_{(A} o_B i_C i_{D)} o^A o^B i^C i^D \\ &= 6\Psi_2 \frac{1}{4!}(\cdots + o_C o_D i_A i_B + o_D o_C i_A i_B + o_C o_D i_B i_A \\ &\quad + o_D o_C i_B i_A + \cdots) o^A o^B i^C i^D \\ &= \Psi_2. \end{aligned} \tag{3.894}$$

where we used $o_A o^A = i_A i^A = 0$ and $o_A i^A = 1 = -i_A o^A$.

We obtain

$$I = 2\Psi_0\Psi_4 - 8\Psi_1\Psi_3 + 6\Psi_2^2 \tag{3.895}$$

$$J = 6\det\begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix}. \tag{3.896}$$

From (3.892) we have that the determinant of $\mathbf{\Psi}$ is $\frac{1}{3}J$

We have

$$I^3 - 6J^2 = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2 \quad \text{subject to } \lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{3.897}$$

which can be shown by verifying:

$$\begin{aligned}(\lambda_1^2 + \lambda_2^2 + (-\lambda_1 - \lambda_2)^2)^3 - 6(\lambda_1^3 + \lambda_2^3 + (-\lambda_1 - \lambda_2)^3)^2 \\ = (\lambda_1 - \lambda_2)^2(\lambda_2 - (-\lambda_1 - \lambda_2))^2((-\lambda_1 - \lambda_2) - \lambda_1)^2.\end{aligned} \tag{3.898}$$

Equation (3.897) establishes that two or more of the λ's are equal is equivalent to $I^3 = 6J^2$.

Eigenbivectors of the Weyl Tensor

$$C_{ab}{}^{cd}X_{cd} = \mu X_{ab}, \tag{3.899}$$

where $X_{ab} = -X_{ab}$. In terms of spinors:

$$X_{ab} = \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\xi_{A'B'} \tag{3.900}$$

where ϕ_{AB} and $\xi_{A'B'}$ are both symmetric. Defing X^*_{ab}:

$$X^*_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}X_{cd}$$

and using

$$\epsilon_{abcd} = i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'})$$

we obtain

$$\begin{aligned}X^*_{ABA'B'} &= \frac{1}{2}i(\epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'}{}^{D'}\epsilon_{B'}{}^{C'} - \epsilon_A{}^D\epsilon_B{}^C\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'})X_{CDC'D'} \\ &= \frac{1}{2}i(\epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'}{}^{D'}\epsilon_{B'}{}^{C'} - \epsilon_A{}^D\epsilon_B{}^C\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'})(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\xi_{C'D'}) \\ &= \frac{1}{2}i(\phi_{AB}\epsilon_{B'A'} + \epsilon_{AB}\xi_{B'A'} - \phi_{BA}\epsilon_{A'B'} - \epsilon_{BA}\xi_{A'B'}) \\ &= i(\epsilon_{AB}\xi_{A'B'} - \phi_{AB}\epsilon_{A'B'}).\end{aligned} \tag{3.901}$$

As such we have

$$X_{ab} + iX^*_{ab} = 2\phi_{AB}\epsilon_{A'B'}, \qquad X_{ab} - iX^*_{ab} = 2\epsilon_{AB}\xi_{A'B'}. \tag{3.902}$$

The Weyl tensor in terms of spinors is

$$C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'} \tag{3.903}$$

Defining

$$C^*_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef}C_{efcd}X^{cd}$$

We obtain

$$\begin{aligned}
C^*_{ABCDA'B'C'D'} &= \frac{1}{2}i(\epsilon_A{}^E\epsilon_B{}^F\epsilon_{A'}{}^{F'}\epsilon_{B'}{}^{E'} - \epsilon_A{}^F\epsilon_B{}^E\epsilon_{A'}{}^{E'}\epsilon_{B'}{}^{F'}) \\
&\quad (\Psi_{EFCD}\epsilon_{E'F'}\epsilon_{C'D'} + \epsilon_{EF}\epsilon_{CD}\overline{\Psi}_{E'F'C'D'}) \\
&= \frac{1}{2}i(\Psi_{ABCD}\epsilon_{B'A'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{B'A'C'D'} \\
&\quad -\Psi_{BACD}\epsilon_{A'B'}\epsilon_{C'D'} - \epsilon_{BA}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'}) \\
&= i(\overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} - \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'})
\end{aligned} \tag{3.904}$$

Then

$$C_{abcd} + iC^*_{abcd} = 2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}, \qquad C_{abcd} - iC^*_{abcd} = 2\overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \tag{3.905}$$

From (3.899) we obtain

$$(C_{ab}{}^{cd} + i\frac{1}{2}\epsilon_{ab}{}^{ef}C_{ef}{}^{cd})X_{cd} = \mu(X_{ab} + i\frac{1}{2}\epsilon_{ab}{}^{ef}X_{ef}) \tag{3.906}$$

which in terms of spinors is

$$2\Psi_{AB}{}^{CD}\epsilon_{A'B'}\epsilon^{C'D'}(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\xi_{C'D'}) = \mu 2\phi_{AB}\epsilon_{A'B'}$$

or

$$\Psi_{AB}{}^{CD}\phi_{CD} = \frac{1}{2}\mu\phi_{AB} \tag{3.907}$$

where we have used $\epsilon^{C'D'}\epsilon_{C'D'} = 2$ and $\epsilon^{C'D'}\xi_{C'D'} = 0$.

From (3.899) we obtain

$$(C_{ab}{}^{cd} - i\frac{1}{2}\epsilon_{ab}{}^{ef}C_{ef}{}^{cd})X_{cd} = \mu(X_{ab} - i\frac{1}{2}\epsilon_{ab}{}^{ef}X_{ef}) \tag{3.908}$$

which in terms of spinors is

$$2\overline{\Psi}_{A'B'}{}^{C'D'}\epsilon_{AB}\epsilon^{CD}(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\xi_{C'D'}) = \mu 2\epsilon_{AB}\xi_{A'B'}$$

or

$$\overline{\Psi}_{A'B'}{}^{C'D'}\xi_{C'D'} = \frac{1}{2}\mu\xi_{A'B'} \tag{3.909}$$

where we have used $\epsilon^{CD}\epsilon_{CD} = 2$ and $\epsilon^{CD}\phi_{CD} = 0$.

3.17.13 Focussing and Shearing of Null Curves

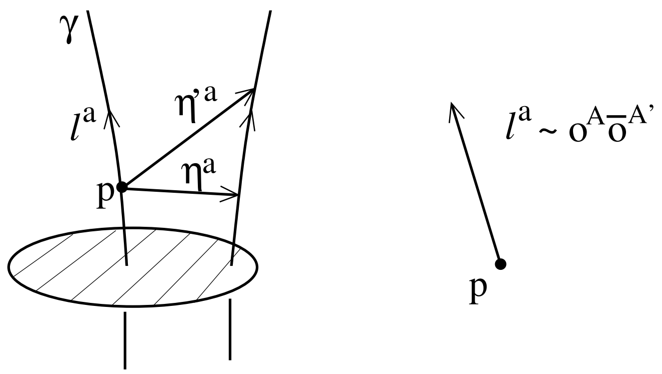


Figure 3.32: visualflownull.

$$[l,\eta]^a = l^b\nabla_b\eta^a - \eta^b\nabla_b l^a = 0. \tag{3.910}$$

$$Do^A = D\iota^A = 0. \tag{3.911}$$

This means that o and $\imath$ are parallelly propergated along γ and so remain a spin basis at each point of γ.

recalling that $D = l^a \nabla_a$

$$D(l^a \eta_a) = l^a D\eta_a = l^a l^b \nabla_b \eta_a = l^a \eta^b \nabla_b l^a = \frac{1}{2}\eta^b \nabla_b (l^2) = 0 \tag{3.912}$$

We now construct the NP tetrad $(l, n, m, \overline{m})$

$$\begin{aligned} \eta^a &= ul^a + \overline{z}m^a + z\overline{m}^a \\ &= uo^A\overline{o}^{A'} + \overline{z}o^A\overline{\imath}^{A'} + z\overline{o}^{A'}\imath^A. \end{aligned} \tag{3.913}$$

Now, since $l^b \nabla_b \eta^a = \eta^b \nabla_b l^a$ we have

$$D\eta^a = l^b \nabla_b \eta^a = \eta^b \nabla_b l^a, \tag{3.914}$$

so that,

$$D\eta^a = uDl^a + \overline{z}\delta l^a + z\overline{\delta} l^a. \tag{3.915}$$

In terms of spinors this reads

$$o^A\overline{o}^{A'}Du + o^A\overline{\imath}^{A'} + \imath^A\overline{o}^{A'} = \overline{z}o^A\overline{\delta}\overline{o}^{A'} + \overline{z}\overline{o}^{A'}\delta o^A + zo^A\overline{\delta}\overline{o}^{A'} + z\overline{o}^{A'}\overline{\delta}o^A. \tag{3.916}$$

Multipling by $o_A \overline{\imath}_{A'}$ gives

$$-Dz = \overline{z}o_A\delta o^A - zo_A\overline{\delta}o^A, \tag{3.917}$$

i.e.,

$$Dz = -\rho z - \sigma\overline{z}. \tag{3.918}$$

The interpretation of z is as follows. Consider the projection of η^a onto the spacelike surface 2-dimensional subspace $T_\perp$ as introduced in subsection?? Recall that this surface was spanned by $m^a, \overline{m}^a$, or equivalently $e_{\hat{1}}{}^a, e_{\hat{2}}{}^a$. Write the projection as

$$
\begin{aligned}
\sqrt{2}(xe_{\hat{1}} - ye_{\hat{2}}) &= x(m+\overline{m})/i + y(m-\overline{m})/i \\
&= \overline{z}m + z\overline{m},
\end{aligned} \tag{3.919}
$$

where $z = x + iy$, which is consistent with (3.913).

Suppose first that $\sigma = 0$ while ρ is real, i.e. $Dz = -\rho z$, or

$$
Dx = -\rho x, \quad Dy = -\rho y. \tag{3.920}
$$

This is isotrpic magnificatoin at a rate of $-\rho$. Next suppose that $\sigma = 0$ while $\rho = -i\omega$, so that $Dz = i\omega z$, or

$$
Dx = -\omega y, \quad Dy = \omega x. \tag{3.921}
$$

This corresponds to a rotation with angular velocity ω. Next consider the case where $\rho = 0$ and σ is real. Then

$$
Dx = -\sigma x, \quad Dy = \sigma y, \tag{3.922}
$$

which represents a volume-preserving shear at a rate with principle axes along the x and y axes.

3.17.14 Goldberg Sachs Theorem

Equivalence relations for geodesic shearfree null congruences

Lemma 3.17.2 *The following three conditions are equivalent*

a) $l^a = o^A\overline{o}^{A'}$ *corresponds to geodesic shearfree null congruences;*

b) $\kappa = \sigma = 0$*;*

c) $o^A o^B \nabla_{AA'} o_B = 0$.

Proof:

b) implies c) and **c) implies b):**

Recall that $\kappa = o^A o^B \overline{o}^{B'} \nabla_{BB'} o_A$ and $\sigma = o^A o^B \overline{\imath}^{B'} \nabla_{BB'} o_A$. We write

$$o^A o^B \nabla_{BB'} o_A = c\overline{o}_{B'} + d\overline{\imath}_{B'} \tag{3.923}$$

then

$$\kappa = o^A o^B \overline{o}^{B'} \nabla_{BB'} o_A = c\overline{o}^{B'}\overline{o}_{B'} + d\overline{o}^{B'}\overline{\imath}_{B'} = -d. \tag{3.924}$$

and

$$\sigma = o^A o^B \overline{\imath}^{B'} \nabla_{BB'} o_A = c\overline{\imath}^{B'}\overline{o}_{B'} + d\overline{\imath}^{B'}\overline{\imath}_{B'} = c. \tag{3.925}$$

so that

$$o^A o^B \nabla_{BB'} o_A = \sigma\overline{o}_{B'} - \kappa\overline{\imath}_{B'} \tag{3.926}$$

This establishes the equivalence of b) and c).

b) implies a)

$$\kappa = o^A D o_A = o^A o^B \overline{o}^{B'} \nabla_{BB'} o_A = 0.$$

and

$$\sigma = o^A \delta o_A = o^A o^B \overline{\imath}^{B'} \nabla_{BB'} o_A = 0.$$

We write

$$\begin{aligned} l^b \nabla_b l_a &= D(o_A \overline{o}_{A'}) \\ &= o_A D\overline{o}_{A'} + \overline{o}_{A'} D o_A \\ &= a o_A \overline{o}_{A'} + b o_A \overline{\imath}_{A'} + \overline{b}\imath_A \overline{o}_{A'} + c\imath_A \overline{\imath}_{A'} \end{aligned} \tag{3.927}$$

Contracting the second and third line on the RHS with $o^A\overline{o}^{A'}$ implies $c = 0$. Next, contracting the second and third line on the RHS with $\imath^A\overline{o}^{A'}$ implies

$$o_A \imath^A (\overline{o}^{A'} D\overline{o}_{A'}) = \kappa = 0 = -b = -\overline{b}.$$

Next, contracting the second and third line on the RHS with $\imath^A\overline{\imath}^{A'}$ implies

$$\bar{\iota}^{A'} D\bar{o}_{A'} + \iota^A Do_A = \epsilon + \bar{\epsilon} = a$$

.

We find

$$l^b \nabla_b l_a = a l_a. \tag{3.928}$$

which the non-affinely parameterised geodesic equation.

a) implies b):

$l^b \nabla_b l_a = a l_a$ says

$$D(o_A \bar{o}_{A'}) = o_A D\bar{o}_{A'} + \bar{o}_{A'} Do_A = a o_A \bar{o}_{A'} \tag{3.929}$$

Contracting with o^A implies

$$\bar{o}_{A'}(o^A Do_A) = 0,$$

or

$$o^A Do_A = \kappa = 0. \tag{3.930}$$

$\square$

Integrability conditions

$$A^a \nabla_a \phi = f, \qquad B^a \nabla_a \phi = g, \tag{3.931}$$

where A^a, B^a are locally tranverse vector fields which are surface forming,

$$A^a \nabla_a B^b - B^a \nabla_a A^b = \alpha A^b + \beta B^b. \tag{3.932}$$

Theorem 3.17.3 *A necessary and sufficient condition for solutions of the system (3.931) to exist is*

$$A^a \nabla_a g - B^a \nabla_a f = \alpha f + \beta g. \tag{3.933}$$

Proof:

Suppose ϕ is a solution of (3.931) then (3.932) implies

$$\begin{aligned}
& A^a\nabla_a(B^b\nabla_b\phi) - B^a\nabla_a(A^b\nabla_b\phi) \\
= \; & (A^a\nabla_a B^b - B^a\nabla_a A^b)\nabla_b\phi + (A^aB^b - B^aA^b)\nabla_a\nabla_b\phi \\
= \; & (\alpha A^a + \beta B^a)\nabla_a\phi \\
= \; & \alpha f + \beta g
\end{aligned} \tag{3.934}$$

No suppose conversely that (3.940) holds. We choose a special coordinate system such that

$$A^a\nabla_a = \frac{\partial}{\partial y^1}$$

Then we may solve $A^a\nabla_a\phi = f$ via

$$\phi(y^1, y^\alpha) = \int_{\tilde{y}}^{y^1} f(s, y^\alpha)ds \tag{3.935}$$

where $\alpha = 2, \ldots, n$. Let $B^a\nabla_a\phi - g = h$. We may choose $\tilde{y} = \tilde{y}(y^\alpha)$ to set $h = 0$ on an intial surface $y^1 = const$. Now

$$\begin{aligned}
A^a\nabla_a h &= A^a\nabla_a(B^b\nabla_b\phi) - A^a\nabla_a g \\
&= B^a\nabla_a(A^b\nabla_b\phi) + (A^a\nabla_a B^b - B^a\nabla_a A^b)\nabla_b\phi - A^a\nabla_a g \\
&= B^a\nabla_a f + (\alpha A^a + \beta B^a)\nabla_a\phi - A^a\nabla_a g \\
&= (\alpha A^a + \beta B^a)\nabla_a\phi - (A^a\nabla_a g - B^a\nabla_a f) \\
&= \alpha f + \beta B^a\nabla_a\phi - \alpha f - \beta g \\
&= \beta h.
\end{aligned} \tag{3.936}$$

Since $h = 0$ intially we see that $h = 0$ and so $B^a\nabla_a\phi = g$, i.e. ϕ solves (3.931).

□

Theorem 3.17.4 *(Sommers 1976) The integrable condition for the equation*

$$\xi^A\nabla_{AA'}x = \alpha_{A'} \tag{3.937}$$

in complex x for ξ^A analytic shearfree null geodesics is

$$\xi^A\xi^B\nabla_A{}^{A'}\alpha_{A'} = \alpha_{A'}\xi^A\nabla_A{}^{A'}\xi^B. \tag{3.938}$$

Proof: We write the equation to be solved in component form relative to some basis (o^A, ι^A),

$$\xi^A\overline{o}^{A'}\nabla_{AA'}x = \overline{o}^{A'}\alpha_{A'}, \qquad \xi^A\iota^{A'}\nabla_{AA'}x = \iota^{A'}\alpha_{A'} \tag{3.939}$$

putting it into the form (3.931) where we identify $\phi = x$, $A^a = \xi^A\overline{o}^{A'}$, $B^a = \xi^A\iota^{A'}$, $f = \overline{o}^{A'}\alpha_{A'}$, and $g = \iota^{A'}\alpha_{A'}$. The integrability condition is then

$$\xi^A\overline{o}^{A'}\nabla_{AA'}(\overline{\iota}^{B'}\alpha_{B'}) - \xi^A\overline{\iota}^{A'}\nabla_{AA'}(\overline{o}^{B'}\alpha_{B'}) = \alpha\overline{o}^{A'}\alpha_{A'} + \beta\overline{\iota}^{A'}\alpha_{A'}. \tag{3.940}$$

where α and β are given by

$$\xi^A\overline{o}^{A'}\nabla_{AA'}(\xi^B\overline{\iota}^{B'}) - \xi^A\overline{\iota}^{A'}\nabla_{AA'}(\xi^B\overline{o}^{B'}) = \alpha\xi^B\overline{o}^{B'} + \beta\xi^B\overline{\iota}^{B'}. \tag{3.941}$$

Transvecting (3.941) with $\alpha_{B'}$ and equating to (3.940), we find

$$\xi^A\xi^B\{\overline{o}^{A'}\nabla_{AA'}(\overline{\iota}^{B'}\alpha_{B'}) - \overline{\iota}^{A'}\nabla_{AA'}(\overline{o}^{B'}\alpha_{B'})\} = \alpha_{B'}\xi^A\{\overline{o}^{A'}\nabla_{AA'}(\xi^B\overline{\iota}^{B'}) - \iota^{A'}\nabla_{AA'}(\xi^B\overline{o}^{B'})\} \tag{3.942}$$

or

$$\xi^A\xi^B\{(\overline{o}^{A'}\overline{\iota}^{B'} - \overline{\iota}^{A'}\overline{o}^{B'})\nabla_{AA'}\alpha_{B'}\} = \alpha_{B'}\xi^A\{(\overline{o}^{A'}\overline{\iota}^{B'} - \iota^{A'}\overline{o}^{B'})\nabla_{AA'}\xi^B)\} \tag{3.943}$$

Using $\overline{o}^{A'}\overline{\iota}^{B'} - \overline{\iota}^{A'}\overline{o}^{B'} = \epsilon^{A'B'}$, (3.943) becomes

$$\xi^A\xi^B\nabla_A{}^{A'}\alpha_{A'} = \alpha_{A'}\xi^A\nabla_A{}^{A'}\xi^B \tag{3.944}$$

□

Some relations

If ξ is geodesic shearfree it satifies

$$\xi^A \xi^B \nabla_{AA'} \xi_B = 0 \tag{3.945}$$

We define $\overline{\eta}_{A'}$ as the proportinality factor in the equation

$$\xi^A \nabla_{AA'} \xi_B = \xi_B \overline{\eta}_{A'} \tag{3.946}$$

Consider

$$\begin{aligned}
\xi_B \xi^A \nabla_{AA'} \overline{\eta}^{A'} &= \xi_B \xi^A \nabla_{AA'} \overline{\eta}^{A'} + \xi^A \overline{\eta}^{A'} \nabla_{AA'} \xi_B \\
&= -\xi^A \nabla_A{}^{A'} (\xi_B \overline{\eta}_{A'}) \\
&= \xi^A \nabla_{AA'} (\xi^C \nabla_C{}^{A'} \xi_B) \\
&= (\nabla_C{}^{A'} \xi_B) \xi^A \nabla_{AA'} \xi^C + \xi^A \xi^C \nabla_{AA'} \nabla_C{}^{A'} \xi_B \\
&= (\nabla_C{}^{A'} \xi_B) \xi^C \overline{\eta}_{A'} + \xi^A \xi^C \nabla_{AA'} \nabla_C{}^{A'} \xi_B \\
&= 0 + \xi^A \xi^C \square_{AC} \xi_B \\
&= \Psi_{ABCD} \xi^A \xi^C \xi^D.
\end{aligned} \tag{3.947}$$

where we used (3.762). Therefore we have the first relation

$$\xi_B \xi^A \nabla_{AA'} \overline{\eta}^{A'} = \Psi_{ABCD} \xi^A \xi^C \xi^D. \tag{3.948}$$

Now taking the derivative of $\xi^A \xi^D \nabla_A{}^{A'} \xi_D = 0$ gives

$$\begin{aligned}
0 &= \nabla_{BA'} (\xi^A \xi^D \nabla_A{}^{A'} \xi_D) \\
&= \xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D + (\nabla_{BA'} \xi^A) \xi^D \nabla_A{}^{A'} \xi_D + (\nabla_{BA'} \xi^D) \xi^A \nabla_A{}^{A'} \xi_D \\
&= \xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D + 2 (\nabla_{BA'} \xi^D) \xi^A \nabla_{(A}{}^{A'} \xi_{D)}
\end{aligned} \tag{3.949}$$

The first term of the last line

$$\begin{aligned}
\xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D &= \frac{1}{2} \xi^A \xi^D (\nabla_{BA'} \nabla_A{}^{A'} \xi_D + \nabla_{DA'} \nabla_A{}^{A'} \xi_B) + \frac{1}{2} \xi^A \xi^D \epsilon_{BD} \nabla_{CA'} \nabla_A{}^{A'} \xi^C \\
&= \frac{1}{2} \xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D + \frac{1}{2} [\xi^A \xi^D \nabla_{DA'} \nabla_A{}^{A'} \xi_B - \xi^A \xi_B \nabla_{DA'} \nabla_A{}^{A'} \xi^D]
\end{aligned}$$

which implies

$$\xi^A\xi^D\nabla_{BA'}\nabla_A{}^{A'}\xi_D = \xi^A\xi^D\nabla_{DA'}\nabla_A{}^{A'}\xi_B - \xi^A\xi_B\nabla_{DA'}\nabla_A{}^{A'}\xi^D \tag{3.950}$$

Using (3.762) and (3.764)

$$\begin{aligned}
\xi^A\xi^D\nabla_{BA'}\nabla_A{}^{A'}\xi_D &= \xi^A\xi^D\nabla_{A'(D}\nabla_{A)}{}^{A'}\xi_B - \xi^A\xi_B(2\nabla_{A'(D}\nabla_{A)}{}^{A'}\xi^D - \nabla_{AA'}\nabla_D{}^{A'}\xi^D) \\
&= \xi^A\xi^D\Box_{AD}\xi_B - 2\xi^A\xi_B\Box_{AD}\xi^D + \xi^A\xi_B\nabla_{AA'}\nabla_D{}^{A'}\xi^D \\
&= \Psi_{ABCD}\xi^A\xi^C\xi^D + \xi^A\xi_B\nabla_{AA'}\nabla_D{}^{A'}\xi^D
\end{aligned} \tag{3.951}$$

We introduce, analogously to (3.946), the proportinality factor $\overline{\zeta}_{A'}$

$$\begin{aligned}
\xi_A\overline{\zeta}_{A'} &:= \xi^B\nabla_{AA'}\xi_B \\
&= \frac{1}{2}\xi^B(\nabla_{AA'}\xi_B + \nabla_{BA'}\xi_A + \epsilon_{DA'}\nabla_{DA'}\xi^D) \\
&= \frac{1}{2}\xi_A(\overline{\zeta}_{A'} + \overline{\eta}_{A'} - \nabla_{DA'}\xi^D)
\end{aligned} \tag{3.952}$$

Hence

$$\overline{\zeta}_{A'} = \overline{\eta}_{A'} - \nabla_{DA'}\xi^D. \tag{3.953}$$

The second term in (3.949) becomes

$$\begin{aligned}
2(\nabla_{BA'}\xi^D)\xi^A\nabla_{(A}{}^{A'}\xi_{D)} &= (\nabla_{BA'}\xi^D)\xi_D(\overline{\eta}^{A'} + \overline{\zeta}^{A'}) \\
&= -\xi_B\overline{\zeta}_{A'}(\overline{\eta}^{A'} + \overline{\zeta}^{A'}) \\
&= \xi_B\overline{\eta}_{A'}\overline{\zeta}^{A'} \\
&= \xi_B\overline{\eta}_{A'}(\overline{\eta}^{A'} - \nabla_D{}^{A'}\xi^D) \\
&= -\xi_B\overline{\eta}_{A'}\nabla_D{}^{A'}\xi^D
\end{aligned} \tag{3.954}$$

The sum of the RHS of (3.951) and (3.954) is equal to zero by (3.949) hence

$$\xi_B\xi^A\nabla_{AA'}\nabla_D{}^{A'}\xi^D - \xi_B\overline{\eta}_{A'}\nabla_D{}^{A'}\xi^D = -\Psi_{ABCD}\xi^A\xi^C\xi^D. \tag{3.955}$$

Goldberg Sachs Theorem

Theorem 3.17.5 *In a spacetime which satifies the vacuum field equations* $R_{\mu\nu} = 0$ *any two of the following conditions imply the third:*

a) The Weyl tensor Ψ_{ABCD} *is algebraically special with* $n-$*fold repeated principal spinor* o, $(n = 2, 3, 4)$;

b) either spacetime is flat or o *generates a geodesic shearfree congruence;*

c) $\nabla^{AA'}\Psi_{ABCD}$ *contracted with* $(5 - n)$ o*'s vanishes.*

Proof:

a) and b) imply c): $n = 2$.

Assume a), then $\Psi_{ABCD} = \xi_{(A}\xi_B\alpha_C\beta_{D)}$. Obviously

$$\xi^A\xi^B\xi^C\Psi_{ABCD} = 0,$$

and so

$$\xi^A\xi^B\xi^C\nabla^{DD'}\Psi_{ABCD} + 3\Psi_{ABCD}\xi^A\xi^B\nabla^{DD'}\xi^C = 0. \tag{3.956}$$

Now we use b). If spacetime is flat then condition c) obviously holds. If instead ξ is geodesic and shearfree

a) and c) imply b): $n = 2$.

Condition a) implies (3.956). If c) holds then (3.956) imjplies

$$\Psi_{ABCD}\xi^A\xi^B\nabla^{DD'}\xi^C = 0,$$

which on substitution of $\Psi_{ABCD} = \xi_{(A}\xi_B\alpha_C\beta_{D)}$ implies

$$\begin{aligned} 0 &= \xi_{(A}\xi_B\alpha_C\beta_{D)}\xi^A\xi^B\nabla^{DD'}\xi^C \\ &= \frac{1}{24}[\cdots + \alpha_A\beta_B(\xi_C\xi_D + \xi_D\xi_C) + \alpha_B\beta_A(\xi_C\xi_D + \xi_D\xi_C) + \ldots]\xi^A\xi^B\nabla^{DD'}\xi^C \\ &= \frac{1}{6}(\alpha_{(A}\beta_{B)}\xi^A\xi^B)\xi_C\xi_D\nabla^{DD'}\xi^C \end{aligned} \tag{3.957}$$

and so

$$\xi_C \xi_D \nabla^{DD'} \xi^C = 0$$

which proves b).

b) and c) imply a): $n = 2$.

Consider the relation

$$\xi^A \xi^B \xi^C \Psi_{ABCD} = x \xi_D.$$

We wish to show that $x = 0$. Taking derivatives we have

$$\xi^A \xi^B \xi^C \nabla^{DD'} \Psi_{ABCD} + 3\Psi_{ABCD} \xi^A \xi^B \nabla^{DD'} \xi^C = \xi_D \nabla^{DD'} x + x \nabla^{DD'} \xi_D. \tag{3.958}$$

Assuming c ($n = 2$), the first term on the LHS vanishes. Assume $\Psi_{ABCD} = \xi_{(A} \alpha_B \beta_C \gamma_{D)}$, then we have for the second term on the LHS

$$\begin{aligned}
3\xi_{(A} \alpha_B \beta_C \gamma_{D)} \xi^A \xi^B \nabla^{DD'} \xi^C &= \frac{3}{4}[\xi_A \alpha_{(B} \beta_C \gamma_{D)} + \xi_B \alpha_{(A} \beta_C \gamma_{D)} \\
&\quad + \xi_C \alpha_{(A} \beta_B \gamma_{D)} + \xi_D \alpha_{(A} \beta_B \gamma_{C)}] \xi^A \xi^B \nabla^{DD'} \xi^C \\
&= \frac{3}{4}[\xi_C \alpha_{(A} \beta_B \gamma_{D)} + \xi_D \alpha_{(A} \beta_B \gamma_{C)}] \xi^A \xi^B \nabla^{DD'} \xi^C
\end{aligned}$$

which on using (3.946) and (3.952) becomes

$$\begin{aligned}
&\frac{3}{4} \xi^A \xi^B [\alpha_{(A} \beta_B \gamma_{D)} \xi_C \nabla^{DD'} \xi^C + \alpha_{(A} \beta_B \gamma_{C)} \xi_D \nabla^{DD'} \xi^C] \\
= &-\frac{3}{4}[\xi^A \xi^B \xi^D \overline{\zeta}^{D'} \alpha_{(A} \beta_B \gamma_{D)} + \xi^A \xi^B \xi^C \overline{\eta}^{D'} \alpha_{(A} \beta_B \gamma_{C)}] \\
= &-3x(\overline{\zeta}^{D'} + \overline{\eta}^{D'})
\end{aligned} \tag{3.959}$$

where we used

$$4x = \xi^A \xi^B \xi^C \alpha_{(A} \beta_B \gamma_{C)} \tag{3.960}$$

which follows from

$$x \xi_D = \xi^A \xi^B \xi^C \Psi_{ABCD} = \xi^A \xi^B \xi^C \xi_{(A} \alpha_B \beta_C \gamma_{D)} = \frac{1}{4} \xi^A \xi^B \xi^C \xi_D \alpha_{(A} \beta_B \gamma_{C)}.$$

(3.958) now reads

$$
\begin{aligned}
-3x(\overline{\zeta}^{D'} + \overline{\eta}^{D'}) &= \xi_D \nabla^{DD'} x + x \nabla^{DD'} \xi_D \\
&= \xi_D \nabla^{DD'} x + x(\overline{\zeta}^{D'} - \overline{\eta}^{D'})
\end{aligned} \tag{3.961}
$$

This becomes

$$
\xi^A \nabla_{AA'}(lnx) = 2\overline{\eta}_{A'} + 4\overline{\zeta}_{A'} = 6\overline{\eta}_{A'} - 4\nabla_{DA'}\xi^D. \tag{3.962}
$$

To check if a solution lnx of this equation exists we substitute its RHS for $\alpha_{A'} = 6\overline{\eta}_{A'} - 4\nabla_{DA'}\xi^D$ into the integrability theorem (3.17.4). This yields

$$
\begin{aligned}
-\xi^A \xi_B \nabla_{AA'}(6\overline{\eta}^{A'} - 4\nabla_D{}^{A'}\xi^D) &= \alpha_{A'}\xi^A \nabla_A{}^{A'}\xi_B \\
&= \xi_B \alpha_{A'} \overline{\eta}^{A'} \\
&= -\xi_B \overline{\eta}_{A'}(6\overline{\eta}^{A'} - 4\nabla_D{}^{A'}\xi^D) \\
&= 4\xi_B \overline{\eta}_{A'} \nabla_D{}^{A'}\xi^D
\end{aligned} \tag{3.963}
$$

Rearanging this gives

$$
-6\xi_B \xi^A \nabla_{AA'}\overline{\eta}^{A'} = -4(\xi_B \xi^A \nabla_{AA'}\nabla_D{}^{A'}\xi^D - \xi_B \overline{\eta}_{A'} \nabla_D{}^{A'}\xi^D) \tag{3.964}
$$

Substitution of the identities (3.948) and (3.955) gives

$$
-6\Psi_{ABCD}\xi^A\xi^B\xi^D = -4(-\Psi_{ABCD}\xi^A\xi^B\xi^D)
$$

or

$$
\Psi_{ABCD}\xi^A\xi^B\xi^D = 0
$$

which says $x = 0$, contary to our assumption.

□

3.17.15 Tetrad Formulism and the Cartan Structure Equations

$$T^{\hat{b}\ldots}_{\hat{a}\ldots} = e_{\hat{a}}{}^{a} e^{\hat{b}}{}_{b} \cdots T^{b\ldots}_{a\ldots} \tag{3.965}$$

$$T^{b\ldots}_{a\ldots} = e^{\hat{a}}{}_{a} e_{\hat{b}}{}^{b} \cdots T^{\hat{b}\ldots}_{\hat{a}\ldots} \tag{3.966}$$

The directional derivative along a tetrad vector is denoted by a comma or by ∂_a

$$T^{\hat{b}\ldots}_{\hat{a}\ldots,\hat{c}} = \partial_{\hat{c}} T^{\hat{b}\ldots}_{\hat{a}\ldots} = e_{\hat{c}}{}^{c} \frac{\partial}{\partial x^c} T^{\hat{b}\ldots}_{\hat{a}\ldots} \tag{3.967}$$

The tetrad components of the covariant derivative are denoted by a semicolon

$$T^{\hat{b}\ldots}_{\hat{a}\ldots;\hat{c}} = e^{\hat{a}}{}_{a} e_{\hat{b}}{}^{b} \cdots e_{\hat{c}}{}^{c} T^{b\ldots}_{a\ldots;c} \tag{3.968}$$

They are given by

$$T^{\hat{b}\ldots}_{\hat{a}\ldots;c} = T^{\hat{b}\ldots}_{\hat{a}\ldots,\hat{c}} - \Gamma^{\hat{d}}{}_{\hat{a}\hat{c}} T^{\hat{b}\ldots}_{\hat{d}\ldots} - \cdots + \Gamma^{\hat{b}}{}_{\hat{d}\hat{c}} T^{\hat{d}\ldots}_{\hat{a}\ldots} + \cdots \tag{3.969}$$

where the $\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$ are the Ricci rotation coefficients

$$\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} = -e^{\hat{a}}{}_{a;b} e_{\hat{b}}{}^{a} e_{\hat{c}}{}^{b} \tag{3.970}$$

and take the place of the Christoffel symbols in the tetradad formulism. The rotation coefficients also appear in the commutator of two directional derivatives along tetrad vectors

$$\begin{aligned}
2T^{\cdots}_{\ldots,[\hat{a}\hat{b}]} &= T^{\cdots}_{\ldots,\hat{a}\hat{b}} - T^{\cdots}_{\ldots,\hat{b}\hat{a}} \\
&= e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a} T^{\cdots}_{\ldots,a})_{,b} - e_{\hat{a}}{}^{a}(e_{\hat{b}}{}^{b} T^{\cdots}_{\ldots,b})_{,a} \\
&= e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a})_{,b} T^{\cdots}_{\ldots,a} - e_{\hat{a}}{}^{a}(e_{\hat{b}}{}^{b})_{,a} T^{\cdots}_{\ldots,b} + e_{\hat{a}}{}^{a} e_{\hat{b}}{}^{b}(T^{\cdots}_{\ldots,ab} - T^{\cdots}_{\ldots,ba}) \\
&= T^{\cdots}_{\ldots,\hat{c}}[e^{\hat{c}}{}_{a}(e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a})_{,b} - e_{\hat{a}}{}^{b}(e_{\hat{b}}{}^{a})_{,b})] \\
&= T^{\cdots}_{\ldots,\hat{c}}[e^{\hat{c}}{}_{a}(e_{\hat{b}}{}^{b}[(e_{\hat{a}}{}^{a})_{,b} + \Gamma^{a}_{bd} e_{\hat{a}}{}^{d}] - e_{\hat{a}}{}^{b}[(e_{\hat{b}}{}^{a})_{,b} + \Gamma^{a}_{bd} e_{\hat{a}}{}^{d}])] \\
&= T^{\cdots}_{\ldots,\hat{c}}[e^{\hat{c}}{}_{a}(e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a})_{;b} - e_{\hat{a}}{}^{b}(e_{\hat{b}}{}^{a})_{;b})] \\
&= 2T^{\cdots}_{\ldots,\hat{c}} \Gamma^{\hat{c}}{}_{[\hat{a}\hat{b}]}
\end{aligned} \tag{3.971}$$

The tetrad vector determine linear differential forms

$$e^{\hat{a}}{}_{a}dx^{a} = e^{\hat{a}}, \tag{3.972}$$

in terms of which the metric form is given by

$$ds^{2} = e_{\hat{a}}e^{\hat{a}} = e_{\hat{a}a}e^{\hat{a}}{}_{b}dx^{a}dx^{b} = g_{ab}dx^{a}dx^{b} \tag{3.973}$$

The exterior product of two linear differential forms $A = A_{a}dx^{a}$ and $B = B_{a}dx^{a}$ is the anti-symmetric multiplication

$$\begin{aligned} A \wedge B &= -B \wedge A \\ &= A_{a}B_{b}dx^{a} \wedge dx^{b} \\ &= A_{[a}B_{b]}dx^{a} \wedge dx^{b} \end{aligned} \tag{3.974}$$

The exterior derivative of a linear differential form is

$$dA = A_{b,a}dx^{a} \wedge dx^{b} = A_{[b,a]}dx^{a} \wedge dx^{b} \tag{3.975}$$

Cartan Structure Equations

We define

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}e^{c} \tag{3.976}$$

$$\mathcal{R}^{\hat{a}}{}_{\hat{b}} = R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}e^{\hat{c}} \wedge e^{\hat{d}}. \tag{3.977}$$

The Cartan structure equations are

$$de^{\hat{a}} = e^{\hat{b}} \wedge \Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}e^{\hat{b}} \wedge e^{\hat{c}} \tag{3.978}$$

$$\frac{1}{2}\mathcal{R}^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{f}} \wedge \Gamma^{\hat{f}}{}_{\hat{b}} \tag{3.979}$$

Proof:

First equation

$$
\begin{aligned}
de^{\hat{a}} &= d(e^{\hat{a}}{}_a dx^a) \\
&= \partial_b e^{\hat{a}}{}_b dx^b \wedge dx^a \\
&= -(\partial_b e^{\hat{a}}{}_a - \Gamma^d_{ba} e^{\hat{a}}{}_d) dx^a \wedge dx^b \\
&= -\nabla_b e^{\hat{a}}{}_a dx^a \wedge dx^b \\
&= -(\nabla_b e^{\hat{a}}{}_a) \delta^a_c \delta^b_d \, dx^c \wedge dx^d \\
&= -(\nabla_b e^{\hat{a}}{}_a) e_{\hat{b}}{}^a e_{\hat{c}}{}^b e^{\hat{b}}{}_c e^{\hat{c}}{}_d dx^c \wedge dx^d \\
&= -(\nabla_b e^{\hat{a}}{}_a) e_{\hat{b}}{}^a e_{\hat{c}}{}^b e^{\hat{b}} \wedge e^{\hat{c}} \\
&= \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}}.
\end{aligned} \tag{3.980}
$$

Second equation:

$$
\begin{aligned}
R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}} &= e^{\hat{a}}{}_a e_{\hat{b}}{}^b R^a{}_{b\hat{c}\hat{d}} e^{\hat{c}}{}_c e^{\hat{d}}{}_d dx^c \wedge dx^d \\
&= e^{\hat{a}}{}_a e_{\hat{b}}{}^b R^a{}_{bcd} dx^c \wedge dx^d \\
&= 2 e^{\hat{a}}{}_a e_{\hat{b}}{}^b (\partial_c \Gamma^a_{bd} + \Gamma^e_{bd} \Gamma^a_{ce}) dx^c \wedge dx^d
\end{aligned} \tag{3.981}
$$

Consider $d\Gamma^{\hat{a}}{}_{\hat{b}}$

$$
\begin{aligned}
d\Gamma^{\hat{a}}{}_{\hat{b}} &= d(\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{c}}{}_b dx^b) \\
&= d[(e^{\hat{a}}{}_c e_{\hat{c}}{}^d \nabla_d e_{\hat{b}}{}^c) e^{\hat{c}}{}_b dx^b] \\
&= d(e^{\hat{a}}{}_c \nabla_b e_{\hat{b}}{}^c dx^b) \\
&= d([e^{\hat{a}}{}_c \partial_b e_{\hat{b}}{}^c + e^{\hat{a}}{}_c \Gamma^c_{bd} e_{\hat{b}}{}^d] dx^b) \\
&= [(\partial_a e^{\hat{a}}{}_c)(\partial_b e_{\hat{b}}{}^c) + e^{\hat{a}}{}_c \partial_a \partial_b e_{\hat{b}}{}^c + (\partial_a e^{\hat{a}}{}_c) \Gamma^c_{bd} e_{\hat{b}}{}^d \\
&\quad + e^{\hat{a}}{}_c \Gamma^c_{bd} (\partial_a e_{\hat{b}}{}^d) + e^{\hat{a}}{}_c (\partial_a \Gamma^c_{bd}) e_{\hat{b}}{}^d] dx^a \wedge dx^b.
\end{aligned} \tag{3.982}
$$

The term $e^{\hat{a}}{}_c \partial_a \partial_b e_{\hat{b}}{}^c dx^a \wedge dx^b$ vanishes. Now

$$
\begin{aligned}
\Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} &= (e^{\hat{a}}{}_c \nabla_a e_{\hat{c}}{}^c dx^a) \wedge (e^{\hat{c}}{}_d \nabla_b e_{\hat{b}}{}^d dx^b) \\
&= [e^{\hat{a}}{}_c \partial_a e_{\hat{c}}{}^c + e^{\hat{a}}{}_c \Gamma^c_{ae} e_{\hat{c}}{}^e] e^{\hat{c}}{}_d [\partial_b e_{\hat{b}}{}^d + \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= [-e_{\hat{c}}{}^c (\partial_a e^{\hat{a}}{}_c) + e^{\hat{a}}{}_c \Gamma^c_{ae} e_{\hat{c}}{}^e] e^{\hat{c}}{}_d [\partial_b e_{\hat{b}}{}^d + \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= [-(\partial_a e^{\hat{a}}{}_d) + e^{\hat{a}}{}_c \Gamma^c_{ad}][\partial_b e_{\hat{b}}{}^d + \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= -(\partial_a e^{\hat{a}}{}_c)(\partial_b e_{\hat{b}}{}^c) - (\partial_a e^{\hat{a}}{}_c) \Gamma^c_{bd} e_{\hat{b}}{}^d + e^{\hat{a}}{}_c \Gamma^c_{ad} (\partial_b e_{\hat{b}}{}^d) \\
&\quad + e^{\hat{a}}{}_c \Gamma^c_{ad} \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b
\end{aligned} \tag{3.983}
$$

Combining (3.982) and (3.983)

$$\begin{aligned} d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} &= [e^{\hat{a}}{}_c \Gamma^c_{bd} (\partial_a e_{\hat{b}}{}^d) + e^{\hat{a}}{}_c \Gamma^c_{ad} (\partial_b e_{\hat{b}}{}^d) \\ & \quad e^{\hat{a}}{}_c (\partial_a \Gamma^c_{bd}) e_{\hat{b}}{}^d + e^{\hat{a}}{}_c \Gamma^c_{ad} \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\ &= [e^{\hat{a}}{}_c (\partial_a \Gamma^c_{bd}) e_{\hat{b}}{}^d + e^{\hat{a}}{}_c \Gamma^c_{ad} \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\ &= e^{\hat{a}}{}_a e_{\hat{b}}{}^b (\partial_c \Gamma^a_{db} + \Gamma^a_{ce} \Gamma^e_{db}) dx^c \wedge dx^d \\ &= \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}}. \end{aligned} \tag{3.984}$$

$\square$

Equations (3.978) determine the anti-symmetric part of $\Gamma^{\hat{a}}{}_{[\hat{b}\hat{c}]}$ of the rotation coefficients. Define

$$\Gamma_{\hat{a}\hat{b}\hat{c}} = g_{\hat{a}\hat{d}} \Gamma^{\hat{d}}{}_{\hat{b}\hat{c}}. \tag{3.985}$$

The vanishing of the covariant derivatives of the metric tensor

$$\begin{aligned} 0 &= \nabla_{\hat{c}} g_{\hat{a}\hat{b}} = g_{\hat{a}\hat{b},\hat{c}} - \Gamma^{\hat{d}}_{\hat{a}\hat{c}} g_{\hat{d}\hat{b}} - \Gamma^{\hat{d}}_{\hat{b}\hat{c}} g_{\hat{a}\hat{d}} \\ &= g_{\hat{a}\hat{b},\hat{c}} - \Gamma_{\hat{b}\hat{a}\hat{c}} - \Gamma_{\hat{a}\hat{b}\hat{c}} \end{aligned}$$

implies the symmetric part $\Gamma_{(\hat{a}\hat{b})\hat{c}}$ of the rotation coefficients

$$\Gamma_{(\hat{a}\hat{b})\hat{c}} = \frac{1}{2} \partial_{\hat{c}} g_{\hat{a}\hat{b}}. \tag{3.986}$$

The expressions for $\Gamma^{\hat{a}}{}_{[\hat{b}\hat{c}]}$ and $\Gamma_{(\hat{a}\hat{b})\hat{c}}$ determine all rotation coefficients, and (3.979) then determine all the components of the curvature tensor.

Specialisation to rigid tetrads

When we limit ourselves to the case of rigid tetrads where the $g_{\hat{a}\hat{b}}$ are constants. The rotation coefficients are the anti-symmetric in the first two indicies

$$\Gamma_{\hat{a}\hat{b}\hat{c}} = -\Gamma_{\hat{b}\hat{a}\hat{c}},$$

and are determined by (3.978)

$$
\begin{aligned}
\Gamma_{\hat{a}\hat{b}\hat{c}} &= \frac{1}{2}(\Gamma_{\hat{a}\hat{b}\hat{c}}+\Gamma_{\hat{c}\hat{a}\hat{b}})+\frac{1}{2}(\Gamma_{\hat{b}\hat{c}\hat{a}}+\Gamma_{\hat{a}\hat{b}\hat{c}})-\frac{1}{2}(\Gamma_{\hat{c}\hat{a}\hat{b}}+\Gamma_{\hat{b}\hat{c}\hat{a}}) \\
&= \frac{1}{2}(\Gamma_{\hat{a}\hat{b}\hat{c}}-\Gamma_{\hat{a}\hat{c}\hat{b}})+\frac{1}{2}(\Gamma_{\hat{b}\hat{c}\hat{a}}-\Gamma_{\hat{b}\hat{a}\hat{c}})-\frac{1}{2}(\Gamma_{\hat{c}\hat{a}\hat{b}}-\Gamma_{\hat{c}\hat{b}\hat{a}}) \\
&= \Gamma_{\hat{a}[\hat{b}\hat{c}]}+\Gamma_{\hat{b}[\hat{c}\hat{a}]}-\Gamma_{\hat{c}[\hat{a}\hat{b}]} \qquad (3.987)
\end{aligned}
$$

The curvature one forms $\Gamma_{\hat{a}\hat{b}}$ satisfy

$$\Gamma_{\hat{a}\hat{b}} = -\Gamma_{\hat{b}\hat{a}} \qquad (3.988)$$

and therefore has six independent components. They are obtained from the first Cartan structure equation (3.978),

$$de^{\hat{a}} = \Gamma^{\hat{a}}{}_{\hat{b}} \wedge e^{\hat{b}}.$$

Once we have calculated the curvature one forms, we obtain the Ricci rotation coefficients using

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{c}}. \qquad (3.989)$$

We calculate the curvature two forms $\mathcal{R}^{\hat{a}}{}_{\hat{b}}$ using the second Cartan structure equation (3.979),

$$\frac{1}{2}\mathcal{R}^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{f}} \wedge \Gamma^{\hat{f}}{}_{\hat{b}}$$

which are related to the Riemann tensor as

$$\mathcal{R}^{\hat{a}}{}_{\hat{b}} = R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}} \qquad (3.990)$$

After calculating the Riemann tensor we obtain the Ricci tensor by

$$R_{\hat{a}\hat{b}} = R_{\hat{b}\hat{a}} = R^{\hat{c}}{}_{\hat{a}\hat{b}\hat{c}} = g^{\hat{c}\hat{d}} R_{\hat{c}\hat{a}\hat{b}\hat{d}}. \qquad (3.991)$$

There are 10 independent components. We then calculate the Rici scalar by

$$R = g^{\hat{a}\hat{b}} R_{\hat{a}\hat{b}}. \qquad (3.992)$$

3.17.16 Specialisation to Null Tetrads

$$e_{\hat{0}} = l, \quad e_{\hat{1}} = n, \quad e_{\hat{2}} = m, \quad e_{\hat{3}} = \overline{m}, \tag{3.993}$$

$$g_{\hat{a}\hat{b}} = g^{\hat{a}\hat{b}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.994}$$

$$e^{\hat{a}} = g^{\hat{a}\hat{b}} e_{\hat{b}} \tag{3.995}$$

$$e^{\hat{0}} = e_{\hat{1}} = n, \quad e^{\hat{1}} = e_{\hat{0}} = l, \quad e^{\hat{2}} = -e_{\hat{3}} = -\overline{m}, \quad e^{\hat{3}} = -e_{\hat{2}} = -m, \tag{3.996}$$

Independent curvature one forms

There are six independent curvature one forms $\Gamma_{\hat{a}\hat{b}}$ as it is anti-symmetric. The non-zero curvature one forms are

$$\Gamma_{\hat{0}\hat{1}}, \Gamma_{\hat{0}\hat{2}}, \Gamma_{\hat{0}\hat{3}}, \Gamma_{\hat{1}\hat{2}}, \Gamma_{\hat{1}\hat{3}}, \Gamma_{\hat{2}\hat{3}} \tag{3.997}$$

However

$$\Gamma_{\hat{0}\hat{1}} = \overline{\Gamma_{\hat{0}\hat{1}}}, \quad \Gamma_{\hat{0}\hat{2}} = \overline{\Gamma_{\hat{0}\hat{3}}}, \quad \Gamma_{\hat{1}\hat{3}} = \overline{\Gamma_{\hat{1}\hat{2}}}, \quad \Gamma_{\hat{2}\hat{3}} = -\overline{\Gamma_{\hat{2}\hat{3}}}.$$

The collection (3.997) is equivalent to the collection

$$\Gamma_{\hat{1}\hat{0}} + \Gamma_{\hat{2}\hat{3}}, \Gamma_{\hat{0}\hat{2}}, \Gamma_{\hat{0}\hat{3}}, \Gamma_{\hat{1}\hat{2}}, \Gamma_{\hat{1}\hat{3}}, \Gamma_{\hat{1}\hat{0}} - \Gamma_{\hat{2}\hat{3}} \tag{3.998}$$

This collection is made up of the one forms

$$\Gamma_{\hat{0}\hat{3}}, \quad \Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}, \quad \Gamma_{\hat{1}\hat{2}} \tag{3.999}$$

and their complex cojugates. The colection (3.999) are taken as our six independent curvature one forms.

Independent rotation coefficients

Recall the Ricci rotation coefficients are given by

$$\Gamma_{\hat{a}\hat{b}\hat{c}} = -e_{\hat{a}a;b} e_{\hat{b}}{}^{a} e_{\hat{c}}{}^{b}$$

$$\begin{aligned}
\kappa &= \Gamma_{\hat{2}\hat{0}\hat{0}} & \epsilon &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{0}} + \Gamma_{\hat{2}\hat{3}\hat{0}}) & \pi &= \Gamma_{\hat{1}\hat{3}\hat{0}} \\
\sigma &= \Gamma_{\hat{2}\hat{0}\hat{2}} & \beta &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{2}} + \Gamma_{\hat{2}\hat{3}\hat{2}}) & \mu &= \Gamma_{\hat{1}\hat{3}\hat{2}} \\
\rho &= \Gamma_{\hat{2}\hat{0}\hat{3}} & \alpha &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{3}} + \Gamma_{\hat{2}\hat{3}\hat{3}}) & \lambda &= \Gamma_{\hat{1}\hat{3}\hat{3}} \\
\tau &= \Gamma_{\hat{2}\hat{0}\hat{1}} & \gamma &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{1}} + \Gamma_{\hat{2}\hat{3}\hat{1}}) & \nu &= \Gamma_{\hat{1}\hat{3}\hat{3}}
\end{aligned} \tag{3.1000}$$

Independent curvature two forms

Before we write down the first Cartan structure equations for these independent one forms we need the following:

$$\begin{aligned}
\Gamma_{\hat{0}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{3}} &= g^{\hat{c}\hat{d}} \Gamma_{\hat{0}\hat{c}} \wedge \Gamma_{\hat{d}\hat{3}} \\
&= g^{\hat{0}\hat{1}} \Gamma_{\hat{0}\hat{0}} \wedge \Gamma_{\hat{1}\hat{3}} + g^{\hat{1}\hat{0}} \Gamma_{\hat{0}\hat{1}} \wedge \Gamma_{\hat{0}\hat{3}} + g^{\hat{2}\hat{3}} \Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{3}\hat{3}} + g^{\hat{3}\hat{2}} \Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{2}\hat{3}} \\
&= \Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{1}\hat{0}} - \Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{2}\hat{3}}
\end{aligned} \tag{3.1001}$$

$$\begin{aligned}
\Gamma_{\hat{2}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{3}} &= g^{\hat{c}\hat{d}} \Gamma_{\hat{2}\hat{c}} \wedge \Gamma_{\hat{d}\hat{3}} \\
&= g^{\hat{0}\hat{1}} \Gamma_{\hat{2}\hat{0}} \wedge \Gamma_{\hat{1}\hat{3}} + g^{\hat{1}\hat{d}} \Gamma_{\hat{2}\hat{1}} \wedge \Gamma_{\hat{0}\hat{3}} + g^{\hat{2}\hat{3}} \Gamma_{\hat{2}\hat{2}} \wedge \Gamma_{\hat{3}\hat{3}} + g^{\hat{3}\hat{2}} \Gamma_{\hat{2}\hat{3}} \wedge \Gamma_{\hat{2}\hat{3}} \\
&= \Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{1}\hat{2}} - \Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{1}\hat{3}}
\end{aligned} \tag{3.1002}$$

$$\begin{aligned}
\Gamma_{\hat{1}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{0}} &= g^{\hat{c}\hat{d}} \Gamma_{\hat{1}\hat{c}} \wedge \Gamma_{\hat{d}\hat{0}} \\
&= g^{\hat{0}\hat{1}} \Gamma_{\hat{1}\hat{0}} \wedge \Gamma_{\hat{1}\hat{0}} + g^{\hat{1}\hat{0}} \Gamma_{\hat{1}\hat{1}} \wedge \Gamma_{\hat{0}\hat{0}} + g^{\hat{2}\hat{3}} \Gamma_{\hat{1}\hat{2}} \wedge \Gamma_{\hat{3}\hat{0}} + g^{\hat{3}\hat{2}} \Gamma_{\hat{1}\hat{3}} \wedge \Gamma_{\hat{2}\hat{0}} \\
&= -\Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{1}\hat{2}} - \Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{1}\hat{3}}
\end{aligned} \tag{3.1003}$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{2}} &= g^{\hat{c}\hat{d}}\Gamma_{\hat{1}\hat{c}} \wedge \Gamma_{\hat{d}\hat{2}} \\
&= g^{\hat{0}\hat{1}}\Gamma_{\hat{1}\hat{0}} \wedge \Gamma_{\hat{1}\hat{2}} + g^{\hat{1}\hat{0}}\Gamma_{\hat{1}\hat{1}} \wedge \Gamma_{\hat{0}\hat{2}} + g^{\hat{2}\hat{3}}\Gamma_{\hat{1}\hat{2}} \wedge \Gamma_{\hat{3}\hat{2}} + g^{\hat{3}\hat{2}}\Gamma_{\hat{1}\hat{3}} \wedge \Gamma_{\hat{2}\hat{2}} \\
&= \Gamma_{\hat{1}\hat{0}} \wedge \Gamma_{\hat{1}\hat{2}} - \Gamma_{\hat{2}\hat{3}} \wedge \Gamma_{\hat{1}\hat{2}} \qquad (3.1004)
\end{aligned}
$$

The first Cartan structure equations are then

$$
\begin{aligned}
d\Gamma_{\hat{0}\hat{3}} + \Gamma_{\hat{0}\hat{3}} \wedge (-\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) &= \frac{1}{2}R_{\hat{0}\hat{3}\hat{a}\hat{b}}e^{\hat{a}} \wedge e^{\hat{b}} \qquad (3.1005) \\
d(\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) - 2\Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{1}\hat{3}} &= \frac{1}{2}(R_{\hat{2}\hat{3}\hat{a}\hat{b}} + R_{\hat{1}\hat{0}\hat{a}\hat{b}})e^{\hat{a}} \wedge e^{\hat{b}} \qquad (3.1006) \\
d\Gamma_{\hat{1}\hat{2}} + (-\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) \wedge \Gamma_{\hat{1}\hat{2}} &= \frac{1}{2}R_{\hat{1}\hat{2}\hat{a}\hat{b}}e^{\hat{a}} \wedge e^{\hat{b}} \qquad (3.1007)
\end{aligned}
$$

Independent components of the Ricci tensor

$$
R_{\hat{a}\hat{b}} = R_{\hat{b}\hat{a}} = R^{\hat{c}}{}_{\hat{a}\hat{b}\hat{c}} = g^{\hat{c}\hat{d}}R_{\hat{c}\hat{a}\hat{b}\hat{d}} \qquad (3.1008)
$$

$$
\begin{aligned}
R_{\hat{3}\hat{3}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{3}\hat{3}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{3}\hat{3}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{3}\hat{3}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{3}\hat{3}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{3}\hat{3}\hat{2}} \\
&= 2R_{\hat{0}\hat{3}\hat{3}\hat{1}}. \qquad (3.1009)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{3}\hat{0}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{3}\hat{0}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{3}\hat{0}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{3}\hat{0}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{3}\hat{0}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{3}\hat{0}\hat{2}} \\
&= -R_{\hat{0}\hat{3}\hat{2}\hat{3}} - R_{\hat{0}\hat{3}\hat{1}\hat{0}}. \qquad (3.1010)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{0}\hat{0}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{0}\hat{0}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{0}\hat{0}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{0}\hat{0}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{0}\hat{0}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{0}\hat{0}\hat{2}} \\
&= -2R_{\hat{0}\hat{3}\hat{2}\hat{0}}. \qquad (3.1011)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{2}\hat{3}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{2}\hat{3}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{2}\hat{3}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{2}\hat{3}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{2}\hat{3}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{2}\hat{3}\hat{2}} \\
&= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{2}\hat{3}\hat{0}} + R_{\hat{0}\hat{2}\hat{3}\hat{1}} \\
&= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{2}\hat{3}\hat{0}} - (R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{0}\hat{1}\hat{2}\hat{3}}) \\
&= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}}. \qquad (3.1012)
\end{aligned}
$$

where we used the cyclic identity $R_{\hat{0}\hat{2}\hat{3}\hat{1}} + R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{0}\hat{1}\hat{2}\hat{3}} = 0$.

$$
\begin{aligned}
R_{\hat{1}\hat{0}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{1}\hat{0}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{1}\hat{0}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{1}\hat{0}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{1}\hat{0}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{1}\hat{0}\hat{2}} \\
&= +R_{\hat{1}\hat{0}\hat{1}\hat{0}} - R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{0}\hat{2}\hat{3}\hat{1}} \\
&= +R_{\hat{1}\hat{0}\hat{1}\hat{0}} - R_{\hat{0}\hat{3}\hat{1}\hat{2}} - (R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{2}\hat{3}\hat{0}\hat{1}}) \\
&= R_{\hat{2}\hat{3}\hat{0}\hat{1}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}} \qquad (3.1013)
\end{aligned}
$$

where we used the same cyclic identity again.

$$
\begin{aligned}
R_{\hat{1}\hat{1}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{1}\hat{1}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{1}\hat{1}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{1}\hat{1}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{1}\hat{1}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{1}\hat{1}\hat{2}} \\
&= 2R_{\hat{1}\hat{2}\hat{1}\hat{3}}. \qquad (3.1014)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{3}\hat{1}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{3}\hat{1}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{3}\hat{1}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{3}\hat{1}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{3}\hat{1}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{3}\hat{1}\hat{2}} \\
&= -R_{\hat{2}\hat{3}\hat{1}\hat{3}} + R_{\hat{1}\hat{0}\hat{1}\hat{3}}. \qquad (3.1015)
\end{aligned}
$$

Altogether

$$
\begin{aligned}
R_{\hat{2}\hat{2}} &= 2R_{\hat{0}\hat{2}\hat{2}\hat{1}}, && (3.1016) \\
R_{\hat{3}\hat{0}} &= -R_{\hat{0}\hat{3}\hat{2}\hat{3}} - R_{\hat{0}\hat{3}\hat{1}\hat{0}}, && (3.1017) \\
R_{\hat{0}\hat{0}} &= -2R_{\hat{0}\hat{3}\hat{2}\hat{0}}, && (3.1018) \\
& && (3.1019) \\
R_{\hat{2}\hat{3}} &= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}}, && (3.1020) \\
R_{\hat{1}\hat{0}} &= R_{\hat{2}\hat{3}\hat{0}\hat{1}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}}, && (3.1021) \\
R_{\hat{1}\hat{1}} &= 2R_{\hat{1}\hat{2}\hat{1}\hat{3}}, && (3.1022) \\
R_{\hat{3}\hat{1}} &= -R_{\hat{2}\hat{3}\hat{1}\hat{3}} + R_{\hat{1}\hat{0}\hat{1}\hat{3}} && (3.1023)
\end{aligned}
$$

where $R_{\hat{0}\hat{0}}$, $R_{\hat{2}\hat{3}}$, $R_{\hat{1}\hat{0}}$, $R_{\hat{1}\hat{1}}$ are real and $R_{\hat{2}\hat{2}}$, $R_{\hat{2}\hat{0}}$, $R_{\hat{2}\hat{1}}$ are complex conjugates of $R_{\hat{3}\hat{3}}$, $R_{\hat{3}\hat{0}}$, $R_{\hat{3}\hat{1}}$, respectively:

$$\begin{aligned} R_{\hat{3}\hat{3}} &= \overline{R_{\hat{2}\hat{2}}} \\ R_{\hat{2}\hat{0}} &= \overline{R_{\hat{3}\hat{0}}} \\ R_{\hat{2}\hat{1}} &= \overline{R_{\hat{3}\hat{1}}} \end{aligned} \tag{3.1024}$$

We have 10 independent terms altogether.

The Ricci scalar is given by

$$\begin{aligned} R &= g^{\hat{a}\hat{b}} R_{\hat{a}\hat{b}} \\ &= g^{\hat{0}\hat{1}} R_{\hat{0}\hat{1}} + g^{\hat{1}\hat{0}} R_{\hat{1}\hat{0}} + g^{\hat{2}\hat{3}} R_{\hat{2}\hat{3}} + g^{\hat{3}\hat{2}} R_{\hat{3}\hat{2}} \\ &= 2(R_{\hat{1}\hat{0}} - R_{\hat{2}\hat{3}}). \end{aligned} \tag{3.1025}$$

The Weyl tensor and Petrov classification

$$\begin{aligned} \Psi_0 &= C_{abcd} l^a m^b l^c m^d \\ &= [R_{abcd} - \frac{1}{2}(R_{ac}g_{db} - R_{ad}g_{bc} - R_{bc}g_{ad} + R_{bd}g_{ac}) + \frac{R}{6}(g_{ac}g_{db} - g_{ad}gcb)] l^a m^b l^c m^d \\ &= R_{abcd} e_{\hat{0}}{}^a e_{\hat{2}}{}^b e_{\hat{0}}{}^c e_{\hat{2}}{}^d \\ &= R_{\hat{0}\hat{2}\hat{0}\hat{2}}. \end{aligned} \tag{3.1026}$$

$$\begin{aligned} \Psi_1 &= C_{abcd} l^a m^b l^c n^d \\ &= [R_{abcd} - \frac{1}{2}(R_{ac}g_{db} - R_{ad}g_{bc} - R_{bc}g_{ad} + R_{bd}g_{ac}) + \frac{R}{6}(g_{ac}g_{db} - g_{ad}g_{cb})] l^a m^b l^c n^d \\ &= R_{abcd} l^a m^b l^c n^d - \frac{1}{2}(-R_{bc} m^b l^c) \\ &= R_{abcd} e_{\hat{0}}{}^a e_{\hat{2}}{}^b e_{\hat{0}}{}^c e_{\hat{1}}{}^d + \frac{1}{2} R_{bc} m^b l^c \\ &= R_{\hat{0}\hat{2}\hat{0}\hat{1}} + \frac{1}{2} R_{\hat{2}\hat{0}}. \end{aligned} \tag{3.1027}$$

$$
\begin{aligned}
\Psi_2 &= C_{abcd} l^a m^b \overline{m}^c n^d \\
&= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} g_{cb})] l^a m^b l^c n^d \\
&= R_{abcd} l^a m^b l^c n^d - \frac{1}{2}(-R_{bc} m^b l^c) \\
&= R_{abcd} e_{\hat{0}}{}^a e_{\hat{2}}{}^b e_{\hat{3}}{}^c e_{\hat{1}}{}^d + \frac{1}{2} R_{bc} e_{\hat{2}}{}^b e_{\hat{3}}{}^c \\
&= R_{\hat{0}\hat{2}\hat{3}\hat{1}} + \frac{1}{2} R_{\hat{2}\hat{3}}.
\end{aligned} \tag{3.1028}
$$

$$
\begin{aligned}
\Psi_3 &= C_{abcd} l^a n^b \overline{m}^c n^d \\
&= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} g_{cb})] l^a n^b \overline{m}^c n^d \\
&= R_{abcd} l^a n^b \overline{m}^c n^d - \frac{1}{2}(-R_{bc} n^b \overline{m}^c) \\
&= R_{abcd} e_{\hat{0}}{}^a e_{\hat{1}}{}^b e_{\hat{3}}{}^c e_{\hat{1}}{}^d + \frac{1}{2} R_{bc} e_{\hat{1}}{}^b e_{\hat{3}}{}^c \\
&= R_{\hat{0}\hat{2}\hat{0}\hat{1}} + \frac{1}{2} R_{\hat{1}\hat{3}}.
\end{aligned} \tag{3.1029}
$$

$$
\begin{aligned}
\Psi_4 &= C_{abcd} \overline{m}^a n^b \overline{m}^c n^d \\
&= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} gcb)] \overline{m}^a n^b \overline{m}^c n^d \\
&= R_{abcd} \overline{m}^a n^b \overline{m}^c n^d \\
&= R_{abcd} e_{\hat{3}}{}^a e_{\hat{1}}{}^b e_{\hat{3}}{}^c e_{\hat{1}}{}^d \\
&= R_{\hat{3}\hat{1}\hat{3}\hat{1}}.
\end{aligned} \tag{3.1030}
$$

3.18 Summary

3.19 Biblioliographical notes

In this chapter I have relied on the following refferences: The review article by Abhay Ashtekar and Badri Krishnan, *Isolated and Dynamical Horizons and Their Applications*; the thesis of Badri Krishnan, *Isolated Horizons in Numerical Relativity*; the book *A Relativist's Toolkit* by Eric Poisson.

3.20 Worked Exercises and Details

Event Horizons

Proposition: If $\eta \in T_\perp$ initially, it remains in this subspace.

Proof:

$$\begin{aligned} l^a \nabla_a \eta^b &= l^a \nabla_a (P^b{}_c \eta^c) \\ &= P^b{}_c l^a \nabla_a \eta^c \quad \text{(since } l^a\nabla_a n^b = 0 \text{ and } l^a \nabla_a l^b = 0) \\ &= P^b{}_c B^c{}_d \eta^d \\ &= P^a{}_b B^b{}_c P^c{}_d \eta^d \\ &= \hat{B}^a{}_b \eta^b. \end{aligned} \tag{3.1031}$$

or

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \tag{3.1032}$$

Raycherdhuri's equation for null geodesic congruences.

In the following we will use:

$$l^c \nabla_c P^a{}_b = l^c \nabla_c (\delta^a{}_b + n^a l_b + l^a n_b) = 0 \tag{3.1033}$$

$l^c \nabla_c n^a = 0$ and $l^c \nabla_c l^a = 0$.

As well as

$$P^b{}_c R_{ab}{}^c{}_d = \tag{3.1034}$$

$$\begin{aligned} \frac{d\theta}{d\lambda} &= l^a \nabla_a (B^b{}_c P^c{}_b) \\ &= P^c{}_b l^a \nabla_a B^b{}_c \quad \text{(since } l^a\nabla_a n^b = 0 \text{ and } l^a \nabla_a l^b = 0) \\ &= \end{aligned} \tag{3.1035}$$

$$
\begin{aligned}
\frac{d\theta}{d\lambda} &= \nabla_l (B^a{}_b P^b{}_a) \\
&= P^b{}_a \nabla_l B^a{}_b \\
&= P^b{}_a l^d \nabla_d \nabla_b l^a \\
&= P^b{}_a l^d \nabla_b \nabla_d l^a + P^b{}_a l^d [\nabla_d, \nabla_b] l^a \\
&= P^b{}_a \left[\nabla_b (\underbrace{l^d \nabla_d l^a}_{=0}) - (\nabla_b l^d)(\nabla_d l^a) \right] + P^b{}_a R_{db}{}^a{}_e l^e
\end{aligned}
\tag{3.1036}
$$

We'll take a breather for second . $B^a{}_d = \nabla_d l^a$

$$
\begin{aligned}
\frac{d\theta}{d\lambda} &= -P^b{}_a B^a{}_d B^d{}_b - l^d R_{de} l^e \\
&= -P^b{}_a B^a{}_d \delta^d{}_e B^e{}_b - l^d l^e R_{de} \ \text{ (inserting a) } \delta \\
&= -P^b{}_a B^a{}_d (P^d{}_e - l^d n_e - n^d l_e) B^e{}_b - l^d l^e R_{de} \\
&= -P^b{}_a B^a{}_d P^d{}_e B^e{}_b + P^b{}_a B^a{}_d l^d n_e B^e{}_b + P^b{}_a B^a{}_d n^d l_e) B^e{}_b - l^d l^e R_{de} B^e{}_b \\
&= \hat{B}^a{}_d \hat{B}^d{}_b - l^d l^e R_{de}
\end{aligned}
\tag{3.1037}
$$

$$
\xi^c \nabla_c \theta = \frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - l^a l^b R_{ab} \tag{3.1038}
$$

$$
\frac{d\mathcal{A}}{d\lambda} = \theta \mathcal{A} \tag{3.1039}
$$

$$
\begin{aligned}
\frac{d^2 \mathcal{A}^{1/2}}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{d\mathcal{A}^{1/2}}{d\lambda}\right) &= \frac{d}{d\lambda}\left(\frac{1}{2}\mathcal{A}^{-1/2}\frac{d\mathcal{A}}{d\lambda}\right) = \frac{1}{2}\frac{d}{d\lambda}\left(\mathcal{A}^{1/2}\theta\right) \\
&= \frac{1}{2}\mathcal{A}^{1/2}\frac{d\theta}{d\lambda} + \frac{1}{4}\theta^2 \mathcal{A}^{1/2} \\
&= \frac{\mathcal{A}^{1/2}}{2}\left[(-\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - l^a l^b R_{ab}) + \frac{\theta^2}{2}\right] \\
&= -\frac{1}{2}(|\sigma|^2 - |\omega|^2 + \frac{1}{2} l^a l^b R_{ab})\mathcal{A}^{1/2}
\end{aligned}
\tag{3.1040}
$$

$$
\frac{d^2 \mathcal{A}^{1/2}}{d\lambda^2} = -\frac{1}{2}(|\sigma|^2 - |\omega|^2 + \frac{1}{2} l^a l^b R_{ab})\mathcal{A}^{1/2} \tag{3.1041}
$$

Surface gravity of event horizons.

(1) Obtain

$$\kappa^2 = -\frac{1}{2}(\nabla^a \chi^b)(\nabla_a \chi_b). \tag{3.1042}$$

(2) Show that κ is constant along the null generators of the event horizon.

(3) Show that κ is constant transversal to the horizon, i.e. that κ does not change from one null generator to another.

Proof:

(1)

$$\begin{aligned} \chi_{[a}\nabla_b\chi_{c]} &= \frac{1}{3!}(\chi_a\nabla_b\chi_c + \chi_b\nabla_c\chi_a + \chi_c\nabla_a\chi_b \\ &- \ \chi_a\nabla_c\chi_b - \chi_b\nabla_a\chi_c - \chi_c\nabla_b\chi_a). \end{aligned} \tag{3.1043}$$

Using Killing's equation $\nabla_{(a}\chi_{b)} = 0$

$$\chi_{[a}\nabla_b\chi_{c]} = \frac{2}{3!}(\chi_a\nabla_b\chi_c - \chi_b\nabla_a\chi_c + \chi_c\nabla_a\chi_b) \tag{3.1044}$$

Setting $\chi_{[a}\nabla_b\chi_{c]} = 0$

$$\chi_c\nabla_a\chi_b = -2\chi_{[a}\nabla_{b]}\chi_c \tag{3.1045}$$

contract (3.1045) with $\nabla^a\chi^b$

$$\begin{aligned} \chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) &= -2(\nabla^a\chi^b)\chi_{[a}\nabla_{b]}\chi_c \quad \text{as } \nabla^a\chi^b = -\nabla^b\chi^a \text{ we can omit the anticommutator} \\ &= -2(\nabla^a\chi^b)(\chi_a\nabla_b\chi_c) \\ &= -2\kappa\chi_b(\nabla^a\chi^b) \quad \text{using } \chi_a\nabla^a\chi^b = -\kappa\chi^b \\ &= -2\kappa^2\chi_c. \end{aligned} \tag{3.1046}$$

(2)

$$\nabla_b\nabla_c\xi_a = R_{abcd}\xi^d \tag{3.1047}$$

We differentiate (3.1042)

$$2\kappa\partial_c\kappa = -\nabla^a\xi^b\nabla_c\nabla_d\xi_a = -\nabla^c\xi^a \tag{3.1048}$$

$$2\kappa\partial_a\kappa = -\nabla^c\xi^d R_{cdab}\xi^b \tag{3.1049}$$

$$2\kappa\xi^a\partial_a\kappa = -\nabla^c\xi^d R_{cdab}\xi^a\xi^b = 0 \tag{3.1050}$$

Examples of normed spaces.

3.20.1 Non-Expanding Horizons

Non-Expanding Horizons.

a)

The second equation in (4.3.4) implies that the vector $-R^a{}_b\ell_b$ is tangential to Δ. The energy condition and the field equations imply this vector must also be future causal. Use these facts to prove that in the Newman-Penrose formalism we have:

$$\Phi_{00} \stackrel{\triangle}{=} 0 \ \text{ and } \ \Phi_{01} = \overline{\Phi}_{10} \stackrel{\triangle}{=} 0 \tag{3.1051}$$

Why do these equations not depend upon the specific choice of null normal ℓ and m?

b)

$$\Psi_0 \stackrel{\triangle}{=} 0 \ \text{ and } \ \Psi_1 \stackrel{\triangle}{=} 0 \tag{3.1052}$$

c)

We derive the important relation between the one-form ω_a the imaginary part Ψ_2

$$\mathcal{D}_{[a}\omega_{b]} \stackrel{\triangle}{=} 2(Im[\Psi_2])\epsilon_{ab} \tag{3.1053}$$

To show this, contract (4.3.4) with n and use $\ell^a n_a = -1$

Answers:

a)

This means that $R^a{}_b$ must be proportional to ℓ^a and hence, $R_{\underleftarrow{a}\,b}\ell^b$

b)

It follows from $R^a{}_b \ell_b \triangleq 0$ and the conformal tensor (C.605)

$$[\mathcal{D}_a \mathcal{D}_b - \mathcal{D}_b \mathcal{D}_a]\ell^c \triangleq -2R_{\underleftarrow{abd}}{}^c \ell^d \triangleq -2C_{\underleftarrow{abd}}{}^c \ell^d \tag{3.1054}$$

Thus, if v is any 1-form on Δ satisfying $v\dot{\ell} \triangleq 0$ Δ‘ , 0, contracting the previous equation with v_c we get

$$C_{\underleftarrow{abd}}{}^c v_c \ell^d \triangleq 0 \tag{3.1055}$$

c)

$$2\mathcal{D}_{[a}\omega_{b]} \triangleq C_{abcd}\ell^c n^d \tag{3.1056}$$

$$2\mathcal{D}_{[a}\omega_{b]} \triangleq 4(\Re[\Psi_2])n_{[a}\ell_{b]} + 2\Psi_3 \ell_{[a}m_{b]} + 2\overline{\Psi}_3 \ell_{[a}\overline{m}_{b]} + 4i(\Im[\Psi_2])m_{[a}\overline{m}_{b]} \tag{3.1057}$$

$$\begin{aligned} q_a{}^c q_b{}^d \mathcal{D}_{[a}\omega_{b]} &\triangleq 2i(Im[\Psi_2])q_a{}^c q_b{}^d m_{[c}\overline{m}_{d]} \\ &\triangleq i(Im[\Psi_2])(q_a{}^c -)(q_b{}^d -)2m_{[c}\overline{m}_{d]} \\ &\triangleq i(Im[\Psi_2])(q_a{}^c -)(q_b{}^d -)(e_1 + ie_2)_{[a}(e_1 - ie_2)_{b]} \\ &\triangleq 2i(Im[\Psi_2])\epsilon_{ab} \end{aligned} \tag{3.1058}$$

we obtain???

$$d\omega \triangleq 2(Im[\Psi_2])\epsilon \tag{3.1059}$$

Examples of normed spaces.

Non-Expanding Horizons.

$$\Psi_2 \to \Psi_2 + 2c\Psi_1 + c^2\Psi_0. \tag{3.1060}$$

Examples of normed spaces.

3.20.2 Weakley Isolated Horizons

Non-Expanding Horizons.

Examples of normed spaces.

3.20.3 Isolated Horizons

Dynamical Horizons.

De-Sitter cosmological horizon is an isolated horizon.

Examples of normed spaces.

De-Sitter cosmological horizon as an isolated horizon.

De-Sitter cosmological horizon is an isolated horizon.

Examples of normed spaces.

Details: Consequences of the boundary conditions:

(a) The condition $o^A D_a o_A \hat{=} 0$ implies

$$\boxed{\nabla_a l_b \hat{=} -2U_a l_b,} \tag{3.1061}$$

where, as in the main text, $U_a \hat{=} \Re\alpha_a$

Proof:

$$D_a o_A = -\alpha_a o_A - \beta_a \imath_A \tag{3.1062}$$

contarcting with o^A this is

$$o^A D_a o_A = -\alpha_a (o^A o_A) - \beta_a (o^A \imath_A) = -\beta_a \tag{3.1063}$$

Therefore $o^A D_a o_A \hat{=} 0$ implies,

$$D_a o_A \hat{=} -\alpha_a o_A. \tag{3.1064}$$

Using $l_b = i\sigma_{bAA'} o^A \overline{o}^{A'}$ and compatibility condition of σ i.e. $\nabla_a \sigma_{bAA'} = 0$,

$$\begin{aligned}
\nabla_a l_b &= i\sigma_{bAA'}\nabla_a(o^A\overline{o}^{A'}) \\
&= i\sigma_{bAA'}[o^A\nabla_a\overline{o}^{A'} + \overline{o}^{A'}\nabla_a o^A] \\
&\hat{=} -i\sigma_{bAA'}[o^A\alpha^*_a + \overline{o}^{A'}\alpha_a o^A] \\
&\hat{=} -(i\sigma_{bAA'}o^A\overline{o}^{A'})(\alpha^*_a + \alpha_a) \\
&\hat{=} -2U_a l_b.
\end{aligned} \tag{3.1065}$$

Equation (4.3.4) implies that l is a geodesic vector field,

$$\boxed{l^a\nabla_a l_b \hat{=} -2(l^a U_a)l_b} \tag{3.1066}$$

and that the Lie derviative with respect to l of the metric induced on Δ vanishes:

$$\mathcal{L}(g_{ac}(g^{cb} + l^a n^b + n^a l^b)) \hat{=} 2\nabla_{(a}l_{b)} = 0. \tag{3.1067}$$

It is twist free as,

$$\boxed{l_{[a}\nabla_b l_{c]} \hat{=} -2l_{[a}U_b l_{c]} = 0, \quad \Rightarrow \hat{\omega} = 0.} \tag{3.1068}$$

so that the congruence with tangent vector l^a is both shear-free and geodesic, (although it need not be affinely parametrized). Furthermore, since $T^{ab}l_a l_b \geq 0$ by condition, and Einstein's equation holds at Δ, we can use Raychaudhuri's equation

$$\mathcal{L}_t\theta_{(l)} = -\frac{1}{2}\theta^2_{(l)} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}l^a l^b \tag{3.1069}$$

to conclude that the shear must vanish if the expansion $\theta_{(l)}$ vanishes.

Finally, note that the Raychaudhuri equation also implies that

$$R_{ab}l^a l^b \hat{=} 8\pi G T_{ab}l^a l^g \hat{=} 0 \tag{3.1070}$$

i.e., that there is no flux of matter energy-momentum across the horizon.

$$\mathcal{D}_a\imath_A \hat{=} \gamma_a\imath_A + if(v)\overline{m}_a o_A \tag{3.1071}$$

$$\mathcal{D}_a(\imath_A o^A) = \mathcal{D}_a(1) = 0, \quad \Rightarrow \imath^A\mathcal{D}_a o_A = -o^A\mathcal{D}_a\imath_A \tag{3.1072}$$

$$
\begin{aligned}
\nabla_a n_b &= \nabla_a(\imath^A \bar{\imath}^{A'}) = \imath^A \nabla_a \bar{\imath}^{A'} + \bar{\imath}^{A'} \nabla_a \imath^A \\
&\hat{=} \imath^A(\alpha_a^* \bar{\imath}^{A'} - if(v) m_a \bar{o}^{A'}) + \bar{\imath}^{A'}(\alpha_a \imath^A + if(v)\overline{m}_a o^A) \\
&\hat{=} (\alpha_a^* + \alpha_a)\bar{\imath}^{A'}\imath^A + if(v)[\overline{m}_a o^A \bar{\imath}^{A'} - m_a \imath^A \bar{o}^{A'}] \\
&\hat{=} 2U_a n_b - 2f(v)\overline{m}_{(a} m_{b)} \qquad (3.1073)
\end{aligned}
$$

$$\boxed{\nabla_a n_b \hat{=} 2U_a n_b - 2f(v)\overline{m}_{(a} m_{b)} \qquad (3.1074)}$$

n_a is twist-free

n_a shear-free

has spherically symetric expansion, $2f$, and vanishing Newman-Penrose coefficient $\pi \hat{=} \overline{m}^a l^b \nabla_a n_b$.

$$\boxed{A_a{}^{AB} \hat{=} -2\alpha_a i^{(A} o^{B)} - \beta_a o^{(A} o^{B)} \qquad (3.1075)}$$

and

$$\beta_a \hat{=} if(v)\overline{m}_a \qquad (3.1076)$$

$$\boxed{F_{ab}{}^{AB} = -4(\partial_{[a}\alpha_{b]}\imath^{(A} o^{B)} - 2(\partial_{[a}\beta_{b]} - 2\alpha_{[a}\beta_{b]})o^A o^B \qquad (3.1077)}$$

Finally, let us establish (4.3.4) . We assume that the slice M intersects Δ in S such that on S the unit normal τ_a to M is given by $\tau_a = (i_a + n_a)/\sqrt{2}$. Therefore, using $\mathcal{D}_a o_A \hat{=} -\alpha_a o_A$

$$\boxed{K_a{}^{AB} = \frac{f(v)}{\sqrt{2}}(m_a \imath^A \imath^B + \overline{m}_a o^A o^A) \qquad (3.1078)}$$

Since the connection A on $\mathcal{M}$ is given by: $A_a{}^{AB} = \Gamma_a{}^{AB} - \frac{i}{\sqrt{2}} K_a{}^{AB}$, using the definition of V we have the required result:

$$\boxed{V_a = -i\Gamma_a{}^{AB}\imath_A o_B. \qquad (3.1079)}$$

$$
\begin{aligned}
R_{AA'BB'CC'DD'} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
&+ \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\
&+ \frac{R}{12}(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}) \qquad (3.1080)
\end{aligned}
$$

Let us consider the Ricci tensor. Since $T_{ab}l^a l^b \geq 0$ the Raychaudhuri equation for l^a implies $T_{ab}l^a l^b \hat{=} 0$, whence by Einstein's equation which holds on Δ,

$$\underline{R}_{ab}l^b - \frac{1}{2}\underline{R}\,\underbrace{\underline{g}_{ab}l^b}_{=0} \hat{=} \kappa\,\underbrace{\underline{T}_{ab}}_{=0}\,l^b \tag{3.1081}$$

, we conclude

$$\Phi_{00} = \frac{1}{2}R_{ab}l^a l^b \hat{=} 0. \tag{3.1082}$$

$$\boxed{\begin{aligned}\Phi_{ABA'B'} \quad \hat{=} \quad & 4\Phi_{11}\iota_{(A}o_{B)}\iota_{A'}o_{B'} + \Phi_{22}o_A o_B \overline{o}_{A'}\overline{o}_{B'} + \Phi_{11}\iota_A\iota_B\overline{o}_{(A'}\overline{o}_{B')} \\ & +\Phi_{20}\iota_A\iota_B\overline{\iota}_{A'}\overline{\iota}_{B'} - 2\Phi_{12}\iota_{(A}o_{B)}\overline{o}_{A'}\overline{o}_{B'} - 2\Phi_{21}o_A o_B\overline{\iota}_{(A}\overline{o}_{B)}\end{aligned} \quad (3.1083)}$$

(We will show later that Φ_{20} and Φ_{02} also vanish.)

(b) Let us now turn to the Weyl tensor. Using (4.3.4) one gets:

$$R_{abcd}l^d = 2\nabla_{[a}\nabla_{b]}l_c = -2(\nabla_{[a}U_{b]})l_c \tag{3.1084}$$

Transvecting this equation with appropriate vectors and using the fact that the trace of the Weyl tensor vanishes, we conclude that

$$\boxed{\Psi_0 \hat{=} 0 \ \text{ and } \Psi_1 \hat{=} 0. \quad (3.1085)}$$

Proof:

we have that $\underline{R}_{ab}l^b = 0$, so that

$$R_{abcd}l^d = C_{abcd}l^d = -2(\nabla_{[a}U_{b]})l_c \tag{3.1086}$$

$$\Psi_0 := C_{abcd}l^a m^b l^c m^d = \tag{3.1087}$$

(c)

$$\nabla_a n_b = 2U_a n_b - 2f m_{(a}\overline{m}_{b)}. \tag{3.1088}$$

(3.1088)

$$
\begin{aligned}
\nabla_a(\nabla_b n_c) &= \nabla_a(2U_b n_c - 2f m_{(b}\overline{m}_{c)}) \\
&= 2U_b \nabla_a n_c - 2(\partial_a f) m_{(b}\overline{m}_{c)} \quad (\text{as } \nabla_a m_b = \nabla_a m_c = 0 \text{ and } \nabla_a U_b = 0) \\
&= 2U_b(2U_a n_c - 2f m_{(a}\overline{m}_{c)}) - 2(\partial_a f) m_{(b}\overline{m}_{c)} \\
&= 4U_a U_b n_c - 2f(\overline{m}_a U_b m_c + m_a U_b \overline{m}_c) - (\partial_a f)(m_b \overline{m}_c + \overline{m}_b m_c)
\end{aligned}
\tag{3.1089}
$$

Hence it follows that

$$
R_{abcd} n^d = 2\nabla_{[a}\nabla_{b]} n_c = -4f(\overline{m}_{[a} U_{b]} m_c + m_{[a} U_{b]} m_c) - (\partial_{[a} f) m_{b]} \overline{m}_c - (\partial_{[a} f) \overline{m}_{b]} m_c \tag{3.1090}
$$

we conclude:

$$
\boxed{Im\Psi_2 \hat{=} 0, \quad \Psi_2 + \frac{R}{12} \hat{=} \frac{2U \cdot l}{r_\triangle}, \quad \text{and} \quad \Psi_3 - \Phi_{21} \hat{=} 0.} \tag{3.1091}
$$

Consequently, the Weyl spinor has the form:

$$
\Phi_{ABCD} \hat{=} 6\Phi_2 \imath_{(A}\imath_B o_C o_{D)} - 4\Phi_3 \imath_{(A} o_B o_C o_{D)} + \Phi_4 o_{(A} o_B o_C o_{D)}, \tag{3.1092}
$$

where Φ_2 and Φ_3 are subject to (3.1092).

(d)

Recall the expression for F in terms of the self-dual part of the Riemann curvature:

$$
F_{ab}^{AB} = -\frac{1}{4} R_{ab}{}^{cd} \Sigma_{ab}{}^{AB}. \tag{3.1093}
$$

Using the decomposition (4.3.4) of the Riemann tensor, we obtain:

$$
F_{abCD} \hat{=} -\frac{1}{2}\Psi_{ABCD}\Sigma_{ab}{}^{AB} - \frac{1}{2}\overline{\Phi}_{ABCD}\overline{\Sigma}_{ab}^{A'B'} - \frac{R}{24}\Sigma_{abCD}. \tag{3.1094}
$$

Finally, using the identity:

$$
\Sigma_{ab}{}^{AB} \hat{=} 4\imath^{(A} o^{B)} m_{[a}\overline{m}_{b]} + 4i o^A o^B n_{[a}\overline{m}_{b]}, \tag{3.1095}
$$

whcih follows from the definition the of null tetrad, and equations (4.3.4) and (4.3.4), we can express the pull-back F_{ab}^{CD} of F to ∇ as:

$$
F_{ab}^{CD} \hat{=} (\Psi_2 - \Phi_{11} - \frac{R}{24})\Sigma_{ab}{}^{CD} - 2i(3\Psi_3 - 2\Phi_{11}) o^C o^D n_{[a}\overline{m}_{b]}, \tag{3.1096}
$$

where we have used $\Psi_3 - \Phi_{21} = 0$. Other Weyl and Ricci components do not appear because F is the pull-back to ∇). This equation can be rewritten in the following simpler form:

$$F_{ab}^{CD} \hat{=} \left[(\Psi_2 - \Phi_{11} - \frac{R}{24})\delta_C^A \delta_D^B - (\frac{3\Psi_3}{2} - \Phi_{11}) o^A o^B \iota_C \iota_D \right] \Sigma_{ab}{}^{CD}. \tag{3.1097}$$

(e) Next, we will use the Bianchi identity ($\mathcal{D} \wedge F \hat{=} \ = 0$):

$$\nabla_a F_{bc} + \nabla_a F_{bc} + \nabla_a F_{bc} = 0 \tag{3.1098}$$

and the equation of motion ($\mathcal{D} \wedge \Sigma \hat{=} \ = 0$)

$$\nabla_a \Sigma_{bc} + \nabla_a \Sigma_{bc} + \nabla_a \Sigma_{bc} \tag{3.1099}$$

to extract further information about Ψ_2, Φ_{11} and R which appear in the above relation between F and Σ. Using (4.3.4) and (4.3.4), we obtain:

$$0 \hat{=} \nabla_{[a} F_{bc]}{}^{AB} = \tag{3.1100}$$

$\sigma = \kappa = 0$. From

$$Do^A = \epsilon o^A - \kappa \iota^A \tag{3.1101}$$

we have $Do^A = \epsilon o^A$. It follows that

$$Dl^a = D(o^A \overline{o}^{A'}) = (\epsilon + \bar{\epsilon}) o^A \overline{o}^{A'} = (\epsilon + \bar{\epsilon}) l^a, \tag{3.1102}$$

Gauge freedom on horizon

$$v \mapsto \tilde{v} = F(v) \text{ with } F'(v) > 0 \tag{3.1103}$$

$$n_a \mapsto \tilde{n} = F'(v) n_a, \tag{3.1104}$$

$$(l^a, n^a) \mapsto (\tilde{l}^a, \tilde{n}^a) = \big((F'(v))^{-1} l^a, F'(v) n^a \big) \tag{3.1105}$$

$l^a = \iota^A \bar{\iota}^{A'}$

$$
\begin{aligned}
\imath^A &\mapsto \tilde{\imath}^A = \left(F'(v)\right)^{-1/2} \imath^A \\
o^A &\mapsto \tilde{o}^A = \left(F'(v)\right)^{1/2} o^A
\end{aligned} \tag{3.1106}
$$

If we also include the freedom to perform a phase rotation $e^{i\theta}$, then we see we can have in general,

$$
(\tilde{\imath}^A, \tilde{o}^A) = \left((\exp\Theta)\imath^A, (\exp -\Theta)o^A\right) \tag{3.1107}
$$

where

$$
\exp Re(2\Theta) = F'(v), \quad \theta := Im\Theta. \tag{3.1108}
$$

A complexification of $U(1)$

3.20.4 Rotating Isolated Horizons

Details: Axisymmetric spacetimes

$$
\epsilon_{abcd}\eta^a\xi^b\nabla^d\xi^c = 0 = \epsilon_{abcd}\xi^a\eta^b\nabla^d\eta^c \tag{3.1109}
$$

$$
\xi_{[a}\eta_b\nabla_c\xi_{d]} = 0 = \xi_{[a}\eta_b\nabla_c\eta_{d]} \tag{3.1110}
$$

Stationary axisymmetric fields admit 2-spaces orthogonal to the group orbits if and only if the conditions

$$
\xi^d R_{d[a}\xi_b\eta_{c]} = 0 = \eta^d R_{d[a}\xi_b\eta_{c]} \tag{3.1111}
$$

are satisfied.

$$
\nabla(\psi^a\omega_a) \tag{3.1112}
$$

that ψ and ξ commute is the same as saying that $\mathcal{L}_\psi\xi = 0$, as $\mathcal{L}_\psi g_{ab} = 0$

$$
\mathcal{L}_\psi\omega_a = 0. \tag{3.1113}
$$

and we are left with proving

$$\psi^a \nabla_{[b}\omega_{a]} = 0. \tag{3.1114}$$

$$\begin{aligned} \epsilon^{abcd}\nabla_c\omega_d &= \epsilon^{abcd}\epsilon_{defg}\nabla_c(\xi^e\nabla^f\xi^g) \\ &= 6\nabla_c(\xi^{[c}\nabla^a\xi^{b]}) \end{aligned} \tag{3.1115}$$

where we used the identity (4.3.4)

$$\xi^{[c}\nabla^a\xi^{b]} = \frac{1}{3}(\xi^c\nabla^a\xi^b + \xi^b\nabla^c\xi^a + \xi^a\nabla^b\xi^c) \tag{3.1116}$$

$$\begin{aligned} \nabla_c(\xi^c\nabla^a\xi^b) &= (\nabla_c\xi^c)\nabla^a\xi^b + \xi^c\nabla_c\nabla^a\xi^b \\ &= -\xi^c R^{ab}{}_{cd}\xi^d \\ &= 0. \end{aligned} \tag{3.1117}$$

$$\begin{aligned} \nabla_c(\xi^b\nabla^c\xi^a + \xi^a\nabla^b\xi^c) &= (\nabla_c\xi^c)(\nabla^c\xi^a) + (\nabla_c\xi^a)(\nabla^b\xi^c) + \xi^b\nabla_c\nabla^c\xi^a + \xi^a\nabla_c\nabla^b\xi^c \\ &= 2\xi[b\nabla_c\nabla^{|c|}\xi^{a]} \\ &= -2\xi^{[b}R^{a]}{}_c\xi^c, \end{aligned} \tag{3.1118}$$

hence it becomes

$$ds^2 = -V^{-1}e^{2\gamma}(dr^2 + dz^2) - V^{-1}r^2 d\phi^2 + V(cdt - Nd\phi)^2 \tag{3.1119}$$

$$V = e^w, \qquad e^{2\gamma} = e^v \tag{3.1120}$$

Rotating Isolated Horizons

Questions

$$\Phi(r,\theta,\phi) = \int dV \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} dr'd\phi'd\theta' \sin\theta' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \tag{3.1121}$$

where (r', ϕ', θ') are the angular coorinates of $\vec{r}'$ of the matter.

$$\bar{\vec{r}} = \frac{\int dV \vec{r}' \rho(\vec{r}')}{\int dV \rho(\vec{r}')} \tag{3.1122}$$

In the axisymmetric case, the centre of mass lies along the axis of symmetry a distance d_{CofM} from the origin, which is given by:

$$d_{CofM} = 2\pi \frac{\int dr' d\theta' \sin\theta' (r' \cos\theta') \rho(r', \theta')}{\int dV \rho(r')} \tag{3.1123}$$

This easily seen, for if θ part of the integral:

$$\int dV \rho(r') \frac{\cos\theta}{r'^2} = 2\pi \int dr' \int_0^\pi d\theta \sin\theta \frac{\cos\theta}{r'^2} = 0 \tag{3.1124}$$

5. Prove

$$\tilde{\mathcal{R}}(\zeta, \phi) = -\frac{1}{R^2} f''(\zeta) = -\frac{1}{R^2} \zeta^b \tilde{D}_b (\zeta^a \tilde{D}_a f(\zeta)). \tag{3.1125}$$

Answers

5.

$$\tilde{q}_{ab} = R^2 (f^{-1} \tilde{D}_a \zeta \tilde{D}_b \zeta + f \tilde{D}_a \phi \tilde{D}_b \phi) \tag{3.1126}$$

$$\tilde{q} = \det \tilde{q}_{ab} = \tag{3.1127}$$

$$\Gamma^a_{ab} = \frac{\partial}{\partial x^b} \ln \sqrt{\tilde{q}} \tag{3.1128}$$

Rotating isolated horizons.

Questions

1. Prove (Eq. 4.3.4)

$$\int r^n P_n(\cos\theta) \vec{\nabla} \cdot (\vec{x} \times \vec{j}) d^3x \longrightarrow -R^{n+1} \oint_S (\tilde{\epsilon}^{ab} \tilde{D}_b P_n(\cos\theta)) \tilde{j}_a d\tilde{V} \tag{3.1129}$$

2.

3.

$$\tilde{\mathcal{D}}_a\zeta = \frac{1}{R^2}\tilde{\epsilon}_{ba}\varphi^b \tag{3.1130}$$

$$\tilde{q}_{ab} = R^2(f^{-1}\tilde{\mathcal{D}}_a\zeta\tilde{\mathcal{D}}_b\zeta + f\tilde{\mathcal{D}}_a\phi\tilde{\mathcal{D}}_b\phi) \tag{3.1131}$$

$$J_n = -\sqrt{\frac{4\pi}{2n+1}}\frac{R_\Delta^{n+1}}{4\pi G}\oint_S Y_n^0(\zeta)Im[\Phi_2]d^2V. \tag{3.1132}$$

$$M_n = -\sqrt{\frac{4\pi}{2n+1}}\frac{M_\Delta R_\Delta^n}{8\pi}\oint_S Y_n^0(\zeta)\tilde{R}d^2V. \tag{3.1133}$$

Answers

2. Integrating by parts on the ζ integration

$$\int_{-1}^{1} d\zeta f''(\zeta)\zeta = [f'(\zeta)\zeta]_{-1}^{1} - \int_{-1}^{1} d\zeta f'(\zeta) = [f'(\zeta)(1-\zeta)]_{-1}^{1} \tag{3.1134}$$

and using the boundary conditions ??$(f'|_{\zeta=-1}) = 0$ and $(f'|_{\zeta=1}) = 1$??, the integral zanishes.

3.20.5 Dynamical Horizons

Dynamical Horizons.

$$\tilde{K} = \tilde{q}^{ab}D_a\hat{r}_b = \frac{1}{2}\tilde{q}^{ab}\nabla_a(\ell_b - n_b) = -\frac{1}{2}\Theta_{(n)} > 0. \tag{3.1135}$$

Examples of normed spaces.

Dynamical Horizons.

Gauss-Codacci equation relating the space-time curvature to the intrinsic curavature of H leads to

$$2\mathcal{G}_{ab}R^aR^b = -\tilde{\mathcal{R}} + \tilde{\mathcal{K}}^2 - \tilde{\mathcal{K}}_{ab}\tilde{\mathcal{K}}^{ab} \tag{3.1136}$$

proof

and the defintion of the Riemann tensor gives

$$\mathcal{R}_{ab}R^aR^b = -2R^a\mathcal{D}_{[a}\mathcal{D}_{b]}R^b = \mathcal{D}_a\alpha^a + \tilde{\mathcal{K}}^2 - \tilde{\mathcal{K}}_{ab}\tilde{\mathcal{K}}^{ab} \tag{3.1137}$$

where we have defined

$$\alpha^a := \tilde{R}^b\mathcal{D}_b\tilde{R}^a - \tilde{R}^a\mathcal{D}_b\tilde{R}^b \tag{3.1138}$$

$$\begin{aligned}
\mathcal{R}_{ab}\tilde{R}^a\tilde{R}^b &= -\tilde{R}^a(\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a)\tilde{R}^b \\
&= \mathcal{D}_a(\tilde{R}^b\mathcal{D}_b\tilde{R}^a - \tilde{R}^a\mathcal{D}_b\tilde{R}^b) + (\mathcal{D}_a\tilde{R}^a)^2 - (\mathcal{D}_a\tilde{R}^b)(\mathcal{D}_b\tilde{R}^a)
\end{aligned} \tag{3.1139}$$

$$\tilde{K}_{ab}\tilde{K}^{ab} = \frac{1}{2}(\tilde{K})^2 + \tilde{\sigma}_{ab}\tilde{\sigma}^{ab}, \tag{3.1140}$$

$$\tilde{K} = 2A + B, \tag{3.1141}$$

$$K_{ab}K^{ab} = \frac{1}{2}(\tilde{K})^2 + 2W_aW^a + B^2, \tag{3.1142}$$

$$\tilde{K} = -2A. \tag{3.1143}$$

(b)

$$\tilde{q}^{ab}\nabla_a\ell_b = \tilde{K} - K_{ab}R^aR^b + K \tag{3.1144}$$

Vaidya spacetime.

Prove the only non-vanishing component of the Einstein tensor is $G_{uu} = -(2/r^2)/(dm/du)$

Answers

$$\nabla_bT^{ab} = 0 \tag{3.1145}$$

$$\nabla_b[\rho\ell^a\ell^b] = 0 \tag{3.1146}$$

$$\ell^b\nabla_b(\rho\ell^a) + \rho\ell^b(\nabla_b\ell^a) = 0 \tag{3.1147}$$

$\ell_a\ell^a = 0$ implies that $\ell_a(\nabla_b\ell^a) = 0$. Contracting (3.1147) with ℓ^a we conclude

$$\nabla_b(\rho \ell^b) = 0 \tag{3.1148}$$

and hence

$$\ell^b \nabla_b \ell^a = 0. \tag{3.1149}$$

i.e. is a geodesic of the timelike curves.

Poor man derivation of Schwarzschild solution.

We assume the spacetime to be static. This means the all the metric components are independent of the time coordinate and the geometry the spacetime is unchanged under time reversal. As the line element is invariant under $t \to -t$ there cannot be a cross term $drdt$.

On the hypersurface of constant time and constant r, it is required that the metric be that of a 2-sphere

$$dl^2 = r_0(d\theta^2 + \sin^2\theta d\phi^2).$$

The metric can be put in the form

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.1150}$$

The metric components are

$$g_{00} = A, \quad g_{11} = -B, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \tag{3.1151}$$

The metric is diagonal so that $g^{00} = 1/g_{00}$. The coordinates are labelled

$$x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi.$$

By definition

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}).$$

As none of the metric components depend on ϕ. There are many other examples that turn out to be zero. We find altogether:

$$
\begin{gathered}
\Gamma^0_{01} = \Gamma^0_{10} = A'/2A \\
\Gamma^1_{00} = A'/2B, \qquad \Gamma^1_{11} = B'/2B \\
\Gamma^1_{22} = -r/B, \qquad \Gamma^1_{33} = -r\sin^2\theta/B \\
\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = 1/r \\
\Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3_{32} = \Gamma^3_{23} = \cot\theta
\end{gathered} \tag{3.1152}
$$

All others being zero. We use these to calculate the components of the Ricci tensor.

$$
R_{00} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} \tag{3.1153}
$$

$$
R_{11} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB} \tag{3.1154}
$$

$$
R_{22} = 1 - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{B} \tag{3.1155}
$$

$$
R_{33} = R_{22}\sin^2\theta \tag{3.1156}
$$

From (3.1153) and (3.1154)

$$
\frac{R_{00}}{A} + \frac{R_{11}}{B} = \frac{(A'/A + B'/B)}{rB} \tag{3.1157}
$$

The vacuum field equations imply

$$
\frac{A'}{A} + \frac{B'}{B} = 0 \tag{3.1158}
$$

or

$$
-AB\left(\frac{1}{AB}\right)' = 0 \tag{3.1159}
$$

whence

$$
AB = Const.
$$

At points remote from the source both A and B tend to unity, so that the constant is unit. Therefore

$$
B = 1/A.
$$

$$
\begin{aligned}
1-\frac{r}{2B}\left(\frac{A'}{A}-\frac{B'}{B}\right)-\frac{1}{B} &= 1-\frac{Ar}{2}\left(\frac{A'}{A}+\frac{A'}{A^2/A}\right)-A \\
1-A'r-A & \qquad\qquad\qquad\qquad (3.1160)
\end{aligned}
$$

$$1-A'r-A=0.$$

Integration with respect to r gives

$$rA-r=Const.$$

and

$$A=1+\frac{Const.}{r}$$

$$d\tau^2=\left(1+\frac{Const.}{r}\right)dt^2-\frac{dr^2}{1+Const./r}-r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (3.1161)$$

Consider the Newtonian limit. A point mass M situated at the origin in Newtonian theory gives rise to a potential

$$\phi=-GM/r$$

Using this in the weak field limit gives

$$g_{00}\simeq 1+2\phi/c^2=1-2GM/c^2r.$$

We see that the constant turns out to be $-2GM/c^2$

$$ds^2=\left(1-\frac{2GM}{c^2r}\right)dt^2-\frac{dr^2}{1-2GM/c^2r}-r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (3.1162)$$

Derivation of the Kerr-Newman solution: a rotating charged blackhole

We consider spacetimes where coordinates exist in which the metric takes the form

$$g_{\mu\nu}=\eta_{\mu\nu}+2hl_\mu l_\nu, \qquad (3.1163)$$

where $\eta_{\mu\nu}$ is the metric of Minkowski space.

It is easy to see that $g=-1$, and

$$g^{\mu\nu}=\eta^{\mu\nu}-2hl^{\mu}l^{\nu} \tag{3.1164}$$

We have

$$g^{\mu\nu}l_{\mu}l_{\nu}=\eta^{\mu\nu}l_{\mu}l_{\nu}=0. \tag{3.1165}$$

$$\sqrt{2}v=z-t,\quad \sqrt{2}u=z+t,,\quad \sqrt{2}\zeta=x+iy,\quad \sqrt{2}\overline{\zeta}=x-iy. \tag{3.1166}$$

Then the metric () gives the line element

$$ds^2=2d\zeta d\overline{\zeta}+2dudv+2h(e^3)^2 \tag{3.1167}$$

Null tetrad

$$e^{\hat{1}}=du+\overline{Y}d\zeta+Yd\overline{\zeta}-Y\overline{Y}dv, \tag{3.1168}$$

A null tetrad, with scalar products given by

$$g_{\hat{a}\hat{b}}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \tag{3.1169}$$

is completed as follows:

$$\begin{gathered}e^{\hat{2}}=d\zeta-Ydv,\qquad e^{\hat{3}}=d\overline{\zeta}-\overline{Y}dv\\ e^{\hat{0}}=dv+he^{\hat{1}}\end{gathered} \tag{3.1170}$$

Proof:

$\square$

We know $e^{\hat{a}}{}_{a}dx^a$, $dx^0=dv, dx^1=du, dx^2=d\zeta, dx^3=\overline{\zeta}$, so we can read off $e^{\hat{a}}{}_{a}$

$$e^{\hat{a}}{}_{a}=\begin{pmatrix}1-hY\overline{Y}&h&h\overline{Y}&hY\\-Y\overline{Y}&1&\overline{Y}&Y\\-Y&0&1&0\\-\overline{Y}&0&0&1\end{pmatrix} \tag{3.1171}$$

Directional derivatives

The inverse of the above matrix is

$$\begin{pmatrix} 1 & -h & 0 & 0 \\ -Y\overline{Y} & 1+hY\overline{Y} & -\overline{Y} & -Y \\ Y & -hY & 1 & 0 \\ \overline{Y} & -h\overline{Y} & 0 & 1. \end{pmatrix} \tag{3.1172}$$

The transpose gives the dual tetrad, $e_{\hat{a}}{}^{a}$,

$$e_{\hat{a}}{}^{a} = \begin{pmatrix} 1 & -Y\overline{Y} & Y & \overline{Y} \\ -h & 1+hY\overline{Y} & -hY & -h\overline{Y} \\ 0 & -\overline{Y} & 1 & 0 \\ 0 & -Y & 0 & 1 \end{pmatrix} \tag{3.1173}$$

Then the directional derivatives $\partial_{\hat{a}} = e_{\hat{a}}{}^{a}\partial_a$ can now be calculated to be

$$\begin{gathered} \partial_2 = \partial_\zeta - \overline{Y}\partial_u, \qquad \partial_3 = \partial_{\overline{\zeta}} - Y\partial_u, \\ \partial_1 = \partial_u - h\partial_0, \\ \partial_0 = \partial_v + Y\partial_\zeta + \overline{Y}\partial_{\overline{\zeta}} - Y\overline{Y}\partial_u. \end{gathered} \tag{3.1174}$$

Equations for gravity coupled to the electromagnetic field

The gravitational equations for interacting gravitational and electromagnetic fields are

$$\begin{aligned} R_{\hat{a}\hat{b}} - \frac{1}{2}g_{\hat{a}\hat{b}}R &= 8\pi T_{\hat{a}\hat{b}} \\ T_{\hat{a}\hat{b}} &= \frac{1}{4\pi}(F_{\hat{a}\hat{c}}F_{\hat{b}}{}^{\hat{c}} + \frac{1}{4}g_{\hat{a}\hat{b}}F_{\hat{c}\hat{d}}F^{\hat{c}\hat{d}}) \end{aligned} \tag{3.1175}$$

These are supplemented by Maxwell's equations

$$\nabla_{\hat{b}}F^{\hat{a}\hat{b}} = j^{\hat{a}}, \qquad \partial_{[\hat{a}}F_{\hat{b}\hat{c}]} = 0. \tag{3.1176}$$

Since $T = 0$, it follows that

$$R = 2(R_{\hat{1}\hat{0}} - R_{\hat{2}\hat{3}}) = 0. \tag{3.1177}$$

The field equations can then be written as

$$R_{\hat{a}\hat{b}} = 2(F_{\hat{a}\hat{c}}F_{\hat{b}}{}^{\hat{c}} + \frac{1}{4}g_{\hat{a}\hat{b}}F_{\hat{c}\hat{d}}F^{\hat{c}\hat{d}}) \tag{3.1178}$$

Restrictions placed on the solutions

We find solutions of the Einstein-Maxwell equations for spaceties with a degenerate metric. We assume that $e_{\hat{0}}$ are the tangents of a congruence of null geodesics, implying

$$\kappa = 0 = -\Gamma_{\hat{0}\hat{2}\hat{0}}. \tag{3.1179}$$

Assume the complex expansion Z

$$Z = \theta + i\omega = -\Gamma_{\hat{0}\hat{3}\hat{2}} \tag{3.1180}$$

is non-zero.

Calculation of the rotation coeficients

We need to express the differentials $dv, d\zeta, d\overline{\zeta}$ in terms of the basis one forms:

$$\begin{aligned} dv &= e^{\hat{0}} - he^{\hat{1}} \\ d\zeta &= e^{\hat{2}} + Ydv = e^{\hat{2}} + Y(e^{\hat{0}} - he^{\hat{1}}) \\ d\overline{\zeta} &= e^{\hat{3}} + \overline{Y}dv = e^{\hat{3}} + \overline{Y}(e^{\hat{0}} - he^{\hat{1}}). \end{aligned} \tag{3.1181}$$

$$\begin{aligned} de^{\hat{2}} &= Y_{,a}dx^a \wedge dv \\ &= -Y_{,\hat{a}}e^{\hat{a}}{}_a dx^a \wedge dv \\ &= -Y_{,\hat{a}}e^{\hat{a}} \wedge dv \\ &= -Y_{,\hat{a}}e^{\hat{a}} \wedge (e^{\hat{0}} - he^{\hat{1}}) \end{aligned} \tag{3.1182}$$

Similarly

$$de^{\hat{3}} = -\overline{Y}_{,\hat{a}}e^{\hat{a}} \wedge (e^{\hat{0}} - he^{\hat{1}}). \tag{3.1183}$$

$$\begin{aligned} de^{\hat{1}} &= Y_{,\hat{a}}e^{\hat{a}} \wedge d\overline{\zeta} + \overline{Y}_{,\hat{a}}e^{\hat{a}} \wedge d\zeta - \overline{Y}(Y_{,\hat{a}})e^{\hat{a}} \wedge dv - Y(\overline{Y}_{,\hat{a}})e^{\hat{a}} \wedge dv \\ &= Y_{,\hat{a}}e^{\hat{a}} \wedge (e^{\hat{3}} + \overline{Y}dv) + \overline{Y}_{,\hat{a}}e^{\hat{a}} \wedge (e^{\hat{2}} + Ydv) - \overline{Y}(Y_{,\hat{a}})e^{\hat{a}} \wedge dv - Y(\overline{Y}_{,\hat{a}})e^{\hat{a}} \wedge dv \\ &= Y_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{3}} + \overline{Y}_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{2}} \end{aligned} \tag{3.1184}$$

$$de^0 = hY_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{3}} + h\overline{Y}_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{2}} + h_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{1}} \tag{3.1185}$$

Now we shall read off the rotation coeficients. Starting with $de^{\hat{0}}$

$$\begin{aligned} de^{\hat{0}} &= \frac{1}{2}\left(h_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{1}} - h_{,\hat{b}}e^{\hat{1}} \wedge e^{\hat{b}}\right) + \frac{1}{2}\left(h\overline{Y}_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{2}} - h\overline{Y}_{,\hat{a}}e^{\hat{2}} \wedge e^{\hat{a}}\right) \\ &+\frac{1}{2}\left(hY_{,\hat{a}}e^{\hat{a}} \wedge e^{\hat{3}} - hY_{,\hat{b}}e^{\hat{3}} \wedge e^{\hat{b}}\right) \end{aligned} \tag{3.1186}$$

which expanded out gives

$$\begin{aligned} de^{\hat{0}} &= \frac{1}{2}\left(h_{,\hat{0}}e^{\hat{0}} \wedge e^{\hat{1}} - h_{,\hat{0}}e^{\hat{1}} \wedge e^{\hat{0}}\right) + \frac{1}{2}\left(h\overline{Y}_{,\hat{0}}e^{\hat{0}} \wedge e^{\hat{2}} - h\overline{Y}_{,\hat{0}}e^{\hat{2}} \wedge e^{\hat{0}}\right) \\ &+\frac{1}{2}\left(hY_{,\hat{0}}e^{\hat{0}} \wedge e^{\hat{3}} - hY_{,\hat{0}}e^{\hat{3}} \wedge e^{\hat{0}}\right) + \frac{1}{2}\left((h\overline{Y}_{,\hat{1}} - h_{,2})e^{\hat{1}} \wedge e^{\hat{2}} - (h\overline{Y}_{,\hat{1}} - h_{,2})e^{\hat{2}} \wedge e^{\hat{1}}\right) \\ &+\frac{1}{2}\left((hY_{,\hat{1}} - h_{,\hat{3}})e^{\hat{1}} \wedge e^{\hat{3}} - (hY_{,\hat{1}} - h_{,\hat{3}})e^{\hat{3}} \wedge e^{\hat{1}}\right) \\ &+\frac{1}{2}\left((hY_{,\hat{2}} - h\overline{Y}_{,\hat{3}})e^{\hat{2}} \wedge e^{\hat{3}} - (hY_{,\hat{2}} - h\overline{Y}_{,\hat{3}})e^{\hat{3}} \wedge e^{\hat{2}}\right) \end{aligned} \tag{3.1187}$$

Comparing this to

$$\begin{aligned} de^{\hat{0}} &= \Gamma^{\hat{0}}{}_{\hat{a}\hat{b}}e^{\hat{a}} \wedge e^{\hat{b}} \\ &= \Gamma^{\hat{0}}{}_{\hat{0}\hat{1}}e^{\hat{0}} \wedge e^{\hat{1}} + \Gamma^{\hat{0}}{}_{\hat{1}\hat{0}}e^{\hat{1}} \wedge e^{\hat{0}} + \dots \\ &= \Gamma^{\hat{0}}{}_{[\hat{0}\hat{1}]}(e^{\hat{0}} \wedge e^{\hat{1}} - e^{\hat{1}} \wedge e^{\hat{0}}) + \dots \end{aligned} \tag{3.1188}$$

we can read off

$$\begin{aligned} &\Gamma^{\hat{0}}{}_{[\hat{0}\hat{1}]} = \frac{1}{2}h_{,\hat{0}}, \quad \Gamma^{\hat{0}}{}_{[\hat{0}\hat{2}]} = \frac{1}{2}h\overline{Y}_{,\hat{0}}, \quad \Gamma^{\hat{0}}{}_{[\hat{0}\hat{3}]} = \frac{1}{2}hY_{,\hat{0}} \\ &\Gamma^{\hat{0}}{}_{[\hat{1}\hat{2}]} = \frac{1}{2}(h\overline{Y}_{,\hat{1}} - h_{,2}), \quad \Gamma^{\hat{0}}{}_{[\hat{1}\hat{3}]} = \frac{1}{2}(hY_{,\hat{1}} - h_{,\hat{3}}) \\ &\Gamma^{\hat{0}}{}_{[\hat{2}\hat{3}]} = \frac{1}{2}(hY_{,\hat{2}} - h\overline{Y}_{,\hat{3}}). \end{aligned} \tag{3.1189}$$

or

$$\begin{aligned} &\Gamma_{\hat{1}[\hat{0}\hat{1}]} = \frac{1}{2}h_{,\hat{0}}, \quad \Gamma_{\hat{1}[\hat{0}\hat{2}]} = \frac{1}{2}h\overline{Y}_{,\hat{0}}, \quad \Gamma_{\hat{1}[\hat{0}\hat{3}]} = \frac{1}{2}hY_{,\hat{0}} \\ &\Gamma_{\hat{1}[\hat{1}\hat{2}]} = \frac{1}{2}(h\overline{Y}_{,\hat{1}} - h_{,2}), \quad \Gamma_{\hat{1}[\hat{1}\hat{3}]} = \frac{1}{2}(hY_{,\hat{1}} - h_{,\hat{3}}) \\ &\Gamma_{\hat{1}[\hat{2}\hat{3}]} = \frac{1}{2}(hY_{,\hat{2}} - h\overline{Y}_{,\hat{3}}). \end{aligned} \tag{3.1190}$$

Using

$$\begin{aligned}
de^{\hat{1}} &= \frac{1}{2}\left(\overline{Y}_{,\hat{0}}e^{\hat{0}}\wedge e^{\hat{2}} - \overline{Y}_{,\hat{0}}e^{\hat{2}}\wedge e^{\hat{0}}\right) + \frac{1}{2}\left(Y_{,\hat{0}}e^{\hat{0}}\wedge e^{\hat{3}} - Y_{,\hat{0}}e^{\hat{3}}\wedge e^{\hat{0}}\right)\\
&+\frac{1}{2}\left(\overline{Y}_{,\hat{1}}e^{\hat{1}}\wedge e^{\hat{2}} - \overline{Y}_{,\hat{1}}e^{\hat{2}}\wedge e^{\hat{1}}\right) + \frac{1}{2}\left(Y_{,\hat{1}}e^{\hat{1}}\wedge e^{\hat{3}} - Y_{,\hat{1}}e^{\hat{3}}\wedge e^{\hat{1}}\right)\\
&+\frac{1}{2}\left((Y_{,\hat{2}} - \overline{Y}_{,\hat{3}})e^{\hat{2}}\wedge e^{\hat{3}} - (Y_{,\hat{2}} - \overline{Y}_{,\hat{3}})e^{\hat{3}}\wedge e^{\hat{2}}\right)
\end{aligned} \tag{3.1191}$$

$$\begin{aligned}
de^{\hat{1}} &= {\Gamma^{\hat{1}}}_{\hat{a}\hat{b}}e^{\hat{a}}\wedge e^{\hat{b}}\\
&= {\Gamma^{\hat{1}}}_{\hat{0}\hat{1}}e^{\hat{0}}\wedge e^{\hat{1}} + {\Gamma^{\hat{0}}}_{\hat{1}\hat{0}}e^{\hat{1}}\wedge e^{\hat{0}} + \dots\\
&= {\Gamma^{\hat{1}}}_{[\hat{0}\hat{1}]}(e^{\hat{0}}\wedge e^{\hat{1}} - e^{\hat{1}}\wedge e^{\hat{0}}) + \dots
\end{aligned} \tag{3.1192}$$

$$\begin{aligned}
&{\Gamma^{\hat{1}}}_{[\hat{0}\hat{1}]} = 0, \quad {\Gamma^{\hat{1}}}_{[\hat{0}\hat{2}]} = \frac{1}{2}\overline{Y}_{,\hat{0}}, \quad {\Gamma^{\hat{1}}}_{[\hat{0}\hat{3}]} = \frac{1}{2}Y_{,\hat{0}}\\
&{\Gamma^{\hat{1}}}_{[\hat{1}\hat{2}]} = \frac{1}{2}\overline{Y}_{,\hat{1}}, \quad {\Gamma^{\hat{1}}}_{[\hat{1}\hat{3}]} = \frac{1}{2}Y_{,\hat{1}}\\
&{\Gamma^{\hat{1}}}_{[\hat{2}\hat{3}]} = \frac{1}{2}(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}).
\end{aligned} \tag{3.1193}$$

or

$$\begin{aligned}
&\Gamma_{\hat{0}[\hat{0}\hat{1}]} = 0, \quad \Gamma_{\hat{0}[\hat{0}\hat{2}]} = \frac{1}{2}\overline{Y}_{,\hat{0}}, \quad \Gamma_{\hat{0}[\hat{0}\hat{3}]} = \frac{1}{2}Y_{,\hat{0}}\\
&\Gamma_{\hat{0}[\hat{1}\hat{2}]} = \frac{1}{2}\overline{Y}_{,\hat{1}}, \quad \Gamma_{\hat{0}[\hat{1}\hat{3}]} = \frac{1}{2}Y_{,\hat{1}}\\
&\Gamma_{\hat{0}[\hat{2}\hat{3}]} = \frac{1}{2}(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}).
\end{aligned} \tag{3.1194}$$

$$\begin{aligned}
de^{\hat{2}} &= Y_{,\hat{a}}e^{\hat{0}}\wedge e^{\hat{a}} - hY_{,\hat{a}}e^{\hat{1}}\wedge e^{\hat{a}}\\
&= \frac{1}{2}\left((Y_{,\hat{1}} + hY_{,\hat{0}})e^{\hat{0}}\wedge e^{\hat{1}} - (Y_{,\hat{1}} + hY_{,\hat{0}})e^{\hat{1}}\wedge e^{\hat{0}})\right) + \frac{1}{2}(Y_{,\hat{2}}e^{\hat{0}}\wedge e^{\hat{2}} - Y_{,\hat{2}}e^{\hat{2}}\wedge e^{\hat{0}})\\
&+\frac{1}{2}(Y_{,\hat{3}}e^{\hat{0}}\wedge e^{\hat{3}} - Y_{,\hat{3}}e^{\hat{3}}\wedge e^{\hat{0}}) + \frac{1}{2}(-hY_{,\hat{2}}e^{\hat{1}}\wedge e^{\hat{2}} + hY_{,\hat{2}}e^{\hat{2}}\wedge e^{\hat{1}})\\
&\frac{1}{2}(-hY_{,\hat{3}}e^{\hat{1}}\wedge e^{\hat{3}} + hY_{,\hat{3}}e^{\hat{3}}\wedge e^{\hat{1}})
\end{aligned} \tag{3.1195}$$

$$
\begin{aligned}
de^{\hat{2}} &= \Gamma^{\hat{2}}{}_{\hat{a}\hat{b}} e^{\hat{a}} \wedge e^{\hat{b}} \\
&= \Gamma^{\hat{2}}{}_{\hat{0}\hat{1}} e^{\hat{0}} \wedge e^{\hat{1}} + \Gamma^{\hat{0}}{}_{\hat{1}\hat{0}} e^{\hat{1}} \wedge e^{\hat{0}} + \dots \\
&= \Gamma^{\hat{2}}{}_{[\hat{0}\hat{1}]} (e^{\hat{0}} \wedge e^{\hat{1}} - e^{\hat{1}} \wedge e^{\hat{0}}) + \dots
\end{aligned} \tag{3.1196}
$$

$$
\begin{aligned}
&\Gamma^{\hat{2}}{}_{[\hat{0}\hat{1}]} = \frac{1}{2}(Y_{,\hat{1}} + hY_{,\hat{0}}), \quad \Gamma^{\hat{2}}{}_{[\hat{0}\hat{2}]} = \frac{1}{2} Y_{,\hat{2}}, \quad \Gamma^{\hat{2}}{}_{[\hat{0}\hat{3}]} = \frac{1}{2} Y_{,\hat{3}} \\
&\Gamma^{\hat{2}}{}_{[\hat{1}\hat{2}]} = \frac{1}{2} - hY_{,\hat{2}}, \quad \Gamma^{\hat{2}}{}_{[\hat{1}\hat{3}]} = \frac{1}{2} - hY_{,\hat{3}} \\
&\Gamma^{\hat{2}}{}_{[\hat{2}\hat{3}]} = 0.
\end{aligned} \tag{3.1197}
$$

or

$$
\begin{aligned}
&\Gamma_{\hat{3}[\hat{0}\hat{1}]} = \frac{1}{2}(Y_{,\hat{1}} + hY_{,\hat{0}}), \quad \Gamma_{\hat{3}[\hat{0}\hat{2}]} = \frac{1}{2} Y_{,\hat{2}}, \quad \Gamma_{\hat{3}[\hat{0}\hat{3}]} = \frac{1}{2} Y_{,\hat{3}} \\
&\Gamma_{\hat{3}[\hat{1}\hat{2}]} = -\frac{1}{2} hY_{,\hat{2}}, \quad \Gamma_{\hat{3}[\hat{1}\hat{3}]} = -\frac{1}{2} hY_{,\hat{3}} \\
&\Gamma_{\hat{3}[\hat{2}\hat{3}]} = 0.
\end{aligned} \tag{3.1198}
$$

The equation for $de^{\hat{3}}$ is the same as that for $de^{\hat{2}}$ but with Y replaced by $\overline{Y}$. Therefore, we immediately have

$$
\begin{aligned}
&\Gamma^{\hat{3}}{}_{[\hat{0}\hat{1}]} = \frac{1}{2}(\overline{Y}_{,\hat{1}} + hY_{,\hat{0}}), \quad \Gamma^{\hat{3}}{}_{[\hat{0}\hat{2}]} = \frac{1}{2} \overline{Y}_{,\hat{2}}, \quad \Gamma^{\hat{3}}{}_{[\hat{0}\hat{3}]} = \frac{1}{2} \overline{Y}_{,\hat{3}} \\
&\Gamma^{\hat{3}}{}_{[\hat{1}\hat{2}]} = -\frac{1}{2} h\overline{Y}_{,\hat{2}}, \quad \Gamma^{\hat{3}}{}_{[\hat{1}\hat{3}]} = -\frac{1}{2} h\overline{Y}_{,\hat{3}} \\
&\Gamma^{\hat{3}}{}_{[\hat{2}\hat{3}]} = 0.
\end{aligned} \tag{3.1199}
$$

or

$$
\begin{aligned}
&\Gamma_{\hat{2}[\hat{0}\hat{1}]} = \frac{1}{2}(\overline{Y}_{,\hat{1}} + h\overline{Y}_{,\hat{0}}), \quad \Gamma_{\hat{2}[\hat{0}\hat{2}]} = \frac{1}{2} \overline{Y}_{,\hat{2}}, \quad \Gamma_{\hat{2}[\hat{0}\hat{3}]} = \frac{1}{2} \overline{Y}_{,\hat{3}} \\
&\Gamma_{\hat{2}[\hat{1}\hat{2}]} = -\frac{1}{2} h\overline{Y}_{,\hat{2}}, \quad \Gamma_{\hat{2}[\hat{1}\hat{3}]} = -\frac{1}{2} h\overline{Y}_{,\hat{3}} \\
&\Gamma_{\hat{2}[\hat{2}\hat{3}]} = 0.
\end{aligned} \tag{3.1200}
$$

The independent curvature one-foms are written interms of the Ricci rotation coefficients

$$
\Gamma_{\hat{0}\hat{3}} = \Gamma_{\hat{0}\hat{3}\hat{0}} e^{\hat{0}} + \Gamma_{\hat{0}\hat{3}\hat{1}} e^{\hat{1}} + \Gamma_{\hat{0}\hat{3}\hat{2}} e^{\hat{2}} + \Gamma_{\hat{0}\hat{3}\hat{3}} e^{\hat{3}} \tag{3.1201}
$$

$$\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}} = (\Gamma_{\hat{2}\hat{3}\hat{0}} + \Gamma_{\hat{1}\hat{0}\hat{0}})e^{\hat{0}} + (\Gamma_{\hat{2}\hat{3}\hat{1}} + \Gamma_{\hat{1}\hat{0}\hat{1}})e^{\hat{1}} + (\Gamma_{\hat{2}\hat{3}\hat{2}} + \Gamma_{\hat{1}\hat{0}\hat{2}})e^{\hat{2}} + (\Gamma_{\hat{2}\hat{3}\hat{3}} + \Gamma_{\hat{1}\hat{0}\hat{3}})e^{\hat{3}} \quad (3.1202)$$

$$\Gamma_{\hat{1}\hat{2}} = \Gamma_{\hat{1}\hat{2}\hat{0}}e^{\hat{0}} + \Gamma_{\hat{1}\hat{2}\hat{1}}e^{\hat{1}} + \Gamma_{\hat{1}\hat{2}\hat{2}}e^{\hat{2}} + \Gamma_{\hat{1}\hat{2}\hat{3}}e^{\hat{3}} \quad (3.1203)$$

We calculate the relavent Ricci rotation coefficients

$$\begin{aligned} \Gamma_{\hat{0}\hat{3}\hat{0}} &= \Gamma_{\hat{0}[\hat{3}\hat{0}]} + \Gamma_{\hat{3}[\hat{0}\hat{0}]} - \Gamma_{\hat{0}[\hat{0}\hat{3}]} \\ &= 2\Gamma_{\hat{0}[\hat{3}\hat{0}]} \\ &= -Y_{,\hat{0}} \end{aligned} \quad (3.1204)$$

$$\begin{aligned} \Gamma_{\hat{0}\hat{3}\hat{1}} &= \Gamma_{\hat{0}[\hat{3}\hat{1}]} + \Gamma_{\hat{3}[\hat{1}\hat{0}]} - \Gamma_{\hat{1}[\hat{0}\hat{3}]} \\ &= -\frac{1}{2}Y_{,\hat{1}} - \frac{1}{2}(Y_{,\hat{1}} + hY_{,\hat{0}}) - \frac{1}{2}hY_{,\hat{0}} \\ &= -Y_{,\hat{1}} - hY_{,\hat{0}} \end{aligned} \quad (3.1205)$$

$$\begin{aligned} \Gamma_{\hat{0}\hat{3}\hat{2}} &= \Gamma_{\hat{0}[\hat{3}\hat{2}]} + \Gamma_{\hat{3}[\hat{2}\hat{0}]} - \Gamma_{\hat{2}[\hat{0}\hat{3}]} \\ &= -\frac{1}{2}(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}) - \frac{1}{2}Y_{,\hat{2}} - \frac{1}{2}\overline{Y}_{,\hat{3}} \\ &= -Y_{,\hat{2}} \end{aligned} \quad (3.1206)$$

$$\begin{aligned} \Gamma_{\hat{0}\hat{3}\hat{3}} &= \Gamma_{\hat{0}[\hat{3}\hat{3}]} + \Gamma_{\hat{3}[\hat{3}\hat{0}]} - \Gamma_{\hat{3}[\hat{0}\hat{3}]} \\ &= 2\Gamma_{\hat{3}[\hat{3}\hat{0}]} \\ &= -Y_{,\hat{3}}. \end{aligned} \quad (3.1207)$$

$$\begin{aligned} \Gamma_{\hat{2}\hat{3}\hat{0}} &= \Gamma_{\hat{2}[\hat{3}\hat{0}]} + \Gamma_{\hat{3}[\hat{0}\hat{2}]} - \Gamma_{\hat{0}[\hat{2}\hat{3}]} \\ &= -\frac{1}{2}\overline{Y}_{,\hat{3}} + \frac{1}{2}Y_{,\hat{2}} - \frac{1}{2}(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}) \\ &= 0. \end{aligned} \quad (3.1208)$$

$$\begin{aligned} \Gamma_{\hat{1}\hat{0}\hat{0}} &= \Gamma_{\hat{1}[\hat{0}\hat{0}]} + \Gamma_{\hat{0}[\hat{0}\hat{1}]} - \Gamma_{\hat{1}[\hat{1}\hat{0}]} \\ &= 2\Gamma_{\hat{0}[\hat{0}\hat{1}]} \\ &= 0. \end{aligned} \quad (3.1209)$$

$$
\begin{aligned}
\Gamma_{\hat{2}\hat{3}\hat{1}} &= \Gamma_{\hat{2}[\hat{3}\hat{1}]} + \Gamma_{\hat{3}[\hat{1}\hat{2}]} - \Gamma_{\hat{1}[\hat{2}\hat{3}]} \\
&= \frac{1}{2}h\overline{Y}_{,\hat{3}} - \frac{1}{2}hY_{,\hat{2}} - \frac{1}{2}(hY_{,\hat{2}} - h\overline{Y}_{,\hat{3}}) \\
&= h\overline{Y}_{,\hat{3}} - hY_{,\hat{2}}.
\end{aligned} \tag{3.1210}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{0}\hat{1}} &= \Gamma_{\hat{1}[\hat{0}\hat{1}]} + \Gamma_{\hat{0}[\hat{1}\hat{1}]} - \Gamma_{\hat{1}[\hat{1}\hat{0}]} \\
&= 2\Gamma_{\hat{1}[\hat{0}\hat{1}]} \\
&= h_{,\hat{0}}.
\end{aligned} \tag{3.1211}
$$

$$
\begin{aligned}
\Gamma_{\hat{2}\hat{3}\hat{2}} &= \Gamma_{\hat{2}[\hat{3}\hat{2}]} + \Gamma_{\hat{3}[\hat{2}\hat{2}]} - \Gamma_{\hat{2}[\hat{2}\hat{3}]} \\
&= 2\Gamma_{\hat{2}[\hat{3}\hat{2}]} \\
&= 0.
\end{aligned} \tag{3.1212}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{0}\hat{2}} &= \Gamma_{\hat{1}[\hat{0}\hat{2}]} + \Gamma_{\hat{0}[\hat{2}\hat{1}]} - \Gamma_{\hat{2}[\hat{1}\hat{0}]} \\
&= \frac{1}{2}h\overline{Y}_{,\hat{0}} - \frac{1}{2}\overline{Y}_{,\hat{1}} + \frac{1}{2}(\overline{Y}_{,\hat{1}} + h\overline{Y}_{,\hat{0}}) \\
&= h\overline{Y}_{,\hat{0}}.
\end{aligned} \tag{3.1213}
$$

$$
\begin{aligned}
\Gamma_{\hat{2}\hat{3}\hat{3}} &= \Gamma_{\hat{2}[\hat{3}\hat{3}]} + \Gamma_{\hat{3}[\hat{3}\hat{2}]} - \Gamma_{\hat{3}[\hat{2}\hat{3}]} \\
&= 2\Gamma_{\hat{3}[\hat{3}\hat{2}]} \\
&= 0.
\end{aligned} \tag{3.1214}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{0}\hat{3}} &= \Gamma_{\hat{1}[\hat{0}\hat{3}]} + \Gamma_{\hat{0}[\hat{3}\hat{1}]} - \Gamma_{\hat{3}[\hat{1}\hat{0}]} \\
&= \frac{1}{2}hY_{,\hat{0}} - \frac{1}{2}Y_{,\hat{1}} + \frac{1}{2}(Y_{,\hat{1}} + hY_{,\hat{0}}) \\
&= hY_{,\hat{0}}.
\end{aligned} \tag{3.1215}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{2}\hat{0}} &= \Gamma_{\hat{1}[\hat{2}\hat{0}]} + \Gamma_{\hat{2}[\hat{0}\hat{1}]} - \Gamma_{\hat{0}[\hat{1}\hat{2}]} \\
&= -\frac{1}{2}h\overline{Y}_{,\hat{0}} + \frac{1}{2}(\overline{Y}_{,\hat{1}} + h\overline{Y}_{,\hat{0}}) - \frac{1}{2}\overline{Y}_{,\hat{1}} \\
&= 0.
\end{aligned} \tag{3.1216}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{2}\hat{1}} &= \Gamma_{\hat{1}[\hat{2}\hat{1}]} + \Gamma_{\hat{2}[\hat{1}\hat{1}]} - \Gamma_{\hat{1}[\hat{1}\hat{2}]} \\
&= 2\Gamma_{\hat{1}[\hat{2}\hat{1}]} \\
&= h_{,\hat{2}} - \overline{Y}_{,\hat{1}}
\end{aligned} \tag{3.1217}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{2}\hat{2}} &= \Gamma_{\hat{1}[\hat{2}\hat{2}]} + \Gamma_{\hat{2}[\hat{2}\hat{1}]} - \Gamma_{\hat{2}[\hat{1}\hat{2}]} \\
&= 2\Gamma_{\hat{2}[\hat{2}\hat{1}]} \\
&= h\overline{Y}_{,\hat{2}}
\end{aligned} \tag{3.1218}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{2}\hat{3}} &= \Gamma_{\hat{1}[\hat{2}\hat{3}]} + \Gamma_{\hat{2}[\hat{3}\hat{1}]} - \Gamma_{\hat{3}[\hat{1}\hat{2}]} \\
&= \frac{1}{2}(hY_{,\hat{2}} - h\overline{Y}_{,\hat{3}}) + \frac{1}{2}h\overline{Y}_{,\hat{3}} + \frac{1}{2}hY_{,\hat{2}} \\
&= hY_{,\hat{2}}.
\end{aligned} \tag{3.1219}
$$

$$
\begin{aligned}
\Gamma_{\hat{0}\hat{3}} &= \Gamma_{\hat{0}\hat{3}\hat{0}}e^{\hat{0}} + \Gamma_{\hat{0}\hat{3}\hat{1}}e^{\hat{1}} + \Gamma_{\hat{0}\hat{3}\hat{2}}e^{\hat{2}} + \Gamma_{\hat{0}\hat{3}\hat{3}}e^{\hat{3}} \\
&= -Y_{,\hat{0}}e^{\hat{0}} - (Y_{,\hat{1}} + hY_{,\hat{0}})e^{\hat{1}} + e^{\hat{2}} - Y_{,\hat{2}}e^{\hat{2}} - Y_{,\hat{3}}e^{\hat{3}} \\
&= -Y_{,\hat{a}}e^{\hat{a}} - hY_{,\hat{0}}e^{\hat{1}} \\
&= -dY - hY_{,\hat{0}}e^{\hat{1}}.
\end{aligned} \tag{3.1220}
$$

$$
\begin{aligned}
\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}} &= (\Gamma_{\hat{2}\hat{3}\hat{0}} + \Gamma_{\hat{1}\hat{0}\hat{0}})e^{\hat{0}} + (\Gamma_{\hat{2}\hat{3}\hat{1}} + \Gamma_{\hat{1}\hat{0}\hat{1}})e^{\hat{1}} + (\Gamma_{\hat{2}\hat{3}\hat{2}} + \Gamma_{\hat{1}\hat{0}\hat{2}})e^{\hat{2}} + (\Gamma_{\hat{2}\hat{3}\hat{3}} + \Gamma_{\hat{1}\hat{0}\hat{3}})e^{\hat{3}} \\
&= (0+0)e^{\hat{0}} + (h\overline{Y}_{,\hat{3}} - hY_{,\hat{2}} + h_{,\hat{0}})e^{\hat{1}} + (0 + h\overline{Y}_{,\hat{0}})e^{\hat{2}} + (0 + hY_{,\hat{0}})e^{\hat{3}}
\end{aligned} \tag{3.1221}
$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{2}} &= \Gamma_{\hat{1}\hat{2}\hat{0}}e^{\hat{0}} + \Gamma_{\hat{1}\hat{2}\hat{1}}e^{\hat{1}} + \Gamma_{\hat{1}\hat{2}\hat{2}}e^{\hat{2}} + \Gamma_{\hat{1}\hat{2}\hat{3}}e^{\hat{3}} \\
&= 0e^{\hat{0}} + (h_{,\hat{2}} - \overline{Y}_{,\hat{1}})e^{\hat{1}} + (h\overline{Y}_{,\hat{2}})e^{\hat{2}} + (hY_{,\hat{2}})e^{\hat{3}}
\end{aligned} \tag{3.1222}
$$

Let us summarise

$$
\begin{aligned}
\Gamma_{\hat{0}\hat{3}} &= -Y_{,\hat{a}}e^{\hat{a}} - hY_{,\hat{0}}e^{\hat{1}} \\
&= -dY - hY_{,\hat{0}}e^{\hat{1}}. \\
\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}} &= h\overline{Y}_{,\hat{0}}e^{\hat{2}} + hY_{,\hat{0}}e^{\hat{3}} + [h_{,\hat{0}} + h(\overline{Y}_{,\hat{3}} - Y_{,\hat{2}})]e^{\hat{1}} \\
\Gamma_{\hat{1}\hat{2}} &= h\overline{Y}_{,\hat{2}}e^{\hat{2}} + hY_{,\hat{2}}e^{\hat{3}} + (h_{,\hat{2}} - h\overline{Y}_{,\hat{1}})e^{\hat{1}}
\end{aligned} \tag{3.1223}
$$

Geodesic assumption

The $e^{\hat{2}} \wedge e^{\hat{0}}$ term in Cartan structure equation for $\Gamma_{\hat{0}\hat{3}}$

$$\begin{aligned}
d\Gamma_{\hat{0}\hat{3}} + \Gamma_{\hat{0}\hat{3}} \wedge (\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) &= d(-dY - hY_{,\hat{0}}e^{\hat{1}}) \\
&\quad + (-Y_{,\hat{a}}e^{\hat{a}} - hY_{,\hat{0}}e^{\hat{1}}) \wedge (h\overline{Y}_{,\hat{0}}e^{\hat{2}} + hY_{,\hat{0}}e^{\hat{3}} + [h_{,\hat{0}} + h(\overline{Y}_{,\hat{3}} - Y_{,\hat{2}})]e^{\hat{1}}) \\
&= \cdots - hY_{,\hat{0}}de^{\hat{1}} + (-Y_{,\hat{0}}e^{\hat{0}}) \wedge (h\overline{Y}_{,\hat{0}}e^{\hat{2}}) + \ldots \\
&= \cdots + 2hY_{,\hat{0}}\overline{Y}_{,\hat{0}}e^{\hat{2}} \wedge e^{\hat{0}} + \ldots \\
&= \cdots + \frac{1}{2}R_{\hat{0}\hat{3}\hat{2}\hat{0}}e^{\hat{2}} \wedge e^{\hat{0}} + \frac{1}{2}R_{\hat{0}\hat{3}\hat{0}\hat{2}}e^{\hat{0}} \wedge e^{\hat{2}} + \ldots \\
&= \cdots + R_{\hat{0}\hat{3}\hat{2}\hat{0}}e^{\hat{2}} \wedge e^{\hat{0}} + \ldots
\end{aligned} \tag{3.1224}$$

We have

$$\frac{1}{2}R_{\hat{0}\hat{0}} = R_{\hat{0}\hat{3}\hat{2}\hat{0}} = 2hY_{,\hat{0}}\overline{Y}_{,\hat{0}}. \tag{3.1225}$$

In a vacuum $R_{\hat{0}\hat{0}} = 0$, and it follows that

$$Y_{,\hat{0}} = -\Gamma_{\hat{0}\hat{3}\hat{0}} = \overline{\kappa} = 0 \tag{3.1226}$$

The expressions () simplify to

$$\begin{aligned}
\Gamma_{\hat{0}\hat{3}} &= -Y_{,\hat{a}}e^{\hat{a}} = -dYe^{\hat{1}}. \\
\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}} &= [h_{,\hat{0}} + h(\overline{Y}_{,\hat{3}} - Y_{,\hat{2}})]e^{\hat{1}} \\
\Gamma_{\hat{1}\hat{2}} &= h\overline{Y}_{,\hat{2}}e^{\hat{2}} + hY_{,\hat{2}}e^{\hat{3}} + (h_{,\hat{2}} - h\overline{Y}_{,\hat{1}})e^{\hat{1}}
\end{aligned} \tag{3.1227}$$

From (3.1206)

$$Z = -\Gamma_{\hat{0}\hat{3}\hat{2}} = Y_{,\hat{2}} \tag{3.1228}$$

Note that $\overline{Z} = \overline{Y}_{,\hat{3}}$. We substitute these into (3.1227) and obtain

$$\begin{aligned}
\Gamma_{\hat{0}\hat{3}} &= -Y_{,\hat{a}}e^{\hat{a}} = -dY. \\
\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}} &= [h_{,\hat{0}} + (\overline{Z} - Z)h]e^{\hat{1}} \\
\Gamma_{\hat{1}\hat{2}} &= h\overline{Y}_{,\hat{2}}e^{\hat{2}} + hZe^{\hat{3}} + (h_{,\hat{2}} - h\overline{Y}_{,\hat{1}})e^{\hat{1}}
\end{aligned} \tag{3.1229}$$

From (3.1207) the shear is given by

$$\Gamma_{\hat{0}\hat{3}\hat{3}} = -Y_{,\hat{3}}. \tag{3.1230}$$

Calculation of components of the Riemann tensor

Recall

$$d\Gamma_{\hat{0}\hat{3}} + \Gamma_{\hat{0}\hat{3}} \wedge (\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) = \frac{1}{2} R_{\hat{0}\hat{3}\hat{a}\hat{b}} e^{\hat{a}} \wedge e^{\hat{b}}$$

Since $d\Gamma_{\hat{0}\hat{3}} = -ddY = 0$

$$(-Y_{,\hat{a}} e^{\hat{a}}) \wedge ([h_{,\hat{0}} + (\overline{Z} - Z)h] e^{\hat{1}}) = \frac{1}{2} R_{\hat{0}\hat{3}\hat{a}\hat{b}} e^{\hat{a}} \wedge e^{\hat{b}}$$

$$-Y_{,\hat{a}}[h_{,\hat{0}} + (\overline{Z} - Z)h] e^{\hat{a}} \wedge e^{\hat{1}} = \frac{1}{2} R_{\hat{0}\hat{3}\hat{a}\hat{b}} e^{\hat{a}} \wedge e^{\hat{b}} \tag{3.1231}$$

$$\begin{aligned}
& -Y_{,\hat{0}}[h_{,\hat{0}} + (\overline{Z} - Z)h] e^{\hat{0}} \wedge e^{\hat{1}} \\
& -Y_{,\hat{2}}[h_{,\hat{0}} + (\overline{Z} - Z)h] e^{\hat{2}} \wedge e^{\hat{1}} \\
& -Y_{,\hat{3}}[h_{,\hat{0}} + (\overline{Z} - Z)h] e^{\hat{3}} \wedge e^{\hat{1}} \\
= \; & R_{\hat{0}\hat{3}\hat{0}\hat{1}} e^{\hat{0}} \wedge e^{\hat{1}} + R_{\hat{0}\hat{3}\hat{0}\hat{2}} e^{\hat{0}} \wedge e^{\hat{2}} + R_{\hat{0}\hat{3}\hat{0}\hat{3}} e^{\hat{0}} \wedge e^{\hat{3}} \\
& + R_{\hat{0}\hat{3}\hat{1}\hat{2}} e^{\hat{1}} \wedge e^{\hat{2}} + R_{\hat{0}\hat{3}\hat{1}\hat{3}} e^{\hat{1}} \wedge e^{\hat{3}} + R_{\hat{0}\hat{3}\hat{2}\hat{3}} e^{\hat{2}} \wedge e^{\hat{3}}.
\end{aligned} \tag{3.1232}$$

From which we read off

$$R_{\hat{0}\hat{3}\hat{0}\hat{2}} = R_{\hat{0}\hat{3}\hat{0}\hat{3}} = R_{\hat{0}\hat{3}\hat{1}\hat{0}} = R_{\hat{0}\hat{3}\hat{2}\hat{3}} = 0 \tag{3.1233}$$

$$R_{\hat{0}\hat{3}\hat{1}\hat{2}} = Y_{,\hat{2}}[h_{,\hat{0}} + (\overline{Z} - Z)h] \tag{3.1234}$$

$$R_{\hat{0}\hat{3}\hat{1}\hat{3}} = Y_{,\hat{3}}[h_{,\hat{0}} + (\overline{Z} - Z)h] \tag{3.1235}$$

For a moment we turn to Maxwell's equations.

Independent components of Maxwell's equations

The field equations for $R_{\hat{0}\hat{0}}$ are

$$\begin{aligned} R_{\hat{0}\hat{0}} &= F_{\hat{0}\hat{c}}F_{\hat{0}}{}^{\hat{c}} + \frac{1}{4}g_{\hat{0}\hat{0}}F_{\hat{c}\hat{d}}F^{\hat{c}\hat{d}} \\ &= g^{\hat{c}\hat{d}}F_{\hat{0}\hat{c}}F_{\hat{0}\hat{d}} \end{aligned} \tag{3.1236}$$

We represent the electromagnetic field by the complex tensor

$$\mathcal{F}_{ab} = -\mathcal{F}_{ba} = F_{ab} + \frac{1}{2}i\epsilon_{abcd}F^{cd} \tag{3.1237}$$

where ϵ_{abcd} is completely anti-symmetric and equal to $(-g)^{1/2}$ when $abcd = 0123$. The corresponding null tetrad components are

$$\mathcal{F}_{\hat{a}\hat{b}} = -\mathcal{F}_{\hat{b}\hat{a}} = F_{\hat{a}\hat{b}} + \frac{1}{2}i\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}F^{\hat{c}\hat{d}} \tag{3.1238}$$

where $\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is completely anti-symmetric and $\epsilon_{\hat{0}\hat{1}\hat{2}\hat{3}} = -i$.

$$\begin{aligned} \mathcal{F}_{\hat{2}\hat{3}} &= F_{\hat{2}\hat{3}} + \frac{1}{2}i\epsilon_{\hat{2}\hat{3}\hat{c}\hat{d}}F^{\hat{c}\hat{d}} \\ &= F_{\hat{2}\hat{3}} + i\epsilon_{\hat{2}\hat{3}\hat{0}\hat{1}}F^{\hat{0}\hat{1}} \\ &= F_{\hat{2}\hat{3}} + F^{\hat{0}\hat{1}} \\ &= F_{\hat{2}\hat{3}} + g^{\hat{0}\hat{c}}g^{\hat{1}\hat{d}}F_{\hat{c}\hat{d}} \\ &= F_{\hat{2}\hat{3}} + F_{\hat{1}\hat{0}}. \end{aligned} \tag{3.1239}$$

$$\begin{aligned} \mathcal{F}_{\hat{1}\hat{0}} &= F_{\hat{1}\hat{0}} + \frac{1}{2}i\epsilon_{\hat{1}\hat{0}\hat{c}\hat{d}}F^{\hat{c}\hat{d}} \\ &= F_{\hat{1}\hat{0}} + i\epsilon_{\hat{1}\hat{0}\hat{2}\hat{3}}F^{\hat{2}\hat{3}} \\ &= F_{\hat{1}\hat{0}} - F^{\hat{2}\hat{3}} \\ &= F_{\hat{1}\hat{0}} - g^{\hat{2}\hat{c}}g^{\hat{3}\hat{d}}F_{\hat{c}\hat{d}} \\ &= F_{\hat{1}\hat{0}} + F_{\hat{2}\hat{3}} \end{aligned} \tag{3.1240}$$

$$\begin{aligned}\mathcal{F}_{\hat{1}\hat{2}} &= F_{\hat{1}\hat{2}} + \frac{1}{2}i\epsilon_{\hat{1}\hat{2}\hat{c}\hat{d}}F^{\hat{c}\hat{d}}\\ &= F_{\hat{1}\hat{2}} + i\epsilon_{\hat{1}\hat{2}\hat{0}\hat{3}}F^{\hat{0}\hat{3}}\\ &= F_{\hat{1}\hat{2}} + F^{\hat{0}\hat{3}}\\ &= F_{\hat{1}\hat{2}} + g^{\hat{0}\hat{c}}g^{\hat{3}\hat{d}}F_{\hat{c}\hat{d}}\\ &= 2F_{\hat{1}\hat{2}} \end{aligned} \tag{3.1241}$$

$$\begin{aligned}\mathcal{F}_{\hat{3}\hat{1}} &= F_{\hat{3}\hat{1}} + \frac{1}{2}i\epsilon_{\hat{3}\hat{1}\hat{c}\hat{d}}F^{\hat{c}\hat{d}}\\ &= F_{\hat{3}\hat{1}} + i\epsilon_{\hat{3}\hat{1}\hat{0}\hat{2}}F^{\hat{0}\hat{2}}\\ &= F_{\hat{3}\hat{1}} + F^{\hat{0}\hat{2}}\\ &= F_{\hat{3}\hat{1}} + g^{\hat{0}\hat{c}}g^{\hat{2}\hat{d}}F_{\hat{c}\hat{d}}\\ &= F_{\hat{3}\hat{1}} + F_{\hat{1}\hat{3}} = 0. \end{aligned} \tag{3.1242}$$

$$\begin{aligned}\mathcal{F}_{\hat{2}\hat{0}} &= F_{\hat{2}\hat{0}} + \frac{1}{2}i\epsilon_{\hat{2}\hat{0}\hat{c}\hat{d}}F^{\hat{c}\hat{d}}\\ &= F_{\hat{2}\hat{0}} + i\epsilon_{\hat{2}\hat{0}\hat{1}\hat{3}}F^{\hat{1}\hat{3}}\\ &= F_{\hat{2}\hat{0}} + F^{\hat{1}\hat{3}}\\ &= F_{\hat{2}\hat{0}} + g^{\hat{1}\hat{c}}g^{\hat{3}\hat{d}}F_{\hat{c}\hat{d}}\\ &= F_{\hat{2}\hat{0}} + F_{\hat{0}\hat{2}} = 0. \end{aligned} \tag{3.1243}$$

$$\begin{aligned}\mathcal{F}_{\hat{3}\hat{0}} &= F_{\hat{3}\hat{0}} + \frac{1}{2}i\epsilon_{\hat{3}\hat{0}\hat{c}\hat{d}}F^{\hat{c}\hat{d}}\\ &= F_{\hat{3}\hat{0}} + i\epsilon_{\hat{3}\hat{0}\hat{1}\hat{2}}F^{\hat{1}\hat{2}}\\ &= F_{\hat{3}\hat{0}} + F^{\hat{1}\hat{2}}\\ &= F_{\hat{3}\hat{0}} + g^{\hat{1}\hat{c}}g^{\hat{2}\hat{d}}F_{\hat{c}\hat{d}}\\ &= F_{\hat{3}\hat{0}} + F_{\hat{0}\hat{3}} = 0. \end{aligned} \tag{3.1244}$$

Altogether we have

$$\begin{gathered}\mathcal{F}_{\hat{2}\hat{3}} = \mathcal{F}_{\hat{1}\hat{0}} = F_{\hat{2}\hat{3}} + F_{\hat{1}\hat{0}}, \quad \mathcal{F}_{\hat{1}\hat{2}} = 2F_{\hat{1}\hat{2}},\\ \mathcal{F}_{\hat{3}\hat{1}} = \mathcal{F}_{\hat{2}\hat{0}} = \mathcal{F}_{\hat{3}\hat{0}} = 0, \end{gathered} \tag{3.1245}$$

Thus the electromagnetic field is described by only two complex components,

$$\mathcal{F}_{\hat{2}\hat{3}}\,(=\mathcal{F}_{\hat{1}\hat{0}}) \quad \text{and} \quad \mathcal{F}_{\hat{1}\hat{2}}. \tag{3.1246}$$

or

$$\mathcal{F}^{\hat{3}\hat{2}}\,(=\mathcal{F}^{\hat{0}\hat{1}}) \quad \text{and} \quad \mathcal{F}^{\hat{0}\hat{3}}. \tag{3.1247}$$

Maxwell's equations

Maxwell's equations are

$$\mathcal{F}^{\hat{a}\hat{b}}{}_{;\hat{b}} = \mathcal{F}^{\hat{a}\hat{b}}{}_{,\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} + \Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{a}\hat{c}} = 0. \tag{3.1248}$$

Now note for $\mathcal{F}^{\hat{2}\hat{b}}{}_{;\hat{b}}$

$$\mathcal{F}^{\hat{2}\hat{b}}{}_{;\hat{b}} = \mathcal{F}^{\hat{2}\hat{0}}{}_{;\hat{0}} + \mathcal{F}^{\hat{2}\hat{1}}{}_{;\hat{1}} + \mathcal{F}^{\hat{2}\hat{2}}{}_{;\hat{2}} + \mathcal{F}^{\hat{2}\hat{3}}{}_{;\hat{3}} = -\mathcal{F}^{\hat{3}\hat{2}}{}_{;\hat{3}} \tag{3.1249}$$

and also

$$\mathcal{F}^{\hat{2}\hat{b}}{}_{,\hat{b}} = -\mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{3}} \tag{3.1250}$$

$$\begin{aligned}
\Gamma^{\hat{2}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} &= \Gamma^{\hat{2}}{}_{\hat{0}\hat{1}}\mathcal{F}^{\hat{0}\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{1}\hat{0}}\mathcal{F}^{\hat{1}\hat{0}} + \ldots \\
&= (\Gamma^{\hat{2}}{}_{\hat{0}\hat{1}} - \Gamma^{\hat{2}}{}_{\hat{1}\hat{0}})\mathcal{F}^{\hat{0}\hat{1}} + \ldots \\
&= 2\Gamma^{\hat{2}}{}_{[\hat{0}\hat{1}]}\mathcal{F}^{\hat{0}\hat{1}} + 2\Gamma^{\hat{2}}{}_{[\hat{0}\hat{2}]}\mathcal{F}^{\hat{0}\hat{2}} + 2\Gamma^{\hat{2}}{}_{[\hat{0}\hat{3}]}\mathcal{F}^{\hat{0}\hat{3}} \\
&\quad +2\Gamma^{\hat{3}}{}_{[\hat{1}\hat{2}]}\mathcal{F}^{\hat{1}\hat{2}} + 2\Gamma^{\hat{2}}{}_{[\hat{1}\hat{3}]}\mathcal{F}^{\hat{1}\hat{3}} + 2\Gamma^{\hat{2}}{}_{[\hat{2}\hat{3}]}\mathcal{F}^{\hat{2}\hat{3}} \\
&= 2(\Gamma^{\hat{2}}{}_{[\hat{0}\hat{1}]} - \Gamma^{\hat{2}}{}_{[\hat{2}\hat{3}]})\mathcal{F}^{\hat{3}\hat{2}} + 2\Gamma^{\hat{2}}{}_{[\hat{0}\hat{3}]}\mathcal{F}^{\hat{0}\hat{3}} \\
&= (Y_{,\hat{1}} + hY_{,\hat{0}})\mathcal{F}^{\hat{3}\hat{2}} + Y_{,\hat{3}}\mathcal{F}^{\hat{0}\hat{3}} \\
&= Y_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} + Y_{,\hat{3}}\mathcal{F}^{\hat{0}\hat{3}}
\end{aligned} \tag{3.1251}$$

The term $\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{2}\hat{c}}$ becomes

$$\begin{aligned}
\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{2}\hat{c}} &= \Gamma^{\hat{0}}{}_{\hat{1}\hat{0}}\mathcal{F}^{\hat{2}\hat{1}} + \Gamma^{\hat{1}}{}_{\hat{0}\hat{1}}\mathcal{F}^{\hat{2}\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{2}\hat{0}}\mathcal{F}^{\hat{2}\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{0}\hat{2}}\mathcal{F}^{\hat{2}\hat{0}} \\
&\quad +\Gamma^{\hat{0}}{}_{\hat{3}\hat{0}}\mathcal{F}^{\hat{2}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{0}\hat{3}}\mathcal{F}^{\hat{2}\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{2}\hat{1}}\mathcal{F}^{\hat{2}\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{1}\hat{2}}\mathcal{F}^{\hat{2}\hat{1}} + \\
&\quad +\Gamma^{\hat{1}}{}_{\hat{3}\hat{1}}\mathcal{F}^{\hat{2}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{1}\hat{3}}\mathcal{F}^{\hat{2}\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{3}\hat{2}}\mathcal{F}^{\hat{2}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{2}\hat{3}}\mathcal{F}^{\hat{2}\hat{2}} \\
&= -(\Gamma^{\hat{0}}{}_{\hat{3}\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{3}\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{3}\hat{2}})\mathcal{F}^{\hat{3}\hat{2}}
\end{aligned} \tag{3.1252}$$

$$
\begin{aligned}
\Gamma^{\hat{0}}{}_{\hat{3}\hat{0}} = \Gamma_{\hat{1}\hat{3}\hat{0}} &= \Gamma_{\hat{1}[\hat{3}\hat{0}]} + \Gamma_{\hat{3}[\hat{0}\hat{1}]} - \Gamma_{\hat{0}[\hat{1}\hat{3}]} \\
&= -\frac{1}{2}hY_{,\hat{0}} + \frac{1}{2}(Y_{,\hat{1}} + hY_{,\hat{0}}) - \frac{1}{2}Y_{,\hat{1}} \\
&= 0
\end{aligned} \tag{3.1253}
$$

$$
\Gamma^{\hat{1}}{}_{\hat{3}\hat{1}} = \Gamma_{\hat{0}\hat{3}\hat{1}} = -Y_{,\hat{1}} - hY_{,\hat{0}}.
$$

$$
\Gamma^{\hat{2}}{}_{\hat{3}\hat{2}} = \Gamma_{\hat{3}\hat{3}\hat{2}} = -\Gamma_{\hat{3}\hat{3}\hat{2}} = 0.
$$

So that $\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{2}\hat{c}} = Y_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}}$. The Maxwell equation

$$
\mathcal{F}^{\hat{2}\hat{b}}{}_{,\hat{b}} + \Gamma^{\hat{2}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} + \Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{2}\hat{c}} = 0
$$

becomes

$$
\mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{3}} - 2Y_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} - Y_{,\hat{3}}\mathcal{F}^{\hat{0}\hat{3}} = 0. \tag{3.1254}
$$

or

$$
\mathcal{F}_{\hat{2}\hat{3},\hat{3}} - 2Y_{,\hat{1}}\mathcal{F}_{\hat{2}\hat{3}} - Y_{,\hat{3}}\mathcal{F}_{\hat{1}\hat{2}} = 0. \tag{3.1255}
$$

□

Note

$$
\mathcal{F}^{\hat{3}\hat{b}}{}_{;\hat{b}} = \mathcal{F}^{\hat{3}\hat{0}}{}_{;\hat{0}} + \mathcal{F}^{\hat{3}\hat{1}}{}_{;\hat{1}} + \mathcal{F}^{\hat{3}\hat{2}}{}_{;\hat{2}} = -\mathcal{F}^{\hat{0}\hat{3}}{}_{;\hat{0}} + \mathcal{F}^{\hat{3}\hat{2}}{}_{;\hat{2}} \tag{3.1256}
$$

and also

$$
\mathcal{F}^{\hat{3}\hat{b}}{}_{,\hat{b}} = -\mathcal{F}^{\hat{0}\hat{3}}{}_{,\hat{0}} + \mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{2}} \tag{3.1257}
$$

The term $\Gamma^{\hat{3}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}}$ becomes

$$
\begin{aligned}
\Gamma^{\hat{3}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} &= 2(\Gamma^{\hat{3}}{}_{[\hat{0}\hat{1}]} - \Gamma^{\hat{3}}{}_{[\hat{2}\hat{3}]})\mathcal{F}^{\hat{3}\hat{2}} + 2\Gamma^{\hat{3}}{}_{[\hat{0}\hat{3}]}\mathcal{F}^{\hat{0}\hat{3}} \\
&= (\overline{Y}_{,\hat{1}} + hY_{,\hat{0}})\mathcal{F}^{\hat{3}\hat{2}} + \overline{Y}_{,\hat{3}}\mathcal{F}^{\hat{0}\hat{3}} \\
&= \overline{Y}_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} + \overline{Y}_{,\hat{3}}\mathcal{F}^{\hat{0}\hat{3}} \\
&= \overline{Y}_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} + \overline{Z}\mathcal{F}^{\hat{0}\hat{3}}
\end{aligned} \tag{3.1258}
$$

The term $\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{3}\hat{c}}$ becomes

$$\begin{aligned}
\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{3}\hat{c}} &= \Gamma^{\hat{0}}{}_{\hat{1}\hat{0}}\mathcal{F}^{\hat{3}\hat{1}} + \Gamma^{\hat{1}}{}_{\hat{0}\hat{1}}\mathcal{F}^{\hat{3}\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{2}\hat{0}}\mathcal{F}^{\hat{3}\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{0}\hat{2}}\mathcal{F}^{\hat{3}\hat{0}} \\
&\quad + \Gamma^{\hat{0}}{}_{\hat{3}\hat{0}}\mathcal{F}^{\hat{3}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{0}\hat{3}}\mathcal{F}^{\hat{3}\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{2}\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{1}\hat{2}}\mathcal{F}^{\hat{3}\hat{1}} + \\
&\quad + \Gamma^{\hat{1}}{}_{\hat{3}\hat{1}}\mathcal{F}^{\hat{3}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{1}\hat{3}}\mathcal{F}^{\hat{3}\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{3}\hat{2}}\mathcal{F}^{\hat{3}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{2}\hat{3}}\mathcal{F}^{\hat{3}\hat{2}} \\
&= (\Gamma^{\hat{0}}{}_{\hat{2}\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{2}\hat{1}} + \Gamma^{\hat{3}}{}_{\hat{2}\hat{3}})\mathcal{F}^{\hat{3}\hat{2}} - \Gamma^{\hat{2}}{}_{\hat{0}\hat{2}}\mathcal{F}^{\hat{0}\hat{3}}
\end{aligned} \tag{3.1259}$$

$$\Gamma^{\hat{0}}{}_{\hat{2}\hat{0}} = \Gamma_{\hat{1}\hat{2}\hat{0}} = 0.$$

$$\begin{aligned}
\Gamma^{\hat{1}}{}_{\hat{2}\hat{1}} = \Gamma_{\hat{0}\hat{2}\hat{1}} &= \Gamma_{\hat{0}[\hat{2}\hat{1}]} + \Gamma_{\hat{2}[\hat{1}\hat{0}]} - \Gamma_{\hat{1}[\hat{0}\hat{2}]} \\
&= -\frac{1}{2}\overline{Y}_{,\hat{1}} - \frac{1}{2}(\overline{Y}_{,\hat{1}} + h\overline{Y}_{,\hat{0}}) - \frac{1}{2}h\overline{Y}_{,\hat{0}} \\
&= -\overline{Y}_{,\hat{1}} - h\overline{Y}_{,\hat{0}}.
\end{aligned} \tag{3.1260}$$

$$\Gamma^{\hat{3}}{}_{\hat{2}\hat{3}} = \Gamma_{\hat{2}\hat{2}\hat{3}} = 0.$$

$$\Gamma^{\hat{2}}{}_{\hat{0}\hat{2}} = \Gamma_{\hat{3}\hat{0}\hat{2}} = -\Gamma_{\hat{0}\hat{3}\hat{2}} = Y_{,\hat{2}} = Z.$$

We have

$$\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{3}\hat{c}} = -\overline{Y}_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} - Z\mathcal{F}^{\hat{0}\hat{3}}$$

The Maxwell's equations

$$\mathcal{F}^{\hat{3}\hat{b}}{}_{;\hat{b}} = \mathcal{F}^{\hat{3}\hat{b}}{}_{,\hat{b}} + \Gamma^{\hat{3}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} + \Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{3}\hat{c}} = 0. \tag{3.1261}$$

$$\mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{2}} - \mathcal{F}^{\hat{0}\hat{3}}{}_{,\hat{0}} + \overline{Y}_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} + \overline{Z}\mathcal{F}^{\hat{0}\hat{3}} - \overline{Y}_{,\hat{1}}\mathcal{F}^{\hat{3}\hat{2}} - Z\mathcal{F}^{\hat{0}\hat{3}} = 0$$

that is

$$\mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{2}} - \mathcal{F}^{\hat{0}\hat{3}}{}_{,\hat{0}} - Z\mathcal{F}^{\hat{0}\hat{3}} = 0 \tag{3.1262}$$

or

$$\mathcal{F}_{\hat{2}\hat{3},\hat{2}} - \mathcal{F}_{\hat{1}\hat{2},\hat{0}} - Z\mathcal{F}_{\hat{1}\hat{2}} = 0 \tag{3.1263}$$

$\square$

Now note for ${\mathcal{F}^{\hat{1}\hat{b}}}_{;\hat{b}}$

$${\mathcal{F}^{\hat{1}\hat{b}}}_{;\hat{b}} = {\mathcal{F}^{\hat{1}\hat{0}}}_{;\hat{0}} + {\mathcal{F}^{\hat{1}\hat{1}}}_{;\hat{1}} + {\mathcal{F}^{\hat{1}\hat{2}}}_{;\hat{2}} + {\mathcal{F}^{\hat{1}\hat{3}}}_{;\hat{3}} = -{\mathcal{F}^{\hat{0}\hat{1}}}_{;\hat{0}} \tag{3.1264}$$

and also

$${\mathcal{F}^{\hat{1}\hat{b}}}_{,\hat{b}} = -{\mathcal{F}^{\hat{0}\hat{1}}}_{,\hat{0}} = -{\mathcal{F}^{\hat{3}\hat{2}}}_{,\hat{0}} \tag{3.1265}$$

The term ${\Gamma^{\hat{1}}}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}}$ becomes

$$\begin{aligned}
{\Gamma^{\hat{1}}}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} &= 2({\Gamma^{\hat{1}}}_{[\hat{0}\hat{1}]} - {\Gamma^{\hat{1}}}_{[\hat{2}\hat{3}]})\mathcal{F}^{\hat{3}\hat{2}} + 2{\Gamma^{\hat{1}}}_{[\hat{0}\hat{3}]}\mathcal{F}^{\hat{0}\hat{3}} \\
&= (\overline{Y}_{,\hat{3}} - Y_{,\hat{2}})\mathcal{F}^{\hat{3}\hat{2}} + Y_{,\hat{0}}\mathcal{F}^{\hat{0}\hat{3}} \\
&= (\overline{Y}_{,\hat{3}} - Z)\mathcal{F}^{\hat{3}\hat{2}}
\end{aligned} \tag{3.1266}$$

The term ${\Gamma^{\hat{b}}}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{1}\hat{c}}$ becomes

$$\begin{aligned}
{\Gamma^{\hat{b}}}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{1}\hat{c}} &= {\Gamma^{\hat{0}}}_{\hat{1}\hat{0}}\mathcal{F}^{\hat{1}\hat{1}} + {\Gamma^{\hat{1}}}_{\hat{0}\hat{1}}\mathcal{F}^{\hat{1}\hat{0}} + {\Gamma^{\hat{0}}}_{\hat{2}\hat{0}}\mathcal{F}^{\hat{1}\hat{2}} + {\Gamma^{\hat{2}}}_{\hat{0}\hat{2}}\mathcal{F}^{\hat{1}\hat{0}} \\
&\quad + {\Gamma^{\hat{0}}}_{\hat{3}\hat{0}}\mathcal{F}^{\hat{1}\hat{3}} + {\Gamma^{\hat{3}}}_{\hat{0}\hat{3}}\mathcal{F}^{\hat{1}\hat{0}} + {\Gamma^{\hat{1}}}_{\hat{2}\hat{1}}\mathcal{F}^{\hat{1}\hat{2}} + {\Gamma^{\hat{2}}}_{\hat{1}\hat{2}}\mathcal{F}^{\hat{1}\hat{1}} + \\
&\quad + {\Gamma^{\hat{1}}}_{\hat{3}\hat{1}}\mathcal{F}^{\hat{1}\hat{3}} + {\Gamma^{\hat{3}}}_{\hat{1}\hat{3}}\mathcal{F}^{\hat{1}\hat{1}} + {\Gamma^{\hat{2}}}_{\hat{3}\hat{2}}\mathcal{F}^{\hat{1}\hat{3}} + {\Gamma^{\hat{3}}}_{\hat{2}\hat{3}}\mathcal{F}^{\hat{1}\hat{2}} \\
&= ({\Gamma^{\hat{1}}}_{\hat{0}\hat{1}} + {\Gamma^{\hat{2}}}_{\hat{0}\hat{2}} + {\Gamma^{\hat{3}}}_{\hat{0}\hat{3}})\mathcal{F}^{\hat{1}\hat{0}} \\
&= -({\Gamma^{\hat{1}}}_{\hat{0}\hat{1}} + {\Gamma^{\hat{2}}}_{\hat{0}\hat{2}} + {\Gamma^{\hat{3}}}_{\hat{0}\hat{3}})\mathcal{F}^{\hat{3}\hat{2}}
\end{aligned} \tag{3.1267}$$

$${\Gamma^{\hat{1}}}_{\hat{0}\hat{1}} = \Gamma_{\hat{0}\hat{0}\hat{1}} = 0.$$

$${\Gamma^{\hat{2}}}_{\hat{0}\hat{2}} = \Gamma_{\hat{3}\hat{0}\hat{2}} = -\Gamma_{\hat{0}\hat{3}\hat{2}} = Y_{,\hat{2}} = Z$$

$$\begin{aligned}
{\Gamma^{\hat{3}}}_{\hat{0}\hat{3}} = \Gamma_{\hat{2}\hat{0}\hat{3}} &= \Gamma_{\hat{2}[\hat{0}\hat{3}]} + \Gamma_{\hat{0}[\hat{3}\hat{2}]} - \Gamma_{\hat{3}[\hat{2}\hat{0}]} \\
&= \frac{1}{2}\overline{Y}_{,\hat{3}} - \frac{1}{2}(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}) + \frac{1}{2}Y_{,\hat{2}} \\
&= \overline{Y}_{,\hat{3}}.
\end{aligned} \tag{3.1268}$$

$$\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{1}\hat{c}} = -(Z + \overline{Y}_{,\hat{3}})\mathcal{F}^{\hat{3}\hat{2}}$$

The Maxwell's equations

$$\mathcal{F}^{\hat{1}\hat{b}}{}_{;\hat{b}} = \mathcal{F}^{\hat{1}\hat{b}}{}_{,\hat{b}} + \Gamma^{\hat{1}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} + \Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{1}\hat{c}} = 0. \tag{3.1269}$$

$$-\mathcal{F}^{\hat{0}\hat{1}}{}_{,\hat{0}} + (\overline{Y}_{,\hat{3}} - Z)\mathcal{F}^{\hat{3}\hat{2}} - (Z + \overline{Y}_{,\hat{3}})\mathcal{F}^{\hat{3}\hat{2}} = 0$$

that is

$$\mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{0}} + 2Z\mathcal{F}^{\hat{3}\hat{2}} = 0 \tag{3.1270}$$

or

$$\mathcal{F}_{\hat{2}\hat{3},\hat{0}} + 2Z\mathcal{F}_{\hat{2}\hat{3}} = 0 \tag{3.1271}$$

□

Now note for $\mathcal{F}^{\hat{0}\hat{b}}{}_{;\hat{b}}$

$$\mathcal{F}^{\hat{0}\hat{b}}{}_{;\hat{b}} = \mathcal{F}^{\hat{0}\hat{0}}{}_{;\hat{0}} + \mathcal{F}^{\hat{0}\hat{1}}{}_{;\hat{1}} + \mathcal{F}^{\hat{0}\hat{2}}{}_{;\hat{2}} + \mathcal{F}^{\hat{0}\hat{3}}{}_{;\hat{3}} = \mathcal{F}^{\hat{3}\hat{2}}{}_{;\hat{1}} + \mathcal{F}^{\hat{0}\hat{3}}{}_{;\hat{3}} \tag{3.1272}$$

and also

$$\mathcal{F}^{\hat{0}\hat{b}}{}_{,\hat{b}} = \mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{1}} + \mathcal{F}^{\hat{0}\hat{3}}{}_{,\hat{3}} \tag{3.1273}$$

The term $\Gamma^{\hat{0}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}}$ becomes

$$\begin{aligned}
\Gamma^{\hat{0}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} &= 2(\Gamma^{\hat{0}}{}_{[\hat{0}\hat{1}]} - \Gamma^{\hat{0}}{}_{[\hat{2}\hat{3}]})\mathcal{F}^{\hat{3}\hat{2}} + 2\Gamma^{\hat{0}}{}_{[\hat{0}\hat{3}]}\mathcal{F}^{\hat{0}\hat{3}} \\
&= (h_{,\hat{0}} - hY_{,\hat{2}} + h\overline{Y}_{,\hat{3}})\mathcal{F}^{\hat{3}\hat{2}} + hY_{,\hat{0}}\mathcal{F}^{\hat{0}\hat{3}} \\
&= (h_{,\hat{0}} - hZ + h\overline{Y}_{,\hat{3}})\mathcal{F}^{\hat{3}\hat{2}}
\end{aligned} \tag{3.1274}$$

The term $\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{0}\hat{c}}$ becomes

$$
\begin{aligned}
\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{0}\hat{c}} &= \Gamma^{\hat{0}}{}_{\hat{1}\hat{0}}\mathcal{F}^{\hat{0}\hat{1}} + \Gamma^{\hat{1}}{}_{\hat{0}\hat{1}}\mathcal{F}^{\hat{0}\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{2}\hat{0}}\mathcal{F}^{\hat{0}\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{0}\hat{2}}\mathcal{F}^{\hat{0}\hat{0}} \\
&\quad + \Gamma^{\hat{0}}{}_{\hat{3}\hat{0}}\mathcal{F}^{\hat{0}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{0}\hat{3}}\mathcal{F}^{\hat{0}\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{2}\hat{1}}\mathcal{F}^{\hat{0}\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{1}\hat{2}}\mathcal{F}^{\hat{0}\hat{1}} + \\
&\quad + \Gamma^{\hat{1}}{}_{\hat{3}\hat{1}}\mathcal{F}^{\hat{0}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{1}\hat{3}}\mathcal{F}^{\hat{0}\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{3}\hat{2}}\mathcal{F}^{\hat{0}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{2}\hat{3}}\mathcal{F}^{\hat{0}\hat{2}} \\
&= \Gamma^{\hat{0}}{}_{\hat{1}\hat{0}}\mathcal{F}^{\hat{0}\hat{1}} + \Gamma^{\hat{0}}{}_{\hat{3}\hat{0}}\mathcal{F}^{\hat{0}\hat{3}} + \Gamma^{\hat{2}}{}_{\hat{1}\hat{2}}\mathcal{F}^{\hat{0}\hat{1}} + \\
&\quad + \Gamma^{\hat{1}}{}_{\hat{3}\hat{1}}\mathcal{F}^{\hat{0}\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{1}\hat{3}}\mathcal{F}^{\hat{0}\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{3}\hat{2}}\mathcal{F}^{\hat{0}\hat{3}} \\
&= (\Gamma^{\hat{0}}{}_{\hat{1}\hat{0}} + \Gamma^{\hat{2}}{}_{\hat{1}\hat{2}} + \Gamma^{\hat{3}}{}_{\hat{1}\hat{3}})\mathcal{F}^{\hat{3}\hat{2}} \\
&\quad + (\Gamma^{\hat{0}}{}_{\hat{3}\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{3}\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{3}\hat{2}})\mathcal{F}^{\hat{0}\hat{3}} \qquad (3.1275)
\end{aligned}
$$

$$
\Gamma^{\hat{0}}{}_{\hat{1}\hat{0}} = \Gamma_{\hat{1}\hat{1}\hat{0}} = 0.
$$

$$
\begin{aligned}
\Gamma^{\hat{2}}{}_{\hat{1}\hat{2}} = \Gamma_{\hat{3}\hat{1}\hat{2}} &= \Gamma_{\hat{3}[\hat{1}\hat{2}]} + \Gamma_{\hat{1}[\hat{2}\hat{3}]} - \Gamma_{\hat{2}[\hat{3}\hat{1}]} \\
&= -\frac{1}{2}hY_{,\hat{2}} + \frac{1}{2}(hY_{,\hat{2}} - h\overline{Y}_{,\hat{3}}) - \frac{1}{2}h\overline{Y}_{,\hat{3}} \\
&= -h\overline{Y}_{,\hat{3}} \qquad (3.1276)
\end{aligned}
$$

$$
\Gamma^{\hat{3}}{}_{\hat{1}\hat{3}} = \Gamma_{\hat{2}\hat{1}\hat{3}} = -\Gamma_{\hat{1}\hat{2}\hat{3}} = -hY_{,\hat{2}} = -hZ
$$

$$
\begin{aligned}
\Gamma^{\hat{0}}{}_{\hat{3}\hat{0}} = \Gamma_{\hat{1}\hat{3}\hat{0}} &= \Gamma_{\hat{1}[\hat{3}\hat{0}]} + \Gamma_{\hat{3}[\hat{0}\hat{1}]} - \Gamma_{\hat{0}[\hat{1}\hat{3}]} \\
&= -\frac{1}{2}hY_{,\hat{0}} + \frac{1}{2}(Y_{,\hat{1}} + hY_{,\hat{0}}) - \frac{1}{2}Y_{,\hat{1}} \\
&= 0. \qquad (3.1277)
\end{aligned}
$$

$$
\Gamma^{\hat{1}}{}_{\hat{3}\hat{1}} = \Gamma_{\hat{0}\hat{3}\hat{1}} = -Y_{,\hat{1}} - hY_{,\hat{0}}
$$

$$
\Gamma^{\hat{2}}{}_{\hat{3}\hat{2}} = \Gamma_{\hat{3}\hat{3}\hat{2}} = 0.
$$

$$
\Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{0}\hat{c}} = -h(\overline{Y}_{,\hat{3}} + Z)\mathcal{F}^{\hat{3}\hat{2}} - Y_{,\hat{1}}\mathcal{F}^{\hat{0}\hat{3}}
$$

The Maxwell's equations

$$
\mathcal{F}^{\hat{0}\hat{b}}{}_{;\hat{b}} = \mathcal{F}^{\hat{0}\hat{b}}{}_{,\hat{b}} + \Gamma^{\hat{0}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{c}\hat{b}} + \Gamma^{\hat{b}}{}_{\hat{c}\hat{b}}\mathcal{F}^{\hat{0}\hat{c}} = 0. \qquad (3.1278)
$$

$$\mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{1}} + \mathcal{F}^{\hat{0}\hat{3}}{}_{,\hat{3}} + (h_{,\hat{0}} - hZ + h\overline{Y}_{,\hat{3}})\mathcal{F}^{\hat{3}\hat{2}} - h(\overline{Y}_{,\hat{3}} + Z)\mathcal{F}^{\hat{3}\hat{2}} - Y_{,\hat{1}}\mathcal{F}^{\hat{0}\hat{3}} = 0$$

That is

$$\mathcal{F}^{\hat{3}\hat{2}}{}_{,\hat{1}} + \mathcal{F}^{\hat{0}\hat{3}}{}_{,\hat{3}} - 2hZ\mathcal{F}^{\hat{3}\hat{2}} - Y_{,\hat{1}}\mathcal{F}^{\hat{0}\hat{3}} = 0 \tag{3.1279}$$

or

$$\mathcal{F}_{\hat{2}\hat{3},\hat{1}} + \mathcal{F}_{\hat{1}\hat{2},\hat{3}} - 2hZ\mathcal{F}_{\hat{2}\hat{3}} - Y_{,\hat{1}}\mathcal{F}_{\hat{1}\hat{2}} = 0 \tag{3.1280}$$

$\square$

Altogether, Maxwell's equations are

$$\begin{aligned}
\mathcal{F}_{\hat{2}\hat{3},\hat{3}} - 2Y_{,\hat{1}}\mathcal{F}_{\hat{2}\hat{3}} - Y_{,\hat{3}}\mathcal{F}_{\hat{1}\hat{2}} &= 0 && (3.1281)\\
\mathcal{F}_{\hat{2}\hat{3},\hat{2}} - \mathcal{F}_{\hat{1}\hat{2},\hat{0}} - Z\mathcal{F}_{\hat{1}\hat{2}} &= 0 && (3.1282)\\
\mathcal{F}_{\hat{2}\hat{3},\hat{0}} + 2Z\mathcal{F}_{\hat{2}\hat{3}} &= 0 && (3.1283)\\
\mathcal{F}_{\hat{2}\hat{3},\hat{1}} + \mathcal{F}_{\hat{1}\hat{2},\hat{3}} - 2Zh\mathcal{F}_{\hat{2}\hat{3}} - Y_{,\hat{1}}\mathcal{F}_{\hat{1}\hat{2}} &= 0. && (3.1284)
\end{aligned}$$

Calculation of components of the Riemann tensor

We calculate $R_{\hat{2}\hat{3}\hat{2}\hat{3}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}}$ and $R_{\hat{2}\hat{3}\hat{1}\hat{0}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}}$ which contribute to $R_{\hat{2}\hat{3}}$ and $R_{\hat{1}\hat{0}}$ respectively. Recall

$$d(\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) + 2\Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{1}\hat{2}} = \frac{1}{2}(R_{\hat{2}\hat{3}\hat{a}\hat{b}} + R_{\hat{1}\hat{0}\hat{a}\hat{b}})e^{\hat{a}} \wedge e^{\hat{b}}$$

$$d([h_{,\hat{0}} + (\overline{Z} - Z)h]e^{\hat{1}}) + 2(-Y_{\hat{a}}e^{\hat{a}}) \wedge (h\overline{Y}_{,\hat{2}}e^{\hat{2}} + hZe^{\hat{3}} + (h_{,\hat{2}} - h\overline{Y}_{,\hat{1}})e^{\hat{1}})$$

$$\begin{aligned}
&d([h_{,\hat{0}} + (\overline{Z} - Z)h]e^{\hat{1}}) + 2(-Y_{\hat{a}}e^{\hat{a}}) \wedge (h\overline{Y}_{,\hat{2}}e^{\hat{2}} + hZe^{\hat{3}} + (h_{,\hat{2}} - h\overline{Y}_{,\hat{1}})e^{\hat{1}})\\
&= [h_{,\hat{0}} + (\overline{Z} - Z)h]_{,\hat{0}}e^{\hat{0}} \wedge e^{\hat{1}} + \cdots + [h_{,\hat{0}} + (\overline{Z} - Z)h](Y_{,\hat{2}} - \overline{Y}_{,\hat{3}})e^{\hat{2}} \wedge e^{\hat{3}}\\
&-2Y_{,\hat{0}}(h_{,\hat{2}} - h\overline{Y}_{,\hat{1}})e^{\hat{0}} \wedge e^{\hat{1}} + \cdots - 2hY_{,\hat{3}}\overline{Y}_{,\hat{2}}e^{\hat{3}} \wedge e^{\hat{2}} + \cdots - 2hZY_{,\hat{2}}e^{\hat{2}} \wedge e^{\hat{3}}
\end{aligned} \tag{3.1285}$$

comparing this to

$$
\begin{aligned}
&\frac{1}{2}(R_{\hat{2}\hat{3}\hat{a}\hat{b}} + R_{\hat{1}\hat{0}\hat{a}\hat{b}})e^{\hat{a}} \wedge e^{\hat{b}} \\
&= \cdots + (R_{\hat{2}\hat{3}\hat{1}\hat{0}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}})e^{\hat{1}} \wedge e^{\hat{0}} + \\
&+(R_{\hat{2}\hat{3}\hat{2}\hat{3}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}})e^{\hat{2}} \wedge e^{\hat{3}} \ldots
\end{aligned} \tag{3.1286}
$$

we read off

$$
R_{\hat{2}\hat{3}\hat{2}\hat{3}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}} = [h_{,\hat{0}} + (\overline{Z} - Z)h](Z - \overline{Z}) + 2hY_{,\hat{3}}\overline{Y}_{,\hat{2}} - 2hZ^2 \tag{3.1287}
$$

$$
R_{\hat{2}\hat{3}\hat{1}\hat{0}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}} = -[h_{,\hat{0}} + (\overline{Z} - Z)h]_{,\hat{0}} + 2Y_{,\hat{0}}(h_{,\hat{2}} - h\overline{Y}_{,\hat{1}}) \tag{3.1288}
$$

Using (3.1234) we find

$$
\begin{aligned}
R_{\hat{2}\hat{3}} &= R_{\hat{2}\hat{3}\hat{2}\hat{3}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}} \\
&= [h_{,\hat{0}} + (\overline{Z} - Z)h](Z - \overline{Z}) + 2hY_{,\hat{3}}\overline{Y}_{,\hat{2}} - 2hZ^2 - 2Z[h_{,\hat{0}} + (\overline{Z} - Z)h] \\
&= -(Z + \overline{Z})[h_{,\hat{0}} + (\overline{Z} - Z)h] - 2Z^2h + 2Y_{,\hat{3}}\overline{Y}_{,\hat{2}}h
\end{aligned} \tag{3.1289}
$$

$$
\begin{aligned}
R_{\hat{1}\hat{0}} &= R_{\hat{2}\hat{3}\hat{1}\hat{0}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}} \\
&= -[h_{,\hat{0}} + (\overline{Z} - Z)h]_{,\hat{0}} + 2Y_{,\hat{0}}(h_{,\hat{2}} - h\overline{Y}_{,\hat{1}}) - 2Z[h_{,\hat{0}} + (\overline{Z} - Z)h] \\
&= -[h_{,\hat{0}} + (\overline{Z} - Z)h]_{,\hat{0}} - 2Z[h_{,\hat{0}} + (\overline{Z} - Z)h]
\end{aligned} \tag{3.1290}
$$

The gravitational field equations are

$$
-(\overline{Z} + Z)[h_{,\hat{0}} + (\overline{Z} - Z)h] - 2Z^2h + 2Y_{,\hat{3}}\overline{Y}_{,\hat{2}} = -\mathcal{F}_{\hat{2}\hat{3}}\overline{\mathcal{F}}_{\hat{2}\hat{3}} \tag{3.1291}
$$

and

$$
-[h_{,\hat{0}} + (\overline{Z} - Z)h]_{,\hat{0}} - 2Z[h_{,\hat{0}} + (\overline{Z} - Z)h] = \mathcal{F}_{\hat{2}\hat{3}}\overline{\mathcal{F}}_{\hat{2}\hat{3}} \tag{3.1292}
$$

Let us now assume $Y_{,\hat{3}} \neq 0$

Solving the field equations for h

Consider

$$Y_{,[\hat{2},\hat{0}]} = Y_{,\hat{c}}\Gamma^{\hat{c}}{}_{[\hat{2},\hat{0}]}$$

First the LHS is

$$Y_{,[\hat{2},\hat{0}]} = \frac{1}{2}[(Y_{,\hat{2}})_{,\hat{0}} - (Y_{,\hat{0}})_{,\hat{2}}] = \frac{1}{2}Z_{,\hat{0}} \tag{3.1293}$$

then the RHS is

$$\begin{aligned} Y_{,\hat{c}}\Gamma^{\hat{c}}{}_{[\hat{2},\hat{0}]} &= Y_{,\hat{0}}\Gamma^{\hat{0}}{}_{[\hat{2},\hat{0}]} + Y_{,\hat{1}}\Gamma^{\hat{1}}{}_{[\hat{2},\hat{0}]} + Y_{,\hat{2}}\Gamma^{\hat{2}}{}_{[\hat{2},\hat{0}]} + Y_{,\hat{3}}\Gamma^{\hat{3}}{}_{[\hat{2},\hat{0}]} \\ &= Y_{,\hat{1}}(-\frac{1}{2}\overline{Y}_{,\hat{0}}) + Y_{,\hat{2}}(-\frac{1}{2}Y_{,\hat{2}}) + Y_{,\hat{3}}(-\frac{1}{2}\overline{Y}_{,\hat{2}}) \\ &= -\frac{1}{2}Z^2 \end{aligned} \tag{3.1294}$$

Therefore we have

$$Z_{,\hat{0}} = -Z^2. \tag{3.1295}$$

Adding the field equations (3.1291) and (3.1292), we obtain

$$\begin{aligned} 0 &= -(\overline{Z}+Z)[h_{,\hat{0}} + (\overline{Z}-Z)h] - 2Z^2h - [h_{,\hat{0}} + (\overline{Z}-Z)h]_{,\hat{0}} - 2Z[h_{,\hat{0}} + (\overline{Z}-Z)h] \\ &= -(\overline{Z}+3Z)h_{,\hat{0}} + (Z^2 - \overline{Z}^2)h - 2Z^2h - 2Z(\overline{Z}-Z)h - [h_{,\hat{0}\hat{0}} + (\overline{Z}-Z)h_{,\hat{0}} + h(\overline{Z}-Z)_{,\hat{0}}] \\ &= -2(\overline{Z}+Z)h_{,\hat{0}} + (\overline{Z}-Z)^2h - h_{,\hat{0}\hat{0}} + h(\overline{Z}-Z)_{,\hat{0}} \\ &= -h_{,\hat{0}\hat{0}} - 2(\overline{Z}+Z)h_{,\hat{0}} + (\overline{Z}-Z)^2h - (\overline{Z}^2 - Z^2)h \\ &= -h_{,\hat{0}\hat{0}} - 2(\overline{Z}+Z)h_{,\hat{0}} - 2Z\overline{Z}h \end{aligned} \tag{3.1296}$$

Thus

$$h_{,\hat{0}\hat{0}} + 2(Z+\overline{Z})h_{,\hat{0}} + 2Z\overline{Z}h = 0. \tag{3.1297}$$

We check, using (3.1295), that $Z\overline{Z}$ and $Z+\overline{Z}$ are particular solutins: first $\overline{Z}$

$$
\begin{aligned}
& (Z\overline{Z})_{,\hat{0}\hat{0}} + 2(Z+\overline{Z})(Z\overline{Z})_{,\hat{0}} + 2Z\overline{Z}(Z\overline{Z}) \\
& = (-Z^2\overline{Z} - Z\overline{Z}^2)_{,\hat{0}} + 2(Z+\overline{Z})(-Z^2\overline{Z} - Z\overline{Z}^2) + 2(Z\overline{Z})^2 \\
= \;& (2Z^3\overline{Z} + 2Z^2\overline{Z}^2 + 2Z\overline{Z}^3) - 2Z^3\overline{Z} - 4Z^2\overline{Z}^2 - 2Z\overline{Z}^3 + 2(Z\overline{Z})^2 \\
= \;& 0. \qquad (3.1298)
\end{aligned}
$$

Now $Z+\overline{Z}$:

$$
\begin{aligned}
& (Z+\overline{Z})_{,\hat{0}\hat{0}} + 2(Z+\overline{Z})(Z+\overline{Z})_{,\hat{0}} + 2Z\overline{Z}(Z+\overline{Z}) \\
& = (-Z^2 - \overline{Z}^2)_{,\hat{0}} + 2(Z+\overline{Z})(-Z^2 - \overline{Z}^2) + 2Z\overline{Z}(Z+\overline{Z}) \\
= \;& 2(Z^3 + \overline{Z}^3) - 2(Z^3 + Z^2\overline{Z} + Z\overline{Z}^2 + \overline{Z}^3) + 2Z^2\overline{Z} + 2Z\overline{Z}^2 \\
= \;& 0 \qquad (3.1299)
\end{aligned}
$$

therefore, the general solution is

$$
h = \frac{1}{2}M(Z+\overline{Z}) + BZ\overline{Z}, \qquad M_{,\hat{0}} = B_{,\hat{0}} = 0, \qquad (3.1300)
$$

where M and B are real.

Maxwell's equation () can be solved for $\mathcal{F}_{\hat{2}\hat{3}}$ and gives

$$
\mathcal{F}_{\hat{2}\hat{3}} = AZ^2, \qquad A_{,\hat{0}} = 0. \qquad (3.1301)
$$

Substitution of these results into (),

$$
\begin{aligned}
-A\overline{A}Z^2\overline{Z}^2 &= -(\overline{Z}+Z)[h_{,\hat{0}} + (\overline{Z}-Z)h] - 2Z^2h \\
&= -(\overline{Z}+Z)h_{,\hat{0}} - (\overline{Z}^2 + Z^2)h \\
&= (\overline{Z}+Z)[\frac{1}{2}M(Z^2+\overline{Z}^2) + B(Z^2\overline{Z} + Z\overline{Z}^2)] \\
&\quad -(\overline{Z}^2 + Z^2)[\frac{1}{2}M(Z+\overline{Z}) + BZ\overline{Z}] \\
&= B(\overline{Z}+Z)(Z^2\overline{Z} + Z\overline{Z}^2) - B(\overline{Z}^2 + Z^2)Z\overline{Z} \\
&= 2BZ^2\overline{Z}^2 \qquad (3.1302)
\end{aligned}
$$

we find that

$$
B = -\frac{1}{2}A\overline{A} \qquad (3.1303)
$$

So that

$$h = \frac{1}{2}M(Z+\overline{Z}) - \frac{1}{2}A\overline{A}Z\overline{Z} \tag{3.1304}$$

Some commutation relations

$$\begin{aligned}
(AZ)_{,\hat{2}\hat{0}} - (AZ)_{,\hat{0}\hat{2}} &= 2(AZ)_{,\hat{c}}\Gamma^{\hat{c}}{}_{[\hat{2}\hat{0}]} \\
&= 2(AZ)_{,\hat{0}}\Gamma^{\hat{0}}{}_{[\hat{2}\hat{0}]} + 2(AZ)_{,\hat{1}}\Gamma^{\hat{1}}{}_{[\hat{2}\hat{0}]} + 2(AZ)_{,\hat{2}}\Gamma^{\hat{2}}{}_{[\hat{2}\hat{0}]} + 2(AZ)_{,\hat{3}}\Gamma^{\hat{3}}{}_{[\hat{2}\hat{0}]} \\
&= (AZ)_{,\hat{0}}h\overline{Y}_{,\hat{0}} + (AZ)_{,\hat{1}}\overline{Y}_{,\hat{0}} - (AZ)_{,\hat{2}}Y_{,\hat{2}} + (AZ)_{,\hat{3}}\overline{Y}_{,\hat{2}} \\
&= -(AZ)_{,\hat{2}}Y_{,\hat{2}} + (AZ)_{,\hat{3}}\overline{Y}_{,\hat{2}} \\
&= -Z(AZ)_{,\hat{2}}
\end{aligned} \tag{3.1305}$$

$$\begin{aligned}
(AZ)_{,\hat{2}\hat{3}} - (AZ)_{,\hat{3}\hat{2}} &= 2(AZ)_{,\hat{c}}\Gamma^{\hat{c}}{}_{[\hat{2}\hat{3}]} \\
&= 2(AZ)_{,\hat{0}}\Gamma^{\hat{0}}{}_{[\hat{2}\hat{3}]} + 2(AZ)_{,\hat{1}}\Gamma^{\hat{1}}{}_{[\hat{2}\hat{3}]} + 2(AZ)_{,\hat{2}}\Gamma^{\hat{2}}{}_{[\hat{2}\hat{3}]} + 2(AZ)_{,\hat{3}}\Gamma^{\hat{3}}{}_{[\hat{2}\hat{3}]} \\
&= (AZ)_{,\hat{0}}h(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}) + (AZ)_{,\hat{1}}(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}) + (AZ)_{,\hat{2}}0 + (AZ)_{,\hat{3}}0 \\
&= (\overline{Z} - Z)(AZ)_{,\hat{1}} + h(\overline{Z} - Z)(AZ)_{,\hat{0}}
\end{aligned} \tag{3.1306}$$

Note

$$Y_{,[\hat{2},\hat{3}]} = \frac{1}{2}[(Y_{,\hat{2}})_{,\hat{3}} - (Y_{,\hat{3}})_{,\hat{2}}] = \frac{1}{2}Z_{,\hat{3}}$$

$$\begin{aligned}
Y_{,[\hat{2},\hat{3}]} &= Y_{,\hat{c}}\Gamma^{\hat{c}}{}_{[\hat{2}\hat{3}]} \\
&= Y_{,\hat{0}}\Gamma^{\hat{0}}{}_{[\hat{2}\hat{3}]} + Y_{,\hat{1}}\Gamma^{\hat{1}}{}_{[\hat{2}\hat{3}]} + Y_{,\hat{2}}\Gamma^{\hat{2}}{}_{[\hat{2}\hat{3}]} + Y_{,\hat{3}}\Gamma^{\hat{3}}{}_{[\hat{2}\hat{3}]} \\
&= Y_{,\hat{1}}\frac{1}{2}(Y_{,\hat{2}} - \overline{Y}_{,\hat{3}}) + Y_{,\hat{2}}0 + Y_{,\hat{3}}0 \\
&= \frac{1}{2}(Z - \overline{Z})Y_{,\hat{1}}
\end{aligned} \tag{3.1307}$$

so that

$$Z_{,\hat{3}} = (Z - \overline{Z})Y_{,\hat{1}} \tag{3.1308}$$

Note

$$Y_{,[\hat{2},\hat{1}]} = \frac{1}{2}[(Y_{,\hat{2}})_{,\hat{1}} - (Y_{,\hat{1}})_{,\hat{2}}] = \frac{1}{2}Z_{,\hat{1}} - \frac{1}{2}Y_{,\hat{1}\hat{2}}$$

$$\begin{aligned}
Y_{,[\hat{2},\hat{1}]} &= Y_{,\hat{c}}\Gamma^{\hat{c}}{}_{[\hat{2}\hat{1}]} \\
&= Y_{,\hat{0}}\Gamma^{\hat{0}}{}_{[\hat{2}\hat{1}]} + Y_{,\hat{1}}\Gamma^{\hat{1}}{}_{[\hat{2}\hat{1}]} + Y_{,\hat{2}}\Gamma^{\hat{2}}{}_{[\hat{2}\hat{1}]} + Y_{,\hat{3}}\Gamma^{\hat{3}}{}_{[\hat{2}\hat{1}]} \\
&= Y_{,\hat{1}}(-\frac{1}{2}\overline{Y}_{,\hat{1}}) + Y_{,\hat{2}}(\frac{1}{2}hY_{,\hat{2}}) + Y_{,\hat{3}}(\frac{1}{2}h\overline{Y}_{,\hat{2}}) \\
&= \frac{1}{2}hZ^2 - \frac{1}{2}Y_{,\hat{1}}\overline{Y}_{,\hat{1}} \qquad (3.1309)
\end{aligned}$$

so that

$$Z_{,\hat{1}} = Y_{,\hat{1}\hat{2}} + hZ^2 - Y_{,\hat{1}}\overline{Y}_{,\hat{1}}. \qquad (3.1310)$$

Maxwell's equations again

(3.1282)

$$(AZ^2)_{,\hat{2}} - \mathcal{F}_{\hat{1}\hat{2},\hat{0}} - Z\mathcal{F}_{\hat{1}\hat{2}} = 0 \qquad (3.1311)$$

The Kerr-Newman solution

$$\begin{aligned}
ds^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\
&\quad + \frac{2mr^3 - e^2r^2}{r^4 + a^2z^2}\left[dt + \frac{z}{r}dz + \frac{r}{r^2+a^2}(xdx + ydy) - \frac{a}{r^2+a^2}(xdy - ydx)\right]^2
\end{aligned} \qquad (3.1312)$$

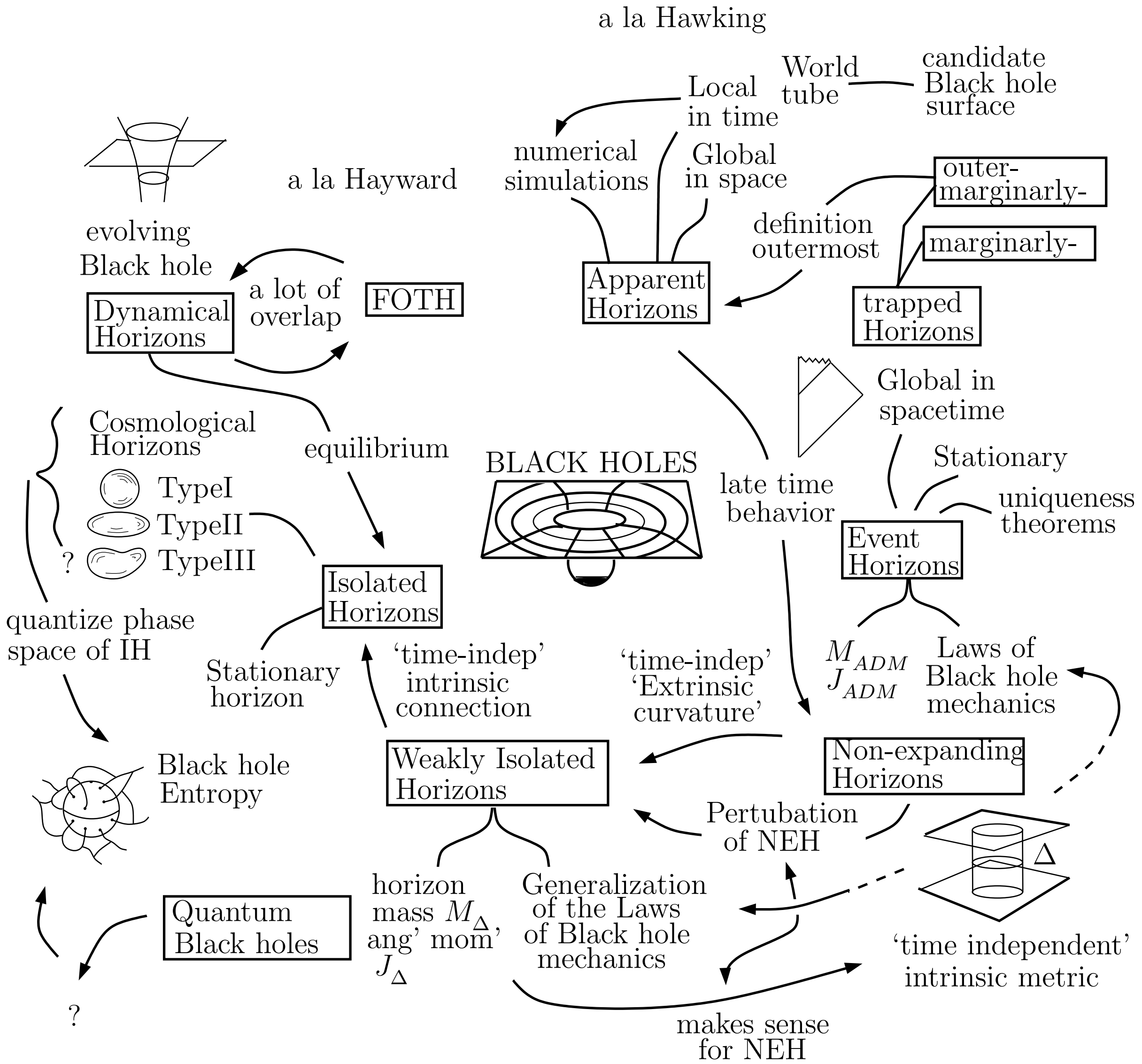


Figure 3.33: Horizons.