Contents:

• Chapter 1: Einstein’s hole argument Complete, Partial and Dirac Observables
• Chapter 2: Introduction to General Relativity and its physical Observables
• Chapter 3: Black holes - Event, Isolated and Dynamical Horizons
• Chapter 4: Cosmology
• Chapter 5: Hawking Penrose Singularity Theorems
• Chapter 6: Consistent Discrete Classical GR
• Chapter 7: Quantum Field Theory on Curved Spacetimes
• Chapter 8: Introduction to Quantum General Relativity
• Glossary
• Many Detailed Appendices with Worked Exercises
Chapter 1: Classical GR, Einstein’s hole argument and Physical geometry

• Einstein’s hole argument.
• Measurements of geometry.
• Conceptual issues.
• Relational mechanics.
• Partial, complete and Dirac observables.

Chapter 2: Introduction to General Relativity and its Physical Observables

• Special Relativity
• Principles of General Relativity
• Spacetime Measurements
• GPS Observables
• Action Principle
• Perturbations Around Exact Solutions

Chapter 3: Black holes - Event, Isolated and Dynamical Horizons

• Event Horizons and Penrose Diagrams
• Non-Expanding Horizons
• Weakly Isolated Horizons and Generalisations of the Laws of Black Hole Mechanics
• Isolated Horizons and Rotating Isolated Horizons
• Dynamical Horizons
Chapter 4: Cosmology

- Classical Cosmology
- Homogeneous and Isotropic Cosmology
- Outline of the Singularity Theorems
- Gauge Invariant Perturbations

Chapter 5: Hawking Penrose Singularity Theorems.

- Basic Definitions
- Strong Causality
- The Space of Causal Curves
- Conjugate Points
- Singularity Theorems
- Collapse of a Star
- The Big Bang
- Initial Value Problem

Chapter 6: Consistent Discrete Classical GR.

- Introduction.
Chapter 7: Quantum Field Theory on Curved Spacetimes

- Introduction
- Quantum Field Theory on Flat Spacetime
- Quantum Field Theory on Curved Spacetime
- Quantum Fields in an Expanding spacetime
- Quantum Fields During Inflation
- Hawking Radiation
- Algebraic Approach
- Back Reaction
- The Need for Quantum Gravity
- Stochastic Gravity

Chapter 8: Introduction to Quantum Gravity.

- Introduction to Quantum General Relativity
- Ashtekar-Barbero Variables
- Quantum Constraints - the Equations of Canonical Quantum Gravity
- The Loop Representation
- Geometric Operators
- Spin Networks
- The Hamiltonian Constraint and the Modern formalism
- Spin Foams
- Semi-Classical Limit
• The Master Constraint Programme

• Physical Applications: Black Hole Entropy, Loop Quantum Cosmology, Quantum Gravity Phenomenology, and Background-independent Scattering Amplitudes

• The problem of Time

• Other Approaches
Appendices

- A Physics Glossary
- B Mathematics Glossary
- C Mathematics
- D Physical Geometry and Derivation of Einstein’s Field Equations
- E Constrained Hamiltonian Systems, Dirac Observables and the Constraint Algebra
- E ADM and First Order Formulation of Einstein’s Equations
- F Basic Functional Analysis
- H Details of Hawking’s Calculation
## Contents

1 Classical GR, Einstein’s hole argument and Physical Geometry 52

1.1 Einstein’s Hole Argument ............................................. 54

1.2 Background Independence - A Farewell to Spacetime ................. 67

1.2.1 Comparison of GR with the Rubber Sheet Analogy ................. 67

1.2.2 The View of the World that Emerges ............................... 69

1.2.3 Common Misunderstandings ........................................ 72

1.2.4 The Blessing of background independence - Non-Perturbative Quantum Gravity Finite and Requires No Renormalization! ............ 73

1.3 Physical Geometry ..................................................... 76

1.3.1 Physical GPS Coordinates ......................................... 77

1.3.2 Physical Area ........................................................ 78

1.3.3 Description of a Measurement of Area .............................. 79

1.3.4 Einstein’s Field Equations ......................................... 81

1.3.5 The velocity composition law ...................................... 82

1.3.6 Dust as Matter Reference System ................................ 82

1.4 Some conceptual issues ................................................ 82

1.5 Relational Mechanics ................................................. 83

1.5.1 Covariant Hamiltonian Formulation ................................ 83

1.5.2 Depamerizable Mechanics: Identification of a “time” variable .... 86

1.5.3 Fully Constrained Hamiltonian Systems .......................... 87
1.6 Partial, Complete and Dirac Observables ........................................ 88
  1.6.1 Infinitely Many Constraints .................................................. 91
  1.6.2 Observables for Canonical General Relativity ......................... 92
  1.6.3 Approximate Observables for Canonical General Relativity .......... 93
1.7 The Problem of Time .................................................................... 94
1.8 The problem of Quantum Cosmology .............................................. 96

2 Introduction to General Relativity and its Physical Observables 97
  2.1 Introduction .............................................................................. 97
  2.2 Special Relativity ...................................................................... 97
    2.2.1 Simultaneity ........................................................................ 98
    2.2.2 Time Dialation ..................................................................... 98
    2.2.3 Length Contraction .............................................................. 98
    2.2.4 Lorentz Transformations ...................................................... 99
    2.2.5 Velocities ............................................................................ 100
    2.2.6 Acceleration ........................................................................ 102
    2.2.7 Doppler Effect ..................................................................... 102
    2.2.8 Relativistic Mass and Energy .............................................. 103
    2.2.9 The Twin Paradox ............................................................... 103
  2.3 The Principles of General Relativity .............................................. 103
    2.3.1 The Principle of Equivalence ............................................... 103
    2.3.2 The Gravitation Red-shift: Warping Time .............................. 104
    2.3.3 The Curvature of Spacetime ................................................. 105
    2.3.4 Curvature in a Weak Uniform Gravitation field .................... 106
    2.3.5 The Principle of General Relativity ...................................... 108
    2.3.6 Background Independent Theories ....................................... 108
    2.3.7 Einstein’s Hole Argument ................................................... 108
3.17.10 Goldberg Sachs Theorem ........................................ 354
3.17.11 Tetrad Formalism and the Cartan Structure Equations .... 364
3.17.12 Specialisation to Null Tetrads .................................. 369
3.18 Summary ................................................................. 374
3.19 Bibliographical notes .................................................. 374
3.20 Worked Exercises and Details ........................................ 375
   3.20.1 Non-Expanding Horizons ....................................... 378
   3.20.2 Weakley Isolated Horizons ..................................... 380
   3.20.3 Isolated Horizons ................................................ 380
   3.20.4 Rotating Isolated Horizons .................................... 386
   3.20.5 Dynamical Horizons .............................................. 389

4 Classical Cosmology .......................................................... 420
  4.1 Classical Cosmology ................................................. 420
     4.1.1 Fluid Flow Equations ....................................... 420
     4.1.2 Newtonian Cosmology ....................................... 421
     4.1.3 Relativistic Cosmology ...................................... 422
     4.1.4 Spaces of Constant Curvature ................................ 423
  4.2 Homogeneous and Isotropic Cosmology ................................ 423
     4.2.1 The Luminosity Distance ..................................... 423
  4.3 The Singularity Theorems ............................................. 424
     4.3.1 Application of the Singularity Theorem: Cosmological Singularity . 426
     4.3.2 Energy Conditions ............................................. 427
     4.3.3 Causality and Chronology .................................... 427
     4.3.4 Existence of maximum geodesic ................................ 429
     4.3.5 The Significance of Conjugate Points: The Singularity Theorems .... 432
  4.4 Backreaction Issues in Relativistic Cosmology and the Dark Energy Debate 432
4.4.1 Cosmological Perturbation Theory ........................................ 432

4.5 Gauge Invariant Perturbations Around Symmetry Reduced Sectors of General Relativity: Applications to Cosmology ................................. 433

4.5.1 Introduction ............................................................... 433

4.6 Approximate Complete Observables ........................................ 435

4.7 Application to Cosmology ..................................................... 439

5 Proof of the Hawking-Penrose Singularity Theorems ........................ 445

5.1 Proof of the Hawking-Penrose Singularity Theorem ....................... 445

5.1.1 Introduction ............................................................... 445

5.1.2 Some Basic Terminology ............................................... 446

5.1.3 The Singularity Theorem of Hawking and Penrose .................... 450

5.1.4 Basic Definitions ....................................................... 452

5.1.5 Achronal Sets ............................................................ 455

5.1.6 Strong Causality ........................................................ 459

5.1.7 The Space of Causal Curves .......................................... 469

5.1.8 Conjugate Points ....................................................... 479

5.1.9 The Singularity Theorems ............................................. 497

5.1.10 Implication of the “Displayed” Singularity Theorem from the Established Version ................................................................. 500

5.2 Black Holes .................................................................. 526

5.2.1 Collapse of a Star ....................................................... 527

5.2.2 The Cauchy Problem - Existence and Uniqueness .................... 531

5.2.3 Non-Linear Hyperbolic Differential Equations ....................... 532

5.2.4 Existence and Uniqueness ............................................... 536

5.2.5 Cauchy-Kowalewski Theorem ......................................... 537

5.2.6 Reduced Einstein Equations .......................................... 538

5.2.7 Sobolev Inequalities ..................................................... 541
5.2.8 Developments for the Empty Space Einstein Equations . . . . . . . 542
5.2.9 Stability of Closed Trapped Surfaces . . . . . . . . . . . . . . . . . . . 543
5.2.10 Apparent Horizons . . . . . . . . . . . . . . . . . . . . . . . . . . . . 543
5.3 The Big Bang . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 543
5.4 Biblioiographical notes . . . . . . . . . . . . . . . . . . . . . . . . . . . . 544
5.5 Sobolev Spaces . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 544
5.5.1 Review of \( L^p \) Spaces . . . . . . . . . . . . . . . . . . . . . . . . 544
5.5.2 Hardy-Littlewood-Sobolev Inequality . . . . . . . . . . . . . . . . . . 546
5.5.3 Sobolev Inequalities . . . . . . . . . . . . . . . . . . . . . . . . . . . . 549
5.5.4 Classical Sobolev Spaces . . . . . . . . . . . . . . . . . . . . . . . . . 555
5.5.5 Fractional \( H^s \) - Sobolev Spaces . . . . . . . . . . . . . . . . . . 558
5.6 Worked Exercises and Details . . . . . . . . . . . . . . . . . . . . . . . . . 558

6 Consistent Discrete Classical GR 563
6.0.1 “Dirac’s” canonical approach to general discrete systems . . . . . . 565

7 Quantum Field Theory on Curved Spacetimes 567
7.1 Motivation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 567
7.2 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 567
7.3 Quantum Field Theory in Flat Spacetime . . . . . . . . . . . . . . . . . . 568
7.3.1 The Simple Harmonic Oscillator . . . . . . . . . . . . . . . . . . . . . 568
7.3.2 Quantisation of the Klein-Gordon field in Flat Spacetime . . . . . . 569
7.3.3 Mode Expansion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 570
7.3.4 Behaviour of Fock Basis Under Lorentz Transformations . . . . . . 570
7.3.5 Fock Particles . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 570
7.4 Quantum Field Theory on Curved Spacetimes . . . . . . . . . . . . . . . 571
7.4.1 Bogolyubov Transformations . . . . . . . . . . . . . . . . . . . . . . . 571
7.4.2 \( a \)-Particles in the \( b \)-Vacuum . . . . . . . . . . . . . . . . . . . 572
7.5 Quantum Fields in an Expanding Universe ............................................. 572
  7.5.1 Particle Interpretation ................................................................. 572
7.6 Quantum Fields During Inflation ......................................................... 573
7.7 The Unruh Effect .................................................................................... 573
7.8 Hawking Radiation .................................................................................. 573
  7.8.1 Hand Wavy Calculation ..................................................................... 573
  7.8.2 Hawking’s Calculation ..................................................................... 574
  7.8.3 Rotating Black Holes and Higher Spin Fields ..................................... 577
  7.8.4 Black Hole Thermodynamics .............................................................. 578
  7.8.5 Information Loss Paradox ................................................................. 578
7.9 Backreaction ........................................................................................... 578
7.10 The Algebraic Approach ......................................................................... 578
  7.10.1 Algebraic Quantum Theory ............................................................. 578
  7.10.2 Wightman Axioms in Minkowski Spacetime ..................................... 578
  7.10.3 Revision ............................................................................................ 579
  7.10.4 Operator Product Expansion ............................................................ 579
  7.10.5 Algebraic Quantum Field Theory ...................................................... 579
  7.10.6 Viewpoint on QFT ......................................................................... 579
  7.10.7 Global and Local Particles ................................................................. 579
7.11 The Need For Quantum Gravity ............................................................. 580
  7.11.1 Backreaction .................................................................................... 580
  7.11.2 Information Loss Paradox ................................................................. 580
7.12 Stochastic Gravity ................................................................................... 580

8 Introduction to Quantum General Relativity .............................................. 581
  8.1 The Problem of Quantising General Relativity ....................................... 581
    8.1.1 The Problem of Time in Canonical Quantum Gravity .................... 582
C.5.4 Summary of Tensor Calculus .............................. 758
C.5.5 Linear operators and Matrices .............................. 759

C.6 Group Theory .................................................. 760
C.6.1 Examples of Groups ...................................... 760
C.6.2 Unitary Representations of Groups ......................... 765
C.6.3 Schur’s First Lemma .................................. 766
C.6.4 Schur’s Second Lemma .................................. 768
C.6.5 Orthogonality relations ................................ 770
C.6.6 The Characters of a Representation ....................... 771
C.6.7 Direct Products ........................................... 773

C.7 Continuous Groups, Lie Groups and Lie algebras .......... 774
C.7.1 Infinitesimal Generating Technique ....................... 775
C.7.2 General Structure of Lie Groups ......................... 778
C.7.3 Rotations SO(3) and SU(2) .............................. 779
C.7.4 Spin Direct Products .................................. 781
C.7.5 Direct Products and Clebsch-Gordan Coefficients .......... 788
C.7.6 Recoupling Theory ...................................... 791
C.7.7 SO(3,1) and SL(2,C) .................................. 791
C.7.8 SO(4) .................................................. 793
C.7.9 Conformal Group ........................................ 794
C.7.10 Group Integration: The Haar Measure .................... 795
C.7.11 Peter-Weyl theorem ..................................... 800
C.7.12 Analogies ............................................... 802
C.7.13 Clebsch-Gordan ......................................... 804
C.7.14 Semi-direct Products ..................................... 804

C.8 Infinite-Dimensional Group Representations .............. 808
C.8.1 Group Actions .............................................. 808
C.8.2 Countable and Locally Compact Topological Groups . . . . . . . . 808
C.8.3 Haar Measure . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 809
C.8.4 Summary of Group theory . . . . . . . . . . . . . . . . . . . . . . . 809
C.9 Manifolds and Elementary Topology . . . . . . . . . . . . . . . . . . . . 810
  C.9.1 Sets and Mappings Between Sets . . . . . . . . . . . . . . . . . . . 810
  C.9.2 Continuity . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 811
C.10 Elementary Tensor Analysis . . . . . . . . . . . . . . . . . . . . . . . . . . 813
  C.10.1 Affine Connection . . . . . . . . . . . . . . . . . . . . . . . . . . . 815
  C.10.2 Affine Geodesic . . . . . . . . . . . . . . . . . . . . . . . . . . . . 817
  C.10.3 The Metric Connection . . . . . . . . . . . . . . . . . . . . . . . . . 818
  C.10.4 Curvature . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 820
  C.10.5 Gaussian Normal Coordinates . . . . . . . . . . . . . . . . . . . . . 822
  C.10.6 Bianchi Identities . . . . . . . . . . . . . . . . . . . . . . . . . . . 823
  C.10.7 Conformal Tensor, Ricci tensor and Ricci Scalar . . . . . . . . . . . 824
  C.10.8 The Weyl Tensor . . . . . . . . . . . . . . . . . . . . . . . . . . . . 824
  C.10.9 Index Free Formulism . . . . . . . . . . . . . . . . . . . . . . . . . 824
C.11 Differential Geometry . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 827
  C.11.1 Tangent Vectors . . . . . . . . . . . . . . . . . . . . . . . . . . . . 827
  C.11.2 Covectors . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 829
  C.11.3 Induced Metric and Other Objects on Sub-manifolds . . . . . . . . . 830
C.12 Active Diffeomorphisms and the Lie Derivative . . . . . . . . . . . . . . . . 832
  C.12.1 Mapping a Manifold to Itself Along Integral Curves . . . . . . . . . 834
  C.12.2 The Lie Derivative . . . . . . . . . . . . . . . . . . . . . . . . . . . 835
  C.12.3 Pull-back and Lie Derivative of a co-vector . . . . . . . . . . . . . . 842
  C.12.4 More on Lie Derivative . . . . . . . . . . . . . . . . . . . . . . . . . 843
  C.12.5 Isometries and Killing Vector Fields . . . . . . . . . . . . . . . . . . 844
  C.12.6 Conserved Quantities . . . . . . . . . . . . . . . . . . . . . . . . . 846
## C.19 Principal Bundle

- C.19.4 Principal Bundle ........................................ 895
- C.19.5 Action of the structure Group on a Principal Bundle ........ 896
- C.19.6 Connections on Principal Bundles ........................ 896
- C.19.7 Gauge Fields ........................................... 900
- C.19.8 Parallel Transport in a Principal Bundle .................... 906
- C.19.9 Curvature on a Principal Bundle .......................... 906
- C.19.10 Extension and Reduction of Principal Bundles ............... 907
- C.19.11 The Complex Line Bundle ................................ 909

## C.20 Summary of Differential Geometry .......................... 909

## C.21 Summary ................................................... 909

## C.22 Bibliographical notes ....................................... 910

## C.23 Worked Exercises and Details .............................. 910

### C.23.1 Dynamical and Non-Dynamical Symmetries ................. 910

## D Constrained Hamiltonian Systems, Dirac observables and the Constraint Algebra .................. 914

- D.0.2 Introduction ............................................. 914

### D.1 Hamiltonian Mechanics ..................................... 915

#### D.1.1 Poisson Brackets ...................................... 917

#### D.1.2 Symplectic Geometry and Phase Space ................... 919

#### D.1.3 Canonical Transformations ............................. 922

#### D.1.4 Infinitesimal Contact Transformations ................... 926

#### D.1.5 Noether’s Theorem .................................... 928

### D.2 Geometry of Configuration Space and Phase Space ............ 929

#### D.2.1 Vector Fields on Configuration Space and Phase Space .... 929

#### D.2.2 The Lie Derivative ..................................... 929

#### D.2.3 Definition of Hamiltonian System ......................... 930
D.2.4 Symplectic Geometry of Phase Space .................................. 931
D.2.5 Canonical Transformations ................................................. 933
D.2.6 The Hamiltonian Framework: Résumé ................................... 934
D.2.7 Connection to quantum mechanics ..................................... 934

D.3 Covariant Phase Space ............................................................. 935
D.3.1 Space of Solutions ............................................................... 935
D.3.2 Field Theory ................................................................. 936
D.3.3 Hamiltonian-Jacobi Theory ................................................ 937
D.3.4 Hamilton principal function ................................................ 940

D.4 Solving for the Dynamics using the HJ Equation ...................... 942
D.4.1 1. Free particle (one-dimension) ........................................ 942
D.4.2 2. The Harmonic oscillator (one-dimension) ......................... 946
D.4.3 Hamiltonian Characteristic Function ................................... 948
D.4.4 ‘Derivation’ of Schrödinger’s Equation ................................. 950

D.5 Constrained Hamiltonian Systems ............................................. 952
D.5.1 Examples ................................................................. 956
D.5.2 Dirac’s Procedure for Constrained Hamiltonian Systems .......... 958
D.5.3 First Class Constraints and Gauge Symmetries ...................... 962
D.5.4 Dirac Method and Electrodynamics .................................... 963
D.5.5 Quantization of Constrained Hamiltonian Systems ................. 964
D.5.6 Dirac Observables ........................................................... 964
D.5.7 Darboux’s Theorem ........................................................ 964
D.5.8 Symplectic Reduction ....................................................... 966
D.5.9 Poisson Reduction ........................................................... 982
D.5.10 Symplectic Group Actions ............................................... 982

D.6 Worked Examples ................................................................. 983

D.7 Open Constraint Algebras ....................................................... 985
E  ADM and First order Formalism of Einstein’s Theory  

E.1 Intrinsic and Extrinsic Curvature  
E.2 ADM Metric Formulation  
E.3 Stuff  
E.4 The Hamiltonian Formulation  
E.5 The Cauchy Problem  
E.6 Gravitational Hamiltonian  
  E.6.1 Boundary Term  
  E.6.2 Constraint Algebra  
E.7 First Order Formulation of Einstein Equations  
E.8 Palatini Method in the Connection Formulation  
  E.8.1 Method I  
  E.8.2 Method II  
E.9 Inclusion of Matter  
  E.9.1 Yang-Mills  
  E.9.2 Klein-Gordan - Scalar Matter Field  
  E.9.3 Fermionic Matter  
  E.9.4 In the Language of Differential Geometry  
E.10 Self-dual Connection Formulation  
  E.10.1 Self-dual Curvature  
  E.10.2 Self-dual Action  
E.11 Ashtekar’s Canonical Formalism  
E.12 Generators of Symmetry Transformations  
  E.12.1 The Gauss-law Constraint Generates Gauge Transformations  
  E.12.2 Incorporating Matter in the Quantum Theory
E.13 Toy Model: Free Particle described using Half-Complex Coordinates. . . . 1071
  E.13.1 Complex Variables and Reality Conditions .......................... 1071
  E.13.2 Quantization in Complex Coordinates. .............................. 1072
E.14 The Holst Action ............................................................. 1073
  E.14.1 3+1 Decomposition of the Holst Action .............................. 1073
  E.14.2 The Diffeomorphism Constraint ....................................... 1073
  E.14.3 The Hamiltonian Constraint ........................................... 1075
  E.14.4 Addition Constraints ................................................... 1075
  E.14.5 Final Total Hamiltonian ............................................... 1077
E.15 Bibliographical notes ....................................................... 1077
E.16 Worked Exercises and Details ............................................. 1078

F Basic Functional Analysis .................................................... 1085
  F.1 Finite Hilbert Space ...................................................... 1085
    F.1.1 The Hamilton-Cayley Theorem ...................................... 1085
    F.1.2 Projection Operators ............................................... 1086
    F.1.3 Spectral Theorem for Finite Spaces .............................. 1089

G Quantum Field Theory ....................................................... 1093
  G.1 Elementary Quantum Mechanics ........................................ 1093
    G.1.1 Path Integrals and Functional Integrals ......................... 1093
    G.1.2 Semi-Classical Limit ............................................... 1094
    G.1.3 Second Quantization ............................................... 1095
    G.1.4 N Real variables ................................................... 1095
    G.1.5 Complex variables .................................................. 1097
  G.2 Grassmann Integration .................................................. 1098
  G.3 Quantization on the Space of Classical Solutions ................. 1106
    G.3.1 Harmonic Oscillators .............................................. 1108
G.3.2 ‘Fock Space’ Quantization ........................................... 1110
G.3.3 The Fock Representation of Field Theory ....................... 1118
G.3.4 The Fock Representation of a Free Scalar Field ............... 1119
G.3.5 The Fock Representation of the Maxwell Field ............... 1120
G.3.6 Quantum Field Theory on Curved Spacetime - The Basics .... 1120

H Details of Hawking’s Calculation ...................................... 1121
H.1 Decomposition into Complete Basis .................................. 1121
H.2 Solution of Klein-Gordon Equation in Schwarzschild Spacetime . 1121
H.3 Bogoliubov Coefficients ................................................ 1124
List of Figures

1.1 Rubbersheet simulation of geodesic motion in special relativity. 53
1.2 Rubbersheet. It does not matter that the coordinates are time-dependent - it still serves as a physical reference system. 53
1.3 $E(x) = Q/x^2$. 58
1.4 $E(y) = Q/y^2$. 59
1.5 $E(y) = Q/y^2$. 59
1.6 Passive spatial diffeomorphism $f : M \to M$ refers to invariance under change of coordinates. The same object in a different coordinate system. Any theory of nature is invariant under passive diffeomorphisms. 61
1.7 An active diffeomorphism $f : M \to M$ drags fields on the manifold while remaining in the same coordinate system. $f$ is viewed as a map that associates one point in the manifold to another one. 62
1.8 The value of $\tilde{\phi}(P)$ at $P$ is equated to the value of $\phi(P_0)$ at $P_0$, i.e. $\tilde{\phi}(P) = \phi(P_0)$. Under this transformation $f$ we identify one point of the manifold $P_0$ to another point $P$ $f : P_0 \to P$. 62
1.9 The value of the metric function $\tilde{g}_{ab}$ at $P$ is defined by the value of the metric function $g_{ab}$ at $P_0$, i.e. $\tilde{g}_{ab}(P) = g_{ab}(P_0)$. We go to a new coordinate system which assigns $P$ the same coordinate values that $P_0$ has in the x-coordinates, so that $\tilde{g}_{ab}(y_1 = u_1, y_1 = u_2) = g_{ab}(x_1 = u_1, x_1 = u_2)$, compare to (1.31). 63
1.10 (a) An active diffeomorphism in which we identify one point of the manifold to another point. (b) We then go to a coordinate system that assigns the newly identified points the original coordinate values. 64
1.11 (a) An active diffeomorphism in which we actively drag the tensor function over the, in doing so indentify one point of the manifold to another. (b) We then go to a coordinate system which assigns the newly identified points their original coordinate values. That is to say - we carry the tensor function over the manifold, keeping the coordinate lines ‘attached’.  

1.12 Einstein’s hole argument.  

1.13 Resolution of Einstein’s hole argument.  

1.14 Illustration of smearing. operator valued distributions.  

1.15 Regime where gravity is very strong so that the non-perturbative and background independence of GR must be taken into account. That spacetime points have no independent physical reality casts doubt on the hand-wavey argument I gave above.  

1.16 Laboratory walls exemplify Newton’s absolute space and the clock absolute time. We can define positions relative to the wall.  

1.17 GPS.  

1.18 GPS3D A spacetime point in Minkoski spacetime can be expressed as a relation amongst 4 measurable variables. This definition of spacetime location retains meaning in the jump to GR.  

1.19 clock time.  

1.20 .  

1.21 measLocation.  

1.22 measVelocity.  

1.23 partialobs. $\tau$ is an unphysical parameter labelling different possible correlations between the time reading $t$ of the clock and the elongation $x$ of the pendulum.  

1.24 partComptDitt1.  

1.25 partComptDitt3. (a) $t = t_1$ when the clock function $T(\alpha_C^t(x))$ assumes the value $\tau$. (b) The function $F_{[f,T]}(\tau, x)$ gives the value that the function $f(\alpha_C^t(x))$ assumes if the function $T(\alpha_C^t(x))$ assumes the value $\tau$. $F_{[f,T]}(\tau, x)$ is a complete observable generated from the partial observables $T(x)$ and $f(x)$.  

2.1 timedilF.  

2.2 rocketEarth.
2.3 rocketEarth

2.4 rocket. The clock at the top seems to run faster than the one on the bottom.

2.5 rocketaccel. The clock at the top seems to run faster than the one on the bottom.

2.6 lightdeflec.

2.7 WeakGrav. geodesic.

2.8 Hole. Einstein’s hole argument.

2.9 Hole3. Einstein’s hole argument. $\Phi : M \rightarrow M' \rightarrow M$.

2.10 Hole4. Einstein’s hole argument. A gauge transformation which does not change the coordinate label system but moves the points on the manifold, and then evaluate the coordinates of the new point.

2.11 We display the geometric interpretation of the curvature tensor. Carry a third vector $Z$, by parallel transport from $p$ to $s$ via $q$, comparing this with transporting this from $p$ to $s'$ via $r$ we find a discrepancy between the two vectors given in terms of the curvature tensor components $R^d_{abc}$ by the formula $\epsilon^2 X^a Y^b Z^c R^d_{abc}$.

2.12 geodesic deviation.

2.13 clock time.

2.14 measLocation.

2.15 tidalforceF.

2.16 worldfunc1.

2.17 measVelocity.

2.18 $\eta$ is the orthogonal connecting vector.

2.19 We find the spatial frame components $\eta^a$ of the orthogonal connecting vector by projecting onto a spatial frame field. This is the precise analogue of the Newtonian connecting vector.

2.20 Different world line passing through $P$ corresponds to different observer with different $v^a$.

2.21 GPScoord. $s_1$ and $s_2$ are the GPS coordinates of the point $p$. $\Sigma$ is a Cauchy surface with $p$ in its future domain of dependence.

2.22 crossParea.
2.23 continuityEM Y and Y. .............................................. 136
2.24 Lcfluid. The Lorentz contraction of a fluid element. .......... 139
2.25 LumDist. Luminosity distance ........................................ 165
2.26 pertCosGauge. A diffeomorphism on the perturbed manifold \( \mathcal{M} \) induces a change in coordinates of the background manifold \( \mathcal{M}_0 \). The issue of perturbative gauge invariance is closely related, though not equivalent to, the coordinate independence of General Relativity. ..................... 169
2.27 pertManifolds. 5-dimensional manifold \( \mathcal{N} \) containing a 1-parameter family of smooth non-intersecting 4-manifolds \( \mathcal{M}_\epsilon \). \( \mathcal{N} = \mathcal{M} \times \mathbb{R} \) ......................... 171
2.28 pertManFlow. The diffeomorphism \( \varphi \). ................................. 173
2.29 pertManPush. The push-forward \( \varphi_{\lambda|\mu}^* \) is the natural linear map between the tangent spaces \( T_p \mathcal{M}_0 \) and \( T_{\varphi(p)} \mathcal{M}_\lambda \) induced by the diffeomorphism \( \varphi \). The push-forward \( \varphi_{\lambda|\mu}^* \) is the linear map between the co-tangent spaces \( T^*_{\varphi(p)} \mathcal{M}_\lambda \). Push-forwards and pull-backs are related by . . . . 174

3.1 eventhorizon. ............................................................... 198
3.2 Penrose diagram of Minkowski spacetime ............................ 220
3.3 Penrose diagram of the Kruskal solution ............................ 221
3.4 Penrose diagram of a black hole. ..................................... 221
3.5 Penrose diagram of a black hole. ..................................... 233
3.6 Rotating blackhole ....................................................... 274
3.7 Normal and tangent vector to a tangent ............................. 275
3.8 (a) Normal to a timelike surface, (b) Normal to a spacelike surface, (c) Normal to a null surface. ......................................... 275
3.9 Coordinatizing a null surface in Minkowsian spacetime - \( \lambda, \theta, \phi \). ..................................................... 275
3.10 Each two sphere. Coordinates \( \lambda, \theta, \phi \). ......................... 276
3.11 A spacial two-sphere \( S \) embedded in the spacial slice \( \Sigma \) (which in turn is embedded in spacetime \( \mathcal{M} \)), with two sets of orthogonal null vector fields. The vector field \( n^a \) is the unit timelike normal to \( \Sigma \), \( R^a \) is the unit spacial normal to \( S \), and \( n^a \) and \( \ell^a \) are, respectively, the outgoing and ingoing null vectors orthogonal to \( S \). .................................................. 276
3.12 A spacial two-sphere \( S \) .................................................. 277
3.13 In Schwarzschild black hole the horizon is generated by the radial light rays, which meet at the center. .................................................. 284

3.14 (a) An open interval of the real line is the set of points between $a$ and $b$ excluding $a$ and $b$. (b) ................................................................. 285

3.15 $q$ lies in the chronological future of $z$............................................. 285

3.16 The chronological future $I^+(p)$ of $p$ is an open set; given any point $q \in I^+(p)$, there exists a sufficiently small neighbourhood $V(q)$ contained in $I^+(p)$. Similar statements hold for the $p$ in the chronological past $I^-(q)$ of $q$. 286

3.17 If two points on the event horizon are timelike separated, we can produce a timelike curve starting inside the black hole which joins to a point outside the event horizon. .................................................. 287

3.18 When neighbouring null geodesics have conjugate points there exists a timelike curve joining the two conjugate points. The dashed line represents a timelike curve joining to the null geodesic. The points $q$ and $q'$ are timelike separated - rounding off the corner. We make this argument more rigorous in the appendix M. .................................................. 288

3.19 In flat spacetime, when a null geodesic curve joins onto a timelike curve, there exists a timelike curve between $p$ and $q$. .................................................. 288

3.20 There exists a timelike curve joining the two conjugate points. A timelike curve joining to the null geodesic. (b) Continuing in this way, we “peel” away a timelike curve that joins $r$ and $p$. .................................................. 289

3.21 There exists a timelike curve joining the two in this way, we a timelike curve that joins $r$ and $p$. .................................................. 290

3.22 There exists a timelike curve joining the two in this way, we a timelike curve that joins $r$ and $p$. .................................................. 290

3.23 Classical boundary conditions for weakly isolated horizons. .................. 292

3.24 Conformal spacetime diagram of a WHI. ............................................. 292

3.25 Construction of a null tetrad for a NEH. Any spatial two surface $S$ determines uniquely, up to rescaling, two null vectors orthogonal to $S$. .................................................. 293

3.26 multipole is the position of the mass density source and $\vec{r}$ is the requested position for the potential $\Phi(\vec{r})$. .................................................. 297

3.27 dipolemass (a) monpole. (b) dipole (c) quadrupole ......................... 298

3.28 magnitude is the position of the current density and $\vec{r}$ is the requested position for the magnetic potential $\Phi_m(\vec{r})$. .................................................. 300
3.29 polarcoo ................................................................. 304
3.30 axicoords. In addaptive coordinates ............................ 304
3.31 ............................................................... 308
3.32 visualflownull. ...................................................... 353
3.33 Horizons. ............................................................. 419
4.1 LumDist. Luminosity distance .................................... 424
4.2 nonmingeodesic. ..................................................... 425
4.3 trapped two-surface such that the areas of pulses of light emitted from each little element of surface decrease in both directions. .... 426
4.4 illustrating a trapped surface. ..................................... 426
4.5 illustrating a past-directed trapped surface, corresponding to a cosmological singularity. ........................................ 427
4.6 $I^+(p)$ is open. ....................................................... 428
4.7 trivialsing. ........................................................... 428
4.8 The points $y_n$ converge to the point $q$ in the boundary of $L^+(S)$. From each $y_n$ there is a past directed timelike curve $\lambda_n$ to $S$. These curves converge to the past directed null geodesic segment $\gamma$ through $q$. ............... 429
4.9 $K = J^+(S) \cap J^-(p)$ $K$ is compact. ......................... 429
4.10 trivialsing. .......................................................... 430
4.11 A timelike curve can be approximated by null geodesic segments. This approximating null curve fails to have a well defined tangent vector anywhere. 431
4.12 The global hyperbolicity of $\mathcal{M}$ is closely related to the future or past development of initial data from a given spacelike hypersurface. ........ 432
5.1 Sir Roger Penrose and Stephen Hawking. Initiated by Penrose, Penrose and Hawking, together with Robert Geroch, contributed much of the work on the existence of spacetime singularities with the use of point-set topological methods. .................................................. 446
5.2 ................................................................. 446
5.3 An example of a non-orientable space. ......................... 447
5.4 (a) A closed achronal set with edge. (b) A closed achronal set without edge.

5.5 The $h$–null cone contains more timelike vectors than the $g$–null cone so there is more likelyhood to find closed timelike curves in $(\mathcal{M}, h)$ than in $(\mathcal{M}, g)$.

5.6 Domains of dependence. (a) The future domains of dependence of the achronal set $S$. (b) The past domain of dependence of the set $S$. (c) The total domain of dependence of $S$.

5.7 closedtrapArea. A closed trapped surface is when the outgoing light cone also converges.

5.8 Focal points. (b) Focal point to a surface.

5.9 A path is a connected set in $\mathbb{R}$ to the space-time manifold $\mathcal{M}$.

5.10 Examples in Minkowski spacetime. (a) The curve $\alpha$ is taken to contain its own past end point $a$ and future end point $b$. (b) The curve $\alpha'$ is now future-inextendable. (c) The curve $\alpha''$ is now past-inextendable. (d) The curve $\gamma$ is not future-inextendable because it cannot be prolonged any further.

5.11 (a) A curve in Minkowski spacetime with point removed. (b) A curve is zig zag that it fails to have a well defined tangent vector at a point. (c) Here we have a time like curve which is null at its end point.

5.12 In flat spacetime, when a null geodesic curve joins onto a timelike curve, there exists a timelike curve between $p$ and $q$.

5.13 Say there are two points $p$ and $q$ connected by a null curve and a point $r$ which is connected to $q$ by a timelike curve. A timelike curve joining to the null geodesic. (b) Continuing in this way, (c), we “peel” away a timelike curve that joins $r$ and $p$.

5.14 Artificial example of how the causal future of a point $p$ will not necessarily coincide with the closure of the chronological future of $p$.

5.15 The point $q$ is conjugate to $p$ along null geodesics, so a null geodesic $\gamma$ that joins $p$ to $q$ will leave the boundary of the uture of $p$ at $q$.

5.16

5.17 The future set cannot be bounded by timelike curves.

5.18

5.19

5.20
5.21 The lines $\ell_1$ and $\ell_2$ have been removed. This is a space-time which is causal but fails to be strongly causal. 460

5.22 (a) (b) Suitable causal basis. 460

5.23 Minkowski space-time: $b$ is joined to $a$ by a null geodesic we consider $x \in I^+(a)$ and $y \in I^-(b)$. In the first case (a) $y \ll x$. In case (b) $y < x$ but $y \not< x$. In case (c) $x$ and $y$ are not causally related, in particular $y \not< x$. 461

5.24 $b$ is joined to $a$ by a null geodesic we consider $x \in I^+(a)$ and $y \in I^-(b)$. In the first case (a) $y \ll x$ there are no closed timelike curves. In case (c) For some $x$ and $y$. 461

5.25 converse. (b) That $y \gg x$ to be true, there must be a timelike curve that enters $U(a)$. 462

5.26 proof of. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 463

5.27 For $i \geq i_0$, $U_i \in I^+(x)$, from which we can conclude $x \ll a_i$. For $i \geq i'_0$, $c_i \in I^-(y)$. If we choose $i > i_0, i'_0$ then we have $x \ll a_i$ and $c_i \ll y$. 463

5.28 $b$ a $x \in I^+(a)$ and $y \in I^-(b)$. In the first case (a) $y \ll x$ timelike curves. (b) Points in the open set of $b$ are also in $I^+(x) \cap I^-(y)$ In case (c). . . . . 464

5.29 $a \in I^-(y) \cap V(y)$ and $b \in I^+(y) \cap V(y)$. In the first case (a) $y \ll x$ timelike curves. (b) Points in the open set of $b$ are also in $I^+(x) \cap I^-(y)$. . . . . . . 464

5.30 $a \in I^-(y) \cap V(y)$ and $b \in I^+(y) \cap V(y)$. In the first case (a) $y \ll x$ timelike curves. (b) Points in the open set of $b$ are also in $I^+(x) \cap I^-(y)$. . . . . . . 465

5.31 (a) There is an endless null geodesic along which strong causality is violated.
(b) Strong causality is violated everywhere in $R$. 466

5.32 $F := I^+(Q)$ and $P := I^-(Q)$. $Q = F \cap P$, $\partial Q = F \partial + \partial P$. 466

5.33 The lines $\ell_1$, $\ell_2$ and $\ell_3$ have been removed. 467

5.34 The future domain of dependence, $D^+(\Sigma)$, of $\Sigma$. $p$ is in $D^+(\Sigma)$, $q$ isn’t because there are past-inextendable causal curve through $q$ that don’t intersect $\Sigma$, e.g. the curve $\gamma$. 467

5.35 The future domain of dependence, $D^+(\Sigma)$, of a closed $\Sigma$ in Minkowski spacetime. 468

5.36 Domains of dependence. 468

5.37 The future domain of dependence. 469
5.38 There exists a subsequence $\gamma_m$ of past inextendible causal curves that do not meet $S$ that converges to a past inextendible $C^0$ null geodesic $\gamma$ starting at $p$. 

5.39 The edge of a closed achronal surface $\Sigma$. 

5.40 A simple example of a closed achronal surface without edge can be given by considering the spacetime $\mathbb{R} \times S$ with light cones locally at 45 degrees. For the open neighbourhood there is no $r \in I^- (p)$ and $q \in I^+ (p)$ with a timelike curve between them that doesn’t intersect $S$. The closed achronal set $\Sigma$ has no edge. 

5.41 We can only get convergence to a cluster point in $K$ if the space wasn’t locally finite. 

5.42 $K = J^+(S) \cap J^-(p)$ $K$ is compact. 

5.43 $K = J^+(S) \cap J^-(p)$ $K$ is compact. 

5.44 $G$ is an open set. 

5.45 approximate a causal curve by a causal trip. 

5.46 Imposition of the condition $I^+ [A_j] \cap A_{j+2} \neq \emptyset$ avoids cases such as the above. 

5.47 The geodesic $\lambda$. 

5.48 The sequence is a Cauchy sequence in $C(a, b)$, but it is not convergent, since there is a point missing. 

5.49 Minkowski space time with a point removed is not globally hyperbolic. The point $q$ is not in $D^+(S)$ as there are non-spacelike curves like $\lambda$ which do not meet $S$ in the past. 

5.50 (a) An upper semi-continuous function. (b) A lower semi-continuous function. 

5.51 Two points joined by a timelike curve can be connected by a broken null geodesic. 

5.52 A sequence of null curves may converge to a timelike curve. 

5.53 If $A$ and $B$ are parallel in Minkowskian spacetime then $\gamma_1$ and $\gamma_2$ are maximal. (b) Here there is only one element in $C_K (A, B)$ which is necessarily maximal. It is not a geodesic. (c) Here the maximal element is a trip. 

5.54 Displacement vectors for a hypersurface. 

5.55 

5.56 Future horozmos.
5.57 “doubling” future horizons. $E^+[S]$ is compact.

5.58 From the fact that $H$ is a Cauchy horizon it follows that through every point of $H$ there passes a maximally extended past-directed null geodesic that remains in $H$. Since $H$ is compact, such a curve would have to come back arbitrarily close to itself - reentering some Alexandrov neighbourhood and so violating strong causality.

5.59 $E^+[S]$ is compact by definition, however its Cauchy horizon is non-compact. As the two subsets can not homeomorphic there must be at least one trajectory $\gamma$ which remains in $\text{int}D^+(E^+[S])$.

5.60 $\gamma$ is a future endless causal geodesic in $\text{int}D^+(E^+(S))$. $H = H^+(E^+[S])$. As the intersection of the closed set $J^-(\gamma)$ with a compact set generated by null geodesic segments from $T$ of some bounded affine length, $E^-[T]$ is compact.

5.61 The geodesic $\lambda$.

5.62 The limit geodesic $\gamma$ contains conjugate points.

5.63 Since $b_n \to b$, $\text{tr}(b_n) < \text{tr}(b)/2$ for all $n > N$.

5.64 $\eta(S)$ is contained in the bounded segment from $\gamma(s_1)$ to $\gamma(s_5)$.

5.65 The Jacobi fields which are zero at $r$ must have expansion $\theta$ which is positive at $p$ otherwise $r$ would lie in the bounded interval from $\gamma(s_1)$ to $\gamma(s_5)$.

5.66 With $R_{abcd}V^bV^c \neq 0$. Non-positive expansion at $p$ ($\theta > 0$) implies there is a point $q$ conjugate to $r$, in the past of $p$. This is just the time reversed version of the focussing theorem.

5.67 The null expansion scalar $\hat{\theta}$.

5.68 Singularity theorems.

5.69 Diagram of collapse of a star.

5.70 Penrose diagram of collapse of a star.

5.71 starCollaps.

5.72 .

5.73 characteristic.

5.74 EnergyIneq1.

5.75 EnergyCond.
7.1 Reflects forward in time by the strong gravitational field outside the event horizon. .................................................. 574

7.2 The antiparticle mode falling into the black hole can be interpreted as a particle travelling backwards in time, forming the singularity down to the horizon. .................................................. 575

7.3 Penrose diagram of a star that collapses to form a black hole. .................. 576

8.1 Graphical representation of the Mandelstam identity (8.109) relating different Wilson loops. ........................................... 608

8.2 The action of the Hamiltonian constraint translated to the ‘path-integral’ or spin foam description. Where \( N(x_n) \) is the value of \( N \) at the vertex and \( H_{\text{nop}} \) are the matrix elements of the operator \( \hat{H} \). ......................... 612

8.3 a) A spherical star of mass \( M \) undergoes collapse. b) Later, a spherical shell of mass \( \delta M \) falls into the resulting black hole. With \( \Delta_1 \) and \( \Delta_2 \) are both isolated horizons, only \( \Delta_2 \) is part of the event horizon. ......................... 618

8.4 Quantum Horizon. Polymer excitations in the bulk puncture the horizon, endowing it with quantized area. Intrinsically, the horizon is flat except at punctures where it acquires a quantized deficit angle. These angles add up to \( 4\pi \). ......................... 619

A.1 Abhay Ashtekar. .................................................. 624

A.2 Julian Barbour. .................................................. 626

A.3 LebRien. .................................................. 627

A.4 Bondi coordinates at future null infinity. ................................... 629

A.5 Cauchy Horizon. .................................................. 630

A.6 We display the geometric interpretation of the curvature tensor. Carry a third vector \( Z \), by parallel transport from \( p \) to \( s \) via \( q \), and compare this with transporting this from \( p \) to \( s' \) via \( r \). We find that the two vectors differ according to a rotation given in terms of the curvature tensor components \( R_{abc}^d \) by the formula \( \epsilon^2 X^a Y^b Z^c R_{abc}^d \). ......................... 634

A.7 Bryce DeWitt. .................................................. 635

A.8 Paul Dirac. .................................................. 636

A.9 Bianca Dittrich. .................................................. 637
A.10 The future domain of dependence, $D^+(\Sigma)$, of $\Sigma$. $p$ is in $D^+(\Sigma)$, $q$ isn’t because there are past-inextendable causal curve through $q$ that don’t intersect $\Sigma$, e.g. the curve $\gamma$.

A.11 Albert Einstein (1879-1955).

A.12 The charge renormalization.

A.13

A.14 Rodolfo Gambini.

A.15 geons.

A.16 Stephen Hawking.

A.17 An expression for the conserved mass, evaluated when the spacial boundary is pushed to infinity. (b) $\partial \Sigma$ is a closed spacelike two-surface surrounding the source.

A.18 Penrose process.

A.19 Sir Roger Penrose.

A.20 PonRegg.

A.21 Jorge Pullin.

A.22 radartimeF. Schematic of the definition of radar time $\tau(x)$.

A.23 ReggeGlos. Gluing flat simplices to get a surface with curvature.

A.24 Regge Lagrangian.

A.25

A.26 Carlo Rovelli.

A.27 Thomas Thiemann.

A.28 John Wheeler.

B.1 bijective

B.2 DiffClass0. A chart on $\mathcal{M}$ comprises an open set $U$ of $\mathcal{M}$, called a coordinate patch, and a map $\phi: U \rightarrow \mathbb{R}^n$.

B.3 Computing the Gauss linking number.

B.4 Computing the Gauss linking number.
B.5 ................................................................. 713
B.6 injective ......................................................... 714
B.7 knots Reidemeister moves. ..................................... 716
B.8 knots Reidemeister moves. ..................................... 716
B.9 Pachner move in d=3. (a) the 1 → 4 move subdivides. ....... 723
B.10 pullbackDef0. Pushing forward a vector X from TM_x to T N_{\phi(x)}. ....... 727
B.11 pullbackDef. \varphi_\ast|_p : T_pM \rightarrow T_{\varphi(p)}N .................................. 727
B.12 Simplices in 3d. .................................................. 735
B.13 surjective .......................................................... 737
B.14 unicoverEx. ....................................................... 741
B.15 An upper semi-continuous function. ......................... 741

C.1 Coordinate induced basis. ....................................... 751
C.2 coset. Suppose H is a subgroup of a finite group G, here the elements of H
are listed first. The shaded box are the elements of the (left) coset of g_r \in G 761
C.3 infintesmal rotation. ................................................. 776
C.4 infintesmal rotation. ............................................... 776
C.5 ................................................................. 793
C.6 ................................................................. 793
C.7 Penrose diagram for Minkowskian spacetime. .......... 795
C.8 Haarmeas1. ......................................................... 796
C.9 HaarmeasSO3. ...................................................... 796
C.10 TranRotGrF. ....................................................... 807
C.11 ................................................................. 811
C.12 ................................................................. 811
C.13 (a) (b) f is discontinuous at p_t. ......................... 811
C.14 Open sets interior points (b) ................................. 812
C.15 $(U, \phi_1)$ and $(V, \phi_2)$ are two coordinate patches on $X$. Transition functions, $\phi_2 \circ \phi_1^{-1}$, are ordinary functions that go from points of one $\mathbb{R}^n$ space onto another, i.e. $\phi_2 \circ \phi_1^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n$. The domain and range of the transition function are the shaded regions in $\mathbb{R}^n$.

C.16 Möbius.

C.17 tangtoM.

C.18 connection.

C.19 coordbasevec. The vector $\xi$ may be thought of as being composed of $\xi = \xi^1 \partial / \partial x^1 + \xi^2 \partial / \partial x^2$. $\xi^1$ and $\xi^2$ are the components of $\xi$ in the $(x^1, x^2)$-coordinate system.

C.20 tangvector. maps the tangent spaces of $M$ linearly into the.

C.21 Two curves $\lambda(t)$ and $\mu(t)$ are tangent at $p$ if and only if their images are tangent at $\phi(p)$ in $\mathbb{R}^n$.

C.22 activeDiffGeom. A pushforward of the tensor $T_{ab}(x)$, i.e. $T_{ab}(x) \rightarrow \tilde{T}_{ab}(y)$.

C.23 activeDiffGeom1. The red dashed lines in (a) are the $x$-coordinate lines of the point $P$. We perform a coordinate transformation back to the original coordinate system. The pushed-forward tensor $\tilde{T}_{ab}(y)$ transforms to $\tilde{T}'_{ab}(x)$, i.e. $\tilde{T}_{ab}(y) \rightarrow \tilde{T}'_{ab}(x)$.

C.24 The tangent vector field resulting from a congruence of curves.

C.25 The local congruence of curves resulting from vector field.

C.26 .

C.27 pullbackDef0. Pushing forward a vector $X$ from $TM_x$ to $TN_{\phi(x)}$.

C.28 The push-forward map $h_\star$ that maps the tangent spaces of $M$ linearly into the tangent spaces of $N$.

C.29 The pullback map $\phi^*$ of a function $f$ from $N$ to $M$ by a map $\phi : M \rightarrow N$ is the composition of $\phi$ with $f$.

C.30 pushLie. The maps the co-tangent spaces of $M$ linearly into the co-tangent spaces of $N$.

C.31 The Killing vector field resulting from a congruence of curves.

C.32 Framefield or tetrad with one spatial dimension suppressed.

C.33 areaofPar.
C.34 areaofPar2. .......................................... 855
C.35 surfElement. ........................................... 856
C.36 stokeExam. ............................................. 871
C.37 stokeExam2. ............................................ 871
C.38 boundery. .............................................. 873
C.39 boundarychain. Boundary of a chain. ................. 874
C.40 A curve through $e$ under the map $h \mapsto ghg^{-1}$, first a right action $R_{g^{-1}}$ as $h \mapsto hg^{-1}$ followed by the left action $L_g$ as $hg^{-1} \mapsto ghg^{-1}$. The identity $e$ is mapped to itself but points $h$ and $f$ near it are generally changed, so that a tangent vector at $e$, in $T_e(G)$, is mapped to another one in $T_e(G)$. 877
C.41 A vector $X \in T_e(G)$ is mapped to another one in $Ad_g(X) \in T_e(G)$. Written formally as $(ad_g)_*: T_e(G) \rightarrow T_e(G)$. 878
C.42 If a Lie group is a direct product of the proper subgroup and some discrete subgroup then each connected component $G_i$ is obtained from the proper subgroup $G_1$ by applying some discrete transformation $\gamma_i$ of a discrete subgroup $\Gamma$. 879
C.43 leftTran. The left translation along $g$ maps a neighbourhood of $e$ onto one of $g$. There is a natural map of a vector at $e$ to one at $g$. 882
C.44 Fibre bundle, $TS^1 = S^1 \times R$. The base manifold $M$ (the real line $R^1$). The circle is the fibre. The fibre bundle consists of a manifold and a projection map $\pi$. $\pi^{-1}(U)$ is the local product space. 888
C.45 The inverse map $\pi^{-1}(U)$ is the local product space. 888
C.46 tangent bundle, $TS^1 = S^1 \times R$. The base manifold $M$ (the circle $S^1$) consists of a manifold and a projection map $\pi$. $\pi^{-1}(U)$ is the local product space. 889
C.47 Möbius. .................................................. 889
C.48 tangent bundle, $TS^1 = S^1 \times R$. The curved line is a section. 890
C.49 Travelling up fibre. ..................................... 891
C.50 Definition of the right action of $G$ on the principal fibre bundle $P$. 895
C.51 The horizontal subspace $H_{gu}P$, defining by the connection in definition (1), is obtained from $H_uP$ by the left action. 897
C.52 ............................................................. 898
D.1 Legendre transform relating Lagrangian formulation to the Hamiltonian formulation. .................................................. 916
D.2 Legendre transform relating Lagrangian formulation to the Hamiltonian formulation. .................................................. 917
D.3 We cannot solve to get $p$ from... .................................................. 918
D.4 Hamilton. ........................................................................ 940
D.5 Hamilton. ........................................................................ 941
D.6 Here a smooth family of 2-planes coincides with the tangent spaces of a nonintersecting space filling surfaces. Suprisingly this is not always the case. 967
D.7 partComptDitt1. ................................................................. 986
D.8 partComptDitt3. (a) $t = t_1$ when the clock function $T(\alpha_C^C(x))$ assumes the value $\tau$. (b) The function $F_{[T]}(\tau, x)$ gives the value that the function $f(\alpha_C^C(x))$ assumes if the function $T(\alpha_C^C(x))$ assumes the value $\tau$. $F_{[T]}(\tau, x)$ is a complete observable generated from the partial observables $T(x)$ and $f(x)$. ................................................................. 987
D.9 comptasGIE. ........................................................................ 987
D.10 partComptDitt4. ................................................................. 990
Terminology and Notation

Here is a list of symbols.

\[
\begin{align*}
\{,\} & \text{commutator} \\
\{,\} & \text{Poisson bracket} \\
\dagger & \text{Hermitian conjugation} \\
\equiv & \text{definition} \\
\delta & \text{only true in a special coordinate system} \\
\text{iff} & \text{If and only if} \\
\eta_{ab} & \text{Minkowski metric} \\
\eta(x) & \text{test function of a variation of action} \\
\mathcal{A} & \text{space of gauge fields or area} \\
A_\mu(x) & \text{Yang-Mills connection} \\
D_\mu & \text{covariant derivative} \\
\mathcal{M} & \text{spacetime manifold} \\
\mathcal{M} & \text{The Master constraint} \\
\hat{\mathcal{M}} & \text{The Master constraint operator} \\
\omega^{\alpha}_{\mu\beta} & \text{spin connection} \\
\mathcal{C} & \text{constraint surface in phase space} \\
S & \text{labels spin-network} \\
s & \text{equivalent class of spin-networks under the action of Diff denoted } s-\text{ knots} \\
s(S) & \text{denotes equivalent class } S \text{ to which belongs} \\
g_{ab} & \text{spacetime metric} \\
K_{ab} & \text{extrinsic curvature of } \Sigma \\
G_{ab} & \text{Einstein tensor} \\
T_{ab} & \text{The energy-momentum tensor} \\
e^a_j, E^a_i & \text{tetrad and triad} \\
\mathcal{L}_t & \text{Lie derivative with respect to } t \\
n_a & \text{unit normal to } \Sigma \\
N, (\tilde{N}) & \text{lapse function (density)} \\
N^a & \text{shift vector on } \Sigma \\
\Omega_{\alpha\beta} & \text{symplectic form} \\
\mathcal{A}/G & \text{space of gauge fields moduli gauge transformations} \\
[A] & \text{gauge equivalence classe of the connection } A \\
\mathcal{H}\mathcal{A} & \text{the holonomy algebra} \\
\mathcal{H}\hat{\mathcal{A}} & \text{the completion of the holonomy algebra in the norm } \|f\| := \sup_{[A]\in\mathcal{A}/G} |f([A])| \\
\mathcal{A}/\mathcal{G} & \text{spectrogram of } \mathcal{H}\hat{\mathcal{A}}
\end{align*}
\]
Preface
Acknowledgments

Discussions with Tong, Pun Wai on parts of the proof of the singularity theorems.
Introduction

the beginning of the revolutionary contributions to physics by Einstein,
Paths through the report

Introductory book on general relativity [?]