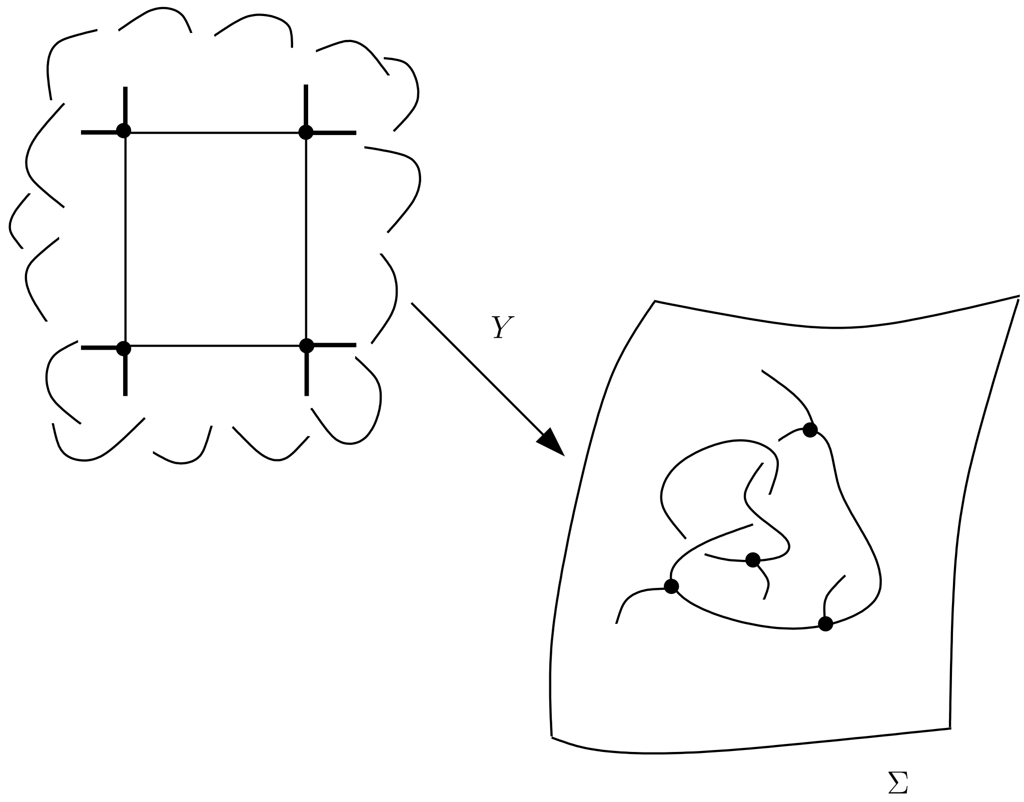
Volume III
Advanced and Recent Topics in Loop Quantum Gravity



Draft version

By

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Terminology and Notation

Here is a list of symbols.

[,]	commutator
{ , }	Poisson bracket
$\dagger$	Hermitian conjugation
$:=$	definition
$\equiv$	identity
$\stackrel{*}{=}$	only true in a special coordinate system
iff	If and only if
η_{ab}	Minkowski metric
$\eta(x)$	test function of a variation of action
$\mathcal{A}$	space of gauge fields or area
$A_\mu(x)$	Yang-Mills connection
D_μ	covariant derivative
$\mathcal{M}$	spacetime manifold
$\mathbf{M}$	The Master consraint
$\hat{\mathbf{M}}$	The Master consraint operator
$\omega^\alpha_{\mu\beta}$	spin connection
$\mathcal{C}$	constraint surface in phase space
S	labells spin-network
s	equivalent class of spin-networks under the action of Diff denoted $s-$ knots
$s(S)$	denotes equivalent class S to which belongs
g_{ab}	spacetime metric
K_{ab}	extrinsic curvature of Σ
G_{ab}	Einstein tensor
T_{ab}	The energy-momentum tensor
e^a_I, E^a_i	tetrad and triad
$\mathcal{L}_t$	Lie derivative with respect to t
n_a	unit normal to Σ_t
N, $(\tilde{N})$	lapse function (density)
N^a	shift vector on Σ
$\Omega_{\alpha\beta}$	sympletic form
$\mathcal{A}/\mathcal{G}$	space of gauge fields moduli gauge transformations
$[A]$	gauge equivalence classe of the connection A
$\mathcal{HA}$	the holonomy algebra
$\overline{\mathcal{HA}}$	the completion of the holonomy algebra in the norm $\|f\| := \sup_{[A]\in\mathcal{A}/\mathcal{G}} \mid f([A]) \mid$
$\overline{\mathcal{A}/\mathcal{G}}$	spectrum of $\overline{\mathcal{HA}}$

Preface

> So how do you go about teaching them something new? By mixing what they know with what they don't know. Then, when they see in their fog something they recognize they think, "Ah, I know that!" And then it's just one more step to "Ah, I know the whole thing." And their mind thrusts forward into the unknown and they begin to recognize what they didn't know before and they increase their powers of understanding"
>
> PICASSO

Warning: We are sure there are lots of mistakes in these notes. Use at your own risk! Corrections and other feedback would be very appreciated.

These notes are meant to be supplementary to Loop quantum gravity literarture. It is explicit (fills in all the gaps normally left out of the calculations);It is a self-contained and bringing together of relavant maths. Blow-by-blow account. Present material in lauguage more familiar to a broarder community of theoretical physists. To which modern coordinate-free differential geometry is unfamiliar and possibly daunting.

Style of the Book:

The report is mainly aimed at theoretic physicists who are non-experts. fill in calculations an details that are skipped over in th litrature. people who we have tried to make it accessible to people who are not familiar with modern methods and will have no need to apply these methods in their field of study. for example coordinate-free diffential gometry

I should stress at the beginning that I am a physicist and not a mathematician will not always be at a level of rigour that would satisfy a proper mathematical physicist.

This report assumes the reader is somewhat familiar with special relativity and it would help if he/she has knowledge with general realtivity at the undergraduate level. The treatment is self-contained in that an a priori knowledge is not assumed. Try to bring together many useful results. Many proofs are added and others expanded on to make them more accessible.

The many excellent expositions of Loop-quantum gravity

The initial idea was to try to present the subject at a level lower than in most of the litrature. It may take the reader rather long time to make his/her way through the report.

the report offer something of value for the mathematically inclined and the not so alike.

[32] T. Thiemann, Introduction to Modern Canonical Quantum General Relativity, [gr-qc/0110034]

I'm not really an expert and I will try to indicate where my understanding is a bit more iffy. At the moment it is an enthusiastic if not entirely reliable account, which I hope to improve on this through critisms, suggestions...

The purpose of these notes, not to replace any book or paper. You should turn to other source for other explainations.

Nevertheless, these notes written to be reasonably self-contained and comprehensible.

I make no claim to originality in these notes.

Often follows quite closely the treatments of

I have used a number of sources.

though it is not indispensable and understanding for the report.

This discussion is intended to provide only a passing familiarity with the scheme to allow the unfamiliar reader to follow certain calculations and to have a general understanding of the results.

Acknowledgments

Dedicated

Introduction

the beginning of the revolutionary contributions to physics by Einstein,

An attempt to apply the principles of quantum mechanics to general relativity.

The problem of merging gravitation and quantum mechanics is extrememly difficult - it has defied solution for over seventy years. Classical theory developed by Einstein took 10 years. the principles of quantum mrchanics and general relativity.

Canonical quantization is the oldest non-perturbative approach to quantization of general relativity.

"Is it just me or is quantum gravity easier nowadays?"

Didn't think he would see the completion of quantum gravity in his own life time. dead and buried. There is now hope (not a word usually assocciated with quantum gravity for a lot of people).

In spite of these compelling features... Quantizing gravity equivalent to finding a general solution to the classical field equations. The techniqual difficulties seemed even worse than non-renormalization of perturbative quantum gravity. Seem hopelessly difficult.

A school of thought has a long history in studies of quantum gravity. The viewpoint here is that it may well be possible to quantise pure general relativity consistently, and in a way that respects the geometrical framework of the classical theory, but to do so requires the use of techniques that are quite different from the weak-field perturbative methods that, for example, have dominated most particle-physics based approaches to quantum gravity. Much effort has been devoted to finding such nonperturbative schemes, and in this paper we will be concerned with a particular one that has evolved from the introduction of a new set of canonical variables to describe the phase space of classical general relativity.

A matter of getting the right perspective on the problem. General relativists have argued for decades that gravity must be quantized non-perturbativly - easily said that done!!

List of acheivements:

(1) Predcition of quantized strectrum for area and volume operators.

(2) Black hole entropy from first principles for physical black holes in that

(i) non-exremal black holes non-rotating black holes with Maxwell charges and dilatonic charges (1998), and recently has even been extended to rotating black holes and black holes with arbitrary disortions, also

(ii) definition of black hole applies restrictions to horizon geometry only and not to the bulk spacetime outside the black hole. As a consequence, we can have time varying dynamics outside black hole; also means results extend to cosmological horizons too.

(3) Resolution of the big bang singularity.

(4) initial boundary conditions for the universe are determined, rather than guessed at.

(5) New mechansim for inflation from quantum geometry.

(6) A mathematically consistent quantum theory of gravity coupled to the standard model - explicitly demonstrated to be finite (1997). However serious worries about whether it is physically consistent - in particular whether it has the correct classical limit.

Progress in conceptual no less dramatic

(1) Relational quantum mechanincs.

If it ultimently the case that it does not lead to the correct theory, we will have leared many things of value along the way.

The book is primary on loop quantum gravity, however: Brian Greene in The Fabric of the Cosmos he says

"If I were to hazard a guess on future developments, I'd imagine that the background-indpendent techniques developed by the loop quantum gravity community will be adapted to string theory, paving the way for a string formulation that is background indepndent. And that spark, I suspect, that will ignite a third susperstring revolution in which, I'm optimistic, many of the remaining deep mysteries will be solved".

Paths through the report

Introductory book on general relativity [?]

T. Thiemann, Introduction to Modern Canonical Quantum General Relativity, [gr-qc/0110034] [32]

Chapter 1

Introduction

Chapter 2

Algebraic Quantum Gravity, Reduced Phase Quantization and the Master constraint Path Integral

2.1 Introduction

In LQG the Hamiltoian or Master constraint operator necessarily changes the number of degrees of freedom on which the semiclassical state, that it acts on, depends. The fluctuations of the degrees of freedom added by the operator are therefore not suppresed by the semiclassical state.

Semiclassical states are graph dependent, no normalizable semiclassical states depending on all graphs.

Try to use only one fundamental graph.

Algebraic quantum gravity is a new approach to canonical quantum gravity suggested by loop quantum gravity. But in ontrast to loop quantum gravity, the quantum kinematics of algebraic quantum gravity is determined by an abstract $^*-$algebra generated by a countable set of elementary operators labeled by a single algebraic graph with countably infinite number of edges, while in loop quantum gravity the elementary operators are labelled by a collection of embedded graphs with a finite number of edges.

The missing information about the topology and differential structure of the spacetime manifold as well as about the background metric to be approximated is supplied by coherent states.

Definition An oriented algebraic graph is an abstract graph specified by its adjacency matrix α, which is an $N \times N$ matrix. One of its entries α_{IJ} stand for the number of edges that start at vertex I and end at vertex J. The valence of the vertex I is given by

$$v_I = \sum_J (\alpha_{IJ} + \alpha_{JI}).$$

$\square$

In the qunatization procedure, one fixes a specific cubic algebric graph with a countably infinite number of edges and with the valence of each vertex $v_I = 2 \times \dim(\Sigma)$.

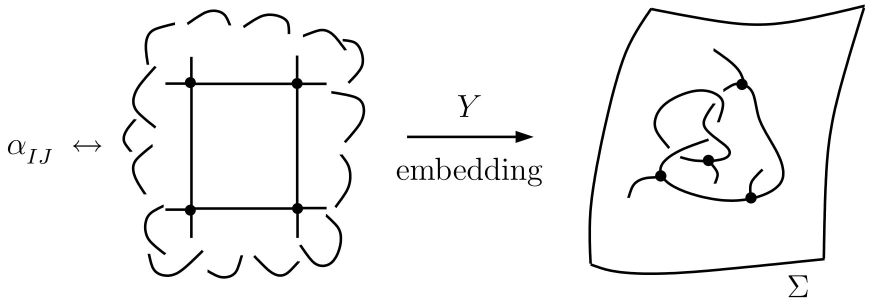


Figure 2.1: The quantum theory is independent of knottedness of a graph, topology and differentiability of a manifold. This information is attained when the abstract graph is embedded into a differential manifold when we do semi-classical analysis.

Given such data, the corresponding coherent state defines a sector in the ITP which can be identified with a usual QFT on the given manifold and background. Thus, AQG contains QFT on all curved spacetimes at once, providing contact with the familiar low energy physics and possibly has something to say about topology change. Topology is semiclassical concept, so AQG may incorporate topology change.

2.1.1 Differences Between AQG and LQG

Topology:

In LQG it must be provided.

In AQG it is absent.

Differenetiable structure:

In LQG it must be provided.

In AQG it is absent.

Size of Hilbert spaces:

For LQG the set Γ of all finite embedded graphs is uncountable.

For AQG the ITP of a countable number of Hilberts spaces of which at least countably infinite many are at least two dimensional is not separable.

The two Hilbert spaces are not directly elated to each other.

In LQG one needs all graphs because the algebra of elementary operators contains the holonomies along all possible paths and those are obtained from a fixed given path through the natural action of the diffeomorphism group.

In AQG the action of the infinitesimal diffeomorphisms preserves the algebraic graph and so there is no need to take all algebraic graphs into account.

AQG possibly can deal with topology change in the sense that it incorporates the semi-classical limits for all topologies while the corresponding states belong to the same Hilbert space.

2.2 Algebraic Quantum Gravity

Given the algebraic graph α, we define a quantum $^*-$algebra by associating with each edge e an element $A(e)$ of a compact, connected, semisimple Lie group G and an element $E_j(e)$ take value in its Lie algebra g. These are subject to the commutation relations

$$[\hat{A}(e), \hat{A}(e')] = 0,$$
$$[\hat{E}_j(e), \hat{A}(e')] = i\hbar\mathcal{Q}^2\delta_{e,e'}\frac{\tau_j}{2}\hat{A}(e),$$
$$[\hat{E}_j(e), \hat{E}_k(e')] = -i\hbar\mathcal{Q}^2\delta_{e,e'}f_{jkl}\hat{E}_l(e),$$

and $^*-$relations

$$\hat{A}(e)^\dagger = [\hat{A}(e)^{-1}]^T, \qquad \hat{E}_j(e)^\dagger = \hat{E}_j(e),$$

where $\mathcal{Q}$ stands for the coupling constant, τ_j are the generators in the Lie algebra and f_{jkl} the structure constants.

When we test the semiclassical limit we should specify an embedding map Y which maps the algebraic graph to an embedded one. With this specific embedding, we will see the correspondence between the classical algeba of elementary observables and the quantum $^*-$algebra.

$$A_0(e) := A_0(Y(e)) := \mathcal{P}\exp(\int_{Y(e)} A_0) \tag{2.1}$$

$$E_0(e) := \int_{S_e} \epsilon_{abc} dx^a \wedge dx^b A_0(\rho_e)\ (E_0)^c(x)\ A_0(\rho_e(x))^{-1} \tag{2.2}$$

which we will refer to as holonomies and electric fluxes respectively.

As one can show, if the classical theory is equipped with the following Poisson brackets

$$\begin{aligned}
\{(A_0)^j_a(x), (A_0)^k_b(y)\} &= 0 \\
\{(E_0)^j_a(x), (E_0)^k_b(y)\} &= 0 \\
\{(E_0)^j_a(x), (A_0)^k_b(y)\} &= Q^2 \delta^a_b \delta^k_j \delta(x,y)
\end{aligned} \tag{2.3}$$

then the quantities () satisfy

2.2.1 AQG Hilbert Space

A natural representation of $\mathfrak{U}$ of AQG is the von Neumann's ITP

$$\mathcal{H}_{AQG} = \otimes_{e \in E(\alpha)} L_2(\mathbf{G}, d\mu_H). \tag{2.4}$$

AQG space of states is the closure of the linear span of elementary states denoted $\otimes_f := \otimes_e f_e$:

$$\otimes_f = \otimes_e f_e, \qquad f_e = f_e(h_e) \in L_2(\mathbf{G}, d\mu_H). \tag{2.5}$$

Two elements $\otimes_f$ and $\otimes_{f'}$ in $\mathcal{H}^{\otimes}$ are said to be strongly equivalent if

$$\sum_e | < f_e, f'_e >_{\mathcal{H}_e} -1|$$

converges. We denote by $[f]$ the strong equivalence class containing $\otimes_f$. It turns out that two elements in $\mathcal{H}^{\otimes}$ are orthogonal if they lie in different strongly equivalence clases. Hence the infinite tensor Hilbert space $\mathcal{H}^{\otimes}$ can be decomposed as a direct sum of the Hilbert subspaces (sectors) $\mathcal{H}^{\otimes}_{[f]}$ which are the closure of strongly equivalence clases $[f]$. Furthermore, even though the Hilbert space $\mathcal{H}^{\otimes}$ is non-separable each sector $\mathcal{H}^{\otimes}_{[f]}$ is separble and has a natural Fock space structure. The basic elements of the quantum algebra are represented on $\mathcal{H}^{\otimes}$ in the obvious way

$$\begin{aligned} \hat{A}(e)\otimes_f &:= [A(e)f_e] \otimes [\otimes_{e'\neq e} f_{e'}], \\ \hat{E}_j(e)\otimes_f &:= [i\hbar\mathcal{Q}^2 X^e_f f_e] \otimes [\otimes_{e'\neq e} f_{e'}]. \end{aligned} \tag{2.6}$$

For each embedding Y for every sector isomrphic with $\mathcal{H}_{Y(\alpha)} \subset \mathcal{H}_{LQG}$.

2.2.2 Quantum Dynamics

diffeomorphism transformations are not meaningful in the algebraic formulation as the algebraic graph is not embedded in a manifold. We can implement operator corresponding to diffeomorphism generators which have meaning when the algebraic graph is embedded in a manifold.

We employ the (extended) master constraint

$$\hat{M} := \sum_{v\in V(\alpha)} [\hat{G}_j(v)^\dagger \hat{G}_j(v) + \hat{D}_j(v)^\dagger \hat{D}_j(v) + \hat{H}_j(v)^\dagger \hat{H}_j(v)]$$

to implememnt algebraic versions of diffeomorphism.

Let $L(v, e_1, e_2)$ denote the set of minimal loops starting at v along e_1 and ending at v along e_2^{-1}. Recall a loop β is said to be minimal provided that there is no other loop within α satisfying the same restrictions with fewer edges transversed.

where $V := \sum_{v\in V(\alpha)} V_v$.

$$\begin{aligned} Q^{(r)}_v &:= \frac{1}{E(v)} \sum_{e_1\cap e_2\cap e_3=v} \epsilon_v(e_1, e_2, e_3) \mathrm{Tr}((\hat{A}(e_1)[\hat{A}(e_1)^{-1}, V^{(r)}_v]) \\ &\times \quad (\hat{A}(e_2)[\hat{A}(e_2)^{-1}, V^{(r)}_v])\ (\hat{A}(e_3)[\hat{A}(e_3)^{-1}, V^{(r)}_v])) \end{aligned} \tag{2.7}$$

Gauss constraint

$$\hat{G}_j(v) := \hat{\mathcal{Q}}^{1/2}_v \sum_{e\ at\ v} \hat{E}_j(e)$$

Spatial diffeomorphism constraint

$$
\begin{aligned}
\hat{D}_j(v) &:= \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \\
&\times \sum_{\beta \in L(v, e_1, e_2)} \mathrm{Tr}(\tau_j [\hat{A}(\beta) - \hat{A}(\beta)^{-1}]\ \hat{A}(e_3)\ [\hat{A}(e_3)^{-1}, \sqrt{\hat{V}_v}])
\end{aligned} \tag{2.8}
$$

Euclidean Hamiltonian constraint (up to overall factor)

$$
\begin{aligned}
\hat{H}_E^{(r)}(v) &:= \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \\
&\times \sum_{\beta \in L(v, e_1, e_2)} \mathrm{Tr}([\hat{A}(\beta) - \hat{A}(\beta)^{-1}]\ \hat{A}(e_3)\ [\hat{A}(e_3)^{-1}, \hat{V}_v^{(r)}])
\end{aligned} \tag{2.9}
$$

Lorentzian Hamiltonian constraint (up to overall factor)

$$
\begin{aligned}
\hat{T}(v) &:= \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \epsilon_v(e_1, e_2, e_3) \\
&\times \mathrm{Tr}((\hat{A}(e_1)[\hat{A}(e_1)^{-1}, [\hat{H}_E^{(1)}, \hat{V}]])\ (\hat{A}(e_2)[\hat{A}(e_2)^{-1}, [\hat{H}_E^{(1)}, \hat{V}]]) \\
&\times (\hat{A}(e_3)[\hat{A}(e_3)^{-1}, \sqrt{\hat{V}_v}]))
\end{aligned} \tag{2.10}
$$

$$
\hat{H}(v) = \hat{H}_E^{(1/2)}(v) + \hat{T}(v); \tag{2.11}
$$

When we test the semiclassical limit of these operators we should specify an embedding map Y which maps an algebraic graph to an embedded one. With this specific embedding, we can see the correspondence between the classical algeba of elementary observables and the quantum $^*-$algebra.

2.3 Semi-Classical Analysis

We want to show that AQG is a canonical quantization of classical General Relativity including matter. now, the classical theory is formulated on manifolds diffeomorphic to $\mathbb{R} \times \sigma$ where σ is a three manifold of arbitrary toploogy.

In AQG, as apposed to LQG, semiclassical tools developed for background independent quantum field theories already available can be aplied to the operators encoding the dynamics, and not just to the kinematic operators.

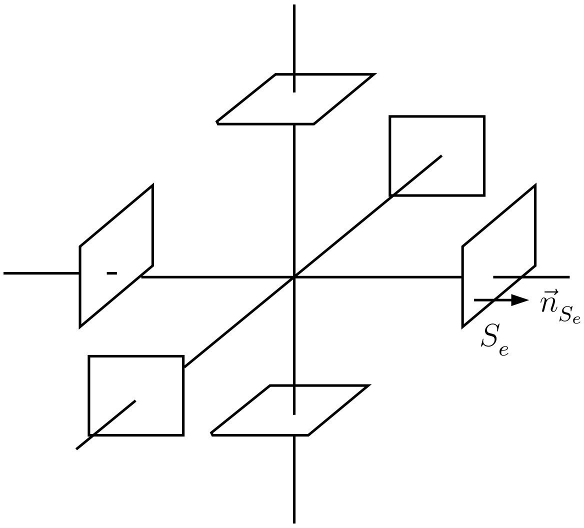


Figure 2.2: Cubic algebraic graph.

Choose Σ, embedding $Y : \alpha \to \Sigma$, surfaces S_e. We choose embedding Y such that γ is dual to a certain triangulation γ^*.

Choose classical field configuration $(A_0(x), E_0(x))$.

The precision with which the semiclassical limit is reached depends on the choice of embedding (its "fineness" with respect to the background metric to be approximated) which is a feature of the state.

Compute

$$g_e(A_0, E_0) := \exp(i\tau_j E_0^j(S_e)) \; A_0(Y(e)) \in G^{\mathbb{C}} \tag{2.12}$$

$$\psi_{e;A_0,E_0} := \psi^{t_e}_{g_e;(A_0,E_0)} \tag{2.13}$$

Hence the coherent state on the whole graph is represented by an infinite tensor product state:

$$\psi_{A_0,E_0} := \otimes_e \psi_e, \qquad \psi_e(h_e) := \sum_\pi \dim(\pi) e^{-t_e \lambda_\pi} \chi(g_e h_e^{-1}) \tag{2.14}$$

It is important to keep in mind that (2.13) is a state in the abstrct graph Hilbert space, we just use all the data σ, Y, A_0, E_0, etc in order to construct specific elements of the abstract ITP Hilbert space.

Minimum uncertainty states for $\mathfrak{U}_{AQG}$

$$< \psi_{A_0,E_0}, \hat{A(e)}\psi_{A_0,E_0} > = A_0(e), \qquad < \psi_{A_0,E_0}, \hat{E_j(S)}\psi_{A_0,E_0} > = E_{j0}(S) \tag{2.15}$$

$$< \Delta\hat{A}(e) >< \Delta\hat{E_j}(S) > = \frac{1}{2}| < [\hat{A(e)}, \hat{E_j(S)}] > | \tag{2.16}$$

With the semiclassical state we just constructed, the expectation value of the above (extended) master constraint operator can be calculated and its semiclassical limit tested.

In the non-Abelian case, except in special cases, the volume operator cannot be diagonalized analytically, which is prohibited so far as any explicit calculations involving the quantum dynamics. The substitution of the correct non-Abelian gauge group $SU(2)$ of the canonical formulation of General Relativity by the incorrect Abelian gauge group $U(1)^3$. This is crucial in order that semiclassical calculations can be carried out. The calculation in [222] shows that the result of the exact non-Abelian calculation matches precisely the results of the Abelian approximation.

The result of the calculation is

$$< \psi^t_{\alpha,A_0,E_0}, \hat{\mathbf{M}}\psi^t_{\alpha,A_0,E_0} > = \lim_{t\to\infty} \hat{\mathbf{M}}^{cubic}(A_0, E_0) = \lim_{\epsilon\to\infty} \hat{\mathbf{M}}(A_0, E_0). \tag{2.17}$$

Consequently, it has been shown that Algebraic Qunatum Gravity is a theory of quantum gravity which has the same infinitesimal generators as General Relativity. Thus the problem of whether the semiclassical sector includes General Relativity, that is still unsolved within the framework of Loop Quantum Gravity, is significantly improved in the context of Algebraic Quantum Gravity. Additionaly, it has been shown that the next-to-leading order term of the expectation value which can be interpreted as fluctuations of $\hat{\mathbf{M}}$ are finite.

2.4 Reduced Phase Space Quantization

2.4.1 Brown-Kuchar Framework

It is based on the introduction of a dust field, whose world-line identifies a preferred time-like direction. This direction plays the role of time. The constraints are modified by terms due to the matter field

2.4.2 Relational Framework

Deparametrized Theories

For deparametrized theories it is possible to find canonical coordinates consisting of two sets of canonical pairs (P^I, T_I) and (q^a, p_a) respectively (where the Poisson brackets between elements of the first and second set vanish) such that the constraints C_I can be rewritten in the equivalent form

$$C_I = P_I + h_I(q^a, p_a) \tag{2.18}$$

that is, they no longer depend on the variables T^I. This is a special case and most gauge systems cannot be written in this form. Even with dust General Relativity is a priori not of this form, however, one can reduce it to this form with additional manipulation.

2.4.3 Motivation for Abstract (Algebraic) Graphs

1. $\mathcal{H}$ too large (not separable) due to graph uncountability
2. vast overcounting of the number of degrees of freedom

In free scalar field theory on Minkowski space the quantum configuration space consists of Schwarz distributions rather than smooth functions. The label set of the fields consists of test functions of rapid decrease which are dense in the Hilbert space of square integrable functions on $\mathbb{R}^3$ and there exists a countable orthonormal basis of that Hilbert space.

Since for spatially diffeomorphism invariant operators (on dust space) such as the Hamiltonian $\hat{\mathbf{H}}$ or any other operationally interesting observable (which does not refer directly to the dust label space) the embedding of a graph is immaterial, we can consider the AQG reformulation as an economic description of the reduced LQG in the sense that diffeomorphism related embeddings would lead to isomorphic sectors superselected by these kind of observables.

2.4.4 Semiclassical Analysis

The semiclassical states depend on a differentiable mainfold χ, an embedding Y of the algebraic graph Γ into χ, a cell complex $Y^*(\Gamma)$ dual to $Y(\Gamma)$ as well as a point (A_0, E_0) in the classical reduced phase space.

2.4.5 Standard Model Hamiltonian on Minkowski Spacetime

It is widely accepted that the framework of QFT on curved spacetimes [11] (touched upon in appendix O) should be an excellent approximation to quantum gravity whenever metric fluctuations are small. In particular, when the background spacetime is Minkowski, the standard model must be reproduced.

Require the construction of a minimum energy eigenstate of $\hat{\mathbf{H}}$ which is simultaneously a minimal uncertainty state Ω for all observables and which is peaked around the flat vacuum (no excitations of observable matter) spacetime. One would study metter excitations of Ω and consider matrix elements of $\hat{\mathbf{H}}$ in such states. Hopefully, the resulting matrix elements of an effective matter Hamiltonian on Minkowski space should be close to the Hamiltonian of the standard model on Minkowski space.

It would be interesting to see whether this background indepndent lattice theory which is manifestly UV finite and non perturbative can explore the non perturbative sector of the standard model such as QCD.

2.4.6 RPSQ Scattering Amplitudes

Physical Hamiltonian defines S-Matrix, scattering theory, Feynman rules.

With a physical Hamiltonian $\hat{\mathbf{H}}$ at our disposal it is possible in principal to perform scattering theory, that is, one can compute matrix elements of the time evolution operator $U(\tau) = \exp(i\tau\hat{\mathbf{H}})$. The analytic evaluation of these matrix elements is too difficult but as in ordinary QFT we may use Fermi's Golden rule and expand, for short time intervals τ, the exponential as

$$U(\tau) = 1_{\mathcal{H}} + i\tau\hat{\mathbf{H}} + \mathcal{O}(\tau^2).$$

There are difficulties in computing the matrix elements of $\hat{\mathbf{H}}$ which invloves the square root of a positive self-adjoint operator. However, since in scattering theory initial and final states are excitations over a ground state which we do not know exactly but presumably can approximate by kinematic coherent states, one can invoke the techniques developed for AGQ to expand the square root of the operator around the square root of its expectation value.

2.5 Algebraic Quantum Gravity and Spin Foams

1. How to make contact with the canonical theory

2. How to rempove the triangulation dependence of the models.

Th extended Master constraint defines a new type of spin foam model which one computes heuristically by

$$\psi_{phys} = \int_{\mathbb{R}} dt \ \exp(it\mathbf{M})\psi \tag{2.19}$$

the physical inner product.

If the expression

$$(\psi, \exp(\pm it\mathbf{M})\psi')$$

is analytic in t (for instance if ψ and ψ' are analytic vectors for M) then it can be considered as the analytic continuation $t \mapsto \mp it$ in t

Notice that $(\psi, \exp(\pm it\mathbf{M})\psi')$ vanishes when ψ, ψ' do not belong to the same sector of the ITP. If we now write

$$\exp(-t\mathbf{M}) = [\exp(-t\mathbf{M}/N)]^N$$

and insert $N-1$ resolutions of unity

$$1_{sector} = \sum_s |s><s|$$

where $|s>$ denotes a countable orthonormal basis for the given sector then we arrive at a path integral formulation of the physical inner product.

$$\begin{aligned} (\eta(\psi), \eta(\psi'))_{phys} &:= \int_{\mathbb{R}} dt \ (\psi, \exp(it\mathbf{M})\psi') \\ &= \int_0^\infty dt \, [(\psi, \exp(it\mathbf{M}\psi')) + (\psi, \exp(-it\mathbf{M}\psi'))] \end{aligned} \tag{2.20}$$

Let us restrict ourselves to the case that the semiclassical theories we want to quantize have compact σ. The appropriate sector of the ITP is then based on the vector $\otimes_{\mathbf{1}} = \otimes_e \mathbf{1}$ where $\mathbf{1}$ is the constant function equal to one. An orthonormal basis for this sector is given by spin ntwork functions defined over all finite subgraphs of the algebraic graph. Then (2.20) defines a concrete spin foam model of General Relativity for which the issue of triangulation dependenceis absent. Details are the subject of future publications by Thiemann et al.

2.6 Master Constraint Path Integral

2.6.1 Introduction

By using the standard technique of skeletonisation and coherent state path-integral, we derive a path-integral

For all the physical models simpler than quantum gravity, the group averaging technique gives the correct physical Hilbert space. We assume this is also the case for the AQG master constraint operator $\hat{\mathbf{M}}$ defined in (), in applying the group averaging technique.

It turns out to be more convenient to use the coherent states in the skeletonisation.

Consider the computation for the group averaging inner product and derive a path-integral formula from the master constraint operator $\hat{\mathbf{M}}$ on the sector $\mathcal{H}_{AL}$.

$$\langle \eta(f) | \eta(f') \rangle \ := \lim_{\epsilon \to 0} \frac{1}{\ell_P^2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \left\langle f \left| \exp[\frac{i}{\ell_P^2} \tau (\hat{\mathbf{M}} - \epsilon)] \right| f' \right\rangle_{Kin} \tag{2.21}$$

2.6.2 Coherent State Path-Integrals

Resolution of identity with coherent states

Single-step amplitude

Evaluation of the overlap function

Computation of the matrix element

The path-integral formula with a classical action in the exponential?

2.6.3 Formulation of the Master Constraint Path-Integral

Resolution of identity with coherent states

insert in the resolution of identity with coherent states

$$
\begin{aligned}
& \left\langle f \left| [1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon)]^N \right| f' \right\rangle_{Kin} \\
= & \int dg_N \ldots dg_1 dg_0 \left\langle \tilde{\psi}^t_{g_N} \left| 1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon) \right| \tilde{\psi}^t_{g_{N-1}} \right\rangle_{Kin} \times \\
& \times \left\langle \tilde{\psi}^t_{g_{N-1}} \left| 1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon) \right| \tilde{\psi}^t_{g_{N-2}} \right\rangle_{Kin} \ldots \left\langle \tilde{\psi}^t_{g_1} \left| 1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon) \right| \tilde{\psi}^t_{g_0} \right\rangle_{Kin} \\
& \times \left\langle f | \tilde{\psi}^t_{g_N} \right\rangle_{Kin} \left\langle \tilde{\psi}^t_{g_0} | f' \right\rangle_{Kin} \qquad (2.22)
\end{aligned}
$$

where the measure

$$
dg = \prod_{e \in E(\gamma)} \frac{d^3p(e)dh(e)}{t^3} + \mathcal{O}(t^\infty)
$$

up to an overall constant.

Single-step amplitude

First let us compute the single-step amplitude

$$
< \tilde{\psi}^t_{g_i} | 1 + \frac{i\tau}{\ell_p^2 N}(\mathbf{M}-\epsilon) | \tilde{\psi}^t_{g_{i-1}} >_{Kin} .
$$

The overlap function $< \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin}$ is sharply peaked at $g_i = g_{i-1}$ in a Gaussian fashion (with width $\sqrt{t}$).

In the semiclassical limit $t \to 0$ we have

$$
\begin{aligned}
& \lim_{t\to 0} < \tilde{\psi}^t_{g_i} | 1 + \frac{i\tau}{\ell_p^2 N}(\mathbf{M}-\epsilon) | \tilde{\psi}^t_{g_{i-1}} >_{Kin} \\
= & \lim_{t\to 0} \left[1 + \frac{i\tau}{\ell_p^2 N} \frac{< \tilde{\psi}^t_{g_i} | \mathbf{M}-\epsilon | \tilde{\psi}^t_{g_{i-1}} >_{Kin}}{< \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin}} \right] < \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin} \\
= & \lim_{t\to 0} \left[1 + \frac{i\tau}{\ell_p^2 N} \frac{< \tilde{\psi}^t_{g_i} | \mathbf{M}-\epsilon | \tilde{\psi}^t_{g_i} >_{Kin}}{< \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_i} >_{Kin}} \right] < \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin}
\end{aligned}
\qquad (2.23)
$$

As the master constraint has the correct semiclassical limit

$$\lim_{t\to 0} < \tilde{\psi}^t_{g_i}|1+\frac{i\tau}{\ell_p^2 N}(\mathbf{M}-\epsilon)|\tilde{\psi}^t_{g_{i-1}} >_{Kin}$$
$$= \lim_{t\to 0}\left\{1+\frac{i\tau}{\ell_p^2 N}(\mathbf{M}[g_i]-\epsilon)\right\} < \tilde{\psi}^t_{g_i}|\tilde{\psi}^t_{g_{i-1}} >_{Kin}$$

$$< \tilde{\psi}^t_{g_i}|\mathbf{M}|\tilde{\psi}^t_{g_i} >$$

Evaluation of the overlap function

Let us evaluate the overlap function

$$< \tilde{\psi}^t_{g_i}|\tilde{\psi}^t_{g_{i-1}} >_{Kin}= \prod_{e\in E(\gamma)} \frac{< \psi^t_{g_i(e)}|\psi^t_{g_{i-1}(e)} >}{\|\psi^t_{g_i(e)}\|\ \|\psi^t_{g_{i-1}(e)}\|}$$

where

$$\psi^t_{g_i(e)} = \sum_{2j_e=0}^{\infty}(2j_e+1)e^{-tj_e(j_e+1)/2}\chi_{j_e}\big(g(e)h^{-1}(e)\big) \tag{2.24}$$

is the complexifier coherent state on the edge e. If we set $n=2j+1$

$$< \psi^t_{g_i}|\psi^t_{g_{i-1}} >_{Kin}= \frac{e^{t/4}}{2\sinh(z_{i,i-1})}\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_{i,i-1}} \tag{2.25}$$

Therefore

$$< \tilde{\psi}^t_{g_i}|\tilde{\psi}^t_{g_{i-1}} >_{Kin} \;=\; \prod_{e\in E(\gamma)} \frac{\sqrt{|\sinh(z_i)\sinh(z_{i-1})|}}{\sinh(z_{i,i-1})}\frac{\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_{i,i-1}}}{\sqrt{\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_i}}\sqrt{\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_{i-1}}}} \tag{2.26}$$

We now use the Poisson resumation formula

$$\sum_{n\in\mathbb{Z}} f(ns) = \frac{1}{s}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} dxe^{2\pi inx/s}f(x) \tag{2.27}$$

with $s = \sqrt{t_e}$. This gives

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} n e^{-tn^2} e^{nz_{i,i-1}} &= \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} \sqrt{t} n e^{tn^2} e^{\sqrt{t} n z_{i,i-1}/\sqrt{t}} \\
&= \frac{1}{t} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} dx e^{2\pi i n x/\sqrt{t}} x e^{-x^2} e^{x z_{i,i-1}/\sqrt{t}} \\
&= \frac{1}{t} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} dx x e^{-x^2} e^{(2\pi i n + z_{i,i-1})x/\sqrt{t}} \\
&= C \frac{1}{t^{3/2}} \sum_{n\in\mathbb{Z}} (2\pi i n + z_{i,i-1})^2 e^{(2\pi i n + z_{i,i-1})^2/4t} \qquad (2.28)
\end{aligned}$$

only the term with $n = 0$

$$\begin{aligned}
< \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin} &\approx \prod_{e\in E(\gamma)} \frac{\sqrt{|\sinh(z_i)\sinh(z_{i-1})|}}{\sinh(z_{i,i-1})} \frac{z_{i,i-1} e^{z^2_{i,i-1}/t}}{\sqrt{|z_i e^{z_i^2/t}|}\sqrt{|z_{i-1} e^{z^2_{i-1}/t}|}} \\
&= \prod_{e\in E(\gamma)} \frac{z_{i,i-1}\sqrt{|\sinh(z_i)\sinh(z_{i-1})|}}{\sqrt{|z_i z_{i-1}|}\sinh(z_{i,i-1})} e^{[z^2_{i,i-1} - z_i^2 - z^2_{i-1}]/t} \qquad (2.29)
\end{aligned}$$

Computation of the matrix element

$$\begin{aligned}
&\left\langle f \left| \exp[\frac{i}{\ell_p^2}\tau(\mathbf{M} - \epsilon)] \right| f' \right\rangle_{Kin} \\
= &\int dg_N \cdots dg_0 < \tilde{\psi}^t_{g_N} | \tilde{\psi}^t_{g_{N-1}} >_{Kin} \cdots < \tilde{\psi}^t_{g_1} | \tilde{\psi}^t_{g_0} >_{Kin} \\
&\times \exp\left[i \frac{\tau}{\ell_p^2 N} \big(\mathbf{M}[g_N] - \epsilon + t F^t(g_N, g_{N-1})\big)\right] \cdots \exp\left[i \frac{\tau}{\ell_p^2 N} \big(\mathbf{M}[g_1] - \epsilon + t F^t(g_1, g_0)\big)\right] \\
&\times \overline{f(g_N)} f'(g_0)
\end{aligned}$$

where $f(g) :=< \tilde{\psi}^t_g | f >_{Kin}$.

The product of overlap functions

$$
\begin{aligned}
& < \tilde{\psi}^t_{g_N} | \tilde{\psi}^t_{g_{N-1}} >_{Kin} < \tilde{\psi}^t_{g_{N-1}} | \tilde{\psi}^t_{g_{N-2}} >_{Kin} \cdots < \tilde{\psi}^t_{g_1} | \tilde{\psi}^t_{g_0} >_{Kin} \\
= & \prod_{e \in E(\gamma)} \frac{z_{N,N-1} \sqrt{|\sinh(p_N)\sinh(p_{N-1})|}}{\sqrt{p_N p_{N-1}}\sinh(z_{N,N-1})} e^{[z^2_{N,N-1} - \frac{1}{2}p_N^2 - \frac{1}{2}p^2_{N-1}]/t} \\
& \times \frac{z_{N-1,N-2} \sqrt{|\sinh(p_{N-1})\sinh(p_{N-2})|}}{\sqrt{p_{N-1} p_{N-2}}\sinh(z_{N-1,N-2})} e^{[z^2_{N-1,N-2} - \frac{1}{2}p_{N-1}^2 - \frac{1}{2}p^2_{N-2}]/t} \ldots \\
& \cdots \times \prod_{e \in E(\gamma)} \frac{z_{1,0} \sqrt{|\sinh(p_1)\sinh(p_0)|}}{\sqrt{p_1 p_0}\sinh(z_{1,0})} e^{[z^2_{1,0} - \frac{1}{2}p_1^2 - \frac{1}{2}p^2_0]/t} \\
= & \prod_{e \in E(\gamma)} \frac{\sinh(p_N)}{p_N} e^{-p_N^2/2t} \frac{\sinh(p_{N-1})}{p_{N-1}} e^{-p^2_{N-1}/t} \ldots \frac{\sinh(p_1)}{p_1} e^{-p_1^2/t} \frac{\sinh(p_0)}{p_0} e^{-p_0^2/2t} \\
& \times \frac{z_{N,N-1}}{\sinh(z_{N,N-1})} e^{z^2_{N,N-1}/t} \ldots \frac{z_{1,0}}{\sinh(z_{1,0})} e^{z^2_{1,0}/t} \sqrt{\frac{p_0 p_N}{\sinh(p_0)\sinh(p_N)}}
\end{aligned}
$$

Combining the results

$$
\begin{aligned}
& \left\langle f \left| \exp[\frac{i}{\ell_p^2}\tau(\mathbf{M} - \epsilon)] \right| f' \right\rangle_{Kin} \\
= & \int \prod_{e \in E(\gamma)} \prod_{i=0}^{N} \frac{dh_i d^3 p_i}{t^3} \frac{\sinh(p_i)}{p_i} e^{-p_i^2/t} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} e^{z^2_{k,k-1}/t} \\
& \times \exp\left[i\frac{\tau}{\ell_p^2 k}\big(\mathbf{M}[g_k] - \epsilon + tF^t(g_k, g_{k-1})\big)\right] \overline{f(g_N)} f'(g_0)
\end{aligned}
\tag{2.30}
$$

Define Lie algebra variables θ_k such that

$$
h_k = e^{\theta_k \cdot \tau/2},
$$

$z_{k,k-1}$ is found to be

$$
\begin{aligned}
\cosh(z_{k,k-1}) & = \frac{1}{2} tr(h_{k-1} h_k^\dagger e^{-i(p_k + p_{k+1})\cdot\tau/2}) \\
& = \frac{1}{2} tr(e^{i\left[-(p_k+p_{k+1}) + i(\theta_k - \theta_{k-1})\right])\cdot \tau/2}) \\
& = \cosh\left(\sqrt{[\frac{-(p_k + p_{k-1})}{2} + \frac{(\theta_k + \theta_{k-1})}{2}]^2} \right)
\end{aligned}
\tag{2.31}
$$

Therefore,

$$z_{k,k-1}^2 - \frac{1}{2}p_k^2 - \frac{1}{2}p_{k-1}^2 = -\frac{1}{4}\left[(p_k - p_{k-1})^2 + (\theta_k - \theta_{k-1})^2 + 2i(p_k + p_{k-1})\cdot(\theta_k - \theta_{k-1})\right] \quad (2.32)$$

We insert this result back into (2.30) and obtain

$$\begin{aligned}
&\left\langle f \left| \exp[\frac{i}{\ell_p^2}\tau(\mathbf{M}-\epsilon)] \right| f' \right\rangle_{Kin} \\
&= \int \prod_{e\in E(\gamma)} \prod_{i=0}^{N} \frac{dh_i d^3p_i}{t^3} \frac{\sinh(p_i)}{p_i} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} \overline{f(g_N)} f'(g_0) \\
&\times \exp\left\{ -i\frac{(p_k + p_{k-1})\cdot(\theta_k - \theta_{k-1})}{2t} - \frac{1}{4t}\left[(p_k - p_{k-1})^2 + (\theta_k - \theta_{k-1})^2\right] + \right. \\
&\left. + i\frac{\tau}{\ell_p^2 k}[\mathbf{M}[g_k] - \epsilon + tF^t(g_k, g_{k-1})] \right\}
\end{aligned} \quad (2.33)$$

The path-integral formula with a classical action in the exponential?

We make the following approximations:

(i) We assume the fluctuation F^t is negligible.

(ii)

With the above approximations, (2.33) simplifies to

$$\begin{aligned}
&\left\langle f \left| \exp[\frac{i}{\ell_p^2}\tau(\mathbf{M}-\epsilon)] \right| f' \right\rangle_{Kin} \\
&= \int \prod_{e\in E(\gamma)} \prod_{i=0}^{N} \frac{dh_i d^3p_i}{t^3} \left[\prod_{j=0}^{N} \frac{\sinh(p_j)}{p_j} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} \right] \overline{f(g_N)} f'(g_0) \\
&\times \prod_{k=1}^{N} \exp\left\{ -i\frac{(p_k + p_{k-1})}{2t}\cdot(\theta_k - \theta_{k-1}) + i\frac{\tau}{\ell_p^2 k}[\mathbf{M}[g_k] - \epsilon + tF^t(g_k, g_{k-1})] \right\}
\end{aligned} \quad (2.34)$$

which is the analogue of the path-integral of the "Master Action" on the continuum

$$
\begin{aligned}
S_{Master} &= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{\tau}{N}\frac{(p_k+p_{k-1})}{2t}\cdot\frac{(\theta_k-\theta_{k-1})}{\tau/N} - \lim_{N\to\infty}\sum_{k=1}^{N}\frac{\tau}{N}\frac{1}{\ell_p^2}\mathbf{M}[p,\theta] \\
&= \int_{t_i}^{t_f} dt \int_{\Sigma} p_j \partial_t \theta^j - \int_{t_i}^{t_f} dt\mathbf{M}[p,\theta] \qquad (2.35)
\end{aligned}
$$

we insert (2.34) back into () and obtain

$$
\begin{aligned}
&< \eta(f)|\eta(f') > \\
&= \lim_{\epsilon\to 0}\frac{1}{\ell_p^2}\int_{-\infty}^{\infty}\int \prod_{e\in E(\gamma)}\prod_{i=0}^{N}\frac{dh_i d^3p_i}{t^3}\left[\prod_{j=0}^{N}\frac{\sinh(p_j)}{p_j}\prod_{k=1}^{N}\frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right]\overline{f(g_N)}f'(g_0) \\
&\quad\times\prod_{k=1}^{N}\exp\left\{-\frac{i}{\ell_p^2}\left[\sum_{e\in E(\gamma)} a^2\frac{(p_k+p_{k-1})}{2}(\theta_k-\theta_{k-1})^a - \frac{\tau}{N}(\mathbf{M}[g_k]-\epsilon)\right]\right\} \qquad (2.36)
\end{aligned}
$$

2.6.4 LQG Master Constraint Path-Integral in terms of the Original Hamiltonian Constraints

The τ integral involved is

$$
\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\prod_{k=1}^{N}\exp\left\{\frac{\tau}{N}(\mathbf{M}[g_k]-\epsilon)\right\} = \delta(\sum_{k=1}^{N}\mathbf{M}[g_k]-\epsilon)
$$

So that

$$
\begin{aligned}
&< \eta(f)|\eta(f') > \\
&= \lim_{\epsilon\to 0} N\int \prod_{e\in E(\gamma)}\prod_{i=0}^{N}\frac{dh_i d^3p_i}{t^3}\left[\prod_{j=0}^{N}\frac{\sinh(p_j)}{p_j}\prod_{k=1}^{N}\frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right]\overline{f(g_N)}f'(g_0) \\
&\quad\times\prod_{k=1}^{N}\exp\left[-\frac{i}{\ell_p^2}\sum_{e\in E(\gamma)} a^2\frac{(p_k+p_{k-1})}{2}(\theta_k-\theta_{k-1})^a\right]\delta(\sum_{k=1}^{N}\mathbf{M}[g_k]-\epsilon) \qquad (2.37)
\end{aligned}
$$

The master constraint has the following expression:

$$\mathbf{M} = \sum_{v\in V(\gamma)} \frac{G_{j,v}}{V_v^{1/2}}\frac{G_{j,v}}{V_v^{1/2}} + \frac{D_{j,v}}{V_v^{1/2}}\frac{D_{j,v}}{V_v^{1/2}} + \frac{H_v}{V_v^{1/2}}\frac{H_v}{V_v^{1/2}} \tag{2.38}$$

We need to deal with

$$\delta(\sum_{k=1}^{N}\sum_{v\in V(\gamma)} \left\{\frac{G_{j,v}}{V_v^{1/2}}[g_k]\frac{G_{j,v}}{V_v^{1/2}}[g_k] + \frac{D_{j,v}}{V_v^{1/2}}[g_k]\frac{D_{j,v}}{V_v^{1/2}}[g_k] + \frac{H_v}{V_v^{1/2}}[g_k]\frac{H_v}{V_v^{1/2}}[g_k]\right\} - \epsilon) \tag{2.39}$$

Fisrt consider the type of integral

$$\begin{aligned}
& \lim_{\epsilon\to 0}\int dx_1 dx_2\cdots dx_N \delta(\sum_{i=1}^{N} x_i x_i - \epsilon) f(\vec{x}) \\
= & \lim_{\epsilon\to 0}\int dx_2\cdots dx_N \frac{1}{2\sqrt{\epsilon - \sum_{i=2}^{N} x_i x_i}}\left[f(x_1 = \sqrt{\epsilon - \sum_{i=2}^{N} x_i x_i}) + f(x_1 = -\sqrt{\epsilon - \sum_{i=2}^{N} x_i x_i})\right] \\
= & \left\{\lim_{\epsilon\to 0}\int dx_2\cdots dx_N \frac{1}{2\sqrt{\epsilon - \sum_{i=2}^{N} x_i x_i}}\right\} f(\vec{x} = 0) \\
\equiv & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon) f(\vec{x} = 0) \\
= & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\int dx_1 dx_2\cdots dx_N \prod_{i=1}^{N}\delta(x_i) f(\vec{x})
\end{aligned} \tag{2.40}$$

Consider the type of integral

$$\begin{aligned}
& \lim_{\epsilon\to 0}\int dt_1\cdots dx_M \delta(\sum_{i=1}^{N} x_i(\vec{t}) x_i(\vec{t}) - \epsilon) f(t_1, \ldots, t_M) \\
= & \lim_{\epsilon\to 0}\int dx_1\ldots dx_N dt_{N+1}\ldots dt_M \frac{1}{\det(\partial x/\partial t)}\delta(\sum_{i=1}^{N} x_i x_i - \epsilon)\tilde{f}(x_1,\cdots,x_N,t_{N+1},\cdots,t_M) \\
= & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\int dx_1\ldots dx_N dt_{N+1}\ldots dt_M \frac{1}{\det(\partial x/\partial t)}\prod_{i=1}^{N}\delta(x_i)\tilde{f}(x_1,\cdots,x_N,t_{N+1},\cdots,t_M) \\
= & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\int dt_1\cdots dx_M \prod_{i=1}^{N}\delta(x_i(\vec{t})) f(t_1,\ldots,t_M)
\end{aligned} \tag{2.41}$$

Therefore we obtain the result:

$$\lim_{\epsilon\to 0}\delta(\sum_{i=1}^{N} x_i(\vec{t})x_i(\vec{t})-\epsilon)=\lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\prod_{i=1}^{N}\delta(x_i) \tag{2.42}$$

$$\begin{aligned}
&\prod_{k=1}^{N}\prod_{v\in V(\gamma)}\delta^3\big(\frac{G_{j,v}}{V_v^{1/2}}[g_k]\big)\delta^3\big(\frac{D_{j,v}}{V_v^{1/2}}[g_k]\big)\delta^3\big(\frac{H_v}{V_v^{1/2}}[g_k]\big)\\
=&\prod_{k=1}^{N}\prod_{v\in V(\gamma)}V_v^{7/2}\delta^3\big(G_{j,v}[g_k]\big)\delta^3\big(D_{j,v}[g_k]\big)\delta^3\big(H_v[g_k]\big)\\
=&\prod_{k=1}^{N}\prod_{v\in V(\gamma)}d^3\Lambda_{v,k}d^3N_{v,k}dN_{v,k}\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)}V_v^{7/2}[g_k]\right]\\
&\times\prod_{k=1}^{N}\exp\left\{-\frac{i}{\ell_p^2}\sum_{v\in V(\gamma)}(\Lambda^i_{v,k}G_{i,v}[g_k]+N^i_{v,k}D_{i,v}[g_k]+N_{v,k}H_kg_k])\right\}
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&<\eta(f)|\eta(f')>\\
=&\lim_{\epsilon\to 0}\mathcal{N}(\epsilon)N\int\prod_{e\in E(\gamma)}\left[\prod_{i=0}^{N}\frac{dh_id^3p_i}{t^3}\right]\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)}d^3\Lambda_{v,k}d^3N_{v,k}dN_{v,k}\right]\\
&\times\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)}V_v^{7/2}[g_k]\right]\left[\prod_{j=0}^{N}\frac{\sinh(p_j)}{p_j}\prod_{k=1}^{N}\frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right]\sqrt{\frac{p_0p_N}{\sinh(p_0)\sinh(p_N)}}\\
&\times\prod_{k=1}^{N}\exp\left\{-\frac{i}{\ell_p^2}\sum_{e\in E(\gamma)}a^2<p>_k\frac{(\theta_k-\theta_{k-1})^a}{\Delta T_k}\Delta T_k\right.\\
&\qquad\left.+\sum_{v\in V(\gamma)}(\Lambda^i_{v,k}G_{i,v}[g_k]+N^i_{v,k}D_{i,v}[g_k]+N_{v,k}H_kg_k])\right\}\times\overline{f(g_N)}f'(g_0)
\end{aligned} \tag{2.44}$$

Here the graph γ is a finite graph with finite number of edges and vertices.

$$
\begin{aligned}
&\mathcal{D}\mu[g,\Sigma,N^a,N] \\
= &\prod_{e\in E(\gamma)}\left[\prod_{i=0}^{N}\frac{dh_i d^3p_i}{t^3}\right]\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)} d^3\Lambda_{v,k}d^3N_{v,k}dN_{v,k}\right]\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)} V_v^{7/2}[g_k]\right] \\
&\times\left[\prod_{j=0}^{N}\frac{\sinh(p_j)}{p_j}\prod_{k=1}^{N}\frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right]\sqrt{\frac{p_0p_N}{\sinh(p_0)\sinh(p_N)}}
\end{aligned}
\tag{2.45}
$$

We choose a reference vector Ω and define the path-integral representation of the group averaging physical inner product

$$
<\eta(f)|\eta(f')>_\Omega = \lim_{\epsilon\to 0}\frac{\int_{\mathbb{R}} dt < f|e^{it(\mathbf{M}-\epsilon)}|f'>}{\int_{\mathbb{R}} dt < \Omega|e^{it(\mathbf{M}-\epsilon)}|\Omega>}
\tag{2.46}
$$

Chapter 3

Semiclassical Analysis

3.1 Twisted Geometries

A basis of states for LQG is given by the spin network states. A spin network state has support on a graph Γ and determines a 3d "quantum geometry"; they are eigenvectors of geometric operators which determine the intrinsic geometry. The extrinsic gemetry (analogous to the conjugate momentum) is completely "spread out", due to the Heisenberg uncertainty principle.

To make contact with a semiclassical description of space, we can consider coherent states peaked (but not sharp) on both the intrinsic and exstrinsic geometry. A twisted geometry is a specific choice of "interpolating geometry", chosen among distcontinuous metrics.

Twisted geometries parametrise the phase space of loop quantum gravity on a fixed graph in terms of quantities describing the intrinsic and extrinsic geometry of a three dimensional triangulation dual to the graph.

To any graph and any holonomy-flux configuration, we can associate a twisted geometry: a discrete discontinuous geometry on a cellular decomposition space into polyhedra. Thanks to this result, the phase space of LQG on a graph can be visualised not only in terms of holonomies and fluxes, but also in terms of a simple geometric picture of adjacent flat polyhedra.

3.1.1 Hopf Map

Consider the space $\mathbb{C}^2$ of pairs of complex numbers (z_0, z_1). Define a bundle space by the equation

$$|z_0|^2 + |z_1|^2 = 1$$

This stands for the equation of a 3-sphere. Now consider the family of complex planes through the origin. Each such plane is given by qn equation of the form

$$Az_0 + Bz_1 = 0$$

where A and B are complex numbers (not both zero). Being 2-dimensional planes it intersects S^3 in a circle S^1. The different planes only meet at the origin, so no distsinct S^1's have no point in common.

We get the same plane $Az_0 + Bz_1 = 0$ if we muliply both A and B by the same complex non-zero number, so it is the ratio A/B that determines the different planes. The space of such ratios is the Riemann sphere.

Chapter 4

Quantum Cosmology from Loop Quantum Gravity

4.1 Introduction

• Quantum fluctuations of the inhomogeneous degrees of freedom are believed to play a majour role in structure formulation.

• Relation between LQC and LQG. Can we derive the LQC assumptions from the more complete LQG model?

4.2 Towards Inhomogeneous Loop Quantum Cosmology

Appraoch put forward by Vidotto and Rovelli. One uses average gravitational degrees of freedom over a triangulation of a 3-shpere representing a closed universe.

4.2.1 Triangularations of the three-sphere

First notice that the two-sphere can be described as two disks glued together along their boundaries (fig 4.2.1). Since the anaolgue of a disk in the next higher dimension is a solid ball, it may not be suprising that the three-sphere can be described by gluing together of two solid balls.

As a second description, it is well known that a two sphere (the Riemann sphere) can be thought of as the two-plane plus a point at infinity. The three-sphere is analogus. We

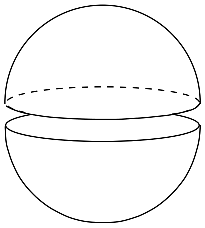


Figure 4.1: The two sphere is formed by gluing two disks along the boundary.

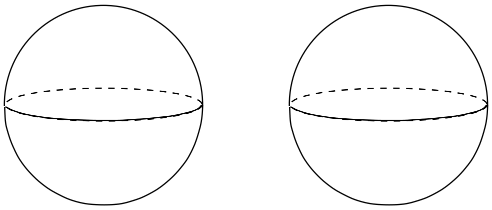


Figure 4.2: The three sphere is formed by gluing two three-balls along the boundary.

take the points of $\mathbb{R}^3$ together with one point, which we think of as infinity, since it is not part of $\mathbb{R}^3$.

A surface was compact if it could be triangulated with finitely many triangles. In the 'triagulation' of a three-manifold we simply replace triangles with tetrahra.

4.2.2 Averaged Classical variables

We introduce the pair (U_f, E_f) of average gravitational degrees of freedom over a triangulation Δ_n of space, where

• U_f: the parallel transport of the ashtekar connection A_a along the link e_f of the dual to the face,

• E_f: flux Φ_f of the Ashtekar's Electric field E^a across the triangle f

$$\Phi_f \equiv \int_f d^2 y E^b(y) n_b,$$

parallel transported to the center of the tetrahedron.

What does it mean to parallel transport the Electric field $\mathbf{E}^b \equiv E^b_i \tau^i$? This means that its covariant derivative along the curve $\gamma^a(t)$ should be zero,

$$\dot{\gamma}^a(t) \mathbf{D}_a \mathbf{E}^b = 0 \tag{4.1}$$

where $\dot\gamma^a(t) = d\gamma^a(t)/dt$ is the tangent curve. This gives the equation

$$\dot\gamma^a(t)\partial_a \mathbf{E}^b(t) = -ig\dot\gamma^a(t)\mathbf{A}_a(t)\mathbf{E}^b(t) \tag{4.2}$$

Integrating this equation we obtain

$$\mathbf{E}^b(t) = \mathbf{E}^b(0) - ig\int_0^t ds\dot\gamma^a(s)\mathbf{A}_a(s)\mathbf{E}^b(s) \tag{4.3}$$

Iterating by repeated substituting the RHS of (4.3) into itself we obtain

$$\begin{aligned}\mathbf{E}^b(t) &= \mathbf{E}^b(0) - ig\int_0^t ds\dot\gamma^a(s)\mathbf{A}_a(s)\mathbf{E}^b(0)\\ &- g^2\int_0^t ds_{a_1}\dot\gamma^a(s_1)\mathbf{A}_{a_1}(s_1)\int_0^t ds_2\dot\gamma^{a_2}(s_2)\mathbf{A}_{a_2}(s_2)\ \mathbf{E}^b(0)\\ &+ \ \ldots\end{aligned} \tag{4.4}$$

Repeating the process indefinitely we obtain

$$\mathbf{E}^b(t) = \sum_{n=0}^{\infty}\left((-ig)^n\int_{t_1\geq\cdots\geq t_n\geq 0}\dot\gamma^{a_1}(t_1)\mathbf{A}_{a_1}(t_1)\cdots\dot\gamma^{a_n}(t_n)\mathbf{A}_{a_n}(t_n)dt_1\cdots dt_n\right)\ \mathbf{E}^b(0) \tag{4.5}$$

so the electric flux parallely transported to the center of the tetrahedron is

$$E_f = U_{e_1}^{-1}\Phi_f U_{e_1}. \tag{4.6}$$

4.2.3 Algebra of coarse averaged variables

We have

$$\{A_a(x), E^b(y)\} = \delta_a^b\delta^3(x-y). \tag{4.7}$$

Let us consider $\{U_f, U_{f'}\}$,

$$\begin{aligned}\{U_f, U_{f'}\} &=\\ &=\end{aligned} \tag{4.8}$$

Let us consider $\{E^i_f, U_{f'}\}$.

$$\begin{aligned}
&\{E^i_f, U_{f'}\}\\
= \;& \{\int_f d^2y E^b_i(y) n_b, \sum_{n=0}^{\infty}(-ig)^n \int_{t_1\geq\cdots\geq t_n\geq 0} \dot\gamma^{a_1}(t_1)\mathbf{A}_{a_1}(t_1)\cdots\dot\gamma^{a_n}(t_n)\mathbf{A}_{a_n}(t_n)dt_1\cdots dt_n\}\\
= \;& \sum_{n=0}^{\infty}\int_f d^2y n_b \int_{t_1\geq\cdots\geq t_n\geq 0} dt_1\cdots dt_n \dot\gamma^{a_1}(t_1)\cdots\dot\gamma^{a_n}(t_n)\{E^b(y), \mathbf{A}_{a_1}(t_1)\cdots\mathbf{A}_{a_n}(t_n)\}\\
= \;& \sum_{n=0}^{\infty}(-ig)^n \int_f d^2y n_b \int_{t_1\geq\cdots\geq t_n\geq 0} dt_1\cdots dt_n \dot\gamma^{a_1}(t_1)\cdots\dot\gamma^{a_n}(t_n)\\
& \Big(\{E^b_i(y), \mathbf{A}_{a_1}(t_1)\}\mathbf{A}_{a_2}(t_2)\cdots\mathbf{A}_{a_n}(t_n) + \mathbf{A}_{a_1}(t_1)\{E^b_i(y), \mathbf{A}_{a_2}(t_2)\}\cdots\mathbf{A}_{a_n}(t_n) + \dots\\
& \mathbf{A}_{a_1}(t_1)\mathbf{A}_{a_2}(t_2)\cdots\{E^b_i(y), \mathbf{A}_{a_n}(t_n)\}\Big)\\
= \;& \tau_i \sum_{n=0}^{\infty}(ig)^n \int_f d^2y n_b \int_{t_1\geq\cdots\geq t_n\geq 0} dt_1\cdots dt_n \dot\gamma^{a_1}(t_1)\cdots\dot\gamma^{a_n}(t_n)\\
& \Big(\delta^b_{a_1}\delta(y-\gamma(t_1))\mathbf{A}_{a_2}(t_2)\cdots\mathbf{A}_{a_n}(t_n)+
\end{aligned} \tag{4.9}$$

If $f \neq f'$ then the domain of the $y-$intergal does not coincide with any $t-$integral and so $\{E^i_f, U_{f'}\} = 0$. Now say $f = f'$. If $\gamma(t_r)$ intersects the face f then we have the volume element $d^2y(dt_r n\cdot\dot\gamma^{a_r}(t_r))$. So that

$$\begin{aligned}
& \int_f\int d^2y(dt_r n\cdot\dot\gamma^{a_r}(t_r))\delta(y-\gamma(t_r))\dot\gamma^{a_1}(t_1)\cdots\dot\gamma^{a_{r-1}}(t_{r-1})\dot\gamma^{a_{r+1}}(t_{r+1})\dot\gamma^{a_n}(t_n)\\
& \times\mathbf{A}_{a_1}(t_1)\cdots\mathbf{A}_{a_{r-1}}(t_{r-1})\mathbf{A}_{a_{r+1}}(t_{r+1})\cdots\mathbf{A}_{a_n}(t_n)\\
= \;& \dot\gamma^{a_1}(t_1)\cdots\dot\gamma^{a_{r-1}}(t_{r-1})\dot\gamma^{a_{r+1}}(t_{r+1})\dot\gamma^{a_n}(t_n)\mathbf{A}_{a_1}(t_1)\cdots\mathbf{A}_{a_{r-1}}(t_{r-1})\mathbf{A}_{a_{r+1}}(t_{r+1})\cdots\mathbf{A}_{a_n}(t_n)
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&\{E^i_f, U_f\}\\
= \;& \tau_i\sum_{n=0}^{\infty}(ig)^n
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\{U_f, U_{f'}\} &= 0,\\
\{E^i_f, U_{f'}\} &= \delta_{ff'}\,\tau^i U_f,\\
\{E^i_f, E^j_{f'}\} &= -\delta_{ff'}\epsilon^{ijk}E^k_f.
\end{aligned} \tag{4.12}$$

4.2.4 Completeness theorem - from averaged connection formulism to restricted space of Wilson loops

The completeness theorems of full Loop quantum gravity say that if you know the trace of the holonomy along all possible loops in a manifold for a given vector potential you can reconstruct from their values all the gauge invariant information in the vector potential.

4.2.5 The definition of the Auxilary Hilbert space

The auxilary Hilbert space is

$$H_{aux} := L^2[SU(2)^{2n}, dU_f] \tag{4.13}$$

and states are $\psi(U_f)$.

A complete, orthonormal basis of states that solve the Guass gauge constraint are $SU(2)$ spin networks on the dual graph Δ_n^*

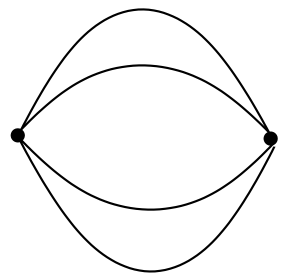


Figure 4.3: The dual simplex Δ_2^*.

4.2.6 Non-graph changing Hamiltonian constraint

4.2.7 Hamiltonian constraints

the gauge constraints are

$$G_t \equiv \sum_{f\in t} E_f \approx 0 \tag{4.14}$$

where the sum is over the four faces of the tetrahedron.

The Hamiltonian constraints are

$$C_t \equiv V_t^{-1} \sum_{ff' \in t} tr[U_{ff'} E_{f'} E_f] \approx 0 \tag{4.15}$$

4.2.8 Good approximation for large universe

$$U_{ff'} \approx \mathbb{I} + |\alpha|^2 F_{ab} + \mathcal{O}(|\alpha|^2 F) \tag{4.16}$$

is also a good approximation as the universe increases in size if F_{ab} is small enougth, and near flat spacetime approximation is a good approximation even for coarse triangulations.

4.2.9 Matter as clock variable

$$S_t = \frac{1}{V_t} C_t + \frac{\kappa}{2V_t} p_{\phi_t}^2 \approx 0. \tag{4.17}$$

where

$$V_t = \sum_{ff'ff''} \sqrt{tr(E_f E_{f'} E_{f''})}. \tag{4.18}$$

4.2.10 Born-Oppenheimer approximation

The Born-Oppenheimer ansatz is

$$\psi(R, r) = \Psi(R)\phi(R; r) \tag{4.19}$$

4.2.11 Recovering Loop quantum cosmology

Chapter 5

Unificiation from Background Independent Quantum Gravity

5.1 The Standard Model from Q-Deformed LoopQuantum Gravity

5.1.1 Noiseless Subsystems and Quantum Gravity

There are also indications, however, that some quantum information processing techniques may inspire new ways to look at some old problems such as quantum gravity []. In particular, the framework of noiseless subsystems has been argued to provide an intuitive picture for the emergence of particles and space-time from a fully quantum system []. In [] they show that the framework of noiseless subsystem can provide a basis for thinking about any (quantum) mechanical system that is subject to constraints.

• The low-energy problem is to find propagating modes in a background-independent quantum theory of gravity (e.g. spin foam, LQG). Necessary, in order to make contact with physics.

Noiseless sub-systems

When an environment has a symmetry to it, there exists a subspace of the Hilbert space which is protected against decoherence effect of the environment - this is called a noiseless subsystem.

• Noiseless Subsystems are useful for describing long-term behaviour of the system because they are conserved.

• If we divide the quantum gravitational field into subsystems, those properties that are **conserved** under interactions between subsystems will characterize the low energy-limit of spacetime geometry.

• The commutant $A^{int\prime}$ should include the symmetries that characterize classical spacetime (e.g. Poincare).

Particles in quantum geometry

It is argued that a particle could be some kind emergent excitation, some subsystem, of microscopic quantum dynamics. The quantum spacetime, being dynamical, is constantly changing. In order for the emergent excitation to behave as if it were a particle moving through a fixed, non-dynamical background spacetime it must be a noiseless sub-system. So Markopoulou identified emergent particles as noiseless subsystems of quantum geometry corresponding to a Lorentz invariance symmetry at the appropriate scale.

5.1.2 The Standard Model - a Reminder

Matter

The Standard Model fermion are the leptons and quarks

(i) Leptons:

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}, \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} \tag{5.1}$$

The electron, muon and tau electron all have $-1e$, their anti particles have charge $+1e$, but the neutrinos are all uncharged.

They form doublets under $SU(2)$

(ii) Quarks:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix} \tag{5.2}$$

Left handed fermions in the Standard Model

Fermion left-handed	symbol	spin	electric charge	colour charge	mass
electron	e	1/2	-1	1	0.510999 Mev
electron neutrino	ν_e	1/2	0	1	~ 0 ev $< 2ev$ Mev
positron	e^c	1/2	$+1$	1	0.510999 Mev
electron antineutrino	μ_e^c	1/2	0	1	~ 0 ev < 2 ev
up quark	u	1/2	$+2/3$	3	~ 3 Mev
down quark	d	1/2	$-1/3$	3	~ 6 Mev
anti-up quark	u^c	1/2	$-2/3$	$\overline{3}$	~ 3 Mev
anti-down quark	d^c	1/2	$+1/3$	$\overline{3}$	~ 6 Mev

Forces

(i) Forces mediated by bosons (spin $0, 1, 2, \dots$)

(ii) The electromagnetic force is mediated by the photon (γ)

(iii) The weak force is mediated by the $W^{\pm}$ (posite, negative charge) and Z^0 (uncharged) bosons

(iv) The strong force is mediated by gluons (g)

Discrete symmetries

$$\begin{aligned} \mathcal{P} &: \mathbf{x} \to -\mathbf{x} \\ \mathcal{C} &: e \to -e \\ T &: t \to -t \end{aligned} \tag{5.3}$$

Parity transfomation:

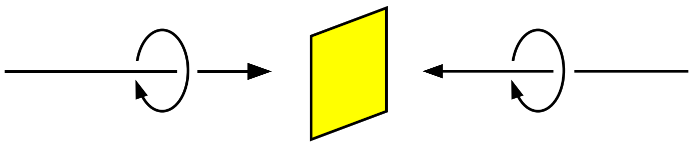


Figure 5.1: parityPartF. The operation $\mathcal{P}$ reverses the momentum of the particle whithout flipping its spin.

Charge conjugation $\mathcal{C}$: is the particle-antiparticle symmetry. Charge conjugation is defined to take a fermion with a given spin orientaion into an antifermion with the same spin orientaion.

• $\mathcal{P}$ converts a left-handed electron into a right-handed electron,

and

• $\mathcal{C}$ converts a left-handed electron into a left-handed positron.

the combination of these two operations interchanges left-handed particles with right-handed antiparticles.

zero-mass fermions

A wave function of negative energy and momentum $-\mathbf{p}$ corresponds to an anti-fermion with positive energy and momentum $\mathbf{p}$. Zero mass fermions and anti-zero mass ferminons with positive chirality both also have helicity (they are "right-handed"). Similarly, fermions and anti-fermions with negative chirality both barry negative helicity (they are "left-handed").

Although strong and electromagnetic forces make no distinction between right-handed or left-handed particles (particle invariance), particles subject to weak forces do make this distinction. (A right-handed particle is a particle spinning in the direction the right-hand fingers curl when the particle is traveling in the direction pointed-to by the right thumb). Thus, left-handed neutrinos are matter, whereas right-handed neutrinos are antimatter.

Gauge symmetry group of fermions

$SU(3) \times SU(2) \times U(1)$

Electroweak

iso-spin doublets

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix} = \tag{5.4}$$

Strong

What is conserved in the Standard Model

(i) Energy, momentum, but not mass

(ii) Angular momentum

(iii) Charge and colour

(iv) Lepton flavour number (and therefore lepton number)

(v) Quark number (but not quark flavour number)

(vi) Charge conjugation ($\mathcal{C}$), parity ($\mathcal{P}$), and time reversal (T) are conserved by the strong and electromagnetic interactions, but not by the weak interaction ($\mathcal{CP}T$ is always conserved).

5.1.3 Conventional Preon Models and their Problems

Unlike hadrons, the light quarks and leptons are much lighter than the inverse of the largest experimentally allowed binding scale for their proposed subcomponents. This made it challenging to bind such hypothetical subcomponents by means of ordinary gauge interactions.

The binding mechanism proposed here operates at Planck scales, below the scales at which effective field theory would be a good description. The states are bound here, not by fields, but by quantum topology, because the configurations that we interpret as quarks and leptons are conserved under the dynamics of the quantum geometry.

5.2 The Standard Model from Loop Qunatum Gravity?

5.2.1 Introduction

These theories already contain elementary particles which can be identified with particles of the standard model so that the symmetries of the standard model are symmetries of the dynamics of the quantum geometry.

When they are continually in interaction with the quantum fluctuations of the microscopic theory - they are protected by symmetries in the dynamics.

5.2.2 The Standard Model from Loop Qunatum Gravity?

Preons:

"Quarks, leptons and heavy vector bosons are suggested to be composed of a stable spin-1/2 preons, existing in three flavours, combined according to simple rules."

Now onto the emergence of (part) of the Standard Model:

Sundance O. Bilson-Thompson (who was working on preons, not quantum gravity) noticed that a simple braiding of ribons that captured precisely the structure of the preon models of particle physics.

From this insight of Bilson-Thompsons', Markopoulou that the different ways of braid and knot the edges of graphs in a quantum spacetime must be different kinds of elementary particles!!! - So loop quantum gravity is not just about quantum spacetime - it already has elemenmtary particle physics in it.

From abstract of "Quantum Gravity and the Standard Model" (hep-th/0603022):

"We show that a class of background independent models of quantum spacetime have local excitations that can be mapped to the first generation fermions of the standard model of particle physics. These states propagate coherently as they can be shown to be noiseless subsystems of the microscopic quantum dynamics[2]. These are identified in terms of certain patterns of braiding of graphs, thus giving a quantum gravitational foundation for the topological preon model proposed in [1]."

5.2.3 Substitution Moves

Ribbon graph theories

Quantum $SU(2)$ group.

There are arguments from both the canonical quantisation and spin foam models that the cosmological constant induces a framing of the graph because the algebra of $SU(2)$ is quantum deformed and to represent the qunatum deformed algebra labeled by the deformation parameter q related to the cosmological constant

$$q = e^{2\pi i/k+2}, \qquad k = 6\pi/G\Lambda$$

the graph has to have a framing - you have to keep track of whether the graph has been twisted or not.

They study theories based on framed graphs in tree spatial dimensions.

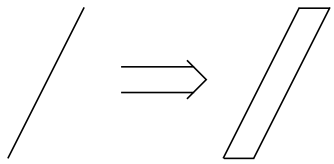


Figure 5.2: The edges are framed. The nodes become trinions.

Dynamical Issues

The dynamics is studied in different ways there are Hamiltonian constraint dynamics and spin foam models. In the Hamiltonian constraint construction one only has the expansion move

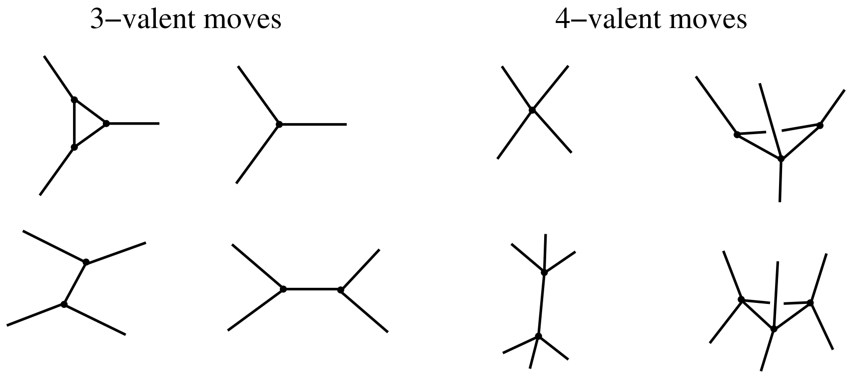


Figure 5.3: Spin foam dynamics.

in spin foams in addition to the exchange and expansion moves there are moves corresponding to local changes in triangualtion of manifolds which leave the topology unchanged called the dual Pachner moves.

The result is a sequence of states, $|0>, |1>, \ldots, |N>$, esch following from the previous one by a substitution move.

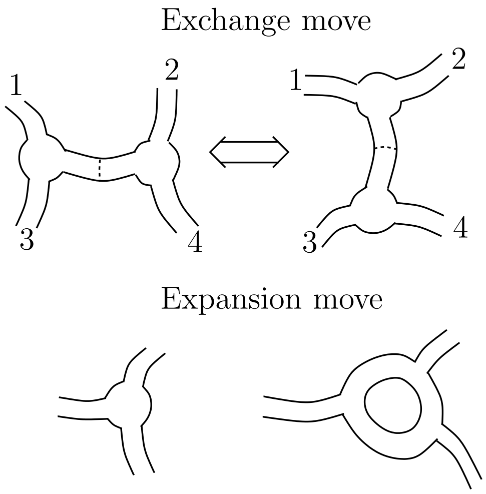
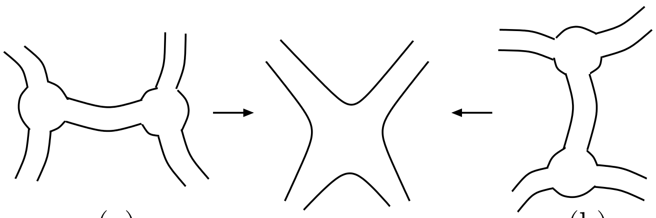


Figure 5.4: substitution move.



(a) (b)

Figure 5.5: Both ribbon graphs (a) and (b), related by an exchange move, have the same embedded topology.

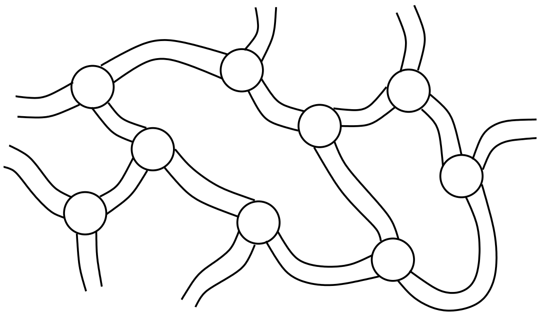


Figure 5.6:

5.2.4 Discrete Symetries of Braids

Reduced graph is otained by removing all unlinked unknotted circles.

The reduced link of the ribbon is a constant of the motion.

Definition of a subsystem: The reduced link disconnects from the reduced link of the whole graph. This gives conserved quantities labeling subsystems.

Properties of these invariants

(i) Distinguish over crossings from under crossings

(ii) Distinguish twists

(iii) Are chiral: distinguish left and rihgt handed structures

These invariants are independent of the choice of algebra G and the evolution amplitudes. They exists for a large class of theories.

Braids on N strands, attached at either or both the top and bottom.

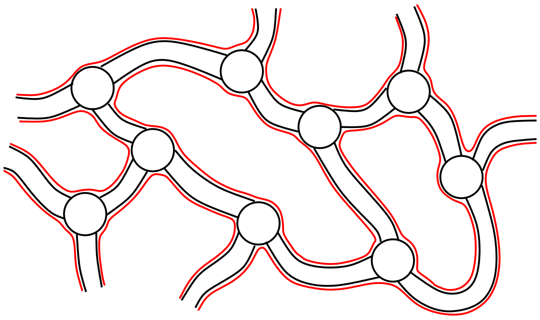


Figure 5.7: The link of a graph is obtained by tracing curves of the graph.

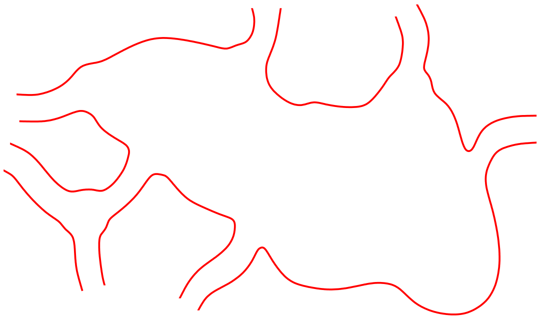


Figure 5.8: The reduced link of a graph is obtained by removing all isolated circles from its link. Any two graphs which differ by expansion moves have the same reduced link and hence invariants of these reduced links (being independent of exchange moves as well) are constants of motion.

The braids and twists are constructed by sequences of moves. The moves form the braid group.

To each braid B there is then a group element $g(B)$ which is a product of braiding and twisting.

Parity. A parity transformation is defined as a reflection of a braid, while not effecting the handedness of the twists on the strands.

Charge conjugation.

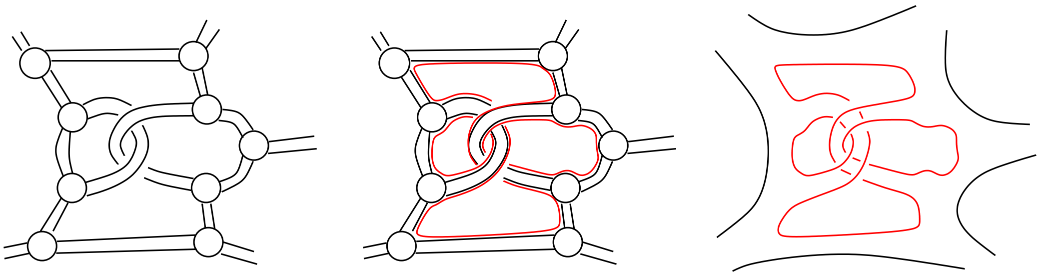


Figure 5.9:

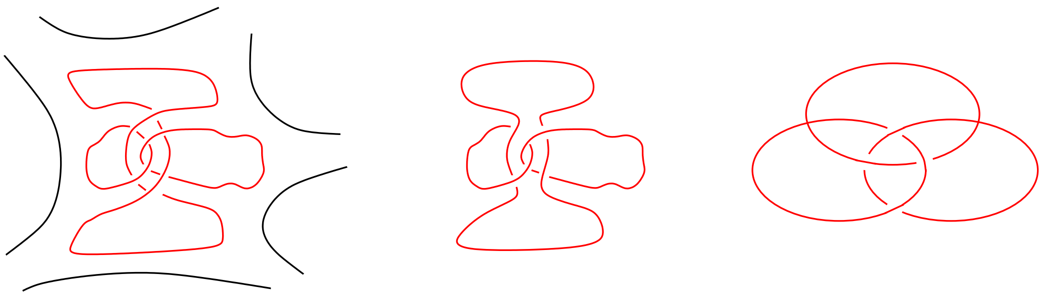


Figure 5.10:

5.2.5 Identification of the First Generation Fermions of the Standard Model

Assumptions

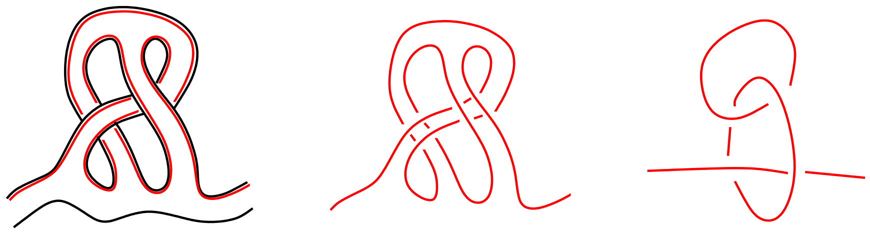
As we will see, in order to arrive at the standard model fermions we have to make two assumptions:

• The lightest states are the simplest non-trivial braids made of ribbons with no twists (no charge) or one full twist (positive or negative charge).

• Quantum numbers are assigned only to braids with no positive and negative charge mixing. Such a rule is necessary in preon models as discussed [275].

These should be justified in a fundamental theory. At the level that an effective dynamics in spacetime emerges a notion of mass of emergent excitations will arise. This requires that there be in the low energy limit an emergent translation invariance in space and time. This will imply the conservation of energy and momentum for small excitations around the ground state. When the effective Hamiltonian H is evaluated on the states described here at zero momentum it will give us a mass matrix.

(1) Given two braids as just described, which have the same number of strands and twistings, but differ by the number of crossings, the mass will increase with the number



(a) (b) (c)

Figure 5.11: (a) Link obtained by taking the trace of the edges. (c) Simplification of the link.

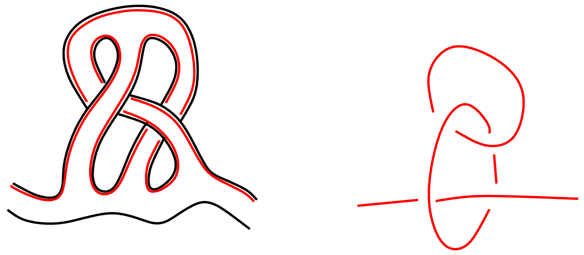


Figure 5.12: Link associated with mirror image.

of crossings.

(2) Non-trivially braided states with both positive and negative twisted strands incident on the same vertex should have a heavy mass M. All other states are light relative to the scale M.

Identification of fermions

These are partially charged states, with one and two +'s, and the rest zeroes. Theses are shown in figures 5.19 and 5.20. These are the quarks, with total charges $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$. In each of these there is an "odd strand out" - one that is different from the other two. The non-trivial nature of the braid means that there are three distinct positions in the braid which the "odd strand out" can occupy. Hence each of the partially charged states comes in three versions. We will equate these permutations with colour.

The correspondence between braids and particles suggests that more properties are waiting to be derived from the theory. The most substantial achievement would be to calculate the masses of the elementary particles from first principles.

It is natural to hypothesize then that the second generation standard model fermions come from the next most complicated states, which have three crossings. There are two kinds of three crossing states. There are three stranded braids, such as shown in Figure

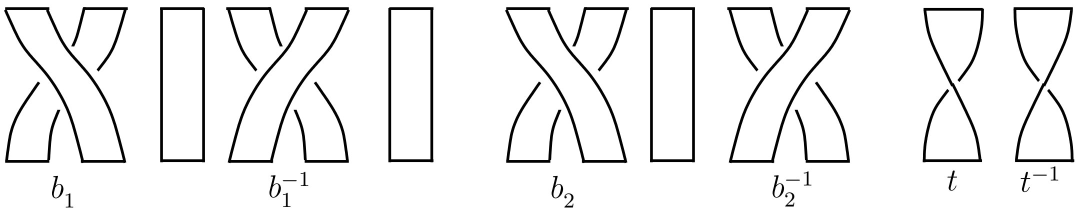


Figure 5.13: basucMovesF. The basic braiding moves on three strands, and the basic twisting moves.

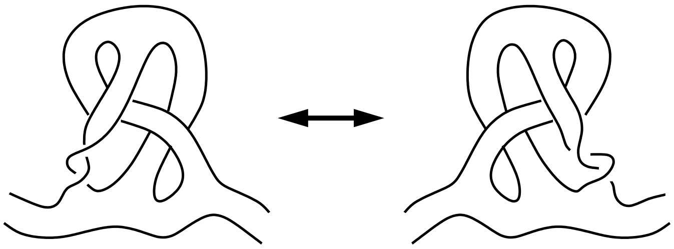


Figure 5.14: paritytranF. The effect of a parity transformation on a braid.

E.7??. It is straightforward to see that by adding twists to this state one gets a repeat of the pattern for the first generation. However, this appears to imply no upper bound on the number of allowed generations.

Are they Fermionic?

It has been shown these degrees of freedom to be chiral but are they fermionic? Graphs could obey non-trivial anyonic statistics as the edges can be anyonic in 3d for the same reason that points are anyonic in 2d because they give representation to the braid rather than the permutation group. In physics terms we are looking for an inverse quantum Hall effect, that is, rather than a fermion decomposing into three anyons, anyons (of the Baez kind) combining into a fermion.

Higher generations

generation = crossings - 1

one new $SU(2) + SU(3)$ singlet appears for all higher generations

Weak vector bosons come from unbraided triplets of lines.

With one more rule to suppress quark-lepton mixing weak interactoins are a consequence of loacl moves.

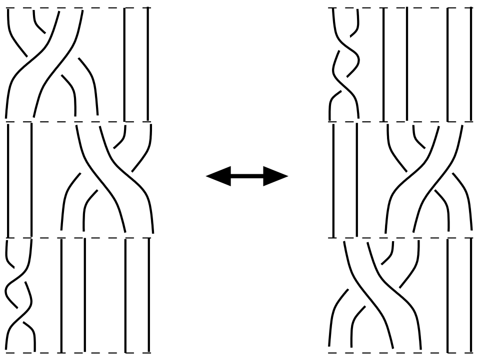


Figure 5.15: chargeDefF. The definition of a charge conjugation transformation on a braid.

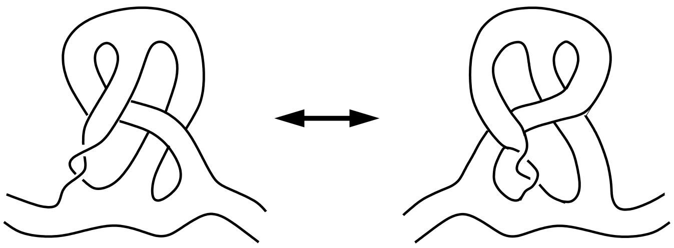


Figure 5.16: chargetranF. The effect of a charge conjugation transformation on a braid.

From all allowed twists we get a copy of the 1st generation.

These give additional states which are $SU(2) + SU(3)$ singlets but come in left and right versions. Could these be the right handed neutrinos.

Vector bosons

Plank scale mass problem

A problem that any theory of quantum gravity which porports to contain the standard model, this includs string theory, is when the only mass available is the Plank mass, how do we get the much lighter masses of the particles of the standard model. The only known way has to do with chirality. Therefore it is interesting that this LQG type model involves chiral degrees of freedom, but nothing has been worked out so far.

5.2.6 Four Valent Framed Braids: Propagation and Interaction

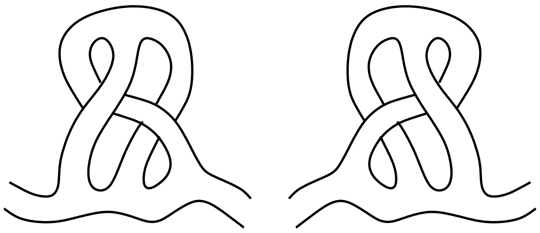


Figure 5.17: elecneutrinoF. The electron neutrino and anti-neutrino - two unchaged states.

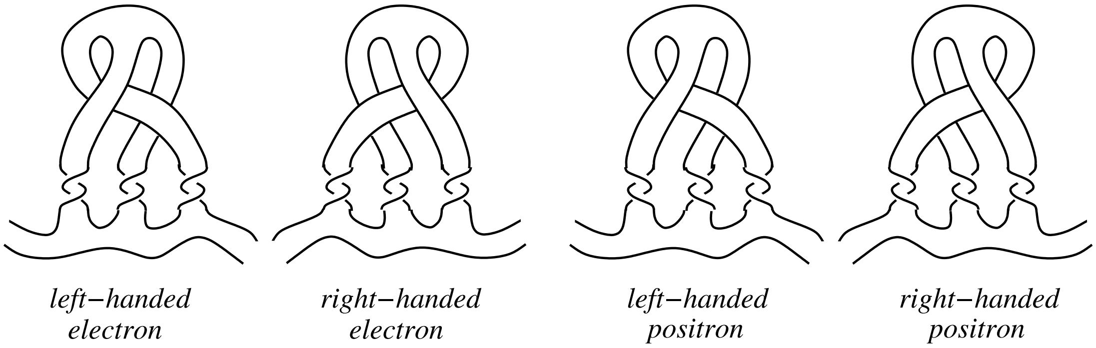


Figure 5.18: elecpositronF. The electron and positron - four maximally charged states.

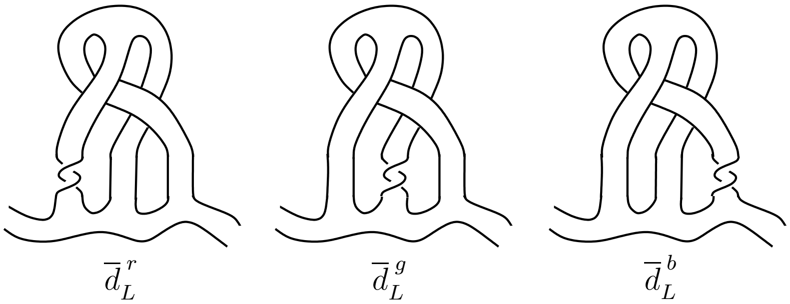


Figure 5.19: dquarkleftF. The left-handed down states - showing tripling of states for fractional charge.

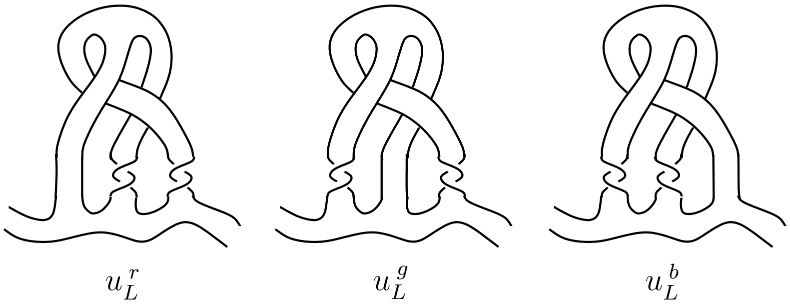


Figure 5.20: uquarkleftF. The three left-handed up states.

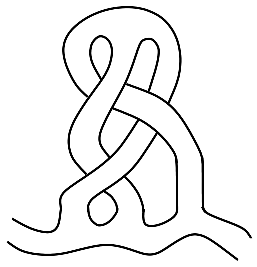


Figure 5.21: muonneutrF. A possible identification of the muon neutrino as a three crossing state.

Appendix A

Physics Glossary

This discussion is intended to provide only a passing familiarity with definitions to allow the unfamiliar reader to follow certain calculations and to have a general understanding of the results.

• **absolute space and time:** Newton's view of space and time according to which they exist, independent of whether anything is in the universe or not.

• **abstract spin network:** They are the equivalence classes of spin networks under spatial diffeomorphisms. GR tells us that the spacetime manifold is a convenient mathematical device devoid of physical meaning. How a spin network is embedded in the spacial three-slice. Once we factorize out the spatial diffeomorphism invariance the only remaining information in the state is the knotedness of the loop. These are objects that don't live in the manifold anymore - they are not embedded in a space. They are not quantum excitations in space, but quantum excitations of space - if there is no spinor network there is no space!

• **achronal:** A subset S of spacetime $\mathcal{M}$ is said to be achronal provided no two of its points can be joined by a timelike curve.

• **active diffeomorphism:** transformation relates different objects in $\mathcal{M}$ in the same coordinate system. This means that the diffeomorphism f is viewed as a map associating one point of the manifold to another one.

Invariants with respect to active diffeomorphisms are obtained by solving away the coordinates x from solutions to the equations of motion.

is a solution of Einstein's equations, namely active diffeomorphisms are dynamical symmetries of Einstein's equations sending solutions to solutions.

• **ADM:** In the Hamiltonian formulation of GR the generator of time translations in an asymptotically flat spacetime.

(1) ADM energy

(2) ADM angular momentum

• **Alexandrofff topology:** For a spacetime $\mathcal{M}$ and any two spacetime points a, b. The chronological sets $\{I^+(a) \cap I^-(b) : a, b \in \mathcal{M}\}$ are open and and form a base for a topology on $\mathcal{M}$, which is called the Alexandroff topology. (Also see maths glossary).

• **algebraic quantum theory:** In the Heisenbergs picture of quantum mechanics the observables play the dominant role, they are governed by an equation of motion while the states are time-independent. In algebraic quantum theory we formulate more absract version of Heisenberg's approach, in which the observables are the relevent objects and the quantum states are secondary.

There are two main ingredients: firstly, a $C\star$-algebra of observables often denoted $\mathcal{A}$, and secondly, states (different meaning from above) ω which act on an observable $A \in \mathcal{A}$ giving a complex number i.e. $\omega : \mathcal{A} \rightarrow C$ (In this report we use the notation E_ψ instead of ω),

$$E_\psi(.) := \frac{<\psi,\ .\ \psi>}{\|\psi\|} \iff \omega(.) \tag{A.1}$$

These states are positive linear functionals ($\omega(A^*, A) \geq 0$ for all $A \in \mathcal{A}$) such that $\omega(I) = 1$, or in our notation

$$E_\psi(A) = \frac{<\psi, A^*\hat{A}\psi>}{\|\psi\|} \geq 0 \text{ for all } A \in \mathcal{A}, \quad E_\psi(I) = \frac{<\psi, \hat{I}\psi>}{\|\psi\|} \equiv 1. \tag{A.2}$$

The value of the state ω acting on an observable A can be interpreted as the expectation value of the operator A on the state ω, i.e. $<A>= \omega(A)$, or

$$E_\psi(A) = \frac{<\psi, \hat{A}\psi>}{\|\psi\|} \tag{A.3}$$

• **affine quantum gravity:** [402]

• **algebraic quantum field theory (AQFT):** Reformulating QFT on an axiomatic basis: that is, starting from what seem to be physically necessary and mathematically precise principles which any QFT would have to satisfy, and then finding QFTs which actually satisfy them, (see [114], and references therein, for extensive discussion of it).

Fields at a point. In the formulation of quantum field theory one traditionally starts with a particle picture and investigate the scattering theory.

takes the collection of all operators localized in a particular region.

Operators of observables localized in an open region of spacetime $\mathcal{O}$ form an algebra $\mathcal{A}(\mathcal{O})$ of bounded operaotrs. It is possible to encode all physically relevant properties in terms of these algebras and their transformation behaviour under Poincare group.

Algebraic quantum field theory (see) is a general approach to quantum field theory based on algebras of local observables, the relations among them, and their representations.

Haag-Kastler axioms

(i) powerful structural analysis proven useful for rigorous treatment of models,

(ii) revision of the particle concept,

(iii) algebraic approach has shed new light on the treatment of quantum field theories on curved spacetime.

the physical concept of locality of nets of local algebras $\mathcal{O} \mapsto A(\mathcal{O})$. For each open region $\mathcal{O}$ one assigns a C^*-algebra $A(\mathcal{O})$. These are mutually (anti)commuting for spacelike separated regions.

In AQFT one cleanly separates two states of quantizing a field theory, namely first to define a suitable algebra A and then study its representations in a second step.

The axiomatic nature of AQFT makes it ideal for foundational study, since we can specify with mathematical precision exactly what the entity is that we are studying.

• **algebraic quantum gravity:** Algebraic qunatum gravity, proposed by Thiemann [??], is a new approach canonical quantum gravity suggeted by loop quantum gravity.

It has been shown that algebraic quantum gravity admits a semiclassical limit whose infintesimal gauge symmetry agrees with that of general relativity.

• **algebra of kinematic observables:** kinematic observables because they are not gauge invariant. These separate points of the unconstrained classical phase space of the system - that is . More complicated composite functions can be expressed in terms of (of limits) them. Dirac constraint approach: We want to ultimately construct a physical Hilbert space of solutions to the constraints. The kinematic observables can be projected onto the physical observables.

In the reduce phase space scheme we first find the physical Dirac observables and then atempt to quantize our system by finding a representation of the Poison algebra of these observables as linear operators on a Hilbert space.

• **almost periodic functions:**

• **alternative theories of quantum gravity:**

Well developed background independent approaches to quantum gravity are causal sets, causal dynamical triangulations and quantum Regge calculus.

• **anomilies:** When we quantize a classical system it is possible that a symmetry of the classical theory is lost. That is, an anomily is a classical symmetry not carried over to the quantum theory. Anomalises arise when the classical brackets cannot be fully realized by the quantum commutators.

• **apparent horizon:** See trapped surface.

i) S, the expantion $\Theta_{(\ell)}$ of the one null normal, ℓ^a, is everywhere zero and that

ii) that of the other null normal, n^a, is negative.

• **Arnowitt-Deser-Misner (ADM):** together, (as well as Dirac), they laid much of the groundwork for canonical quantum gravity. They expressed GR in Hamiltonian form by splitting spacetime into $R \times \sigma$ where R (the real line) represents "time" and σ is an three-dimensional space at a given "time". As their "configuration" variable they used the 3-metric that the spacial slice picks up from the metric of the spacetime, and as their "conjugate momentum" the extrinsic curvature(-to do with the time derivative of the 3-metric). This Hamiltonian construction is known as the ADM formulism.

• **Ashtekar, Abhay:**

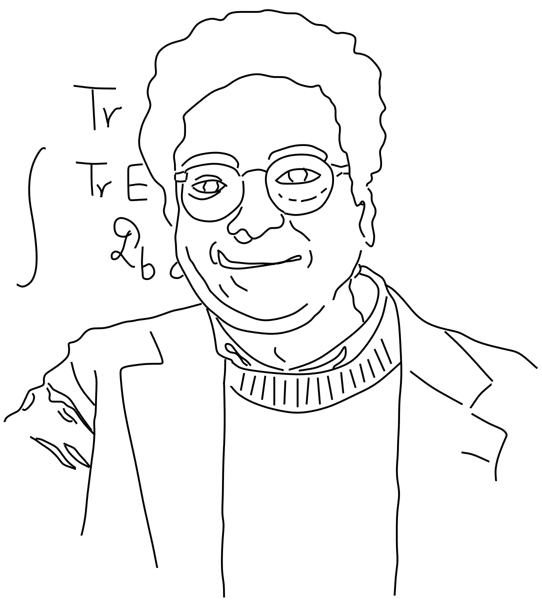


Figure A.1: Abhay Ashtekar.

• **Ashtekar's new variables:** A major obstacle to progress in the canonical approach had been the complicated nature of the field equations in the traditional variables, (q_{ab}, p^{ab}). This obstacle was removed in 1984 with the introduction of new canonical variables - Astekar's new variables. In terms of these, *all* equations of the theory become polynomial, in fact, at worst quartic, [93].

Self-dual connection. This odd choice leads to constraint equations that are polynomial in the basic variables - the connection A and the "electric" field E. The Hamiltonian constraint, that was non-polynomial in the ADM treatment, is quadratic in A and E.

• **Ashtekar-Isham-Lewendowski state:**

Haar measure on cylindrical functions.

• **asymtotically flat gravitational field:**

an asymtotically flat gravitational field: in this case the Hamiltonian is given by suitable boundary terms and the observables at infinity evolve in the Lorentz time of the asymptotic metric.

• **asymptotic flatness:** There exists coordinates $x = (x1, x2, x3)$ defined outside a compact set on Σ such that, for some $\delta > 1$,

$$h_{ij} - (1 + \frac{2M}{r})\delta_{ij} = \mathcal{O}(r^{-\delta}),)$$

as $r \to \infty$ with $M \geq 0$ (positive mass theorem).

• **automrphisms:** generalized Heisenberg picture one-parameter group of automorphisms of the algebra of the algebra of observables. Take any $A \in \mathcal{A}$an automorhism is

$$\gamma_t A = e^{it\hat{\mathcal{H}}/\hbar} A e^{-it\hat{\mathcal{H}}/\hbar} \tag{A.4}$$

where $\gamma_t A \in \mathcal{A}$. Obviously

$$\gamma_s \gamma_t A = \gamma_{s+t} A \tag{A.5}$$

$$\frac{d}{dt} A(t) = [A(t), \mathcal{H}]. \tag{A.6}$$

• **background independent:** There are various ways of expressing this property - it means that how a particle or a field is localization with respect to a spacetime manifold has no physical significance.

The theroy does not depend on a background metric, thus all distances are gauge equivalent.

There is no fixed geometric component to the Einstein-Hilbert action.

• **back-reaction:** in some background spacetime affect the dynamics of the background. This back-reaction can be described in terms of an effective energy-momentum tensor τ_{ab}.

$$G_{\mu\nu} = \kappa < \hat{T}_{\mu\nu} > \tag{A.7}$$

Gravitat

Dynamical theory over curved spacetime not a dynamical theory of spacetime.

see quantum field theory on curved spacetime.

• **Baez, John**

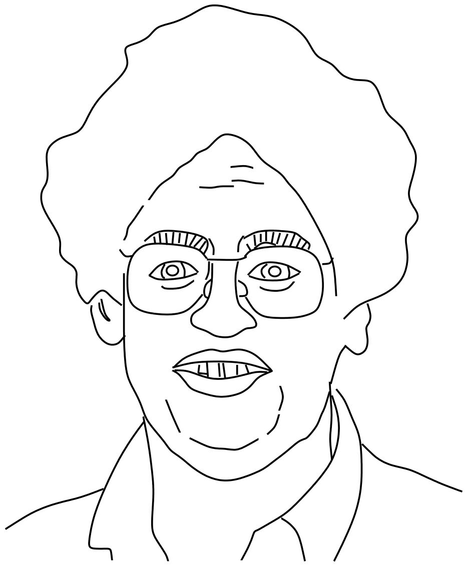


Figure A.2: John Baez.

• **Barbour, Julian:** Reinterpreted Einstein's general theory of relativity as a relational theory in which space and time are nothing but a system of relations. Since receiving his doctorate in 1968 from the University of Cologne, has never held an academic job but became an honored member of the quantum gravity community as it was Barbour who taught people what it means to make a background-independent theory.

• **Bargmann representation:**

• **Barrett, John**

• **Barrett-Crane model:** The Barrett-Crane model is one of the most extensively studied spin foam models for quantum gravity. It can be viewed as a spin foam quantization of the $SO(4)$ Plebanski formulation of general relativity. Specifically, one quantizies the Topological BF theory then imposes the simplicity constraints (that convert BF theory into GR) at the quantum level.

Figure A.3: Julian Barbour.

The original strategy followed by Barrett and Crane the starting point was with the quantization of a 4-simplex.

There are problems with the Barret-Crane models from its ultra-locality which follows from the way in which the simplicity constraints are imposed. Various proposals have been made recently to address these problems, [200], [286] . From [286]: "... that a new exciting period of the development of the subject of spin foam models may be opening."

• **baryonic matter:** Baryonic matter or simply ordinary matter is anything made of atoms and their constituents, and this includes all of stars, planents, gas and dust in the universe. Ordinary baryonic matter, it turns out, is not enough to account for the observed matter density; Baryons are present in the amount preredicted by the Big Bang Nucleosynthesis, some percent of the density required to close the universe.

• **Benny Hill effect:** There is a negative force between a particle of negative mass and a particle of positive mass, whereas by $F_n = m_n a_n$ (F_n is the force exerted on the negative mass particle and a_n is the acceleration of the negative particle mass particle) the negative mass particle accelerates towards the positive mass particle, while the positive particle mass accelerates away.

• **Bergmann-Komar group:** Transformations between different covariant gauge fixings form a group - Bergmann-Komar group.

• **beta function:**

• **Bekenstein bound:**

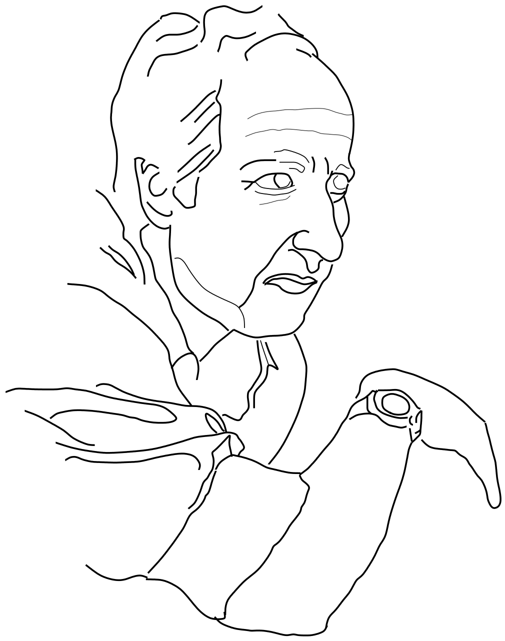


Figure A.4: John Barrett.

$$S < 8\pi\zeta RE \tag{A.8}$$

• **Big bang:**

journal Advances in Theoretical and Mathematical Physics:

"The question of whether the universe had a beginning at a finite time is now 'transcended'. At first, the answer seems to be 'no' in the sense that the quantum evolution does not stop at the big-bang. However, since space-time geometry 'dissolves' near the big-bang, there is no longer a notion of time, or of 'before' or 'after' in the familiar sense. Therefore strictly, the question is no longer meaningful. The paradigm has changed and meaningful questions must now be phrased differently, without using notions tied to classical space-times."

• **Biot-Savart law:**

$$B = -\frac{1}{4\pi} \int d^3y \, \epsilon_{\alpha\beta\gamma} \, j^\beta(y) \, \frac{(x-y)^\gamma}{|x-y|^3} \tag{A.9}$$

• **black-body radiation:** Thermal radiation emmited by black-bodies, black-bodies being perfect emmiters and absorbers of radiation. photons moving around randomly, without any discernible source.

• **black hole entropy:**

• **black hole:** colapsing star forming an object so dense that nothing can escape from it. A region of spacetime hidden behind a horizon.

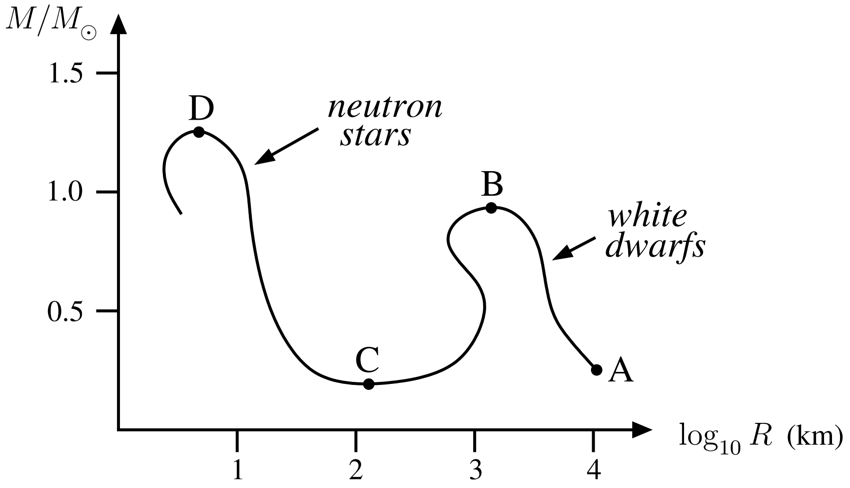


Figure A.5: LebRien.

• **black hole uniqueness theorems:** If a black hole is stationary, then it must be either static or axially symmetric. This is the Kerr family of black holes

• **Bochner-Minlos theorem:** Fourier transform corresponds a unique positive-definite measure.

• **Bogoliubov transforms:** Two different Fock space representations are related by a transformation to new operators.

In ordinary quatum field theory in flat spacetime, one quantizes ϕ by first decomposing the field into Fourier modes,

$$\phi = \sum_{\mathbf{k}} \left(a_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(t, \mathbf{x}) \right)$$

with

$$u_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{\mathbf{k}} t}, \qquad \omega_{\mathbf{k}} = (|\mathbf{k}|^2 + m^2)^{1/2}$$

and replacing the coefficients $a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}$ by annihilation and creation operators respectively. The Fourier modes $u_{\mathbf{k}}$ are a set of orthonormal functions satisfying

$$(\Box + m^2) u_{\mathbf{k}}(t, \mathbf{x}) = 0, \qquad \partial_t u_{\mathbf{k}}(t, \mathbf{x}) = -i\omega_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}),$$

where the second condition determines what one means by positive and negative frequency, and thus allows to distinguish creation and annihilation operators.

In a curved spacetime, or noninertial reference frame in flat spacetime, standard Fourier modes are no longer available. However, with a choice of time coordinate t one can still find a decomposition to obtain creation and annihilation operators. Given two different reference frames with time coordinates t and $\bar{t}$, two such decompositions exist:

$$\phi = \sum_i \left(a_i u_i + a_i^\dagger u_i^*\right) = \sum_i \left(\overline{a}_i \overline{u}_i + \overline{a}_i^\dagger \overline{u}_i^*\right)$$

and since (u_i, u_i^*) are a complete set of functions, we can write

$$\overline{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*).$$

This relation is known as a Bogoliubov transformation, and the coefficients α_{ji} and β_{ji} are called Bogoliubov coefficients.

• **Bojowald, Martin:**

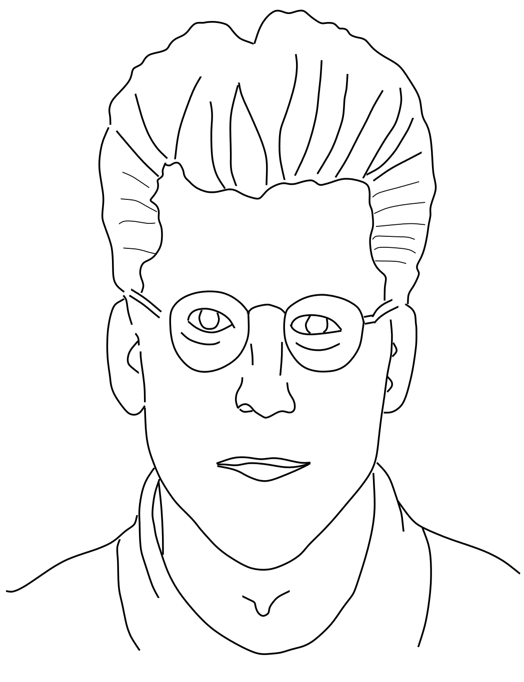


Figure A.6: Martin Bojowald.

• **Boltzmann statistical mechanics:**

• **Bondi coordinates:** $\mathcal{I}^+$ can be foliated into a one parameter family of two dimensional non-intersecting spacelike surfaces, each difeomorphic to S^2, parametrized by a variable

u. On each of these surfaces one can introduce complex stereographical coordinates ζ and $\overline{\zeta}$. The coordinates $(u, \zeta, \overline{\zeta})$ on $\mathcal{I}^+$ are called the Bondi coordinates on $\mathcal{I}^+$.

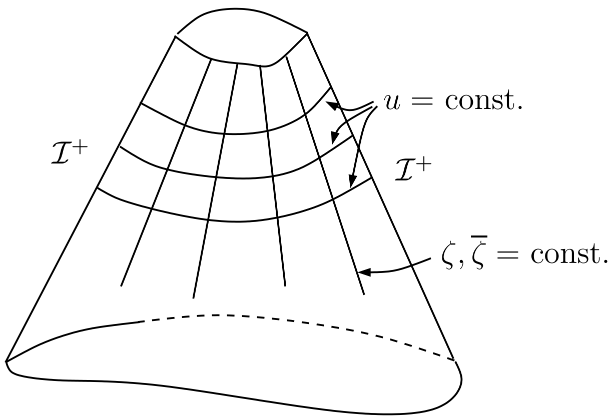


Figure A.7: Bondi coordinates at future null infinity.

• **Bondi-Sachs mass:** Both the energy of the radiation and the energy of the leftover system are included in the total ADM energy. The Bondi energy is the gravitating mass as seen by light rays propagating out to infinity in the lightlike direction, rather than the spacelike direction. For a spacetime of an isolated body emitting graviational the Bondi-Sachs mass decreases, while the ADM mass stays constant.

• **Born-Oppenheimer approximation:**

• **bottom-up:** 'bottom-up' approach one attempts to formulate the full theory and then make some approximation to get the low energy description. Hopeful, in doing so we recover the correct classical theory. When there are an infinite number of degrees of freedom, the passage from the quantum theory to the classical regime does not always have to give the classical theory that was quantized!

• **BRST:** The central idea of BRST theory is to replace the original gauge symmetry by a rigid fermionic symmetry acting on an appropriately extended space containing new variables, "ghosts". Ghosts first appeared in quatum field theory - they ensured that the theory would be unitary and independent of the gauge choice. They did this through contributions to virtual process only - they have the wrong spin statistics to correspond to physical particles. They violate the spin statistic relation which requires physical fields with half-integer spin to be fermionic and fields with integer spin to be bosonic.

• **bubble formulism:** Kuchař's bubble time formulism

• **canonical:** Simplest or standard form.

• **canonical quantisation:** Classical mechanics admits two different kinds of formulation: Hamiltonian and Lagrangian. So does quantum mechanics. The one corresponding to the Hamiltonian formulation of classical mechanics is called canonical quantisation and involves a Hilbert space and operators.

The basic idea is

(i) to take the states of the system to be described by wavefunctions $\Psi(q)$ of the configuration variables,

(ii) to replace each momentum variable p by differentiation with respect to the conjugate configuration variable, and

(iii) to determine the time evolution of Ψ via the Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{\mathcal{H}}\Psi,$$

where $\hat{\mathcal{H}}$ is an operator corresponding to the classical Hamiltonian $\mathcal{H}(p,q)$.

General relativity can be cast in Hamiltonian form, however the Hamiltonian is a constraint on phase space. Dirac developed the general method for the canonical quantisation of systems with constraints. It involves the imposition of the constraints as an additional condition on the Hilbert space.

• **canonical transformations:** A canonical transformation in a classical theory relates descriptions by different 'generalised coordinates' on phase space with the same sympletic structure. These classical theories are equivalent, but the resulting quantum theories need not be equivalent.

• **Cartan's structure equations:** See maths glossary.

• **Cauchy Horizon:** $D^-(p)$

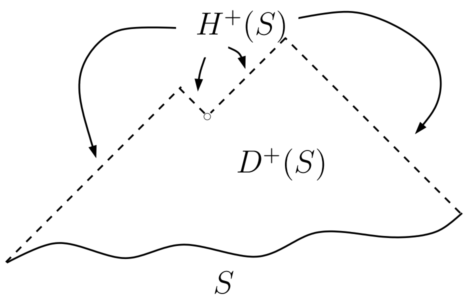


Figure A.8: Cauchy Horizon.

• **Cauchy surface:** A Cauchy surface is a spacelike or null surface that intersects every timelike curve in a spacetime $\mathcal{M}$ once and only once. (Not every spacetime has a Cauchy surface! - see global hyperbolicity).

• **causal** As the very concepts of time and spacial location are not fundamental at all in quantum gravity and only emerges in a semiclassical approximation, causality should too only emerge and be applicable only in a semi-classical limit.

??but no background causal structure in the classical theory - the notion of causality only emerges when the dynamics of gravity can be neglected??

• **causal future:** events are influenced by those in their past.

• **causality conditions:** Avoid paradoxes traveling into the past. Heierarky of conditions:

1) Chronology: no closed timelike curves;

2) Causality: no closed causal curves;

3) Strong causality: no "almost closed" causal curves;

4) Stably causality: close metrics are causal.

• **causal set theory:** The causal approach to quantum gravity is the proposal that the structure of spacetime at the quantum scale should be modelled after a locally finite partially ordered set (poset) of elements, the *causal set*, whose partial order is the relation that defines past distinction between events.

See quantum causal histories.

[gr-qc/0507078]. This approach does not attempt a quantization of classical theory. Rather, it is a construction of a quantum theory of causal sets based on two features of classical genral relativity that it taakes as fundamental: the causal structure, which is replaced by discrete causal set;

• **causal spin networks:**

Do not necessarily need to build in causality as it could be an emergent property??

• **CBR, cosmic background radiation:** Thermal radiation. photons moving around randomly, without any discernible source, corresponding to a temperature of about $2.7K$.

• **charge spin networks:** $U(1)$ gauge group labelled by integers.

• **Chern-Simons:**

Relevant in condensed-matter physics - description of excitations with fractional statistics, *anyons*, in the fractional Hall effect (FQHE) and a proposal for a mechanism of superconductivity based on anyons Y-H. Chen, F. Wilczek, E. Witten, and B.I. Halerin, Int. J. Mod. Phys. B **3**, 1001 (1989).

• **Chern-Simons-Witten gravity:** The local Poincare $ISO(2,1)$ symmetry is, on shell, equivalent to the usual diffeomorphism [387]. This $(2+1)$ dimensional gravity is based on the non-degenerate, invariant and associative inner product on $iso(s,1)$ on which the conventional Killing metric is degenerate [387]:

How do we define the trace operation in Chern-Simons define

$$< P^a, J^a >= \eta^{ab}, \quad < J^a, J^b >=< P^a, J^b >= 0, \tag{A.10}$$

$$[P^a, J^a] = \epsilon_{abc} J^c, \quad [J^a, J^b] = \epsilon_{abc} P^c \quad [P^a, P^b] = 0, \tag{A.11}$$

where $\eta^{ab} = diag(-1, 1, 1)$ and P^a, J^a are the generators of $iso(2,1)$ (here Roman indices are for internal coordinates and Greek indices for spacetime coordinates). The gauge connection is an $iso(2,1)$ valued one form

$$A = \omega_a J^a + e_a P^a, \tag{A.12}$$

where ω and e are the homogenious (Lorentz) and the inhomogeneous (translational) part of the connection components respectively.

ISO(2,1) Gauge theory with the gravitational Chern-Simons term

If we define Tr(. .) :=< .,. > so that (A.11)

$$\mathrm{Tr}(P^a J^a) = \eta^{ab}, \quad \mathrm{Tr}(J^a J^b) = \mathrm{Tr}(P^a J^b) = 0, \tag{A.13}$$

$$\int \mathrm{Tr} \left\{ A \wedge dA + \frac{3}{2} A \wedge A \wedge A \right\} \tag{A.14}$$

• **chromodynamics:**

• **closed systems:** The universe as a whole. while the standard Copenhagen interpretation works well in the laboratory, it cannot be applied to closed systems.

• **classical configuration space:** The classical configuration is generally taken to be the space of all smooth functions which dcay rapidly at infinity an a t = constant slice.

• **clock variable:** A clock variable is a function on the extended phase space of a system which strictly monotonous along the evolution orbits...

• **closed trapped surface:** See trapped surfaces.

• **coherent states:** They provided the closest approximation to classical physics as uncertainty is minimum and the states are peaked in both configuration and momentum representations.

• **collapse of the wave function:**

• **Coleman-Mandula theorem:** The theorem states that a group that non-trivially combines both the Lorentz group and a compact Lie group cannot have finite dimensional, unitary representations. This presens a problem to constructing unified theory of gravity and the other forces. A way to evade the Coleman-Mandula theorem is to use super symmetry.

• **comparative properety:**a comparative property and to express it we need a way to introduce scale into our system.

• **constraints:**

we can take these constraints to say that small automorphisms of the bundle $P|_S$

• **constructive quantum field theory:**

Geared to scalar fields propagating on Minkowski space. Key axioms are called the Osterwalder-Schrader axioms.

• **common eigenstate:**

• **complete set of commuting operators:** In three-dimensions, three coordinate operators $\hat{x}, \hat{y}$ and $\hat{z}$. As they commute with each other we can measure one then measure the next without changing the . and uniquely specifies the state of the system.

The outcomes of the measurement of a complete set of commuting observables characterizes the state.

When we have spherical symmetry potentials in three-dimensions, the set $\hat{\mathcal{H}}$, $\hat{L}^2$ and $\hat{L}_z$

We can define an orthonormal basis in this Hilbert space by diagonalizing a complete set of commuting self-adjoint operators.

Measuring A first, we would find a value a_n and would leave the system in an eigenstate of A.

$$A|\psi_i >= a_i|\psi_i >, \quad B|\psi_i >= b_i|\psi_i > \tag{A.15}$$

$$(AB - BA)|\psi_i >= 0 \quad \Longrightarrow \; AB|\psi_i >= a_i B|\psi_i > \tag{A.16}$$

$$< \psi_i|(AB - BA)|\psi_j >= (a_i - a_j) < \psi_i|B|\psi_j >= 0, \tag{A.17}$$

So $< \psi_i|B|\psi_j >$ vanishes unless $a_i = a_j$.

$|\psi_i >$ are **common** eigenstates of $\hat{A}$ and $\hat{B}$. We denote them as $| >$

Measuring A first we find it has the value a_i and the system would be left in the eigenstate of A. As shown above, this eigenstate is also an eigenstate of B. Thus a measurement of B yields a value b_i

Definition: A set of self-adjoint operators is called a CSCO if the operators all commute with each other and if the set of their common eigenstates is complete and not degenerate.

• **complexifier:** The "complexifier" was introduced by Thiemann in [243] as a powerful tool that can give rise to kinematical coherent states for a general quantuum system.

the complexification of $SU(2)$ is $SL(2,\mathbf{C})$.

$$u = \exp[i\tau^j_j/2] \tag{A.18}$$

• **Compton scattering:** When a photon scatters off a free electron, in the relativistic case $\hbar f >> m_e c^2$.

• **conformally flat spacetimes:**

$$g_{ab} = \Omega\eta_{ab} \tag{A.19}$$

• **conformal transformations:**

• **congruence:** A congruence is a family of curves such that precisely one curve of the family passes through each point. It is a geodesic congruence if the curves are geodesics. A vector field v^a determines a congruence whose curves are tangent to v^a. We call such curves streamlines of v^a.

• **conjugate momentum:**

$$p_i := \frac{\partial\mathcal{L}}{\partial q^i}. \tag{A.20}$$

$$\{q,p\} = 1 \tag{A.21}$$

In the Hamilton formulation of GR, the conjugate momenta of the induced metric is related to the extrinsic curvature K^{ab} by

$$\pi^{ab} = q^{-1/2}(K^{ab} - Kq^{ab}). \tag{A.22}$$

• **connection:** self-consistent way of comparing vectors in two different vector spaces.

We define connection and curvature variables $A^{IJ}_{(\gamma)\mu}$ and $F^{IJ}_{(\gamma)\mu\nu}$

$$A^{IJ}_\mu = \omega^{IJ}_\mu + \frac{\gamma}{2}\epsilon^{IJ}{}_{KL}\omega^{IJ}_\mu \tag{A.23}$$

$$F^{IJ}_{\mu\nu} = R^{IJ}_{\mu\nu} + \frac{\gamma}{2}\epsilon^{IJ}{}_{KL}R^{IJ}_{\mu\nu} \tag{A.24}$$

If one chooses the value of γ such that $\gamma^2\sigma = 1$, then the variables A^{IJ}_{μ} and $F^{IJ}_{\mu\nu}$ are self-dual with respect to the internal indices.

• **Connes-Kreimer's Hopf algebra of Feynman's diagrams:** renormalization via Hopf algebra of rooted trees. Applied by Fontini Markopoulou to renormalization of spin foams background-independent models of quantum gravity.

• **consistent discretizations:** Rodolfo Gambini Jorge Pullin based on approximating the continuum theory by a discrete theory. It allows to bipass difficult conceptual problems turning them into technical ones.

relational quantum theories unfortunately one would have to disregard all the results of LQG

consistent discrete quantum gravity *a la* Pullin, Gambini *et al*: A discrete approach to quantum gravity that is free of constraints translating difficult conceptual problems into problems of computational nature.

• **consistent-histories approach to quantum theory** The consistent-histories approach to quantum theory was originally introduced to provide a novel way of re-interpreting standard quantum theory, particularly in regard to the role played by measurement. However, because of the novel way in which time is handled, consistent-history theory also has the potential for providing new and powerful ways of studying quantum theories of gravity. Most recently, in [2] the formalism was applied to construct a history version of the canonical form of classical general relativity. The possibility also arises to use this formalism in the context of generalised ideas of time and space: for example, in models where spacetime is not represented by a differentiable manifold.

[?] 28 Oct 2001, *Quantising the Foliation in History Quantum Field Theory C.J. Isham, K. Savvidou*, [quant-ph/0110161].

• **constants of motion:**

$$\frac{d}{dt} < \hat{A} >= \frac{1}{i\hbar} < [\hat{\mathcal{H}}, \hat{A}] > + < \frac{\partial \hat{A}}{\partial t} > . \tag{A.25}$$

If $\hat{A}$ commutes with the Hamiltonian and does not explicitly depend on time, the observable A is said to be a constant of motion.

$$[\hat{\mathcal{H}}, \hat{A}] = 0 \tag{A.26}$$

• **constraint:** [89]: In addition to the Hamiltonian equations of motion, the theory will exhibit constraint equations. The constraints play a double role in a Hamiltonian theory.

On the one hand they generate a group of symmetries of the phase space referred to as the gauge transformations, on the other hand the set of solutions of the constraints defines the constraint surface of the phase space.

The simplest example for such a kind of theory is certainly Maxwell theory,where the structure group is $U(1)$. A more general example is Yang Mills theory, where the structure group may be an arbitrary compact Lie group G. In this case the group of the gauge transformations is the group of the fiber preserving automorphisms of the given bundle, homotopic to the identity. The group is often referred to as the Yang-Mills gauge transformations.

• **constraint algebra:**

quantum constraint algebra - is mathematically consistent but seems to fail to reproduce the classical constraint algebra - however in calculating the righthand side of the classical Hamilton-Hamilton Poisson bracket involves... So not obvious if it does fail??

• **constructive quantum field theory:** The program of constructive quantum field theory was initiated in the early seventies. This approach has had remarkable success in certain 2 and 3 dimensional models. From a theoretical physics perspective, the underlying ideas may be summarized roughly as follows. Consider, for definiteness, a scalar field theory. The key step then is that of giving meaning to the Euclidean functional integrals by defining a rigorous version $d\mu$ of the heuristic measure "$[\exp(S(\phi))] \prod_x d\phi(x)$" on the space of histories of the scalar field, where $S(\phi)$ denotes the action governing the dynamics of the model. The appropriate space of histories turns out to be the space $\mathcal{S}'$ of (tempered) distributions on the Euclidean space-time and regular measures $d\mu$ on this space are in one to one correspondence with the so called generating functionals, which are functionals on the Schwarz space $\mathcal{S}$ of test functions satisfying certain rather simple conditions. (Recall that the tempered distributions are continuous linear maps from the Schwarz space to complex numbers.)

Thus, the problem of defining a quantum scalar field theory can be reduced to that of finding suitable measures $d\mu$ on $\mathcal{S}'$, or equivalently, appropriate functionals χ on $\mathcal{S}$. Furthermore, there is a succinct set of axioms due to Osterwalder and Schrader which spells out the conditions that χ must satisfy for it to be appropriate, i.e., for it to lead to a consistent quantum field theory which has an associated

i. Hilbert space of states,
ii. a Hamiltonian,
iii. a vacuum and
iv. an algebra of observables.

(from Integration on the space of Connections Modulo Gauge Transformations)

For introduction see for example [397].

• **coordinate singularity** Are locations in coordinate space where calculations become degenerate or divergent, due solely to "bad coordinates". An example of this is the

Schwarzschild radius $r = 2M$ in Schwarzschild coordinates. A radially-infalling observer takes an infinite amount of time in the Schwarzschild time coordinate to cross the Schwarzschild radius. The proper time of the observer would measure although he would see the outside universe speed up??????

One can avoid the detection of unreal coordinate singularities by maximally extending the space-time.

• **Copenhagen interpretation:** ??? (quant-ph/9803052) an a priori classical part of the world is assumed to exist from the outset. Such a structure is there thought to be necessary for the "coming into being" of observed measurement results. ???

In the absence of a background spacetime geometry, the Copenhagen interpretation is of little use - it is definitely physically incomplete.

[338] "The usual well defined quantum operators like the area and volume will exist and be well defined, except that in the discrete theory the total volume of the slice will be an observable since there are no further constraints."

no external observer of the universe as a whole throws into doubt the instrumentalism of the Copenhagen interpretation; as does an attempt to construct a quantum gravity theory with no background spacetime in which an 'observer' could be placed. [362].

standard Copenhagen divide the physical world into the microscopic regime that obeys the laws of quantum mechanics and the macroscopic regime which obeys the laws of classical mechanics e.g. Maxwell's equations, Einstein's equations. Observers are classical. artificial partition not clear where to put this quantum/classical border. (quant-ph/0103161)

• **Corichi, A:**

• **cosmic censorship:** The principle that singularities are never "naked", that is, do not occur unless surrounded by a shielding event horizon.

• **cosmological constant:** A constant introduced into Einstein's field equations of general relativity in order to supplement to gravity. If positive (repulsive), it counteracts gravity, while if negative (attractive), it augments gravity. It can be interpreted physically as an energy density associated with space itself.

• **cosmological constant problem:** zero point energy of quantum fields - why is Λ not infinite? With a Plankian cut-off the value obtained from calculations performed in quantum field theory on the vacuum energy density corresponding to quantum fluctuations of the fields we observe in nature, imply values that are over 120 orders of magnitude greater than the observed value.

In his model, he was able to show he cosmological constant paradox appears only if spacetime is regarded as fundamental rather than emergent, [266]. The excitations are part of the pre-geometry and cannot curve the effective spacetime they simulate. The fields we observe in nature emerge as in the low energy, course grained limit of a more fundamental theory.

• **cosmological principle:** this states that the universe must appear the same whatever the point from which it is observed. It implies that the universe is homogeneous and isotropic on the largest scales and forms the foundations of all of present day cosmology.

• **course graining:** A nice example of a Hopf algebra.

background-independent course graining *a la* Fontini-Markopoulou [122], [?]

• **covariant quantum statistical mechanics:**

intensive thermodynamical quantities extensive thermodynamical quantities

• **Covariant Loop Quantum Gravity:** (CLQG) is a loop quantization of a Lorentz covariant formulation. It uses the full Lorerentz gauge group and produces results independent of the Immirzi parameter.

• **Crane, L:**

• **crossing-symmetry:** In gravity this has a relation to the following: [151], [23]. 4d-diff invariant spinfoam. A problem of the canonical formalism is that it is necessary to make a $(3+1)$ decomposition of spacetime which breaks the manifest 4-diffeomorphism invariance. The vector constraint, generated space diffeomorphisms, and the scalar constraint, generating time reparametrization, arise from the underlying 4-diffeomorphism invariance. It reduces the full, 4-dimensional invariance - the canonical formalism preserves spatial diffeomorphism invariance within each hypersurface of the foliation.

There is a suggestion for curing the slicing dependence.

the spacetime covariant formalism in [151] naturally suggest a "covariantisation" of the operator, described in [151] under the name of "crossing symmetry".

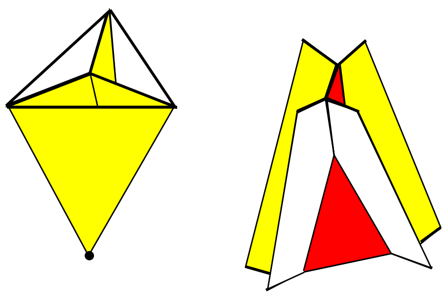


Figure A.9: crossing. The crossing symmetry vertices.

Actually, a well known property of standard QFT is that the vertex amplitutde does not depend on which states are "in" and which are considered "out" - this is related to the crossing symmetry mentioned here.

• **curvature:**

• **curvature tensor:** The vector X defines a curve through the point p via parael transport, The vector Y defines another curve through p. We can form and attempted parallelagram.

$$\epsilon^2 X^a Y^b Z^c R_{abc}{}^d$$

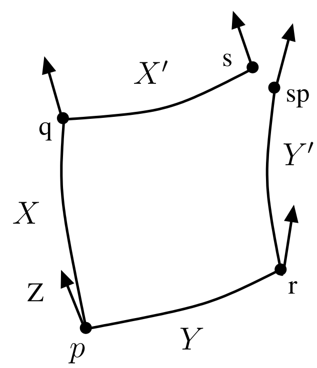


Figure A.10: We display the geometric interpretation of the curvature tensor. Carry a third vector Z, by parallel transport from p to s via q, comparing this with transporting this from p to s' via r we find a discrepancy between the two vectors given in terms of the curvature tensor components $R_{abc}{}^d$ by the formula $\epsilon^2 X^a Y^b Z^c R_{abc}{}^d$.

• **cut-off coherent states:** The coherent state ψ_m, produced by the complexifier method, with respect to a finite graph as a graph dependent coherent state in $\mathcal{H}_{Kin}$ is called a cut-off coherent state. These are then normalizable, graph dependent states, however they can not be used to test the semiclassical properties of graph changing operators, such as the Hamiltonian constraint, as their expectation values with respect to these cut-off states is always zero.

• **dark matter:** A kind of particle that may exist in our universe. This may not be part of the standard Model of particle physics.

conclusion comes from galaxy rotation curves,

[**?**] A proper General relativist calculation shows a large departure from Newtonian approximation,

A galaxy is modelled as a stationary axially symmetric pressure free-fluid in general relativity. For weak gravitational fields under consideration, the field equations and the equations of motion ultimately lead to one linear and one non-linear equation relating the angular velocity to the fluid density. It is shown that the rotation curves for the Milky Way, NGC 3031, NGC 3198 and NGC 7331 are consistent with the mass density distribution of the visible matter cocentrated in flattened disks. Thus the need for massive *halo of exotic dark matter is reamovedd. For these glaxies we determine the mass density for the luminous threshold as* $10^{-21.75} kg{\cdot}m^{-3}$.

"It is understandable that the convectional gravity approach has focused upon Newtonian theory in the study of galactic dynamics as the galactic field is weak (apart from the deep core regions where black holes are said to reside) and the motions are non-relativistic ($v \ll c$)."

"We have seen that the non-linearity for the computation of density inherent in the Einstein field equations for a stationary axially-symmetric pressure-free mass distribution, even in the case of weak fields, leads to the correct galactic velocity curves as opposed to the incorrect curves that had been derived on the basis of Newtonian gravitational theory."

• **decoherence:** what decoherence does for you is make any discrepencies minute so we don't have to worry about conceptual difficulties. aleviates a practical problem - it is not clear if it elevates conceptual difficulties.

kent dowker (1995): one might at least hope to predict the gross features: the existence and persistence of (what appears to us to be) a classical world with very large-scale structure; the fact that matter clumps and fields in this classical world generally obey classical equations of motion to a very good degree of approximation;

interacting with the classical world, follow the probabilistic laws of Copenhagen quantum theory.

• **decoherent histories:** Taken from - Quantum logic and decohering histories, [quant-ph/9506028]:

In recent years, much attention has been devoted to the so-called decoherent histories approach to quantum theory. A major motivation for this scheme is a desire to replace the traditional Copenhagen interpretation of quantum theory with one that avoids any fundamental split between observer and system and the associated concept of state-vector reduction induced by a measurement. The key ingredient of the new approach is an assertion that, under certain conditions, a probability can be ascribed to a complete history of a quantum system without invoking any external state-vector reductions in the development of the history. Any such scheme would clearly be particularly attractive in quantum cosmology where a fundamental observer-system split seems to be singularly inappropriate.

Dowker and Kent 1 have raised some serious doubts in the context of their penetrating analysis. [365], see also L Smolin [13].

the possibility that the decoherent histories programme could provide a framework for solving certain technical or structural problems that arise in quantum gravity. An example is the problem of time that features prominently in canonical quantum gravity (see [364]).

• **decoupling:** this is the instant in the history of the universe where matter (baryons and electrons) and the electromagnetic radiation cease to be coupled. After this point . This took place when the temperature was about 4000 K.

• **deformed special relativity:**

Deformed special relativity provides an arena in which to build an effective theory for quantum gravity. [256].

• **density matrix:** mixed state described by a density matrix ρ. Expectation values of observables are given by

$$\rho[A] = \mathrm{tr}\,[A\rho]\,. \tag{A.27}$$

• **de Sitter universe:** The de Sitter universe plays an important role in cosmology as it can be used as a good approximation for the stage of accelerated expansion - inflation.

• **DeWitt, Bryce:** Made majour contributions to classical and quantum field theory, in particular, to the theory of gravitation. Through application of Dirac's theory of constraints in development of a canonical approach to quantum gravity, DeWitt was led to what became to be known as the Wheeler-DeWitt equation (the Hamilton constraint in 'metric variables'), which has since been applied many times to problems in quantum cosmology.

By the end of 1965 he had found the rules for quantising the gravitational and non-Abelian gauge fields to all orders nearly two years befor a paper by Fad'deev and Popov deriving the same rule was published.

See BRYCE SELIGMAN DEWITT 1923-2004 - A Biographical Memoir by STEVEN WEINBERG.

• **Dirac conjecture:** All first class constraints generate gauge transformations at a given time.

• **Dirac, Paul:** Developed the hamiltonian formulation of GR independently of Bergmann and his group, with the long term goal to quantize gravity. The main tool for this, the hamiltonian theory of contsrained systems, was developed for this purpose.

• **Dirac-Bergman algorithm:** Procedure for generating a finite set of constraints consistent under evolution.

• **Dirac observable:** An observable is a function on the constraint surface that is invariant under gauge transformations generated by *all* the first class constraints. Equivalently, an observable is a function on the phase space that has weakly vanishing Poisson brackets with the first class constraints.

It is often stated that there are no Dirac observables known for General Relativity, except for the ten Poincar?e charges at spatial infinity in situations with asymptotically at boundary conditions.

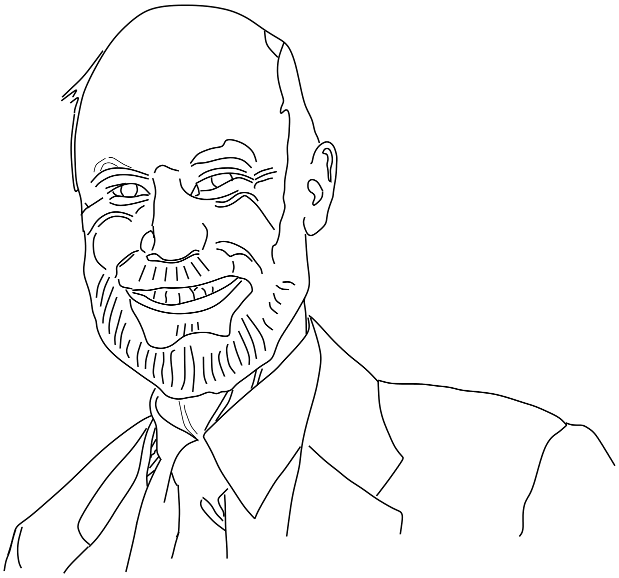


Figure A.11: Bryce DeWitt.

• **dispersion relations:** The form of the dispersion relations is dictated by Lorentz invariance.

• **Dittrich, Bianca:**

• **domain of dependence:** The future domain of dependence, $D^{+}(\Sigma)$, is the set of points p in $\mathcal{M}$ for which every past-inextendable causal curve through p intersects Σ.

• **Doplicher, Haag, Roberts DHR:** for classification of available representations in order to know which features are tied to a specific representation and which are not.

• **double harmonic oscillator model:** The double harmonic oscillator was first studied by Rovelli [277] as a toy model to help understand the "problem of time".

The Hamiltonian for the double harmonic oscillator

$$\mathcal{H} = \lambda \left(\frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) - E \right) \tag{A.28}$$

One of the oscillators can be thought of as a "clock" for the other oscillator.

Other toy models that share some essential features of GR are .

• **doubly special relativity:** Doubly special relativity theories are relativistic theories in which the transformations between inertial observers are characterized by two observer-independent scales - the light speed and the Planck length l_p. The Planck length seems to have a crucial role to play in quantum theories of gravity as a threshold to quantum

Figure A.12: Paul Dirac.

effects of spacetime. It is argued that the planck length must have the same value in all inertial frames which is in contradiction with special relativity. Double special relativity is a proposal to solve this perceived problem.

A phenomenological description of the signatures of Lorentz invariance violation could be doubly special relativity (DSR) [??], a theory in which not only the speed of light but also the Planck energy is (inertial) frame independent.

• **Dreyer, Olaf:** [19] "...argues that the incompatibility between quantum theory and general relativity can be resolved only if we give up the idea that space is fundamental. He proposes that space itself emerges from a more fundamental description that is quite different."

• **dust:** Matter that does not interact on one another by mechanical forces but interacts with itself only via the gravitational field.

• **dynamical horizons:** Isolated horizons provide a quasi-local description of black holes in equilibrium. There has also been developed a framework for the quasi-local treatment of black holes that are not in equilibrium (because of infalling matter or interaction with external bodies), namely *dynamical horizons*.

The formal definition of a dynamical horizon is as follows,

i) S, the explanation $\Theta_{(\ell)}$ of the one null normal, ℓ^a, is everywhere zero and that

ii) that of the other null normal, n^a, is negative.

The second condition merely says that n^a is inward pointing null normal. none of the

Figure A.13: Bianca Dittrich.

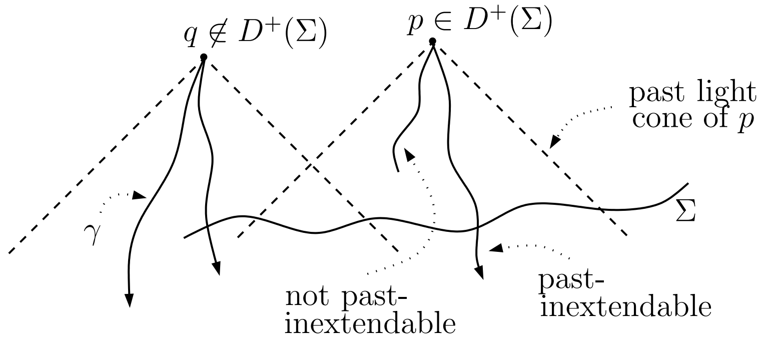


Figure A.14: The future domain of dependence, $D^+(\Sigma)$, of Σ. p is in $D^+(\Sigma)$, q isn't because ther are past-inextendable causal curve through q that don't intersect Σ, e.g. the curve γ.

light rays emerging from *any* point on S are directed towards the 'outside' region.

how black-holes grow

• **dynamical symmetry:** e.g. invariance under active diffeomorphisms inGR.

• **dynamically triangulated quantum gravity:** dynamical triangulations which edge lengths are fixed and the path integral is represented as a sum over triangulation.

A weak coupling phase with quantum fluctuations around a "semiclassical" dynamically generated background geometry despite the fact that the formulation is explicitly background-independent, [264].

• **edge:** Let S be a closed achronal set. The edge of S is defined as a set of points $x \in S$ such that every neihbourhood of x contains $y \in I^+(S)$ and $z \in I^-(S)$ with a timelike curve from z to y which does not meet S.

Link between vertices of a graph.

• **Ehernfest's theorem:**

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}_x\rangle}{m}, \tag{A.29}$$

$$\frac{d\langle\hat{p}_x\rangle}{dt} = -\left\langle\frac{\partial\hat{V}}{\partial x}\right\rangle. \tag{A.30}$$

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{x}], \quad \frac{d\hat{p}_x}{dt} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{p}_x] \tag{A.31}$$

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{p_x}^2 + \hat{p_y}^2 + \hat{p_z}^2) + \hat{V}(x, y, z) \tag{A.32}$$

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}_x}{m}, \quad \frac{d\hat{p}_x}{dt} = -\frac{\partial\hat{V}}{\partial x} \tag{A.33}$$

$$\frac{d}{dt}\int\psi^*\hat{x}\psi dx = \frac{1}{m}\int\psi^*\hat{p}_x\psi dx, \qquad \frac{d}{dt}\int\psi^*\hat{p}_x\psi dx = -\int\psi^*\frac{\partial\hat{V}}{\partial x}\psi dx \tag{A.34}$$

• **Einstein, Albert-(1878-1955):** Began his scientific work at the beginning of the 20th century. During the years 1905-1906 he published three articles: (1) The special theory of relativity; (2) an explanation of the photo-electric effect; (3) Brownian motion. The three corner stones of modern physics. Considered to be very clever.

• **Einstein's hole argument:** As explained by Rovelli *"Assume the gravitational field equations are covariant. Consider a solution of these equations in which the gravitational field is* e [basically the metric] *and there is a region H of the universe without matter (the "hole"). Assume that inside H there is a point A where* e *is flat and the point B where it is not flat. Consider a smooth map $\phi : \mathcal{M} \to \mathcal{M}$ which reduces to the identity outside H, and such that $\phi(P) = Q$, and let $\tilde{e} = \phi_* ei$ be the pull back of* e *under ϕ* [the metric induced by the diff transformation ϕ]. *Notice that* e *is flat around the event A while $\tilde{e}$ is not. The two fields* e *and $\tilde{e}$ have the same past, are both solutions of the field equations, but have different properties at the event A, therefore they are not deterministic. But we know that (classical) gravitational physics is deterministic, therefore, either the field equations can or cannot be generally covariant, or there is no meaning to the physical event A."*

• **eikonal approximation:** an approximation for the scattering amplitude valid in the high energy, low momentum transfer regime,

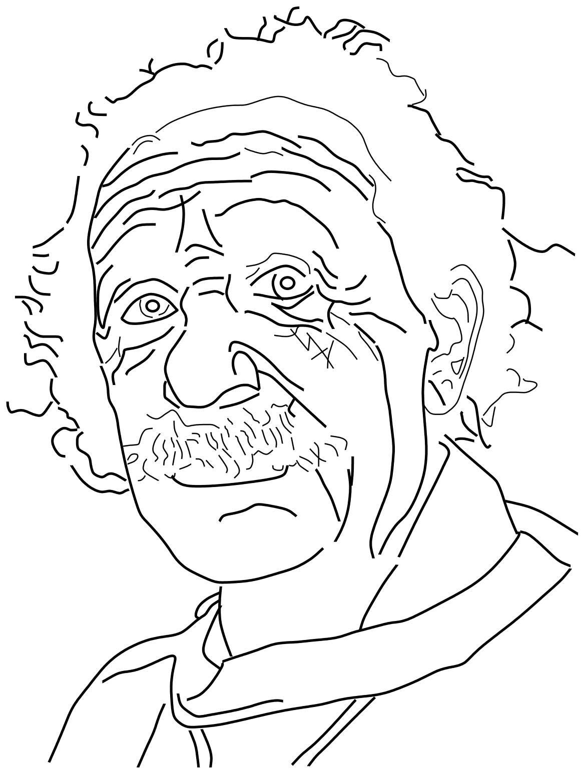


Figure A.15: Albert Einstein (1879-1955).

$$\mathbf{p}' = \mathbf{p} - \frac{\mathbf{k}}{2}, \quad \mathbf{p}'' = \mathbf{p} + \frac{\mathbf{k}}{2}, \quad \mathbf{p}^2 \to \infty, \quad \mathbf{p}^2 > \mathbf{k}^2 \tag{A.35}$$

The dominant contributions come from classical trajectories.

• **eikonal equation:**

• **electroweak theory:** combing theory of QED and weak-interactions.

• **energy conditions:**

The weak energy condition:

$$T^{ab} u_a u_b \geq 0, \tag{A.36}$$

for all timelike vectors u^a. This condition requires that the local energy density be non-negative in every observer's frame. Again this seems to be a very reasonable condition in classical theory.

The strong energy condition:

$$((T^{ab} - \frac{1}{2}g^{ab}T)u_a u_b \geq 0, \tag{A.37}$$

for all timelike vectors u^a. The frame in which T^{ab} is diagonal, this condition implies that the local energy density ρ plus the sum of local pressures p^i is non-negative: $\rho + \sum_i p^i \geq o$, and that $\rho + p^i$ for each ^{i}. This condition certainly holds for ordinary forms of matter, although it can be violated of classical physics.

The null energy condition:

$$T^{ab}k_a k_b \geq 0, \tag{A.38}$$

This condition is implicit in the weak energy condition. That is, if we assume the weak energy condition, then the null energy condition follows by continuity as u^a approaches a null vector.

The dominant energy condition:

The dominant energy condition implies that matter cannot travel faster that the speed of light.

$$R_{ab}W^a W^b \geq 0 \tag{A.39}$$

for any null vector $\mathbf{W}$. The importance of the weak energy condition is that it implies that in presence of matter a congruence of null geodesics that are initially diverging will begin to .

$$\frac{d}{d\lambda}\hat{\theta} = -R_{ab}K^a K^b - 2\hat{\sigma}^2 - \frac{1}{2}\hat{\theta}^2 \leq 0. \tag{A.40}$$

$\hat{\theta}$ monotonically decreases along the null geodesics if the first term on the right hand side if the weak energy condition holds.

• **energy conservation:** Notion of energy and the law of energy conservation have played a key role in analyzing behaviour of physical theories. However, it is not an independent physical requirement, but rather a consequence of time translation symmetry of the background metric and the dynamical equations.

Energy conservation is just a consequence of invariance under time translations, which, it turn is a feature of the homogeneity of Minkowski spacetime. However, when we come to general relativity the homogeneity of the Minkowski solution is not feature invariant under active diffeomorphisms. Hence there is no fundamental energy conservation in nature - there is nothing sacred about energy conservation!

When we restrict possible spacetimes to be those that are asymptotically flat, diffeomorphisms at infinity are just those of time and space translation we recover our notions of energy and momentum conservation.

In the absence of fixed a spacetime we must learn how to do physics in the absence of energy or momentum.

Define $E = v^a K_a$, where K_a is a killing vector,

$$\begin{aligned} \nabla_v E &= v^a \nabla_a (v^c K_c) \\ &= v^a v^c \nabla_a K_c \quad \text{since } v \text{ is the tangent of a geodesic,} \\ &= v^a v^c \nabla_{(a} K_{c)} = 0 \end{aligned} \tag{A.41}$$

• **Euclidean geometry:** Flat geometry based upon the geometries axioms of Euclid.

• **energy momentum tensor T_{ab}:**

• **entropy:** A quantitative measure of disorder of a system. The greater the disorder, the higher the entropy. It is defined as the amount of information about the microscopic motion of the constituents making up the system which are not determined by a description of the macroscopic state of that system.

• **epistemology:** Philosophic subject concerned with the significance of what we can know.

• **EPR:**

Rovelli's book [20]

The EPR apparent paradox might be among these. The two observers far from each other are physical systems. The standard account neglects the fact that each of the two is in a quantum superposition with respect to the other, until the moment they physically communicate. But this communication is a physical interaction and must be strictly consistent with causality.

• **entanglement entropy:** Black hole entanglement entropy is a measure of the information loss due to separation between

inside and outside the event horizon.

• **equations of motion (EOM):** They are a prescription for calculating the acceleration from the state of the system (from initial position and velocity). The initial state yields the initial acceleration, and the integration leads from the initial acceleration to later sates.

• **event:** A point in four-dimensional: a location in both space and time.

• **Event horizon:** A surface that divides spacetime into two regions: that which can be observed and that which can not. The Schwartzschild radius of a nonrotating black hole is an event horizon.????????

• **evolving Constants of motion:** Introduced by rovelli in [279] (an idea that goes back to DeWitt, Bergmann nad Einstein). The Hamiltonian constraints generate the dynamics, dynamics is pure gauge, and the observables of the theory are constant along the gauge orbits - these are the evolving constants of the motion. They are formed as follows: in a totally constrained theory, the values of fields are not physically observable. On the other hand, if one chooses a one-parameter family of observables such that their value coincides with the value of a dynamical varaible when the parameter takes the value of another dynamical variable, which one uses to characterize the evolution.

• **exceptional edges:** Exceptional edges are characterized by not being effected by action of the Scalar constraint. gr-qc/0409061

Application of the Hamiltonian constraint on a vertex v will produce a new edge a_{12} fig.(A.16) attached to (two) the edges e_1 and e_2. The edges incident to the vertices v_1 and v_2 only have two linearly independent tangent directions, namely those of e_1 and a_{12}. An exceptional edge, here a_{12}, verticies are annihilated by a second application of the Hamiltonian constraint. A failure to 'propagate'.

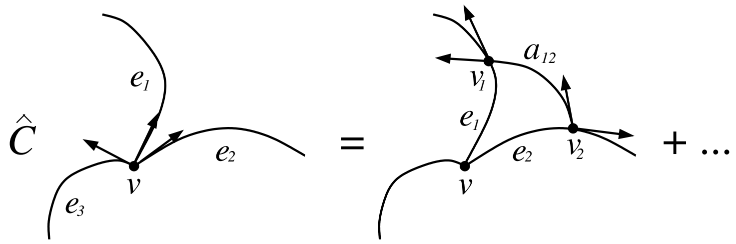


Figure A.16: exceptional. The Hamiltonian constraint's action on the trivalent node v with linearly independent tangent vectors produces two new vertices v_1 and v_2 - each of which has only two linearly independent tangent vectors.

• **Extended Master Constraint Operator $\hat{M}_E$:**

$$\hat{\mathbf{M}}_{\mathbf{E}} := \int_\sigma d^3x \frac{C^2 + q^{ab} C_a C_b}{\sqrt{\det(q)}} \tag{A.42}$$

Both constraints are spatially diffeomorphism invariant. However, $\hat{\mathbf{M}}_{\mathbf{E}}$ allows us to implement both the Hamiltonian and the spatial diffeomorphism constraint on $\mathcal{D}_{kin}$ provided we implement the corresponding operators in a non graph changing fashion. In [77] it was shown how to do that using the notion of a minimal loop which is a loop (average) within the graph on which the constraint acts. It follows that instead of using dual operators we can directly work with operators on $\mathcal{H}_{kin}$ and their adjoints so that the construction of the quadratic form can be sidestepped.

There is the Extended Master Constraint Operator $\hat{\mathbf{M}}_{\mathbf{EE}}$ which also includes the Gauss constraint in addition:

$$\hat{\mathbf{M}}_{\mathbf{EE}} := \int_\sigma d^3x \frac{C^2 + q^{ab} C_a C_b + C_j C_j}{\sqrt{\det(q)}}. \tag{A.43}$$

See page ?? of the Feynman lectures.

• **extentive thermodynamical quantities:**

• **extremal black holes** Black holes having the maximum possible charge for a given mass and angular momentum. Thought to be astronomically extremely rare. Ruled out by the third law of black hole mechanics.

• **extrinsic curvature:** We can describe this curvature by looking at how the normal vectors to the surface change along the surface. One uses the derivative operator of the larger space to measure the changes in the normal vector field, and then project back into the surface.

extrinsic curvature of spacetime $\mathcal{M}$ to measure the changes in the timelike normal vector field of the slice Σ. Roughly speaking, it is to do with the time derivative of the 3-metric.

Contains the information on how a hypersurface is embedded in the enveloping spacetime.

• **faithful:**

faithful representation:

faithful state:

• **false vacuum:** A metastable state in which a quantum field is zero, but its corresponding vacuum energy density is not zero.

• **falsifiable:** The property of a scientific hypothesis that it is not possible to perform an experiment that would disprove, or falsify, the hypothesis.

• **Fell's theorem:** (From [11] Quantum Field Theory in Curved Sapcetime and Black Hole Thermodynamics, Robert M. Wald) For an algebra of observables $\mathcal{A}$ of a field theory, although two representations of $\mathcal{A}$ may be unitarily inequivalent, the determination of a finite number of expectation values of observables in $\mathcal{A}$ - each made with fininte accuracy - cannot distinguish between different representations.

Quote:

whether the resulting physical Hilbert space admits a sufficient number of physical semi-classical states - in some sense this is granted by Frell's theorem [114] once we have a representation of the physical Hilbert space of the C$*$ algebra of our preferred algebra of bounded Dirac observables. [208]

• **Fermions:** The metric formulation cannot incorporate fermionic equations of motion. The reason for this is there is no finite-dimensional spinorial representation of the 4×4 general-linear real matrix group, however, there is a finite-dimensional spinorial representation of the Lorentz group - which is what Dirac discovered. We use a tetrad frame to establish an instantaneous inertial frame, in which the laws of special relativity apply. It can be introduced in a generally covariant way. When we fix the tetrad to Minkowski spacetime we recover Dirac's equation.

• **Fermi-Walker derivative:**

• **Fermi-Walker propagated:** propagation along a non-geodesic curve.

• **Feynmann diagrams:**

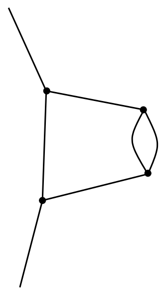


Figure A.17: The charge renormalization.

Feynman graphs yield integrals that diverge when momentum becomes indefinitely large (or, equivalently, when distances become indefinitely small).

one-particle irreducible (1PI) diagrams

• **Feynmann-Kac formula:** Instead of considering matrix elements of the unitary operator $U(t) = \exp(itH/\hbar)$, consider the analytic continuation $S^{A_1 \ldots A_n}(t_1, \ldots, t_n)$, of the Wightman functions $W^{A_1 \ldots A_n}(t_1, \ldots, t_n)$ is

$$S^{A_1 \ldots A_n}(t_1, \ldots, t_n) := W^{A_1 \ldots A_n}(t_1, \ldots, t_n).$$

These are the so-called Schwinger functions. A path integral for the Schwinger functions have better chance to be rigoroursly defined because the measure has a damping factor rather than an oscillating one and so one can give a rigorours defination is by backwards analytic continuation (when possible). The path integral for the Schwinger functions (when it can be proved) is called the Feynmann-Kac formula.

• **first class constraints:** restricting the fields to the constraint surface.

infinitesimal "gauge transformation"

$$(1 + a\frac{\partial}{\partial x})f(x) = f(x + a) + \mathcal{O}(a^2) \tag{A.44}$$

Functions invariant under this gauge transformation satisfy the constraint equation,

$$\frac{d}{dx}f(x) = 0. \tag{A.45}$$

The first class property is that

$$\{C_m, C_n\} = C_{mn}{}^l C_l \tag{A.46}$$

where the $C_{mn}{}^l$ are called the structure constants if they are independent of on the phase space coordinates and structure functions if they are functions over phase space; that is the first class constraints satisfy, in general, a "non-Lie" algebra

$$C_{mn}{}^l = C_{mn}{}^l(q^i, p_i) \tag{A.47}$$

as it is possible that from the canonical variables.

The first class property of the constraints guarantees, that the flow of the constraints is integrable to an n dimensional surface - the gauge orbit.

they generate gauge transformations.

in the Hamiltonian approach based on Dirac's conjecture [72], where a suitable combination of the first class constraints is shown to to be a generator of local symmetries of the Lagrangian.

as apposed to a structure *constant* we can have a structure *function* depending on the phase space coordinates.

Dirac's conjecture in the Hamiltonian approach the symmetries are generated by the first class constraints.

In the Lagrangean approach local symmetries are reflected in the existence of so called gauge identities.

• **first class constraints:** When all the constraints are first class we then have a closed constraint algebra.

• **flatness problem:** The observed fact that the geometry of of the universe is nearly very nearly flat, a very special condition, without an explanation of why it should be.

• **Fleischhack, Christian:**

Figure A.18: Christian Fleischhack.

• **Fock representation:** The most commonly used representations of canonical commutation relations/canonical anti-commutation relations are the so called Fock representations, defined in bosonic/fermioinic Fock spaces. These spaces have distinguished vector Ω called the vacuum killed by the annihilation operators and cyclic with repsect to creation operators (i.e., repeated applications and complex linear combinations of creation operators on the Ω generates a dense subset of the Hilbert space). They were introduced to describe systems of many particle systems with Bose/Fermi statistics.

Fock space is considered the prototypical QFT state space by some

fundamentally the Fock representation is not a valid representation for interacting quantum field theories.

States in LQG are fundamentally different from Fock states of Minkowskian quantum field theories. The main reason is the underlying diffeomorphism invariance: In the absence of a background geometry, it is not possible to introduce the familiar Gaussian measures and associated Fock spaces.

• **foliation of space time:** We have a spacial three-space stacked one above the other, one for every "moment in time".

• **form factor:** Given a loop γ, its form factor $F^a(\gamma, \vec{x})$ is defined by

$$F^a(\gamma, \vec{x}) := \oint_\gamma ds \dot{\gamma}^a \delta^3(\vec{x}, \vec{\gamma}(s)) \tag{A.48}$$

• **focussing theorem:** It is easy to show that the Raychaudhuir equation implies the following inequality for the expansion $\theta(\lambda)$

$$\theta^{-1}(\lambda) \geq \theta^{-1}(0) + \frac{\lambda}{4} \tag{A.49}$$

From this inequality we see that if the congruence is initially converging ($\theta(0) < 0$), then $\theta(\lambda) \to -\infty$ within an affine parameter $\lambda \leq 2/|\theta(0)|$. Demonstrates attractive instability of gravity. This result is employed in the area increase law of black holes and in the singularity theorems.

• **frame of reference:** The coordinate system to which a particular observer refers his or her measurements.

• **free-fall:** a particle is said to be in free-fall when its motion is affected by no forces except gravity. Unrestrained motion under the influence of the gravitational field.

or

A body in free fall does not feel any acceleration, and hence behaves locally as an inertial reference system of SR.

It is meaningless to say whether or not the gravitational field is flat around some point p of $\mathcal{M}$. It is meaningful, however, to ask whether or not the gravitational field is flat around the place where two particles coincide.

• **free data:** isolated horizons. reconstruct from...

• **Friedmann models:** A class of cosmological models that are isotropic and homogeneous, contain specified matter-energy density, conserve matter, and admit no cosmological constant. Also called standard models.

[307]

All textbooks on classical GR incorrectly describe the Friedmann equations as physical evolution equations rather than what they really are, namely gauge transformation equations. The true evolution equations acquire possibly observable modifications to the gauge transformation equations whose magnitude depends on the physical clock that one uses to deparametrise the gauge transformation equations.

• **functional analysis:** is the branch of mathematics which is concerned with the study of spaces of functions.

• **functional integration:**

In the Hilbert space language, this is the origin of field theoretic infinities, e.g., the reason why we can not naively multiply field operators.

• **future-distinguishing spacetime:** A spacetime is said to be future-distinguishing if $a \neq b$ implies $I^+(a) \neq I^+(b)$. (past-distinguishing is similarly defined: if $a \neq b$ implies $I^-(a) \neq I^-(b)$). (See also chronology condition, strongly causality, stable causality).

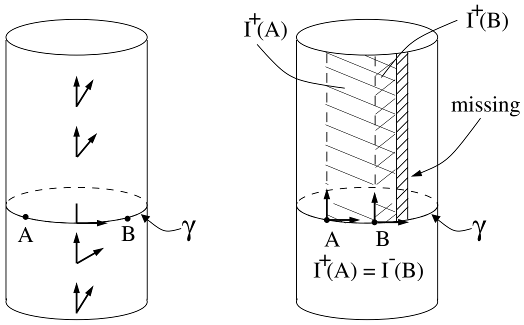


Figure A.19:

• **Gambini, Rodolfo:**

Figure A.20: Rodolfo Gambini.

• **gamma ray burst:** A $\gamma-$ray burst is a light signal of extremely high energetic photons (up to 1 TeV) that travells over cosmological distances ($\approx 10^9$ years). Could have applications in quantum gravity phenomenology.

They are enormous explosions that for a few seconds can produce as much light as that emitted by a whole galaxy. Signals from these explosions reach Earth on average once about once a day.

• **gauge fixing:** One method to eliminate the superfluous degrees of freedom of a gauge theory is through the introduction of a new set of constraints.

leads to second class constraints. null components? Poisson bracket is replaceds by the Dirac bracket

$$\{f,g\}^* := \{f,g\} - \{f,C_i\}B^{-1}_{ij}\{C_j,g\} \tag{A.50}$$

where

$$B_{ij} := \{C_i, C_j\} \tag{A.51}$$

The geometric interpretation of this is: the symplectic structure induced on the hypersurface, defined by the condition $C_j = 0$??, from the symplectic structure of the phase space $\mathcal{M}$ is given by the Dirac bracket $\{\cdot,\cdot\}^*$.

Interpretation of gauge invariant variables is based on the coincidence of the variables in an appropriate gauge choice. For example, blah defined by blah describes blah , because the variable blah coincides with blah, defined in blah in the gauge choice blah.

• **gauge invariance:**

• **gauge invariant coherent states:** Projections of the complexifier coherent states onto the gauge-invariant subspace, [253], [254].

• **gauge potential:** Gauge potentials can be identified as the connection on a principal fibre bundle in a certain gauge.

• **gauge symmetry:**

A gauge symmetry is a redundancy in the description of the system and not a symmetry of a system. For example in a situation with rotational symmetry, such as a planet moving about the sun, although there is a symmetry that changes the angle in the plane of motion there is still physically a difference between two different values of the angular variable. In a gauge theory this is not the case. If two field configurations are related by a gauge transformation then they are physically identical.

Dirac's definition of a gauge symmetry is when two solutions with the same initial data set separate at some point.

• **gauge transformations:** Gauge transformations are maps from solutions of the equations of motion into other solutions which are physically equivalent.

There is a choice of a gauge potential in a certain chart but there is also a choice of local charts. This other flexibility means that a gauge does not just refer to a particular gauge potential.

In the physics literature gauge transformations often refer to passing from one chart to another in a overlapping region - a passive gauge transformation. In the physics literature gauge transformations always refer to passing from one gauge potential to a physically equivalent one in the same chart - an active gauge transformation. It is active gauge transformations that have physical meaning.

electromagnetism the gauge transformation of the gauge potential is

$$\delta A_\mu = \{A_\mu, G\} = \partial_\mu \Lambda, \tag{A.52}$$

for an arbitrary function Λ.

In general relativity the gauge transformations (diffeomorphisms) for the metric field

$$\delta g_{\mu\nu} = \epsilon^\rho \partial_\rho + g_{\mu\rho}\partial_\nu \epsilon^\rho + g_{\rho\nu}\epsilon^\rho, \tag{A.53}$$

for some arbitrary function ϵ^ρ.

• **Gauss-Codaza equations:** Gauss-Codacci equation relating the space-time curvature to the intrinsic curvature of a sub-manifold. The relation can be written:

$$R^p_{qrt} = \Gamma^p_{qr|t} - \Gamma^p_{qt|r} + \Gamma^s_{qr}\Gamma^p_{st} - \Gamma^s_{qt}\Gamma^p_{sr}. \tag{A.54}$$

See maths glossary.

This is not the same as saying that "General covariance is the idea that the laws of nature must be the same in all reference frames, and hence all coordinate systems".

Gelfand triple: In LQG the nice quantum states are typically taken to be finite linear combinations of spin network states whose space is denoted by Cyl^* (the space of cylindrical functions of connections). Solutions to the quantum Einstein equations belong to its dual, Cly^*.

• **generating functional:**

$$W[J] = \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}[\mathcal{L}(\phi, \partial_\mu\phi) + J\phi]d\tau\right) \tag{A.55}$$

$$N^{-1}\exp(i\int\int \mathcal{L}[\phi])\prod_x [d\phi(x)] \tag{A.56}$$

in QFT the generating functional (A.55) is introduced as formal extrapolation of measures in infinite-dimensional spaces

$$\prod_x [d\phi(x)]. \tag{A.57}$$

This fails to be a true measure because the Lebesgue measure in infinite-dimensional spaces fails to be defined in general.

Given a connection A all its gauge invariant information is contained in all graphs with a finite number of degrees of freedom. We have a genuine field theory with an infinite number of degrees of freedom.

called the vertical polarization.)

• **general boundary formulation:** The so-called *general boundary formalism* extends the usual formulation of dynamics in terms of space-like hypersurfaces to one with closed boundary surfaces.

Homepage of Robert Oeckl http://www.matmor.unam.mx/ robert/

"Quantum gravity is the search for a theory that may unify general relativity with quantum mechanics and ultimately with the standard model of elementary particle physics. This problem has motivated me to think about the foundations of quantum theory. In particular, I am trying to extend the standard formulation of quantum mechanics in such a way as to make it more compatible with general relativistic concepts. It turns out that this is best done directly in the context of quantum field theory rather than non-relativistic quantum mechanics. I call this extension tentatively the general boundary formulation."

• **general covariance:** The principle that the laws of nature must be the same in all reference frames. From [17] (p.469): there should not be special coordinates that have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates.

• **general relativity:** Often described as a theory that conceives gravity not as a force between masses but as a change in the geometry of spacetime due to the presence of the matter and energy. However, the full content of GR is that it *discards the very notion of spacetime*. Instead, "spacetime geometry" is conceived as certain aspects of the relationships that exist between physical objects that live in the world (this what general relativists are referring to when they say that GR is a *background independent* theory). This is the context for Einstein's remark *"Beyond [my] wildest expectations"* (see Einstein's hole argument, active diffeomorphisms)

• **generalized fields:** in the passage from the classical field to the quantum field.with classical fields there is no classical particle picture space of distributional fields - smooth classical fields have measure zero in the quantum configuration space.

In quantizing a finite dimensional system one defines states as functions on a configuration space and then defines an inner product of two such functions ψ and ϕ through

$$(\psi, \phi) = \int_{\mathcal{Q}} d\mu \psi^* \phi \tag{A.58}$$

In the quantum theory systems with a finite number of degrees of freedom - the classical and quantum configuration spaces agree. In field theory the quantum configuration space is a substantial enlargement of the classical; in scalar field theories, for example, although the classical configurations are smooth functions on a t =const slice, the quantum configuration space consists of all tempered distributions.

The set of smooth configurations is of zero measure with respect to the Gaussian measure that determines the inner product! That is, for two smooth wavefunctions Ψ and Φ

$$\int_{\mathcal{Q}} d\mu \Psi^* \Phi = 0 \tag{A.59}$$

The regularization problems of quantum field theory can be traced back to this fact.

It is the distributional field fluctuations at the horizon that are the microscopic source for black hole entropy.

• **generalized Wick transformation:** which is arising by relating simpler Riemannian quantum theory to the more complicated physical Lorentzian theory. Allows one to obtain physical states in Lorentzian gravity from Wick transforming solutions of Riemannian quantum theory.

• **generic condition:**

(i) The strong energy condition holds,

(ii) Every timelike or null geodesic contains a point where

$$\ell_{[a} R_{b]dc[e} \ell_{f]} \ell^c \ell^d \neq 0, \tag{A.60}$$

(ℓ_a being the tangent vector of the geodesic). The condition serves to exclude certain pathological spacetimes ... I think?

see energy conditions - strong and weak.

• **geodesic:** A geodesic is the closest thing there is to a straight line curved space time. The shortest distance between two points. A more general definition of a geodesic is its velocity vector is parallel transported along the curve it traces out in spacetime. In other words, the parallely propagated vector at any point of the curve is parallel, that is, proportional to the vector at this point:

$$\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = \lambda \frac{dx^a}{du}. \tag{A.61}$$

• **geodesically convex:** A set U of spacetime $\mathcal{M}$ is geodesically convex if any two points of U may be joined by a unique geodesic lying entirely in U.

• **geometric quantization:** Geometric quantization provides a geometric, general framework for the quantization of a sympletic manifold M which is not neseccarily a cotangent bundle, for example when M it is compact.

arXiv:quant-ph/0006037 v1 7 Jun 2000 HARMONIC ANALYSIS WITH RESPECT TO HEAT KERNEL

Roughly a polarization is a choice of d coordinates on the $2d-$phase space $\mathcal{M}$, with the idea that the functions in our quantum Hilbert space will be independent of these d variables.

For example if $\mathcal{M} = \mathbb{R}^{2d}$, then we may take the usual position and momentum variables $x_1, \ldots, x_d, p_1, \ldots, p_d$ and then consider functions that depend only on $x_1, \ldots, x_d$ and are independent of $p_1, \ldots, p_d$. (This is called the vertical polarization.)

Alternatively, one may consider complex variables $z_1, \ldots z_d, \overline{z}_1, \ldots, \overline{z}_d$ and then consider the functions that are independent of $?z_k$, that is, holomorphic. (This is a complex polarization.) In that case our Hilbert space is the Segal-Bargmann space. To be more precise, in geometric quantization the elements of the quantum Hilbert space are not functions, but rather sections of a certain complex line bundle with connection. The sections are required to be covariantly constant in the directions corresponding to the polarization.

Apparent dependence of the act of quantization on the choice of coordinates.

• **geons:**

Lines of force can disappear at one point and reappear at the other - one points acts as sink and the other a source. The field equations of gravity would determine the motion of the sources and sinks and therefore how the charged particles represented by them. Purely gravitational effects.

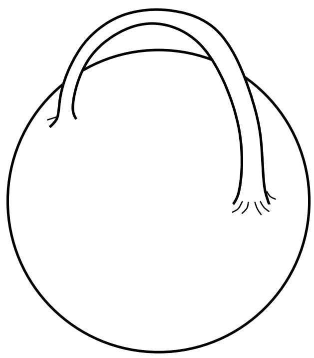


Figure A.21: geons.

• **Gibbs state:**

$$\exp(\mathcal{H}/kT) \tag{A.62}$$

• **Giesel, Kristina:**

Figure A.22: Kristina Giesel.

• **Gleason's theorem:**

• **globally hyperbolic:** Globally hyperbolicity is a proper of some spacetimes with the topology $\mathbf{R} \times \Sigma$, it ensures that there exists a universal time function whose gradient is everywhere timelike.

The physical significance of global hyperbolicity comes from the fact that there is a family of Cauchy surfaces $\Sigma(t)$ for a region of spacetime $\mathcal{M}$. One can predict what will happen in from data on the Cauchy surface.

In a globally hyperbolic region of spacetime, there is a geodesic of maximum length joining any pair of points that can be joined by a timelike or null curve. This fact is used in the singularity theorems: one establishes there is a region of spacetime which is globally hyperbolic, then one demonstrates that there is a geodesic which has a pair of conjugate points and hence is not a of doesn't maximum length (involves focussing theorem). The way out of this contradiction is that this geodesic comes to a full stop indicating the presence of a singularity.

Equivalent definition of global hyperbolicity: A spacetime $\mathcal{M}$ is said to be globally hyperbolic if the sets $J^+(x) \cap J^-(y)$ are compact for all x, y in $\mathcal{M}$

The global hyperbolicity of $\mathcal{M}$ is closely related to the future or past development of initial data from a given spacelike hypersurface.

• **"global time problem"**

"multiple choice problem" - that one could use different time functions and one must prove the quantum theories are equivalent.

[?] Arlen Anderson, *Evolving Constants of Motion*, [gr-qc/95070708].

• **Gowdy spacetime:** an example of a that can be quantized as a midisuoerspace model.

• **gravitational backgrounds:**

$$\mathcal{L} = \sqrt{-g}\{g^{\mu\nu}\nabla\phi^*\nabla_\nu\phi - m^2\phi^*\phi\} \tag{A.63}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + m^2\phi = 0 \tag{A.64}$$

and the conserved current density is:

$$J_\mu = i(\phi^*\nabla_\mu\phi - (\nabla_\mu\phi^*)\phi) \equiv i\phi^* \overleftrightarrow{\nabla}_\mu \phi \tag{A.65}$$

the inner product:

$$< \psi|\phi >= i \int_\Sigma \sqrt{-g}\psi^* \overleftrightarrow{\nabla}_\mu \phi \, d\Sigma^\mu \tag{A.66}$$

is independent of Σ.

• **"gravitational bound":** the gravitational boound proble in GR is an intrinsically non-linear problem even when the conditions are such that the field is weak and the motions are non-relativistic, at least in the time-dependent case. [**?**]

• **gravitational wave:** A gravitational wave is a linear perturbation of the gravitational field characterised by an oscillating curvature tensor which causes the immediate neighbourhood of space-time region through which it passes to oscillate.

• **gravitons:** Gravitons don't play a fundamental role in the loop quantum gravity. Instead, they only serve as a kind of approximate way of dealing with small perturbations of a weave state in which a superposition of spin networks mimic a given classical spacetime metric.

"There is a general belief, reinforced by statements in standard textbooks, that: (i) one can obtain the full non-linear Einstein's theory of gravity by coupling a massless, spin-2 field h_{ab} self-consistently to the total energy momentum tensor, including its own; (ii) this procedure is unique and leads to Einstein-Hilbert action and (iii) it only uses standard concepts in Lorentz invariant field theory and does not involve any geometrical assumptions. After providing several reasons why such beliefs are suspect and critically re-examining several previous attempts we provide a detailed analysis aimed at clarifying the situation. First, we prove that it is impossible to obtain the Einstein-Hilbert (EH) action, starting from the standard action for gravitons in linear theory and iterating repeatedly. Second, we use the Taylor series expansion of the action for Einstein's theory, to identify the tensor $\mathcal{S}^{ab}$, to which the graviton field h_{ab} couples to the lowest order. We show that the second

rank tensor $\mathcal{S}^{ab}$ is not the conventional energy momentum tensor T^{ab} of the graviton and provide an explanation for this feature. Third, we construct the full nonlinear Einstein's theory with the source being spin-0 field, spin-1 field or relativistic particles by explicitly coupling the spin-2 field to this second rank tensor $\mathcal{S}^{ab}$ order by order and summing up the infinite series. Finally, we construct the theory obtained by self consistently coupling h_{ab} to the conventional energy momentum tensor T^{ab} order by order and show that this does not lead to Einstein's theory. (condensed)." There is more to gravity than gravitons and this will be elaborated in a separate publication, [in preparation].

Fock states are not available in the absence of background spacetime - so what are particles? ????

• **Groenwald-van Hove theorem:** A collection f_i of functions on the symplectic space $(\mathcal{M}, \omega)$ which Poisson commute with each other (are in involution) is said to be complete if the vanishing of $\{f_i, g\}$ for all i implies that g is a function of the form $g(x) = h(f_1(x), \ldots, f_n(x))$. A collection of operators $\{A_j\}$ is said to be complete if any operator B which commutes with each A_j is a multiple of the identity. Consider a linear map ρ from the space of observables $C^\infty(\mathcal{M})$ on the symplectic space $(\mathcal{M}, \omega)$ such that

(1) $\rho(1) = \mathbb{I}$,

(2) $\rho(\{f, g\}) = i[\rho(f), \rho(g)]/\hbar$,

(3) for some complete set of functions $f_1, \ldots, f_n$ in involution, the operators $\rho(f_1), \ldots, \rho(f_n)$ form a complete commuting set.

The Groenwald-van Hove theorem states that such a map does not exist in general.

• **group averaging:** Projecting it onto the physical Hilbert space by gauge-orbit smearing. We are using the finite - rather than the infinitesimal - form of the constraints.

you have to always first of all you should define a subspace in the Hilbert space which should have stronger topology that the topology of the Hilbert space.

• **Gribov problem:** An obstruction occurs when the gauge fixing term can not be defined globally or that the gauge fixing function intersects a gauge orbit more than once.

• **Haag-Kastler axioms:** [32] [393] [394] Axioms of algebraic quantum field theory.

• **Haag-Ruelle theory:** In standard scattering theory one makes the physical assumption that in the far future $t_f \to \infty$ and far past $t_i \to -\infty$ any outgoing and ingoing particles respectively do not interact. This is not really true. However, using the methods of of local quantum physics, assuming that te theory has a mass gap (the four momentum squared operator should have a pure point spectrum which is separeted from the continuum) one can prove that the vacuum correlators of the asymptotic fields reduce to those of free field, where the vacuum really means the interacting vacuum.

• **Haag's theorem:** In quantum field theory, as apposed to quantum mechaincs, there exist unitary inequivalent representations (that is, representations which cannot be connected by a unitary transformation) of the canonical commutation relations (CCR). Haag's theorem shows that the representation of the interacting theory differs from that of the free theory. This poses problems for perturbative techniques used in standard textbooks on quantum field theory. The free part is used to define an orthonormal basis of states to which the interaction applies. Haag's theorm says that this is not possible. Haag's theorem forbids us from appling perturbation theory we learned in quantum mechanics to quantum field theory. We are left with the question of why perturbation theory works as well as it does.

• **habitat:** Let $\mathcal{D}$ be the finite linear span of SNWF's which is dense in the kinematic Hilbert space and let $\mathcal{D}^*$ denote its algebraic dual. A habitat is a subspace of $\mathcal{D}^*$ containing $\mathcal{D}^*_{Diff}$ with the minimal requirement that is preserved by the dual action of the Hamiltonian constraints.

raises questions about the Hamiltonian constraints introduced by Thiemann see [].

habitat, **??**.

• **Hamilton constraint:** Hamilton constraint generates "time" translations and hence encodes the dynamics of the theory.

$$\hat{\mathcal{H}}_\epsilon(N)|f>= 0 \tag{A.67}$$

$< \Psi| \, . \, > \leftrightarrow \; \Psi(\, . \,)$

$$\Psi(\, . \,) \in (\mathcal{H}^*_{Kin})_{Diff} \tag{A.68}$$

$$< \Psi|\hat{\mathcal{H}}_\epsilon(N)f > \leftrightarrow \; \Psi(\hat{\mathcal{H}}_\epsilon(N)f) = 0 \tag{A.69}$$

it is independent of the value of ϵ! The underlying reason is understood to be the diffeomorphism covariance of the graph-dependent triangularization prescription. Proved in section ??? The same reasoning applies to the matter coupled case. The limit $\epsilon \to 0$ is therefore already performed; hence the theory, including coupling to the standard model, is manifestly finite and does not require renormalization!

• **Hamilton function:** The Hamilton function of a finite dimensional dynamical system is the value of the action of a solution of the equations of motion, viewed as a function of the initial and final coordinates.

The

gr-qc/0408079

• **Hamilton vector field:** Hamiltonian vector field - Given a manifold X with a symplectic structure ω, any smooth function $f : X \to R$ can be thought of as a "Hamiltonian", meaning physically that we think of it as the energy function and let it give rise to a flow on X describing the time evolution of states. Mathematically speaking, this flow is generated by a vector field $v(f)$ called the "Hamiltonian vector field" associated to f

$$\omega(., v(f)) = df??? \tag{A.70}$$

In other words, for any vector field u on X we have

$$\omega(u, v(f)) = df(u) = uf??? \tag{A.71}$$

The vector field $v(f)$ is guaranteed to exist by the fact that ω is nondegenerate.

• **Hamilton-Jacobi equation:**

$$\begin{aligned}\int_0^t \left(p\frac{dq}{d\tau} - \mathcal{H}\right) d\tau &= \int_0^t \left(P\frac{dQ}{d\tau} - \mathcal{H}' + \frac{dF(q,P)}{dt}\right) d\tau \\ &= \int_0^t \left(P\frac{dQ}{d\tau} - \mathcal{H}'\right) d\tau \\ &+ F(q'', P'', t) - F(q', P', 0)\end{aligned} \tag{A.72}$$

$$\frac{\partial(P,Q)}{\partial p, q} = 1 \tag{A.73}$$

consider the transformation for which the new Hamiltonian is zero.

one gets the Hamilton-Jacobi equation

$$H\left(q, \frac{\partial F}{\partial x}, t\right) + \frac{\partial F}{\partial t} = 0 \tag{A.74}$$

Since $\mathcal{H}' = 0$, the new coordinates Q and P are constants of motion.

• **Hawking, Stephen:**

• **Hawking radiation:** $T = \kappa$

• **Heisenberg representation:** A Heisenberg state is can be viewed as the value Schroödinger state at some fixed time.

• **Heisenberg uncertaiinty principle:** From "THE LIFE OF THE COSMOS" by Lee Smolin p.306:

Figure A.23: Stephen Hawking.

“Let us imagine that we have a friend who lives in Quantumland, where big things can be simple. To tempt us with the beauty of her world, she has seny us a present. We go to the airport to receive it and are given a sealed box with a door on each end. On top of is written, ‘QUANTUM PET CARRIER - ONLY OPEN END AT A TIME.’ In Quantumland no one would have t be told that you can only open one end of a box at a time, but we will see that the reason to be grateful for this advice.

Bringing the box home, we hurriedly open it to see what is inside. Opening one end, we see the head of a cat peer out! Lovely, but the cat stays inside. It seems that one property of pets in Qunatumland is that they can never come out of their box, one can only interact with them by opening one of the doors. OK, we can live with this, but we become curious to at least know the sex of our cat. Well, we can use the door at the other end for this. We try to open it, but we find it is closed tight. Remembering what was written on top, we close the first door. Immediately the back door comes open. By looking in, we are able ascertain that our pet is a boy.

This done, we go back to the first door to play with our cat. To do this we must first close the back door. We then open the front door to find a jolly looking puppy gazing back at us!

After some trials and examinations, we discover that we are in an interesting situation. When we open the first door we discover that our quantum pet is either a cat or a dog. If we open the second door, we discover that our pet is either male or female. Howeveer, by the peculiar properties of the box, our vision is obscured so we cannot be sure, when gazingin the front, what sex our cat or dog is. And when we look in the back door we can ascertain the sex, but we cannot judge reliably whether it is a dog or a cat.

We cannot have both doors open at once, so we can never be sure of both species of our pet and its sex. Ascertaining one destroys knowledge of the other. If all we do is look ot the front end, then once we have seen a cat there, we will always see a cat. If we wish, we may at any time close that door and peer in the other side to learn the pet's sex. Whether it is a cat or a dog, we discover that there is a fifty percent probability that it is male and a fifty percent probability that it is female. But, once we have done that, if we go back to the front, we will not necessarily find a cat, as we did before. For once we have ascertained the sex, the species again gets scrambled, and half the time it will be a cat, and half the time it will be a dog.

What we are experiencing is exactly the Heisenberg Uncertainty Principle. It is happeneing because a complete description of our quantum pet would include its species and its sex. According to classical science, we ought to be able to take the animal out of the box and see what it is. But a quantum pet can never be removed from its box and, for reasons that are perhaps mysterious, we can only ever observe one aspectg at a time.

Perhaps the reader thinks I'm being facetious, or teasing. But no, I am describing what we believe is the general situation we are in when we observe any physical system. The Heisenberg Uncertainty Principle limits the information we can have about any system to always exactly half of the information we need to have a complete description. We always have some choice of which inforamtion we would like to have. But, try as we may, we cannot exceed this limit."

• **holographic principle:**

• **holonomic equivalence:** two loops α and β are said to be holonomic equivalent if they have the same holonomy with respect to every connetion, i.e.

$$H_A(\alpha) = H_A(\beta) \quad \text{for all } A \text{ in the space of connections } \mathcal{A}. \tag{A.75}$$

The set of all holonomic equivalence classes is called the **holonomic loop space** and is denoted $\mathcal{HL}$.

• **holonomy algebra:** denoted $\mathcal{HA}$.

• **holonomy-flux $*$-algebra:** Holonomies of connections lead to well defined operators on $\mathcal{H}^0$.

$$h_\alpha(A) = \mathcal{P}\ \exp\left(\oint_\alpha A_a ds^a\right) \tag{A.76}$$

The connection is smeared in *1-dimensions.* In LQG the classical electric flux $E_k(S)$ through a surface S is the integral of the densitised triad $\tilde{E}^a_k$ over a two surface

, where n^S_a is the conormal vector with respect to the surface S.

The conjugate momentum is dual to a 2-form

$$e_{abi} := \frac{1}{2}\eta_{abc}\tilde{E}^c_i$$

geometrically it is natural to smear in 2-dimensions. It again turns out that these 2-dimensionally smeared fields lead to well defined operators on $\mathcal{H}^0$.

$$[E_i]_f := \int_S e_{abi} f^i dS^{ab}$$

$$\begin{aligned} \frac{\delta h_\alpha(A)}{\delta A_i^a(x)} &= \int_0^1 \delta^3(x^j - \alpha^j(t)) \frac{d\alpha^i}{dt} (U_\alpha)_0^t(A)\ \tau_a\ (U_\alpha)_t^1(A)\ dt && \text{(A.77)} \\ \frac{\delta E(S)}{\delta E_a^i(x)} &= \int_S dudv\ n_i \delta(x^j - X^j(u,v))\ \tau_a, && \text{(A.78)} \end{aligned}$$

By virtue of the canonical commutation relations, the Poisson brackets turn out to be

$$\{E_a(S), h_\alpha(A)\} = \int d^3x \frac{\delta E_a(S)}{\delta E_b^i} \frac{\delta h_\alpha(S)}{\delta A_i^b}. \tag{A.79}$$

From these last three relations we see that if S and α have no common points the integration vanishes. The same happens if α belongs to S since in this case vectors tangent to the graph are orthogonal to n^i. We are led to consider loops with a finite number of intersections with the surface. Consider the case of one edge e intersecting the surface S

$$\begin{aligned} \{[E_S]_f, (h_e)_{mn}\} &= \lim_{\epsilon \to 0} [(h_{e_1}^\epsilon)_{ml}\ \{[E_S]_f, (h_{e(\epsilon)})_{lp}\}\ (h_{e_2}^\epsilon)_{pn}] \\ &= \lim_{\epsilon \to 0} \left[(h_{e_1}^\epsilon)_{ml} \left\{ \int_S f_b(y) \tilde{E}^b(y), \int_{e(\epsilon)} A^a(x) \tau_{lp}^a \right\} (h_{e_2}^\epsilon)_{pn} \right] \\ &= \mp 8\pi G\gamma f_a(P) (h_{e_1}\ \tau_a\ h_{e_2})_{mn} \end{aligned}$$

$$\{E_a(S), h_\alpha(A)\} = 8\pi G\gamma \tau_a \sum_{e \subset \alpha} h_e(A) o(e,S) \tag{A.80}$$

where $o(e,S)$ is $+1$, -1 i the orientations of e and S agree or disagree, respectively.

Hence one can define the action of $E(S)$ ona generic cylindrical function by the expression

$$E_a(S)[f] = \{E_a(S), f(h_\alpha(A))\} \tag{A.81}$$

with $E_a(S)$ acting as a vector field.

The Poisson bracket between the strip functionals vanishes unless the two strips intersect. If they do, the bracket is given by a sum of slightly generalized strip functionals.

If one attempts to smear connections and triads in 3-dimensions the resulting operators fail to be well defined on $\mathcal{H}^0$.

The new algebra provides a distributional extension of the old one.

There is uniqueness theorem on this algebra, see section.

• **Holst action:** The Palatini action plus an extra term multiplied by the inverse of the Barbero-Immirzi parameter which does change the classical equations of motion:

$$S_H(e,A) = \frac{1}{2\kappa}\int_{\Sigma\times\mathbb{R}} (\frac{1}{2}\epsilon_{IJKL}e^I \wedge e^J \wedge F^{KL} + \frac{1}{\gamma}e^I \wedge e^J \wedge F^{KL})$$

• **hoop group:** the quotient of the loop group with respect to holonomic equivalence.

• **horismos:** The future horismos of a set S is defined to be $E^+[S] := J^+[S]/I^+[S]$.

• **Husain, :**

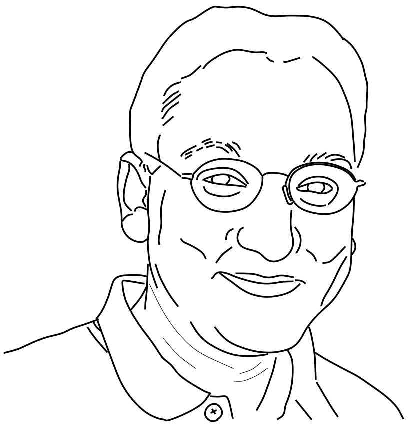


Figure A.24: Husain.

• **ideal clock limit:**

• **Immirzi-Barbero parameter:** A certain free parameter, γ, in the canonical quantization of gravity. This parameter is not present in the classical theory - In the classical theory, different values of the Immirzi parameters label equivalent classical theories. However, in the quantum thoery, representations with different Immirzi parameters are not unitarily equivalent - arises from an ambiguity in the quantization procedure... It can be fixed by the requirement that the quantum gravity computation reproduce the Hawking value for the entropy of a black hole.

A conflicting assertion Alexandrov *et.al*: [153] [154]. *"It was thought that the anomaly in question is a physical one so that the Immirzi parameter becomes a new fundamental constant. However, in a series of works [153] it was shown that this is not the case because* *it is actually a consequence of a diffeomorphism anomaly, whereas there is a quantization which preserves all classical symmetries and leads to results independent of the Immirzi parameter.".*

• **induced metric:** The induced metric the metric q a hypersurface picks up from the metric g of the spacetime the hypersurface is embedded. The induced metric of a hypersurface Σ is obtained by restricting the line element to displacements within the hypersurface Σ.

The induced metric is also called the **first fundamental form**.

Some jargon: The induced metric q is the pullback of the spacetime metric g, denoted $\phi^* g$, where ϕ is the map from the spacetime manifold $\mathcal{M}$ to the hyperspace manifold N, i.e., $\phi : \mathcal{M} \to N$.

• **inertial frames:** The set of all inertial systems is the set that can be derived from a given one by a Lorentz transformations.

• **inextendable curve:**

• **inflation:** An accelerated expansion of the very early universe. Eliminates the need for contrived and highly special initial conditions.

• **inner product for quantum gravity:** In LQG the inner product on spacially diffeomorphism invariant states is known in closed form. One now wishes to have an inner product on the subset of states that satisfy all the constraints. Such a can be written formally in terms of a "projection operator" by means of spin foams - one of the promises of the Master constraint programme is the construction of a well defined physical inner product. - i.e., the physical inner product, ??from which we can calculate physical expectation values??.

• **integrable:**

integrability of of the system in the sense of Lioville:

(1) the number of independent conserved quantities equals the number of degrees of freedom, and that

(2) these conserved quantities are in involution.

are in involution means F_i

$$\{F_i, F_j\} = 0 \tag{A.82}$$

There are standard methods available for solving the system completely (Hamilton-Jacobi theory, action-angle varaibles, ...).

• **intensive thermodynamical quantities:**

• **interaction picture:** In the Schr?odinger picture, time evolution is governed by the states and their equations. In the Heisenberg picture, time evolution is governed by the observables and the equation $\frac{dA}{dt} = \{B, \mathcal{H}\}$.

In the interaction picture we divide the time dependence between the states and the observables. This is suitable for systems with a Hamiltonian of the form B H = B H0 + B H1 where $A_H 0$ is time independent. The interaction picture has many uses in perturbation theory.

• **interpretations:** Some seek to interpret quantum mechanics in a way which accords more with our intuition derived from classical mechanics.

reduction of the wave-packet of a microscopic system and a specification of its quantum state by a macroscopic instrument

• **interval:** In the general theory of relativity, the square of the interval is given by:

$$ds^2 = g_{ab}\, dx^a\, dx^b, \tag{A.83}$$

where g_{ab} are the components of the metric tensor, and dx^a is the differential of the coordinate x^a. Time-like interval $ds^2 < 0$; Space-like interval $ds^2 > 0$.

• **inverse loop transformation:**

• **irreduciable representation:**

• **Chris Isham:** In a seminal paper with Ashtekar rigorouly constructed the quantum configuration space of distributional connections using GNS theory for C^* algebras. Also later apllied topos methods to develop of a theory of quantum gravity.

• **ISO (3) group:** $ISO(3)$ denotes the Euclidean group in three dimensions.

The translation group is an abelian group.

Obviously

$$\hat{R}\hat{T}\hat{R}^{-1} = \hat{T}' \tag{A.84}$$

consequently the translation group is an invariant abelian subgroup of the Euclidean group, (translation-rotation group). $SO(3)$ is another subgroup such that the only element they have in common with the translation group is the identity element $\mathbf{I}$. Every element of $ISO(3)$ can be written in a unique way as $g = RT$, $R \in SO(3), T \in \mathbf{R}^3$ then $ISO(3)$ is the semi-direct product of $\mathbf{R}^3$ and $SO(3)$,

$$SO(3) \otimes_S \mathbf{R}^3. \tag{A.85}$$

Figure A.25: Chris Isham.

We introduce translation generators P_a, $a = 1, 2, 3$, which satisfy

$$[J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = 0, \tag{A.86}$$

• **ISO (2,1) group:** $ISO(2,1)$ denotes the Poincare group in three dimensions.

• **isolated horizons:**

Isolated horizons are generalizations of the event horizon of stationary black holes to physically more realistic situations. The generalization is in two directions. First, while one needs the entire spacetime history to locate an event horizon, isolated horizons are defined using properties of spacetime *at the horizon*. Second, although the horizon itself is stationary, the outside space-time can contain time-dependent fields and geometry.

Restriction to spacetimes which admit an internal boundary which has certain boundary conditions.

Type **I** the isolated horizon geometry is spherical

Type **II** the isolated horizon geometry is axi-symmetric. This class includes rotating isolated horizons as well as distorted (due to exterior matter), see [136].

Type **III** Most general include arbitrary distortions

Note that the symmetries refer *only* of the horizon geometry. This weak set of boundary conditions give rise to the zeroth and first laws of black hole mechanics, in fact, isolated horizons can be characterized this way.

Black hole entropy [132].

The action principle and the Hamiltonian description is well defined and the resulting phase space has an infinite number of degrees of freedom. The full spacetime metric need not admit any geometric symmetries even in a neighbourhrood of the horizon. They quantize this sector of the full phase space using techniques developed for the full theory.

The problem was reduced to counting the number of ways to puncture a two-dimensional sphere, representing the horizon, which gave the horizon area close to a given value.

Isolated Horizons in Numerical Relativity [131].

• **Kaluza-Klein:** The radius of the extra dimension is taken to be fixed and non-dynamical and not subject to Einstein's field equations. As such there are no processes in which gravitational and electromagnetic effects convert into each other. For the thoery to be a true unification, gravity and electro-magnetism should be aspects of one phenomenon.

Another objection is that the theory cannot incorporate ferminons of the gauge theory.

• **Kauffman, Louis:**

• **kernel:** the set of (eigenstates) solutions of an operator which have zero as their eigenvalue. or

• **Killing horizon:**

$$\xi^a \xi_a = -1 - \frac{2M}{r} \tag{A.87}$$

Thus outside the horizon ($r > 2M$) ξ^a is timelike, at the horizon ($r = 2M$) ξ^a is null.

• **Killing vector field:** Geometry does not change along a Killing vector field. An isometry of the spacetime. A Killing vector field satisfies the Killing equation

$$\nabla_a K_b + \nabla_b K_a = 0. \tag{A.88}$$

• **Kinematic Hilbert space:**

There are several complementary characterisations of the Kinematic Hilbert space $\mathcal{H} := \mathcal{H}_\omega$ which are useful in different contexts.

1. Positive linear functional characterisation

2. C^*- algebraic characterisation

3. Projective limit characterisation

• **KMS condition:** Kubo-Martin-Swinger condition for thermal equilibrium.

$$< 0,\beta|A(t)B(t)|0,\beta >=< 0,\beta|B(t')A(t+i\beta)|0,\beta > \tag{A.89}$$

$$\omega((\gamma_t A)B) = \omega(B(\gamma_{t+i\beta}A)) \tag{A.90}$$

Haag *et al* [53] have shown that this KMS condition reduces to the well known Gibbs condition

$$\omega(.) = \frac{\sum_n e^{-\beta E_n}|m\psi_n >< \psi_n|}{(\sum_n e^{-\beta E_n} < \psi_n|\psi_n >)} \tag{A.91}$$

• **KMS states:** The relevant states describing thermal equilibrium are the so-called KMS-states.

• **Kochen-Specker theorem:** [363] [361] Implies it is not possible to assign actual, *possessed*, values to all quantum mechanical physical quantities of an individual system. (See section contextual interpretations)

It is the one of the no-go theorem against hidden variables theories, (the other being Bell's theorem which states that, given the premise of locality of physical theories, a hidden variables theory cannot match the statistical predictions of quantum mechanics).

better version: [362] Kochen-Specker theorem asserts the impossibility of associating real values $V(\hat{A})$ to all physical quantities in a quantum theory (if $\dim\mathcal{H} > 2$) whilst preserving the '$FUNC$' rule that $V(f(\hat{A})) = f(V(\hat{A}))$ - i.e., the value of a function f of a physical quantity A is equal to the function of the value of the quantity. Equivalently, it is not possible to assign true-false values to all the propositions in a quantum theory in a way that respects the structure of the associated lattice of the projection operators.

Before we can state the theorem we first need to introduce the idea of the valuation of an operator $\hat{A}$ representing a physical quantity, $V_s(A)$. Valuation is assigning truth-values to propositions '$A \in \Delta$' asserting that the quantity A lies in the Borel subset Δ of the spectrum of the operator $\hat{A}$ that represents A.

The Kocher-Specker theorem asserts that, in the case when the dimension of Hilbert space is greater than 2, one *cannot* assign real numbers to all quantum quantities in a way that preserves functional relations between them. By 'preserves functional relations' is meant for any operator $\hat{A}$ and any function of it $f(\hat{A})$ (f a function from $\mathbb{R}$ to $\mathbb{R}$), the value of $f(\hat{A})$ is the corresponding function of the value of $\hat{A}$, i.e., $V_s(f(A)) = f(V_s(A))$.

(1)

??? (2) ideal measurement of an observable $\hat{A}$ in a state $\psi \in H$ yields an eigenvalue of $\hat{A}$ and, immediately after the measurement, the state is thrown into corresponding eigenstate. ???

• **Komar integral:** An expression for the conserved mass and angular momentum for stationary spacetime evaluated when the spacial boundary is pushed to infinity.

$$E = \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x\sqrt{q}n_a\sigma_b\nabla^a K^b, \quad J = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{q}n_a\sigma_b\chi^a K^b \tag{A.92}$$

where $\partial\Sigma$, σ_b (timelike and future pointing) and n^a timelike orthogonal unit normal vectors to $\partial\Sigma$. where K^b time translations for energy.

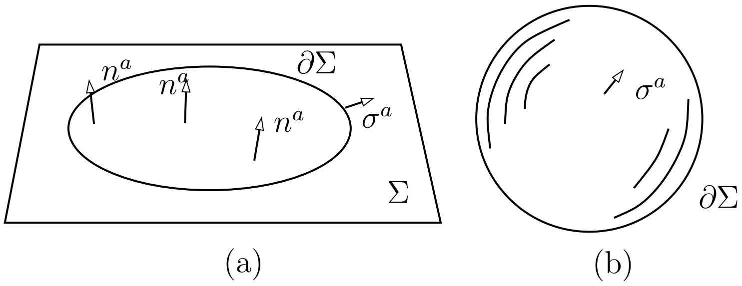


Figure A.26: An expression for the conserved mass, evaluated when the spacial boundary is pushed to infinity. (b) $\partial\Sigma$ is a closed spacelike two-surface surrounding the source.

• **Krasnov, Kirill:**

Figure A.27: Kirill Krasnov.

• **Kruskal coordinates:** These corrdinates cover the entire spacetime manifold of a maximally extended solution of classical general relativity.

• **Kuchar's "reduction ad absurdum":** We require a "global time coordinate"

• **the landscape:**

• **lapse function** N**:** N, together with the shift function N^a, only serves to specify the foliation of $\mathcal{M}$ and as such it is not itself a dynamical field.

• **lattice gauge theories:**

The difference between lattice approaches in Minkowskian field theories and background independent field theories that use floating graphs. An approximation to the continuum theory limit . Whereas in background independent field theories

continuum limit used in lattice approaches in Minkowskian field theories, whereas in background independent field theories allow all possible graphs in $\mathcal{M}$.

lattice gauge theory formulations applied to GR conflict with its invariance under diffeomorphisms as it introduces a fixed background geometric structure.

• **Laws of black hole mechanics:**

The zeroth law of black hole mechanics stationary black holes have constant surface gravity on the entire horizon.

The first law of black hole mechanics tells us how the area of a black hole increases when we make a transition from an initial stationary solution to a nearby stationary solution.

$$\delta M = \frac{\kappa}{8\pi G}\delta a + \Omega \delta J, \tag{A.93}$$

where κ is the surface gravity.

The second law of black hole mechanics states that the area of an event horizon can never decrease.

A system in equilibrium will have settled to a stationary state.

Isolated horizons: all quantities which enter the statement of the law now refer to the horizon itself.

• **Laws of thermodynamics:**

The zeroth law of thermodynamics states that in thermal equilibrium the temperature is constant throughout the system.

The first law of thermodynamics tells us how the

$$\delta E = T\delta S + \delta W. \tag{A.94}$$

The second law of thermodynamics states that the entropy of an isolated system can never decrease.

• **Lax pairs:**

$$\frac{dL}{dt} \equiv \dot{L} = [M, L]. \tag{A.95}$$

$$L(t) = g(t)L(0)g(t)^{-1} \tag{A.96}$$

where the invertible matrix is determined by the equation

$$M = \frac{dg}{dt}g^{-1}. \tag{A.97}$$

• **Lewandowski, Jerzy:**

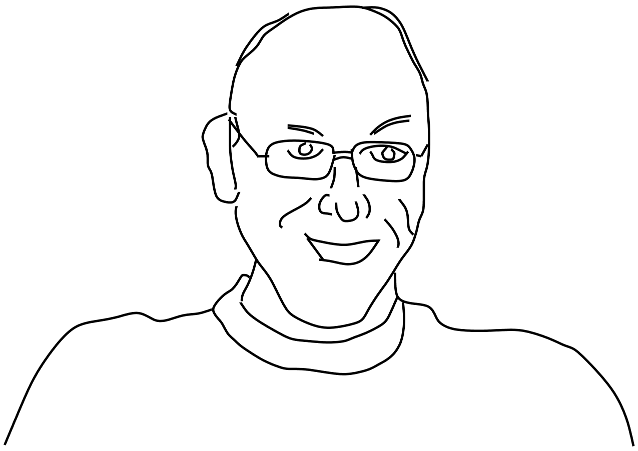


Figure A.28: Jerzy Lewandowski.

• **Lie derivative:** Lie derivative.

see maths glossary.

• **linear 2-cell spin-foam models:**

• **Livine, Etera:**

• **local particles:** Realistic particle detectors are finitely extended. Physical states detected by a localized detector (eigenstates of local operators describing detection) are local particles states. Local particles have a chance of retaining meaning in the fully general relativistic context where Fock states are not available.

• **Loll, Renate:** Loll works on developing a theory of quantum gravity, reconciling the beautiful geometric description of space and time laid out in Einstein's theory of General Relativity with the insight that all of physics at its most fundamental level must

Figure A.29: Etera Livine.

be described by quantum laws of motion. She is one of the pioneers of a new approach to the nonperturbative (background independent) quantization of gravity, that of Causal Dynamical Triangulations which recently has produced a number of remarkable results. These include a dynamical derivation of the fact that space-time is four-dimensional (something that can be taken for granted only in classical gravity) and that it has the shape of a de Sitter Universe (like our own universe in the absence of matter), and of the so-called wave function of the universe which plays an important role in understanding the quantum behaviour of the very early universe. Remarkably, one also finds that the dimensionality of spacetime reduces smoothly to two at short distances, indicative of a highly nonclassical behaviour of spacetime geometry near the Planck scale. These results are obtained by superposing elementary quantum excitations of geometry which have a notion of causality ("cause preceding effect") built into them at the very smallest scale.

• **loop algebra:**

given a connection the identities satisfied by Wilson loops may be implemented on the free vector space of formal sums of *single* loops by requiring that if

$$\sum_i c_i T[\alpha_i, A] = 0 \tag{A.98}$$

for *all* connections, then these loops are linearly dependent

$$\sum_i c_i \alpha_i = 0 \tag{A.99}$$

on the free vector space of single loops.

Figure A.30: Renate Loll.

When the product on the free vector space is defined by

$$\left(\sum_j c_j \alpha_j\right) \cdot \left(\sum_k d_k \alpha_k\right) = -\sum_{j,k} c_j d_k \left(\alpha_j \# \beta_k + \alpha_j \# \beta_k^{-1}\right) \tag{A.100}$$

these Mandelstam relations define an ideal. The quotient space by the ideal defines an algebra, which is called the holonomy algebra...

[gr-qc/9512020]

• **loop quantum cosmology:** Mini-superspace but which appropriate incorporation of the discrete nature of quantum geometry. Very good accessible account can be found in [241].

As yet it makes no attempt to address the problem of time or interpretation of the wavefunction of the universe.

• **loop quantum gravity LQG:**

A background independent approach to quantum gravity. In many ways conservative based only on the experimentally well confirmed principles of general relativity and quantum mechanics.

a canonical quatization of a Hamiltonian description of classical general relativity, whereby spacetime is sliced into a stack of 3-d spacial hypersurfaces Arnowitt-Deser-Misner (ADM). In LQG the basic variable used is, rather than a metric, a connection and its "conjugate momentum" a triad field (can be thought of as the square root of the 3-d spatial metric).

The spacial 3-metric on a hypersurface is constructed from the triad field and the connection contains information about extrinsic curvature of the slice of the slice sitting in the spacetime, establishing contact with the metric ADM formulation.

Wilson loops, polymer nature

• **loop representation:** import machinery from particle physics... [54].

• **loop transformation:** the Fourier transformation [54]

$$\tilde{\psi}(k) = \int dk e^{ik \cdot x} \psi(x), \tag{A.101}$$

which relates the position representation with the momentum representation.

$$\psi(\gamma) := \int_{\mathcal{A}/\mathcal{G}} d\mu(A) T^0(\gamma)[A] \Psi(A), \tag{A.102}$$

where the Wilson loop functionals T^0 plays the role of an integral kernal, (as does $e^{ik \cdot x}$ in the Fourier transformation), and wave functions $\psi(\gamma)$ in the loop representation are labelled by closed spacial curves γ. Once the loop transform is rigourously defined the loop representation is unitarily equivalent to the connection representation.

• **Lorentz covariant loop quantum gravity (CLQG):** Standard loop quantization is based on the Astekar-Barbero formulism and leads to the theory in which $SU(2)$ gauge group in the tangent space. It starts with the first order formulation of general relativity in 3+1 dimensions with the Lorentz gauge group in the tangent space. Then as a result of partial gauge fixing, the canonical formulation possesses, besides the usual diffeomorphism invariance, only a local $SU(2)$ symmetry.

• **LOST uniqueness theorem:** Named after the people who developed it: Lewandowski, Okolow, Sahlmann, and Thiemann. In LQG once one decides to base quantization on the holonomy-flux algebra and have the spacial diffeomorphism group implememnted unitarily, there is only one choice for the Kinematical Hilbert space. Once the Kinematical representation has been choosen in which the constraints can be defined as operators, the physical representation follows by a rather tight procedure. Hence, the whole quanization programme is put on a very theoretical robust footing leading to a falsifiable theory of quantum gravity. One predicition of the theory being the discreteness of the eigenvalues of geometric operators.

• **luminosity distance:**

• **LSZ reduction formulas:** using the LSZ reduction formulas it is more easily demonstrated that the a scattering amplitude is unchanged by making in states $|in>$ into out states $|out>$ and vice versa. so we have a symmetry that implies we can exchange preparation of states and the measurement proper.

• **Mach's principle:**

• **macroscopics:**

• **Major, Seth:**

• **marginally trapped surfaces:** See trapped surface.

• **Markopoulou, Fontini:**

Figure A.31: Fontini Markopoulou.

• **Master constraint:** The multiplicity of smeared Hamiltonian constraints combined into one constraint.

since only a non-anomalous constraint algebra usually leads to a sufficiently large enough physical Hilbert space $\mathcal{H}_{Phys}$

the question of anomaly freeness is translated into the size of the physical Hilbert space $\mathcal{H}_{Phys}$.

the infinite dimensional constraint algebra, whether it closes with structure functions or structure constants, is replaced by a one-dimensional Abelean Lie algebra.

• **Master equation:** Any complete observable $\mathcal{O}$ must be constant under a transformation generated by the Hamilton constraint $C_j(x)$ i.e.,$\{\mathcal{O}, C_j(x)\}|_{C=0} = 0$. This condition is equivalent to

$$\{\mathcal{O}, \{\mathcal{O}, \mathbf{M}\}\}_{\mathbf{M}=0} = 0 \iff \{\mathcal{O}, C_j(x)\}|_{\mathcal{C}=0} = 0. \tag{A.103}$$

see Eq (**??**).

• **Master Constraint Operator:** Replaces the Hamiltonian constraints $\hat{C}_a$ by a single Master constraint $\hat{\mathbf{M}}$ which is a spacially diffeomorphism invariant integral of squares of the individual Hamiltonian constraints

$$\hat{\mathbf{M}} := \frac{1}{2}\hat{C}_a Q^{ab}\hat{C}_b \tag{A.104}$$

which encodes all the necessary information about the constraint surface and the associated invariants (Dirac observables).

• **Master Constraint Programme:**

Extended Master Constraint Programme:

• **maximal extension of a spacetime:** in the context of the Schwarzschild spacetime coordinate systems that allow the continuation of a metric across the event horizon. There also is an inner horizon. is maximally extended if the coordinate system allow for continuation of the metric across every horizon.??

• **matrix models:** field theories over groups.

• **maximally extended curve:** A timelike curve is maximally extended in the past if it has no past endpoint. (Such a curve is also called past-inextendible). The idea behind this is that such a curve is fully extended in the past direction, and not merely a segment of some other curve.

• **mesoscopics:** gravity experiments quantum mesoscopics and structural foundations of quantum mechanics.

• **microscopics:**

• **moments:**

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \frac{1}{2r^5}(3(\mathbf{x}\cdot\mathbf{x}') - r^2 r'^2) + \ldots \tag{A.105}$$

With this expansion, we can rewrite $\Phi(\mathbf{x})$ as

$$Q_{ij} = \int d^3x' \rho(\mathbf{x}')(3x_i' x^{j'} - r'^2 \delta_{ij})$$

• **Montevideo intepretation:** In this interpretation enviromental decoherence is supplemented with loos of coherence due to the use of realistic clocks to measure time to solve the measurement problem. The resulting formulation is framed entirely in terms of quantum objects without having to invoke the existence of measurable classical quantities like time in ordinary quantum mechanics. The formulation eliminates any privileged role to the measurment process giving an objective definition of when an event occurs in a system.

• **M-theory:** background independent M-theory for strings

does not presume that they are vibrating in an preexisting spacetime. a formulation in the absence of spacetime in which instead spacetime emerges from the collective behaviour of the strings.

• **M-theory conjecture:** There is a background independent formulation of string theory which unifies all the known string theories, 11 dimensional supergravity and the 11 dimensional supermembrane theory.

C. M. Hull and P.K. Townsend, Unity of superstring dualities, Nucl. Phys. B348 (1995) 109; P. Townsend, (M)embrane theory on T 9, Nucl. Phys. (Proc. Suppl) 68 (1998) 11-16; hep-th/9507048, in Particles, Strings and Cosmology, ed J. B agger et al (World Scienti c,1996); hep-th/9612121; I. Bars, hep-th/9608061; hep-th/9607122; Petr Horava, MTheory as a Holographic Field theory, hep-th/9712130, Phys.Rev.D59 (1999)046004.

E. Witten, String theory in various dimensions, Nucl. Phys. B443 (1995) 85.

Once (or if) one finds this theory, one must find the classical solutions to this theory and show that the different perturbative string theories arise from expansions around them.

• **multipoles:**

field multipoles

source multipoles

• **naked singularities:** . In recent paper [398] Harada ... *We propose the concept of 'effective naked singularity', which will be quite helpful because general relativity has the limitation of its application for the high-energy end. The appearance of naked singularities is not detestable but can open window for new physics of strongly curved spacetime.* - which of course includes quantum gravity.

• **negative mass:** (the so-called Benny Hill effect).

• **noiseless subsystems:** The framework of noiseless subsystems has been developed as a tool to preserve the fragile quantum information against decoherence []. In brief, when a quantum register (a Hilbert space) is subjected to decoherence due to an interaction with an external and uncontrollable environment, information stored in the register is, in general, degraded. It has been shown that when the source of decoherence exhibits some symmetries, certain subsystems of the quantum register are unaffected by the interactions with the environment and are thus noiseless. These noiseless subsystems are therefore very natural and robust tools that can be used for processing quantum information.

• **net of operator algebras:** an open region of sacetime $\omega \subset \mathcal{M}$

$$\omega \subset \omega' \implies \mathcal{A}_{\omega} \subset \mathcal{A}_{\omega'} \quad \text{for all } \omega, \omega' \subset \mathcal{M},\ \mathcal{A}_{\omega}, \mathcal{A}_{\omega'} \subset \mathcal{A}, \tag{A.106}$$

• **Newmann-Penrose null tetrads:** frames that consisted of two real null vectors and two "complex null vectors". seemingly strange choice well suited for analysis of null geodesics. Intimately related to spinor basis

• **Noether's theorem:** If the equations of motion possesses some continuous symmetry, then there will be a conservation law associated with that symmetry. The most familiar examples of conservation laws are:

(i) if there is invariance under some *spatial translation*, then *momentum* is conserved;

(ii) if there is invariance under *time translation*, then *energy* of the system is a conserved quantity;

(iii) if there is invariance under *rotation* about some axis, then *angular momentum* about the same axis is conserved.

Noether's gave a proof of the general result. Most physical theories implement dynamics as a result of a variation principle, by means of a Lagrangian. Continuous transformations that leave the action invariant - except for boundary terms.

(i) Spatial translation: using the Lagrangian $\mathcal{L}(x,t)$.

$$\mathcal{L}(x+\delta x,t) - \mathcal{L}(x,t) = 0 \;\Rightarrow \frac{\partial \mathcal{L}}{\partial x}(x,t) = 0 \tag{A.107}$$

using this in the Euler-Lagrange equations (E.7),

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,t) = 0 \;\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}}(x,t) =: p = Const'. \tag{A.108}$$

(ii) Time translation: That the Lagrangian does not explicitly depend on t,

$$\frac{\partial \mathcal{L}}{\partial t}(x,t) = 0 \tag{A.109}$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \dot{x}\frac{\partial \mathcal{L}}{\partial x} + \ddot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{A.110}$$

Making use of the Euler-Lagrange equations

$$\frac{d\mathcal{L}}{dt} = \dot{x}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \ddot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\left(\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right), \tag{A.111}$$

$$\frac{d}{dt}\left(\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L}\right) = 0. \tag{A.112}$$

$$E = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}(x, \dot{x}) - \mathcal{L}(x, \dot{x}) \tag{A.113}$$

• **non-commutative geometry:** The idea is to describe geometry in algebraic terms. One deals with a function algebra over the space one is interested in and the geometry of a manifold is reformulated in terms of this algebra. A topological space $\mathcal{M}$ is completely described by the commutative algebra $\mathcal{C}^0(\mathcal{M})$ of continuous functions over it. Once the algebraic description of the geometry is obtained, the structure algebra can be made noncommutative.

Noncommutative geometry is the study of noncommutative algebras as if they were algebras of functions on spaces. noncommutative algebras should be studied as if they were the function algebras of "noncommutative spaces"

See for example [401] for introduction to mathematical constructions.

Connes proved that there is an equivalence between a spinorial compact manifold and a Spectral Triple $(\mathcal{C}(\mathcal{M}), \mathcal{D}, \Gamma)$, which contains the algebra, the Dirac operator $\mathcal{D}$ encoding the differential structure and the metric, and finally the chiral operator Γ.

Connes. quantum mechanics does not permit the measurement of distance smaller than the Plank length and time intervals smaller than the Plank time. Non-commutative spacetimes are adopted as *background* spacetimes.

• **non-commutative spacetime:** non-commutative spacetime can be derived from a quantum deformation which replaces the enveloping algebra of the Poinare Lie algebra by a non-commutative Hopf algebra(quantum group).

[]

• **non-commutative theories:** hepth/9808192

"One success of noncommutative geometry is that the standard model of particle physics of particle physics. As Alain and his colleagues discovered, if you take Maxiwell's theory of electrodynamics and write down the simplest possible noncommutative geometry, out pops the Weinberg-Salaman model unifying electromagnetism with the weak nuclear force. In other words, the weak interactions, together with the Higgs field, show up automatically and correctly."

Non-commutative geometry is a third approach to quantum gravity. It is by definition a background independent approach, as the basic idea is to replace the background manifold by algebraic generalisations of a certain set of diffeomorphism invariant observables, which are the spectrum of the Dirac operator. []

• **non-expending horizon (NEH):** A horizon is a compact, spacelike 2-surface (usually a sphere) expanding at the speed of light, however, not changing its are element.

The zeroth and first law of Black Hole thermodynamics still hold

A three dimensional sub-manifold Δ of spacetime is said to be non-expending horizon (NEH) if it satisfies the following conditions:

(i) Δ is topologically $S^2 \times \mathcal{I}$ and null where $\mathcal{I}$ is an interval on the real line;

(ii) The expansion $\theta_{(\ell)} := q^{ab}\nabla_a \ell_b$ of ℓ vanishes on Δ, where ℓ is any null normal to Δ and q_{ab} is the degenerate metric on Δ;

(iii) All equations of motion hold at Δ.

• **non-perturbative:**doe not describe dynamics of linear perturbations around some fixed background spacetime.

compare with background independence: does not distinguish any particular background metric from the outset.

• **non-renormalizability:**

Part of the reason why some people consider non-renormalizability unacceptable as a physical theory may lie in what one considers the meaning of "quantization" to be.

• **null-tetrads:** Newman-Penrose null-tetrads.

$$\ell^a := \frac{1}{\sqrt{2}}(r^a + t^a), \quad n^a := \frac{1}{\sqrt{2}}(r^a + t^a) \tag{A.114}$$

$$\ell^2 = 0, \quad n^2 = 0, \tag{A.115}$$

complex null vectors

$$m := \frac{1}{\sqrt{2}}(e_1 + ie_2), \quad \overline{m} := \frac{1}{\sqrt{2}}(e_1 - ie_2) \tag{A.116}$$

• **normal region:** A normal region

• **observables:** Observables in Hamiltonian mechanics are real functions defined on phase space, $\mathbb{R}^{2n}$. Some examples of observables are position, momentum, energy, angular momentum.

Observables is a property of a physical system that can in principle be measured. In quantum mechanics, an observable is represented by a *self-adjoint operator*.

In quantum mechanics, however, an observable is represented by a self-adjoint operator defined on a dense subspace of a Hilbert space. Work out mean values, mean deviations - the operator to any power should be a finite quantity for it to have a physical meaning.

In the classical GR we compare observations with coordinate-independent quantities. The coordinate time (as well as the spatial coordinates) can in principle be discarded from the formulation of the theory without loss of physical content, because results of real gravitational experiments are always expressed in coordinate-free form.

• **Oeckl, Robert:** In particular, he works on an extension of the standard formulation of quantum mechanics, precisely to make it compatible with general relativistic concepts. This program is called the general boundary formulation of quantum theory.

Figure A.32: Robert Oeckl.

• **off-shell:**

• **one-partial irreducable (1PI):** See Feynmann diagrams.

• **on-shell:** Equations of motion are imposed.

off-shell

• **operator:** An operator is a linear map taking vectors to vectors

$$\mathbf{A} : |\psi> \rightarrow \mathbf{A}|\psi>$$

linear meaning

$$\mathbf{A}(a|\psi> + b|\phi>) = a\mathbf{A}|\psi> + b\mathbf{A}|\phi>$$

• **operator algebra:**

$$[\hat{q}, \hat{p}] = i. \tag{A.117}$$

Define

$$U = 1 + i\delta q' \hat{p}, \tag{A.118}$$

where $\delta q'$ is an infinitesimal parameter. To infinitesimal order this is a unitary operator,

$$U^{\dagger}U = (1 - i\delta q'\hat{p})(1 + i\delta q'\hat{p}) = 1 + \mathcal{O}((\delta q')^2), \tag{A.119}$$

Hence U preserves the norm of operators it acts on. It is easily seen that U is the operator which performs infinitesimal translations on the coordinate operator $\hat{q}$:

$$UqU^{-1} = q + i[p, q]\delta q' = q + \delta q', \tag{A.120}$$

therefore

$$< q'|Uq =< q'|UqU^{-1}U =< q'|(q + \delta q')U = (q + \delta q') < q'|U, \tag{A.121}$$

which implies that

$$< q'|U =< q + \delta q'|. \tag{A.122}$$

• **operator ordering ambiguity:** Hamiltonian $\mathcal{H}(q, p) = p^2 q$. The classical variables replaced by their operators $p \to \hat{p} = -i\hbar\frac{\partial}{\partial q}$ and $[\hat{p}, \hat{q}] = i\hbar$. There are two possible hermitian Hamiltonian operators corresponding to the classical Hamiltonian,

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{1}{2}\left(\hat{p}^2\hat{q} + \hat{q}^2\hat{p}\right) \quad \text{and} \quad \hat{\mathcal{H}}' = \hat{p}\hat{q}\hat{p}, \\ \hat{\mathcal{H}} &= \hat{\mathcal{H}}' - i\hbar\hat{p}. \end{aligned}$$

Operator ordering ambiguities is an important issue in LQG as it relates to the size of the physical Hilbert space, which in turn relates to the question of whether LQG has the correct semi-classical limit.

• **operator valued distribution:**

$$\hat{A}^a(x) = \int_{-\infty}^{\infty} dk e^{ix \cdot k} \hat{a}_k \tag{A.123}$$

A quntumm field $\Phi(x)$ at a point cannot be a proper observable. Physically this appears evident because a measurement at a point would necessitate infinte energy. It may be regarded as a sesquilinear form on a dense domain $\mathcal{D} \subset \mathcal{H}$; this means that the matrix element $< \Psi_2|\Phi(x)|\Psi_1 >$ is a finite number when both Ψ_1 and Ψ_2 are in $\mathcal{D} \subset \mathcal{H}$ characterized by the property that the probability amplitudes for particle configurations decrease fast with increasing momenta and increasing particle number.

and that it depends linearly on Ψ_1, conjugate linearly on Ψ_2. To obtain an operator defined on the vectors in $\mathcal{D}$ one has to average ("smear out") Φ with a smooth function f in Minkowski space i.e. take

$$\Phi(f) = \int \Phi(x) f(x) d^4x.$$

• **Osterwalder-Schrader Reconstrusction:** An integral of the form "$\int \mathcal{D}\phi e^{iS[\phi]/d}$" ocsilates wildly. We instead first consider the integration "$\int \mathcal{D}\phi e^{iS[\phi]/d}$"; If the Lagrangian is positive then the integral will fall off rapidly. If one can define this and it satisfies certain conditions one can go back to the Lorentzian theory with the Osterwalder-Schrader reconstrusction.

, most importantly, requires that the measure $d\mu(\phi)$ be invariant with respect to Euclidean transformations and that

reflection with respect to the hyperplane

$$(x^0 = 0, x^1, \ldots, x^d). \tag{A.124}$$

The Osterwalder-Schrader Reconstrusction theorem allows one to obtain the Hilbert space of the canonical quantum field theory from the Euclidean one.

The Osterwalder-Schrader reconstruction of Euclidean QFT was generalized to diffeomorphism invariant theories. Replacing the Euclidean group with the diffeomorphism group. [250]

M. Rainer, *Is Loop Quantum Gravity a QFT?*, [gr-qc/9912011].

• **outer marginally trapped surface** See trapped surface.

• **Page-Wooters constrauction:** Proposed by Page-Wooters [47] to deal with an aspect of the 'problem of time". It consists in promoting all variables of the system to quantum operators and choosing one of the variables to be a "clock" and computing conditional probabilities for the other variables to take certain values when the "clock" takes a certain value. This proposal runs into technical difficulties when applied in detail to constrained systems, as emphasized by Kuchar [41]. In constrained systems the physically observable variables are those that have vanishing Poisson brackets with the constraints. If one of the constraints is the Hamilonian then this candidte for a clock does not evolve. Page-Wooters tried circumvent this difficulty by considering "kinematical" variables, i.e. variables that dont commute with the constraints and hence appaer to "evolve". This brings about other problems. Such variables can be promoted to quantum operators acting on the space of wavefunctions that are not necessarily annihilated by the contraints (the kinematic Hilbert space). When the zero eigenvalue of a constraint operator lies in the continuous spectrum (as is the case with GR), the states that are annihilated by the contraints are distributional. Therefore it is difficult to see how they can admit a probablistic interpretation.

• **Palentini first order formulism of GR:**

This condition, which is imposed *a priori* in Einstein's formulation of general relativity, is seen to seen to be part of the equations of motion.

$$\mathcal{D}e_a = \partial_b e_a + \epsilon_{abc} A^b e^c = 0. \tag{A.125}$$

• **partial Cauchy surface:** A partial Cauchy surface, Σ, for a spacetime $\mathcal{M}$ is a hypersurface which no causal curve intersects more than once. An important example is that of the hypersurface Σ extending from a black hole horizon to spacial infinity.

• **partial observables:** These are measurable quantities not determined by the theory, however, correlations between them are. These correlations constitute the dynamics of the theory. We understand dynamics as the relative evolution of partial variables with respect to one another - in which all these variables are treated on an equal footing.

By choosing one of these variables to be an *independent* ("time") and the rest to be *dependent* variables.

e.g. area as described in chapter 1.

doesn't commute with constraints Kuchar.

see complete observables.

• **partial truths:** cosmology. When it will be in the causal past of the point p, once it is it is remains in the past, there are some points that will never be in the past of p.

• **particle:** Fock states: with no interactions present we can expand $\hat{\psi}(x)$ as a complete set of normal modes

$$\{u_i(x)a_i + v_i(x)b_i^\dagger\} \tag{A.126}$$

particles are objects that are trigger readings of detector or that produce bubbles in bubble chambers.

• **particle horizon:** Two different regions of the universe that are separated from each other by a distance greater than $2R_H(t)$ at epoch t can not by causally related. This boundary is called the "particle horizon".

• **partition function:** The partition function is

$$Z(\beta,\mu) = \mathrm{Tr} e^{-\beta(\hat{\mathcal{H}}-\mu\hat{N})} \tag{A.127}$$

where $\hat{\mathcal{H}}$ is the many-body Hamiltonian, $\hat{N}$ the particle number operator, and μ the chemical potential.

$$\mathcal{E} = -\frac{1}{V}\frac{\partial \log Z}{\partial \beta} = \frac{\partial \beta \mathcal{F}}{\partial \beta}, \quad p = \frac{1}{\beta}\frac{\partial \log Z}{\partial V} = -\mathcal{F} \tag{A.128}$$

• **passive diffeomorphism transformation:** Refers to a change of coordinates, i.e. the same object is represented in a different coordinate system.

• **peeling theorem:** asymptotic expansions.

• **Penrose abstract tensor notation:**

• **Penrose inequality:** The Penrose inequality relates the area of cross-sections of an event horizon A_E on a Cauchy surface with the ADM mass M_{ADM} at infinity:

$$\sqrt{A_E/16\pi} \leq M_{ADM}. \tag{A.129}$$

The Penrose inequality has been one of major open questions in classical general relativity, closely tied to an even bigger open question, the cosmic censorship conjecture. It was attempts at finding violations of this that led Penrose to formulate this inequality. The derivation relies so heavily on the cosmic censorship conjecture that a violation of this inequality would go a long way towards contradicting it.

• **Penrose-Carter diagram:** Let $\tilde{\mathcal{M}}$ denote physical space-time with metric $\tilde{g}_{ab}$. The idea was to construct another "unphysical" manifold $\mathcal{M}$ with boundary ζ and metric g_{ab}, such that $\tilde{\mathcal{M}}$ is conformal to the interior of $\mathcal{M}$ with $g_{ab} = \Omega^2\tilde{g}_{ab}$, and so that the "infinity" of $\tilde{\mathcal{M}}$ is represented by the finite hypersurface ζ. We realise the whole physical spacetime $\tilde{\mathcal{M}}$ as a subset of the larger spacetime $\mathcal{M}$.

Asymptotic properties of $\tilde{\mathcal{M}}$ and of fields in $\tilde{\mathcal{M}}$ can be investigated by studying ζ, and the local behaviour of the fields at ζ provided the relevant information is conformally invariant.

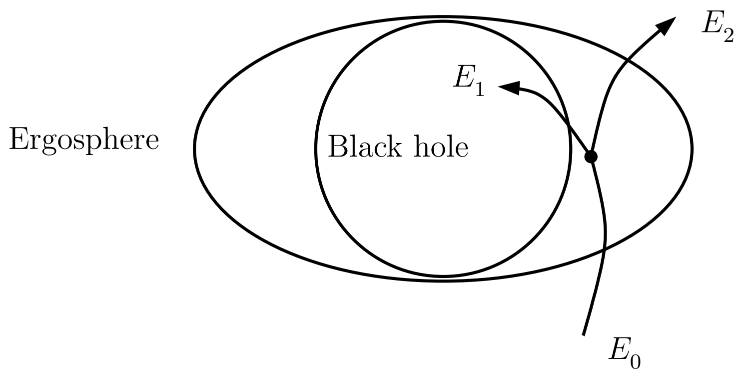


Figure A.33: Penrose process.

• **Penrose process:** extraction of energy from a rotating black hole.

• **Sir Roger Penrose:** modern techniques in GR e.g. use of spinors plus elegant topological methods. Invented twistor theory as a theory of quantum gravity. Ingredients spinor networks which were later found an very important place in LQG by Smolin and Rovelli.

Figure A.34: Sir Roger Penrose.

• **perennial:** Perennials are classical variables that are observables, that is, have vanishing Poisson brackets with all the constraints.

One of the constraints, the Hamiltonian constraint, generates you can not expect any of them to work as a clock??

• **Petrov classification:** number of principle spinors

• **perturbative quantum gravity:**

"However, even if it is accepted that we have finite amplitudes for each fixed topology, we are far from finished. The expressions then have to be summed up. Now there is a problem that the sum apparently actually diverges. The intended finite theory is actually not finite after all! This particular divergence does not seem to worry the string theorists, however because they take this series as an improper realization of the total amplitude. This amplitude is taken to be some analytic quantity, with the power series attempting to find an expression for it by 'expanding around the wrong point', i.e. about some point which is singular for the amplitude ....????? This could be OK, although the divergence encountered here has been shown to be of a rather uncontrollable kind ('not Borel-summable")....... Moreover, if string-theoretic (perturbational) calculations are actually expansions 'about the wrong point', then it is unclear what trust we may place in all these perturbative calculations in any case! Thus we do not yet know whether or not string QFT is actually finite, let alone whether string theory, with all its undoubted attractions, really provides us with a quantum theory of gravity.

The amplitudes they are calculating are not invariant under active diffeomorphisms and hence cannot be an observables of the theory. So can't have any meaning beyond the semiclassical limit. Cannot be applied to extreme situations that we would like to access in quantum gravity.

• **phenomenology of quantum gravity:** Gamma-ray bursts,

• **physical coherent states:** Physical coherent states are needed in order to test the classical limit of gauge-invariant observables.

• **Plebanski action:** Used in a construction of spinfoam models.

• **point holonomies:** Scalar fields e.g. Higg's particle, quantum cosmology transforms as a scalar so not well defined on an edge - instead we consider the exponential at a point

$$\exp\left(i\mu A^{\varphi}(x)\right) \tag{A.130}$$

• **point-identification:** If one manifold is a replica of another we need a point-identification map which relates points in the two manifolds which are to be regarded as the "same". The correspondence of the points of these two spacetime manifolds is arbitrary and so is a "gauge freedom".

• **Poisson bracket:**

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \tag{A.131}$$

• **polymer particle representation:** [241] If one drops the requirement of weak continuity we can circumvent the Stone-von Neumann theorem, one finds that there are unitarily inequivalent representation of CCR algebra (Weyl algebra).

The standard Schrödinger representation representation of the Weyl algebra plays the role of the Fock representation of low energy quantum field theories and the new, unitarily inequivalent representation -called the polymer particle representation- in which states are mathematically analogoous to the polymer-like excitations of quantum egometry.

It mimics the loop representation - e.g. it has discrete - momentum not a well defined operator in this representation.

Appears in the mathematics of loop quantum cosmology.

• **Ponzano-Regge:** A partion function for 3d GR.

$$Z_{PR} = \sum_j \prod_f \dim(j_f) \prod_v$$

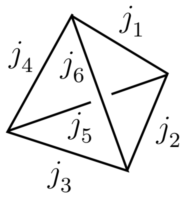


Figure A.35: PonRegg.

$$W(\mathbf{j}) = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \tag{A.132}$$

Witten's $ISO(3)$ Chern-Simons theory [E. Witten, Nucl. Phys. **B311**, 46 (1988).]

• **pre-geometry:** More fundamental substratum from which our familiar continuum physics is assumed to emerge as a course grained and secondary model theory. The dynamics of pre-geometry in the appropriate physical regime, give rise to the appearance of a background spacetime with matter fields or particles whose dynamics are played out over it. Examples of pre-geometric theories, spinfoam of LQG,...

• **preons:** Quarks, leptons and heavy vector bosons are suggested to be composed of a stable spin$-1/2$ preons, existing in three flavours, combined according to simple rules.

• **prequantization:** A prequantisation of a sympletic manifold (M, ω) is a pair (L, D) where L is complex Hermitian line bundle over M and D a compatible connection with curvature $(1/\hbar)\omega$.

• **primary constraints:**

• **principal fibre bundle:** Loosely speaking, in the context of gauge theory in spacetime, it is a manifold that locally looks like cartesian product of spacetime manifold and the parameter space of the gauge group. A fibre bundler whose fibre is a Lie group in a certain special way.

Principal bundles are equivalent, if the fibres are transform $g \to hg$. This corresponds to a gauge transformation for the connection 1-form, $A_\mu(x)$. The curvature of this bundle connection then turns out to be the Maxwell field tensor F_{ab}.

• **principle of complementarity:** The predictions of general relativity will coincide with the Newtonian gravity when there particles have small mass and speeds are small compared to the speed of light c.

• **problem of time:** The problem of time in quantum gravity is a bit tricky to describe, since it takes different guises in different approaches to quantum gravity. (beaz week 41)

In a time reparameterization invariant system of General Relativity there is no time evolution - "nothing happens" in the theory in obvious contradiction to what we observe.

Newton's absolute time is an apparently indispensable notion in standard quantum mechanics. But GR teaches us that there is no external time - no true time by which mechanics evolves. There is only relative evolution of variables. Time must be promoted to the same status with other variables.

In the first chapter we saw that reference bodies are necessarily part of this dynamical system under study - so system is closed and all dynamical variables behaves quantum mechanically - all variables at the same level. In the context of closed systems where everything behaves quantum mechanically. also in quantum cosmology close to the big bang.

• **projected spin network:**

• **proper length:** a proper length of an object is its length as measured in the rest system of the object????????

• **proper time:**The label of "the problem of time" is often given to a number of related, but slightly different isues.

Briefly put, the problem of time is as follows: how is one to apply quantum mechanics to general relativity in which a classical non-dynamical background time is missing?

the proper time of an object is the time measured by a clock located in the rest system of the object.

• **Pullin, Jorge**

• **qef representation (quantized electric Flux representation):** Analogue of spin network states for background independent Maxwell theory. Electric field flux is quantized. The analogous result in $SU(2)$ LQG theory , with its geometric interpretation, corresponds to quantized area.

• **quadrupole moment:** Einstein already in 1917 worked out for quadrupole

Figure A.36: Jorge Pullin.

$$Q_{ij} = \int d^3x' \rho(\mathbf{x}')(3x_i' x^{j'} - r'^2 \delta_{ij})$$

See moments.

• **quantization:** [arXiv: quant-ph/ 0412015]. Quantisation is the problem of deriving the mathematical framework of a quantum mechanical system from the mathematical framework of the corresponding classical mechanical system. A method of quantisation must contain a map $\mathcal{A}$ from the set of classical observables to the set of quantum observables with the following properties:

The theories exist in their own right and perturbation methods serve as approximation techniques to extract answer to "physically interesting questions."? hepth9408108

• **quantum causal histories:** [367]

???Sorkin. Postulating that the fundamental underlying structure is a discrete causal. Attempts to recover the continuum picture by suitable coarse graining.???

• **quantum configuration space:** In quantum mechanics systems with a finite number of degrees of freedom, states are represented by functions on the classical configuration space. However, in field theories quantum states are on a larger space - the *quantum configuration space.*

For systems with a finite number of degrees of freedom, the classical configuration space also serves as the space of independent variables of the wavefunction of the quantum theory; we say that the configuration space is then the quantum configuration space.

For systems with an infinite number of degrees of freedom, on the other hand, the quantum configuration space is an enlargement of the classical configuration space. The free field theory in Minkowski spacetime while the classical configuration space can be built from suitably continuous fields, the quantum configuration space includes all tempered distributions.

$$\int [\mathcal{D}\varphi] e^{iS} \Phi(x)\Phi(x) \tag{A.133}$$

Typical measures used in quantum field theories are. The classical configuration spaces are contained in sets of measure zero. As a result, the action S inside the functional integral is ill-defined

In the Hilbert space language, this is the origin of field theoretic infinities, e.g., the reason why we can not naively multiply field operators.

The presence of an infinite number of degrees of freedom causes only one major modification: the classical configuration space $\mathcal{C}$ of smooth fields is enlarged to the quantum configuration space $\mathcal{S}'$ of (tempered) distributions. Quantum theoretical difficulties associated with defining products of operators can be directly traced back to this enlargement [34].

This enlargement from $\mathcal{A}$ to $\overline{\mathcal{A}}$ which occurs in the passage to the quantum theory is very similar to the enlargement from $\mathcal{C}$ to $\mathcal{S}'$ in the case of scalar fields. This enlargement plays a key role in the quantum theory (especially in the discussion of surface states of the quantum horizon).

• **quantum cosmology:** A theory that attempts to describe the whole universe in quantum-mechanical terms.

• **quantum field theory: Background *dependent* quantum field theory:** In spite its many practical successes where predictions from calculations have been shown to agree with experiment with phenomenal accuracy, QFT is mathematically a problematic construction.

very naturally defined, that we are however unable to calculate without incurring infinities.

can be reformulated in terms of classical statistical field theory.

Background *independent* quantum field theory:

• **quantum field theory on curved spacetime:** There is often confusion between a dynamical theory *on* a given curved spacetime with the dynamic theory *of* the spacetime, which is what GR is really about.

From [304]:

"Approximation methods for gauge invariant observables provide an explicit way to construct observables and may also help to test (quantum) interpretation of these as well as to discuss phenomenological implications [8, 9]. The approximation scheme developed in this work shows explictly how to express local observables in a gauge invariant way and therefore indicates how one could understand quantum field theory on curved space time as an approximation to the full theory of quantum gravity."

• **quantum gauge fixing:** An approximation scheme

all the degrees of freedom are fluctiating but the unphysical ones' fluctuations are suppressed by the semiclassical state.

• **quantum mechanics:**

• **quantum logic reconstruction:** Quantum logic reconstruction attempts

From the single premise that the "experimental propositions" associated with a physical system are encoded by projections in the way indicated above, one can reconstruct the rest of the formal apparatus of quantum mechanics. The first step, of course, is Gleason's theorem, which tells us that probability measures on $L(H)$ correspond to density operators.

Taken directly from http://plato.stanford.edu/entries/qt−quantlog/ (1) implication ($Q_1 \Rightarrow Q_2$),

(2) union ($Q_3 = Q_1 \vee Q_2$), and

(3) intersection ($Q_3 = Q_1 \wedge Q_2$).

(1) always false (Q_0) and always true (Q_∞)

(2) negation of question ($?Q$)

(3) orthogonality: if $Q_1 \Rightarrow ?Q_2$ then Q_1 and Q_2 are orthogonal denoted ($Q_1 \perp Q_2$).

• **quantum tetrahedron:** A quantum tetrahedron is a single tetrahedron whose geometric variables are considered as dynamical and represented by operators on a quatum state space $\mathcal{H}$.

• **quintessentian dark matter:**

"Reinterpreting quintessential dark energy through averaged inhomogeneous cosmologies", [astro-ph/0609315].

"...Einstein General Relativity: the cosmic quintessence emerges in the process of interpreting the real Universe in a homogeneous context."

"...It consists in defining cosmologies that are homogeneous on large scales without supposing any local symmetry, thanks to a spatial averaging procedure. It results in equations for a volume scale factor that not only include an averaged matter source term, but also additional terms that can be interpreted as the effects of the coarse-grained inhomogeneities on the large scales dynamics. These additional terms are commonly named backreaction."

• **RAQ:** Refined algebraic quantization.

• **radar time:** Popularized by Bondi, *radar time* is used to define hypersurfaces of simultaneity in terms of physical clocks and light pulses, as such it should be independent of the choice of coordinates.

Consider an observer travelling on path $\gamma : x^a = x^a_{(\gamma)}(\tau)$ with proper time τ. Define

$\tau^+(x)$:= (earliest possible) proper time at which light ray (null geodesic) leaving the point x could intercept γ.

$\tau^-(x)$:= (latest possible) proper time at which a light ray could leave γ, and still reach the point x.

$\tau(x) := \frac{1}{2}(\tau^+(x) + \tau^-(x))$ is the *radar time.*

$\tau(x) := \frac{1}{2}(\tau^+(x) - \tau^-(x))$ is the *radar distance.*

$\Sigma_{\tau_0} := \{x : \tau(x) = \tau_0\}$ is the observer's *hypersurface of simultaneity at time* τ_o.

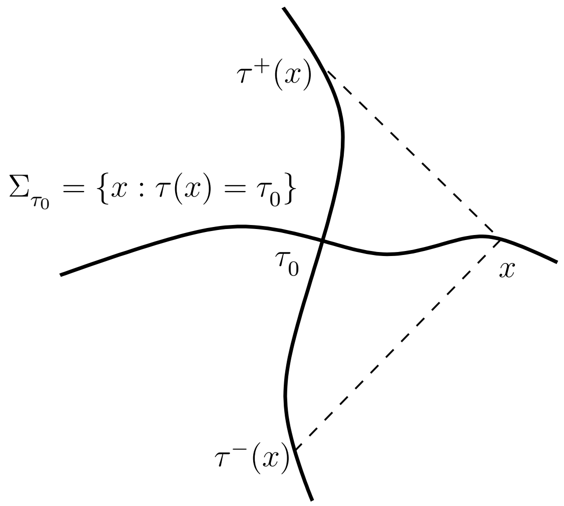


Figure A.37: radartimeF. Schematic of the definition of radar time $\tau(x)$.

• **Raychauduri equation:** Equation that governs the expansion of an infinitesimal bundle of geodesics. There are separate versions for time-like and null geodesics.

Play a central role in the proof of the singularity theorems.

• **real formalism:** With the introduction of a generalized Wick transform to map the constraint equations of Riemannian general relativity to those of the Lorentzian theory.

• **recombination:** This is the moment, in the history of the universe, when free electrons recombine with protons to form hydrogen atoms. Matter ceases to be ionized, and decouples from electromagnetic radiation. This defines the period of decoupling, a million years or so after the Big Bang.

• **reconstruction problem:** how do we extract local geometry and dynamics from the global nature of observables. Observables are to be invariant under active diffeomorphisms. local from non-local observables.

• **reconstruction of quantum mechanics:** See quantum logic reconstruction.

• **reduced phase space:**

• **reduced phase space quantization:** Solve the constraints at the classical level. Quantize without any constraints. Given two 3-metrics one would need to know if one is a time-evolved version of the other, this requires solving for the dynamics. A reduced phase space quantization of gravity would require finding the general solution of the classical field equations.

In the reduced phase space quantization one first performs a gauge fixing and then quantizes. That is one has to represent teh algebra of gasuge fixed functions equipped with teh Dirac bracket as (self-adjoint) operators on a Hilbert space.

"In her recent work, Dittrich generalized Rovelli's idea of partial observables to construct Dirac observables for constrained systems to the general case of an arbitrary first class constraint algebra with structure functions rather than structure constants. Here we use this framework and propose how to implement explicitly a reduced phase space quantization of a given system, at least in principle, without the need to compute the gauge equivalence classes."

[301] based on the framework constructed in [300]

the ultimate Hilbert space must carry a representation of the Poisson algebra of Dirac observables.

• **refined algebraic quantization (RAQ):**

RAQ, as it so far formulated, cannot rigorously deal with constraints which involve nontrivial structure functions since the group averaging really requires an honest (Lie) group structure.

RAQ has actually two implementations: A heuristic version, called group averaging, and a rigorous version, using Rigged Hilbert Spaces.

• **reflection positivity:** Let θ denote the action of the time reflection, i.e. reflection with respect to the hyperplane

$$(x^0 = 0, x^1, \ldots, x^d), \tag{A.134}$$

then θg is the time reflection of the function g. Consider functions of the form

$$f(\phi) = \sum_{j=1}^{N} c_j e^{i\phi(e_j^+)}, \tag{A.135}$$

where c_j is a complex number and e_j^+ are arbitrary Schartz functions that are non-zero in the $x^0 > 0$ half-space. Then reflection positivity of the measure $\mu(\phi)$, is the condition that

$$< \theta f, f >_{L_2} = \int_{S'} (\theta f)(\phi)]^\dagger f(\phi) d\mu(\phi) \geq 0. \tag{A.136}$$

It is the axiom of Osterwalder-Schrader theorem that allows one to construct a physical Hilbert space of the canonical quantum field theory from the Euclidean one with a non-negative Hamiltonian acting on it.

• **Regge manifold:** A d-dimensional Regge manifold is a metric space composed by glued flat d-simplices, with matching geometry at the intersections.

• **Regge model:** One approximates a curved spacetime by a large collection of simplices glued together. The interior of every simplex is flat, the spacetime curvature resides in how the simplices are glued to each other.

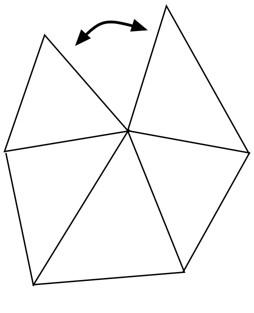


Figure A.38: ReggeGlos. Gluing flat simplices to get a surface with curvature.

Regge Lagrangian:

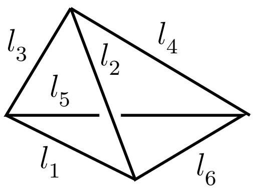


Figure A.39: Regge Lagrangian.

denote the lengths of its edges by l_i ($I = 1, 2, \ldots, 6$) (Fig E.7). The Regge action for the tetrahedron is

$$S_{Regge} = \sum_{I=1} l_i \theta_i, \tag{A.137}$$

where θ_i is the angle between the outward normals of two faces sharing the I-th edge.

• **reheating:** Process whereby the period of inflationary expansion gives way to the standard hot big bang scenerio.

• **relational notion of space:** Aristotle to Decartes picture of space where described by the conjunuity of objects: there is no notion of space without matter.

• **relational interpretation of quantum mechanics:** Rovelli's relational quantum mechanics. resolves the paradox of superpositions by making it a consequence of one' point of view.

It does not try to append a reasonable interpretation to the quantum mechanics formalism, nor to postulate as yet unknown new physics, rather it attempts to uncover the physical meaning quantum mechanics. One drops the assumption that quantum states are observer independent.

In the relational interpretation, the very notion of state of a physical system should be considered meaningless unless it is understood to be relative to another system that plays the role of one possible observer. No implication that the observer is human or has any other a priori special properties only that it has the possibility of interacting with the 'observered ' system.

It is the specification of such an observer that allows the ascription of a state to a system to make sense. So the description the theory provides is complete because it exhausts everything that can be said about the world.

A physical quantity has value with respect to certain observer.

The cat finds himself in one or other of the possible outcomes while at the very same time according to the person outside the box the cat is in a quantum superposition. People say the two can't be true at the same time so we have a paradox. However this discord does not impinge on you because when to different observers can ask the same question they get the same answer.

That the photon has been absorbed (or not) according to the cat isn't to say that it has been (or has not been) absorbed as far as anyone else is concerned - it is not observer independent, that is it is not an absolute property of the system. In this interpretation there is no need for special systems that collapse the wavefunction for everyone else's benefit.

There is a long history in physics of progress being made by abandoning the belief that there are preferred systems and by 'relationalizing' notions that were previously thought to be absolute. The description provided by the theory exhausts everything that can be said about the world.

What is stopping us from taking this as a description of reality is a philosophical prejudice and nothing more. We should adjust our philosophical prejudices to what nature is telling us. We shouldn't be trying to fit reality into our preconceived notions about the world.

http://plato.stanford.edu/entries/qm-relational/

consistent discretizations

context created by the presence of an observer

different descriptions as there are as there are contexts, but with interrelations between them so that if two observers can ask the same question they get the same answer.

• **Relational quantization *a la* Page-Wootters:** [47]

• **relational quantum cosmology:**

[29] Slogan: "Many quantum states to describe one universe, not one state describing many universes".

• **relativistic spin networks:**

• **renormalization:**

renormalized mass and charge values are inserted into the theory as unexplained parameters. Coupling constants of various kinds and masses of the basic quarks and leptons, the Higgs particle, etc. need to be specified.

Any acceleration of a charged particle entails the acceleration of part of the charged particles surrounding it, so the mass m that you observe is the mass m_0 of the particle plus a correction Δm, due to its interaction with the cloud of charged particles. Procedure of replacing infinities with experimentally determined values. a quantum field theory whose Lagrangian has a few free parameters - masses and charges and so. These quantities are refered to as the "coupling constants".

with obvious physical meaning cannot be calculated without incurring divergencies.

Renormalization occurs in evaluating physical observable quantities which in simple terms can be written as formal functional integrals of the form

$$\int [\mathcal{D}\phi] e^{\mathcal{L}(\phi,\partial\phi)} A(\phi,\partial\phi) \tag{A.138}$$

where $\mathcal{L}$ is composed by two parts, a part quadratic in ϕ and $\partial\phi$ term, $\mathcal{L}_0$, and an 'interaction' term, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$. Computing such an integral in perturbative terms leads to a formal power series, each term G of which is obtained by integrating a polynomial under a Gaussian $e^{\mathcal{L}_0}$:

$$\int [\mathcal{D}\phi] e^{\mathcal{L}_0(\phi,\partial\phi)} (1 + g\mathcal{L}_{int}(\phi,\partial\phi) + \frac{g^2}{2!}[\mathcal{L}(\phi,\partial\phi)]^2 + \ldots) A(\phi,\partial\phi) \tag{A.139}$$

Such Feynmann diagrams G are given by multiple integrals over spacetime or, upon Fourier transformation, over momentum space, and are typically divergent in either case. The renormalizatoin technique consists in adding counterterms $\mathcal{L}_G$ to the original Lagrangian $\mathcal{L}$, one for each diagram G, whose role is to cancel the undesired divergencies.

In the case of a renormalizable theory, all the necessary counterterms can be obtained by modifying the numerical parameters of the original Lagrangian.

considered to be ad-hoc

Kreimer there is a Hopf algebra underlies quantum field theory renormalization, and has led to a new approach to perturbative renormalization under development.

The antipode of a diagram is related to the counterterms necessary to subtract its divergencies.

renormalization - background independent: The regularized Hamiltonian operator is independent of the regularization parameter ϵ to first order - the regularization parameter has disappeared from the expresion! Hence no renormailization is involved! The reason why the regularization parameter does not appear is, roughly speaking, because the shrinking process, $\epsilon \to 0$, can be compensated for by an active spacial diffeomorphism.

Tentative ideas have been formulated by Markopoulou [122], [123] and Oeckl [124].

renormalization *a la* Fontini-Markopoulou [122], [123]

$\gamma_1 \subset \gamma_2$ $\gamma_2 \subset \gamma_1$ $\gamma_1 \cap \gamma_2 = \emptyset$

let γ denote a proper sublattice of Γ, namely $\gamma \neq e$ and $\gamma \neq \Gamma$. We call the lattice that is left after we "cut out" γ the **remainder** and denote it Γ/γ.

$$\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p \tag{A.140}$$

These are the **primitive elements** of the Hofp algebra.

The **counit** is an operation which annihilates every lattice except e.

renormalization *a la* R. Oeckl [124]

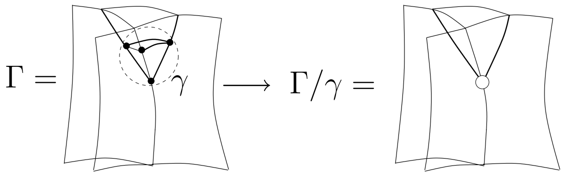


Figure A.40:

• **renormalization group:** A method designed to describe how the dynamics of a system change as we change the scale at which we probe it.

• **renormalized charge:** bare charges divergent, absorbtion of infinites into a redefinition of the charges. screening effects (see effective charges).

• **Ricci identity:**

$$2\nabla_{[a}\nabla_{b]}V_c = R_{abcd}V^d \tag{A.141}$$

• **rigged Hilbert space - also called the Gel'fand triple:** In infinite dimensional Hilbert space there can be operators that have a continuous spectrum, (the most simple example would be that of a free electron which has a continuous momentum spectrum). The corresponding eigenvectors are not normalizable, and hence cannot lie inside the Hilbert space $\mathcal{H}$, (the eigenvector of the momentum operator of a free electron is a plane wave e^{ikx}, which isn't normalizable). The rigged Hilbert space approach to dealing with such operators on or a Gel'fand triple $\{\Phi, \mathcal{H}, \Phi'\}$, in which the Hilbert space $\mathcal{H}$ appears as a linear subspace Φ' which is not itself a Hilbert space. Eigenvectors associated with a continuous spectrum belong to Φ' but are no members of its subspace $\mathcal{H}$. The space Φ' is constructed as a type of dual of another vector space Φ which is a subspace of $\mathcal{H}$. Thus we have $\Phi \subset \mathcal{H} \subset \Phi'$.

• **Riemannian geometry:** A generalized geometry that has the property of being locally flat; that is, in a sufficiently small region, a Riemann geometry can be approximated by a Euclidean or Minkowski geometry.

• **Robertson-Walker metric:** The metric that describes an isotropic and homogeneous cosmological space time.

• **Rovelli, Carlo:**

• **Sandwich Conjecture:** The conjecture that, given two spatial metrics q and q' on two hypersurfaces then there is unique metric that will interpolate between them, up to gauge.

• **scale factor:** The quantity that describes how length change in the expanding (or contracting) universe.

Figure A.41: Carlo Rovelli.

• **scattering amplitudes:**

There is no given *a priori* geometry here, as is the case in background dependent theories. The geometry is encoded in the quantum state itself, not given a priori as in conventional QFTs.

$$\int \mathcal{D}\varphi_I \int \mathcal{D}\varphi \tag{A.142}$$

how can any physics possibly come out of the setup here if the amplitude doesn't change when deforming the boundary surface - that is, until you realize that the metric is one of the boundary fields!

$$< \Psi_1|\Psi_2 >= \int \mathcal{D}\varphi_m \Psi_1^*[\varphi_m]\Psi_2[\varphi_m], \quad |\Psi_1 >, |\Psi_2 >\in \mathcal{H}_{\Sigma_m}. \tag{A.143}$$

The physical information is contained in the spectrum of the boundary operators of partial observables.

• **scattering matrix:** The scattering matrix allows the to calculate experimentally observable scattering cross sections.

• **Schödinger equation:** The equation that describes the evolution of a nonrelativistic wavefunction.

• **Schödinger picture:**

• **Schwarzschild radius:** The radius of the event horizon of a nonrotating black hole of mass M, equal to $2GM/c^2$.

• **secondary constraint:** Dirac bracket substituting for the Poison bracket

• **second-quantization:** The passage from a one particle system with an arbitrary number of particles subject to the Bose/Fermi statistics is called second-quantization.

$\hat{\mathbf{\Psi}}(\mathbf{x})$ does not represent the actual quantum state it is the operator that creates a new particle having the one-particle wavefunction $\psi(\mathbf{x})$, by application on the ground state $|0>$ (the state representing the absence of any particles).

• **self organized criticality:** Self organized critical systems are statistical systems that naturally evolve without fine tuning to critical states in which correlation functions are scale invariant. The earliest example of such a system is the sandpile of Bak, Tang and Wiesenfeld [258]. Since then, many such systems have been studied, including models of phenomena as diverse as biological evolution, earthquakes, astrophysical phenomena and economics [259].

It has been suggested that the emergence of classical spacetime from a discrete microscopic dynamics may be a self-organized critical process, [257].

• **semi-classical quantum gravity:** Semi-classical quantum gravity refers to the setting in which a classical background geometry is coupled to the quantum matter field through the semiclassical Einstein equations:

$$G_{\mu\nu} = -8\pi < T\mu\nu > .$$

It is a perturbative approach where there is a well-defined classical background geometry about which the quantum fluctuations are occuring.

• **separating:** Functions of phase space $\mathbb{R}^{2n}$ - observables. the energy and momentum functions on phase space is sufficient know to find out a which particular point in phase space the system is at.

these functionals form a separating set on $\mathcal{A}/\mathcal{G}$: if all the T_α assume the same values at two connections, they are necessarily gauge related.

• **shadow states:** It would help in assesing the viability of LQG providing the correct semi-classical limit if we closed the gap between its one dimension excitation description with the usual Fock space description, in the low energy limit cutoff.

The problem with the polymer represenation is that, Cly^*, the space containing the solutions to the quantum constraint equations, does not have a physically justified inner product; a definite Hilbert space structure is not yet available. While the kinematic states belong to Cyl, the physical states belong to Cly^*. Therefore, semi-classical states, capturing the low energy physics, should also be in Cly^*. Can we still test a candidate state in Cyl^* for semi-classicality at this stage without acces to expectation values? One uses the notion of *shadow states* to do semiclassical analysis in the polymer represenation.

Shadow states use graphs as 'probes' to extract physical information from elements $(\Psi|$ of Cly^*. More precisely, one 'projects' each $(\Psi|$ to an element $|\Psi_\gamma^{shad}>$ in Cyl_γ, a so-called shadow state, and analyses properties of $(\Psi|$ in terms of its shadows $|\Psi_\gamma^{shad}>$. Each shadow captures only a part of the information contained in our state, but the collection of shadows can be used to determine the properties of the full state in Cly^*.

• **simplicial spin-foam:** Standard spin-foam models as formulated up to now are piecewise flat geometries defined on piecewise linear manifolds. For LQG canonical theory to match these models it would have to be restricted to the category of piecewise linear manifolds and piecewise linear spin-networks. A suitable generalisation of simplicial spin-foam models to lift this restriction is linear 2-cell spin-foam models.

• **simplicity constraints:** Imposing the simplicity constraints turns the topological BF theory into (non-topological) 4-dimensional gravity.

• **simple regions:** A simple region of spacetime $\mathcal{M}$ is a geodesically convex set with compact closure, whose boundary is diffeomorphic to the three shpere S^3.

• **singularity:** the inevitable presence of singularities in general relativity of the classical gravitational being always attractive.

A singularity is a point in a spacetime which the curvature becomes infinite. Robrtson-Walker, Schwarschild, Reissner-Nordstrom and Kerr are examples of solutions contain points of inifinite curvature. The presence of infinite curvature lets itself know by timelike or null geodesics coming to a full stop. In the context of the singularity theorems, the working definition of a singularity is a kind of incompleteness of the space-time under consideration, more precisely, it is an obstruction of some sort to timelike or null geodesics from being indefinitely extendible. Investigate the nature of singularities.

We require a smooth Lorentz metric. You cannot have point on the manifold with singular metric. So the points where this can happen are excised.

• **singularity theorems:** Proved by Hawking and Penrose. The singularity theorems are based on very powerful indirect arguments which show that black hole and cosmological singularities are generic in classical general relativity. Gives us confidence in the big bang. The classical theory predicts its own breakdown. One of the main motivations for attempting to quantize general relativity.

A feature of the singularity theorems is that they do not directly show the existence of black holes. Instead they show that spacetime is geodesically incomplete so that the worldline of an observer comes to an end and cannot be extended. The obstruction to extending the worldline is some kind of singularity, but this might be rather mild and need not correspond to a black hole.

If there is generically in spacetime a closed, trapped region, energy certain positivity and if gravity remains attractive a singularity is inevitable. the conclusion is that there is to be some some sort of obstruction to timelike or null geodesics from being indefinitely extendible.

• **slow-roll approximation:**

from gr-qc/0511007:

The mechanism, proposed in [], considers so called slow-roll approximation, when the inflation rolls down its potential hill towards the potential minimum, but, at the same time, remaining far away from the minimum. It is exactly in the vicinity of the minimum of the potential, where the inflation stops.

• **Smarr's formula:** Smarr's formula translates black hole mass M to its angular momentum J, angular velocity Ω_H, surface gravity κ, and surface area A:

$$M = 2\Omega_H J + \frac{\kappa A}{4\pi}. \tag{A.144}$$

• **Smolin, Lee:**

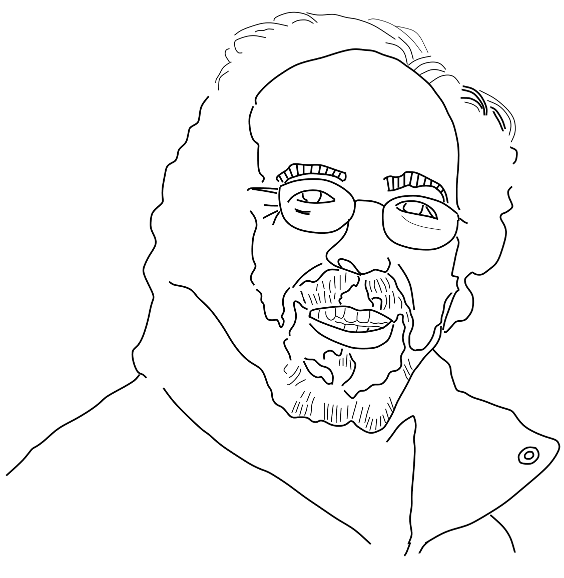


Figure A.42: Lee Smolin.

• **Smolin's Darwinian hypothesis:** Smolin put forward the proposal that the universe fine tunes the values of its physical constants through a Darwinian selection process [184]. Every time a black hole forms, a new universe is developed inside it that has different values for its physical constants from the one in its progeitor.

[125] not just based on *"authors intuition"*

• **space of solutions:** The space of solutions are taken to be the phase space or space of states. This definition is independent of any special time choice so that it is manifestly covariant.

• **spacial averaging problem:**

• **spacial diffeomorphism invariance:**

• **spacetime singularity** The working definition of a spacetime singularity is a point in spacetime beyond which null or timelike geodesics cannot be extended. We mean the singularity

• **special relativity:**

(i) **The principle of relativity**. The laws of physics are the same for all inertial reference frames.

(ii) **The constancy of the speed of light.** The speed of light in a vacuum is the same for all inertial observers irrespective of the motion of the source.

• **Speziale, Simone:** Work on spin foams and the graviton propagator in loop quantum gravity.

Figure A.43: Simone Speziale.

• **spin foam:** Spin foam models are an attempt to describe the geometry of spacetime at the quantum level. They give a construction of transition amplitudes between initial and final spacial geometries labelled by spin networks.

???the spin foam is independent of any special time choice so that it is manifestly covariant???

Spin foams can be understood to be dual to triangularizations.

In this sense the physical state is associated with an entire history of the system, rather than any point on that history.

• **spin network:** A spin network is a generalization of a knot or link. It is a graph embedded in space, with edges carrying labells corresponding to an irreducible representation of a Lie group, and vertices with labells corresponinding to interwining operators.

• **spin network state:** They diagonalize geometric operators and describes quantized three geometry. They provide a complete orfonormal basis for the kinematic Hilbert space. Using spin networks we are working with finite dimensional spaces that have the structure of Hilbert space of spin systems - the same mathematics at work as in ordinary quantum mechanics of angular momentum but with completely different interpretation. They are natural candidates for cylindrical functions in the construction of well defined measures of integration. The scalar product has a very simple form $< \Phi_\alpha \Phi_\beta >= \delta_{\alpha\beta}$. This make easy at the time of calculations.

Spin networks states provide a natural decomposition of gauge-independent kinematic Hilbert space, $\tilde{\mathcal{H}}^0$, into finite dimensional subspaces each of which can be identified with the space of states of a spin-system. This simplifies various constructions and calculations enormously.

Using spin networks we are working with finite dimensional spaces that have the structure of Hilbert spaces of spin systems.

• **spinor:** It is well known that the Pauli matrices satisfy the commutation relations of the rotation algebra,

$$[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k \tag{A.145}$$

They therefore are a representation of the rotation algebra; they generate infinitesimal rotations on a two component vector - this is the 2-component spinor. Finite rotations are obtained by the exponentiating of (E.7),

$$\begin{aligned} \hat{U}_R &= \exp(-i\phi\cdot\tau) = \exp(-i\frac{1}{2}\phi\mathbf{n}\cdot\tau) \\ &= \mathbf{I}\cos(\frac{1}{2}\phi) - i\mathbf{n}\cdot\tau\sin(\frac{1}{2}\phi). \end{aligned} \tag{A.146}$$

The defining property of a spinor is the way it transforms under rotations (or Lorentz transformations depending on the context).

• **Standard model:** particle physics. $SU(3)\times SU(2)\times U(1)_Y$. The Standard Model is not a mathematically consistent theory. It has had undeniable success perturbative calculations of collision cross section, the numerical analysis of particle spectra etc.

• **State:**

There is a second use of this term. A collection 'observables' and is capable of being in certain 'states'. We can define the state of a system as knowledge of the expected values of the observables, that is, a state is an assignment of an expected value of each observable.

A second use of this term is *an expectation value functional.* This is to be chosen at some point in the quantization.

$$F_{\mathrm{Sch}}(W(\zeta)) := e^{-\frac{1}{2}|\zeta|^2}. \tag{A.147}$$

• **state sum model:** [411]

• **Stone-von Neumann uniqueness theorem:**

$$\begin{aligned} \{q, p\} &= 1, \\ \{q, q\} = \{p, p\} &= 0. \end{aligned} \tag{A.148}$$

$$[q, p] = i\hbar, \ [q, q] = [p, p] = 0. \tag{A.149}$$

Now we find concrete repesentation of this algebraic relation. Scrödinger's wavefunction representation

$$\hat{q}\psi(q) = q\psi(q), \quad \hat{p}\psi(q) = -i\hbar\frac{d\psi(q)}{dq} \tag{A.150}$$

$$[\hat{q}, \hat{p}]\psi(q) = (-i\hbar q\frac{d}{dq} + i\hbar\frac{d}{dq}q)\psi(q) = i\hbar\psi(q) \tag{A.151}$$

Heinsenberg's matrix mechanics every observable physical quantity, that is energy, position, momentum, angular momentum, is described by an operator represented as a matrix.

Scrödinger's wavefunction representation and Heinsenberg's matrix mechanics describe the same physics. Are essentially unitary equivalent.

In quantum field theory there is no analog of the Stone-von Neumann uniqueness theorem. Hence, there are an infinite number of inequivalent representations of the Poisson bracket algebra. These inequivalent representations can be thought of as different "phases", which have different physics.

The Fock representation the natural generalization of the Schrödinger representation to infinitely many degrees of freedom. Quantum field theories written in the Fock representation by theoretical physicists.

• **string theory:** all known particles and their interactions are supposed to emerge as certain modes of excitation of and interactions between quantized strings.

String theory attempts to produce a theory of everything, including quantum gravity that will have general relativity as part of the classical limit.

a little “far-out”. String theory is a perturbative approach depending on the choice of background metric. background independent M-theory for strings , Witten using twister space methods in which physics happens in twister space and spacetime is a secondary construction.string field theory.

We find that the massless closed quantum states include one-particle graviton states, making string theory a quantum theory of gravity!

Except string theory can only be defined over stationary spacetimes, which have measure zero in the space of solutions to Einstein’s equations - a timelike Killing vector field is needed to have spacetime suppersymmetry, with out it you get unphysical tachions in the spectrum. dynamical theory over spacetime confused with a dynamical theory of spacetime.

A necessary condition for a perturbative theory to be consistent is that the two dimensional world sheet quantum field theory that defines the theory be conformally invariant. This means that the conformal anomaly on the two dimensional worldsheet vanishes. To leading order in l_{string} this condition is equivalent to the Einstein equations of the background manifold.

The so called Pohlmayer string can be quantized in any number of dimensions, including four dimensional Minkowski space [**?**].

Background dependency: (arXiv:hep-th/9808192)

in which the definitions of the states, operators and inner product of the theory require the specification of a classical metric geometry. The quantum theory then describes quanta moving on this background. The theory may allow the description of quanta fluctuating around a large class of backgrounds, but nevertheless, some classical background must be specified before any physical situation can be described or any calculation can be done. All weak coupling perturbative approaches are background dependent, as are a number of non-perturbative developments. In particular, up to this point, all successful formulations of string theory are background dependent.

String theory only studies semiclassical effects in fixed classical spacetime backgrounds.

• **strongly causal:** There are several types of causality violations that can be are considered, other than the existence of closed timelike curves. Strong causality is an important example. Roughtly speaking, strong causality is said to fail if an arbitrarily small perturbation of the metric will result closed timelike curves, that is, the chronology condition is almost violated. Strong causality plays a key role in the Hawking-Penrose singulairt theorem. Important there is the result: Let $\mathcal{M}$ be a spacetime satisfying: (1) There are no closed trips. (2) Every endless null geodesic in $\mathcal{M}$ contains a pair of conjugate points. Then $\mathcal{M}$ is strongly causal.

see future-distinguishing spacetimes, stablely causal

• **supergravity:**

Bosonic and fermionic Feynman diagrams cancel each other out.

• **superselection:** A representation is irreducible if the Hilbert space does not split into orthogonal subspaces preserved by the action of $\mathcal{A}$. Equivalently, there must exist in the Hilbert space a cyclic vector whose image under the action of $\mathcal{A}$ is dense in the Hilbert space. Physically this requirement rules out the existence of superselection rules: superselection sectors can be identified with the irreducible sectors of a reducible representation.

The representation should be irreducible on physical grounds (otherwise we have superselection sectors implying that the physically relevant information is already contained in a closed subspace).

• **superspace:** the collection of all possible three-metrics (nothing to do with supersymmetry)

• **supersymmetry:**

Supersymmetry requires fermions and bosons to come in pairs consisting of one of each, with the same mass. This is not observed in nature.

• **surface gravity: Surface gravity** is the acceleration of a static particle near the horizon as a measured at spacial infinity.

$$\kappa = -\frac{1}{2}(\nabla^a \xi^b)(\nabla_a \xi_b)|_{\mathcal{N}} \tag{A.152}$$

• **symmetry:** In quantum mechanics group transformations that leaves invariant key structures in the theory - Hilbert space, algebra of observables, etc.

An unitary operator on Hilbert space preserves all scalar products, i.e.,

$$< \hat{U}\psi, \hat{U}\phi > = < \psi, \phi > \tag{A.153}$$

• **symplectic form:** $\omega_{\mu\nu}$ is antisymmetric and also has the properties

$$\omega^{\mu\nu} = \{\xi^\mu, \xi^\nu\} \tag{A.154}$$

(i) It is antisymmetric $\omega_{\mu\nu} = -\omega_{\nu\mu}$,

(ii) is non-degenerate, i.e., $\omega_{\mu\nu} v^\nu = 0$ implies $v^\mu = 0$, that means that the inverse matrix $\omega^{\mu\nu}$ exists,

(iii) it is closed , i.e., $\partial_\mu \omega_{\nu\gamma} + \partial_\nu \omega_{\gamma\mu} + \partial_\gamma \omega_{\mu\nu} = 0$ which is equivalent to the fact that the Poisson bracket satisfies the Jacobi identity.

$$\omega = \frac{1}{2}\omega_{\mu\nu} d\xi^\mu \wedge d\xi^\nu \tag{A.155}$$

• **tempered distributions:** they are continuous (or equivalently - bounded) linear maps from the Schwarz space of functions to complex numbers. In other words, it is the topological dual space to the of $\mathcal{S}(\mathbf{R}^n)$

• **tensors:** The point about tensors is that when we want to make statement we do not wish it to hold in just one coordinate system but rather all coordinate systems.

• **tensor calculus:**

• **test particle:** is an idealized point particle with energy and momentum so small that its effects on spacetime are negligible.

• **thermal clocks:** Clocks that change linearly with respect to thermal time. (Rovelli)

• **thermal field theories:**

• **thermal time hypothesis:** The dynamical laws of generally covariant systems determine correlations observables, but they don't single out any of the observables as that we would call "time".

two equivalent ways of describing the time flow: either the flow in state space (Schrödinger picture), or (generalized Heisenberg picture)

So we say time flow is as a one-parameter group of automorphisms of the algebra of observables.

the physical basis of time [290] [291].

$$\mathcal{H} = -kT \ln D \tag{A.156}$$

The thermal states of a field theory are characterized by have a correlation function, the KMS condition (A.89)

• **thermodynamics:** Thermodynamics deals with large systems in terms of macroscopic observables alone.

know as the zeroth, first and second laws

zeroth:

first:

second:

Bekenstein suggested that the second law of thermodynamics should be extended in the presence of black holes.

• **Thiemann, Thomas:**

Figure A.44: Thomas Thiemann.

• **Thomson scattering:** When a photon scatters off a free electron, in the nonrelativistic case $\hbar f << m_e c^2$.

• **tidal force:**

• **top-down:** the is required. Loop quantum gravity is an example. One of the biggest challenges for the theory, most be established before it can be claimed to be a viable theory of quantum gravity.

• **time:** ??

• **timelike membranes:** Their definition is the same as a dynamical horizons except that they are timelike instead of spacelike. [149]

• **topological quantum field theories:**

They have a finite number of degrees of freedom so are really mechanical systems disguised as field theories. they posses certain features one expects of a quantum theory of gravity.

• **trace class operator:** nuclear operator.

• **trapped surfaces:** A trapped surface $\theta_{(\ell)} < 0$ and $\theta_{(n)} < 0$

marginally trapped surface: A marginally trapped surface $\theta_{(\ell)}, \theta_{(n)} \leq 0$

An **outer marginally trapped surface** $\theta_{(\ell)} \leq 0$. is the boundary of a three-dimension volume whose expansion of the outgoing family of null geodesics orthogonal to S is everywhere non-positive, $\theta_{(\ell)} \leq 0$.

An apparent horizon

A closed, spacelike, two-surface is an apparent horizon if it is the outermost marginally trapped surface.

trapped boundary a la Hayward:

trapped surface a la Penrose:

closed trapped surface:

• **twisted geometry:** A basis of states for LQG is given by the spin network states. A spin network state has support on a graph Γ and determines a 3d "quantum geometry"; they are eigenvectors of geometric operators which determine the intrinsic geometry. The extrinsic gemetry (analogous to the conjugate momentum) is completely "spread out", due to the Heisenberg uncertainty principle. To make contact with a semiclassical description of space, we can consider coherent states peaked (but not sharp) on both the intrinsic and exstrinsic geometry. A twisted geometry is a specific choice of "interpolating geometry", chosen among distcontinuous metrics. To any graph and any holonomy-flux configuration, we can associate a twisted geometry: a discrete discontinuous geometry on a cellular decomposition space into polyhedra. Thanks to this result, the phase space of LQG on a graph can be visualised not only in terms of holonomies and fluxes, but also in terms of a simple geometric picture of adjacent flat polyhedra.

• **twister theory:** invented by Sir Roger Penrose. : entire light-rays are represented as points, and events by entire Riemann spheres. Twistors defined in terms of a pair of spinors. Twisters are the coordinates of twister space. In twister theory spacetime is a secondary concept. Witten is keen on twister theory in developing background-independent string (M-)theory. Witten found that a lot of the beautiful results of twistor theory did not extend from four dimensions to higher dimensions. If Thiemann's LQG-string theory stands up to the critical examination, in four dimensions there could be possibilty of aplying the powerful results of twister theory to a background independent string theory?? [16]. Seminar given by Penrose (2004) *Goals and Achievements of Twistor Theory*, [444].

• **2+1 quantum gravity:** Witten Chern-Simons. See **??**.

In a vacuum in three-dimensions the vanishing of the Ricci tensor implies the Reinmann tensor vanishes

$$R_{ab} = 0 \implies R^a{}_{bcd} = 0 \tag{A.157}$$

so the solution of the equations of motion is that curvature vanishes.

This condition, which is imposed *a priori* in Einstein's formulation of general relativity, is seen to be part of the equations of motion.

$$\mathbf{R}^3 \otimes_S SO(3). \tag{A.158}$$

• **uncertainty relations:**

$$\Delta A \Delta B \geq \frac{1}{2}| < [\hat{A}, \hat{B}] > |, \tag{A.159}$$

where $\Delta A = \sqrt{< \hat{A}^2 > - < \hat{A} >^2}$.

• **unification:** String theory.

Could degrees of freedom corresponding to matter arise in some natural way from those of geometry?

Chakraborty-Peldan unified model [338]

"...If this difficulty could be overcome and the model could be made to include Ferimons via supersymmetry it would become a viable, elegant and mathematically well defined way of having a unified theory of quantum fundamental interactions."

Quantum Gravity and the Standard Model

Discrete quantum gravity: a mechanism for selecting the value of fundamental constants [328].

• **uniform distretization:** A particularly promising version of "consistent discretizations".

Several approaches to the dynamics of loop quantum gravity involve discretizing the equations of motion. Discretized theories are problematic since the first class algebra of constraints of the continuum theory become second class upon discretization. If one treats the second class constraints normaly, the resulting theories have very different dynamics and number of degrees of freedom than those of the continuum theory. It is therefore questionable how these theories could be considered an appropriate starting point for quantization and the definition of a continuum theory through a continuum limit. The uniform discretization approach is a proposal for the quantization of constrained systems which could overcome these difficulties and construct the correct quantum continuum limit.

• **uniqueness theorems:** See black hole uniqueness theorems.

• **unitarity:** to preserve probabilities

Unitarity implies that expectation values of gauge invariant observables does not depend on the gauge or frame of reference.

• **universe:** That which contains and subsumes all the laws of nature, and everything subject to those laws; the sum of all things that exists physically, including matter, energy, physical laws, spacetime.

• **Unruh effect:** An observer who accelerates uniformly through flat empty space will observe a thermal bath of particles, at a temperature given by their acceleration. This means that a state which is empty according to one observer will not be empty according to an accelerating observer, and hence demonstrates that the concept of "vacuum" must be observer dependent. Discovered by Unruh [385] and Davies [386].

• **vacuum:**

$$|0> \tag{A.160}$$

define the vacuum state as follows

$$a(k)|0>=0 \tag{A.161}$$

where

$$W(q_1, q_2, \dots q_n) =< q_1, q_2, \dots q_n|0> \tag{A.162}$$

where $< q_1, q_2, \dots q_n|$ are eigenstates of observable quantities.

$$\lim_{t\to\infty} W(\alpha, -it, \alpha', 0) \to H_0(\alpha) e^{-E_0 t} \overline{H_0(\alpha')} \tag{A.163}$$

$$\lim_{t\to\infty} e^{E_0}|0_{-it}>= |0_M><0_M| \tag{A.164}$$

• **vacuum expectations:**

$$G(x,y) =< 0|a(y)a^\dagger(x)|0 > \tag{A.165}$$

• **Vaidya:** The spherically solution to Einstein's equations with a null fluid as source.

describe the formation of a black hole through infalling null dust.

Simplest examples of spacetimes admitting dynamical horizons.

• **Varadarjan, :**

• **Varadarjan isomorphism:**

$$\sum_i a_i W_{\alpha_i}[A] = 0. \tag{A.166}$$

$$\sum_i a_i \alpha_i = 0. \tag{A.167}$$

• **Vidotto, Francesca:** Work inhomogeneous quantum cosmology and spin foam cosmology.

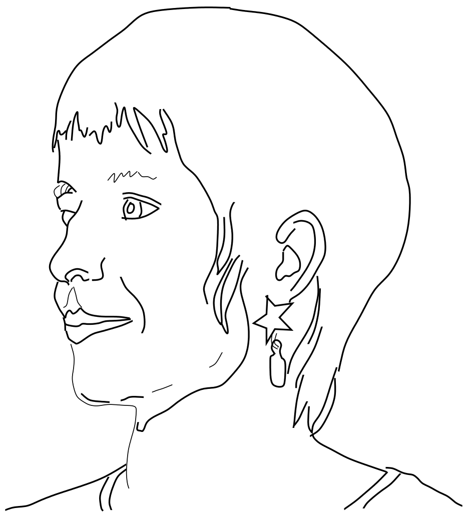


Figure A.45: Francesca Vidotto.

• **volume operator:** $\hat{\mathcal{V}}_{AL}$ $\hat{\mathcal{V}}_{RS}$

• **von Neumann, John:** Developed rigourous mathematical foundations to quantum mechanics. Improtant in quantum gravity where experimental guidence, so far, is missing. There is the Stone-von Neumann uniqueness theorem - a version of which was developed for LQG to prove the uniqueness of the kinematic representation of the holonomy-flux

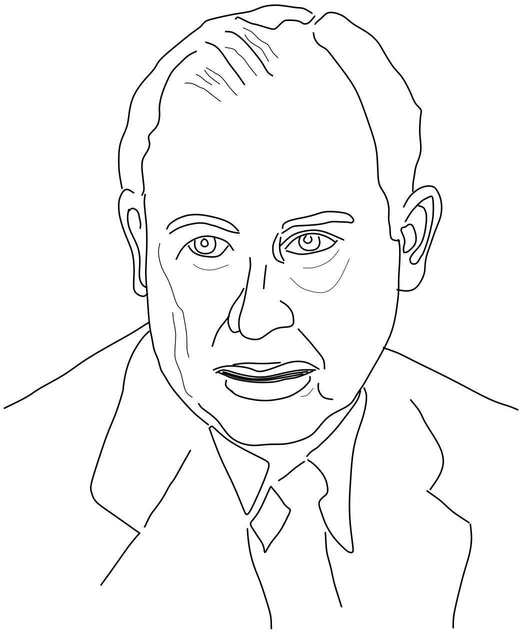


Figure A.46: John von Neumann.

algebra. Developed the infinite tensor product Hilbert space, applied by Thieman et al to quantum gravity in the semiclassical analysis and AQG.

[99], [100]

• **weak anthropic principle:** The weak anthropic principle is the principle whereby the existence of life explained by random selection from an emsemble of universes with differing properties.

• **weave states:** A superposition of spin networks that mimic a given classical spacetime metric. Spin network states are eigenstates of E, so they are like "momentum" eigenstates. So a weave state is maximally spread in the configuration variable $A^a(x)$. Hence they can not represent semiclassical states, just as $e^{ik\cdot x}$ can not represent a semiclassical wavefuntion of a particle because measuring momentum precisely results in complete uncertainty in position.

• **weak cosmic censorship:**

1. $\mathcal{I}^+$ and $\mathcal{I}^-$ are complete;

2. $I^-(\mathcal{I}^+)$ is globally hyperbolic.

• **Weiner-measure:** Brownian

• **Weyl algebra:** Translation $V(\mu)$

• **Weyl tensor:** The Weyl tensor is the conformaly independent part of the curvature tensor. It is the part of the curvature tensor describes the local degrees of freedom of the gravitational field.

• **Wheeler, John:**

Figure A.47: John Wheeler.

• **Wheeler-DeWitt equation:** the Hamiltonian constraint

all the dynamics is contained in constraints.

• **Wick:**

We take spacetime (or manifold?) to Riemannian, but physically, the spacetime is Lorentzian. We pass from one to the other through a Wick rotation.

See the generalized Wick rotation.

• **Wightman axioms:** A list of axioms taken as a blue-print of the general concept of a Quantum Field Theory ([gr-qc/0512053]). Here for special relativistic field theory. Generalised by Thiemann *et al* in [] to generally covariant theories such as Quantum gravity.

i) There exists a Hilbert space $\mathcal{H}$ with a dense domain $\mathcal{D} \subset \mathcal{H}$, so that all $\Phi(F)$ are well defined operators on $\mathcal{D}$, and $\Phi(F)^* = \Phi(\overline{F})$.

ii) $F \mapsto \Phi(F)$ is complex linear and suitably continuous.

iii) Covariance:
There is on $\mathcal{H}$ a unitary representation

$$U(L)\Phi(F)\Phi(L)^* = \Phi(F \circ L^{-1}) \tag{A.168}$$

with $U(L)\mathcal{D} \subset \mathcal{D}$.

iv) Locality, or relativistic causality:
If the *supports* of the test function F_1 and F_2 are causally separated, the corresponding field operators commute

$$[\Phi(F_1), \Phi(F_2)] = 0. \tag{A.169}$$

v) spectrum condition/positivity of the total energy:
Writing $U(1,a) = e^{iP_\mu a^\mu}$, it holds (for expectation values) that

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 \geq 0, \qquad P_0^2 \geq 0. \tag{A.170}$$

vi) Existence and uniqueness of the vacuum:
There exists $\Omega \in \mathcal{D}$, $\|\Omega\| = 1$, so that $U(L)\Omega = \Omega$ and this vector is uniquely determined up to phase factor.

vii) Cyclicity of the vacuum:
The domain $\mathcal{D}$ is spanned by vectors of the form

$$\Omega,\ \Phi(F)\Omega,\ \Phi(F_1)\Phi(F_2)\Omega,\ \ldots. \tag{A.171}$$

• **Wigner function:** The Wigner function of a semi-classical state is peaked along the phase-space trajectory.

• **Wilson loop:** The trace of the holonomy.

• **Wilson renormalization:**

Can be used even if theory not renormalizable.

$$\int \mathcal{D}[\phi] \tag{A.172}$$

• **Winkler, Oliver**

• **world function:** The world function $\Omega(x_A, x_B)$ is defined as half the squared geodesic distance between two points x_A and x_B.

• **Winicour solutions:** Winicour J 1975 *J. Math. Phys.* **16** 1805

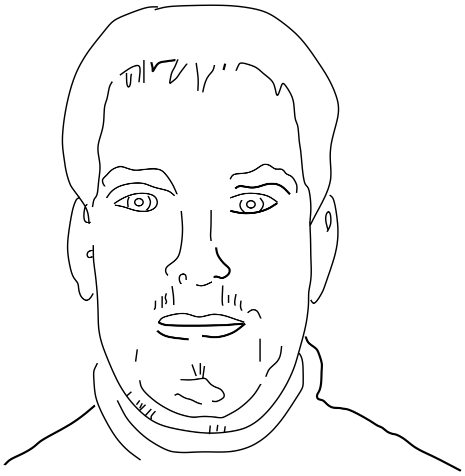


Figure A.48: Oliver Winkler.

• **Witten model:**

[387] E. Witten, "$(2+1)$-dimensional gravity as an exactly solvable system", Nucl. Phys. **B311**: 46, (1988).

• **Yang-Mills:** There is a $\$1,000,000$ mathematical prize for the consolidation of Yang-Mills and special relativity - does consolidation within a background-independent theory with quantum gravity coupled to the standard model count? Or specifically, background dependent, Minkowski spacetime Yang-Mills theory?

Smolin said

"someone might earn a Clay prize by rigorously constructing quantum Yang-Mills within LQG. It will certainly not be me, but there are people working on exactly that program. The conjecture is that background independent QFTs are more likely to exist rigorously in 3+1 dimensions than Poincare invariant QFTs."

$$\emptyset,\ \circ\bigcirc,\ \forall,\ \exists,\ \ell,\ \wp,\ \Rightarrow,\ \uparrow,\ \cong, \star, \simeq, \propto, \subseteq, \gg, \ll, \supset, \not\subset, \imath, \jmath, \wp \tag{A.173}$$

$$\|, \prod, \aleph, \surd, \ddagger, \mapsto, \ni \tag{A.174}$$

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Appendix B

Mathematics Glossary

• **absolute continuous functions:** These are functions of the form:

$$u(x) = \int_{x_0}^{x} v(y)dy + c$$

where $v(y)$ is considered as of u and the fundamental formula

$$u(x) = u(x_0) + \int_{x_0}^{x} \frac{d}{dy}u(y)dy + c$$

still holds. [from http://www.math.ku.dk/ grubb/distribution.htm]

• **absolute continuous measure:** We say that a Borel measure μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ if there is a function, f, locally L^1 (that is, $\int_a^b |f(x)|dx < \infty$ for any finite interval (a,b)) so that

$$\int g d\mu = \int g f dx$$

for any Borel function $g \in L^1(\mathbb{R}, d\mu)$. This generalizes to to measures on topological spaces.

• **absolute continuous operator:**

• **absolute convergence:** The series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

• **algebra:** An algebra is simply a vector space over $\mathbb{C}$ (or more generally over a field k) in which there is defined a distributive and, (in a certain sense), associative multiplication:

(i) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(ii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every scalar α, (a complex number).

Note that an algebra need not be associative in the sense $x \cdot (y \cdot z) \neq (x \cdot y) \cdot z$ where x, y, z are elements of the algebra.

Example An example of a **non-associative algebra** is the vector space of 3-d Euclidean vectors with the multiplication relation taken to be the crossed product $\times$ of two vectors

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}). \tag{B.1}$$

An **associative** algebra is an algebra whose multiplication also satisfies;

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{B.2}$$

A **commutative** algebra is an algebra whose multiplication also satisfies the condition:

$$x \cdot y = y \cdot x. \tag{B.3}$$

It is unital if there is defined a unit **1** which satisfies

$$a\mathbf{I} = \mathbf{I}a = a, \quad \text{for all } a \in A. \tag{B.4}$$

Intro into $*$algebras.

A **$*$-algebra** if there is defined an **involution** satisfying

$$(xy)^* = y^* x^* \quad \text{and} \quad (x^*)^* = x \tag{B.5}$$

which reduces to complex conjugation on the scalars $\alpha \in \mathbb{C}$, i.e.,

$$(\alpha x)^* = \alpha^* x^*. \tag{B.6}$$

A **Banach algebra** is an algebra with norm $\|a\| \geq 0$ which satisfies the conditions $\|x+y\| \leq \|x\| + \|y\|$, $\|xy\| \leq \|x\| \ \|y\|$, $\|\alpha x\| = |\alpha| \ \|x\|$, $\|x\| = 0 \Leftrightarrow x = 0$ and with respect to which it is complete; it contains its limit points$\|x_n - x\| \rightarrow 0$.

A **C*-algebra** is a Banach $*$-algebra whose norm satisfies the C^*-property:

$$\|a * a\| = \|a\|^2, \quad \text{for all } a \in A. \tag{B.7}$$

A familiar example of a C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$.

Example An example would be an associative algebra of the space real-valued of continuous functions formed by ordinary pointwise addition $(f+g)(x) := f(x) + g(x)$ and multiplication $fg(x) := f(x)g(x)$. It is obvious that the addition of two continuous functions is continuous. The product of two continuous functions is continuous. To prove that $fg(x)$ is continuous we must show that given any $\epsilon > 0$ there exists $n > N$ such that

$$|fg(x) - fg(x_n)| < \epsilon. \tag{B.8}$$

Given any $\epsilon > 0$ define $\epsilon' := (|g(x)| + |f(x_n)|)/\epsilon$. As $f(x)$ and $g(x)$ are continuous for $\epsilon' > 0$ there exists $n > N$ such that

$$|f(x) - f(x_n)| < \frac{\epsilon'}{2} \quad \text{and} \quad |g(x) - g(x_n)| < \frac{\epsilon'}{2}. \tag{B.9}$$

continuity of $fg(x)$ follows from

$$\begin{aligned} |fg(x) - fg(x_n)| &= |f(x)g(x) - f(x_n)g(x) + f(x_n)g(x) - f(x_n)g(x_n)| \\ &\leq |f(x) - f(x_n)||g(x)| + |f(x_n)||g(x) - g(x_n)| \\ &< \epsilon'(|g(x)| + |f(x_n)|) \\ &= \epsilon \end{aligned} \tag{B.10}$$

• **adjoint representation:** $[t_a, t_b] = C^c_{ab} t_c$. The commutation constants C^c_{ab} form a representation of the group algebra.

• **algebraic dual** The algebraic dual $\mathcal{D}^*$ is the space of linear functionals on $\mathcal{D}$ without continuity assumptions. See dual spaces.

• **algebraic graph theory:** http://www.utm.edu/departments/math/graph/glossary.html

• **almost everywhere:**

• **almost periodic functions:** Almost periodic functions are functions that can be written as

$$f(x) = \sum_j f_j e^{\frac{\mu_j x}{2}} \tag{B.11}$$

where the sum contains a finite number of terms ($j = 1, 2, \ldots, N$ and $N < \infty$), $f_j \in \mathbb{C}$, and $\nu_j \in \mathbb{R}$.

For loop quantum cosmology models of spaces that have homogeneity and isotropy the holonomy algebra consists of the set of almost periodic functions.

The fundamental theorem for almost periodic functions is a generalisation of the Parseval identity for Fourier series. This result lead Bohr to a result on the uniform approximation to almost periodic functions by exponential functions.

• **analytic function:** A function is called real analytic at a point if it possesses derivatives of all orders and given by a convergent power series locally. For example, a function on the real line $\mathbb{R}$ is analytic at the point p if there exists an interval (a, b) containing p such that in this interval the function can be expanded as a convergent series

$$f(x) = a_0 + a_1(x-p) + a_2(x-p)^2 + a_3(x-p)^3 + \ldots, \tag{B.12}$$

where

$$a_0 = f(p),\ a_1 = f'(p),\ a_2 = \frac{f''(p)}{2!},\ a_3 = \frac{f'''(p)}{3!}, \ldots \tag{B.13}$$

A function is analytic if it is analytic at each point in its whole domain. The set of all analytic functions is contained in the set of smooth functions. Analytic functions are also referred to as C^ω-smooth functions.

• **analytic continuation:** If we have two complex functions $f(z)$ and $g(z)$ satisfying the following properties:

(a) $f(z)$ is defined on a set U of the z complex plane $\mathbb{C}$;

(b) $g(z)$ is analytic in the domain V containing U;

(c) $g(z)$ coincides with $f(z)$ on U;

then $g(z)$ is said to be the analytic continuation of $f(z)$ to the domain V.

• **analytic curve:** A curve in Euclidean space $\mathbb{R}^n$ is piecewise analytic if it can be expanded as a Taylor series locally. A curve in a manifold $\mathcal{M}$ is analytic if and only if its image under a chart is an analytic curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval (a, b) to $\mathbb{R}^n$ in Fig.(**??**) is a analytic map.

• **analytic structure:** a covering homeomorhic to open sets in a fixed Euclidean space, C^ω The coordinate transforms are analytic in both directions, i.e.,

$$\begin{aligned}(\phi_1 \circ \phi_2^{-1})(x_1, x_2, \ldots, x_n) &= a_0 + a_1^{(i)} x_i + a_2^{(ij)} x_i x_j + a_3^{(ijk)} x^3 + \ldots,\\ (\phi_2 \circ \phi_1^{-1})(y_1, y_2, \ldots, y_n) &= b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \ldots.\end{aligned} \tag{B.14}$$

in some interval containing p.

• **associated bundles:** Given a particular principal bundle $(P, \pi, \mathcal{M})$ with structure group G, we can form a fibre bundle F for each space F on which G acts as a group of transformations.

• **atlas:** Two charts ϕ_1, ϕ_2 are $C^\infty-$related if both the map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and its inverse are $C^\infty-$related. A collection of related charts such that every point of $\mathcal{M}$ lies in the domain of at least one chart forms an atlas.

• automorphisms: An isomorphism $X \to X$ is called an automorphism of X. see group homomorphism.

[403] **Automorphisms of groups:** Let G be a group. An isomorphism $G \to G$ is called an automorphism of G. Let $x, y \in X$ and $g \in G$,

$$g(xy)g^{-1} = (gxg^{-1})(gyg^{-1}), \quad i_g(xy) = i_g(x)i_g(y) \tag{B.15}$$

The set $\mathrm{Aut}(G)$ of such automorphisms becomes a group under composition:

(i) the composite of two automorphisms is again an automorphism;

(ii) composition of maps is always associative;

(iii) the identity map $g \mapsto g$ is an identity element;

(iv) an automorphism is one-to-one and onto, and therefore has an inverse (we just change the direction of the arrows), which is again an automorphism.

Automorphisms of algebras: Let $\mathcal{A}$ be an algebra. An isomorphism $\mathcal{A} \to \mathcal{A}$ is called an automorphism of $\mathcal{A}$. Let $a, b \in G$

$$\begin{aligned} g(ab)g^{-1} &= (gag^{-1})(gbg^{-1}), \quad i_g(ab) = i_g(a)i_g(b) \\ g(a+b)g^{-1} &= gxg^{-1} + gyg^{-1}, \quad i_g(a+b) = i_g(a) + i_g(b). \end{aligned} \tag{B.16}$$

$*-$**automorphisms:**

Time automorphisms of operator algebras in quantum mechanics: $\hat{\mathcal{O}}_i \in \mathcal{A}$ where $\mathcal{A}$ is the quantum operator algebra of a system and $e^{i\hat{\mathcal{H}}t} \in G$ where G is the group of time-evolution operators of parameterized by time t, $-\infty < t < \infty$.

$$e^{-i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_i(t_0)e^{i\hat{\mathcal{H}}t} = \hat{\mathcal{O}}_i(t_0 + t) \tag{B.17}$$

it is an algebra homomorphism because

$$
\begin{aligned}
e^{i\hat{\mathcal{H}}t}(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2)e^{-i\hat{\mathcal{H}}t} &= (e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_1 e^{-i\hat{\mathcal{H}}t})(e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_2 e^{-i\hat{\mathcal{H}}t}) \\
e^{i\hat{\mathcal{H}}t}(\hat{\mathcal{O}}_1+\hat{\mathcal{O}}_2)e^{-i\hat{\mathcal{H}}t} &= e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_1 e^{-i\hat{\mathcal{H}}t} + e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_2 e^{-i\hat{\mathcal{H}}t}
\end{aligned} \tag{B.18}
$$

it is one-to-one and onto because

$$
e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_i(t_0+t)e^{-i\hat{\mathcal{H}}t} = \hat{\mathcal{O}}_i(t_0) \tag{B.19}
$$

is an inverse (obtained by simply replacing t with $-t$).

is an example of an inner automorphism of the algebra of observables. two equivalent ways of describing the time flow: either the flow instate space(Schrödinger picture), or (generalized Heisenberg picture). So we say time flow is as a one-parameter group of automorphisms of the algebra of observables. A mathematic theorem called the Tomita-Takikey that states in quantum field theory there is a unique one-parameter automorphism of the operator algebra of - the physical basis of time - this is the thermal time hypothesis [290] [291].

• **axiom of choice:** Say we have a family of nonempty sets $\{X_\alpha\}$. Then there is a set X which contains exactly one element from each set $\{X_\alpha\}$. For finite collection of sets, this is obvious and isn't really an axiom. It is when you It applies as an axiom when there are an infinite number (countable and uncountable) number of sets.

• **Banach algebra:** A Banach algebra is a complex Banach space which is also an algebra with identity 1, and in which

(i) $\|xy\| \leq \|x\| \, \|y\|$,

(ii) $\|1\| = 1$.

Also a normed vector ring.???????

If the norm satisfies the parallelogram law, its is also a Hilbert space.

A **Banach subalgebra** is a closed subalgebra of A which contains 1; they are precisely those subsets of A which themselves are Banach algebras with the same identity, and the same norm.

• **Banach space:** A Banach normed space complete normed space which is a complete metric space with respect to the metric induced by norm, $d(x-y) := \|x-y\|$.

A unital Banach space: A banach space containing the identity with respect to multiplication.

• **Bargmann transform:** The Bargmann transform is the unitary transformation that takes an $L^2(\mathbb{R}^n)$ function of the coordinates q_i to a holomorphic square integrable (with respect to a certain measure) function of n complex variables. It can be interpreted as the transform that takes

$$f \to \text{ Analytic continuation of } e^{-t\Delta/2} f,$$

that is, one obtains the representatioon by heat kernel evolution followed by analytic continuatiooon from the usual position space representation.

• **bijective:** A function is bijective if it is one-to-one and onto, that is injective and surjective.

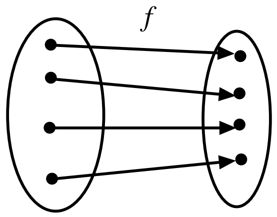


Figure B.1: bijective

See injective, surjective.

bilinear form: A bilinear form (or sesquilinear form) $F(u, v)$ on a Hilbert space H is an assignment of a scalar to each pair of vectors u, v of a subspace $D(F)$ of H in such a way that

(i) $F(\alpha u + \beta v, w) = \overline{\alpha} F(u, w) + \overline{\beta} F(v, w)$, and
(ii) $F(u, \alpha v + \beta w) = \alpha F(u, v) + \beta F(u, w)$

for $u, v, w \in D(F)$. The subspace $D(F)$ is called the domain of the bilinear form.

Bohr compactification: Bohr compactification $\overline{\mathbb{R}}$ of the real line $\mathbb{R}$ is the dual group of $\mathbb{R}$ equipped with the "discrete" topology in which the real line is totally atomized as if no point is near any other point.

The dual of the dual of a group G is the same group back again But suppose you take the dual of G and replace the usual topology with the discrete topology (the power set, same as saying that singletons are open) so you say H is equal to G' as a group but atomized as a topological space and all functions from it are continuous. *Then* you take the dual of H, and that is the Bohr compactification of G.

Bohr group: Dual group to the discrete line. Is compact.

Boolean algebra: A distributive lattice in which every element has a complement is a Boolean lattice or a *Boolean algebra*. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Boolean logic: Formulises in algebraic terms for statements that are either true or false.

See non-Boolean logic

Borel function:

Borel sets: One wants to define a probability distribution on the real line $\mathbb{R}$, that is, assign a real number to any subset Δ of $\mathbb{R}$ that is the probability of some quantity taking its value in Δ. It turns out that it is impossible to construct a probability measure defined on all subsets. one must restrict oneself to a particular family of sets - the Borel subsets, $\mathcal{B}(\mathbb{R})$. This is the smallest family of subsets of $\mathbb{R}$ that includes all the open sets and which is closed under complements and countable intersections.

Borel $\sigma-$algebra: Let X be a topological space. The smallest $\sigma-$algebra on X that contains all open sets of X is called Borel $\sigma-$algebra of X.

• **Borel-summable:** Borel-summation does not pick up on non-perturbative effects.

$$\sum_n a_n x^n \tag{B.20}$$

with $a_n \approx Cn!$

$$\int_0^\infty e^{-t/x}\; t^n dt = n!\; x^{n+1} \tag{B.21}$$

$$\int_0^\infty e^{-t/x} \left(\sum_n \frac{a_n}{n!} x^n \right) \frac{dt}{x} = \sum_n a_n x^n \tag{B.22}$$

$$g(t) := \sum_n \frac{a_n}{n!} x^n \tag{B.23}$$

• **bounded:**

(i) bounded linear operator:

$$\|A\| := \sup_{x \in X} \frac{\|Ax\|}{\|x\|} \equiv \sup_{\|x\|=1} \|Ax\| \tag{B.24}$$

then operator A is bounded if $\|A\| \leq C$. Note: boundedness and continuity are equivalent for linear operator.

- bounded set:

We call a nonempty subset $M \subset X$ a bounded set if its diameter

$$\delta(M) := sup_{x,y\in M}\, d(x,y) \tag{B.25}$$

is finite.

• **bounded contraction semigroup:** imaginary-time path integral has kernel

$$e^{\frac{-\hat{\mathcal{H}}t}{\hbar}} \tag{B.26}$$

• **bounded convergence theorem:**

If the sequence $\{f_n\}$ of measurable functions is uniformally bounded and if $f_n \to f$ in measure as $n \to \infty$, then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

• **bounded inverse theorem:** The range of a bounded operator A is closed in $\mathcal{H}$ implies that $A - \lambda\mathbb{I}$ has in inverse in $\mathcal{B}(\mathcal{H})$.

• **bounded linear functional:** A linear functional satisfying

$$|F(x)| \leq M\|x\|, \quad x \in X.$$

• **bounded linear operator:** An operator A is called bounded if its domain is the whole of X, i.e. $D(A) = X$, and there is a constant M such that

$$\|Ax\| \leq M\|x\|, \quad x \in X.$$

The norm of such an operator is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

• **bounded variation:** A function $f(x)$ defined on an interval $[a,b]$ is said to be of bounded variation if it satisfies

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < C \tag{B.27}$$

for any partition $a = x_0 < x_1 < \cdots < x_n = b$.

- **braid:**
- **braided moinodal category:** A 3-category with one object.
- **braid group B_n:** a restriction of the permutation group.

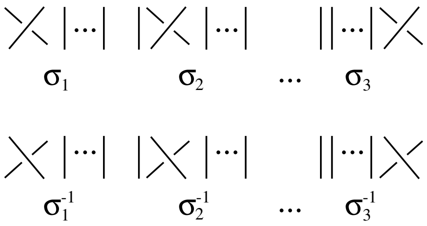


Figure B.2: Braid group generators

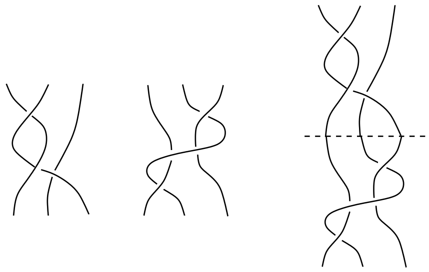


Figure B.3: Multiplication of braids.

$$\begin{aligned} \sigma_i \sigma_i^{-1} &= 1, && i = 1, \ldots, n-1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_i. && i = 1, \ldots, n-2 \\ [\sigma_i, \sigma_j] &= 0 && |i-j| > 1. \end{aligned} \tag{B.28}$$

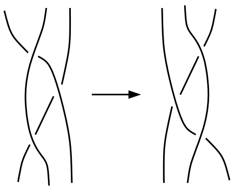


Figure B.4: A Reidermeister move on a three-string braid.

Quantum groups were invented largely to provide solutions for the Yang-Baxter equation and hence solvable models in 2-dimensional statistical mechanics and one-dimensional quantum mechanics. why i-d quantum???????????

- **Caley transform:**

$$a \mapsto u := \frac{a-i}{a+i}$$

• **canonical local trivialisation:** There are local sections $s_i : U_i \to \pi^{-1}(U_i)$ canonically associated to the trivialisation, eefined so that for every $p \in U_i$, $\phi_i(s_i(p)) = (p, e)$. In other words, the map from U_i to G is the constant function sending every point to the identity.

• **Cantor Set:** The Cantor set is

$$C = \bigcap_{k=1}^{\infty} C_k. \tag{B.29}$$

where

$$\begin{aligned} C_0 &= [0,1] \\ C_1 &= [0,\frac{1}{3}]\bigcup[\frac{2}{3},1]. \\ C_2 &= [0,\frac{1}{9}]\bigcup[\frac{2}{9},\frac{1}{3}]\bigcup[\frac{2}{3},\frac{7}{9}]\bigcup[\frac{8}{9},1], \end{aligned} \tag{B.30}$$

and in general C_{n+1} consists of the intervals of C_n with their open middle thirds removed.

• **cardinality:**

• **Cartan's structure equations:**

$$d\theta_i = -\frac{1}{2} c^i_{kl} \theta_k \wedge \theta_l. \tag{B.31}$$

• **category:** A category $\mathcal{C}$ consists of "objects", C, and "morhisms", f, between them, such that

(i) If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morhisms, then there exists a morphism $g \circ f : C_1 \to C_3$.

(ii) It is assumed that the identity map for C, $\mathrm{id} : C \to C$, is a morphism for every object C of $\mathcal{C}$.

The set of objects is usually denoted ob$\mathcal{C}$.

Examples:

The category of open sets in Euclidean spaces, where the morphisms are the smooth maps.

The category of abelian groups, where the morphisms are homomorphisms.

n-category:

n-categorical group: A is by definition a group-object of the category od groupoids.

• **category theory:** TAKEN DIRECTLY FROM mathworld.wolfram.com

Category Theory

The branch of mathematics which formalizes a number of algebraic properties of collections of transformations between mathematical objects (such as binary relations, groups, sets, topological spaces, etc.) of the same type, subject to the constraint that the collections contain the identity mapping and are closed with respect to compositions of mappings. The objects studied in category theory are called categories.

http://mathworld.wolfram.com/

an abstraction of many concrete concepts in diverse branches of mathematics.

• **CAR:** stands for canonical anti-commutation relations. CAR algebra....

• **Cat:** The category of all small categories.

• **Cauchy-Kowalewski:**

• **Cauchy sequence:** A sequence $\{a_l\}$ satisfying

$$\|a_n - a_m\| \to 0 \quad \text{as } m, n \to \infty.$$

• **CCR:** stands for canonical commutation relations.

• **Cech compactification:**

• **center:** An abelian subgroup of a ring R is the **center**

$$X = Z(R) \;\; X = \{a \in R : b \in R, \; ab = ba \;\; \text{for all } b\} \tag{B.32}$$

The center of a Lie group is the set of commuting elements $\{x \in \mathbf{g} : [x, y] = 0$, for all $y \in \mathbf{g}\}$.

X is the centralizer of Y $X = C_R(Y)$

$$X = Z(R) \;\; X = \{a \in R : b \in Y, \; ab = ba \;\; \text{for all } b\} \tag{B.33}$$

Alternative

The kernel of the $G-$action is the subgroup of G defined by

$$K := \{g \in G : gp = p \text{ for all } p \in \mathcal{M}\}.$$

The kernel measures the part of the group that is not represented at all in the $G-$action on $\mathcal{M}$. An example is given by the adjoint action of G on itself in which

$$\mathrm{Ad}_g(g') := gg'g^{-1}.$$

The kernel of this action is the centre $C(G)$ of G.

• **central extension:** A central extension of a Lie algebra Lie(G) is a Lie algebra E together with a homomorphism $\pi : E \to \mathrm{Lie}(G)$ such that $\mathrm{Ker}(\pi) \subset Z(E)$ where $Z(E) = \{A \in E : [A, B] = 0 \text{ for all } B \in E\}$ is the centre of E.

• **characters:**

Characters of a finite group:

$$\chi(A) = \mathrm{tr}A = \sum_i A_{ii}. \tag{B.34}$$

These characters are invariant under simultaneity transformations because of the cyclic symmetry of matrices

$$\chi(U^{-1}AU) = \mathrm{tr}(U^{-1}AU) = \mathrm{tr}A \tag{B.35}$$

Characters of algebras:

Let $\mathcal{U}$ be an abelian C^*-algebra. A **character** χ, of A, is a nonzero linear map, $\omega; A \in \mathcal{U} \mapsto \chi(A) \in \mathbb{C}$, of $\mathcal{U}$ into the complex numbers $\mathbb{C}$ such that

$$\chi(AB) = \chi(A)\chi(B) \tag{B.36}$$

for all $A, B \in \mathcal{U}$. As it preserves multiplication (B.36) it is a homomorphism. See the spectrum $\sigma(\mathcal{U})$.

• **chart:** Given a topological space $\mathcal{M}$, a chart on $\mathcal{M}$ is a one-to-one map ϕ from an open subset $U \subset \mathcal{M}$ to an open subset $\phi(U) \subset \mathbb{R}^n$, i.e., a map $\phi : \mathcal{M} \to \mathbb{R}^n$. A chart is often called a coordinate system.

• **Chern classes:** cohomology

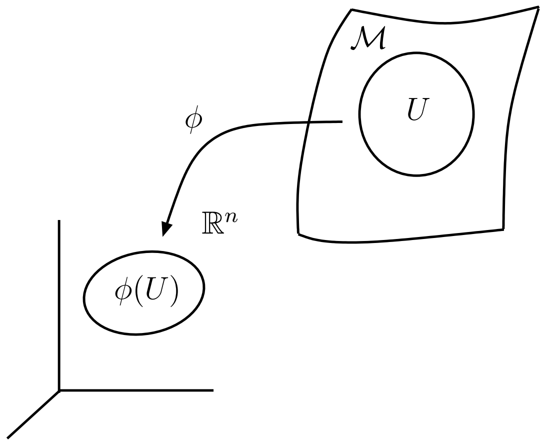


Figure B.5: DiffClass0. A chart on $\mathcal{M}$ comprises an open set U of $\mathcal{M}$, called a coordinate patch, and a map $\phi : U \to \mathbb{R}^n$.

$$\det(t\mathbf{I} + a_c T^c) \tag{B.37}$$

substituting

second *Chern class*

$$c_2(P) = \frac{1}{4\pi^2}\left(\frac{1}{2}(TrF) \wedge (TrF) - \frac{1}{2}Tr(F \wedge F)\right) \tag{B.38}$$

• **chromatic evaluation:** The chromatic evaluation of the spin network is equivalent to the Temperley-Lieb trace of the spin network, or, equivalent to the Kauffman bracket.

• **class function:** A function of elements $x, y, z, ..$ of a group G is said to be a class function if

$$f(x, y, \ldots, z) = f(g^{-1}xg, g^{-1}yg, g^{-1}\ldots zg) \tag{B.39}$$

where $g \in G$.

• **closable operator:** Let A be a linear operator from a normed vector space X to a normed vector space Y. It is called closable if for $\{x_k\} \subset D(A)$, $x_k \to 0$, $Ax_k \to y$ imply that $y = 0$.

• **closed bilinear form:** A bilinear form $F(u, v)$ is called closed if $\{u_n\} \subset D(F)$, $u_n \to u$ in H, $F(u_n - u_m) \to 0$ as $m, n \to \infty$ imply that $u \in D(F)$ and $F(u_n - u) \to 0$ as $n \to \infty$.

• **closed graph theorem:** The closed graph theorem implies that if $A : X \to Y$ is closed and has the domain of A is equal to X then A is bounded.

See bounded inverse theorem.

• **closed surface:** Examples of closed surface are the sphere, the torus, the Klein bottle. They are classified by the genus and their orientability. An examples of a non-closed surface is a disk which is a sphere with a puncture.

• **closure property:** If certain set of operations on sets in $\mathcal{F}$ again produce sets in $\mathcal{F}$, we say $\mathcal{F}$ is closed under these operations.

• **cobordism:** see Fig.(J.1) - referring to wrong figure for some reason!

• **co-cycle:**

• **cocycle Radon-Nikodym theorem:** The Cocycle Radon-Nikodym [] theorem states that two modular automorphisms defined by two states of a von Neumann algebra are inner-equivalent.

see rovelli thermal time hypothesis

• **codimension:** $\text{codim}X = \text{dim}X^{\perp}$.

• **cohomology:** Roughly speaking, the cohomology $H^p(M, \mathbb{R})$ counts the number of noncontractable p-dimensional surfaces in M.

we consider all closed forms modded out by exact forms. In other words two forms are said to be equivalent if

$$\lambda_1 = \lambda_2 + d\Phi \quad \rightarrow \quad [\lambda_1] = [\lambda_2] \tag{B.40}$$

for any Φ. two closed forms are called cohomology.

• **commutant:** The commutant of a collection of operators $\mathcal{A}$ is defined as

$$\mathcal{A}' := \{A \in \mathcal{A} : [A, B] = 0 \text{ forall } B \in \mathcal{A}\}. \tag{B.41}$$

• **compact:** Every open cover has a finite subcover.

• **compactification:** The process of adding points to a given topological space in order to make it compact. The simplist compactifiction is adding just one point, for exmple the process of adding a point to a plane to make a sphere.

• **compact Lie group:**

Each group element is uniquely related by the parameters.

• **compact manifold:** A manifold is compact every open cover has a finite subcover. “A town is compact when it can be policed by a finite number of arbitrary short-sighted policemen”.

• **compact operators:** An operator A is said to be a compact operator if there exists a sequence of finite rank operators that converge to A.

• **comparable:** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms defined on the same linear space Φ. These two norms are called **comparable** if if for every $\varphi \in \Phi$ there exists a constant $C > 0$ such that

$$\|\varphi\|_1 \leq \|C\varphi\|_2, \quad \text{for all } \varphi \in \Phi. \tag{B.42}$$

• **compatible:** Two norms are called compatible if and only if every sequence $(\varphi_n)_{n=1}^{\infty} \subset \Phi$ which is Cauchy with respect to both norms and which converges to 0with respect to one of them, also converges to 0 with respect to the other.

• **completion:** complicates matters: it would be like doing real analysis in the rational numbers instead of the real line R.

Completeness is an important property since it allows us to perform limit operations which arise frequently in our constructions.

• **completely regular topological space:** Let X be a topological space. If for each open neighbourhood $\mathcal{U}$ of x, there is a continuous function $0 \leq f(x) \leq 1$ such that $f(x) = 0$ and f is identically one on the complement $X - \mathcal{U}$ of $\mathcal{U}$ in X, i.e.

$$f(x) = \begin{cases} 0 & \text{for } x \in X \\ 1 & \text{for } x \in \mathcal{U} - X \end{cases} \tag{B.43}$$

(Completely regular spaces are also called Tychonoff spaces). are able to support sufficiently many continuous functions: for two distinct points x and y of a completely regular space X, there is a continuous function on X taking distinct values at x and y.

Third class of spaces contains, for example, all normal and all Hausdorff spaces.

• **complete set:** A collection f_i of functions on the symplectic space $(\mathcal{M}, \omega)$ which Poisson commute with each other (are in involution) is said to be complete if the vanishing of $\{f_i, g\}$ for all i implies that g is a function of the form $g(x) = h(f_1(x), \ldots, f_n(x))$.

A collection of operators $\{A_j\}$ is said to be complete if any operator B which commutes with each A_j is a multiple of the identity. This condition is equivalent to the irreducibility of $\{A_j\}$, that is, there is no non-trivial subspace that is invariant under each A_j.

• **complete space:** all Cauchy sequences (defined wrt a norm??) converge to an element in the space.

• **complexification of a Lie group:** The complexification of a Lie group $\mathbf{g}$ denoted $\mathbf{g}_{\mathbb{C}}$.

Recall the defining properties of a Lie algebra: antisymmetric $[X, Y] = -[Y, X]$, bilinearity $[\alpha X, Y] = \alpha[X, Y] = [X, \alpha Y]$ for any real number α, and satisfies the Jacobi identity $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] \equiv 0$. $X_1 + iX_2$ where $X_1, X_2 \in \mathbf{g}$

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1]). \tag{B.44}$$

It is clear that (B.44) is *real* bilnear and skew-symmetric. If we prove that it is *complex* linear in the first factor, it will be complex linear in the second because of the skew-symmetry. As we already know it is real linear in the first factor, it suffices to show that it is *imaginary* linear. This is not difficult to prove, all we need to do is verify that

$$[i(X_1 + iX_2), Y_1 + iY_2] = i[X_1 + iX_2, Y_1 + iY_2] \tag{B.45}$$

is true. This is easily done by expanding each side and seeing they are indeed equal.

It remains to check the Jacobi identity. First consider Y and Z to be in $\mathbf{g}$ but take X to be in $\mathbf{g}_{\mathbb{C}}$. Now, $X = X_1 + iX_2$ is linear in the Jacobi identity and the Jacobi identity holds separately for X_1 and X_2,

$$[[X_1, Y], Z] + [[Z, X_1], Y] + [[Y, Z], X_1] + i([[X_2, Y], Z] + [[Z, X_2], Y] + [[Y, Z], X_2]) \equiv 0, \tag{B.46}$$

and so the Jacobi identity holds for $X \in \mathbf{g}_{\mathbb{C}}$ and $Y, Z \in \mathbf{g}$. Similarly for Y and Z. Therefore we have shown that the elements of the complexification $\mathbf{g}_{\mathbb{C}}$ satisfy the Jacobi identity.

• **complexification:** - We can tensor a real vector space with the complex numbers and get a complex vector space; this process is called complexification. For example, we can complexify the tangent space at some point of a manifold, which amounts to forming the space of complex linear combinations of tangent vectors at that point. (from Geometric Quantization John Baez August 11, 2000).

• **complex manifold:** $\mathcal{M}$ is a complex manifold if

(i) $\mathcal{M}$ is a topological space;

(ii) $\mathcal{M}$ is provided with a family of pairs $\{(U_i, \varphi_i)\}$;

(iii) $\{U_i\}$ is a family of open sets which covers $\mathcal{M}$, φ_i is a homomorphism from U_i to an open subset U'_i of $\mathbb{C}^m$;

(iv) Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_j \varphi_j^{-1}$ is holomorphic.

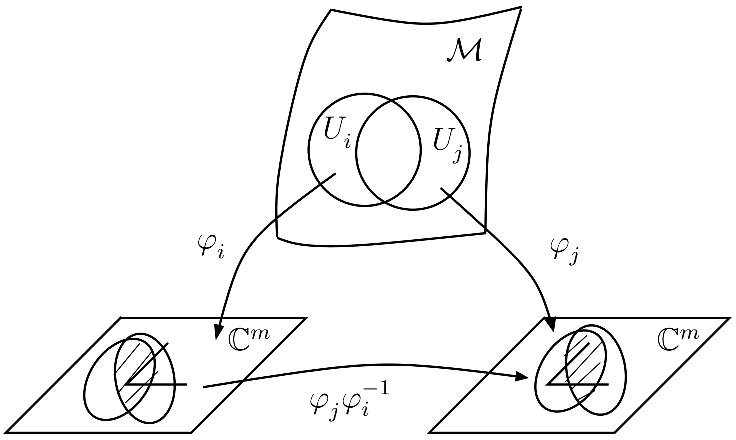


Figure B.6: Complex manifold.

• **congruence:** A congruence is a set of curves which fill a manifold, or part of it, without intersecting. Through every point there passes one and only one curve.

• **congruence relation:**

• **connected:** In the topological sense: a topological set not able to be partitioned into non-empty open subsets each of which has no points in common with the closure of the other.

• **connection:** Roughly, comparison of objects in two different spaces is made by a prescribed mapping, and the mappings that connect the various spaces are called connections.

A connection on a principal bundle is an assignment to each local trivialisation $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ (a choice of gauge in physics terms) a Lie algebra one-form ω_i on U_i which satisfies the following rule between different local trivialisations, (for simplicity here in the case of matrix groups G):

$$\omega_j = t_{ij}\omega_i t_{ij}^{-1} + t_{ij}^{-1}dt_{ij}.$$

A connection is an example of a gauge field.

• **continuity:**

− uniformaly continuous when for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x, x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$

− pointwise continuous when for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. A function which is pointwise continuous at every point is pointwise continuous.

− sequentially continuous when it preserves limits of convergent sequences: if $(a_i)_{i\in N}$ converges to a in X then $(f(a_i))_{i\in N}$ converges to $f(a)$ in Y.

• **continuous function:** A function f is continuous at a point p if whenever we can force the distance between $f(x)$ and $f(p)$ to be as small as desired by taking the distance between x and p to be small enough.

A function is said to be absolutely continuous if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon \tag{B.47}$$

whenever (a_i, b_i), $i = 1, 2, \ldots, n$, are non-overlapping subintervals of I with $\sum_{i=1}^{n} |b_i - a_i| < \delta$

• **contractible open cover:** A collection $\{U_i\}$ is a contractible open cover of M if each U_i and each non-empty finite intersection $U_i \cap U_j \cap \cdots$ is contractible to a point.

• **contraction:** Suppose there is a map A from the space M to itself, this is a contraction mapping if there is a positive constant $K < 1$ such that $\|Ax - Ay\| < K\|x - y\|$ for all $x, y \in M$.

• **convergence:**

(x_n) converges to x if and only if the sequence (x_n) is eventually in every neighbourhood of x.

(From Weak convergence of inner superposition operators) modes of convergence:

The precise definitions. Let $A_\nu : X \to Y$, $A : X \to Y$, where $\nu \in \mathbb{N}$ are the mappings between two Banach spaces X and Y . One says that the sequence A converges to A

(S) strongly (pointwise), if $A_\nu x \to Ax$ in Y for all $x \in X$;

(C) continuously, if $A_\nu x \to Ax$ in Y for any norm converging sequence $x_\nu \to x$ in X;

(W) weakly (pointwise), if $A_\nu x \rightharpoonup Ax$ in Y for all $x \in X$;

(CW) continuously weakly, if $Ax \rightharpoonup Ax$ in Y for any weakly converging sequence $x_\nu \rightharpoonup x$ in X.

• **countable:** (denumerable) A set is countably infinite if there is a one-to-one onto function from the set A to the set $\{1, 2, 3, \ldots, \}$.

• **countably Hilbert space:** is a complete linear topological space whose topology is defined by a countable family of Hilbert spaces, Φ_n. The topology. Countable set of norms $\|\varphi\|_n$ where the norms are associated with a scalar product - $\|\varphi\|_n := \sqrt{(\varphi, \varphi)_n}$.

First property is that Φ_n is the Cauchy completion of Φ in the norm $\|.\|_n$???. Then for any m, n it is required that if (ϕ_k) is both a $\|.\|_m$ convergent sequence and an $\|.\|_n$ Cauchy sequence in Φ then is also $\|.\|_n$ convergent.

Second definition:

A space Φ is a countably Hilbert space (or a countably scalar product space) if an increasing denumerable number of scalar products

$$(\varphi, \varphi)_1 \leq (\varphi, \varphi)_2 \leq \cdots \leq (\varphi, \varphi)_p \leq \cdots \tag{B.48}$$

are defined on Φ such that the norms

$$\|\varphi\|_p := \sqrt{(\varphi, \varphi)_p}, \;\; p = 0, 1, 2, \ldots \tag{B.49}$$

are comparable and compatible.

• **countably neighbourhood base:**

• **cover:** Given a set X with $A \subseteq X$ is said to be regular if, for a **cover** for A is a family of subsets $\mathcal{U} = \{U_i : \in I\}$ of X such that $A \subseteq \cup_{i \in I} U_i$. A subcover $\mathcal{U}$ for A is a subfamily $\mathcal{V} \subset \mathcal{U}$ which still forms a cover for A.

• **cubulations:** Cubic triangulations of a four manifold are called cubulations. The first motivation for considering cubulations in spin foam models is that current semiclassical states used in LQG do not assign good classical behaviour to the volume operator of LQG (which plays a pivitol role in the dynamics) unlessthe underlying graph has cubic topology.

Cubulations also nicely fit in with the framework of Algebraic Quantum Gravity which in its minimal version is also formulated in terms of algebraic graphs of cubic topology only.

• **curve:** We refer to Fig.(**??**). A curve in a manifold $\mathcal{M}$ is a map λ of the open interval $I = (a, b) \in \mathbb{R} \to \mathcal{M}$ such that for every coordinate system of $\mathcal{M}$ $\phi \circ \lambda : I \to \mathbb{R}^n$. We say the curve smooth if $\phi \circ \lambda : I \to \mathbb{R}^n$ is smooth. The set of curves is denoted $\mathcal{C}^\infty$.

Alternative kinds of curves such as piecewise analytic, continuous, oriented, an embedding (does not come arbitrarily close to itself) are defined in the obvious way.

See piecewise-analytic

• **cyclic state:** GNS "vacuum state". For example for the SHO $|0>$ for which $\hat{\mathcal{H}}|0>= \frac{1}{2}\hbar|0>$. $\hat{a}|0>= 0$. This, with $|n>:= (\hat{a}^\dagger)^n|0>$.

$$< 0|[\hat{a}, \hat{a}^\dagger]|0 >= \left(\frac{1}{2} + 1\right)\hbar \tag{B.50}$$

Any representation with a cyclic state, can be constructed from the "vacuum" expectation value of the algebra of operators.

the precise definition of a cyclic vector is: a vector Φ is cyclic for a C^*algebra A acting in a Hilbert space $\mathcal{H}$ if and only if the linear space $\mathcal{A}\Phi = \{A\Phi, A \in \mathcal{A}\}$ is dence in $\mathcal{H}$.

(In particular the vacuum) is cyclic and separating for the algebras $\mathcal{A}$, i.e. $\mathcal{A}\Phi$ is dense in $\mathcal{H}$ (Φ is cylic) and $A\Phi = 0$, $A \in \mathcal{A}$ implies $A = 0$ (Φ is separating).

More precisely, a vector Φ is separating for $\mathcal{A}$ acting in $\mathcal{H}$ if and only if it is cyclic for the commutant $\mathcal{A}'$ of $\mathcal{A}$; the commutant is the set of operators in $\mathcal{H}$ which commute with all $A \in \mathcal{A}$. If $[A, A'] = 0$ then

Let $\mathcal{R}$ be a von Neumann algebra generated by $\pi_\omega(\mathcal{A})$, i.e., $\mathcal{R} = ()''$ A cyclic and separating vector in the Hilbert space $\mathcal{H}_\omega$ (ξ is separating in $\mathcal{A}$ if it is cyclic in $\mathcal{A}'$).

• **cylindrical function:** Fake infinite functions: although they depend on a field, say a scalar field $\phi(x)$, they only really depend on a finite number of variables. One begins by introducing a space $\mathcal{S}$ of 'probes'. Elements of $\mathcal{S}$ probe the structure of the scalar field $\phi \in \mathcal{C}$, ($\mathcal{C}$ being the classical configuration space of scalar functions), through linear functions h_e on $\mathcal{C}$:

$$h_e(\phi) = \int_{\mathcal{M}} d^3x \; e(x)\phi(x)$$

which capture a small part of the scalar field. Given a set α of probes, $h_{e_1}, \ldots, h_{e_n}$ and a (suitably regular) complex-valued function ψ of n real variables, we can now define a more general function Ψ on $\mathcal{C}$,

$$\Psi_\alpha(\phi) := \psi(h_{e_1}(\phi), \ldots, h_{e_n}(\phi))$$

These are examples of cyclindrical functions. Their linear span is usually denoted Cyl_α.

???term cylindrical borrowed from free fields in Minkowski???

Kolmogorov used this special class of functions to define measures for infinite dimensional integration theory.

• **cylindrical measure theory:** The idea of cylindrical measure theory is to reduce the integration over infinite dimensional spaces to a series of finite dimensional subspaces by making use of a special class of function(-al)s - cylindrical function(al)s. However, because of nesting, overlapping, (as well as freedom in choice of basis) of the finite dimensional subspaces, integration over them are subject to certain non-trivial consistency conditions. When these consistency conditions are satisfied, we say the infinite dimensional function space is said to be equipped with a cylindrical measure.

• **Darboux theorem:** Let $\mathcal{Q}$ be a configuration space (manifold). In local coordinates $(q^1, \ldots, q^n)$ one may identify the canonical momentum variables $(p_1, \ldots, p_n)$ with the cotangent vectors in the coordinate basis of $(q^1, \ldots, q^n)$. The Darboux theorem asserts there is a sympletic form Ω on the cotangent space $T^*\mathcal{Q}$ (phase space Γ) given by

$$\Omega_{ab} = \sum_{\mu=1}^{n} 2\nabla_{[a} dp_\mu \nabla_{b]} dq^\mu \tag{B.51}$$

and the pair (Γ, Ω) is a symplectic manifold.

• **Dehn surgery:** Dehn surgery is a procedure by which one can construct all three-manifold topologies. One begins with drawing a knot or link (i.e., a set of knots) on a given manifold, then we thicken the knot into a tube. One then removes this tubular region, give it a twist and then glue it back.

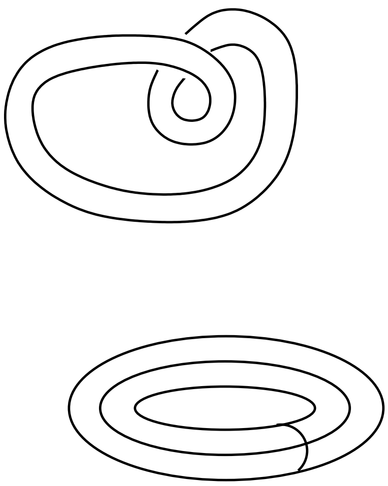


Figure B.7: Dehn surgery.

The three-sphere is two torus glued together. It can all be reduced to working with Chern-Simons on a torus. Once we solve Chern-Simons on a torus we have the exact closed solution for Chern-Simons on any three-manifold. Chern-Simons is related to $2+1$ quantum gravity, hence Witten showed that $2+1$ quantum gravity is exactly solvable, [387].

• **DeMorgan'a Laws:** Let A and B be subsets of X then

1) $X\backslash(A \cup B) = (X\backslash A) \cap (X\backslash B)$ and

2) $X\backslash(A \cap B) = (X\backslash A) \cup (X\backslash B)$.

• **dense:**

• **De'Ram cohomology:**

• **dihedral angles:** Angle subtended where to planes intersect???

• **diffeomorphic:** Two manifolds connected by a diffeomorphism are said to be diffeomorphic. From the differential-geometric point of view, diffeomorphic manifolds not distinguished by some othe structure (e.g. a metric) are effectively the same.

• **diffeomorphism:** A $f : M \to N$ from one manifold M to a manifold N is a smooth map whose inverse is also smooth.

• **differentiable manifold:** If $\mathcal{M}$ is a space and Φ its maximal atlas, the set $(\mathcal{M}, \Phi)$ is a differentiable manifold. We can have C^∞, C^k, analytic, and semianalytic manifolds.

• **differential forms:** A $p-$form is defined to be a completely antisymmetric tensor of type $\begin{pmatrix} 0 \\ p \end{pmatrix}$. A one-form is a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor and a scalar function is a zero form. The number p is the degree of the form.

• **Dirac operator:** a "square root" of the D'Alambert operator in flat Minkowski spacetime

• **direct sum:**

$$\underline{\underline{A}} \oplus \underline{\underline{B}} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{B}} \end{pmatrix}, \quad \vec{v} \oplus \vec{w} = \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \tag{B.52}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \oplus \begin{pmatrix} p & q \\ r & t \end{pmatrix} = \begin{pmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & r & t \end{pmatrix} \tag{B.53}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} \tag{B.54}$$

• **discrete counting measure:**

• **discrete topology:** Let $\mathcal{T}$ be a topological space. A poit $p \in \mathcal{T}$ is called isolated if $\{p\}$ is open in $\mathcal{T}$. The unique topology in which every point is isolated is called the discrete topology. All functions are contiuous in the discrete topology.

• **distribution:** A distribution V on a manifold M is a choice of a subspace V_x of each tangent space $T_p(M)$, where the choice depends smoothly on x.

See integrable distribution.

• **division ring:** A ring with identity is called a *division ring* if all its non-zero elements are regular (invertible).

• **domain of an operator:** $\mathcal{D}(A) \subset \mathcal{H}$ such that $||\hat{}\psi > |^2 < \infty$

• **dominating convergence theorem:** The (Lebesgue) dominating convergence theorem is concerned with when the integral of a limit functions is equal to the limit of integrals, i.e. , when

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n. \tag{B.55}$$

• **dual complex:** Let us fix a triangulation Δ of a $d-$dimensional spacetime manifold $\mathcal{M}$. This triangulation defines another decomposition of $\mathcal{M}$ into cells called a dual complex. There is a one-to-one correspondence between $k-$simplicies of the triangulation and the $(d-k)-$cells in the dual complex.

• **dual group:** The dual group $\hat{G}$ is the set of characters $\{\gamma\}$ of G, i.e., homorphisms of G into the circle group $\{z \in \mathbb{C},\ |z| = 1\}$, with group multiplication defined by

$$< \gamma_1\gamma_2, g >=< \gamma_1, g >< \gamma_2, g > \tag{B.56}$$

[157]

$$(\lambda_1 g_1 + \lambda_2 g_2) \cdot g_3 = \lambda_1 g_1 \cdot g_3 + \lambda_2 g_2 \cdot g_3 \tag{B.57}$$

group algebra $\mathbb{R}(G)$.

As this has the structure of a vector space we may form a dual vector space via some inner product Fun(G) of functions on G. This bilinear inner product will take values from $< .,. >$: $\mathbb{R}(G)\otimes$Fun$(G) \to \mathbf{R}$. A natural choice is simply as functions on the group space $< g, f >:= f(g)$.

given the inner product between these two vector spaces and given an operator on one of them we can define its dual action, that is its adjoint, acting on the other.

generators of the group vector space e_i and the generators of the dual space e^i.

• **dual space:**

the algebraic dual The set of all functionals defined on a vector space X is can itself be made into a vector space. This is called the alghebraic dual of X and is denoted X^*.

the topological dual The set of all linear bounded (that is, continuous) functionals on X is called the topological dual of X and is denoted X'. It is called topological dual because ... continuous transformation preserves the topology the topological dual [208]:

Choose a dense and invariant domain Φ for a .is defined as the algebraic dual of Φ, i.e. the set of all linear functionals on Φ equipped with the weak *topology of pointwise convergence.

• **embedding:** doesn't come arbitrarily close to itself.

• **empty set:** Set which has no elements, denoted $\emptyset$.

• **entire function:** A complex function $f(z)$ analytic at all points of any open set of the complex plane.

OR

A complex function $f(z)$ analytic everywhere in the complex plane within a finite distance of the origin, it is an entire function. For example polynomials $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ are entire functions. They diverge at infinity...

• **enveloping algebra:** see universal enveloping algebra

If I is an ideal of a Lie algebra $\mathcal{G}$, the equivalence classes in $\mathcal{G}$

$$[A] = [A + I], \quad A \in \mathcal{G}, I \in I, \tag{B.58}$$

forms a Lie algebra - the quotient algebra, $\mathcal{G}/I$.

$$[[A], [B]] = [[A, B]] \tag{B.59}$$

$$[A + I_1, B + I_2] = [A, B] + [i_1, B] + [A, i_2] + [i_1, I_2] \tag{B.60}$$

$\psi : \mathcal{G}$

the sums and products in $im(\psi) \subset$ form an algebra called the **enveloping algebra** of $\mathcal{G}$

• **epimorphism:** A surjective group homomorphism.

• **epsilon net:** A finite or infinite number of points on a metric space such that each point on the space is with a distance of ϵ of some point on the net.

• **equivalence classes:** We transfer our attention from objects and products of objects to consideration of equivalence classes of objects and the induced multiplication between these classes.

• **essentially self-adjoint:** An operator which has a unique self-adjoint extension is said to be essentially self-adjoint, having a canonically defined Friedrichs extension and canonical functional calculus.

• **ergodic:**

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\omega(\tau^t(A)B)dt=\omega(A)\omega(B), \tag{B.61}$$

and weak-mixing

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}|\omega(\tau^t(A)B)-\omega(A)\omega(B)|^2dt=0. \tag{B.62}$$

• **Euler's theorem:**

$$F-E+V \tag{B.63}$$

• **extension of an operator:** An opeator B is an extension of an operator A if $D(A)\subset D(B)$ and $Bx=Ax$ for $x\in D(A)$.

• **extended diffeomorphisms:** Consider maps $\phi:\Sigma\to\Sigma$, that are continuous, invertible, and such that the map and its inverse are smooth everywhere, except, possibility, at a finite number of isolated points. The group formed by these maps is usually denoted $Diff^*$. (A strict diffeomorphism is a map that is smooth everywhere and whose inverse is also smooth everywhere).

See diffeomorphisms.

• **15-j symbols:** An elementary vertex of BF theory. A 4-simplex with 15 quantum numbers, the 10 irreducible representations associated with the 10 edges and the five interwiner labels of the 5 nodes (tetdrahedra) of the 4-simplex. See fig (B.8).

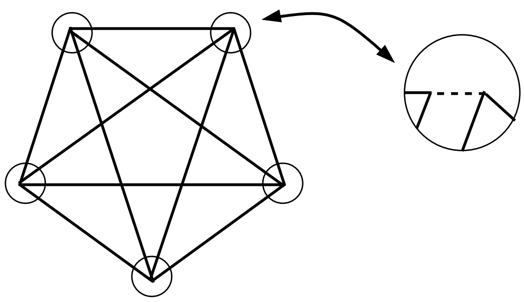


Figure B.8: 15JSymbol.

• **faithful:**

faithful representation: A representation is said to be faithful if Ker$\pi = \{0\}$ and non-degenerate if $\pi(a)\psi = 0$ for all $a \in A$ implies $\psi = 0$.

The representation () is said to **faithful** if and only if π is a $*$-morphism between $\mathcal{A}$ and $\pi(\mathcal{A})$, i.e., if and only if ker $\pi = \{0\}$.

• **Fell's theorem:** From [11]: Let $(\mathcal{F}_1, \pi_1)$ and $(\mathcal{F}_2, \pi_2)$ be possibly unitary inequivalent representations of the Weyl algebra $\mathcal{A}$. Let $A_1, \ldots, A_n \in \mathcal{A}$ and let $\epsilon_1, \ldots, \epsilon_n > 0$. Let ω_1 be an algebraic state corresponding to a density matrix on $\mathcal{F}_1$. Then there exists a state ω_2 corresponding to a density matrix $\mathcal{F}_2$ such that for all $i = 1, \ldots, n$ we have

$$|\omega_1(A_i) - \omega_2(A_i)| < \epsilon_i. \tag{B.64}$$

See physics glossary.

• **fibre bundles:** A bundle is a triplet $(E, \pi, \mathcal{M})$, where E and $\mathcal{M}$ are manifolds of some differentiability class and $\pi : E \to \mathcal{M}$ is the projection map. The inverse image $\pi^{-1}(x)$, $x \in \mathcal{M}$, is the fibre, F_x, over x. Fibre bundles are those bundles whose fibres over all of $\mathcal{M}$ are homoemorphic to a common space F, the typical fibre. They are the proper mathematical notion for introducing internal symmetries in a field theory.

• **fibre metric:** Let s and s' be sections over U_I. The inner product between s and s' at p is defined by

$$(s, s')_p := h_{IJ}(p) s^I(p) s'^J(p)$$

if the fibre is in $\mathbb{R}^k$. If the fibre is $\mathbb{C}^k$ we define

$$(s, s')_p := h_{IJ}(p) \overline{s^I(p)} s'^J(p).$$

• **field:** (b) A commutative division ring. Are the "number systems" in maths. *over a field* means $\alpha\mathbf{x} + \beta\mathbf{v}$.

• **filter:** A filter in a set X is a system $\mathcal{F}$ of non-empty subsets of X satisfying the conditions:

i) $A \cap B \in \mathcal{F}$ for all A and B in $\mathcal{F}$,
ii) if $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

• **first countable:** A topological space is first countable space if it has a countable if it has a countable base at each point.

• **first homotopy group:**

• **foliation:** A foliation consists of an integrable sub-bundle of a tangent bundle.

• **folium:** A state's **folium** is the set of states ω_ρ on $\mathcal{U}$ defined by

$$\omega_\rho := \frac{\mathrm{tr}_{\mathcal{H}_\omega}(\rho\pi_\omega(a))}{\mathrm{tr}_{\mathcal{H}_\omega}(\rho)} \tag{B.65}$$

where ρ is a positive trace class operator on the GNS Hilbert space $\mathcal{H}_\omega$.

• **folium of states:** The folium of states is the set of states determined by the density matrices on the Hilbert space of the given representation.

• **forgetful functor:** MOSTLY TAKEN DIRECTLY FROM mathworld.wolfram.com

In many categories the objects are sets equipped with some kind of additional structure. For example the category of groups, where an object is a group, and an arrow is a group homomorphism $f : G_1 \to G_2$ from G_1 to G_2. However a category need not have structured sets as its objects.

In a mere set elements are either the same or different; and that's all.

A forgetful functor (also called underlying functor) is defined from a category of algebraic gadgets (groups, Abelian groups, modules, rings, vector spaces, etc.) to the category of sets. A forgetful functor leaves the objects and the arrows as they are, except for the fact they are finally considered only as sets and maps, regardless of their algebraic properties.

Other forgetful functors neglect only part of the algebraic properties, e.g., the commutative law when passing from Abelian groups to groups, or multiplication when passing from rings to Abelian groups.

• **free action:** The group action on a manifold (moving one point of the manifold to another in a way that has the structure of a group G) is said to be free if the every element that is not the identity of G has no fixed points.

• **free group:** see ???

• **free associative algebra:**

• **Fréchet space:** complete metric space???

• **Friedrichs extension:** quadratic form $Q_{\hat{\mathbf{M}}}$ whose closure is the quadratic form of a unique self-adjoint operator $\overline{\hat{\mathbf{M}}}$, called the Friedrichs extension of $\hat{\mathbf{M}}$.

Master constraint

The Friedrichs extension is a self-adjoint extension of a non-negative densely defined symmetric operator

Let A be a semi-bounded symmetric operator, that is,

$$q_A(\psi) = <\psi, A\psi> \geq \gamma \|\psi\|^2, \qquad \gamma \in \mathbb{R}.$$

Then there is a self-adjoint extension $\tilde{A}$ whihc is also bounded from below by γ and which satisfies $D(\tilde{A}) \subset= \mathcal{H}_{A-\gamma}$.

• **functional calculus:**

allows one to construct all kinds of operators.

finite dimensional Hilbert space $\mathcal{H}$

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

Define $\Psi : C(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ by

$$\Psi(f) = f(\lambda_1)P_1 + \cdots + f(\lambda_n)P_n.$$

It is not hard to see that Ψ is an isometric $*-$isomorphism into its range: i.e.,

$$\begin{array}{ll} (a) & \|\Psi(f)\| = \|f\|_\infty \\ (b) & \Psi(f+g) = \Psi(f) + \Psi(g) \\ (c) & \Psi(fg) = \Psi(f)\Psi(g) \\ (d) & \Psi(\overline{f}) = \Psi(f)^*. \end{array} \tag{B.66}$$

The process of passing from $f \in C(\sigma(T))$ to $f(T)$ is called functional calculus. It allows us to construct, for example, square roots, logs, and exponentials of operators.

• **functionals:**

Positive linear functionals correspond precisely to the set of density matrices.

• **functions:**

functions of compact support: function is non-zero only within a compact region - a function whose domain is a compact space. Interesting things about these are....

• **function spaces:**

Examples. Let $U \subset \mathbb{R}^n$ be an open set of $\mathbb{R}^n$.

(1) $P(U)$ is the space of all polynomials of n variables as functions on ω;

(2) $C(U)$ the space of all continuous functions on U;

(3) $C^k(U)$ the space of all functions with continuous partial derivatives of order k on U;

(4) $C^\infty(\omega)$ the space of all smooth (infinitely differentiable) functions on U.

• **functor:** Functors are the structure preserving assignments between categories.

A **contravariant functor** $F : \mathcal{C} \to \mathcal{V}$ between two categories

(i) maps every object $C \in \mathrm{ob}\mathcal{C}$ to an object $F(C) \in \mathrm{ob}\mathcal{V}$, and

(ii) every morphism $f : C_1 \to C_2$ in $\mathcal{C}$ to a morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathcal{V}$, such that

$$F(g \circ f) = F(f) \circ F(g), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}. \tag{B.67}$$

A **covariant functor** $F : \mathcal{C} \to \mathcal{V}$ that maps every morphism in $\mathcal{C}$ to $F(f) : F(C_1) \to F(C_2)$, and

$$F(g \circ f) = F(g) \circ F(f), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}. \tag{B.68}$$

Contravariant ones change the direction of arrows, the covariant ones preserve directions.

Examples:

Let A be a vector space and $F(C) = \mathrm{Hom}(C, A)$, the linear maps from C to A.

$$C_1 \xrightarrow{\phi} C_2 \qquad \mathrm{Hom}\,(C_1,A) \xleftarrow{\mathrm{Hom}\,(\phi,A)} \mathrm{Hom}\,(C_2,A)$$

Figure B.9: A contravariant functor.

• **Gauss-Bonnet theorem:**

$$\int_{\mathcal{M}} K dS = 2\pi\chi(M) \tag{B.69}$$

• **Gauss-Codaza equations:** The torsion is the anti-symmetric part of the connection and the curvature in terms of the connection:

• **Gaussian curvature:** a disk D_ϵ centered at the point $\mathbf{p}$ with area $A(D_\epsilon)$

$$K(\mathbf{p}) = \frac{12}{\pi} \lim_{\epsilon \to 0} \frac{\pi\epsilon^2 - A(D_\epsilon)}{\epsilon^4} \tag{B.70}$$

• **Gauss linking number:**

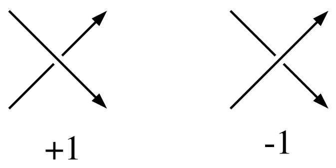


Figure B.10: Computing the Gauss linking number.

$$L = \frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\nu \epsilon_{\mu\nu\beta} \frac{(z^\beta - y^\beta)}{|z-y|^3} \tag{B.71}$$

Consider two closed loops γ and γ', as for example in Fig E.7 . If we think of first loop γ to be a wire carrying a current I, then by law it will generate a magnetic field $\mathbf{B}$ around the closed curve γ'.

$$B[\alpha] = \int_{S^\alpha} B^a d^2 S_a = \oint_\alpha A_a dl^a, \tag{B.72}$$

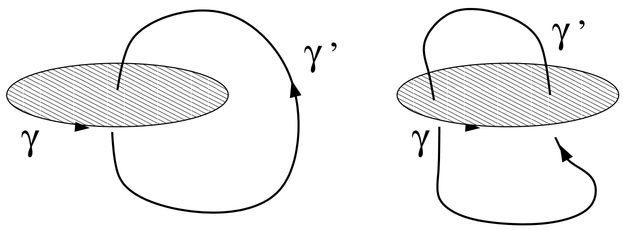


Figure B.11: Computing the Gauss linking number.

n is the number of times that the current passes through the closed loop. Let us set $I = 1$, we have

$$L = n = \frac{1}{4\pi} \oint_{\gamma'} \mathbf{B}(x') \cdot d\mathbf{x}. \tag{B.73}$$

We can calculate the magnetic field $\mathbf{B}(x')$ produced by wire γ with the Biot-Savart law, Eq(B.73). Substituting this into E.7, we get an explicit equation for the Gauss linking number:

$$L = \frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\nu \epsilon_{\mu\nu\beta} \frac{(z^\beta - y^\beta)}{|z-y|^3} \tag{B.74}$$

started off knot theory. knot theory of atoms. Irony that that the basic constituents of the fundamental physical description of nature put forward by LQG are knots (well actually links - a link being a set of knots).

We say that two spin networks are **isotopic**. So by abstract spin networks we mean the isotopic type rather than a particular way of realizing the spin network in "space".

• **Gauss self-linking number:**

$$\frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_i} dy^\nu \epsilon_{abc} \frac{(z^c - y^c)}{|\gamma(s) - \gamma(t)|} \tag{B.75}$$

Figure B.12: Gauss self-linking number.

• **Gel'fand map:** A map from the commutative $C*-$algebra $\mathcal{A}$ onto the space of continuous functions on the spectrum of the algebra, $\mathcal{C}(\Delta(\mathcal{A}))$.

$$\begin{aligned} Alg &\rightarrow C(\Delta) \\ a &\mapsto \overset{\vee}{a} \end{aligned} \tag{B.76}$$

• **Gel'fand-Naimark-Segal construction:**

• **Gel'fand-Neimark theorem:** Every C^*-algebra with identity is isomorphic to the C^*-algebra of all continuous bounded functions on a compact Hausdorff space called the *spectrum* of the algebra.

• **Gel'fand spectral theorem:**

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with C.

• **Gel'fand topology:**

$$\begin{array}{rcl} Alg & \to & C(\Delta) \\ a & \mapsto & \overset{\vee}{a} \end{array} \tag{B.77}$$

• **Gel'fand spectral theorem:**

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with C.

• **Gel'fand topology:**

weak* convergence of a weak* convergence of a sequence (rather, a generalization of a sequence called a net) of functionals. Then weak* convergence of (f_n) means that there is an $f \in X'$ such that $f_n(x) \to f(x)$ for all $x \in X$.

Every character is a bounded linear functional on $\mathcal{A}$, that is, $\Delta(\mathcal{A}) \subset \mathcal{A}'$. The Gel'fand topology on the spectrum of a unital, Abelean Banach algebra is the weak * topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$.

• **Gel'fand transformation:** Let $\mathcal{A}$ be a Banach algebra?? Given any $A \in \mathcal{A}$ we can define a function $\hat{A} : \Delta \to \mathbb{C}$ by

$$\hat{A}(h) = h(A). \tag{B.78}$$

$\hat{A}$ is called the Gel'fand transform of A.

• **generalized functions:** distributions

• **generalized knot theory:** knots with intersections...

• **germs:** Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds and $x \in \mathcal{M}$. Consider all smooth mappings $f : \mathcal{U}_f \to \mathcal{N}$, where $\mathcal{U}_f$ is some open neighbourhood of x in $\mathcal{M}$. We say two such functions f, g are equivalent and we put $f \sim_x g$ if there exists an open neighbourhood $\mathcal{V}$, which contains x, such that $f|\mathcal{V} = g|\mathcal{V}$. This is an equivalence relation on the set of mappings considered. The equivalence class of a mapping f is called the germ of f at x, sometimes denoted by $\mathrm{germ}_x\, f$. The set of all these germs is denoted by C.

We may also consider the composition of germs: $\mathrm{germ}_{f(x)} \circ \mathrm{germ}_x\, f = \mathrm{germ}_x\, (g \circ f)$.

Germ of an edge: Let $x \in \Sigma$ be given. The germ $[e]_x$ of an entire analytic edge e with $b(e) = e(0) = x$ is defined by the infinite number of Taylor coefficients $e^{(n)}(0)$ in some parameterisation. The germ $[e]_x$ encodes the orientation of e and its knowledge allows us to reconstruct $e(t)$ from x up to reparametrisation due to analyticity.

• **Gleason's theorem:** Let H have dimension greater than 2. Then every countably additive probability measure on the lattice $L(H)$ has the form $\mu(P) = Tr(WP)$, for a density operator W on H.

Stanford encyclopedia of philosophy.

• **GNS construction:**

• **graph**

For a linear operator $A : X \to Y$ the set of points $\{x \in X, Ax \in Y\}$ is called the graph of the operator A.

See partial function.

• **greatest lower bound:** the greatest lower bound is usually called its **infimum** and denoted inf A.

• **Green's function:** Roughly, a Green's function $G(\vec{r}, t)$ is the solution of a differential equation subject to the initial condition $G(\vec{r}, t) = \delta(\vec{r})$.

• **Groenewold and van Howe theorem:**

• **group:** A collection G of objects g_i upon which we associate a 'multiplication' operation (techniqually know as a binary operation) which we write as "$\cdot$". By definition a group satisfies the following properties

(i) identity e such that $e \cdot g = g \cdot e = g$;
(ii) closed under multiplication i.e. for any $g_1, g_2 \in G$ their product is also an element of the group i.e., $g_1 \cdot g_2 \in G$;
(iii) every element has an inverse;
(iv) associativity $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$;

• **group homomorphism:** A homomorphism between two groups is, roughly speaking, a function between them which preserves the (respective) group operations. Let G_1, G_2 be groups. A function f that maps the elements of G_1 in to G_2 ($f : G_1 \to G_2$) is a homomorphism if and only if, for all $a, b \in G_1$ we have

$$f(ab) = f(a)f(b). \tag{B.79}$$

$f : G_1 \to G_2$ is an isomorphism if it is a bijection (one-to-one and onto). We write $G_1 \cong G_2$. An isomorphism from a group G to itself is called an automorphism.

• **groupoid:** closure applying the binary operator to two elements of a given set S returns a value which is itself a member of S. Associativity, existence of identity, and inverse of each element are not required. An example of a groupoid is for a, b positive definite real numbers

$$a \star b = \sqrt{(a^2 + b)} \tag{B.80}$$

There is no identity element because we require $a, b > 0$, and correspondingly no inverse. It is easy to check that associativity does not always hold:

$$\begin{aligned} a \star (b \star c) &= a \star (\sqrt{(b^2 + c)}) \\ &= \sqrt{(a^2 + \sqrt{(b^2 + c)})} \end{aligned}$$

and

$$\begin{aligned} (a \star b) \star c &= \sqrt{((a \star b)^2 + c)} \\ &= \sqrt{(a^2 + b + c)}, \end{aligned}$$

for these to be equal requires

$$\sqrt{(b^2 + c)} = b + c$$

or

$$b^2 + c = (b + c)^2 = b^2 + 2bc + c^2$$

or

$$c[c + (2b - 1)] = 0$$

or

$$c = 0 \quad or \quad c = 1 - 2b$$

so unless $b < 1/2$, c wont be positive definite.

• **group theory:**

• **Haag's theorem:** Two significant consequences of Haag's theorem are:

i) Scalar filed theories with different masses correspond to unitarily inequivalent representations of the Weyl algebra.

ii) Representations of interacting and free field theories are unitarily inequivalent and hence means that the interaction picture underlying perturbative QFT of Wightman fields strictly speaking does not exist, it only exists if there are no interactions.

The formal statement of Haag's theorem is:

Suppose that

(1) two weakly continuous and irreducible representations $(\pi_I, \mathcal{H}_I), I = 1, 2$ of the Weyl algebra $\mathfrak{A}$ of a scalar field theory are given,

(2) the Euclidean group E of spacial transaltions and rotations is implemented unitarily and weakly continuously by representations u_I on $\mathcal{H}_I$ such that $u_I(e)\pi_I(a)u_I^{-1}(e) = \pi_I(\alpha_e(a))$ for all $e \in E, a \in \mathfrak{A}$ and

(3) there is a unique Euclidean invariant state $\Omega_I \in \mathcal{H}_I$, that is, $u_I(e)\Omega_I = \Omega_I$.

If the two representations of the Weyl algebra are unitary equivalent, that is, there exista unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that $W\pi_1(a)W^{-1} = \pi_2(a)$ for all $a \in \mathfrak{A}$, then $Wu_1(a)W^{-1} = u_2(e)$ for all $e \in E$ and $V\Omega_1 = c\Omega_2$ where c is a complaex number of modulus one.

• **Haar measure:** For any compact group G the Haar measure is the (unique) measure dU that is group invariant.

Let G be a locally compact abelian group. A Haar measure on G is a positive regular Borel measure μ having the following two properties:

(1) $\mu(E) < \infty$ if E is compact;
(2) $\mu(E + x) = \mu(E)$ for all measurable $E \subset G$ and all $x \in G$.

One can prove that the Haar merasure always exists and that it is unique up to multiplication by a positive constant.

• **Hahn-Banach theorem:** The Hahn-Banach theorem is a central tool in functional analysis. It allows for the extending bounded linear functionals defined on a subspace of some vector space to the full space, in a norm-preserving manner, and shows that there are "enough" continuous linear functionals defined on every normed vector space to make the study of dual spaces interesting.

• **Hall transform:**

• **Harmonic polynomials:** homomgeneous polynomials P annihilated by the Laplace operator on $\mathbb{R}^d$:

$$\Delta P(x^1, \ldots, x^d) = 0, \tag{B.81}$$

for

$$\Delta = \partial_1^2 + \cdots + \partial_d^2 \tag{B.82}$$

where $\partial_i = \partial/\partial x_i$.

• **Hausdorff (or T_2):** A topological space is called Hausdorff if and only if for any two distinct points x and y, $x \neq y$, there are open sets O_1, O_2 such that $x \in O_1$, $y \in O_2$, and the two open sets do not overlap, i.e., $O_1 \cap O_2 = \emptyset$.

• **heat kernal:**

the coherent states of the simple harmonic oscillator coherent states can be obtained as analytic continuation of the heat kernal on $\mathbb{R}^n$:

$$\psi_z^t(x) = e^{-t\Delta}\delta_{x'}(x)\big|_{x'\to z} \quad x \in, z \in \mathbb{C}, \tag{B.83}$$

the Laplacian Δ playing the role of a *complexifier*???

It was shown by Hall [242] that coherent states on a connected compact Lie group G can analogously be defined as an analytic continuation of the heat kernel

$$\psi_g^t(x) = e^{-t\Delta_G}\delta_{h'}^{(G)}(h)\Big|_{h'\to u}, \tag{B.84}$$

to an element u of the complexification $G^{\mathbb{C}}$ of G.

• **heat kernal measure:**

$$\frac{d\mu}{dt} = \frac{1}{4}\Delta K_C \mu_t \tag{B.85}$$

let μ_t denote the associated heat kernel measure

$$d\mu_t(g) := \mu_t(g)dg. \tag{B.86}$$

• **Heisenberg group:** A matrix representation of the Heisenberg Lie algebra is

$$m(p,q,t) = \begin{pmatrix} 0 & p_1 & \cdots & p_n & t \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It is easily verified that

$$m(p,q,t)m(p',q',t') = m(0,0,pq')$$

and so

$$[m(p,q,t), m(p',q',t')] = m(0,0,pq'-qp').$$

Using

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

we have

$$\exp m(p,q,t)\exp m(p',q',t') = \exp m\big(p+p',q+q',t+t'+\frac{1}{2}(pq'-qp')\big).$$

One identifies a point $X \in \mathbb{R}^{2n+1}$ with the matrix $e^{m(X)}$, and makes $\mathbb{R}^{2n+1}$ into a group with group law

$$(p,q,t)(p',q',t') = \big(p+p',q+q',t+t'+\frac{1}{2}(pq'-qp')\big).$$

This is called the Heisenberg group and is denoted $\mathbf{H_n}$. The element $(0,0,0)$ is the identity and the inverse of the element (p,q,t) is $(-p,-q,-t)$.

• **Hellinger-Toeplitz theorem:** If A is a symmetric operator whose domain of dependence $D(A)$ is the whole of the Hilbert space $\mathcal{H}$, $D(A) = \mathcal{H}$, then A is bounded. Note that symmetric everywhere defined operators are self-adjoint, so this theorem can also be stated that a self-adjoint operator is bounded. This shows that unbounded operators are not everywhere defined on the Hilbert space.

• **Heyting algebra:** ??? [367] Internal observables satisfy a Heyting algebra, which is a weak version of the Boolean algebra of ordinary observables.

Open sets are the standard example of a Heyting algebra. Given an open set $U \subset O$, the complement of U, which we call U^c, is a closed set. If we want an algebra of open sets, we need, instead of U^c, to use the interior of the set complement of U, $Int(U^c)$. But then, clearly, $U \cap Int(U^c) \subset O$ since the closure of U has been left out. (example comes from [366]).

A poset $\mathbf{H}$ is a Heyting algebra if we have elements

$$T \in \mathbf{H}, \qquad \perp \in \mathbf{H} \tag{B.87}$$

and operations

$$T \in \mathbf{H} \tag{B.88}$$

See sections I.7.

• **higher-dimensional algebra:** [400]

• **Hilbert-Schmidt operators:** The space of Hilbert-Schmidt operators consists of those compact operators A such that the trace $\mathrm{Tr}(A^*A)$ exists. These operators naturally appear in Bogol'ubov transformations.

• **Hilbert space:** A complete inner product space which is a complete metric space with respect to the metric induced by its inner product (compare to a Banach space).

(see also a pre-Hilbert space).

Hilbert space completion: Consider the completion of an inner product space V as the metric space completion, $\mathcal{H}$, of V by taking equivalence classes of Cauchy sequences in V. It can be shown that the inner product structure of V naturally extends to $\mathcal{H}$ in such a way as to provide $\mathcal{H}$ with the structure of a Hilbert space, with V naturally identified with a dense subspace of $\mathcal{H}$. See Reed and Simon.

• **Hölder inequality:** integral (or sum) inequality

$$\left(\int |fg|\right) \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q} \tag{B.89}$$

where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$.

$$\left(\sum_i |f_i g_i|\right) \leq \left(\sum_i |f_i|^p\right)^{1/p} \left(\sum_i |g_i|^q\right)^{1/q} . \tag{B.90}$$

• **homeomorphic:** Related by a homeomorphism.

• **homeomorphism:** A one-to-one correspondence that is continuous in both directions between the points of two topological spaces.

• **homogeneous function:** A function $f(x_1, \ldots, x_n)$ is a homogeneous function of degree D if

$$f(\rho x_1, \ldots, \rho x_n) = \rho^D f(x_1, \ldots, x_n) \tag{B.91}$$

Define $x_1' = \rho x_1, \ldots x_n' = \rho x_n$. Then differentiating both sides of (B.91) by ρ we find

$$
\begin{aligned}
D\rho^{D-1} f(x_1, \ldots, x_n) &= \frac{\partial f}{\partial x_1'}\frac{\partial x_1'}{\partial \rho} + \cdots + \frac{\partial f}{\partial x_n'}\frac{\partial x_n'}{\partial \rho} \\
&= \left[x_1 \frac{\partial}{\partial(\rho x_1)} + x_2 \frac{\partial}{\partial(\rho x_2)} + \cdots + x_n \frac{\partial}{\partial(\rho x_n)}\right] f(\rho x_1, \ldots, \rho x_n)
\end{aligned}
$$

Setting $\rho = 1$ gives

$$
\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) = D f(x_1, \ldots, x_n). \tag{B.92}
$$

• **homeomorphism:** A homeomorphism is a one-to-one correspondance that is continuous in both directions between two topological spaces. It is an equivalence relation and preserves topological properties. If it also preserves distances it is an isometry. See also diffeomorphism.

• **Hom(A,B):** $\mathrm{Hom}(A, B)$ denotes the collection of morphisms from A to B.

• **homogeneous simultaneous equations:**

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z + d_1 &= 0 \\
a_2 x + b_2 y + c_2 z + d_1 &= 0 \\
a_3 x + b_3 y + c_3 z + d_1 &= 0
\end{aligned} \tag{B.93}
$$

$$
\begin{aligned}
a_1 \tilde{x} + b_1 \tilde{y} + c_1 \tilde{z} &= 0 \\
a_2 \tilde{x} + b_2 \tilde{y} + c_2 \tilde{z} &= 0 \\
a_3 \tilde{x} + b_3 \tilde{y} + c_3 \tilde{z} &= 0
\end{aligned} \tag{B.94}
$$

$$
X = \frac{\tilde{x}}{\tilde{z}}, \quad Y = \frac{\tilde{y}}{\tilde{z}} \tag{B.95}
$$

$$
\begin{aligned}
a_1 X + b_1 Y + c_1 &= 0 \\
a_2 X + b_2 Y + c_2 &= 0 \\
a_3 X + b_3 Y + c_3 &= 0
\end{aligned} \tag{B.96}
$$

These have the trivial solution $X = 0, Y = 0, Z = 0$. If the condition

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0 \tag{B.97}$$

is met there exist non-trivial solutions to the homogeneous equations. In this case we can start with a solution to the non-homgeneous system of equations equations and obtain another one by adding arbitrary linear combination of the solutions to the homogeneous system of equations:

$$\begin{aligned} x &\to x + v_x X \\ y &\to y + v_y Y \\ z &\to z + v_Z Z \end{aligned} \tag{B.98}$$

• **homomorphism:** A homomorphism θ is a mapping from one algebraic structure to another under which the structual properties of its domain are preserved in its range in the sense that if $*$ is te operation on the domain, and $\circ$ is the operation on the range, then

$$\theta(x * y) = \theta(x) \circ \theta(y).$$

In particular, a group homomorphism is a mapping θ such that both domain and range are groups, and

$$\theta(xy) = \theta(x)\theta(y)$$

for all x and y in the domain. A ring homomorphism is a mapping θ from one ring to another such that

$$\theta(x + y) = \theta(x) + \theta(y) \quad \text{and} \quad \theta(xy) = \theta(x)\theta(y).$$

• **hoop group:** 'holonomy equivalence class of a loop based at x_0'.

• **Hopf algebra:** [190], [192], [191]

Algebra

multiplication $\mu : A \otimes A \to A$ $m(a \otimes b) = ab$. m is **associative**, $(ab)c = a(bc)$.

$$[m(m \otimes \mathbf{1})](a \otimes b \otimes c) = [m(\mathbf{1} \otimes m)](a \otimes b \otimes c) \tag{B.99}$$

mathematicians write this in the short-hand notation

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m) \tag{B.100}$$

and depict it as a diagram (I.2.4(a)).

unit η sends every number to the same number times the identity element, i.e.

$$\eta(k) := \mathbf{1}k, \quad \text{for all } k \in \mathbb{C}. \tag{B.101}$$

$$m(a \otimes \eta(k)) = ak = ka = m(\eta(k) \otimes a), \quad \text{for all } a \in A, \text{ and for all } k \in \mathbb{C}. \tag{B.102}$$

$$m \circ (\eta \otimes \mathrm{id})(a) = a = m \circ (\mathrm{id} \otimes \eta)(a) \tag{B.103}$$

Coalgebra

comultiplication $\Delta : C \to C \otimes C$. Δ is **coassociative**.

$$(\mathbf{1} \otimes \Delta)\Delta = (\Delta \otimes \mathbf{1})\Delta \tag{B.104}$$

counit

Coalgebra

A bialgebra structure on a vector space $A = C$ is a quadruple of objects $(\mu, \eta, \Delta, \epsilon)$ which satisfy all of the commutative diagrams as well as the following compatibility equations:

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1) = 1 \otimes 1, \quad \epsilon(hg) = \epsilon(h)\epsilon(g), \quad \epsilon(1) = 1, \quad \text{for all } g, h \in A. \tag{B.105}$$

linear antipode $S : H \to H$

$$\mu(S \otimes \mathrm{id}) \circ \Delta = \mu(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{B.106}$$

the antipode is a kind of inverse.

In component form using generators:

$$
\begin{aligned}
e_i e_j &= m_{ij}^k e_k \\
\Delta(e_k) &= \mu_k^{ij} e_i \otimes e_j
\end{aligned}
\tag{B.107}
$$

$$
\epsilon_i \mu_k^{ij} = \delta_k^j \tag{B.108}
$$

Definition: Let C and D be coalgebras, with comultiplication Δ_C and Δ_D, and counits ϵ_C and ϵ_D respectively.

(i) A map $f : C \to D$ is a **coalgebra morphism** if $\Delta_D \circ f = (f \otimes f)\Delta_C$.

(ii) A subspace $I \subseteq C$ is a **coideal** if $\Delta \subseteq I \otimes C + C \otimes I$ and if $\epsilon(I) = 0$.

if I is a coideal then, the quotient C/I is a coalgebra with comultiplication induced from Δ. Proof in Appendix Q.

• **Hopf fibration:**

Hopf fibration of S^3. The base space is a 2-dimensional sphere S^2 and fibres are circles S^1

• **horizontal distribution:** Given a principal fibre bundle a horizontal distribution is the assignment of a subspace $V_p(P)$ to the tangent space $T_p(P)$ at each point p of P tat is tangent to the fibre.

• **horizontal vector fields:** Horizontal vector fields are fields whose flow lines move from one fibre into another.

• **hyperbolic differential equations:** wave equation. Consider the partial differential equation

$$
g^{ab} \nabla_a \nabla_b \phi + A^a \nabla_a + B\phi + C = 0 \tag{B.109}
$$

where A^a is an arbitrary smooth vector field, B and C are arbitrary smooth functions, and g_{ab} is an arbitrary smooth Lorentz metric such that the spacetime $(\mathcal{M}, g_{ab})$ is globally hyperbolic. A second order partial differential equation is said to be hyperbolic if and only if it can be expressed in the form

• **hyphs:** Hyphs are a type of collections of loops which have the advantage of being independent of the differentiablity category of the graphs under consideration and in particular includes the analytical and smooth category.

• **ideal:** An **ideal** a in A is a subset such that

(i) a is a subgroup of A regarded as a group under addition;

$(ii)\quad a \in a,\ r \in A \Rightarrow ra \in A$. ??????????????? left-ideal

right-ideal

two-sided ideal

Ideals play the same role in Lie algebras as the normal subgroups play in Lie group theory.

The set of commutators of a Lie algebra, $\mathcal{G}$, denoted by $[\mathcal{G}, \mathcal{G}]$, is a subalgebra of $\mathcal{G}$.

It is also a two-sided ideal of $\mathcal{G}$, for any $A, A_1, A_2 \in \mathcal{G}$,

$$[[A_1, A_2], A] = [A_3, A] \in [\mathcal{G}, \mathcal{G}], \tag{B.110}$$

where $A_3 = [A_1, A_2]$.

$$[A, [A_1, A_2]] = [A, A_3] \in [\mathcal{G}, \mathcal{G}], \tag{B.111}$$

The set $[\Delta A, \Delta A_2]$. Two-sided coideal

The ideal of a Lie algebra $[h_i, g_k] = \sum a_{ikl} h_l$ for all $g_i \in \mathcal{L}(G)$

• **immersion:** A differentiable map $\phi : \mathcal{M} \to \mathcal{N}$ between finite-dimensional manifolds $\mathcal{M}, \mathcal{N}$ is called an immersion when ϕ has everywhere rank $\dim(\mathcal{M})$. An immersion need not be injective (se fig B.13) but when it is,it is called an embedding.

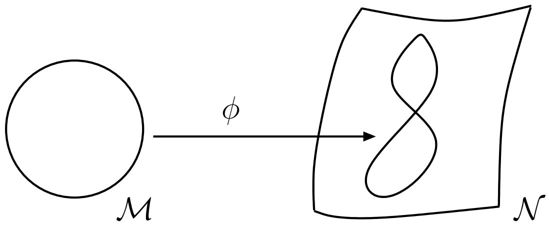


Figure B.13: .

• **implicit function theorem:** The simplest version of the implicit function theorem can be stated as follows. Let f be a continuous real-valued function on an open subset of $\mathbb{R}^2$ that contains the point (a, b), with $f(a, b) = 0$. Suppose that $\partial f/\partial y$ exists and is continuous on the given open subset and that $\partial f/\partial y(a, b) \neq 0$. Then there exist open intervals $U, V \in \mathbb{R}$, with $a \in U$ and $b \in V$, such that there exists a unique function $\rho : U \to V$ such that

$$f(x, \rho(x)) = 0$$

for all $x \in U$, and such that this function is continuous.

• **inclusion map:** IF $U \subseteq V$, the inclusion map i sends U to V, i.e., $i : U \rightarrow$.

• **if and only if:** This means necessary and sufficient. A statement is made of the form: "A is true if and only B is true". To establish A is true only if B is true, we assume A and then show it follows that B must be true. To establish A is true if B is true, we assume B and then show it follows that A must be true.

• **Infeld-van der Waerden symbols:**

• **infinite tensor product:**

• **injective:** A function that is one-to-one. Equivalently, a function is injective when no two distinct inputs give the same output.

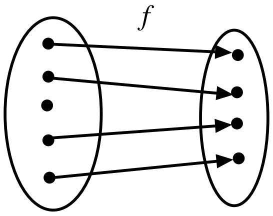


Figure B.14: injective

• **inner product space:** vector space equipped with an inner product

• **insertion operator:** Given an $r-$form

$$\omega = \frac{1}{r!}\omega_{\nu_1\nu_2\ldots\nu_r} x^{\nu_1} \wedge x^{\nu_2} \cdots \wedge x^{\nu_r}, \tag{B.112}$$

the insertion operator is defined by the operation

$$i_X\omega := \frac{1}{(r-1)!}\omega_{\nu\nu_2\ldots\nu_r} X^{\nu} \wedge x^{\nu_2} \cdots \wedge x^{\nu_r}. \tag{B.113}$$

• **integrable distribution:** A distribution on a manifold is said to be integrable if at least locally, there is a foliation of M by submanifolds such that V_x is the tangent space of the submanifold containing the point x.

• **interal subgroup:** Let G be a Lie group. An integral subgroup of G is a subgroup H with a connected Lie group structure such that the canonical injection of H into G is an immersion.

• **interchange of limit operations:** If a sequence of Riemann integral real-valued functions $f_1(x), f_2(x), \ldots$ converges to the function $f(x)$, can we assert that

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx \tag{B.114}$$

is true?

If a series of

$$\frac{d}{dx}\lim_{n\to\infty}\sum_{n=1}^{N} e_n(x) = \lim_{n\to\infty}\sum_{n=1}^{N}\frac{d}{dx}e_n(x) \tag{B.115}$$

• **interwiner:** A trivalent interwiner is a map $I : V^i \otimes V^j \otimes V^k \to \mathbb{C}$ invarinat under the diagonal action of the group on the tensor product.

• **intuituinistic logic:** Comes from constructive mathematics: No proof by contradiction allowed, i.e. *"for all x, either x or not x*

• **invariant polynomial:** consider polynomials in n^2 variables. Let us label these n^2 variables as the entries of an $n-$dimensional matrix A_{ij}, We write for the polynomial: $P(A)$. The polynomial $P(A)$ is said to be **invariant** when

$$P(gAg^{-1}) = P(A) \tag{B.116}$$

for all $g \in GL_n(C)$.

For example

$$\det\left(\mathbf{1} + \frac{\lambda}{2\pi i}A\right) = \sum_{n=0}^{N}\lambda^n P_n(A) \tag{B.117}$$

the coefficients $P_n(A)$ are invariant polynomials of degree n in A.

$$\begin{aligned}\sum_{n=0}^{N}\lambda^n P_n(gAg^{-1}) &= \det\left(\mathbf{1} + \frac{\lambda}{2\pi i}gAg^{-1}\right) = \det\left(g(\mathbf{1} + \frac{\lambda}{2\pi i}A)g^{-1}\right)\\ &= \det(g)\det\left(\mathbf{1} + \frac{\lambda}{2\pi i}A\right)\det(g^{-1})\\ &= \sum_{n=0}^{N}\lambda^n P_n(A).\end{aligned} \tag{B.118}$$

• **invariant subspace:** a subspace $\mathcal{M}$ is invariant under a set of operators if $A\psi$ is in $\mathcal{M}$ for every operator A in the set and every vector ψ in $\mathcal{M}$.

• **inverse function theorem:**

• **irreducible representation:** An representation of operator relations on $\mathcal{M}$, for example the Weyl relations, is irreducible if no proper subspace of $\mathcal{M}$ is invariant under a set of operator relations. Equivalently, given any $\Psi \in \mathcal{M}$, the span of all vectors under the operator relations forms a dense subspace of $\mathcal{M}$. The representation should be irreducible on physical grounds otherwise we have superselection sectors implying that the physically relevant information is already contained in a closed subspace.

• **ISO:**

• **isometrically isomorphic:** Let N and N' be normed linear spaces. These spaces are said to be isometrically isomorphic if the linear transformation T is a one-to-one from N to N' is a one-to-one such that

$$\|x\| = |\ T(x)\| \tag{B.119}$$

for all x in N. So that the normed linear space N' is essentially the same as N.

• **isometric operator:** An operator V defined on the whole of a Hilbert space $\mathcal{H}_1$ and mapping $\mathcal{H}_1$ on to the whole of another Hilbert space $\mathcal{H}_2$ is said to be isometric if, for all $f, g \in \mathcal{H}_1$,

$$(Vf, Vg)_2 = (f, g)_1,$$

where $(.\,,.\,)_1$ and $(.\,,.\,)_2$ denote the inner product on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

In particular, $\mathcal{H}_1$ and $\mathcal{H}_2$ may be subspaces of a single space (the subscrips on the inner products are then unnecessary).

A unitary operator is a particular case of an isometric operator; this case occurs if $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide.

• **isomorphic:** Two groups G and G' are said to isomorphic if their elements can be put into one-to-one correspondence which is preserved under multiplication.

• **isomotopic:** If two objects can be deformed into each other they are said to be isomotopic. For example knots that can be deformed into each other are called isomotopic.

• **isomotopy:** There are two kinds regular and ambient isotopy.

• **Jones polynomial:** The Jones polynomial is defined by:

its value on the unknot,
its value on the disjoint union of a knot with the unknot,
and a "Skein relation".

value on the unknot is chosen to be 1:

$$p(U) = 1. \tag{B.120}$$

Let $U \sqcup K$ be the disjoint union of U and K

$$p(U \sqcup K) = -(x + x^{-1})p(K). \tag{B.121}$$

$$x^{-2}p(K^{+}) - x^{2}p(K^{-1}) + (x^{-1} - x)p(K^{0}) = 0. \tag{B.122}$$

• **Jones-Witten invariant:** won Witten his Fields medal.

• **Kauffman bracket:**

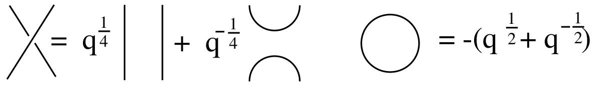


Figure B.15: Skien relation for the Kauffman bracket.

• **kernel:** The kernel of a group homomorphism $\varphi : G \to H$ is defined by

$$\ker\varphi := \varphi^{-1}(\{e_H\}) = \{x \in G : \varphi(x) = e_H\}.$$

• **Killing vector field:** isometries of the spacetime.

• **Kirby calculus:** must be related through sequence of operations. reminiscent of the Reiemeister moves.

• **knots and knot theory:** ... The formal statement of this intuitive idea is: a knot is a smooth embedding of a circle in three-dimensional Euclidean space. Two knots are said to be topologically equivalent if one can be deformed continuously into the other, without crossings.

Knots can be displayed as projections onto a 2-d plain surface.

Two knots or links are topologically equivalent if they can be transformed into each other by repeated use of the three Reidemeister moves.

The aim of knot theory is to characterize knots by a topological invariant.

• **lattice:** A lattice (a mathematical term) is a partially ordered set $(L, \leq)$ such that any two elements a and b possess a minimum in L, denoted $a \wedge b$ ("meet") and a maximum in L, denoted $a \vee b$ ("join"). That is, there exists respectively an element $a \wedge b$ satisfying

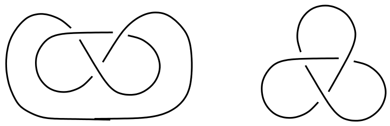


Figure B.16: knots Reidemeister moves.

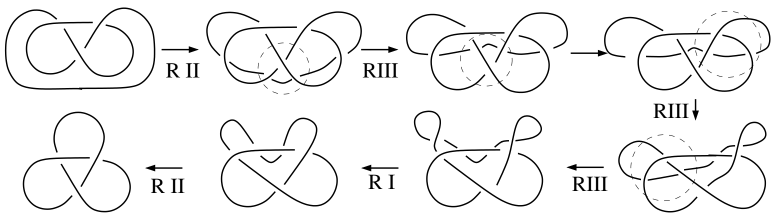


Figure B.17: knots Reidemeister moves.

$a \wedge b \leq a$, $a \wedge b \leq b$ such that any other element c which smaller than both a and b satisfies $c \leq a \wedge b$, and one element satisfying $a \vee b \geq a$, $a \vee b \geq b$ and any other element c with the same property is greater, i.e., $a \vee b \geq c$. For examples of lattices see section I.7.

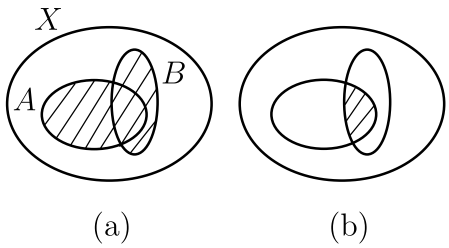


Figure B.18: If $A \subseteq B$ then $A \leq B$. partially ordered set of subsets of X is a lattice. (a) Their union $A \cap B$ is the l.u.b. (b) $A \cup B$ is the g.l.b.

a lattice is called complete if and

Orthomodular lattice

$$(a \vee (a^{\perp} \wedge (a \vee b))) = (a \vee b) \tag{B.123}$$

• **least upper bound:** 3 is an upper bound and its least upper bound the least upper bound is usually called its **supremum** and denoted sup A.

• **Lebesgue integral:** The Reimann integral doesn't deal with all functions we need for our purposes in formulating quantum mechanics and quantum field theory. Disturbing fact that the limit function $f(x) := \lim n \to \infty f_n(x)$ of a sequence of continuous functions $\{f_n(x)\}$ can be discontinuous. A function $\mu(x)$ we would hope has the following properties:

$$\begin{aligned} \mu((a,b]) &= b-a \\ &= \end{aligned} \tag{B.124}$$

Instead of splitting the integration domain into small parts, we decompose the range of the function (see Fig(**??**)).

$$\sum_{n}^{N} c_n \mu(f^{-1}(J_n)) \tag{B.125}$$

maximal invariant subspace:

• **left invariant vector field:**

$$g(t+s) = g(t)g(s) \tag{B.126}$$

Differentiating with respect to s and setting $s = 0$,

$$g'(t) = g(t)g'(0) \tag{B.127}$$

$$g(t) := L_g \exp(tX_e) = g\exp(tX_e) \tag{B.128}$$

Or the push-forward of the vector $X(g_e)$ at g_e by multiplication on the left by any g produces a vector $g_*[X(g_e)]$ at gg_e that coincides with the vector $X(gg_e)$ already at that point. So it is the natural definition of a 'constant' vector field on G.

• **left invariant form:** A differential $p-$form ω is called left invariant provided

$$L_x^* \omega = \omega.$$

If ω_0 is any given $p-$form at e, then a left invariant form is defined by

$$\omega_x = L_{x^{-1}}^* \omega_0.$$

One of the most important applications of left invariant forms is in the theory of connections in fibre bundles - featuring in theoretical physics in relation to the general mathematical framework for Yang-Mills theories.

• **Lie derivative:** Generates infinitesimal active difffeomorphisms.

• **Lie Group:**

one definition: A **Lie group** G is a group which is also a smooth manifold such that the multiplication $G \times G \to G$, $(a, b) \mapsto ab$,and the inverse $G \to G$, $a \mapsto a^{-1}$, are smooth.

• **limit curve lemma:** Introduce a background Riemannian (positive definite) metric on spacetime $\mathcal{M}$. A future inextendible causal curve will have infinite length to the future, as measured in the metric h, parameterization defined on the interval $[0, \infty)$. The limit curve lemma can then be stated:

Let $\gamma_n : [0, \infty) \to \mathcal{M}$ be a sequence of future inextendible causal curves, parameterized with respect to h-arc length, and suppose that $p \in \mathcal{M}$ is an accumulation point of the sequence $\{\gamma_n(0)\}$. Then there exists a future inextendible C^0 causal curve $\gamma : [0, \infty) \to \mathcal{M}$ such that $\gamma(0) = p$ and a subsequence $\{\gamma_m\}$ which converges to γ uniformaly with respect to h on compact subsets of $[0, \infty)$.

There are analogous versions of the limit curve lemma for past inextendible, and (past and future) inextendible causal curves.

• **linear:**

(i) $\alpha(x + y) = \alpha x + \alpha y$;

(ii) $(\alpha + \beta)x = \alpha x + \beta x$;

(iii) $(\alpha\beta)x = \alpha(\beta x)$;

(iv) $1 \cdot x = x$

A linear space is called a **real** linear space or a complex linear space according to whether the scalars are the real numbers or complex numbers.

Some jargon: A complex vector space will often be referred to as a vector space over *over* the complex numbers. This phrase isn't just reserved for vector fields, and will also be used in reference to say rings, groups. Also, the scalars may not just refer to numbers, there are more general "number systems" used in mathematics called *fields.* The real and complex numbers are special cases of fields.

see physics glossary.

• **linear analysis:**

Uniform boundedness

interior mapping principle

Hahn-Banach theorem

• **line bundle:** A line bundle L is a complex vector bundle with one-dimensional fibres.

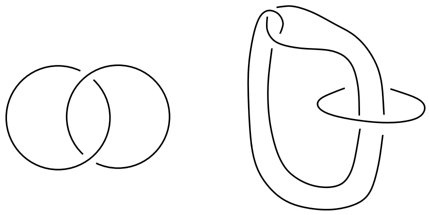


Figure B.19: links Whitehead link.

• **links:**

• **link invariants:**

invariants for three-manifolds Edward Witten topological field theory Chern-Simons

$$\int \mathcal{D}A e^{\int \mathcal{L}_{CS}} W_\alpha[A] \tag{B.129}$$

Knots and links can be obtained by the closure of braids (Alexander theorem)

• **linear independence:**

$$\sum_{i=1}^{N} a_i \mathbf{e_i} = 0 \implies \tag{B.130}$$

• **Lioville's theorem:** A bounded entire function is constant.

• **locally compact:** A topological space is said to be *locally compact* if every point $x \in X$ has an open neighbourhood whose closure is compact.

• **locally finite:**

• **locally trivial:** Given a fibre bundle P with typical fibre F and base space $\mathcal{M}$. Locally trivial means that for each $x \in \mathcal{M}$, there is a neighbourhood U of x and an isomorphism,

$$\phi : \pi^{-1}(U) \to U \times F,$$

sending each fibre $\pi^{-1}(x)$ to $\{x\} \times F$. Intuitively, a bundle looks locally as a product of the base manifold and the fibre. We call ϕ a local trivialisation. If the bundle space E is globally $\mathcal{M} \times F$ the bundle is said to be trivial.

• **logic:**

Conjunction

$$\frac{\begin{array}{l}p\\q\end{array}}{p \wedge q}$$

$$\frac{p \wedge q}{p}$$

$$\frac{p}{p \vee q} \qquad \frac{q}{p \vee q} \qquad \frac{q \vee p \qquad p \rightarrow r \qquad q \rightarrow r}{r} \tag{B.131}$$

Addition

$$\frac{p}{p \vee q}$$

Simplification

$$\frac{p \wedge q}{p}$$

Conjunction

$$\frac{\begin{array}{l}p\\q\end{array}}{p \wedge q}$$

$$\frac{p \qquad q}{p \wedge q} \tag{B.132}$$

Modus ponens

$$\frac{\begin{array}{l}p\\p \rightarrow q\end{array}}{q}$$

Modus tollens

$$\frac{\begin{array}{l}\neg q\\p \rightarrow q\end{array}}{\neg p}$$

Hypothetical syllogism

$$\frac{\begin{array}{l}p \rightarrow q\\q \rightarrow r\end{array}}{p \rightarrow r}$$

Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

See intuituinistic logic. See sections I.7.1 and J.2.

• **Mandelstam identity:** $SL(2, \mathbb{C})$ A satisfies $\det A = 1$. for $A, B \in SL(2, \mathbb{C})$

$$\mathrm{tr} A \mathrm{tr} B = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) \tag{B.133}$$

$SU(2) \subset SL(2, \mathbb{C})$ so the identity. Holonomies H_α are $SU(2)$ matrices.

• **manifold:** In simply terms, a manifold is a space $\mathcal{M}$ which locally looks like an n-dimensional Euclidean space $\mathbb{R}^n$.

Formally, a topological space $\mathcal{M}$ is an $n-$dimensional manifold if there is a locally finite open cover, $\{U_\lambda : \lambda \in \Lambda\}$, of $\mathcal{M}$ such that, for each λ, there is a map ϕ_λ that maps U_λ homeomorphically onto an open subset of $\mathbb{R}^n$.

(i) differentiable (or smooth manifolds, on which one can do calcus;

(ii) Riemannian manifolds, on which distances and angles can be defined;

(iii) symplectic manifolds, which serve as the phase space of dynamical systems;

(iv) 4D pseudo-Riemanniam manifolds which are used in general relativity .

• **maximal atlas:** We take as an example the definition of a maximal atlas with the $C^\infty-$property. Two charts ϕ_1, ϕ_2 are $C^\infty-$related if both the map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and its inverse are $C^\infty-$related. A collection of related charts such that every point of $\mathcal{M}$ lies in the domain of at least one chart forms an atlas. The collection of all such $C^\infty-$related charts forms a maximal atlas.

• **maximal ideal:** A *maximal left ideal* in A is a proper left ideal which is not properly contained in any other proper left ideal.

• **measure:** A measure

• **measurable functions:** Let X, Y be metric spaces, and let $f : X \to Y$ be a function.

A function $f : E \to \mathbb{R}$ is (Lebesgue-)measurable if for any interval $I \subseteq \mathbb{R}$

$$f^{-1}(I) = \{x \in \mathbb{R} \; : \; f(x) \in I\} \in \text{ collection of measurable sets.????} \tag{B.134}$$

• **measure space:** A measure space $(\Omega, \mathcal{B}, \mu)$ is a set Ω together with a $\sigma-$algebra $\mathcal{B}$ of subsets of Ω and μ is $\sigma-$additive, that is,

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) = \sum_{n=1}^{\infty} \mu(U_n) \tag{B.135}$$

for all disjoint measurable sets U_n.

• **minimal loop:** Given a graph γ and a vertex v of this graph, and two different edges e and e' outgoing from v, a loop $\alpha(\gamma, v, e, e')$ within γ with outgoing along e and incoming along e' is said to be minimal if there is no other loop within γ with the same properties and fewer edges tranaversed.

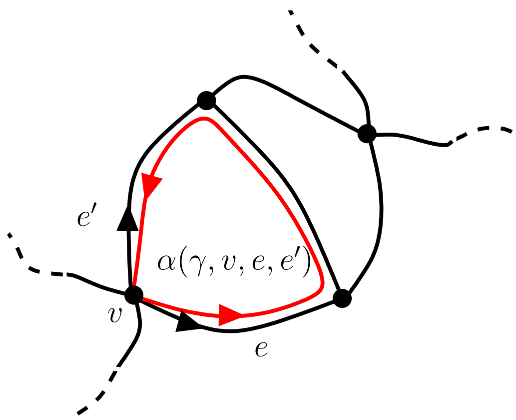


Figure B.20: minimal loop.

• **Minkowski's inequality:**

$$\left(\sum_i |x_i + y_i|^2\right)^{1/2} \leq \left(\sum_i |x_i^2\right)^{1/2} \left(\sum_i |y_i|^2\right)^{1/2} \tag{B.136}$$

• **module:** Modules are also referred to as representations: for instance, representations of a group are essentially the same as modules over the group algebra. Even if you have not come across the term "module" you surely have come across some examples. Vector spaces are (rather simple) examples, as are abelian groups. We elaborate on some of the examples below after giving the formal definition of a module.

A commutative group on which there is defined an *exterior multiplication* (left or right) by elements of a *ring* R, such that multiplication is associative and distributive, and a group element multiplied by an element of the ring is a group element.

1) $(u + v) + w = u + (v + w)$ for all $u, v, w \in M$

2) $u+v=v+u$ for all $u,v \in M$

3) There exists an element $0 \in M$ such that $u+0=u$ for all $u \in M$

4) For any $u \in M$, there exists an element $v \in M$ such that $u+v=0$

5) $a \cdot (b \cdot u) = (a \cdot b) \cdot u$ for all $a,b \in R$ and $u \in M$

6) $a \cdot (u+v) = (a \cdot u) + (a \cdot v)$ for all $a \in R$ and $u,v \in M$

7) $(a+b) \cdot u = (a \cdot u) + (b \cdot u)$ for all $a,b \in R$ and $u \in M$

A vector space is a module where complex numbers (so the ring is the field of complex numbers). A more complex example would be a set of operators acting on a vector space. This is a left module where the ring is the collection of operators.

$$Lu = x_1 L_{1i} + x_2 L_{2i} + \cdots + x_2 L_{ni} = \sum_{m=1}^{n} x_m L_{mi}$$

The action of L on a basis can be represented by a matrix L_{ij}. Any vector, u say, can be represented in the basis

$$u = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{m=1}^{n} a_m x_m$$

$$Lv = L(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) = a_1 L x_1 + a_2 L x_2 + \cdots + a_n L x_n$$

so the coefficients of Lv are

$$(Lv)_m = \sum_{m'=1}^{n} L_{mm'} a_{m'}$$

• **moniod:** A set M with a binary operation $\cdot$ that is

(i) associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and

(ii) there is a unit element $\mathbf{1}$ in M such that $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in M$.

A moniod is a semigroup that has an *identity*.

A monoid is a category with one object.

• **moniodal category:** A moniodal category is a 2-category with one object.

• **morphism:**

A $*$-**morphism** between two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is a mapping $\pi : A \in \mathcal{A} \in \mathcal{A} \mapsto \pi(A) \in \mathcal{B}$, defined for all $A \in \mathcal{A}$ such that

$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,

$\pi(AB) = \pi(A)\pi(B)$,

$\pi(A^*) = \pi(A)^*$.

• **neighbourhood:** A set N is a neighbourhood of a point x in a topological space $(X, \mathcal{T})$ if and only if there is a open set U in $\mathcal{T}$ such that $x \in U \subseteq N$. Note that N need not be open itself.

• **neighbourhood base:** a collection of neighbourhoods, so that every open set of the topology can be expressed as a union of some of these neighbourhoods.

• **net:** Generalization of a the concept of a sequence to permit talk of convergence in non-meterizable topological spaces. Nets are often called Moore-Smith sequences.

• **normal operator:** A normal operator commutes with its self-adjoint.

If N is normal then,

$$N = \sum_{i=1}^{N} \lambda_i |f_i >< f_i|,$$

where $\lambda_1, \ldots, \lambda_N$ are igenvalues and $\{|f_i >\}$ are orthonormal eigenvectors of N. This is known as a strectral decomposition.

• **normal topological space:** A topological space is normal when it is T_1 (see separation conditions), and given any two disjoint closed sets V_1 and V_2, there are disjoint open sets U_1 and U_2 such that $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$.

• **non-Boolean logic:** Non-Boolean logic does not assume every statement can be judged true or false, there are some statements upon which one cannot decide. An example of a physical situation where the underlying logic would be non-Boolean is in cosmology where observers can only make judgements upon statements that have to do with their backwards light cone.

relational type quantum mechanics.

see topos , Heyting algeras

• **norm:**

(i) $\|zx\| = |z|\|x\|$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

• **normed space:** A normed linear space in which every vector x there corresponds a **norm** $\|x\|$ of x, such that

(i) $\|x\| \geq 0$, and $\|x\| = 0 \;\Rightarrow\; x = 0$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$,

(iii) $\|x\| = |\alpha| \|x\|$.

• **normed space:** A vector space endowed with a norm.

• **nuclear spectral theorem:**

• **nuclear topology:**

• **open algebras:**

The smeared Hamiltonian Poisson brackets are an example, see Eq(E.7). Very little is known about the representation theory of open algebras.

• **open mapping theorem:** A fundamental result of functional analysis which states that if a continuous linear operator between Banach spaces is surjective (onto) then it is an open map (i.e. if U is an open set in the intial Banach space then $A(U)$ open set in the Banach space which the operator A maps to).

• **open sets:** Intuitively a set U is open if any point x in U can be moved in any "direction" and still be in the set. For example the set (a, b) of the real line $\mathbb{R}$ is open in $\mathbb{R}$, any point in (a, b) is arbitrarily close to the boundaries a or b, but cannot be a or b, as such if moved a sufficiently small amount are still contained in (a, b).

Subets understood with the properties

(1) Let A be a topological space and $\{\mathcal{U}_\alpha\}$, be any collection (possibly infinite) of open sets in A.

Then

$$\cup_\alpha \mathcal{U}_\alpha \tag{B.137}$$

is open

(2) Let be a finite collection of open sets in A

$$\mathcal{U}_1 \cap \mathcal{U}_2 \cap \cdots \cap \mathcal{U}_n \tag{B.138}$$

is open.

The concept of an open set is fundamental to amny areas of mathemetics, we see them in operator theory as well as in point-set topolgy (singularity theorems for example).

• **operator topologies:** The set of all bounded, linear operators acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. A subset $S \subset \mathcal{B}$ is called an algebra if $\alpha A + \beta B \in S$ and $AB \in S$

Topology needs a definition of what we mean by a neighbourhood of an element.

The norm topology: the ϵ-neighbourhood of A is the set of operators B with $\|A - B\|$. **norm topology** or **uniform topology** in $\mathcal{B}(\mathcal{H})$.

Other important topologies in $\mathcal{B}(\mathcal{H})$ are defined by means of seminorms.

For topologies that which are defined by seminorms it is not enough to consider only the convergence of sequences. The closure of a set is obtained by adding the limit points of em nets.

generally, if a subset is closed in one topology it is automatically closed in every stronger topology one gets even more limit points.

weak operator topology: topology such that are all continuous.

the weak topology is the topology such that all functionals on X^{**} are continuous.

weak star operator topology: The weak star topology is obtained if we use the absolute values of $| < \Phi|A|\Psi > |$ between arbitrary state vectors as a system of seminorms. Thus a sequence of operators converges weakly if all matrix elements converge.

• **oriented manifold:** Intuitively, in the case of a 2-manifolds, a surface is oriented if it is two-sided, and non-oriented if it is 1-sided. The cylinder is oriented, but the Möbius strip is not.

• **orbit:** Let G act on a set X. A subset $\subset X$ is said to be **stable** under the action of G if

$$g \in G \quad x \in S \Rightarrow gx \in S. \tag{B.139}$$

• **outer measure:** Let a be an algebra of subsets of X and μ a measure on it. For $A \subset X$ is defined by

$$\mu^*(A) = \inf \sum_{i=1}^{\infty} \mu(E_i) \tag{B.140}$$

where the infimum is taken over all $\mathcal{U}$-coverings of the set A all collections (E_i), $E_i \in \mathcal{U}$ with $\cup_i E_i \supset A$. to extend μ to as many elements of the powerset as possible.

• **Pachner moves:** Two finite triangulations related by a finite sequence of local modifications, called Pachner moves. Sequences of Pachner moves are the combinatorial equivalent of applying spacetime diffeomorphisms.

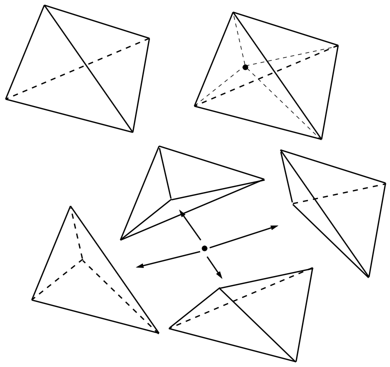


Figure B.21: Pachner move in d=3. (a) the $1 \to 4$ move subdivides.

?????????is the lattice as being embedded in some continuum manifold or whether one regards the lattice itself as existing independently of any background embedding.????????

• **paracompact:** A topological space is said to be paracompact if evry open cover admits an open locally finite refinement.

• **parallelogram law:** A normed space

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{B.141}$$

• **parallel transport:** Parallel transport of a vector is defined as transport without change.

• **parseval's formula:** The generalized Fourier series

$$f(x) = \sum_{-\infty}^{\infty} f_n e_n(x)$$

with Fourier coefficients

$$f_n = \int_a^b e_n^*(x) F(x) dx$$

$$
\begin{aligned}
E_{min} &= \int_a^b |F(x) - f(x)|^2 dx \\
&= \int_a^b F^2(x)dx - \sum_{-\infty}^{\infty} f_n^2 \geq 0.
\end{aligned} \tag{B.142}
$$

gives an average measure of convegence. The Fourier series $f(x)$ of a function $F(x)$ is said to converge in the mean to $F(x)$ if $E_{min} = 0$. When this happens, the Bessel inequality becomes the Parseval equation

$$
\int_a^b F^2(x)dx = \sum_{-\infty}^{\infty} f_n^2. \tag{B.143}
$$

• **partial function:** A partial function is the triple $f = (A, G, B)$ where A and B are sets (possibly empty) and G is a functional relation (possibly empty) between them, called the **graph** of f.

$f : A \to B$ is a **total function** if and only if domain$(f) = A$. A total function is often just referred to as a function.

• **partial isometry:** An operator U in the space of linear operators $\mathcal{L}(\mathcal{H})$ from a separible Hilbert space $\mathcal{H}$ to itself is called a partial isometry if $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$ when restricted to the closed subspace $(\text{Ker}U)^{\perp}$ (Ker$U$ being the set of $x \in \mathcal{H}$ for which $Ux = 0$).

See polar decomposition.

• **partially ordered set:** Let P be a non-empty set. A partial order relation in P is a relation denoted by $\leq$ which has the following properties:

(i) $a \leq a$ for every a (reflexivity);

(ii) $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry);

(iii) $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

• **partition of unity:** Take an open covering $\{U_i\}$ of $\mathcal{M}$ such that each point of $\mathcal{M}$ is covered by a finite number of U_i (if this is always possible, $\mathcal{M}$ is called paracompact, which we assume to be the case). If a family of differentiable functions $\epsilon_i(p)$ satisfies

(i) $0 \leq \epsilon_i(p) \leq 1$

(ii) $\epsilon_i(p) = 0$ if $p \neq U_i$

(iii) $\epsilon_1(p) + \epsilon_2(p) + \cdots = 1$ for any point $p \in \mathcal{M}$

the family $\{\epsilon_i(p)\}$ is called a partition of unity subordinate to the covering $\{U_i\}$. It follows from (iii) that

$$f(p) = \sum_i f(p)\epsilon_i(p) = \sum_i f_i(p)$$

where $f_i(p) \equiv f(p)\epsilon_i(p)$ vanishes outside of U_I by (ii).

• **path ordered:** $\mathcal{P}\left(A(x(s_2))\ A(x(s_1))\ A(x(s_4))\ldots\right) = A(x(s_1))\ A(x(s_2))\ \ldots\ A(x(s_n))$, where $s_1 \geq s_2 \geq \cdots \geq s_n$.

• **Penrose's abstract index notation:** In this notation, the index 'a' of a vector v^a is to be seen as a label indicating that v is a vector (very much like the arrow in $\vec{v}$), and it does not take values in any set. That is, 'a' is not the component of v on any basis.

• **permutation group:**

• **Peter-Weyl Theorem:** The Peter-Weyl theorm asserts that the representaion matries, $D^J_{mn}(g)$, form a complete set of orthogonal functions on the group manifold.

$$f(g) = f(UgU^{-1}) \Rightarrow \quad f(g) = \sum_i a_i \chi_i(g) \tag{B.144}$$

where

$$a_i = \sum \tag{B.145}$$

The *Peter-Weyl Theorem* applied to $U(1)$ gives the Fourier series theory:

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \tag{B.146}$$

where $f(\theta) \in L^2(U(1))$.

• **p-form:** An anti-symmetric covariant tensor field over a manifold $\mathcal{M}$.

• **piecewise:** We say something piecewise, with respect to some property, if it is made up of a finite number of pieces, each of which shares this property.

• **piecewise-analytic curve:** A curve is piecewise analytic if it is made up of a finite number of pieces, each of which is analytic.

See semianalytic curve.

piecewise-smooth curve: A curve is piecewise smooth if it is made up of a finite number of pieces, each of which is smooth.

• **piecewise-linear (PL-) manifolds:** The transition functions are maps between polyhedra which map simplices to simplices.

• **Poincare lemma:** Say we have a $p-$form ω. Recall the exterior derivative, denoted d. If $d\omega = 0$, then ω is said to be closed. If $\omega = d\alpha$, then ω is said to be exact. Exactness implies closure, since $\omega = d\alpha \Rightarrow d\omega = d^2\alpha = 0$. The converse is in general not true. The Poincare lemma states that every closed form is locally exact. That is, if $d\omega = 0$, then $\omega = d\alpha$ in some local region. In general, this will not hold globally.

• **Poincare algebra in** $(2+1)$ **dimensions** $iso(2,1)$**:** Poincare algebra $iso(2,1)$ $(a, b, c = 0, 1, 2)$:

$$[J_a, J_b] = \epsilon^c{}_{ab} J_c, \quad [J_a, P_a] = \epsilon^c{}_{ab} P_c, \quad [P_a, P_b] = 0. \tag{B.147}$$

• **Poisson resummation formula:** As

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

is periodic in x we may write

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} a_k \exp(2\pi i k x)$$

then via the inverse Fourier transform we have

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \exp(2\pi k x) \int_{-\infty}^{\infty} dy f(y) \exp(-2\pi i k y). \tag{B.148}$$

$$\sum_n e^{-\epsilon(n-N)^2} f(n) = \sum_n e^{-\epsilon(y-N)^2} f(y) e^{2\pi I y n} \tag{B.149}$$

• **polar decomposition:** A polar decomposition is the analogue to the decomposition $z = |z|e^{iargz}$ for positive operators on a Hilbert space. Let $\mathcal{H}$ be a Hilbert space. If $A \in \mathcal{B}(\mathcal{H})$ then (by the polar decomposition theorem) it can be uniquely expressed as $A = W|A|$ where $|A|$ is the positive square root of A^*A and W is a partial isometry with initial space equal to the closure of the range of $|A|$ and final space equal to the range of A. The expression $W|A|$ is called the polar decomposition of A.

left polar decomposition

• **polyhedron:** A subset $X \subseteq \mathbb{R}^n$ is said to be a polyhedron if eveery point $x \in X$ has a neighbourhood in X of the form

$$\{\alpha x + \beta y : \alpha, \beta \geq 0, \alpha + \beta = 1, y \in Y\}$$

where $Y \subseteq X$ is compact.

• **Pontyagin duality:**

• **positive elements:** An element of a C^*algebra $\mathcal{A}$ is said to be positive if $A = A^*$ and $\sigma(A) \geq 0$, or equivalently, $A = B^*B$ for some $B \in \mathcal{A}$.

• **positive operator:** An operator $B \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called positive if $(Bx, x) \geq 0$ for all $x \in \mathcal{H}$

• **power set:** Let X be a set. The power set $\mathcal{P}(X)$ is the collection of all subsets of X.

• **presheaf:** TAKEN DIRECTLY FROM mathworld.wolfram.com

For X a topological space, the presheaf $\mathcal{F}$ of Abelian groups (rings, ...) on X is defined such that

1. For every open subset $U \subseteq X$, an Abelian group (ring, ...) $\mathcal{F}(U)$, and

2. For every inclusion $V \subseteq U$ of open subsets of X, a morphism of Abelian groups (rings, ...) $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$

subject to the conditions:

1. If $\emptyset$ denotes the empty set, then $\mathcal{F}(\emptyset) = 0$,

2. ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and

3. If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

In the language of categories, let $\mathbf{Top}(X)$ be the category whose objects are the open subsets of X and the only morphisms are the inclusion maps (these map V to U if $V \subseteq U$). Thus, $\mathbf{Hom}(V, U)$ is empty if $V \not\subseteq U$ and $\mathbf{Hom}(V, U)$ has just one element if $V \subseteq U$. Then a presheaf is a contravariant functor from the category $\mathbf{Top}(X)$ to the category $\mathbf{Ab}$ of Abelian groups (Ring of rings, ...).

As a terminology, if $\mathcal{F}$ is a presheaf on X, then $\mathcal{F}(U)$ are called the sections of the presheaf over the open set U, sometimes denoted as $\Gamma(U, \mathcal{F})$. The maps ρ_{UV} are called the restriction maps. If $s \in \mathcal{F}(U)$, then the notation $\rho_{UV}(s)$ is usually used instead of $s|_V$.

Sheaf: Important in twistor theory and other applications of algebraic geometry and topology in physics. A presheaf $\mathcal{F}$ is called a *sheaf* if for every collection U_i of open subsets of X with $U = \cap_{i \in I} U_i$ the following conditions hold:

1. If $s, t \in \mathcal{F}(U)$ and $\rho_{UU_i}(s) = \rho_{UU_i}(t)$ for all i, then $s = t$;

2. If $s_i \in \mathcal{F}(U_i)$ and if for $U_i \cap U_j \neq \emptyset$ we have

$$\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j) \qquad \text{(B.150)}$$

for all i, then there exists an $s \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s) = s_i$ for all i.

The example of germs of functions on a differentiable manifold is a familiar example of a sheaf.

(definition taken from Topi and computation)

patching condition: Let an open set U be covered by open subsets U_i; then any given sections over the U_i's, such that the sections over U_i and U_j have the same restriction to $U_i \cap U_j$, are restrictions of a single section over U. (for example letting $\mathcal{F}(U)$ be the set of all continuous real-valued functions on U). The crucial thing about this definition is that the space X enters the picture only through its poset of open subsets and the notion of a cover.

• **primative elements of a Hopf algebra:**

• **principal bundle:** A principal bundle is a fibre bundle $\pi : P \to E$ with fibre F equal to the structure group G and having the property that for all U_a and U_b with $U_a \cap U_b \neq \emptyset$,

$$\varphi_{ba} : U_a \cap U_b \to Left(F) \subset Diff(F),$$

where $Left(F) = \{L_g | L_g(h) = gh, \text{ for all } h \in G, g \in G\}$. In other words, changing coordinates corresponds to multiplying the fibre on the left by some element of G.

As an example consider the frame bundle $B(\mathcal{M})$. The total space consists of the set of basis vectors v_i^a of the tangent space for all points over the manifold $\mathcal{M}$. Here i is an index $i = 1, \ldots, n$ labeling the n basis vectors v^a. There is a natural free right action by $GL(n, \mathbb{R})$ on $B(\mathcal{M})$: If we have a basis $\{v_i^a, i = 1, \ldots, n\}$, we know that

$$(v_1^a, v_2^a, \ldots, v_n^a) g := (v_j^a g^j{}_1, v_j^a g^j{}_2, \ldots, v_j^a g^j{}_n),$$

where g in $GL(n, \mathbb{R})$. We can also see the array of vectors $(v_1^a, v_2^a, \ldots, v_n^a)$ as a $n \times n$ matrix with non-zero determinant.

The frame bundle is an example of a principal bundle. In a principal bundle $(P, \pi, \mathcal{M}, G)$ each fibre is diffeomorphic to the structure group G. Principal bundles are in a sense more fundamental than vector bundles, since one can always regard vector bundles as associated bundles to a particular principal bundle. In this example, the tangent bundle $T\mathcal{M}$ is the associated vector bundle to the frame bundle $B(\mathcal{M})$.

• **product spaces:** A Cartesian product equipped with a product topology is called a product space.

• **product topology:** Let $\mathcal{S}$ and $\mathcal{T}$ be topological spaces, and form the product $\mathcal{S} \times \mathcal{T} = \{(u, v) : u \in \mathcal{S} \text{ and } v \in \mathcal{T}\}$ of the two sets $\mathcal{S}$ and $\mathcal{T}$. The product topology on $\mathcal{S} \times \mathcal{T}$ consists of all subsets that are unions of sets of the form $U \times V$, where U is open in $\mathcal{S}$ and V is open in $\mathcal{T}$. Thus these open rectangles form a basis for the product topology.

• **projective family:** Let L be a partially ordered directed set with an pre-order relation denoted $\preceq$ for $S, S', S'' \cdots \in L$. A *projective* family $(\chi_S, p_{SS'})_{S,S' \in L}$ consists of sets χ_S indexed by elements of L, together with a family of onto *projections*,

$$p_{SS'}\, \chi_{S'} \to \chi_S, \tag{B.151}$$

such that

$$p_{SS} = id_{\chi_S}, \tag{B.152}$$

and

$$p_{SS'} \circ p_{S'S''} = p_{SS''} \tag{B.153}$$

for ay triple $S \preceq S' \preceq S''$.

• **projective limit:** Unfortunately the projective family itself does not have a largest element from which one can project to any other. However, such an element can in fact be obtained by a standard procedure called the “projective limit”. It emerges as an appropriate limit, defined as follows. The projective limit $\overline{\chi}$ of a projective family $(\chi_S, p_{SS'})_{S,S' \in L}$ is the subset of the Cartesian product $\times_{S \in L} \chi_S$ that satsfies certain consistency conditions:

$$\overline{\chi} = \{(x_S)_{S \in L} \in \times_{S \in L} \chi_S : S \preceq S' \text{ implies } p_{SS'} x_{S'} = x_S\}. \tag{B.154}$$

This limit is naturally equipped with a family of projections:

$$p_S : \overline{\chi} \to \chi_S, \quad p_S((x_{S'})_{S' \in L}) := x_S. \tag{B.155}$$

• **projective limit construction:**

A graph is a collection of edges such that if two edges meet. Consider the space A_γ, each element of which assigns to every edge in γ

The projective limit of these spaces are precisely the spaces $\mathcal{A}_\gamma$, $\mathcal{G}_\gamma$ and $\mathcal{A}_\gamma / \mathcal{G}_\gamma$.

What about operators that act on this Hilbert space? All operators that are well defined on that Hilbert space arise from consistent family of operators. These operators on each of these individual finite Hilbert spaces fit together in a certain way. If it is well defined on here then it can be shown that that they come from something that fits together in this way.

• **projection mappings:**

$$\hat{P}_k^2 = \hat{P}_k \tag{B.156}$$

$$\sum_{k=1}^{N} \hat{P}_k = \mathbf{I}. \tag{B.157}$$

group averaging Rovelli projection onto physical states not strictly projection operators.

• **pull-back:** The diffeomorphism ϕ maps points in $\mathcal{M}$ to points in $\mathcal{N}$. The push-forward $\phi^*|_p$ is the natural map between the co-tangent spaces $T_p^*\mathcal{M}$ and $T_{\phi(p)}^*\mathcal{N}$ induced by the diffeomorphism ϕ.

$$[\phi^*\omega](X_1, X_2, \dots, X_p) = \omega(\phi_* X_1, \phi_* X_2, \dots, \phi_* X_p). \tag{B.158}$$

• **pure point spectrum:** Let a be a self-adjoint operator. The pure point spectrum $\sigma_{pp}(a)$ is the set of eigenvalues of a.

• **pure states:** Those states which cannot be written as convex linear combinations of other states, i.e. states which cannot be written as $\alpha a + (1-\alpha) b$ for some pair of states a, b.

• **push-forward:** map head-to-head and tail-to-tail. If the vector has components X^μ and the map takes the point with coordinates x^μ to one with coordinates $\xi(x)$, the vector ϕ_* has components

$$(\phi_* X)^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} X^\nu. \tag{B.159}$$

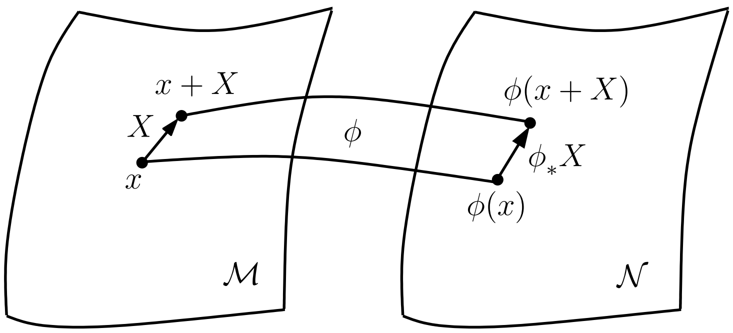


Figure B.22: pullbackDef0. Pushing forward a vector X from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$.

This looks like the transformation formula for contravariant vector components under a coordinate transformation. We are doing an active transformation, changing a vector into a different one.

The diffeomorphism φ maps points in $\mathcal{M}$ to points in $\mathcal{N}$. The push-forward $\varphi_*|_p$ is the natural map between the tangent spaces $T_p\mathcal{M}$ and $T_{\varphi(p)}\mathcal{N}$ induced by the diffeomorphism φ.

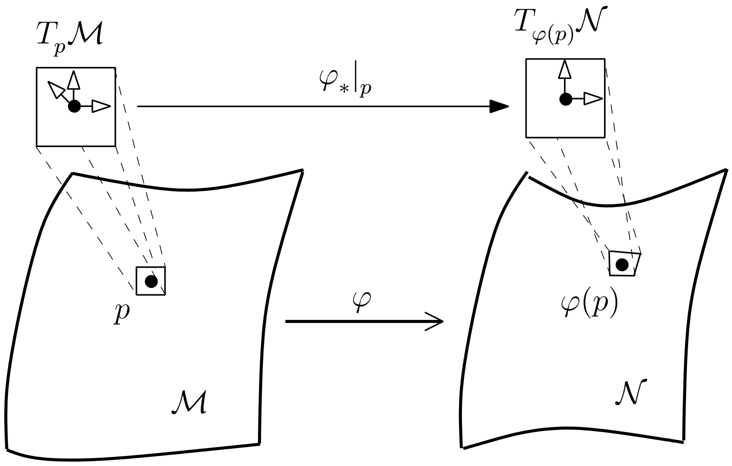


Figure B.23: pullbackDef. $\varphi_*|_p : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$

• **q-deformation:** Introduces a non-commutative to a mathematical structure that was absent in the first place. The non-commutativity is measured by a deformation q parameter. The limiting case gives back the results afforded by the ordinary situation. Some examples are $su_q(2)$ which defines a q-analogue of angular momentum.

• **q-series:**

by L'Hopital's rule

$$\lim q \to 1 \frac{1-q^a}{1-q} = a \tag{B.160}$$

q-shifted factorial

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)(1-aq^{n-1}), & n = 1, 2, \ldots \end{cases} \tag{B.161}$$

q-binimial expansion

$$\sum_{m=0}^{n} y^m q^{\frac{m(m+1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix} \tag{B.162}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-m+1})}{(1-q)(1-q^2)\ldots(1-q^m)}, \tag{B.163}$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1. \tag{B.164}$$

• **quantum groups:** non-co-commutative quasi-triangular Hopf algebras. deform any classical Lie algebra. The initial Hopf algebra is the universal enveloping algebra of the Borel subalgebra of a Lie algebra.

The limiting case - as q goes to 1, the quantum algebra $su_q(2)$ Quantum $SL(2)$ group and quantum $SU(2)$ group.

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B.165}$$

$$U\tilde{\epsilon}U^T = \tilde{\epsilon}, \quad \text{where } \tilde{\epsilon} = \begin{pmatrix} 0 & -A \\ A^{-1} & 0 \end{pmatrix} \tag{B.166}$$

Figure B.24: links.

$$\begin{aligned} &ba = qab \;\; dc = qcd \\ &ca = qac \;\; db = qbd \\ &bc - cb = 0 \\ &ad - da = -(q - q^{-1})bc \\ &ad - q^{-1}bc = 1 \end{aligned} \tag{B.167}$$

where $q = aA^2$. complex non-commuting components a, b, c, d. $q = e^{in}$

$su_q(2)$ algebra

$$
\begin{aligned}
[j_3, j_\pm] &= \pm j_\pm, \\
\left[j_+, j_-\right] &= \frac{q^{2j_3} - q^{-2j_3}}{q - q^{-1}}
\end{aligned}
\tag{B.168}
$$

In a similar way we can define the quantum group $GL_q(2)$,

$$
UQU^T = z_1 Q, \;\; U^T QU = z_2 Q, \quad z_1, z_2 \in \mathbf{C}^* \tag{B.169}
$$

$z_1 = z_2 = z$

• **quasilinear differential equations:** Differential equations linear in the highest derivative terms. For second order quasilinear differential equations, many of the results on linear systems apply locally.

• **quasi-triangular Hopf algebras:** See Hopf algebras. A non-co-commutative quasi-triangular Hopf algebra is called a quantum group.

$$
\Delta'(a) = \sigma \circ \Delta(a) \tag{B.170}
$$

It is non-co-commutative, but the non-co-commutativity is controlled by a matrix $\mathcal{R}$,

$$
\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \quad \text{for all } a \in A. \tag{B.171}
$$

and

$$
\begin{aligned}
(\mathbf{1} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} &= \sum_{i,j} A_i A_j \otimes B_j \otimes B_i, \\
(\mathbf{1} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} &= \sum_{i,j} A_i \otimes A_j \otimes B_i B_j
\end{aligned}
\tag{B.172}
$$

and

$$
(\gamma \otimes \mathbf{1})\mathcal{R} = (\mathbf{1} \otimes \gamma^{-1})\mathcal{R} = \mathcal{R}^{-1} \tag{B.173}
$$

Where is called the universal $\mathcal{R}$-matrix. We write $\mathcal{R} = \sum_i A_i \otimes B_i$ and let

$$\mathcal{R}_{12} = \sum_i A_i \otimes B_i \otimes \mathbf{1},$$

$$\mathcal{R}_{13} = \sum_i A_i \otimes \mathbf{1} \otimes B_i, \tag{B.174}$$

$$\mathcal{R}_{23} = \mathbf{1} \otimes \sum_i A_i \otimes B_i. \tag{B.175}$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{B.176}$$

• **quotient map:**

• **quotient topology:** Let

$$\mathcal{T}' = \{V : V \subseteq Y \text{ and } p^{-1}(V) \text{ is open in } X\}.$$

It is immediate that $\emptyset, Y \in \mathcal{T}'$ as $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(Y) = X$.

• **radical:** A *radical* R of an algebra A is the intersection of all its maximal left ideals. R itself is obviously a proper left ideal.

• **Randon-Nikodym derivative:**

• **Randon-Nikodym theorem:**

• **(real infinite) sequence** is a map a $a : N \to R$

Of course if is more usual to call a function f rather than a; and in fact we usually start labelling a sequence from 1 rather than 0; it doesn't really matter. What the definition is saying is that we can lay out the members of a sequence in a list with a first member, second member and so on. If $a : N \to R$, we usually write a_1, a_2 and so on, instead of the more formal $a(1)$, $a(2)$, even though we usually write functions in this way.

• **reconstruction Theorem:** Does it? We say it *separates* points in $\mathcal{A}/\mathcal{G}$ in that , if A_1 and A_2 are not related by a gauge transformation, there exists a loop γ such that: $T_\gamma(A_1) \neq T_\gamma(A_2)$.

Suppose Σ is a connected manifold with basepoint x_0 and the map $H : \Omega_{x_0} \to G$ satisfies the following conditions:

(i) H is a homomorphism of the composition law of loops, $H(\gamma_1 \circ \gamma_2) = H(\gamma_1)H(\gamma_2)$,

(ii) H takes the same values on thinly equivalent loops: $\gamma_1 \sim \gamma_2$ if $\gamma_1 \circ \gamma_2^{-1}$ is thin??,

(iii) For any smooth finite-dimensionale family of loops $\bar{\psi} : U \to \Omega_{x_0}\Sigma$, the composite map $H\psi : U \to \Omega_{x_0}\Sigma \to G$ is smooth.

Then there exists a differentiable principle fibre bundle ..R. Lool hep-th/9309056

• **regular Borel measure:** A non-negative countably additive set function μ defined on $\mathcal{B}$ is called a *regular Borel measure* if for every Borel set B we have:

$$\begin{aligned} \mu(B) &= \inf\{\mu(O) \,:\, O \text{ open}, O \supset B\}, \\ \mu(B) &= \sup\{\mu(F) \,:\, F \text{ closed}, F \subset B\}. \end{aligned} \tag{B.177}$$

taken from Measure, Integral and Probability, M Capiński and E. Kopp, [**?**].

Notice that regularity of μ on a compact Hausdorff space X reduces to the fact that the measure of every measurable set can be approximated arbitrarily well open or compact (and hence closed since in a Hausdorff space every compact subset is closed, see ??) sets respectively.

• **regular measure:**

• **regular embedding:** For an embedding $\phi : \mathcal{M} \to \mathcal{N}$, the map $\phi : \mathcal{M} \to \phi(\mathcal{M})$ is a bijection and the manifold structure induced by ϕ on $\phi(\mathcal{M})$ is given by the atlas $\{\phi(U_I), \varphi_I \circ \phi^{-1}\}$ where $\{U_I, \varphi_I\}$ is an atlas of $\mathcal{M}$. This structure need not be equivalent to the submanifold structure of $\phi(\mathcal{N})$ which is given by the atlas $\{V_J \cap \phi(\mathcal{M}), \phi_J\}$ where $\{V_I, \phi_I\}$ is an atlas of $\mathcal{N}$. When both differential structures are equivalent the embedding is called regular.

• **regular representation of a C^*−algebra:** If we wish to require that a Weyl algebra is represented weakly continuously, states whose GNS representation have this property are said to be regular.

• **regular representation of a finite group:** It is a matrix representation of the group. We construct the matrices as follows:

	e	a	b	c
e	e			
a^{-1}		e		
b^{-1}			e	
c^{-1}				e

To establish that this is a representation we must prove that

$$A^{(i)}_{jk} = \begin{cases} 1 \text{ if } a_j^{-1}a_k = a_i \\ 0 \text{ otherwise.} \end{cases} \tag{B.178}$$

Proof: It is easily established that

$$\sum_k A_{jk}^{(i)} A_{kl}^{(m)} = \begin{cases} 1 \text{ if and only if } a_k = a_j a_i = a_l a_m^{-1} \\ 0 \text{ otherwise.} \end{cases} \tag{B.179}$$

• **regular representation of a Lie group:** Provides a systematic procedure to construct irreducible representations of a group.

$$\int d\mu(c) \mathcal{D}^{\text{reg}}(a; b, c) f(c) \tag{B.180}$$

$$\mathcal{D}^{\text{reg}}(a; b, c) = \frac{\delta^p(c - \phi(a, b))}{\rho(c)}, \tag{B.181}$$

• **Reidermeister Moves: ??**

• **Reiz lemma:** A vector in the normed space uniquely defines a continuous functional via

$$F_f(V) =< f|V > \tag{B.182}$$

• **Riesz representation theorem:**

Application: [96] - since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, the Riesz representation theorem ensures that there is a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma(f) = \tag{B.183}$$

• **Reiz representation theorem:**

An application ([96]): Now, since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, the ensues that there is a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu([A]) \; \tilde{f}([A]) \tag{B.184}$$

where $\tilde{f} \in C^0(\overline{\mathcal{A}/\mathcal{G}})$ corresponds to f in $\overline{\mathcal{HA}}$.

• **relative topology:** (or induced topology) Let X be a topological space, and let Y be a non-empty subset of X. The relative topology on Y is defined to be the class of all intersections with Y of open sets in X.

• **repeated bisection argument:**

• **representation:** A representation of a group G over a field k (often. $\mathbb{R}$ or $\mathbb{C}$) is a homomorphism $\pi : G \to GL(V)$, where V is a (usually finite dimension) vector space over k. That is, a representation π of G "compares" the "abstract" group G with a concrete group $GL(V)$. There are similar defintions for C^*-algebras and other such ab.

• **representation theory:** Representation theory studies how any given abstract group can be realized as a group of matrices.

operators on the vector space??? concrete example often matrices and operators on a Hilbert space.

• **resolvent set:** The resolvent set $\rho(A)$ is defined as the subset of $\mathbb{C}$ defined as $\{z \in \mathbb{C} : A - z\mathbf{1} \text{ has bounded inverse } \}$.

• **Riemann integral:**

$$\overline{\int} f d\alpha = \underline{\int} f d\alpha \tag{B.185}$$

• **Reimann's crition:** We first give the crition of the most elementary case of a real function f of a closed interval, $[a, b]$, of the real, $f : [a, b] \to \mathbb{R}$. Such Reimann-integrable if and only if for every $\epsilon > 0$ there exists a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

The Reimann integral doesn't deal with all functions we need for our purposes in formulating quantum mechanics and quantum field theory in a mathematically rigorous way.

• **Riemann-Stieltjes integral:**

$$s(P_n) = \sum_{j=1}^{n} x(t_j)[w(tj) - w(t_{j-1})] \tag{B.186}$$

$$\int_a^b x(t) dw(t) \tag{B.187}$$

• **ring:**

In the set algebraic sense:

If groups are roughly thought of as collections of elements that can be added together, then rings are collections of elements for which there is addition and multiplication. To be more specific, a ring is an additive abelian group whose elements can be multiplied as well as added, and in which multiplication is

(i) associative, that is, if $x,\ y,\ z$ are any three elements in R, then $x(yz) = (xy)z$; and

(ii) is distributive, that is, if $x,\ y,\ z$ are any three elements in R, then $x(y+z) = xy+xz$ and $(x+y)z = xz+yz$.

If an element x of R has an inverse, then x is said to be **regular** (or invertible).

a division ring: A ring with identity is called a division ring if all its non-zero elements are regular.

In the set theoretical sense:

Let X be a set, let $R \subseteq \mathcal{P}(X)$. Then R is a ring of subsets if

(i) $\emptyset \in R$;

(ii) if $A, B \in R$ then $A \cap B$, $A \cup B$ and $A \backslash B$ are all in R.

An an alternative definition of a ring, equivalent to the above, R is a ring if

(i) $\emptyset \in R$;

(ii) if $A, B \in R$ then $A \cap B$ is in R, and $A \triangle B \in R$, (where $A \triangle B$ is the symmetric difference $(A\backslash B) \cup (B\backslash A)$).

With operations $\cap$ as multiplication, $\triangle$ as addition, R is a ring in the algebraic sense.

• **right action of a group:**

$$R_g(g') := g'g. \tag{B.188}$$

$g = e^{tX_g}\big|_{t=1}$

$$\frac{d}{dt}(g'e^{tX_g}) \tag{B.189}$$

Vectors of the tangent space at g, $T_g(G)$, are related to vectors in the tangent space at e, $T_e(G)$. This map is denoted $(R_{g^{-1}})_*$ and is called the **pull-back** of the right action $R_g(G)$, the notation and name will be clarified in a moment. We can find this relationship. For clarity we will derive the matrix element of this map

There is a map which acts on the tangent space $T_g(G)$ and takes it to $T_e(G)$

$$(R_{g^{-1}})_* : T_g(G) \to T_e(G) \tag{B.190}$$

• **rooted tress:**

• **saddle-point approximation:** The idea of the saddle point method (or method of steepest descent) is to deform the contour in such a way that the main contribution to the integral comes from a neighborhood of a single point, (or finite number of points).

Let C be a contour in the complex plane and the functions g and S are holomorphic in a neighborhood of this contour. We will consider the asymptotics as $\lambda \to \infty$ of the Laplace integrals

$$G(\lambda) = \int_C g(z) \exp[\lambda S(z)] dz : \tag{B.191}$$

$$f(z) = \int d\tau g(\tau) e^{f(\tau)} \tag{B.192}$$

$$f(z) \approx \sum_i g(\tau_i) e^{f(\tau_i)} \tag{B.193}$$

τ_i determined by

$$\left. \frac{df(\tau)}{d\tau} \right|_{\tau=\tau_i} = 0. \tag{B.194}$$

• **Schwartz space $\mathbb{S}(\mathbb{R}^n)$:** - also called the space of **test functions**. The set of functions that are infinitely differentiable and whose derivatives

$$\frac{\partial^k \varphi(x)}{\partial x^k} \to 0 \ \text{ as } x \to \infty \tag{B.195}$$

faster than any power of $1/|x|$, for $k = 0, 1, 2, \ldots$.

natural duality between Schwartz space and space of tempered distributions??

• **Schwarz's inequality:** For two square-integrable functions $f(x), g(x) \in L^2[dx]$

$$\int |f(x)g(x)|^2 dx \leq \int |f(x)|^2 dx \int |g(x)|^2 dx \tag{B.196}$$

holds. Easily follows from

$$\int |f(x) + \alpha g(x)|^2 dx \geq 0 \tag{B.197}$$

where α is real, as this implies (when $\int |g(x)|^2 dx \neq 0$)

$$\left(\alpha + \frac{\int |f(x)||g(x)|dx}{\int |g(x)|^2 dx}\right)^2 + \frac{\int |f(x)|^2 dx}{\int g(x)^2 dx} - \frac{\int |f(x)g(x)|dx}{(\int |g(x)|^2 dx)^2} \geq 0. \tag{B.198}$$

Used in derivation of the uncertainty principle. For a Banach space X with norm $\|\cdot\|$ for any $A, B \in \mathcal{B}(X)$

$$\|AB\| \leq \|A\| \cdot \|B\|. \tag{B.199}$$

• **section:** It is a smooth assignment to each point in the base space of a point in the fibre over it. As an example, the section of the tangent bundle of a manifold $\mathcal{M}$ is a vector field. Note that a section is globally defined and will not alway exist. In the case of a principal fibre bundle, such as a frame bundle, it has a section if and only if it is trivial. This is not necessarily the case for general fibre bundles, such as a tangent fibre bundle. Notice, however, that local sections always exist as bundle spaces are locally trivial.

• **Segal-Bargmann transformation:** The Segal-Bargmann transformation B_t in ordinary quantum mechanics on the phase space $\mathbb{R}^2$ gives a representation in which wave functions are holomorphic, square integrable (with respect to the Liouville measure) functions of the complex variable $z = q - ip \in \mathbb{C}$, given by the convolution of f with a Gaussian,

$$B_t f(z) = \frac{1}{(2\pi t)^{-d/2}} \int_{\mathbf{R}^d} e^{-(z-x)^2/2t} f(x) dx, \quad z \in \mathbb{C}. \tag{B.200}$$

• **self-adjoint:**

An Unbounded operator are self-adjoint if it is a densely defined operator a with domain $D(A)$ is called self-adjoint if $a^\dagger = a$ and $D(a^\dagger) = D(a)$ where

$$D(a^\dagger) := \{\psi \in \mathcal{H}; \sup_{0 \neq \psi' \in D(a)} | < \psi, a\psi' > | / \|\psi'\| < \infty\} \tag{B.201}$$

..... definition not finished this taken from Thiemann Intro to Modern...

• **self-adjoint extensions:**

• **semianalytic function:** A function $f : U \to \mathbb{R}^m$, where U is some open subset of $\mathbb{R}^n$, is called semianlytic

• **semianalytic partition:** Let $U \subset \mathbb{R}^n$ be open and $h := \{h_1, \ldots, h_N\}$ be a system of real-valued analytic functions h_k defined on a neighbourhood of U. The subsets defined by the 3^N sequences of inequalities/inequalities

$$h_1(x) > 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0;$$
$$h_1(x) < 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0;$$

$$h_1(x) > 0, \quad h_2(x) = 0, \ldots, h_N(x) < 0;$$

$$h_1(x) = 0, \quad h_2(x) = 0, \ldots, h_N(x) = 0 \tag{B.202}$$

satisfy the conditions of a partition (i.e. the union of these subsets covers U and the intersection between any two such distinct subsets is empty), this is called the semianalytic partition of U subordinate to h. These subsets can vary in dimension from $n-$dimensional to zero-dimensional.

• **semianalytic structures:** Semianalytic structures are objects such as paths or surfaces which are analytic except possibly on lower-dimensional subsets. On these subsets there is again a notion of semianalyticity. These objects, unlike analytic ones, have a local character since knowing the object on an arbitrary open subset only determines it up to where it fails to be analytic.

• **semi-continuous function:**

• **semi-direct product:** A group G is said to be a semidirect product of the subgroups N and Q, denoted $N \otimes_S Q$, if

(i) $N \triangleleft G$ ($\triangleleft$ denoting that N is a normal subgroup);

(ii) $NQ = G$; (meaning that every element of G can be written as a product nq where n is some element of N and q is some element of Q);

(iii) $N \cap Q = \mathbf{1}$. (Milne Group theory)

For example the translation-rotation group G, for Q as the rotation group and N the translation group. As

$$\hat{R}\hat{T}\hat{R}^{-1} = \hat{T}',$$

where $\hat{R}$ is a rotation and $\hat{T}$ a translation, is again a translation. Hence, translations form an (abelean) normal subgroup of G. (iii) is obvious.

Equivalent condition: and $G \to G/N$ induces an isomorphism $Q \xrightarrow{\approx} G/N$.

$+ + + + + + + ++$

When a group B acts on another group A as a subgroup of the automorphisms of A, a larger group $A \triangleleft B$ can be constructed, whose elements are all pairs $\{(a, b) \ : \ a \in A, b \in B\}$,

• **semi-group:** elements with an associative multiplication, which is closed under multiplication.

• **semi-norm:** same as norm except that we do not demand that only the zero element has zero norm.

in the study of topology defined by a system of semi-norms requires nets instead of sequences??

• **semi-semianalytic partition:** A semi-semianalytic partition of an open set is analagous to a semianalytic partition, except that the functions h are not required to be analytic, just semianalytic.

• **semisimple group:**

Any compact Lie algebra is semisimple.

• **separating:** A collection of functions $\mathcal{C}$ on a (topological) space X is said to separate its points if and only if for any $x_1 \neq x_2$ we can find $f \in \mathcal{C}$ such that $f(x_1) \neq f(x_2)$.

The only if part of the definition says given the values assumed by each and every function in the collection $\mathcal{C}$ exists a unique point $p \in X$ for which the functions take their given values.

They encode all the information about the (topological) space X. This is the starting point for non-commutative geometry...

An important example is that of gauge theory when connection representation to the loop representation. It separates points of $\mathcal{A}/\mathcal{G}$ in the sense that, if $[A_1] \neq [A_2]$, there exists a loop α such that: $T_\alpha(A_1) \neq T_\alpha(A_2)$. The set of configuration variables is sufficiently large in that they encode all the information that is contained in a connection

these functionals form a separating set on $\mathcal{A}/\mathcal{G}$: if all the T_α assume the same values at two connections, they are necessarily gauge related.

• **separation conditions:** There is a whole hierarchy of separation conditions, here we mention a few of them. Let $\mathcal{T}$ be a topological space, and let P and Q be two distinct points of $\mathcal{T}$. $\mathcal{T}$ is called

(i) T_0 if at least one of the points has a neigbourhood excluding the other,

(ii) T_1 if each point has a neigbourhood excluding the other,

(iii) T_2 is the Hausdorff condition holds.

One more separation condition of note is a normal topological space.

• **sesquileanr forms:**

(i) $F(\alpha u + \beta v, w) = \overline{\alpha} F(u, w) + \overline{\beta} F(v, w)$, and
(ii) $F(u, \alpha v + \beta w) = \alpha F(u, v) + \beta F(u, w)$.

sesquileanr form with....

• **sets:** a **set** is a collection of "things".

Standard notation for often-used sets

$\emptyset = \{\}$ = set with no elements

$\mathbb{Z}$ = the integers

$\mathbb{Q}$ = the rational numbers

$\mathbb{R}$ = the real numbers

$\mathbb{C}$ = the complex numbers

• **shadow state:** A shadow state $|\Psi_\gamma^{shad}>$ is an element $(\Psi|$ of Cyl^* projected to the subspace Cyl_γ by the projection operator $\hat{p}_\gamma$:

$$(\Psi|\hat{p}_\gamma : \sum_{\vec{x}_i \in \gamma} \Psi(\vec{x}_i)|\vec{x}_i > \equiv |\Psi_\gamma^{shad}>, \tag{B.203}$$

with $(\Psi| = \sum_{\vec{x}} \overline{\Psi}(x_i)(\vec{x}|$.

See physics glossary.

• **sheaf:** See presheaf.

• **sieve:** closed under post composition.

• $\sigma-$**additive:** the measure of a *countable* union of non-intersecting measurable sets is equal to the sum of their measures:

$$\mu\left(\bigcup_i^\infty A_i\right) = \sum_i^\infty \mu(A_i) \quad \text{where} \quad A_i \cap A_j = \emptyset \text{ for any} i \neq j. \tag{B.204}$$

• $\sigma-$**algebra:**

The word "sigma" refers to sum, meaning union, while the word "algebra" indicates that $\mathcal{M}$ is defined in terms of certain operations, in this case unions and complements of sets

• **simple functions:** Let X be a non-empty set. Then a *simple function* s is a mapping from X to the real line, i.e. $s :\rightarrow R$, such that s only takes finitely many different values.

• **simple representations:** The representation π of G in the vector space V over k is said to be simple if no proper subspace of V is stable under G. That is, π is simple if the following property holds: if U is a subspace of V such that

$$g \in G, \quad u \in U \Rightarrow gu \in U$$

then either $U = 0$ or $U = V$.

• **simplex:** the most elementary geometric figure of a given dimension. For zero dimension it is a point, in two dimensions it is the line, in three it is the tetrahedron in, the 4-simplex in four dimensions, etc. B

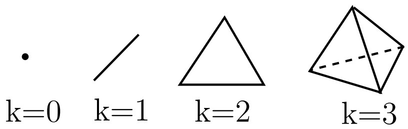


Figure B.25: Simplices in 3d.

• **simplicial complex:**

• **Skien relation:** A recursive relation which allows us to assign a polynomial to a knot which is a topological invariant of the knot. A skien relation is an equation that relates the polynomial of links obtained by changing the crossings in the projection of the original link. Skien relations allows us to calculate the invariants by decomposing the link step by step to a union of unkotted, unlikned loops.

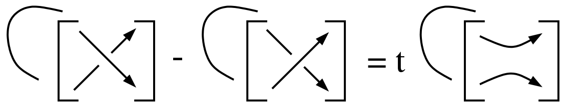


Figure B.26: Skien relation for the Conway polynomial.

If two knots have different polynomials then they are topologically inequivalent. Two different knots can have the same polynomial.

We want a way of obtaining the bracket polynomial of a link in terms of the bracket polynomials of simpler links. Skein relations.

The holy grail of knot theory is to have a recursive relation that distinguishes between all topologically inequivalent knots. The standard example is evaluation of the Conway polynomial for the Trefoil knot

(see knot invariants, Kauffman bracket, knot polynomials,...)

• **smooth curve:** A curve in Euclidean space $\mathbb{R}^n$ is smooth if and only if it is infinitely differentiable. A curve in a manifold $\mathcal{M}$ is smooth if and only if its image under a chart

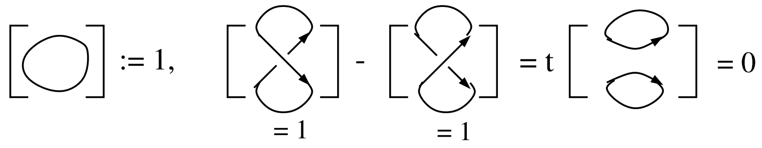


Figure B.27: Skien relation for the Conway polynomial.

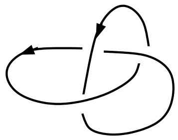


Figure B.28: The Trefoil knot.

is a smooth curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval (a, b) to $\mathbb{R}^n$ in Fig.(**??**) is a analytic map. These curves are denoted as $C^\infty-$curves.

Finite differentiable curves (C^r-curves) are defined in the obvious way.

• **smooth function:** A smooth function is a function that is infinitely differentiable, that is, it does not matter how many times you differentiate the function, the resulting functions are always continuous. Such functions are denoted C^∞.

• **Sobolev embedding theorem:** It shows that $C^k(\omega) \subset H^s(\omega)$ (with continuous inclusion - i.e. the C^k-norm is bounded in terms of the H^s-norm) provided that $s > k + n/2$.

• **Sobolev inequalities:** One can relate the Sobolev norm to more usual norm via Sobolev inequalities.

• **Sobolev norm:** The Sobolev norm is used in the standard formulation of well-posed initial value problems in a general globally hyperbolic spacetime [7] - stability of closed trapped surfaces away from spherical symmetry (I think). Energy inequalities. Relate initial conditions to the solution at a time t later. Can relate the Sobolev norm to more usual norm via Sobolev inequalities.

E.g. Consider the Klein-Gordon equation in $(1 + 1)$- dimensional Minkowski spacetime, $(-\partial_t^2 + \partial_x^2)f = 0$ defined on a suitable region of spacetime. (Who's Afraid of Naked Singularities? gr-qc/9907009)

$$\|f\| := \left(\frac{q^2}{2} \int dx |f|^2 + \frac{1}{2} \int dx \left| \frac{df}{dx} \right|^2 \right)^{\frac{1}{2}}, \tag{B.205}$$

where q^2 is a positive constant.

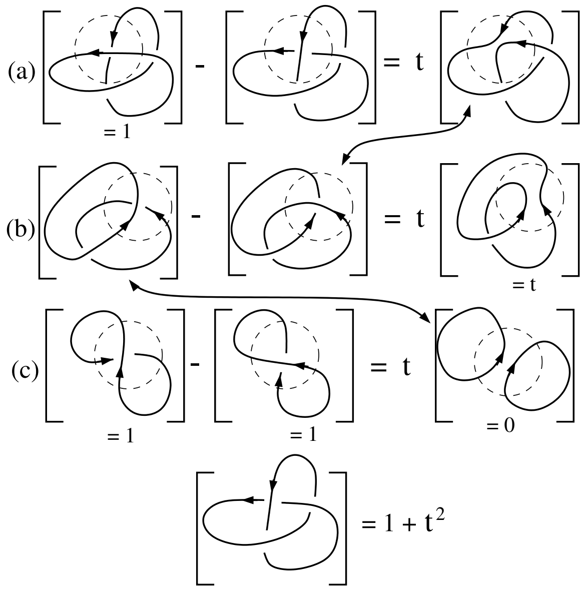


Figure B.29: Evaluation of the Conway polynomial for the Trefoil knot $1 + t^2$.

$$\|\psi, \mathcal{N}\|_m = \left[\sum_{i=0}^{m} \int_{\mathcal{N}} |D^p \psi|^2 d\sigma \right]^{1/2} \tag{B.206}$$

$$\|\psi\| < Const \times \|\psi, \tilde{\mathcal{N}}\|_m \tag{B.207}$$

• **Sobolev space:** The **Sobolev space or** H^1 is the function space $\mathcal{H} = \{f \| f\| < \infty\}$. with inner product

$$(f, g) := \left(\frac{q^2}{2} \int dx f^* g + \frac{1}{2} \int dx \frac{df^*}{dx} \frac{dg}{dx} \right)^{\frac{1}{2}}, \tag{B.208}$$

so that $\|f\|^2 = (f, f)$.

• **span:** For a nonempty subset $M \subset X$ the set of all linear combinations of vectors of $x_i \in M$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, \tag{B.209}$$

is called the span of M, written span M. The completion of span M is denoted $\overline{\text{span } M}$.

• **spectral mapping theorem:** If a $\mathcal{A}$ is a C^*-algebra and if $A \in \mathcal{A}$ is normal then $\sigma_{\mathcal{A}}(f(A)) = f(\sigma_{\mathcal{A}}(A))$ for all continuous functions f. This result is known as the spectral mapping theorem. Can be generalized to f measurable function.

• **spectral measure:**

• **spectral radius:**

$$\sigma(A) = \lim_{n\to\infty} \|A^n\|^{1/n} \tag{B.210}$$

• **spectral representation:** A self-adjoint operator in a (finite) Hilbert space $\mathcal{H}$ has a spectral representation - it's eigenstates form a complete orthonormal basis in $\mathcal{H}$. We can express a self-adjoint operator $\mathbf{A}$ as

$$\mathbf{A} = \sum_n a_n \mathbf{P_n},$$

where a_n is an eigenvalue of $\mathbf{A}$ and $\mathbf{P_n}$ is the corresponding orthogonal projection onto the space of eigenvectors with eigenvalues a_n.

• **spectral theorem:** Every Hertitian matrix is unitarily equivalent to a diagonal one. The spectral theorem is the generalisation of this assertion to operators on Hilbert space.

We first review the spectral theorem for $\mathcal{H}$ finite dimensional:

Suppose that $\mathcal{H}$ is finite dimensional. If $T \in \mathcal{B}(\mathcal{H})$, then $\sigma(T) \neq \emptyset$. Furthermore, $\lambda \in \sigma(T)$ if and only if λ is an eigenvalue of T. Let $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$. Let M_i the corresponding eigenspaces and P_i be the orthogonal projections on these eigenspaces. The spectral theorem then states that the following three conditions are equivalent:

(a) The M_i's are pairwise orthogonal and span $\mathcal{H}$;

(b) The P_i's are pairwise orthogonal

$$I = P_1 + \cdots + P_n.$$

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n;$$

(c) T is normal.

This decomposition is usually called the Spectral Theorem in finite dimensions.

In general, the classical Spectral Theorem says that each normal $T \in \mathcal{B}(\mathcal{H})$ is associated to a (unique) $\mathcal{H}-$projection valued measure on $\sigma(T)$, so that

$$T = \int_{\sigma(T)} d\lambda P(\lambda).$$

• **spectrum:** Infinite dimensional spaces there are operators that have no eigenvalues. For example, $(x_1, x_2, x_3, \dots)$ the operator that translates the components by one place - $(0, x_1, x_2, \dots)$ has no eigenvectors. The next best thing that is useful... is the spectrum of an operator A. The spectrum is defined as the set of values λ for which the operator $(A - \lambda\hat{I})$ is not invertible.

Let $\mathcal{U}$ be a C^*-algebra. The spectrum $\sigma(\mathcal{U})$, of $\mathcal{U}$ is the set of all characters on $\mathcal{U}$.

Any Abilian C*-algebra with identity is naturally isomorphic with the C*-algebra of all continuous functions on a compact, Hausdorff space, called the spectrum of the algebra.

• **spinor group:** The spinor group Spin(n) is a particular double cover of the rotation group $SO(n)$.

$Sp(2n)$ is generated by $2n \times 2n$ matrices X that satisfy

$$XG + GX^T = 0, \tag{B.211}$$

where $G = -G^T$ is some non-degenerate antisymmeteric matrix. We can write G, and the generators X, as tensor products 2×2 and $n \times n$ matrices. One takes

$$G = \sigma_2 \otimes \mathbf{1}, \tag{B.212}$$

where $\mathbf{1}$ is the $n \times n$ unit matrix, and σ_2 the second Pauli matrix. Explicitly G is

$$G = \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix}$$

The generators X are Hermitian matrices, and in addition they must satisfy (B.211). With G given by (B.212), the set of all X can be obtained from the following sets of matrices:

$$\mathbf{1} \otimes A, \quad \sigma_1 \otimes S_1, \quad \sigma_2 \otimes S_2, \quad \sigma_3 \otimes S_3.$$

Here A is an arbitrary $n \times n$ imaginary antisymmetric matrices, S_1, S_2 and denote S_3 denote arbitrary $n \times n$ real symmetric matrices. Explicitly one has

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} 0 & S_1 \\ S_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -iS_2 \\ iS_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} S_3 & 0 \\ 0 & -S_3 \end{pmatrix}.$$

• **stable:** Let G act on a set X. A subset $S \subset X$ is said to be **stable** under the action of G if

$$g \in G \quad x \in S \Rightarrow gx \in S. \tag{B.213}$$

• **stablizer:** The **stabalizer** of an element $x \in X$ is

$$Stab\ (x) = \{g \in G : gx = x\}. \tag{B.214}$$

It is a group - It's obviously associative and closed. $\mathbf{1} \in Stab\ (x)$ as $\mathbf{1}x = x$. If $gx = x$ then $g^{-1}gx = g^{-1}x$ so that $g^{-1}x = x$, hence $g^{-1} \in Stab\ (x)$.

• **states:** A state on a $^*-$algebra is a linear functional $\omega : \mathbb{U} \to \mathbb{C}$ which is positive, that is, $\omega(A^*A) \geq 0$ for all $A \in \mathbb{U}$. The states that physicists are most familiar with are vector states, that is, if we are given a representation $(\mathcal{H}, \pi)$ and an element ψ in the common domain of all the $A \in \mathbb{U}$ then $a \mapsto < \psi, \pi(A)\psi >_{\mathcal{H}}$ evidently defines a state.

• **stratification:** [90] or [102] Let $\mathbb{X}$ be a manifold of differentiability p, and U be a subset of $\mathbb{X}$. Then:

$\mathcal{M}$ is called a **stratification** of $\mathbb{X}$ is a locally finite, disjoint decomposition of $\mathbb{X}$ into connected embedded C^p manifolds $\mathbb{X}_i$ of $\mathbb{X}$, such that:

$$\mathbb{X}_i \cap \mathbb{X} \neq \emptyset \Rightarrow \mathbb{X}_i \subseteq \partial \mathbb{X}_j \ \text{ and } \ dim \mathbb{X}_i < dim \mathbb{X}_j. \tag{B.215}$$

for all $\mathbb{X}_i, \mathbb{X}_j \in \mathcal{M}$.

The elements of the decomposition are called **strata**. $\mathcal{M}$ is called a stratification of U, if and only if U is the union of some elements of $\mathcal{M}$.

• **stratified diffeomorphism:** a stratified map f is a stratified diffeommorphism if and only if $f|_{\mathbb{X}_i}$ is a injective and the restriction of each f_i to the restriction of each f_i to the respective U_i is a C^p-diffeomorphism.

• **stratified isomorphism:** A map f is a stratified isomorphism if and only if in addition to being a stratified mononmorphism, f is a homeomorphism and each $f_i : U_i \to f_i(U_i)$ is a C^p diffeomorphism.

• **stratified map:** Let f be a continuous map from C^p-manifold $\mathbb{X}$ to a $\mathbb{Y}$-manifold. The map f is called a stratified map if and only if:

There is a pair of stratifications $\mathcal{M}$,$\mathcal{N}$ of $\mathbb{X}$ respectively $\mathbb{Y}$, and for each stratum $\mathbb{X}_i$ there exists an open neighbourhood U_i and a C^p map $f_i : \mathbb{X}_i \subseteq U_i \to \mathbb{X}$ with:

$$\overline{\mathbb{X}_i} \subseteq U_i, \quad f_i|_{\mathbb{X}_i} = f|_{\mathbb{X}_i}, \quad \mathbb{X}_i \in \mathcal{N} \quad \text{and} \quad rank f|_{\mathbb{X}_i} = dim f(\mathbb{X}_i).$$

• **stratified monomorphism:** A map f is a stratified monomorphism if and only if in addition to being a stratified map, $f|_{\mathbb{X}_i}$ is injective.

• **strongly continuous one-parameter unitary group:** An operator valued function satisfying

(i) For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$.

(ii) If $\varphi \in$ and $t \to t_0$, then $U(t)\varphi \to U(t_0)\varphi$,

is called a strongly continuous one-parameter unitary group.

• **subcover:** See cover.

• **subgroup:** A subgroup of G is a subset $H \subset G$ such that

(a) $E_G \in H$

(b) $xy \in H$ for all $x, y \in H$

(c) $x^{-1} \in H$ for every $x \in H$.

• **submanifold:**

This class of subsets of a manifold should exclude anything with sharp corners, such as the surface of a cube or of a cone, or anything whose dimensioon could be said to vary.

• **subobject:** A subobject is the analogue of the set-theoretical idea of a subset.

• **subobject classifier:** [360] encodes the possible answers to natural multiple choice question one can ask of the 'elements' of which objects in the topos are built.

While the category **Set** has just the two truth-values , $\{0, 1\}$, so also in any topos the collection of truth-values is an object in the topos - the subobject classifier, written Ω.

• **subring:** Let A be a ring. A **subring** of A is a subset containing I that is closed under addition, multiplication, and the formation of negatives.

• **sujective:** A function is surjective if it is onto.

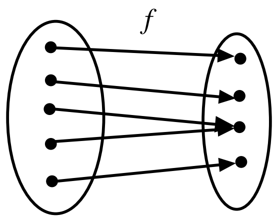


Figure B.30: surjective

See injective, bijective.

• **symmetric operator:** An operator A is called symmetric if its domain of dependence $D(A)$ is contained in the domain of dependence, $D(A^\dagger)$, of its adjoint operator $A^\dagger$, i.e. $D(A) \subset D(A^\dagger)$, and if $A^\dagger = A$ for all $A \in D(A)$.

A linear operator on a Hilbert space H is calledsymmetris if the condition

$$< Ax|y =< x|Ay > \tag{B.216}$$

for all elements x and y in the domain of A. A symmetric every-where- defined operator s self-adjoint.

• **sympletic manifold:** A sympletic manifold (M, ω) is a smooth real $n-$dimensional manifold M without boundary, equipped with a closed non-degenerate two-form ω. The most familiar example of a sympletic manifold is a cotangent bundle $M = T^*Q$. This is nothing but the traditional phase space, Q being the configuration space.

• **tangles:**

• **Temperly-Lieb algebra:**

• **tempered distributions:** The tempered distributions are continuous linear maps from the Schwarz space to complex numbers.

• $10j$ **symbol:** The $10j$ symbol depends on ten spins and appears as the vertex amplitude in the Barrett-Crane model.

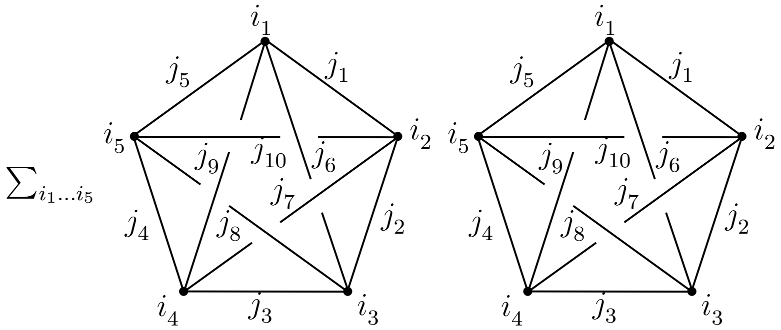


Figure B.31: tenjsymbol. This depends on ten spins and is called the $10j$ symbol.

• **tensor:** covariance etc.

• **tensor algebra:** It is the direct product $\otimes$ that is the binary operation or product that make the tensor algebra an algebra,

The tensor product of any number of elements can be taken as many times as one likes The tensor algebra $T(\mathcal{L})$ is the collection of (including up to an infinite number) collectively as direct summation:

$$T(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \ldots. \tag{B.217}$$

• **Theta function:**

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \tag{B.218}$$

$$\theta(x) = \frac{1}{\sqrt{x}} \theta \left(\frac{1}{x} \right). \tag{B.219}$$

• **three-manifolds:** All three-dimensional topologies can be constructed by starting with a three-sphere S^3, drawing all possible links on it, and performing all possible surgeries on them, (although one may need to perform an infinite number of surgeries to construct some manifolds). The classification of three-manifolds is related to the classification of all links.

Theorem (Lickorish and Wallace): Every closed, orientable, connected three-manifold, M^3 can be obtained by surgery on an unoriented framed knot or link $[L, f]$ in S^3. [hep-th/99071119] topological quantum field theories - a meeting ground for physicists and mathematicians

• **Tomita theorem:** Given any state ρ over a von Neumann algebra, there is always a flow α_t, called the Tomita flow of ρ, such that

$$\rho_0[\alpha_t(A)B] = \rho_0[\alpha_{(-t-i\beta)}(B)A] \tag{B.220}$$

• **topological dual:** See dual spaces.

• **topological vector space:** A topological vector space is a vector space with a topology defined on it such that the operations of addition and scalar multiplication are continuous.

• **topological structure:** In order to have a notion of convergence of points in a space X and a notion of continuity for functions $f : X \to Y$ into some space Y, one has to give X a topological structure.

• **topology:** A very far from being a comprehensive account of the subject of topology.

topology as determined by which functions are continuous. Weakest topology so that a given set of mappings are continuous.

Example: projection mappings $p_\gamma \to$ Hausdorff topology

• **topology of pointwise convergence:** Topology which arises from the seminorm given by

$$\|f\|_x = |f(x)|.$$

The space of functions with this topology is called the space of pointwise convergence.

• **topological space:** A topological space X is a set, with a specified family of subsets...

• **topos:** Very roughly, a topos is a category which behaves much like the category of sets; in fact this category, which is denoted **Set**, is itself a topos.

OR: topos is a categorical model of constructive set theory.

a branch of category theory.

applications to interpretive problems in quantum theory and quantum gravity, see [361].

Non-Boolean logic - that does not assume every statement can be judged true or false, there are statements upon which one cannot decide. Instead a Boolean algebra one has a Heyting algebra. The notion of a topos plays provides a good framework for making sense of the idea of partial truth.

Fotini Markopoulou should be modified to take into account the fact that observers can only give truth values to observables that have to do with their backwards light cone.

topos: To find out more about the subject to John Baez homepage (http://math.ucr.edu/home/baez/README.html) "fun stuff" and then to the link topos.

1. an initial object (an object like the empty set)
2. a terminal object (an an object like a set with one element)
3. binary coproducts (something like the disjoint union of two sets)
4. binary products (something like the Cartesian product of two sets)
5. equalizers (something like the subset of X consisting of all elements x such that $f(x) = g(x)$, where $f, g : X \to Y$)
6. coequalizers (something like the quotient set of X where two elements $f(y)$ and $g(y)$ are identified, where $f, g : Y \to X$)

(3) analogy of $\{0, 1\}$ denoted Ω, intuitively, the elements of Ω are the answers to a natural 'multiple-choice questions' about objects in the objects in the topos, just as "$x \in X$" is a natural for sets.

coming from [361]

See section J.2.

• **total function:** See partial function.

• **totally bounded:** A set X for which all $\epsilon > 0$ there are a finite number of points $x_1, \dots, x_n$ such that

$$X \subset \bigcup_{i=1}^{n} B_\epsilon(x_i). \tag{B.221}$$

• **trace:** Let $\{e_n\}_{n\in\mathbb{N}}$ be a basis for a Hilbert space $\mathcal{H}$ and let A be an operator on $\mathcal{H}$. The trace $\mathrm{Tr}(A)$ is defined as $\sum_{n=1}^{\infty} < Ae_n, e_n >$, whnever this limit exists.

• **trace class operators:** Let $\mathcal{L}_1$ denote the space of, necessarily compact, operators A such that $\mathrm{Tr}(|A|)$ exists. Then $A \mapsto \mathrm{Tr}(|A|)$ defines a norm on $\mathcal{L}_1$. The space $\mathcal{L}_1$ is called the space of trace class operators. The name trace class comes from its property that if A is trace class, then for any orthononormal basis $\{\varphi_n\}$

$$tr(A) = \sum_n < e_n, Ae_n > \tag{B.222}$$

is finite and independent of the orthononormal basis.

• **transition functions:** Whenever we define an object by use of local coordinates, it must have the same meanig in all coordinate systems. For it to have a basis-free significance, one must require the object's coefficients to have a special transformation property under a chgange of basis, this is acheived by the transition functions. As an example, the components of a vector field in one coordinate system must be related to those in another overlapping coordinate system by the Jacobian matrix which is the corresponding transition function. In practical situations transition functions are the gauge transformations required for pasting local charts together.

• **transitively:** given any two points x, y, in a group G there is at least one $g \in G$ that takes x to y, i.e. $gx = y$.

• **triangulable:** A space X is said to be triangulable if there exists a simplicial complex K_X that is exactly homeomorphic to X.

• **triangulation:** When a space X is triangulable the pair (X, K_X) is called a triangulation of X. The triangulation of a space is not unique.

• **trivialisation:** Say P is a fibre bundle with typical fibre F and base space $\mathcal{M}$. Trivialisations are global maps and only apply to trivial bundles. For each $x \in \mathcal{M}$, a trivialisation is an isomorphism,

$$\phi : P \to \mathcal{M} \times F$$

sending each fibre $\pi^{-1}(x) \in P$ to $\{x\} \times F$.

• **twists and writhe:**

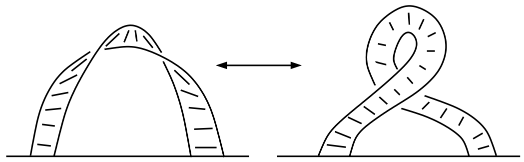


Figure B.32: Twists going to writes.

$$L = Tw + Wr \tag{B.223}$$

• **Tychonov topology:** The Tychonov topology on the direct product $X_\infty = \prod_{l \in \mathcal{L}} X_l$ of topological spaces X_l is the weakest topology such that all projections

$$p_l : X_\infty \to X_i;\ (x_{l'})_{l' \in \mathcal{L}} \to x_l \tag{B.224}$$

are continuous, that is, a net $x^\alpha = (x_l^\alpha)_{l \in \mathcal{L}}$ converges to $x = (x_l)_{l \in \mathcal{L}}$ if and only if $x_l^\alpha \to x_l$ for every $l \in \mathcal{L}$ pointwise (not necessarily uniformally) in $\mathcal{L}$.

• **uniform boundedness theorem:** A fundamental theorem of functional analysis. Let X be a Banach space and Y a normed space. Let $\Phi \subseteq B(X, Y)$ be the set of bounded operators from X to Y which is pointwise bounded, in the sense that, for each $x \in X$ there is some $c \in \mathbb{R}$ so that

$$\|Tx\| \leq c$$

for all $T \in \Phi$. Then Φ is uniformaly bounded: There is some constant C with

$$\|T\| \leq C$$

for all $T \in \Phi$.

• **uniform convergence:** A series of functions $\{f_n(x)\}$ is said to converge uniformally to $f(x)$ if when we put an $\epsilon-$tube around the function $f(x)$, the functions $f_n(x)$ eventually fit inside this.

• **uniform Rovelli-Smolin topology:** The Hamiltonian constraint operator $\hat{S}^\epsilon(N)$ does not converge with respect to the weak operator topology in $\mathcal{H}_{kin}$ when $\epsilon \to 0$. The convergence of $\hat{S}^\epsilon(N)$ holds with respect to the uniform Rovelli-Smolin topology, where one defines $\hat{S}^\epsilon(N)$ to converge if and only if $\Psi_{Diff}[\hat{S}^\epsilon(N)\phi]$ converge for all $\Psi_{Diff} \in Cyl^*_{Diff}$ and $\phi \in Cly(\overline{\mathcal{A}/\mathcal{G}})$

• **uniqueness theorems:**

Some important examples:

Stone-von Neumann theorem

• **unitary transformations:** We must make a distinction between unitary and isometric transformations which do not arise in finite-dimensional vector spaces. An isometry is defined in vector spaces of any dimension as a linear transformation U satisfying

$$(Uf, Ug) = (f, g)$$

for all in the vector space. This implies that $U^\dagger$ is a left inverse of U:

$$U^\dagger U = I.$$

now in finite-dimensional space, the existence of a left inverse guarantees the existence of a right inverse, but on an infinite-dimensional space this is not always the case. If a right inverse also exists, U is said to be unitary. This right inverse must be equal to the left inverse, since if $BA = I = AC$, then $B = C$. Thus for unitary transformations, we write

$$U^\dagger = U^{-1}.$$

• **universal cover:**

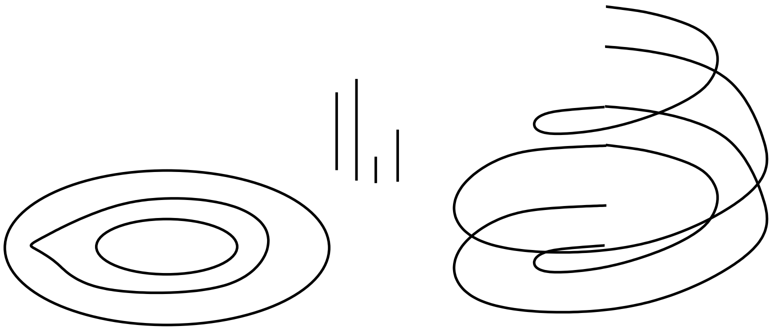


Figure B.33: unicoverEx.

• **universal enveloping algebra:**

$U_q(\mathbf{g})$ denotes the universal enveloping algebra - i.e., all formal powers and linear combinations of the elements of the deformed algebra modulo the standard Lie algebra relations.

Let g be a Lie algebra.

We know $T(\mathcal{L})$ is an associative algebra.

$$[x, y] - (x \otimes y - y \otimes x) \tag{B.225}$$

this collection forms a two-sided ideal I. $z \in I$

described as "universal" because any Lie algebra homomorphism $\psi\mathbf{g} \to A$, where A is a *unital* associative algebra, extends uniquely to a unital algebra homomorphism.

$$z(x \otimes y - y \otimes x) \tag{B.226}$$

$$T(\mathcal{L})/I \equiv U(\mathcal{L}) \tag{B.227}$$

$$\begin{aligned} \Delta\xi &= \xi \otimes 1 + 1 \otimes \xi \quad \text{coroduct} \\ \epsilon\xi &= 0 \quad \text{counit} \\ S\xi &= -\xi \quad \text{antipode} \end{aligned} \tag{B.228}$$

It is a **biideal** The quotient B/I is a bialgebra. An element of a general Hopf algebra which has this linear form

$$\Delta\xi = \xi \otimes 1 + 1 \otimes \xi \tag{B.229}$$

is called **primitive**.

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c \\ \Delta(b) &= a \otimes b + b \otimes d \\ \Delta(c) &= c \otimes a + d \otimes c \\ \Delta(d) &= c \otimes b + d \otimes d \end{aligned} \tag{B.230}$$

• **universal home:** a certain completion, $\overline{\mathcal{A}/\mathcal{G}}$, of $\mathcal{A}/\mathcal{G}$ the universal home for measures

• **universal net:**

• **upper semi-continuous:** A function $f(x)$ is said to be upper semi-continuous if for any x in the domain of f and for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(x) - f(x_0) < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

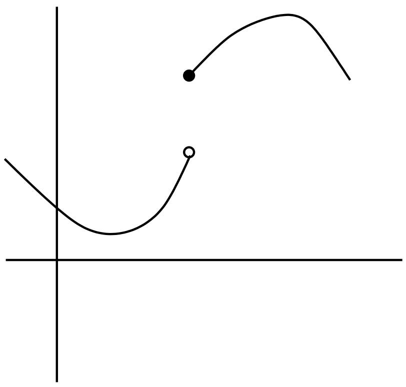


Figure B.34: An upper semi-continuous function.

• **Vassiliev Invariants:**

• **weakly continuous:**

$$(\dot{f}, x) \tag{B.231}$$

• **weakness of a topology:**

Roughly, one topology is weaker that another if it has fewer open sets, and stronger than another if it has more open sets. Let X be a non-empty set. $\emptyset, X$ is the weakest topology and the discrete topology is the strongest topology on X. The more open sets there are, the more continuous functions the space has.

• **weak operator topology:** The *weak operator topology* on $\mathcal{B}(H)$ is the weak topology generated by all functions of the form $T \to (Tx, y)$; that is, it is the weakest topology with respect to which all these functions are continuous. It is easy to see from the inequality $|(Tx, y) - (T_0 x, y)| \leq \|T - T_0\| \, \|x\| \, \|y\|$ that this topology is weaker than the usual norm topology, so that its closed sets are also closed in the usual sense. A C^*-algebra with the further property of being closed in the weak operator topology is called a W^*-algebra. Algebras of this kind are also called *rings of operators, or von Neumann algebras.*

(from intro topology and modern analysis)

• **weak topology:** The weak topology on X^* is the topology such that all functionals on X^{**} are continuous.

• **weak star operator topology:** The weak $*$topology with respect to a Hilbert space $Y = X$: this is similar to the weak topology, however, insted of $X' = X$ we now take a subspace $\mathcal{D}$ of X equipped with a finer topology and as $\mathcal{D}'$ teh topological dual of that topological space. Physical applications are the topology in which the Hamitonian constraint converges and Refined algebraic quantization (RAQ).

The weak star topology is obtained if we use the absolute values of $| < \Phi|A|\Psi > |$ between arbitrary state vectors as a system of seminorms. Thus a sequence of operators converges weakly if all matrix elements converge.

• **webs:** From [arXiv:math-ph/0304002]: For technical purposes, one assumed in the very beginning that these graphs are formed by piecewise analytic paths only. Namely, only in this case two finite graphs are always both contained in some third, bigger graph being again finite. This restriction has the drawback that only analyticity preserving diffeomorphisms can be implemented into that framework. In order to guarantee the inclusion of all diffeomorphisms, at least, piecewise smooth and immersive paths have to be considered as well. For the first time, this has been done by Baez and Sawin [q-alg/9507023] introducing so-called webs. A web is defined as a piecewise smooth graph determined by the union of a finite number of smooth curves that intersect in a controlled way, albeit possibly a countably infinite number of times.

• **Weierstrass approximation theorem:** Any real-valued continuous function f on $[a, b]$ can be arbitrary well approximated by a finite polynomial: given any $\epsilon > 0$, there is a polynomial P such that $\|f - P\| < \epsilon$:

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x,\ a \leq x \leq b. \tag{B.232}$$

We can restate the theorem as: polynomials are dense in the space of real-valued continuous functions on $[a, b]$ (dense in the same kind of way that the rationals $\mathbf{Q}$ are dense in $\mathbb{R}$).

• **well-posedness of initial value problem:** An initial value problem compose by a differential equation together with initial conditions on an suitable boundary. Well-posedness of an initial value problem requires

(i) existence of solutions,

(ii) uniqueness of solutions,

(iii) continuous dependence of solutions on initial conditions.

• **Whitehead's theorem:** a smoothing. For each smooth manifold M, there exists a PL-manifold M_{PL}, called its Whitehead triangulation, so that M is diffeomorphic to a smoothing of M_{PL}. [396]

• **Whitehead's triagulation:** Whitehead's triagulation provide us with a way of "discretizing" spacetime which is not merely some approximation nor introduces a physical cut-off, but [396]

• **Wigner's $6-j$ symbol:**

$$W(j) = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \tag{B.233}$$

• **Wigner transform:**

• **Von-Neumann algebra:** A von Neumann algebra is a *-algebra of bounded operators on a Hilbert space which is closed in the weak operator topology, or more explicitly:

A bounded operator B is a **weak limit** of a set of bounded operators if for each choice of positive number ϵ, positive integer n, and vectors $\psi_1, \psi_2, \ldots \psi_n$ and $\phi_1, \phi_2, \ldots \phi_n$ there is an operator A in the set such that

$$|(\phi_k, A\psi_k) - (\phi_k, B\psi_k)| < \epsilon \tag{B.234}$$

for $k = 1, 2, \ldots, n$. The extension of a set of bounded operators to its weak limits is its **weak closure**. A set of bounded operators is **weakly closed** if it contains its weak limits. A symmetric ring of bounded operators which is weakly closed is a von Neumann algebra.

Applications are:

(i) in Algebraic Quantum Field theory. In algebraic Quantum Field theory associates a von Neumann algebra to each causally complete region of spacetime.

(ii) Tomita-Takesaki theorem - thermal states and thermal time hypothesis.

(iii) Quantum causal histories

(iv) Noiseless subsystems

(v) Infinite tensor product Hilbert spaces.

• **Voronoi Diagram:**

http://www.lepp.cornell.edu/spr/2001-06/msg0033516.html

• **Zorn's lemma** Let $X \neq \emptyset$ be a partially ordered set with the property that every linearly ordered subset $Y \subset X$ (i.e., $y \leq y'$ or $y' \leq y$ for all $y, y' \in Y$) has an upper bound $x_Y \in X$ (i.e., $y \leq x_Y$ for all $y \in Y$). Then X has a maximal element $m \in X$ (i.e., $m \leq x$ for $x \in X$ implies $x = m$) which is a common upper bound for all linearly ordered subsets.

Appendix C

Algebraic Quantum Gravity

C.1 Introduction

The natural Hilbert space representaion aquires the structure of an infinite tensor product (ITP) whose separable strong equivalence class Hilbert subspaces (sectors) are left invraint by the quantum dynamics.

M preserves all the strong equivalence class Hilbert spaces. This follows from the fact that **M** is a countable sum of operators each of which changes only a finite number of entries in a vector of the form $\otimes_f$, hence we get a countable sum of vectors in the same equivalence class (see lemma), which remains normalizable if $\otimes_f$ is in the domain of **M**.

and $t_e := \ell_p^2/a_e^2$ is the classicallity parameter.

C.2 The Infinite Tensor Product

We first consider the tensor product of a finite number of Hilbrt spaces. Say $f_k, g_k \in \mathcal{H}_k$ with inner product $(\cdot,\cdot)_k$ on $\mathcal{H}_k$. If an element of $\otimes_k \mathcal{H}_k$ is f, the inner product of the tensor product is defined as

$$(f, g) = \prod_{k=1}^{n} (f_k, g_k)_k$$

and the norm

$$\|f\| = \sqrt{\prod_{k=1}^{n} (f_k, f_k)_k} = \sqrt{\prod_{k=1}^{n} \|f_k\|_k^2} = \prod_{k=1}^{n} \|f_k\|_k.$$

We are lead to the consideration of the mathematics of products of arbitrary complex numbers.

Now when one forms the infinite tensor product of a collection of Hilbert spaces, a physical requirement is that this product must not depened on the order of the individual Hilbert spaces (whether the collection is countably or uncountably infinite).

Hence we are interested in the convergence properties of a countable or uncountable product of complex numbers which are independent of the ordering of the product. This was developed in the paper by von Neumann (availble at http://www.numdam.org/item?id=1939 0).

As we will see, convergence of products is related to convergence of corresponding summations. Now, it is a remarkable fact that whether an infinite series converges or not can depend on the ordering of the terms of that series. From which it follows that whether or not the coresponding product of complex numbers converges depends on the ordering of the terms of that product.

Absolutely and conditionally convergent series have completely different behaviours under rearrangement

Theorem C.2.1 *(***Riemann's Rearrangement Theorem***) Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. For each real number s, there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges and has sum s.*

Proof:

The nonnegative series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty}(-q_n)$ diverge. In fact, if both were to converge, it would follow that $\sum_{n=1}^{\infty} |a_n|$ converges, that is, $\sum_{n=1}^{\infty} a_n$ would be absolutely convergent. On the other hand, if one of these series converged and the other diverged, it would follow that the partial sums of $\sum_{n=1}^{\infty} a_n$ diverge to either $+\infty$ or to $-\infty$. The convergence of $\sum_{n=1}^{\infty} a_n$ itself implies that both $\{p_n\}$ and $\{q_n\}$ have limit zero.

Now we construct the rearrangement. Choose terms $p_1, p_2, \dots$ up to the first index k_1 such that

$$p_1 + p_2 + \cdots + p_{k_1} > s.$$

This will occur , because $\sum_{n=1}^{\infty}$. Note

$$|p_1 + p_2 + \cdots + p_{k_1} - s| < p_{k_1}.$$

Next, we choose $q_1, q_2, \dots$ up to the first index l_1

such that

$$(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) < s.$$

Note

$$|(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) - s| < \max\{p_{k_1}, |q_{l_1}|\}.$$

We then add just enough new p's to make the left hand side greater than s, followed by just enough q's to make it less than s, and continue. At each phase of the $2n-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_n}, |q_{l_n}|\}$, and at each phase of the $2n+1-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_{n+1}}, |q_{l_n}|\}$. As these have the limit 0, the rearranged series has sum s.

□

Corollary C.2.2 *Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. It has a rearrangement that diverges to $+\infty$.*

Proof:

□

Theorem C.2.3 *Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement. Then $\sum_{n=1}^{\infty} b_n$ converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.*

Proof: Let $\{s_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$, i.e,

$$s_n = \sum_{k=1}^{n} a_k,$$

and let s be the limit. Let $\{t_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} b_k$. Given $\epsilon > 0$, choose M so large that

$$\sum_{k=M+1}^{\infty} |a_k| < \epsilon/2. \tag{C.1}$$

It follows from this that $|s - s_M| < \epsilon/2$,

$$|s - s_M| = |\sum_{k=M+1}^{\infty} a_k| \leq \sum_{k=M+1}^{\infty} |a_k| < \epsilon/2.$$

Choose N so large that every one of the first M terms of $\{a_k\}$ occurs among the first N terms of $\{b_k\}$. So that for any $n \geq N$

$$|t_n - s_M| = |t_n - \sum_{k=1}^{M} a_k| \leq \sum_{k=M+1}^{\infty} |a_k|.$$

Therefore

$$|t_n - s| \leq |t_n - s_M| + |s_M - s| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem C.2.4 *If $b_n \geq 0$ for all n, then $\sum_{n=1}^{\infty} b_n$ converges if and only if the sequence of partial sums is a bounded sequence.it is bounded*

Proof:

□

C.2.1 Infinite Products of Complex Numbers

A sequence of numbers $a_1, a_2, a_3, \ldots$ the infinite product

$$\prod_{n=1}^{\infty} a_n = a_1 \; a_2 \; a_3 \cdots$$

is defined to be the limit of the partial products $a_1 a_2 \ldots a_n$ as $n \to \infty$. That is, let P_n be the partial product

$$P_n = \prod_{k=1}^{n} a_k$$

then

$$\prod_{n=1}^{\infty} a_n := \lim_{n\to\infty} P_n.$$

The product is said to converge when the limit exists. If the product converges, then the limit of the sequence a_n as $n \to \infty$ must be 1. Proof is easy. Assume that an infinite product $\prod a_n$ is convergent.

of this infinite product. Then for each a_n of the series we have $a_n = P_n/P_{n-1}$. Since the product is convergent, there exists a such that as $n \to \infty$, $P_n = a$ and $P_{n-1} = a$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} P_n/P_{n-1} = a/a = 1.$$

The contrary to this is in general not true. Therefore, the logarithm a_n will be defined for all but a finite number of n, and for those we have ($a_n \neq 0$)

$$\ln \prod_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln a_n$$

with the product on the left converging if and only if the sum on the right converges. This allows the translation of convergence criteria for infinite sums into convergence criteria for infinite products.

Lemma C.2.5 *By the defintion of convergence , if $\prod_n^\infty a_n$, $\prod_n^\infty a'_n$ converge to a, a' respectively then $\prod_n^\infty a_n a'_n$ converges to aa'.*

Proof: Note that if $\{P_n\}$ and $\{P'_n\}$ are converge sequences, converging to P, P' respectively, then

$$\lim_{n\to\infty} (P_n + P'_n) = \lim_{n\to\infty} P_n + \lim_{n\to\infty} P'_n$$

as

$$\begin{aligned} |(P_n + P'_n) - (P + P')| &= |(P_n - P) + (P'_n - P')| \\ &\leq |(P_n - P)| + |(P'_n - P')|. \end{aligned}$$

Therefore, with $P_n = \sum_k^n \ln a_k$, $P'_n = \sum_k^n \ln a'_k$, we have

$$
\begin{aligned}
\ln \prod_n^{\infty} (a_n a_n') &= \sum_n^{\infty} \ln(a_n a_n') \\
&= \sum_n^{\infty} (\ln a_n + \ln a_n') \\
&= \sum_n^{\infty} \ln a_n + \sum_n^{\infty} \ln a_n' \\
&= \ln a + \ln a' \\
&= \ln(ab).
\end{aligned}
$$

□

A criterion for a product to converge.

Theorem C.2.6 *For $\rho_k > 0$, if $\prod_{n=1}^{\infty} |\rho_n - 1|$ converges then $\prod_{n=1}^{\infty} \rho_n$ converges.*

Proof: Set

$$
P_n = \prod_{k=1}^{n} (1 + a_k), \qquad \tilde{P}_n = \prod_{k=1}^{n} (1 + |a_k|)
$$

Then

$$
\tilde{P}_n \leq \exp(|a_1| + \ldots |a_n|)
$$

If $P_n - 1 < 0$ then, expanding the product, we see

$$
-(\tilde{P}_n - 1) < P_n - 1
$$

and if $P_n - 1 \geq 0$ then

$$
P_n - 1 \leq (\tilde{P}_n - 1),
$$

therefore

$$
|P_n - 1| \leq \tilde{P}_n - 1
$$

Hence if the product

$$\prod_{n=1}^{\infty}(1+|a_n|)$$

converges, then so does

$$\prod_{n=1}^{\infty}(1+a_n).$$

If $\sum_{n=1}^{\infty}|a_n|<\infty$ then

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges. Putting $\rho_n = 1+a_n$ $(\rho_n > 0)$ we have $\sum_{n=1}^{\infty}|\rho_n-1|$ converges then $\prod_{n=1}^{\infty}\rho_n$ converges.

□

In fact the implication can be reversed. To prove this we need to first establish a couple of lemmas.

Lemma C.2.7 *For $r_n > 0$, $\sum_{n=0}^{\infty} r_n$ converges if and only if $\prod_{n=0}^{\infty}(1+r_n)$ converges.*

Proof:

□

Lemma C.2.8 *If $0 < r_n < 1$, then $\sum_{n=0}^{\infty} r_n$ converges if and only if $\prod_{n=0}^{\infty}(1-r_n)$ converges.*

Proof: Note the product obviously is finite, it only diverges if it equals zero. First say the product $\prod_{n}^{\infty} r_n$ converges. As $0 < x <$ then

$$\ln(1-x) = -x - x^2/2 - \cdots < -x,$$

so

$$x < -\ln(1-x).$$

This implies

$$\sum_{n}^{\infty} r_n < -\sum_{n}^{\infty} \ln(1-r_n) = \ln\left(1/\prod_{n=0}^{\infty}(1-r_n)\right) < \infty,$$

i.e., the summation converges.

Now say the summation $\sum_{n=0}^{\infty} r_n$ converges. Then there exists finite N such that $r_n < \delta$ for all $n \geq N$. Since $\ln(1-x)$ is convex, it follows that for $x < \delta$

$$\ln(1-x) > -kx$$

where $k = -(\ln(1-\delta))/\delta$. To see this more clearly draw the function $\ln(1-x)$. Replacing x by r_n, summing over n and then exponentiating gives

$$\prod_{n=N}^{\infty}(1-r_n) > \exp(-k\sum_{n=N}^{\infty} r_n) > 0,$$

i.e., the product is non-zero, and so by definition convergent.

□

Theorem C.2.9 *For $\rho_n > 0$, $\sum_{n=0}^{\infty} |\rho_n - 1|$ converges if and only if $\prod_{n=0}^{\infty} \rho_n$ converges.*

Proof: Consider the product

$$\prod_{n=0}^{\infty} \rho_n = \prod_{n=0}^{\infty}[(1+(\rho_n-1)]$$

for $\rho_n > 1$. It follows from lemma (C.2.7) that $\sum_{n=0}^{\infty} |\rho_n - 1|$ converges if and only if $\prod_{n=0}^{\infty}(1+|\rho_n-1|)$ converges. This proves the theorem for the case of $\rho_n > 1$ for all n.

Consider the product

$$\prod_{n=0}^{\infty} \rho_n = \prod_{n=0}^{\infty}[(1-(1-\rho_n)]$$

for $0 < \rho < 1$. It follows from lemma (C.2.8), by replacing r_n by $1-\rho_n$, the theorem is proved for the case of $\rho_n < 1$ for all n.

Now we consider the general case. We factor the partial product

$$P_N = \prod_{n=1}^{N} \rho_n$$

into the product of terms for which $0 < \rho < 1$ and into the product of terms for which $\rho > 1$, which we denote respectively as Q_N and R_N (obviously we can ignore terms for which $\rho_n = 1$). Then

$$P_N = Q_N \cdot R_N.$$

We then use the fact that if $\prod_n^\infty \rho_n$, $\prod_n^\infty \rho'_n$ converge to ρ, ρ' respectively then $\prod_n^\infty \rho_n \, \rho'_n$ converges to $\rho\rho'$ to complete the proof of the theorem.

□

Now we consider products of complex numbers.

Definition The infinite product of complex numbers

$$\prod_{n=0}^{\infty} z_n \tag{C.2}$$

is said to **converge** to the number z provided that for each positive number $\delta > 0$ there exists $N < \infty$ such that for all $n \geq N$

$$|z - \prod_{k=0}^{n} z_k| < \delta.$$

□

Lemma C.2.10 *If z_n are arbitrary complex numbers, then $\sum_n^\infty z_n$ converges if and only if $\sum_n^\infty |z_n|$ converges.*

Proof: Convegrence of $\sum_n^\infty z_n$ is equivalent to the convegrence of both $\sum_n^\infty \mathrm{Re} z_n$ and $\sum_n^\infty \mathrm{Im} z_n$. Similarly, as

$$|\mathrm{Re} z_n|,\ |\mathrm{Im} z_n| \leq |z_n| \leq |\mathrm{Re} z_n| + |\mathrm{Im} z_n|.$$

the convergence of $\sum_n^\infty |z_n|$ is equivalent to the convegrence of both $\sum_n^\infty |\mathrm{Re} z_n|$ and $\sum_n^\infty |\mathrm{Im} z_n|$.

Sufficiency. As we have for real series

$$\left|\sum_k^n a_k\right| \leq \sum_k^n |a_k|,$$

$|\sum_k^n a_k|$ will be bounded as $\sum_k^n |a_k|$ is bounded by hypothesis, this proves sufficiency.

Now we prove necessity. Suppose $\sum_n^\infty z_n$ converges, and let a be its value.

□

in general it is not guatanteed we can write

$$\lim_{n\to\infty}(\prod_k^n |z_k|\, e^{i\varphi_k}) = \lim_{n\to\infty}\prod_k^n |z_k| \lim_{n\to\infty}\prod_k^n e^{i\varphi_k}$$

However we have: if $\{\xi_n\}$ and $\{\zeta_n\}$ are two complex convergent sequences with limits a and b, then

$$\lim_{n\to\infty}(\xi_n\zeta_n) = \xi\zeta$$

Proof:

$$\begin{aligned} |\xi_n\zeta_n - \xi\zeta| &= |(\xi_n\zeta_n - \xi\zeta_n) + (\xi\zeta_n - \xi\zeta)| \\ &\leq |(\xi_n\zeta_n - \xi\zeta_n)| + |(\xi\zeta_n - \xi\zeta)| \\ &= |\xi_n - \xi|\,|\zeta_n| + |\xi|\,|\zeta_n - \zeta| \end{aligned}$$

For $n \geq$ to some N, $|\zeta_n| = |\zeta + (\zeta_n - \zeta)| \leq |\zeta| + |(\zeta_n - \zeta)| \leq |\zeta| + 1$. Given $\epsilon > 0$ we can choose N so large that $n \geq N$ implies $|\zeta_n| \leq |\zeta| + 1$ and also

$$|\xi_n - \xi| < \frac{\epsilon}{2|\zeta| + 2}, \qquad |\zeta_n - \zeta| < \frac{\epsilon}{2|\xi|}.$$

It follows that $n \geq N$ implies $|\xi_n\zeta_n - \xi\zeta| < \epsilon$.

Theorem C.2.11 *Let $z_n = \rho_n e^{i\varphi_n} \in \mathbb{C}$ where $\rho_n = |z_n|$, $\varphi_n \in [-\pi, \pi]$. Then $\prod_\alpha z_n$ converges if and only if*

i) either $\prod_n \rho_n$ converges to zero in which case $\prod_n z_n = 0$, prod ii) or $\prod_n \rho_n$ converges to $\rho > 0$ and $\sum_n |\varphi_n|$ converges in which case

$$\prod_n z_n = \rho e^{i \sum_n \varphi_n}.$$

Proof: the partial product can be written

$$P_n = \prod_{k=1}^n a_k = (\prod_{k=1}^n \rho_k)\ e^{i\sum_{k=1}^n \varphi_k}$$

$$\lim_{n\to\infty} e^{i\sum_{k=1}^n \varphi_k}$$

$$|z - e^{i\sum_{k=1}^n \varphi_k}| < \delta.$$

We require

$$\lim_{n\to\infty} P_n/P_{n-1} = 1$$

that is

$$\varphi_n \to 0$$

We show that when $\prod_n^\infty |z_n| > 0$, then it implies the convergence of $\sum_n^\infty |\varphi_n|$ also.

As $\prod_n^\infty z_n$, $\prod_n^\infty |z_n|$ converge, with the latter > 0

$$\prod_n^\infty \frac{z_n}{|z_n|} = \prod_n^\infty e^{i\varphi_n}$$

converges also.

□

Corollary C.2.12 *Quasi-convergence of $\prod_n z_n$ to a non-vanishing value $\neq 0$ is equivalent with convergence with such a value (this is by definition of quasi-convergence). It holds if and only if all $z_n \neq 0$, and $\sum_n^\infty |z_n - 1|$*

Proof: As it is quasi-convergent we have that $\sum_n^\infty | \, |z_n| - 1|$

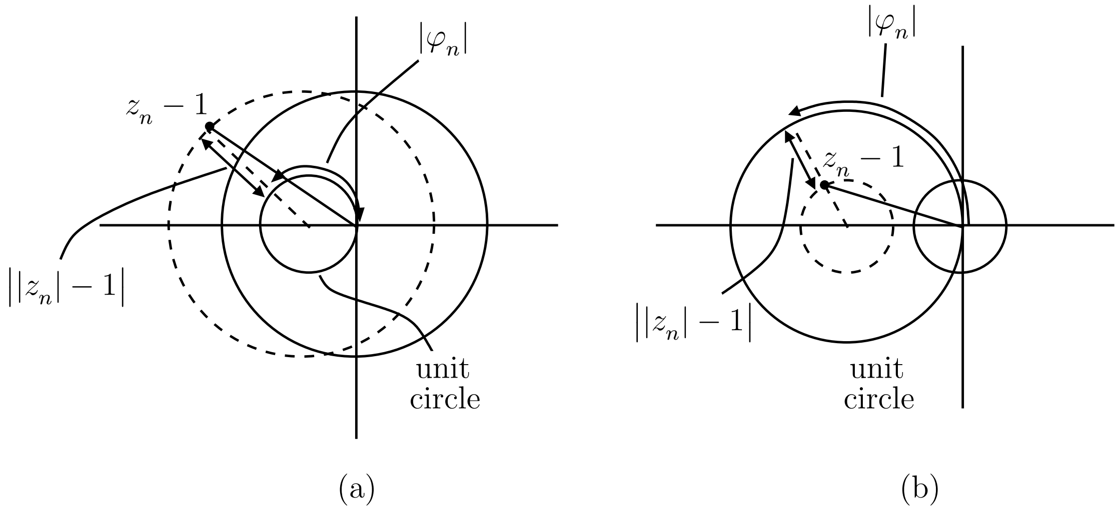


Figure C.1: Proof of the inequalities $\big||z_n| - 1\big| \leq |z_n - 1| \leq \big||z_n| - 1\big| + |\varphi_n|$. (a) Case where $|z_n| > 1$. (b) Case where $|z_n| < 1$.

$$\big||z_n| - 1\big|, \ \frac{1}{\pi}|\varphi_n| \leq |z_n - 1| \leq \big||z_n| - 1\big| + |\varphi_n|$$

Hence the convergence of the sum $\sum_n^\infty |z_n - 1|$ is equivalent to the convergence of both $\sum_n^\infty \big||z_n| - 1\big|$ and $\sum_n^\infty |\varphi_n|$.

Lemma C.2.13 *By the defintion of convergence, if*

1) $\prod_n^\infty z_n$, $\prod_n^\infty z'_n$ *converge to* z, z' *respectively then* $\prod_n z_n z'_n$ *converges to* zz'*,*

2) if $\prod_n^\infty z_n$ *converges to* z*, then* $\prod_n^\infty z_n^*$ *converges to* z^**.*

Proof: 1)

$$\begin{aligned}
\ln \prod_n^\infty (z_n z'_n) &= \sum_n^\infty (\ln \rho_n + \ln \rho'_n + i\varphi_n + i\varphi'_n) \\
&= \sum_n^\infty (\ln \rho_n + i\varphi_n) + \sum_n^\infty (\ln \rho'_n + i\varphi'_n) \\
&= \ln \prod_n^\infty z_n + \ln \prod_n^\infty z'_n \\
&= \ln(zz')
\end{aligned}$$

□

Lemma C.2.14 *For any complex numbers, the convergence of one of* $\sum_{n=0}^{\infty} |\,|z_n| - 1|$, $\sum_{n=0}^{\infty} |\,|z_n|^2 - 1|$ *implies the convergence of the other.*

Proof: If one or other of the series converges, then $\lim_{n\to\infty} |z_n| = 1$. So

$$\begin{aligned}
\lim_{n\to\infty} \frac{|\,|z_n|^2 - 1|}{|\,|z_n| - 1|} &= \lim_{n\to\infty} |\,|z_n| + 1| \\
&= 2 > 0.
\end{aligned}$$

Hence there exists finite N such that

$$1 < \frac{|\,|z_n|^2 - 1|}{|\,|z_n| - 1|} < 3 \qquad \text{for all } n \geq N.$$

Equivalently

$$|\,|z_n| - 1| < |\,|z_n|^2 - 1| < 3|\,|z_n| - 1| \qquad \text{for all } n \geq N.$$

If the series $\sum_{n=0}^{\infty} |\,|z_n|^2 - 1|$ converges then $\sum_{n=0}^{\infty} |\,|z_n| - 1|$ also converges as

$$\sum_{n=N}^{\infty} |\,|z_n| - 1| < \sum_{n=N}^{\infty} |\,|z_n|^2 - 1|.$$

If the series $\sum_{n=0}^{\infty} |\,|z_n| - 1|$ converges then $\sum_{n=0}^{\infty} |\,|z_n|^2 - 1|$ also converges as

$$\sum_{n=N}^{\infty} |\,|z_n|^2 - 1| < \sum_{n=N}^{\infty} 3|\,|z_n| - 1|.$$

The result also follows by applying theorem C.2.9: $\sum_{n=0}^{\infty} |\,|z_n| - 1|$ converges if and only if $\prod_{n=0}^{\infty} |z_n|$ converges, which converges if and only if $\prod_{n=0}^{\infty} |z_n|^2$ converges, which in turn converges if and only if $\sum_{n=0}^{\infty} |\,|z_n|^2 - 1|$ converges.

□

It is understood that the sum is meaniful only if at most a countable number of terms are different from zero.

Recall that for a series $\sum_{\alpha} z_{\alpha}$ to converge absolutely it is necessary that $z_{\alpha} = 0$ for all but countably infinitely many $\alpha \in \mathcal{I}$.

Definition Let $\{z_\alpha\}_{\alpha\in\mathcal{I}}$ be a collection of complex numbers. The infinite product

$$\prod_{\alpha\in\mathcal{I}} z_\alpha \tag{C.3}$$

is said to **converge** to the number z provided that for each positive number $\delta > 0$ there exists a finite set

$$\mathcal{I}_0 \subset \mathcal{I}$$

such that for any other finite $\mathcal{J}$ with

$$\mathcal{I}_0(\delta) \subset \mathcal{J} \subset \mathcal{I}$$

it holds that

$$|z - \prod_{\alpha\in\mathcal{J}} z_\alpha| < \delta.$$

□

In contrast to the case of an infinite series, absolute converge of an infinite product does not imply converge, the phases of the factors could fluctuate wildly. This motivates the following definition.

Definition We say that $\prod_{\alpha\in\mathcal{I}} z_\alpha$ is **quasi-convergent** if $\prod_{\alpha\in\mathcal{I}} |z_\alpha|$ converges. If $\prod_{\alpha\in\mathcal{I}} z_\alpha$ is quasi-convergent but not convergent we define $\prod_{\alpha\in\mathcal{I}} z_\alpha := 0$.

□

This definition assigns a value to the infinite product of numbers which converge absolutely but not necessarily non-absolutely.

Theorem C.2.15 *1) Let $\rho_\alpha \geq 0$.*

i) If for every $\alpha_0 \in \mathcal{I}$ it holds that $\rho_{\alpha_0} = 0$ then $\prod_\alpha \rho_\alpha = 0$.

ii) If for $\rho_\alpha > 0$ for all α then $\prod_\alpha \rho_\alpha$ converges if and only if

$$\sum_\alpha \max(\rho_\alpha - 1, 0)$$

converges.

iii) If for $\rho_\alpha > 0$ for all α then $\prod_\alpha \rho_\alpha$ converges if and only if

$$\sum_\alpha |\rho_\alpha - 1|$$

converges.

2) Let $z_\alpha = \rho_\alpha e^{i\varphi_\alpha} \in \mathbb{C}$ where $\rho_\alpha = |z_\alpha|$, $\varphi_\alpha \in [-\pi, \pi]$. Then $\prod_\alpha z_\alpha$ converges if and only if

i) either $\prod_\alpha \rho_\alpha$ converges to zero in which case $\prod_\alpha z_\alpha = 0$,

ii) or $\prod_\alpha \rho_\alpha$ converges to $\rho > 0$ and $\sum_\alpha |\varphi_\alpha|$ converges in which case

$$\prod_\alpha z_\alpha = \rho e^{i\sum_\alpha \varphi_\alpha}.$$

Proof:

□

Corollary C.2.16 *Quasi-convergence*

Proof:

□

After having introduced convergence for infinite products of complex numbers we can now turn to ITP Hilbert spaces.

C.2.2 $C-$vectors of the ITP

Definition Let $\mathcal{H}_\alpha$, $\alpha \in \mathcal{I}$ be an arbitrary collection of Hilbert spaces. For a sequence $f := \{f_\alpha\}_{\alpha\in\mathcal{I}}$, $f_\alpha \in \mathcal{H}_\alpha$ the object

$$\otimes_f := \otimes_\alpha f_\alpha \tag{C.4}$$

is called a $C-$**vector** provided that $\prod_\alpha \|f_\alpha\|_\alpha$ converges, where $\|\cdot\|_\alpha$ denotes the Hilbert norm of $\mathcal{H}_\alpha$. The set of vectors will be denoted V_C.

$\square$

Lemma C.2.17 *For two $C-$vectors $\otimes_f = \otimes_\alpha f_\alpha$, $\otimes_g = \otimes_\alpha g_\alpha$ the inner product*

$$< \otimes_f, \otimes_g > := \prod_\alpha < f_\alpha, g_\alpha >_\alpha \tag{C.5}$$

is a quasi-convergent product of the individual inner products $< f_\alpha, g_\alpha >$ on $\mathcal{H}_\alpha$.

Proof: We prove that if $f_\alpha, g_\alpha \in \mathcal{H}_\alpha$ for all $\alpha \in \mathcal{I}$, and if $\prod_\alpha \|f_\alpha\|_\alpha$, $\prod_\alpha \|g_\alpha\|_\alpha$ are convergent, then so is $\prod_a |(f_\alpha, g_\alpha)_\alpha|$, that is, $\prod_a (f_\alpha, g_\alpha)_\alpha$ is quasi-convergent.

Since $\|f_\alpha\|_\alpha = 0$ or $\|g_\alpha\|_\alpha = 0$ implies $(f_\alpha, g_\alpha)_\alpha = 0$

by theorem C.2.15 we need only show that $\sum_\alpha \max(|(f_\alpha, g_\alpha)_\alpha| - 1, 0)$ converges as a consequence of the convergence of $\sum_\alpha \max(\|f_\alpha\|^2_\alpha - 1, 0)$, $\sum_\alpha \max(\|g_\alpha\|^2_\alpha - 1, 0)$.

Now as $|(f_\alpha, g_\alpha)_\alpha| \leq \frac{1}{2}\|f_\alpha\|^2_\alpha + \frac{1}{2}\|g_\alpha\|^2_\alpha$,

$$|(f_\alpha, g_\alpha)_\alpha| - 1 \leq \frac{1}{2}(\|f_\alpha\|^2_\alpha - 1) + \frac{1}{2}(\|g_\alpha\|^2_\alpha - 1)$$

and hence

$$\max(|(f_\alpha, g_\alpha)_\alpha| - 1, 0) \leq \frac{1}{2}\max(\|f_\alpha\|^2_\alpha - 1, 0) + \frac{1}{2}\max(\|g_\alpha\|^2_\alpha - 1, 0).$$

$\square$

There are $C-$vectors $\otimes_f$ such that $\prod_\alpha \|f_\alpha\|_\alpha = 0$ although $\|f_\alpha\|_\alpha > 0$ for all $\alpha \in \mathcal{I}$. Thus, it is conceivable that it happens that $< \otimes_f, \otimes_g > \neq 0$ for some $C-$vector $\otimes_g$. If that were the case, the Schwarz inequality is violated for the inner product (C.5) for $C-$vectors.

Definition We say that for $f_\alpha \in \mathcal{H}_\alpha$ the ITP $\otimes_f := \otimes_\alpha f_\alpha$ is a C_0-**vector** (and $f = \{f_\alpha\}$ a C_0 sequence) if $\| \otimes_f \| := \prod_{\alpha\in\mathcal{I}} \|f_\alpha\|_\alpha$ converges to a non-vanishing number. The set of C_0-vectors will be denoted V_0.

$\square$

To distinguish trivial $C-$vectors from non-trivial ones we define

Definition A sequence $\{f_\alpha\}$ defines a C_0-**vector** $\otimes_f = \otimes_\alpha f_\alpha$ if and only if

$$\sum_\alpha |\ \|f_\alpha\|_\alpha - 1| \tag{C.6}$$

converges. The set of C_0-vectors will be denoted V_0.

$\square$

Lemma C.2.18 *For any complex numbers,* $\sum_\alpha |\ |z_\alpha| - 1|$ *converges if and only if* $\sum_\alpha |\ |z_\alpha|^2 - 1|$.

Proof:

$\square$

For a C_0-vector $\|f_\alpha\|_\alpha \neq 0$.

The norm of a C_0-vector does not zanish. By the previous lemma C_0-vector can be defined equivalently as a $\otimes_f = \otimes_\alpha f_\alpha$ such that

$$\sum_\alpha |\ \|f_\alpha\|_\alpha^2 - 1|$$

converges. By theorem C.2.15 this implies that

$$\prod_\alpha \|f_\alpha\|_\alpha^2 = \ (\otimes_f, \otimes_f) \ = \| \otimes_f \|^2$$

converges to a non-zero number, i.e. $\| \otimes_f \| > 0$.

C.2.3 Strong Equivalence Classes

Definition We will call two C_0-sequences $f = \{f_\alpha\}, g = \{g_\alpha\}$ **strongly equivalent**, denoted $f \approx g$, provided that

$$\sum_\alpha |(f_\alpha, g_\alpha)_\alpha - 1| \tag{C.7}$$

converges.

$\square$

Lemma C.2.19 *Strong equivalence of C_0-sequences is an equivalence relation, i.e., a relation such that*

1) (Reflexivity) for any $\otimes_f \in V_0$, we have $\otimes_f \approx \otimes_f$.

2) (Symmetry) for all $\otimes_f, \otimes_g \in V_0$, if $\otimes_f \approx \otimes_g$ then $\otimes_g \approx \otimes_f$.

3) (Transitivity) for all $\otimes_f, \otimes_g, \otimes_h \in V_0$, if $\otimes_f \approx \otimes_g$ and $\otimes_g \approx \otimes_h$ then $\otimes_g \approx \otimes_h$.

Proof: Condition 1) follows from the fact that a C_0-vector is defined by the condition that $\sum_\alpha | \, \|f_\alpha\|_\alpha^2 - 1|$ converges.

$$|(f_\alpha, g_\alpha)_\alpha - 1| = |\overline{(f_\alpha, g_\alpha)_\alpha} - 1| = |\overline{(f_\alpha, g_\alpha)_\alpha - 1}| = |(g_\alpha, f_\alpha)_\alpha - 1|.$$

Condition 3) We prove that, apart from a possible finite number of exceptions, the following

$$|(f_\alpha, h_\alpha)_\alpha - 1| \leq D \left(\left| \|f_\alpha\|_\alpha - 1 \right| + \left| \|g_\alpha\|_\alpha - 1 \right| + |(f_\alpha, g_\alpha)_\alpha - 1| + |(g_\alpha, h_\alpha)_\alpha - 1| \right)$$

holds for some constant D.

We put

$$\begin{gathered} \|f_\alpha\|_\alpha = 1 + \lambda_f, \quad \|g_\alpha\|_\alpha = 1 + \lambda_g, \quad \|h_\alpha\|_\alpha = 1 + \lambda_h, \\ (f_\alpha, g_\alpha) = 1 + \lambda_{fg}, \quad (g_\alpha, h_\alpha) = 1 + \lambda_{gh} \end{gathered}$$

So $|\lambda_f|$, $|\lambda_g|$, $|\lambda_h|$, $|\lambda_{fg}|$, $|\lambda_{gh}| \leq C$, and except for a finite number of α's, $|\lambda_g| \leq \frac{1}{2}$ (or $-1/2 \leq \|g_\alpha - 1\|_\alpha \leq 1/2$).

We express $g_\alpha, f_\alpha, h_\alpha \in \mathcal{H}_\alpha$ as:

$$\begin{aligned} g_\alpha &= a_{11} x_\alpha, \\ f_\alpha &= a_{21} x_\alpha + a_{22} x'_\alpha, \\ h_\alpha &= a_{31} x_\alpha + a_{32} x''_\alpha + a_{33} x'''_\alpha, \end{aligned}$$

where

$$\|x_\alpha\|_\alpha = \|x'_\alpha\|_\alpha = \|x''_\alpha\|_\alpha = 1, \qquad (x_\alpha, x'_\alpha) = (x_\alpha, x''_\alpha) = (x'_\alpha, x''_\alpha) = 0.$$

Then

$$\begin{aligned}
\|g_\alpha\|_\alpha^2 &= |a_{11}|^2, \\
\|f_\alpha\|_\alpha^2 &= |a_{21}|^2 + |a_{22}|^2, \\
\|h_\alpha\|_\alpha^2 &= |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2,
\end{aligned}$$

$$\begin{aligned}
(f_\alpha, g_\alpha)_\alpha &= a_{21}\overline{a_{11}} \\
(g_\alpha, h_\alpha)_\alpha &= a_{11}\overline{a_{31}} \\
(f_\alpha, h_\alpha)_\alpha &= a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}}
\end{aligned}$$

Now

$$\begin{aligned}
|(f_\alpha, g_\alpha)_\alpha - 1| &= |a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}} - 1| \\
&= \left|(a_{21}\overline{a_{11}} \cdot a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1) + a_{22}\overline{a_{32}}\right| \\
&\leq \left|a_{21}\overline{a_{11}}a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1\right| + |a_{22}\overline{a_{32}}|.
\end{aligned}$$

$$\begin{aligned}
\left|a_{21}\overline{a_{11}}a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1\right| &\leq (1 + |\lambda_{fg}|)\,(1 + |\lambda_{gh}|)\,(1 - |\lambda_g|)^{-2} - 1 \\
&\leq (1 + |\lambda_{fg}| + |\lambda_{gh}| + |\lambda_{fg}|\,|\lambda_{gh}|)\,(1 + 2|\lambda_g|) - 1 \\
&\leq \left(1 + |\lambda_{fg}| + |\lambda_{gh}| + \frac{C}{2}(|\lambda_{gh}| + |\lambda_{fg}|)\right)(1 + 2|\lambda_g|) - 1 \\
&\leq D_1(|\lambda_g| + |\lambda_{fg}| + |\lambda_{gh}|)
\end{aligned}$$

where in the second line we used that $1/(1-x)^2$ is convex.

$$|a_{22}|^2 = (|a_{21}|^2 + |a_{22}|^2) - \frac{|a_{21}\overline{a_{11}}|^2}{|a_{11}|^2} \leq (1 + |\lambda_f|)^2 - \frac{(1 - |\lambda_{fg}|)^2}{(1 + |\lambda_g|)^2}$$

$$|a_{32}|^2 = (|a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2) - \frac{|a_{11}\overline{a_{31}}|^2}{|a_{11}|^2} \leq (1 + |\lambda_h|)^2 - \frac{(1 - |\lambda_{gh}|)^2}{(1 + |\lambda_g|)^2}$$

$$
\begin{aligned}
(1+|\lambda_f|)^2 - \frac{(1-|\lambda_{fg}|)^2}{(1+|\lambda_g|)^2} &= 1+2|\lambda_f|+|\lambda_f|^2 - \frac{1-2|\lambda_{fg}|}{(1+|\lambda_g|)^2} - \frac{|\lambda_{fg}|^2}{(1+|\lambda_g|)^2} \\
&\leq (2+C)|\lambda_f| - (1-2|\lambda_{fg}|)[1-2|\lambda_g|] \\
&\leq (2+C)|\lambda_f| + 2|\lambda_{fg}| + 2|\lambda_g| - 4|\lambda_{fg}|\,|\lambda_g| \\
&\leq (2+C)|\lambda_f| + 2|\lambda_{fg}| + 2|\lambda_g| \\
&\leq (2+C)(|\lambda_f| + |\lambda_g| + |\lambda_{fg}|)
\end{aligned}
$$

where in the second line we used $1/(1+x)^2 \geq 1-2x$ for $x \geq 0$. Obviously we also have

$$|a_{32}|^2 \leq (2+C)(|\lambda_g| + |\lambda_h| + |\lambda_{gh}|)$$

Thus

$$|a_{22}a_{32}| \leq D_2(|\lambda_f| + |\lambda_g| + |\lambda_h| + |\lambda_{fg}| + |\lambda_{gh}|).$$

Combining all the inequalities, we obtain

$$|(f_\alpha, h_\alpha) - 1| \leq D(|\lambda_f| + |\lambda_g| + |\lambda_h| + |\lambda_{fg}| + |\lambda_{gh}|).$$

with a finite number of exceptions.

□

An equivalence relation will split the set V_0 into disjoint subsets, the equivalence classes.

Definition The strong equivalence class of a C_0-sequence f is denoted by $[f]$. The set of strong equivalence classes of C_0-sequences will be called $\mathcal{S}$.

□

Theorem C.2.20 *If two C_0-sequences f_α, g_α belong to two different equivalence classes, then*

$$(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = 0.$$

If they belong to the same equivalnce class, then $(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = 0$ if and only if some $(f_\alpha, g_\alpha) = 0$.

Proof: Clearly

$$(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = \prod_\alpha (f_\alpha, g_\alpha)$$

in the sence of quasi-convergence,

□

C_0-vectors from different strong equivalence classes are always orthogonal and those from the same strong equivalence class are orthogonal if and only if they are orthogonal in at least one tensor product factor.

Theorem C.2.21 *$[f] = [g]$ if and only if both $\sum_\alpha \|f^0_\alpha - g^0_\alpha\|^2_\alpha$ and $\sum_\alpha |Im(f^0_\alpha, g^0_\alpha)_\alpha|$ converge*

Proof: In othere words, the combined convergence is equivalent to that of $\sum_\alpha |(f^0_\alpha, g^0_\alpha) - 1|$. Note that if $|z_{\alpha_1}| + |z_{\alpha_2}| + \cdots + |z_{\alpha_n}|$ is bounded then so is $|\zeta_{\alpha_1}| + |\zeta_{\alpha_2}| + \cdots + |\zeta_{\alpha_n}|$ if $|\zeta_{\alpha_1} - z_{\alpha_1}| + |\zeta_{\alpha_2} - z_{\alpha_2}| + \cdots + |\zeta_{\alpha_n} - z_{\alpha_n}|$ is bounded.

Now as $\sum_\alpha \big|\|f_\alpha\| - 1\big|$ converges if and only if $\sum_\alpha \big|\|f_\alpha\|^2 - 1\big|$, we can prove the theorem by demonstrating the convergence of

$$\sum_\alpha |(f^0_\alpha, g^0_\alpha) - \frac{1}{2}\|f^0_\alpha\|^2 + \frac{1}{2}\|g^0_\alpha\|^2|.$$

Now

$$\begin{aligned} \mathrm{Re}\big((f^0_\alpha, g^0_\alpha) - \frac{1}{2}\|f^0_\alpha\|^2 + \frac{1}{2}\|g^0_\alpha\|^2\big) &= -\frac{1}{2}\big(\|f^0_\alpha\|^2 + \|g^0_\alpha\|^2 - 2\mathrm{Re}(f^0_\alpha, g^0_\alpha)\big) \\ &= -\frac{1}{2}\|f^0_\alpha - g^0_\alpha\|^2_\alpha \end{aligned}$$

and

$$\mathrm{Im}\big((f^0_\alpha, g^0_\alpha) - \frac{1}{2}\|f^0_\alpha\|^2 + \frac{1}{2}\|g^0_\alpha\|^2\big) = \mathrm{Im}(f^0_\alpha, g^0_\alpha)$$

Recall that the convergence of $\sum_\alpha z_\alpha$ is equivalent to the combined convergence of $\sum_\alpha |\mathrm{Re} z_\alpha|$, $\sum_\alpha |\mathrm{Im} z_\alpha|$. This completes the proof.

□

Lemma C.2.22 *for each $[f] \in \mathcal{S}$ there exists $f^0 \approx f$ such that $\|f^0_\alpha\|_\alpha = 1$ for all $\alpha \in \mathcal{I}$.*

Proof: Choose $f \in \mathcal{S}$. As $\sum_\alpha \big|\|f'_\alpha\|_\alpha - 1\big|$ converges, apart from a finite number of α's, we have

$$\big|\|f_\alpha\|_\alpha - 1\big| \leq \frac{1}{2}, \qquad \|f_\alpha\|_\alpha \geq \frac{1}{2}.$$

Recall that $f \approx f'$ if $f_\alpha \neq f'_\alpha$ for a finite number of α's. So we replace f_α by any f'_α such that $1/2 \leq \|f'_\alpha\|_\alpha \leq 3/2$ for the exceptional α's.

As $1/x$ is convex, for $1/2 \leq x \leq 1$ we have $(1/x) - 1 \leq 2(1-x)$ and for $1 \leq x \leq 3/2$ we have $1 - (1/x) \leq 2(x-1)$, that is,

$$\left|\frac{1}{x} - 1\right| \leq 2|x-1|$$

for $1/2 \leq x \leq 3/2$. So we have

$$\left|\frac{1}{\|f'_\alpha\|_\alpha} - 1\right| \leq 2\big|\|f'_\alpha\|_\alpha - 1\big|,$$

therefore

$$\sum_\alpha \left|\frac{1}{\|f'_\alpha\|_\alpha} - 1\right|$$

converges. By theorem C.2.15 ii), replace $|z_\alpha|$ by $1/\|f'_\alpha\|_\alpha$, as f'_α is a C_0-sequence so is

$$f^0_\alpha = \frac{1}{\|f'_\alpha\|_\alpha} f'_\alpha.$$

By theorem C.2.15 iv), replace z_α by $1/\|f'_\alpha\|_\alpha$, the C_0-sequences f'_α and f^0_α are equivalent. $\|f^0_\alpha\|_\alpha = 1$ is obvious, therefore the proof is complete.

□

By $\otimes_\alpha f_\alpha$ we denote the functional on X defined by

$$\otimes_\alpha f_\alpha(\{x_\alpha\}) = \prod_\alpha (f_\alpha, x_\alpha)_\alpha$$

We need to prove

$$\otimes_\alpha f_\alpha \neq 0 \quad \Leftrightarrow \quad \prod_\alpha \|f_\alpha\|_\alpha \neq 0$$

But $\otimes_\alpha f_\alpha \neq 0$ implies $\prod_\alpha \|f_\alpha\|_\alpha \neq 0$.

C.2.4 ITP Hilbert Space

1) All the $\mathcal{H}^\otimes_{[f]}$ are isomorphic and mutually orthogonal.

2) Every $\mathcal{H}^\otimes_{(f)}$ is the closed direct sum of all the $\mathcal{H}^\otimes_{[f']}$ with $[f'] \in \mathcal{S} \cap (f)$.

3) The ITP $\mathcal{H}^\otimes$ is the closed direct sum of all the $\mathcal{H}^\otimes_{(f)}$ with $(f) \in \mathcal{W}$.

4) Every $\mathcal{H}^\otimes_{[f]}$ has an explicitly known orthonormal von Neumann basis.

5) If s, s' are two different strong equivalence classes in the same weak equivalence class then there exists a unitary operator on $\mathcal{H}^\otimes$ that maps $\mathcal{H}^\otimes_s$ to $\mathcal{H}^\otimes_{s'}$, otherwise such an operator does not exist, the two Hilbert spaces are unitarily inequivalent subspaces of $\mathcal{H}^\otimes$.

Notice that two isomorphic Hilbert spaces can always be mapped into each other such that scalar products are preserved (just map some orthonormal bases) but here the question is whether this map can be extended unitarily to all of $\mathcal{H}^\otimes$. Intuitively then, strong classes within the same weak classes describe the same physics, those in different weak classes describe different physics such as an infinite difference in energy, magnetization, volume etc.

Definition By $\mathcal{H}_C$ we denote the completion of the complex vector space of finite linear combinations of elements from V_C equipped with the sesquilinear form $< \cdot, \cdot >$ obtained by extending (C.5) from V_C to $\mathcal{H}_C$ by sesquilinearity,

$$\begin{aligned} < \alpha\xi + \beta\chi, \zeta > &= \overline{\alpha} < \xi, \zeta > + \overline{\beta} < \chi, \zeta > \\ < \xi, \alpha\chi + \beta\zeta > &= \alpha < \xi, \chi > + \beta < \xi, \zeta > \end{aligned}$$

for all $\xi, \chi, \zeta \in \mathcal{H}_C$ and $\alpha, \beta \in \mathbb{C}$.

$\square$

We can now give the defintion of the ITP.

Definition We denote by

$$\mathcal{H}^{\otimes} := \otimes_\alpha \mathcal{H}_\alpha \tag{C.8}$$

the Cauchy completion of the pre-Hilbert space $\mathcal{H}_C$. It is called the ITP of the $\mathcal{H}_\alpha$.

$\Box$

The strong equivalence classes provide the basic tool to analyze the structure of $\mathcal{H}^{\otimes}$.

Definition For a strong equivalence class $[f] \in \mathcal{S}$ we define the closed subspace $\mathcal{H}_{[f]}$ of $\mathcal{H}^{\otimes}$ by the closure of the finite linear combinations of $\otimes_{f'}$'s with $f' \in [f]$, i.e., the closure of

$$\left\{\sum_{k=1}^{N} z_k \otimes_{f^k} : z_k \in \mathbb{C},\ f^k \in [f],\ N < \infty\right\}. \tag{C.9}$$

It is called the $[f]-$**adic incomplete ITP** of the $\mathcal{H}_\alpha$'s.

$\Box$

Theorem C.2.23 *The complete ITP decomposes as the direct sum over strong equivalence classes $[f]$ of $[f]-$adic ITP's,*

$$\mathcal{H}^{\otimes} = \overline{\otimes_{[f]\in\mathcal{S}} \mathcal{H}_{[f]}} \tag{C.10}$$

Proof:

$\Box$

C.2.5 Non-Associativity of ITPs

The associative law of tensor products is false. By this we mean the following: Let us subdivide the index set $\mathcal{I}$ into mutually disjoint index sets

$$\mathcal{I} = \cup_\beta \mathcal{I}_\beta$$

where β runs over some other index set $\mathcal{L}$. One can now form the different ITP

$$\mathcal{H}^{\otimes\prime} = \otimes_\beta \mathcal{H}^{\otimes}_\beta, \qquad \mathcal{H}^{\otimes}_\beta = \otimes_{\gamma\in\mathcal{I}_\beta}\mathcal{H}_\gamma.$$

Unless the index set $\mathcal{L}$ is finite, a generic C_0-vector of $\mathcal{H}'^{\otimes}$ is orthogonal to all of $\mathcal{H}^{\otimes}$. Scalar multiplication is not multi-linear. That is, if f and $z\cdot f$ are C_0-sequences where

$$(z\cdot f)_\alpha = z_\alpha f_\alpha$$

for some complex numbers z_α then $\otimes_{z\cdot f} = (\prod_\alpha z_\alpha)\otimes_f$.

Lemma C.2.24 *Let $\prod_\alpha z_\alpha$ be quasi-convergent. Then*

i) If f is a $C-$sequence, so is $z\cdot f$ with $(z\cdot f)_\alpha := z_\alpha f_\alpha$.

ii) If moreover $\sum_\alpha \big||z_\alpha|-1\big|$ converges and f is a C_0-sequence, so is $z\cdot f$.

iii) The product formula

$$\otimes_{z\cdot f} = [\prod_\alpha z_\alpha]\otimes_f \tag{C.11}$$

fails to hold only if

1) $\prod_\alpha z_\alpha$ is (quasi-convergent but) not convergent and

2) $(\otimes_f,\cdot\,)\neq 0$ considered as a linear functional over $C-$vectors.

In this case, $\{z_\alpha\}$, f satisfy the assumptions of ii), and all $z_\alpha\neq 0$.

iv) If $\{z_\alpha\}$, f satisfy the assumptions of ii), then $[z\cdot f]=[f]$ if and only if $\sum_\alpha |z_\alpha - 1|$ converges. If all $z_\alpha\neq 0$, then this is equivalent to the convergence of $\prod_\alpha z_\alpha$ (beyond mere quasi-convergence).

Proof: i): As both $\prod_\alpha |z_\alpha|$, $\prod_\alpha \|f_\alpha\|_\alpha$ converge, so does

$$\prod_\alpha \|z_\alpha f_\alpha\|_\alpha = \prod_\alpha |z_\alpha|\cdot\prod_\alpha \|f_\alpha\|_\alpha$$

ii) $|z_\alpha|\leq C$

$$\begin{aligned}
\big|\|z_\alpha f_\alpha\|_\alpha - 1\big| &= \big||z_\alpha|\;\|f_\alpha\|_\alpha - 1\big| \\
&= \big|(|z_\alpha|-1) + |z_\alpha|(\|f_\alpha\|_\alpha - 1)\big| \\
&\leq \big|(|z_\alpha|-1) + C(\|f_\alpha\|_\alpha - 1)\big|
\end{aligned}$$

and thus $\sum_\alpha \left| \|z_\alpha f_\alpha\|_\alpha - 1 \right|$ converges.

$\square$

C.2.6 Von-Neumann Algebras on ITPs

A von Neumann algebra over a Hilbert space is weakly (equivalently strongly) closed sub$-^*$ algebra of the algebra of bounded operators on that Hilbert space.

Given a bounded operator a_α on $\mathcal{H}_\alpha$ (notice that closed unbounded operators have a polar decomposition into a unitary and self-adjoint piece and that the self-adjoint operator is completely determined by its bounded spectral projections so that restriction to bounded operators is no loss of generality) we can extend it in the natural way to $\mathcal{H}^\otimes$ by defining $\hat{a}_\alpha$ densely on C_0-vectors through

$$\hat{a}_\alpha \otimes_f = \otimes_{f'}$$

with

$$f'_{\alpha'} = f_{\alpha'} \quad \text{for} \quad \alpha' \neq \alpha$$

and

$$f'_\alpha = a_\alpha f_\alpha$$

As we will see, it turns out that the algebra of these extended operators for a given label is automatically a von Neumann algebra for $\mathcal{H}^\otimes$ and we call the weak closure of all these algebras the von Neumann algebra $\mathcal{R}^\otimes$ of local operators.

Definition We denote by $\mathcal{B}(\mathcal{H}_\alpha)$ the set of bounded operators on $\mathcal{H}_\alpha$ and by $\mathcal{B}^\otimes := \mathcal{B}(\mathcal{H}^\otimes)$ the set of bounded operators on the ITP $\mathcal{H}^\otimes$.

$\square$

Definition We denote by $\mathcal{B}_\alpha$ the extension of $\mathcal{B}(\mathcal{H}_\alpha)$ to the ITP, that is,

$$\mathcal{B}_\alpha = \{\hat{A}_\alpha : A_\alpha \in \mathcal{B}(\mathcal{H}_\alpha)\} \tag{C.12}$$

$\square$

Lemma C.2.25 *For all $\alpha \in \mathcal{I}$, the algebra $\mathcal{B}_\alpha$ is a von Neumann algebra over $\mathcal{H}^\otimes$.*

Proof: To prove the theorem one uses the von Neumann density theorem, proved in the next chapter. This states that a set $\mathcal{U}$ of the space of bounded operators of a Hilbert space $\mathcal{H}$ is a von Neuman algrba if it is equal to its double commutant, the commutant of a set $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ being

$$\mathcal{U}' = \{B \in \mathcal{B}(\mathcal{H}) : [A, B] = 0 \text{ for all } A \in \mathcal{U}\},$$

Let us write

$$\mathcal{B}^\otimes = \mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_{\overline{\alpha}})$$

where $\overline{\alpha} = \mathcal{I} - \alpha$.

$$\mathcal{B}'_{\overline{\alpha}} = \{\hat{B} \in \mathcal{B}^\otimes : [\hat{A}, \hat{B}] = 0 \text{ for all } \hat{A} \in \mathcal{B}_{\overline{\alpha}}\},$$

the commutant of $\mathcal{B}_{\overline{\alpha}}$.

□

Definition Two C_0-sequences f, g are said to be weakly equivalent, denoted $f \sim g$ provided that there are complex numbers z_α such that $z \cdot f$ and g are strongly equivalent, that is, $z \cdot f \approx g$.

□

Lemma C.2.26 *The definition of weak equivalenceis unchanged if we restrict to complex numbers with $|z_\alpha| = 1$.*

Proof: We prove that if f and $z \cdot f$ are C_0-sequences we can find z'_α with $|z'_\alpha| = 1$ such that $(z \cdot f) \approx (z' \cdot f)$.

As $\sum_\alpha \left| \|z_\alpha f_\alpha\|_\alpha - 1 \right|$ converges, $z_\alpha f_\alpha = 0$ can occur only for a finite number of α's. For these we may replace z_α by 1 and f_α by non-zero f^0_α. As this so constructed C_0-sequence differes from f by only a finite number of terms, it is srtongly equivalent to it (this follows from theorem C.2.21). So we may assume that $z_\alpha f_\alpha \neq 0$ for all α.

As $\sum_\alpha \left| \|f_\alpha\|_\alpha - 1 \right|$, $\sum_\alpha \left| \|z_\alpha f_\alpha\|_\alpha - 1 \right|$ converge, and all $\|f_\alpha\|_\alpha, \|z_\alpha f_\alpha\|_\alpha \neq 0$, by theorem C.2.15, $\prod_\alpha \|f_\alpha\|_\alpha$, $\prod_\alpha \|z_\alpha f_\alpha\|_\alpha$ converge and have non-zero values. Thus

$$\prod_\alpha \frac{1}{|z_\alpha|}$$

converges too, and has a non-zero value, as

$$\frac{\|f_\alpha\|_\alpha}{\|z_\alpha f_\alpha\|_\alpha} = \frac{1}{|z_\alpha|}.$$

Therefore

$$\sum_\alpha \left|\frac{1}{|z_\alpha|} - 1\right|$$

converges by theorem C.2.15. Now it follows from lemma, (ii), (iv), that $(z \cdot f)$ and $(\frac{z}{|z|} \cdot f)$ are equivalent. Thus

$$z'_\alpha = \frac{z_\alpha}{|z_\alpha|}.$$

Lemma C.2.27 *Weak equivalence is an equivalence relation.*

Proof: Reflextivity: $f \sim f$ as $(1 \cdot f) \approx f$.

Symmetry: If $|z_\alpha| = 1$ we have

$$(f_\alpha, (1/z_\alpha)g_\alpha)_\alpha = z_\alpha(f_\alpha, g_\alpha)_\alpha = (z_\alpha f_\alpha, g_\alpha)_\alpha$$

and $\|(1/z_\alpha)g_\alpha\|_\alpha = \|g_\alpha\|_\alpha$. These together with the symmetry property of strong equivalence implies that if $(z \cdot f) \approx g$ with $|z_\alpha| = 1$ then $(1/z) \cdot g \approx f$ with $(1/z_\alpha)g_\alpha$ a C_0-sequence, that is, $g \sim f$.

Transitivity: If $z \cdot f \approx g$ and $z' \cdot g \approx h$, $|z_\alpha|, |z'_\alpha| = 1$, then $z \cdot f \approx (1/z') \cdot h$ and $z' \cdot z \cdot f \approx h$. So $f \sim g$.

$\square$

Lemma C.2.28 *f and g are weakly equivalent if and only if*

$$\sum_\alpha \big||(f_\alpha, g_\alpha)_\alpha| - 1\big|$$

converges.

Proof: By definition, $f \sim g$ if we can find complex numbers z_α with $|z_\alpha| = 1$ such that

$$\sum_\alpha \left|(z_\alpha f_\alpha, g_\alpha)_\alpha - 1\right|$$

converges. If

$$\sum_\alpha \left||(f_\alpha, g_\alpha)_\alpha| - 1\right|$$

converges, then $f \sim g$ with $z_\alpha = (f_\alpha, g_\alpha)^*_\alpha / |(f_\alpha, g_\alpha)_\alpha|$.

Now, from fig (), it is easy to see that the minimum value of $|(z_\alpha f_\alpha, g_\alpha) - 1| = |z_\alpha (f_\alpha, g_\alpha)_\alpha - 1|$ is $\left||(f_\alpha, g_\alpha)_\alpha| - 1\right|$. Thus $\sum_\alpha \left|(z_\alpha f_\alpha, g_\alpha)_\alpha - 1\right|$ converges for some complex numbers z_α, when $f \sim g$, only if

$$\sum_\alpha \left||(f_\alpha, g_\alpha)_\alpha| - 1\right|$$

converges.

□

Lemma C.2.29 *Assume that a z_α with $|z_\alpha| = 1$ is given for each $\alpha \in \mathcal{I}$. Then there exists one and only one closed, linear operator U, such that*

$$U \otimes_f = \otimes_{z \cdot f}$$

for every C_0-vector f. The operator U is unitary.

Proof:

Proof of uniqueness. If U' is another closed, linear operator meeting the requirements, then

□

C.3 Outline of Semiclassical Analysis

The $U(1)^3$ coherent states are given by

$$\Psi^{t}_{\alpha,(A_0,E_0)} = \prod_{e\in E(\alpha)} \prod_{j=1,2,3} \Psi^{t_e}_{g_e(A_0,E_0)}, \tag{C.13}$$

where

$$\Psi^{t_e}_{g_e(A_0,E_0)} = \sum_{n\in\mathbb{Z}} e^{-t_e n^2/2} \left(g_e h^{-1}\right)^n \tag{C.14}$$

Appendix D

Quantum Electrodynamics

An open problem in quantum gravity is to compute particle scattering amplitudes from the background-independent theory and recover the low energy physics.

Calculations should agree with low energy conventional field theory. Here we introduce conventional scattering theory.

Feynman derived his rules in a non-rigorous fashion but it still incorporated all QED processes. These rules were shown to follow from a systematic treatment within the framework of quantum field theory. In this appendix we follow the route taken by Feynmann, we breifly demonstrate its equivalence to the more rigorous quantum field theory in the next appendix.

D.1 The Electromagnetic Field and the Photon

Light behaves as a wave as it demonstrates interference and diffraction. Maxwell's theory seemed to confirm the wave theory of light.

But then the development following the discovery of the photoelectric effect led to the realisation that sometimes light behaves like particles.

D.1.1 Review of Special Relativistic Notation

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{D.1}$$

$$\eta^{\mu\nu} = (\eta^{-1})_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{D.2}$$

The world four vector

$$x^\mu = \{x^0, x^1, x^2, x^3\} = \{t, x, y, z\} \tag{D.3}$$

describes the spacetime coordinates. The covariant four vector is

$$x_\mu = \eta_{\mu\nu} x^\nu = \{t, -x, -y, -z\} = \{x_0, x_1, x_2, x_3\} \tag{D.4}$$

We have

$$x \cdot x = x^\mu x_\mu \tag{D.5}$$
$$= t^2 - x^2 - y^2 - z^2. \tag{D.6}$$

The definition of four-momentum vector is analogous,

$$p^\mu = \{E, p_x, p_y, p_z\}, \tag{D.7}$$

and the scalar product $p_1 \cdot p_2$ is

$$p_1 \cdot p_2 = p_1^\mu p_{2\mu} = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 \tag{D.8}$$

and the scalar product $x \cdot p$ is

$$x \cdot p = x^\mu p_\mu = x_\mu p^\mu = Et - \mathbf{x} \cdot \mathbf{p}. \tag{D.9}$$

We use the general notions for four vectors

$$a = \{a_0, a_1, a_2, a_3\}. \tag{D.10}$$

We denote three-vectors by bold type as in

$$\mathbf{a} = \{a_1, a_2, a_3\}. \tag{D.11}$$

The components

$$a^\mu = \{a^0, a^1, a^2, a^3\}$$

D.1.2 Maxwell's Equations

Classical electromagnetism is described by Maxwell's equations. In the presence of a charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{j}(\mathbf{x}, t)$, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ satify the equations

$$\nabla \cdot \mathbf{E} = \rho \tag{D.12}$$

$$\nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \tag{D.13}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{D.14}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{D.15}$$

From the second pair of Maxwell's equations follows the existence of scalar and vector $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ potentials defined by

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \cdot \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{D.16}$$

These equations do not determine the potential uniquely, since for an aribtrary function $\Lambda(\mathbf{x}, t)$ the transformation

$$\phi \to \phi' = \phi + \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\Lambda \tag{D.17}$$

Expressed in terms of potentials the first Maxwell equation becomes

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \Box\phi - \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}\right) = \rho \tag{D.18}$$

For the second we need $\nabla \times (\nabla \times \mathbf{A})$

$$\begin{aligned}[\nabla \times (\nabla \times \mathbf{A})]_i &= \epsilon_{ijk}\partial_j(\epsilon_{kj'k'}\partial_{j'}A_{k'}) \\ &= \epsilon_{ijk}\epsilon_{ki'j'}\partial_j\partial_{i'}A_{j'} \\ &= (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'})\partial_j\partial_{i'}A_{j'} \\ &= \partial_j\partial_{i'}A_j - \nabla^2 A_i \end{aligned} \tag{D.19}$$

$$\Box \mathbf{A} + \nabla \left(\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} \right) = \mathbf{j} \tag{D.20}$$

In four-vector notation the gauge transformations (D.17) read

$$A^\mu \to A^\mu + \partial^\mu \Lambda(x).$$

The first (D.18) and second (D.20) Maxwell equations can be combined into one equation

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \tag{D.21}$$

D.1.3 Transverse Gauge field

It is always possible to find a function $\Lambda(x)$ such that the transformed potential satisfies the Lorentz gauge

$$\partial_\mu A^\mu = 0. \tag{D.22}$$

Only in this gauge does the wave equation have the simple form

$$\Box A^\mu = 0. \tag{D.23}$$

For

$$k^\mu k_\mu = 0 \tag{D.24}$$

its solutions are plane waves

$$A^\mu(x,k) = \epsilon^\mu N_k(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{D.25}$$

In the Lorentz gauge, we can make futher gauge transformations $A^\mu \to A^\mu + \partial^\mu \Lambda(x)$ provided Λ satisfies

$$\Box \Lambda(x) = 0.$$

Such regauging does obviously does not change the Lorentz condition. The radiation gauge

$$A^0 = 0, \quad \nabla \cdot \mathbf{A} = 0. \tag{D.26}$$

can be chosen. To see this consider aribtrary $\mathbf{A}'(x)$, and postulate $\Lambda(x)$ such that

$$\nabla \cdot \mathbf{A}(x) = \nabla \cdot \mathbf{A}'(x) - \nabla^2 \Lambda(x) = 0 \tag{D.27}$$

We obviously need to solve the equation

$$-\nabla^2 \Lambda(x) = \nabla \cdot \mathbf{A}'(x) \tag{D.28}$$

Notice this is just the equation

$$-\nabla^2 \Lambda(x) = f(x) \tag{D.29}$$

which we know has the solution

$$\Lambda(x) = \int d^3 r' \frac{f(r')}{4\pi |r' - r|} \tag{D.30}$$

Therefore this gauge can always be choosen. In this gauge the timelike component of ϵ^μ vanishes. ϵ satisfies

$$\epsilon \cdot \mathbf{k} = 0 \tag{D.31}$$

and normalised such that

$$\epsilon \cdot \epsilon = 1 \tag{D.32}$$

$$\begin{aligned} \epsilon^\mu k_\mu &= 0 \\ \epsilon^\mu \epsilon_\mu &= -1 \end{aligned} \tag{D.33}$$

D.1.4 Blackbody Radiation, the Photoelectric Effect and the Compton Effect

$$E = hf \tag{D.34}$$

where f is the frequency and h is Plank's constant.

Blackbody Radiation

Boltzmann statistics for a gas of free particles is

$$p(\vec{v}) = Ne^{-E/kT}$$

Classical physics can be used to derive an equation which describes the intensity of the blackbody radiation as a function of frequency for a fixed temperature - the result is known as the Rayleigh-Jeans law. Although the Rayleigh-Jeans law works for low frequencies, it diverges as f^2; this divergence for high frequencies is called the ultraviolet catastrophe.

Planck's law states that

$$I(f,T) = \frac{2hf^3}{c^2}\frac{1}{e^{\frac{hf}{kT}} - 1} \tag{D.35}$$

We analyse using Bose-Einstein statistics: a "gas" of photons. There is no constrainton the number of photons. We find that for a system of photons, that the number of photons $n(\epsilon)d\epsilon$ in a (small) energy range ϵ to $\epsilon + d\epsilon$ is given by

Consider an energy level ϵ_i with degeneracy g_i, containing n_i bosons. This can be represented by $g_i - 1$ lines, and the bosons by n_i circles. The number of distinct orderings of lines and circles is

$$\frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \tag{D.36}$$

The total number of microstates for a given distribution is

$$\prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \tag{D.37}$$

For $g_i >> 1$ this can be replaced by

$$t(\{n_i\}) = \prod_i \frac{(n_i + g_i)!}{n_i! g_i!}. \tag{D.38}$$

If we assume that both g_i and n_i are large enough for Stirling's approximation to hold for $\ln g_i!$ and $\ln n_i!$, we find that $\ln t$ is given by

$$\ln t \approx \sum_i [(n_i + g_i)\ln(n_i + g_i) - g_i \ln g_i - n_i \ln n_i] dn_i \tag{D.39}$$

We want to maximise $\ln t$ subject to the constraint

$$\sum_i \epsilon_i n_i = U. \tag{D.40}$$

If $\ln t$ were maximal the change in $\ln t$ resulting from changes dn_i in each of the n_i's would zanish:

$$d\ln t \approx \sum_i [\ln(n_i + g_i) - \ln n_i] dn_i = 0. \tag{D.41}$$

If all the dn_i's were independent from each other, each coefficient in (D.41) would have to zanish. However because of the constraint (D.40) it no longer follows that all the dn_i's are independent from each other as they have to satisfy

$$dU = \sum_i \epsilon_i dn_i = 0. \tag{D.42}$$

Adding this multiplied by the Lagrange multiplier β to (D.41) we obtain

$$\sum_i [\ln(n_i + g_i) - \ln n_i + \beta\epsilon_i] dn_i = 0 \tag{D.43}$$

which is a condition for a maximal value for $\ln t$ subject to the constraint (D.40). For appropriate value of β we can consider all dn_i independent from each other. Therefore, each coefficient must zanish separately:

$$\ln\left(\frac{n_i + g_i}{n_i}\right) + \beta\epsilon_i = 0. \tag{D.44}$$

We then find that the most probable distribution is

$$n_i = \frac{g_i}{e^{-\beta\epsilon_i} - 1}. \tag{D.45}$$

This is the Bose-Einstein distribution. β is related to the temperature, so the Bose-Einstein distribution takes the form.

$$n_i = \frac{g_i}{e^{\frac{\epsilon_i}{kT}} - 1}. \tag{D.46}$$

We consider a "gas" of photons.

$$n(\epsilon)d\epsilon = \frac{g(\epsilon)d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{D.47}$$

where $g(\epsilon)$ are the density of states. We first derive the density of states as a function of the wavevector $\vec{k}$.

In order to determine the available wavevectors, we ask what standing waves can propagate within the box subject to boundary condition that the amplitude is zero at the boundaries. Using a cube with a side of length L, we see that there must be an integer number of half wavelengths in L for each of the directions. Hence if the vector is $\vec{k}$, with Cartesian components (k_x, k_y, k_z), we must have

$$k_x = \frac{n_x\pi}{L}, \quad k_y = \frac{n_y\pi}{L}, \quad k_z = \frac{n_z\pi}{L} \tag{D.48}$$

where n_x, n_y and n_z are from the set of positive non-zero integers - these define"elementary cells". The total number of states with wavwenumber, $|\vec{k}|$, less than some value k is found by counting the number of triples (n_x, n_y, n_z) such that

$$\frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} < k \tag{D.49}$$

We can find this by considering the octant of a sphere of radius k in $\vec{k}-$space. The volume of the sphere in $\vec{k}-$space is

$$\frac{4}{3}\pi k^3$$

and the volume of an elementary cell is

$$\left(\frac{\pi}{L}\right)^3$$

therefore the number of states satisfying (D.49) is 1/8 the volume of the sphere divided by the volume of an elementary cell,

$$N(k) = \frac{1}{8}\frac{4}{3}\pi k^3 \left(\frac{L}{\pi}\right)^3 = \frac{1}{3}4\pi\frac{V}{(2\pi)^3}k^3$$

where V is the volume of the container. We must multiply this by a factor of 2 from the fact that there are two polarisations for the photons. Using $g(k) = 2 \times dN(k)/dk$ we obtain for waves in a box

$$g(k)dk = 2 \times 4\pi\frac{V}{(2\pi)^3}k^2dk. \tag{D.50}$$

Now we use the quantum relationship between photon energy and wave number:

$$\epsilon = hf = \frac{hc}{\lambda} = hc\frac{k}{2\pi} = \hbar ck. \tag{D.51}$$

$$g(\epsilon)d\epsilon = 8\pi\frac{V}{(2\pi)^3}\frac{\epsilon^2}{(\hbar c)^3}d\epsilon \tag{D.52}$$

Thus, the number of photons $n(\epsilon)d\epsilon$ in the energy range from ϵ to $\epsilon + d\epsilon$ is

$$n(\epsilon)d\epsilon = \frac{8\pi V}{(hc)^3}\frac{\epsilon^2 d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{D.53}$$

The energy $u(\epsilon)$ in the range ϵ and $\epsilon + d\epsilon$ is given by:

$$u(\epsilon)d\epsilon = n(\epsilon)\epsilon d\epsilon = \frac{8\pi V}{(hc)^3}\frac{\epsilon^3 d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{D.54}$$

Since the energyand frequency are related by the quantum formula,

$$\epsilon = hf,$$

we find that the density is:

$$u(f)df = \frac{8\pi Vh}{c^3}\frac{f^3 df}{e^{\frac{hf}{kT}} - 1} \tag{D.55}$$

The total energy, U. This the area under the graph of the energy spectrum

$$U = \int_0^\infty u(f)df = \frac{8\pi V}{h^2c^3}\int_0^\infty \frac{(hf)^3 df}{e^{\frac{hf}{kT}} - 1} \tag{D.56}$$

Define

$$y = \frac{hf}{kT}$$

in terms of which, the expression for the total energy becomes

$$U = \int_0^\infty u(f)df = \frac{8\pi V}{(hc)^3}(kT)^4 \int_0^\infty \frac{y^3 dy}{e^y - 1} \tag{D.57}$$

The integral is $\pi^4/15$, hence

$$\frac{U}{V} = \frac{8\pi^5}{15(hc)^3}(kT)^4. \tag{D.58}$$

The Photoelectric Effect

The photoelectric effect is the ejection of electrons from a metal surface exposed to electromagnetic radiation. The energy of the emitted electrons is given by the frequency of the irradiating light.

An increase in the intensity of the radiation leads to the emision of more electrons, but does not change their energy. This clearly contradict the view of Maxwell's wave theory where the energy of a wave is given by its intensity.

There is no smaller quantity of energy in radiation of a certain frequency f than the energy of a single photon. The radiation is regarded as a stream of photons, each having an energ hf.

The Compton Effect

The successes of blackbody radiation and the photoelectric effect were not sufficient to convince all scientists of the idea that radiation is quanatised. Further evidence for the photon concept came from the so-called Compton effect.

In 1923 Compton was studying the the scattering of x-rays off graphite. Classically, the charges should oscillate at the same frequency as of the incoming radiation and then give off radiation of the same frequency. However he found that radiation was being emmitted at a longer wavelength. This is called the Compton effect. Specifically, if the incoming radiation is scattered by an angle θ and if λ and λ' are wavelengths of the incident and scattered radiation, respectively, we find that

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta). \tag{D.59}$$

where m_0 is the rest mass of the electron. Thomson scattering, the classical theory of an electromagnetic wave scattered by charged particles, cannot explain the results of the experiment and demonstrates that light cannot be explained purely as a wave phenomenon.

The photon idea provides a clear explanation and provides additional direct confirmation of the quantum nature of radiation. The results can be analyzed in terms of a collision between a photon and an electron (in the experiment the energy of the photon was very much larger than the binding energy of the electron and could therefore be considered as a free electron). The incident photon collides with an at rest electron, which then recoils as a result of the impact, the scattered photon has less energy, smaller frwquency, and longer wavelength than the incident photon.

In fact we can derive (D.59) by a simple calculation. Classically we know from the equation $E^2 - c^2p^2 = m_0^2c^4$ that for a photon implies $p = E/c$. Since the energy of a photon is hf, its momentum is

$$p = \frac{hf}{c} = \frac{h}{\lambda}$$

Part of the energy of the radiation is transferred to the recoiling electron, we have

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + E_{kin} \tag{D.60}$$

where λ' is the wavelength after scattering and $E_{kin} = (\gamma - 1)m_0c^2$ is the relativistic kinematic energy of the recoiling electron. We consider the collision in the $x - y$ plane, where the incoming photon is scattered by an angle θ and the electron, initially at rest, is deflected by an angle ϕ. Conservation of momentum in the x and y directions respectively gives

$$\begin{aligned} \frac{h}{\lambda} &= \frac{h}{\lambda'}\cos\theta + p_e\cos\phi \\ 0 &= \frac{h}{\lambda'}\sin\theta - p_e\sin\phi \end{aligned} \tag{D.61}$$

where $p_e = mv = \gamma m_0 v$. By noting from (D.61) that

$$p_e^2 \sin^2 \phi^2 = \frac{h^2}{\lambda'} \sin^2 \theta, \qquad p_e^2 \cos^2 \phi^2 = h^2 \left(\frac{1}{\lambda} - \frac{1}{\lambda'} \cos \theta \right)^2$$

and adding these together we can eliminate ϕ and after some manipulation obtain

$$p_e^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'} \cos \theta.$$

We can obtain another expression for p_e^2 by using $E^2 = p_e^2 c^2 + m_0^2 c^4 = (E_{kin} + m_0 c^2)^2$ and (D.60)

$$\begin{aligned}
p_e^2 &= \left(\frac{E_{kin}}{c} + m_0 c \right)^2 - m_0^2 c^2 \\
&= \left(\frac{h}{\lambda} - \frac{h}{\lambda'} + m_0 c \right)^2 - m_0^2 c^2 \\
&= \left(\frac{h}{\lambda} - \frac{h}{\lambda'} \right)^2 + 2 \left(\frac{h}{\lambda} - \frac{h}{\lambda'} \right) m_0 c.
\end{aligned}$$

Equating these two expressions for p_e^2 we obtain after cancellation of terms

$$-\frac{2h^2}{\lambda\lambda'} \cos \theta = -\frac{2h^2}{\lambda\lambda'} + 2 \left(\frac{h}{\lambda} - \frac{h}{\lambda'} \right) m_0 c^2$$

which after simplifying gives the final result

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos \theta).$$

the quantity $h/m_0 c = 2.43 \times 10^{-12} m$ is called the Compton wavelength. The wavelength shift $\lambda' - \lambda$ is at most twice the Compton wavelength (for $\theta = 180^0$).

D.1.5 Photons

The formula

$$E = hf$$

means that the energy E carried by a photon and the frequency of the photon's electromagnetic vibration are directly proportional, the constant of proportionality being Planck's constant, h.

We find the energy of the electromagnetic field of a plane wave

$$\vec{\epsilon} N_k () \tag{D.62}$$

by using

$$E_{photon} = \frac{1}{8\pi} \int d^3x < \mathbf{E}^2 + \mathbf{B}^2 >= \frac{1}{4\pi} \int d^3x < \mathbf{B}^2 > \tag{D.63}$$

and substituiting in

$$\mathbf{B} = \nabla \times \mathbf{A} = i N_k \, \mathbf{k} \times \epsilon \, (e^{-ik\cdot x} - e^{ik\cdot x}) = 2 N_k \, \mathbf{k} \times \epsilon \, \sin(k \cdot x) \tag{D.64}$$

and using

$$\begin{aligned} (\mathbf{k} \times \epsilon) \cdot (\mathbf{k} \times \epsilon) &= \epsilon \cdot \epsilon \, \mathbf{k} \cdot \mathbf{k} - (\mathbf{k} \cdot \epsilon)^2 \\ &= (\epsilon_0^2 - \epsilon \cdot \epsilon)\mathbf{k}^2 - (k_0 \epsilon_0 - k \cdot \epsilon)^2 \\ &= \epsilon_0^2 \mathbf{k}^2 + \mathbf{k}^2 - k_0^2 \epsilon_0^2 \\ &= \mathbf{k}^2 = \omega^2. \end{aligned} \tag{D.65}$$

We find the energy to be

$$E_{photon} = \frac{4\omega^2}{4\pi} N_k^2 \int d^3x < \sin^2(\omega t - \mathbf{k} \cdot \mathbf{x}) >= \frac{2\omega^2}{4\pi} N_k^2 V \tag{D.66}$$

where V the volume of the box. The condition $E_{photon} = \hbar\omega$ (where $\omega = 2\pi f$ is the angular frequency and $\hbar = h/2\pi$) leads to the normalisation constant

$$N_k = \sqrt{\frac{4\pi}{2\omega V}}. \tag{D.67}$$

We write

$$A_\mu(x, k) = \sqrt{\frac{4\pi}{2\omega V}} \, \epsilon_\mu(k, \lambda)(e^{-ik\cdot x} + e^{ik\cdot x}). \tag{D.68}$$

D.2 Dirac Spinors

As a precursor to the Dirac equation, we introduce the Klein-Gordon equation which describes relativistic scalar particles.

D.2.1 Klein-Gordon Equation

From quantum mechanics we know about the correspodance between Schrodinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m_0}\nabla^2 + V(x)\right]\psi(\mathbf{x},t) \tag{D.69}$$

and the non-relativistic energy relation,

$$E = \frac{\mathbf{p}^2}{2m_0} + V(\mathbf{x}). \tag{D.70}$$

The former can be obtained from the latter via the substitutions

$$E \to \hat{E} \;=\; i\hbar\frac{\partial}{\partial t} \tag{D.71}$$

$$\mathbf{p} \to \hat{\mathbf{p}} \;=\; -i\hbar\nabla. \tag{D.72}$$

Now consider the classical relativistic equation

$$E^2 = \mathbf{p}^2 + m_0^2, \tag{D.73}$$

Make the same substitutions as before (that is D.72 and D.72). In terms of these operators the Einstein relation between energy, momentum, and mass can be written as

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -\hbar^2\nabla^2\phi + m_0^2\phi \tag{D.74}$$

Current density

Multiply Schrodinger's equation from the left by ψ^* and its conjugate by the left ψ then substract. One obtains

$$i\hbar \frac{\partial |\psi|^2}{\partial t} = -\frac{\hbar^2}{2m_0} \left[\psi^*(\mathbf{x},t)\nabla^2\psi(\mathbf{x},t) - \psi(\mathbf{x},t)\nabla^2\psi^*(\mathbf{x},t) \right] \tag{D.75}$$

This is the continuity equation in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{D.76}$$

where

$$\rho = |\psi|^2 \tag{D.77}$$

is the probability density and

$$\mathbf{j} = -\frac{i\hbar}{2m_0} \left[\psi^*(\mathbf{x},t)\nabla^2\psi - \psi(\mathbf{x},t)\nabla^2\psi^* \right] \tag{D.78}$$

is the current density.

$$\int_V \frac{\partial \rho}{\partial t} d^3x = \frac{\partial}{\partial t} \int_V \rho d^3x = -\int_V \nabla \cdot \mathbf{j} d^3x = \int_S \mathbf{j} \cdot d\mathbf{S} = 0. \tag{D.79}$$

Hence,

$$\int_V \frac{\partial \rho}{\partial t} d^3x = Const. \tag{D.80}$$

By similar reasoning (see section) obtain for the Klein-Gordon equation the four-current density

$$j_\mu = \frac{i\hbar}{2m_0} (\phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^*) \tag{D.81}$$

The probability density is

$$\rho = j_0 = \frac{i\hbar}{2m_0}(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}) \tag{D.82}$$

Since the Klein-Gordon equation is second order in time, at a given time both ϕ and $\partial\phi/\partial t$ may have arbitrary values and so ρ could be negative! Also if ϕ is real then the probability density is zero. The Klein-Gordon equation also has negative energy states. This and the problem with the probability interpretation was the reason for a long time, the Klein-Gordon equation was considered physically meaningless.

D.2.2 Dirac's Equation

This difficulty led Dirac to search for a first order differential equation in t with positive definite probability density. Dirac wanted to construct a Hamiltonian that is linear in spatial derivatives aswell so that time and space are put on the same footing. He postulated an equation of the form

$$i\hbar\frac{\partial \psi}{\partial t} = [-i\hbar(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}) + \hat{\beta} m_0]\psi \tag{D.83}$$

where $\hat{\alpha}, \hat{\beta}$ are $N \times N$ matrices and ψ is a column vector

$$\begin{pmatrix} \psi_1(\mathbf{x}, t) \\ \psi_2(\mathbf{x}, t) \\ \vdots \\ \psi_N(\mathbf{x}, t) \end{pmatrix} \tag{D.84}$$

To find the concrete form of this equation we follow the natural requirements:

- Energy-momentum relation for relativistic free particle

$$E^2 = \mathbf{p}^2 + m_0^2, \tag{D.85}$$

- continuity equation for the density
- Lorentz covariance

Energy-momentum relation for relativistic free particle

Every component ψ_σ of the spinor must satify the Klein-gordon equation

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left(-\hbar^2 \nabla^2 + m_0^2\right) \psi_\sigma \tag{D.86}$$

$$\begin{aligned}
-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} &= i\hbar \frac{\partial}{\partial t}(i\hbar \frac{\partial}{\partial t} \psi) = i\hbar \frac{\partial}{\partial t} \hat{\mathcal{H}} \psi = \hat{\mathcal{H}}^2 \psi \\
&= \left[-i\hbar \hat{\alpha}_i \frac{\partial}{\partial x^i} + \hat{\beta} m_0\right] \left[-i\hbar \hat{\alpha}_j \frac{\partial}{\partial x^j} + \hat{\beta} m_0\right] \psi \\
&= -\hbar^2 \sum_{i,j=1}^{3} \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j} - i\hbar m_0 \sum_{i=1}^{3} (\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i) \frac{\partial \psi}{\partial x^i} + \hat{\beta}^2 m_0^2 \psi
\end{aligned} \tag{D.87}$$

Comparison with (D.86) implies the following requirements

$$\begin{aligned}
\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= 2\delta_{ij} 1, \\
\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= 0, \\
\hat{\alpha}_i^2 = \hat{\beta}^2 &= 1.
\end{aligned} \tag{D.88}$$

For the Hamiltonian to be Hermitian, the matrices $\hat{\alpha}_i, \hat{\beta}$ have to be Hermitian

$$\hat{\alpha}_i^\dagger = \hat{\alpha}_i, \quad \hat{\beta}^\dagger = \hat{\beta}. \tag{D.89}$$

Therefore the eigenvalues are real. Since $\hat{\alpha}_i^2 = 1$ and $\hat{\beta}^2 = 1$, it follows that the eigenvalues can only take the values ± 1. The eigenvalues are independent of the representation. Consider the diagonal representation of $\hat{\alpha}_i$, for example, we have

$$\hat{\alpha}_i = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & A_N \end{pmatrix} \tag{D.90}$$

with eigenvalues $A_1, \ldots, A_N$, and $\hat{\alpha}_i^2 = 1$ yields

$$\hat{\alpha}_i^2 = I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & 0 & \cdots & 0 \\ 0 & A_2^2 & 0 & \cdots & 0 \\ 0 & 0 & A_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & A_N^2 \end{pmatrix} \tag{D.91}$$

from which

$$A_k^2 = 1, \quad i.e. \quad A_k = \pm 1. \tag{D.92}$$

Now from the anticommutation relations we have

$$\hat{\alpha}_i = -\hat{\beta}\hat{\alpha}_i\hat{\beta}$$

and the identity

$$Tr\hat{A}\hat{B} = Tr\hat{B}\hat{A}$$

we conclude

$$Tr\hat{\alpha}_i = -Tr\hat{\beta}\hat{\alpha}_i\hat{\beta} = -Tr\hat{\alpha}_i\hat{\beta}^2 = -Tr\hat{\alpha}_i = 0. \tag{D.93}$$

We see that the matrices $\hat{\alpha}_i, \beta$ must even dimensional.

The smallest even dimension for which the (D.88) can be fulfilled is $N = 4$. In fact it is easily shown that the following is a representation

$$\hat{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\hat{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{D.94}$$

To see this note that

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \tag{D.95}$$

where $\hat{\sigma}_i$ are Pauli's 2×2 matrices which satify the relation

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij}\mathbf{1}, \tag{D.96}$$

as this means

$$\begin{aligned}
\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} \hat{\sigma}_i\hat{\sigma}_j & 0 \\ 0 & \hat{\sigma}_i\hat{\sigma}_j \end{pmatrix} + \begin{pmatrix} \hat{\sigma}_j\hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_j\hat{\sigma}_i \end{pmatrix} \\
&= \begin{pmatrix} \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i \end{pmatrix} \\
&= 2\delta_{ij} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \qquad (D.97)
\end{aligned}$$

and also

$$\begin{aligned}
\hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} = 0. \qquad (D.98)
\end{aligned}$$

Continuity equation for the density

We need to construct the four-current density and the equation of continuity. Let us multiply (D.83) from the left by $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$

$$i\hbar\psi^\dagger \frac{\partial}{\partial t}\psi = -i\hbar \sum_{k=1}^{3} \psi^\dagger \hat{\alpha}_k \frac{\partial}{\partial x^k}\psi + m_0\psi^\dagger\hat{\beta}\psi \qquad (D.99)$$

Take the Hermitian conjugate of (D.83)

$$i\hbar\frac{\partial\psi^\dagger}{\partial t} = i\hbar \sum_{k=1}^{3} \frac{\partial\psi^\dagger}{\partial x^k}\hat{\alpha}_k^\dagger + m_0\psi^\dagger\hat{\beta}^\dagger \qquad (D.100)$$

and multiply from the right by ψ, taking into account the Hermiticty of $\hat{\alpha}_i, \hat{\beta}$, we get

$$-i\hbar\frac{\partial\psi^\dagger}{\partial t}\psi = i\hbar \sum_{k=1}^{3} \frac{\partial\psi^\dagger}{\partial x^k}\hat{\alpha}_k\psi + m_0\psi^\dagger\hat{\beta}\psi \qquad (D.101)$$

Then, subtraction of (D.101) from (D.99) yields

$$i\hbar\frac{\partial}{\partial t}(\psi^\dagger\psi) = -i\hbar\sum_{k=1}^{3}\frac{\partial}{\partial x^k}(\psi^\dagger\hat{\alpha}_k\psi) \tag{D.102}$$

which can be seen as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{D.103}$$

where

$$\rho = \psi^\dagger\psi \tag{D.104}$$

is the positive definite density and

$$j^k = \psi^\dagger\hat{\alpha}^k\psi \tag{D.105}$$

We still have to show that $\rho(x)$ is the temporal component of a four-vector so that the spatial integral $\int\rho d^3x$ becomes constant in time. Only the the probability interpretation of $\rho(x)$ ensured.

Lorentz covariance

D.2.3 Free Motion of a Dirac Particle

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\psi = \left(\hat{\alpha}\cdot\hat{\mathbf{p}} + m_0\hat{\beta}\right)\psi \tag{D.106}$$

Stationary states can be found by substituting

$$\psi(\mathbf{x},t) = \psi(\mathbf{x})\exp[-(i/\hbar)\epsilon t] \tag{D.107}$$

into the Dirac equation. We get

$$\epsilon\psi(\mathbf{x}) = \hat{\mathcal{H}}\psi(\mathbf{x}) \tag{D.108}$$

Split the four-component spinor into two two-component spinors ϕ and χ, i.e.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{D.109}$$

$$\varphi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad \chi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{D.110}$$

or

$$\begin{aligned} \epsilon\psi &= \hat{\alpha}\chi \cdot \hat{\mathbf{p}} + m_0\varphi \\ \epsilon\chi &= \hat{\alpha}\varphi \cdot \hat{\mathbf{p}} - m_0\chi \end{aligned} \tag{D.111}$$

The states

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} \exp[(i/\hbar)\mathbf{p} \cdot \mathbf{x}] \tag{D.112}$$

This results in

$$\begin{aligned} (\epsilon - m_0)I\varphi_0 - \hat{\sigma} \cdot \hat{\mathbf{p}}\chi_0 &= 0, \\ -\hat{\sigma} \cdot \hat{\mathbf{p}}\,\varphi_0 + (\epsilon + m_0)I\chi_0 &= 0. \end{aligned} \tag{D.113}$$

This linear homogeneous system of equations for φ_0 and χ_0 has notrivial solutions only in the case of a vanishing determinant of the coefficients, that is

$$\begin{vmatrix} (\epsilon - m_0)I & -\hat{\sigma} \cdot \hat{\mathbf{p}} \\ -\hat{\sigma} \cdot \hat{\mathbf{p}} & (\epsilon + m_0)I \end{vmatrix} = 0. \tag{D.114}$$

Using the relation

$$(\hat{\sigma} \cdot \mathbf{A})(\hat{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B}I + i\hat{\sigma} \cdot (\mathbf{A} \times \mathbf{B}) \tag{D.115}$$

(D.114) becomes

$$\begin{aligned} (\epsilon^2 - m_0^2)I - (\hat{\sigma} \cdot \hat{\mathbf{p}})(\hat{\sigma} \cdot \hat{\mathbf{p}}) &= (\epsilon^2 - m_0^2)I - \mathbf{p} \cdot \mathbf{p}I - i\hat{\sigma} \cdot (\mathbf{p} \times \mathbf{p}) \\ &= (\epsilon^2 - m_0^2)I - \mathbf{p} \cdot \mathbf{p}I = 0 \end{aligned} \tag{D.116}$$

or

$$\epsilon^2 = m_0^2 + \mathbf{p}^2 \tag{D.117}$$

from which it follows

$$\epsilon = \pm E_p, \quad E_p = \sqrt{\mathbf{p}^2 + m_0^2} \tag{D.118}$$

D.2.4 Positive and Negative Energy Eigenvectors

with solutions

$$\begin{aligned} \varphi(t) &= \varphi(0)e^{-im_0 t} \\ \chi(t) &= \chi(0)e^{-im_0 t} \end{aligned} \tag{D.119}$$

φ represents a particle, while χ represents an antiparticle.

$$\chi_0 = \frac{(\hat{\sigma} \cdot \hat{\mathbf{p}})}{m_0 + \epsilon}\varphi_0. \tag{D.120}$$

Let us denote the two-spinor φ_0 in the form

$$\varphi_0 = U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{D.121}$$

with the normalisation

$$U^\dagger U = U_1^* U_1 + U_2^* U_2 = 1, \tag{D.122}$$

where U_1, U_2 are complex numbers.

D.2.5 Helicity

There is another quantum number, the helicity, can be used to classify the free one-particle states. Its operator should commute with the operators whose eigenvalues have already been introduced to label our free solutions.

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\mathbf{\Sigma}} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \tag{D.123}$$

The helicity commutes with the Hamiltonian

$$\begin{aligned}
[\hat{\mathcal{H}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}] &= \left[\hat{\alpha} \cdot \hat{\mathbf{p}} + m_0 \hat{\beta}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] \\
&= \begin{pmatrix} 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \\ \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \end{pmatrix} \begin{pmatrix} \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix} - \begin{pmatrix} \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \\ \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 \\ (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 \\ (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 & 0 \end{pmatrix} \\
&= 0
\end{aligned}$$

Hence

$$\left[\hat{\mathcal{H}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] = 0 \tag{D.124}$$

and obviously we have

$$\left[\hat{\mathbf{p}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] = 0 \tag{D.125}$$

the helicity operator

$$\hat{\Lambda}_S = \frac{\hbar}{2}\hat{\mathbf{\Sigma}} \cdot \frac{\hat{\mathbf{p}}}{|\mathbf{p}|} = \hat{\mathbf{S}} \cdot \frac{\hat{\mathbf{p}}}{|\mathbf{p}|} \tag{D.126}$$

Helicity is the projection of the spin onto the direction of momentum.

If the electron wave propagates into th edirection of the $z-$axis, we have

$$\mathbf{p} = \{0, 0, p\}$$

and because of (D.126),

$$\hat{\Lambda}_S = \hat{S}_z = \frac{\hbar}{2}\hat{\Sigma} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{D.127}$$

with eigenvalues $\pm\hbar/2$. Clearly, the eigenvectors of $\hat{\Lambda}_S$ are

$$\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_1 \end{pmatrix} \tag{D.128}$$

with

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{D.129}$$

If in any particular direction a quantum state take one of two values it is likely to do with spin half particles. In fact Lorentz-covariance of Dirac's equation will imply that these two-state systmes transform under rotations as two-spinors.

D.2.6 Coupling of Dirac spinors to the Electromagnetic Field

Under a gauge transformation the wavefunction $\psi(x)$ transforms as

$$\psi(x) \to e^{i\Lambda(x)}\psi(x) \tag{D.130}$$

and

$$A^\mu(x) \to A^\mu(x) - \frac{1}{e}\partial^\mu\Lambda(x) \tag{D.131}$$

thus

$$\left(i\hbar\frac{\partial}{\partial x^\mu} - eA^\mu\right) \tag{D.132}$$

is gauge covariant. The minimal coupling prescription

$$i\hbar\frac{\partial}{\partial x^\mu} \to \left(i\hbar\frac{\partial}{\partial x^\mu} - eA^\mu\right) \tag{D.133}$$

results in the equation

$$(i\hbar\gamma^\mu\frac{\partial}{\partial x^\mu} - eA_\mu\gamma^\mu - m_0)\psi = 0. \tag{D.134}$$

D.2.7 Lorentz Transformations for Dirac Spinors

How do we construct the wave function $\psi'(x')$ in one inertial frame if we know the wave function $\psi(x)$ in another frame, where the two frames are related by the Lorentz transformation a_μ^ν? Here we construct the Lorentz transformation law between $\psi(x)$ and $\psi'(x')$.

We start from the principle of special relativity which states that the laws of physics should take the same form in all inertial systems, $\psi'(x')$ must be a solution of a Dirac equation which has the form

$$\left(i\hbar\gamma^{\mu'}\frac{\partial}{\partial x'^\mu} - m_0\right)\psi'(x') = 0 \tag{D.135}$$

in the primed system, where $\gamma^{\mu'}$ satisfy the same anti-commutation relations as γ^μ:

$$\gamma^{\mu'}\gamma^{\nu'} + \gamma^{\nu'}\gamma^{\mu'} = 2\eta^{\mu\nu}I \tag{D.136}$$

and

$$\gamma'^{0\dagger} = \gamma'^{0} \tag{D.137}$$

$$\gamma'^{i\dagger} = -\gamma'^{i} \quad i = 1, 2, 3. \tag{D.138}$$

It can be shown that $\gamma^{\mu'}$ that satify the above relations are identical to γ^μ up to a unitary transformation $\hat{U}$, i.e.

$$\gamma^{\mu'} = \hat{U}^\dagger\gamma^\mu\hat{U}, \quad \hat{U}^\dagger = \hat{U}^{-1}. \tag{D.139}$$

Since unitary transformations do not change the physics, we may use the same γ matrices in both Lorentz systems. From now on we just take $\gamma^\mu = \gamma^{\mu'}$.

$$\left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x^\mu} - m_0\right)\psi(x) = 0 \quad \text{and} \quad \left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x'^\mu} - m_0\right)\psi'(x') = 0$$

Let $\hat{a}$ denote the matrix of the Lorentz transformation a_μ^ν. We write

$$\psi'(x') = \psi'(\hat{a}x') = \hat{S}(\hat{a})\psi(x) = \hat{S}(\hat{a}^{-1}x') \tag{D.140}$$

We must have an inverse transformation

$$\psi(x) = \hat{S}^{-1}(\hat{a})\psi'(x') = \hat{S}^{-1}(\hat{a})\psi'(\hat{a}x)$$

Start with Dirac equation

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - m_0\right)\psi(x) = 0$$

expressing $\psi(x)$ by $\hat{S}^{-1}(\hat{a})\psi'(x')$ yields

$$\left(i\hbar\gamma^\mu \hat{S}^{-1}(\hat{a})\frac{\partial}{\partial x^\mu} - m_0\hat{S}^{-1}(\hat{a})\right)\psi'(x') = 0.$$

We the multiply by $\hat{S}(\hat{a})$

$$\left(i\hbar\hat{S}(\hat{a})\gamma^\mu \hat{S}^{-1}(\hat{a})\frac{\partial}{\partial x^\mu} - m_0\right)\psi'(x') = 0 \tag{D.141}$$

Now we transform $\partial/\partial x^\mu$ to x' coordinates is given by

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = a^\nu{}_\mu \frac{\partial}{\partial x'^\nu} \tag{D.142}$$

So that () becomes

$$\left(i\hbar(\hat{S}(\hat{a})\gamma^\mu \hat{S}^{-1}(\hat{a})a^\nu{}_\mu)\frac{\partial}{\partial x'^\nu} - m_0\right)\psi'(x') = 0. \tag{D.143}$$

Comparing this to Dirac's equation in the x' coordinates we see that $\hat{S}(\hat{a})$ must satisfy

$$\hat{S}(\hat{a})\gamma^\mu \hat{S}^{-1}(\hat{a})a^\nu{}_\mu = \gamma^\nu \tag{D.144}$$

or equivalently

$$\hat{S}(\hat{a})\gamma^\nu \hat{S}^{-1}(\hat{a}) = a_\mu{}^\nu \gamma^\mu \tag{D.145}$$

D.2.8 Infintesimal Generating Technique for Lorentz Transformations

We first give the idea of the infintesimal generating technique with a couple of simple examples.

Example 2: Lorentz transformation in x_1−direction for $2d$-spacetime

We derive the Lorentz transformation formula for boosts in the x_1−direction. Consider two inertia frames, the 'primed' frame one moving away from the 'unprimed' frame at an infintesimal velocity δv along the $x_1$direction. For an infintesimal relative velocity the spacetime transformation is Galilean:

$$x_1' = x_1 - \delta v t. \tag{D.146}$$

How is special relativity brought into the calculation? This is done by requiring that

$$x_1^2 - t^2 = x_1^{'2} - t^{'2}. \tag{D.147}$$

From this we see that $t' \neq t$, and so t should transform some way as well. Let us write

$$t' = t + a\delta v x_1. \tag{D.148}$$

Using this in (D.147) we find $a = -1$. The two transformation equations can be combined in the matrix equation

$$\begin{aligned}\begin{pmatrix} t' \\ x' \end{pmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta v \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix} t \\ x \end{pmatrix} \\ &= (\mathbf{1} + \delta v \hat{I}_2)\begin{pmatrix} t \\ x \end{pmatrix}\end{aligned} \tag{D.149}$$

where $\hat{I}_x = -\mathbf{1}$. Now we repeat the transformation N times to generate a finite transformation with velocity pararmeter $\theta = N\delta v$. Then

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \left(\mathbf{1} + \frac{\theta}{N}\hat{I}_x\right)^N \begin{pmatrix} t \\ x \end{pmatrix} \tag{D.150}$$

In the limit $N \to \infty$,

$$\lim_{N\to\infty}\left(\mathbf{1}+\frac{\theta}{N}\hat{I}_x\right)^N = \exp(\theta\hat{I}_2). \tag{D.151}$$

Noting $\hat{I}_x^2 = \mathbf{1}$, we expand the expontential

$$\begin{aligned}
\exp(\theta\hat{I}_x) &= \mathbf{1}+\theta\hat{I}_x+\frac{\theta^2\hat{I}_x^2}{2!}+\frac{\theta^3\hat{I}_x^3}{3!}+\ldots \\
&= \mathbf{1}[1+\frac{1}{2!}\theta^2+\ldots]+\hat{I}_x[\theta-\frac{\theta^3}{3!}+\ldots] \\
&= \\
&= \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}
\end{aligned} \tag{D.152}$$

$\cosh\theta$ and $\sinh\theta$ can be identified by considering the origin of the primed coordinate system, $x' = 0$, or $x = vt$. Substituiting this into () we have

$$0 = x\cosh\theta - t\sinh\theta. \tag{D.153}$$

So

$$\tan\theta = v$$

Using $1-\tanh^2\theta = (\cosh^2\theta)^{-1}$,

$$\cosh\theta = \frac{1}{(1-v^2)^{1/2}}, \qquad \sinh\theta = \frac{v}{(1-v^2)^{1/2}}. \tag{D.154}$$

We finally obtain the known Lorentz transformations

$$t' = \frac{t-vx}{(1-v^2)^{1/2}}, \qquad x' = \frac{x-vt}{(1-v^2)^{1/2}} \tag{D.155}$$

□

This result can easily be generalised to 4-minkowski space time. We just use the generator

$$I_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{D.156}$$

We end up with the answer

$$t' = \frac{t - vx}{(1 - v^2)^{1/2}}, \qquad x' = \frac{x - vt}{(1 - v^2)^{1/2}}, \qquad y' = y, \qquad z' = z. \tag{D.157}$$

If we had wanted to do boost in the direction given by the unit vector

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma) \tag{D.158}$$

we would use the generator

$$I_n = \begin{pmatrix} 0 & -\cos\alpha & -\cos\beta & -\cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{pmatrix}. \tag{D.159}$$

Note that, to first order, $t^2 - \vec{r}^2 = (t')^2 - \vec{r}'^2$ is satified for an infintesimal relative velocity.

The reader is invited to do the full calculation and derive the Lorentz transformation formula.

Example 2: Rotations about the $z-$direction for $3d$-spacetime

It is easily seen, drawing a diagram, that under an infintesimal rotation $\delta\phi$ around the $z-$axis results in

$$x' = x + y\delta\phi, \qquad y' = y - x\delta\phi \tag{D.160}$$

The two transformation equations can be combined in the matrix equation

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (\mathbf{1} + \delta\phi i \hat{\sigma}_2) \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned} \tag{D.161}$$

where $\hat{\sigma}_2$ is a Pauli matrix. Let us now do the expontentiation using $\hat{\sigma}_2^2 = 1$:

$$
\begin{aligned}
\exp(\phi i\hat{\sigma}_2) &= \mathbf{1} + \phi i\hat{\sigma}_2 - \frac{\phi^2}{2!}\hat{\sigma}_2^2 - i\frac{\phi^3}{3!}\hat{\sigma}_2^3 \\
&= \mathbf{1}(1 - \frac{\phi^2}{2!} + \dots) + i\hat{\sigma}_2(\phi - \frac{\phi^3}{3!} + \dots) \\
&= \mathbf{1}\cos\phi + i\hat{\sigma}_2\sin\phi \\
&= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}
\end{aligned} \tag{D.162}
$$

So that

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{D.163}
$$

□

In 4d-minkowski space time we use the generator

$$
I_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{D.164}
$$

We end up with the answer

$$
t' = t, \qquad x' = x\cos\phi + y\sin\phi, \qquad y' = -x\sin\phi + y\cos\phi, \qquad z' = z. \tag{D.165}
$$

Proper Lorentz Transformations

$$
a^{\nu}{}_{\mu} = \delta^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu} \tag{D.166}
$$

We denote the inverse Lorentz Transformation as $a_{\nu}{}^{\sigma}$. Then, neglecting terms quadtratic in $\Delta\omega$,

$$
\begin{aligned}
a^{\mu}{}_{\nu} a_{\mu}{}^{\sigma} &= \delta^{\sigma}_{\nu} = (\delta^{\mu}_{\nu} + \Delta\omega^{\mu}{}_{\nu})(\delta^{\sigma}_{\mu} + \Delta\omega_{\mu}{}^{\sigma}) \\
&\approx \delta^{\mu}_{\nu}\delta^{\sigma}_{\mu} + \delta^{\mu}_{\nu}\Delta\omega_{\mu}{}^{\sigma} + \delta^{\sigma}_{\mu}\Delta\omega^{\mu}{}_{\nu} \\
&= \delta^{\sigma}_{\nu} + \Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu}
\end{aligned} \tag{D.167}
$$

Hence,

$$\Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu} = 0$$

or

$$\eta^{\mu\nu}\left(\Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu}\right) = 0 = \Delta\omega^{\mu\sigma} + \Delta\omega^{\sigma\mu}.$$

so we must have

$$\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu} \tag{D.168}$$

Consequently, there are six independent non-vanishing parameters $\Delta\omega^{\mu\nu}$.

D.2.9 The $\hat{S}$ Operator for Infintesimal Lorentz Transformations

We aim to determine the operator $\hat{S}$ by assertaining its infintesimal form by finding its expansion to linear order in the generators $\Delta\omega^{\mu\nu}$. We write

$$\hat{S}(\Delta\omega^{\mu\nu}) \;\;=\;\; \mathbf{1} - \frac{i}{4}\hat{\sigma}_{\mu\nu}\Delta\omega^{\mu\nu} \tag{D.169}$$

where $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$. The inverse operator being

$$\hat{S}^{-1}(\Delta\omega^{\mu\nu}) \;\;=\;\; \mathbf{1} + \frac{i}{4}\hat{\sigma}_{\mu\nu}\Delta\omega^{\mu\nu} \tag{D.170}$$

By finding $\sigma_{\alpha\beta}$ we can find $\hat{S}$. By substituting (D.169) and (D.170) into the defing equation for $\hat{S}$:

$$(\delta^{\nu}_{\mu} + \Delta\omega_{\mu}{}^{\nu})\gamma^{\mu} = \hat{S}(\Delta\omega^{\mu\nu})\gamma^{\nu}\hat{S}^{-1}(\Delta\omega^{\mu\nu})$$

we can find an equation that determines $\sigma_{\alpha\beta}$.

$$(\delta^{\nu}_{\mu} + \Delta\omega_{\mu}{}^{\nu})\gamma^{\mu} = \left(\mathbf{1} - \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)\gamma^{\nu}\left(\mathbf{1} + \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)$$

or omitting the quadratic terms in $\Delta\omega_{\mu}{}^{\nu}$,

$$\Delta\omega_\mu{}^\nu\gamma^\mu = -\frac{i}{4}\Delta\omega^{\alpha\beta}(\hat{\sigma}_{\alpha\beta}\gamma^\nu - \gamma^\nu\hat{\sigma}_{\alpha\beta}) \tag{D.171}$$

Using the antisymmetry of $\Delta\omega_\mu{}^\nu$, the LHS becomes

$$\begin{aligned}
\Delta\omega_\mu{}^\nu\gamma^\mu &= \eta^\nu{}_\sigma\Delta\omega_\mu{}^\sigma\gamma^\mu \\
&= \Delta\omega_\beta{}^\alpha(\eta^\nu{}_\alpha\gamma^\beta) \\
&= -\Delta\omega^{\alpha\beta}(\eta^\nu{}_\alpha\gamma_\beta) \\
&= -\frac{1}{2}\Delta\omega^{\alpha\beta}(\eta^\nu{}_\alpha\gamma_\beta - \eta^\nu{}_\beta\gamma_\alpha)
\end{aligned} \tag{D.172}$$

Comparing this with (D.171), we end up with the relation

$$-2i(\eta_\nu{}^\alpha\gamma_\beta - \eta_\nu{}^\beta\gamma_\alpha) = [\hat{\sigma}_{\alpha\beta}, \gamma^\nu] \tag{D.173}$$

It is shown in section D.14.4 that this is solved by

$$\hat{\sigma}_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta].$$

The operator $\hat{S}(\Delta\omega^{\mu\nu})$ is now

$$\hat{S}(\Delta\omega^{\mu\nu}) = \mathbf{1} + \frac{1}{8}[\gamma_\mu, \gamma_\nu]\Delta\omega^{\mu\nu} \tag{D.174}$$

The problem of finding $\hat{S}$ for finite proper Lorentz transformations has now been essentially solved! To construct $\hat{S}$ we now sucessively apply the infintesimal operators (D.174). In order to do this we write

$$\Delta\omega^\nu{}_\mu = \Delta\omega(\hat{I}_\mathbf{n})^\nu{}_\mu \tag{D.175}$$

Here $\Delta\omega$ is an inintesimal parameter of the Lorentz group around an axis in the $\mathbf{n}$ direction.

D.2.10 The $\hat{S}$ Operator for Proper Lorentz Transformations

$$\begin{aligned}
\psi'(x') = \hat{S}(\hat{a})\psi(x) &= \lim_{N\to\infty}\left(\mathbf{1} - \frac{i}{4}\frac{\omega}{N}\hat{\sigma}_{\mu\nu}(\hat{I}_\mathbf{n})^{\mu\nu}\right)^N\psi(x) \\
&= e^{-(i/4)\omega\hat{\sigma}_{\mu\nu}(\hat{I}_\mathbf{n})^{\mu\nu}}\psi(x)
\end{aligned} \tag{D.176}$$

Example: Lorentz boost along x-axis

$$(I_x)^0{}_1 = (I_x)^1{}_0 = -(I_x)^{01} = +(I_x)^{10} = -1.$$

So (D.176) becomes

$$\begin{aligned}
\psi'(x') &= \exp\left\{-\frac{i}{4}\omega\left[\hat{\sigma}_{01}(I_x)^{01} + \hat{\sigma}_{10}(I_x)^{10}\right]\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{4}\omega\left[\hat{\sigma}_{01}(+1) + \hat{\sigma}_{10}(-1)\right]\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{2}\omega\hat{\sigma}_{01}\right\}\psi(x) \qquad \text{(D.177)}
\end{aligned}$$

□

Example: Rotation around z-axis

Recall that

$$\Delta\omega^\nu{}_\mu = \delta\phi(\hat{I}_3)^\nu{}_\mu, \qquad \text{(D.178)}$$

where $\hat{I}_3$ is given by (). Thus only the elements $(\hat{I}_3)^{12} = -(\hat{I}_3)^{21}$ are non-zero, and we get

$$\begin{aligned}
\psi'(x') &= \exp\left\{-\frac{i}{4}\phi\hat{\sigma}_{\mu\nu}(\hat{I}_3)^{\mu\nu}\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{4}\phi\left[\hat{\sigma}_{12}(I_x)^{12} + \hat{\sigma}_{21}(I_3)^{21}\right]\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{4}\phi\left[\hat{\sigma}_{12}(-1) + \hat{\sigma}_{21}(+1)\right]\right\}\psi(x) \\
&= \exp\left\{\frac{i}{2}\phi\hat{\sigma}_{12}\right\}\psi(x) = \exp\left\{\frac{i}{2}\phi\hat{\sigma}^{12}\right\}\psi(x) \qquad \text{(D.179)}
\end{aligned}$$

□

Spinor for spacial rotations

D.2.11 The Four-Current Density

$$j_\mu(x) = \psi^\dagger(x)\gamma^0\gamma^\mu\psi(x). \tag{D.180}$$

This current density transforms under the Lorentz transformation as

$$\begin{aligned} j'^\mu(x') &= \psi^{\dagger'}(x')\gamma^0\gamma^\mu\psi'(x') \\ &= \psi^\dagger(x)\hat{S}^\dagger\gamma^0\gamma^\mu\hat{S}\psi(x) \\ &= \psi^\dagger(x)\gamma^0(\gamma^0\hat{S}^\dagger\gamma^0)\gamma^\mu\hat{S}\psi(x) \\ &= \psi^\dagger(x)\gamma^0\hat{S}^{-1}\gamma^\mu\hat{S}\psi(x) \\ &= \psi^\dagger(x)\gamma^0(a^\nu{}_\mu\gamma^\mu)\psi(x) \\ &= a^\nu{}_\mu j^\mu(x) \end{aligned} \tag{D.181}$$

and as such is identified as a four-vector.

D.2.12 Plane Waves in Arbitrary Directions

Free solutions have the form

$$\psi^r = \omega^r(0)e^{-\epsilon_r(m_0/\hbar)t} \tag{D.182}$$

We have

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega^4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{D.183}$$

$$\omega^r(p) = \hat{S}(-\mathbf{v})\omega^r(0) = e^{-(\omega/2)\alpha\cdot\mathbf{v}/v}\omega^r(0) \tag{D.184}$$

$$\hat{\sigma}_{\mu\nu}(\hat{I}_\mathbf{n})^{\mu\nu} = 2\big(\hat{\sigma}_{01}(\hat{I}_\mathbf{n})^{01} + (\hat{I}_\mathbf{n})^{02} + (\hat{I}_\mathbf{n})^{03}\big) \tag{D.185}$$

$$\frac{\mathbf{v}}{v} = (\cos\alpha, \cos\beta, \cos\gamma) \tag{D.186}$$

Also

$$\begin{aligned}\hat{\sigma}_{0i} &= \frac{i}{2}(\gamma_0\gamma_i - \gamma_i\gamma_0)\\ &= i\gamma_0\gamma_i\\ &= -i\gamma^0\gamma^i = -i\gamma^0\gamma^0\alpha_i = -i\alpha_i \end{aligned} \tag{D.187}$$

With this the spinor transformation for Lorentz transformations to interial systems with direction of velocity $\mathbf{v}/v$ now becomes

$$\hat{S}(-\mathbf{v}) = \hat{S}\left(-\frac{\mathbf{p}}{E}\right) = e^{-(\omega/2)\hat{\alpha}\cdot\mathbf{v}/v} \tag{D.188}$$

When we expand $\hat{S}$ we will need the following

$$\begin{aligned}(\hat{\alpha}\cdot\mathbf{v})^2 &= \hat{\alpha}^i\hat{\alpha}^j v_i v_j\\ &= \gamma^0\gamma^i\gamma^0\gamma^j v_i v_j\\ &= -\gamma^i\gamma^j v_i v_j\\ &= -\frac{1}{2}(\gamma^i\gamma^j + \gamma^j\gamma^i)v_i v_j\\ &= -\frac{1}{2}2\eta^{ij}\mathbf{1}v_i v_j = +v^2\mathbf{1}\end{aligned} \tag{D.189}$$

We expand $\hat{S}$

$$\begin{aligned}\hat{S}(-\mathbf{v}) &= \mathbf{1} - \frac{\omega}{2}\frac{\hat{\alpha}\cdot\mathbf{v}}{v} + \frac{1}{2!}\frac{\omega^2}{4v^2}(\hat{\alpha}\cdot\mathbf{v})^2 - \frac{1}{3!}\frac{\omega^3}{8v^3}(\hat{\alpha}\cdot\mathbf{v})^3 + \ldots\\ &= \mathbf{1}(1 + \frac{1}{2}\frac{\omega^2}{4}) - \frac{\hat{\alpha}\cdot\mathbf{v}}{v}(\frac{\omega}{2}+)\\ &= \mathbf{1}\cosh\frac{\omega}{2} - \frac{\hat{\alpha}\cdot\mathbf{v}}{v}\sinh\frac{\omega}{2}\end{aligned} \tag{D.190}$$

The matrix written out $\hat{\alpha}\cdot\mathbf{v}/v$

$$
\begin{aligned}
\frac{\hat{\alpha}\cdot\mathbf{v}}{v} &= \hat{\alpha}_x\frac{v_x}{v}+\hat{\alpha}_y\frac{v_y}{v}+\hat{\alpha}_z\frac{v_z}{v} \\
&= \frac{p_x}{p}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{ip_y}{p}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&\quad + \frac{p_z}{p}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{p}\begin{pmatrix} 0 & 0 & p_z & p_- \\ 0 & 0 & p_+ & -p_z \\ p_z & p_- & 0 & 0 \\ p_+ & -p_z & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{D.191}
$$

where $p_\pm = p_x \pm ip_y$. We obtain

$$
\hat{S}(-\mathbf{v}) = \cosh\frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \cosh\frac{\omega}{2}\frac{\tanh\frac{\omega}{2}}{p}\begin{pmatrix} 0 & 0 & p_z & p_- \\ 0 & 0 & p_+ & -p_z \\ p_z & p_- & 0 & 0 \\ p_+ & -p_z & 0 & 0 \end{pmatrix} \tag{D.192}
$$

To find expressions for $\cosh\frac{\omega}{2}$ and $\tanh\frac{\omega}{2}$ we consider only motion in the x direction. We convert the rotation angle ω with the aid of

$$
-v_x = \tanh\omega, \tag{D.193}
$$

or

$$
\omega = \tanh^{-1}(-v_x) = -\tanh^{-1}(v_x) \tag{D.194}
$$

We need the equations

$$
\begin{aligned}
\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} &= \frac{1}{2}\sinh\omega, \\
\cosh\frac{\omega}{2}\cosh\frac{\omega}{2} &= \frac{1}{2}(\cosh\omega+1), \\
\sinh\frac{\omega}{2}\sinh\frac{\omega}{2} &= \frac{1}{2}(\cosh\omega-1).
\end{aligned}
\tag{D.195}
$$

Therefore

$$\tanh\frac{x}{2}=\frac{\sinh x}{\cosh x+1}=\frac{\tanh x}{1+1/\cosh x}=\frac{\tanh x}{1+\sqrt{1-\tanh^2 x}} \tag{D.196}$$

With (D.194)

$$\begin{aligned}
-\tanh\frac{\omega}{2} &= \frac{-\tanh\omega}{1+\sqrt{1-\tanh^2\omega}}\\
&= \frac{v_x}{1+\sqrt{1-v_x^2}}\\
&= \frac{m_0/\sqrt{1-v_x^2}}{m_0/\sqrt{1-v_x^2}}\frac{v_x}{1+\sqrt{1-v_x^2}}\\
&= \frac{p_x}{E+m_0}
\end{aligned} \tag{D.197}$$

taking into account that we are considering only motion in the x direction, we may write

$$-\tanh\frac{\omega}{2}=\frac{p}{E+m_0}. \tag{D.198}$$

And

$$
\begin{aligned}
\cosh\frac{\omega}{2} &= \frac{1}{\sqrt{1-\tanh\frac{\omega}{2}}} \\
&= \frac{1}{\sqrt{1-[\tanh\omega/(1+\sqrt{1-\tanh^2\omega})^2]}} \\
&= \frac{1}{\sqrt{1-[v_x/(1+\sqrt{1-v_x^2})^2]}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{(1+\sqrt{1-v_x^2})^2+v_x^2}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{(1+2\sqrt{1-v_x^2}+1-v_x^2)-v_x^2}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{2}\sqrt{1-v_x^2+\sqrt{1-v_x^2}}} \\
&= \frac{[1/\sqrt{1-v_x^2}+1]m_0}{\sqrt{1+[1/\sqrt{1-v_x^2}]}\sqrt{2}m_0} \\
&= \frac{E+m_0}{\sqrt{m_0+E}\sqrt{2m_0}} \\
&= \sqrt{\frac{E+m_0}{2m_0}} \qquad \text{(D.199)}
\end{aligned}
$$

substituting this result and(D.198) into (D.192) we obtain

$$
\begin{aligned}
\hat{S}(-\mathbf{v}) &= \sqrt{\frac{E+m_0}{2m_0}}\begin{bmatrix} 1 & 0 & \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} \\ 0 & 1 & \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} \\ \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} & 1 & 0 \\ \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} & 0 & 1 \end{bmatrix} \\
&= [\omega^1(\mathbf{p}), \omega^2(\mathbf{p}), \omega^3(\mathbf{p}), \omega^4(\mathbf{p})]. \qquad \text{(D.200)}
\end{aligned}
$$

D.2.13 Bilinear Covariants

Linear independence

$$
\begin{aligned}
\hat{\Gamma}^S &= \mathbf{I}, \quad \hat{\Gamma}^V_\mu = \gamma_\mu, \quad \hat{\Gamma}^T_{\mu\nu} = \hat{\sigma}_{\mu\nu} = -\hat{\sigma}_{\nu\mu} \\
\hat{\Gamma}^P &= i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma^5, \quad \hat{\Gamma}^a_\mu = \gamma^5\gamma_\mu \qquad \text{(D.201)}
\end{aligned}
$$

i) $(\hat{\Gamma}^n)^2 = \pm\mathbf{I}$.

Proof: Proved explicitly.

ii) To each $\hat{\Gamma}^n$ except $\hat{\Gamma}^S$ there exists at least one $\hat{\Gamma}^m$ such that

$$\hat{\Gamma}^n\hat{\Gamma}^m = -\hat{\Gamma}^m\hat{\Gamma}^n \tag{D.202}$$

Proof: Proved explicitly.

iii) $Tr(\hat{\Gamma}^n) = 0$

By (D.202)

$$-\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m = +\hat{\Gamma}^n(\hat{\Gamma}^m)^2$$

As $(\hat{\Gamma}^m)^2 = \pm\mathbf{I}$

$$\pm Tr(\hat{\Gamma}^n) = Tr(\hat{\Gamma}^n(\hat{\Gamma}^m)^2) = -Tr(\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m) = -Tr(\hat{\Gamma}^n\hat{\Gamma}^m\hat{\Gamma}^m) = 0.$$

iv) For given $\hat{\Gamma}^a$ and $\hat{\Gamma}^b$ ($a \neq b$) there exists a $\hat{\Gamma}^n \neq \hat{\Gamma}^S$ such that

$$\hat{\Gamma}^a\hat{\Gamma}^b = f^n_{ab}\hat{\Gamma}^n. \tag{D.203}$$

Proof: Proved explicitly.

v) The $\hat{\Gamma}^n$ are linearly independent. Suppose

$$\sum_n a_n\hat{\Gamma}^n = 0. \tag{D.204}$$

Multiply from the right by $\hat{\Gamma}^m \neq \hat{\Gamma}^S$ we get

$$\begin{aligned}
0 &= \sum_n a_n Tr(\hat{\Gamma}^n\hat{\Gamma}^m) \\
&= a_m(\hat{\Gamma}^m)^2 + \sum_{n\neq m} a_n Tr(\hat{\Gamma}^n\hat{\Gamma}^m) \\
&= a_m(\hat{\Gamma}^m)^2 + \sum_{n\neq m} a_n Tr(f^\nu_{nm}\hat{\Gamma}^\nu) \\
&= \pm 4a_m.
\end{aligned} \tag{D.205}$$

Thus $a_m = 0$ for all $m \neq S$. Now in the case of $\hat{\Gamma}^m = \hat{\Gamma}^S$

$$0 = Tr\left(\sum_n a_n \hat{\Gamma}^S \hat{\Gamma}^n\right) = a_S Tr(\mathbf{I}) + \sum_{n \neq S} a_n Tr(\hat{\Gamma}^n) = 0, \tag{D.206}$$

i.e. $a_S = 0$.

□

Lorentz transformations

Under Lorentz transformations

$$\psi \to \hat{S}\psi, \qquad \overline{\psi} \to \overline{\psi}\hat{S}^{-1} \tag{D.207}$$

This is proved by

$$\begin{aligned} \overline{\psi}'(x') &= \psi'^\dagger(x')\gamma^0 \\ &= \psi^\dagger(x)\hat{S}^\dagger\gamma^0 \\ &= \psi^\dagger(x)\gamma^0(\gamma^0\hat{S}^\dagger\gamma^0) \\ &= \overline{\psi}(x)\hat{S}^{-1}. \end{aligned} \tag{D.208}$$

□

We now prove

$$\gamma_5\hat{S} = det|a|\ \hat{S}\gamma_5. \tag{D.209}$$

This is easily proven that for proper Lorentz transformations (here $det|a| = 1$), first

$$[\gamma_5, \hat{\sigma}_{\mu\nu}] = \frac{1}{2}(\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) - (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\gamma_5) = 0$$

where we have used $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$. So from the formula for proper Lorentz transformations

$$\hat{S}(\hat{a}) = \exp\left(-\frac{i}{4}\hat{\sigma}_{\mu\nu}(I_{\mathbf{n}})^{\mu\nu}\right),$$

we have

$$[\hat{S}(\hat{a}), \gamma_5] = 0. \tag{D.210}$$

We know prove (D.209) for spacial reflections, which are given by

$$\mathbf{x}' = -\mathbf{x}, \qquad t' = t, \tag{D.211}$$

with corresponding transformation matrix

$$a^{\nu}{}_{\mu} = \eta^{\mu\nu}. \tag{D.212}$$

The relation

$$\hat{S}^{-1}\gamma^{\mu}\hat{S} = a^{\mu}{}_{\nu}\gamma^{\nu}$$

holds for improper Lorentz transformations as well. Let us dente the parity operator by $\hat{P}$. We can then write

$$a^{\nu}{}_{\mu}\gamma^{\mu} = \hat{P}\gamma^{\nu}\hat{P}^{-1}$$

or

$$a^{\sigma}{}_{\nu}a^{\nu}{}_{\mu}\gamma^{\mu} = \hat{P}a^{\sigma}{}_{\nu}\gamma^{\nu}\hat{P}^{-1}$$

This is equivalent to

$$\delta^{\sigma}_{\mu}\gamma^{\mu} = \hat{P}(\sum_{\nu=0}^{3}\eta^{\sigma\nu}\gamma^{\nu})\hat{P}^{-1} \tag{D.213}$$

which in turn is equivalent to

$$\hat{P}^{-1}\gamma^{\sigma}\hat{P} = \eta^{\sigma\sigma}\gamma^{\sigma}. \tag{D.214}$$

This has the simple solution

$$\hat{P} = e^{i\varphi}\gamma^{0}, \qquad \hat{P}^{-1} = e^{-i\varphi}\gamma^{0} \tag{D.215}$$

For this operator we easily have

$$\hat{P}\gamma_5 = -\gamma_5 \hat{P}. \tag{D.216}$$

$\square$

i) $\overline{\psi}\psi$ is a scalar:

$$\begin{aligned}\overline{\psi}\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\hat{S}\psi \\ &= \overline{\psi}\psi\end{aligned}$$

ii) $\overline{\psi}\gamma_5\psi$ is a pseudoscalar:

$$\begin{aligned}\overline{\psi}\gamma_5\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}\psi \\ &= det|a|\ \overline{\psi}\hat{S}^{-1}\hat{S}\gamma_5\psi \\ &= det|a|\ \overline{\psi}\gamma_5\psi\end{aligned}$$

iii) $\overline{\psi}\gamma^\mu\psi$ is a vector:

$$\begin{aligned}\overline{\psi}\gamma^\mu\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma^\mu\hat{S}\psi \\ &= a^\mu{}_\nu\overline{\psi}\gamma^\nu\psi\end{aligned}$$

iv) $\overline{\psi}\gamma_5\gamma^\mu\psi$ is a pseudovector:

$$\begin{aligned}\overline{\psi}\gamma_5\gamma^\mu\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma_5\gamma^\mu\hat{S}\psi \\ &= \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}(\hat{S}^{-1}\gamma^\mu\hat{S})\psi \\ &= \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}(a^\mu{}_\nu\gamma^\nu)\psi \\ &= det|a|a^\mu{}_\nu\ \overline{\psi}\gamma^\nu\psi \\ &= det|a|a^\mu{}_\nu\ \overline{\psi}\gamma^\nu\psi\end{aligned}$$

v) $\overline{\psi}\hat{\sigma}^{\mu\nu}\psi$ is a pseudovector:

$$
\begin{aligned}
\overline{\psi}\hat{\sigma}^{\mu\nu}\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\hat{\sigma}^{\mu\nu}\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\hat{S}^{-1}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\hat{S}^{-1}(\gamma^{\mu}\hat{S}\hat{S}^{-1}\gamma^{\nu} - \gamma^{\nu}\hat{S}\hat{S}^{-1}\gamma^{\mu})\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\{(a^{\mu}{}_{\rho}\gamma^{\rho})(a^{\nu}{}_{\tau}\gamma^{\tau}) - (a^{\mu}{}_{\rho}\gamma^{\rho})(a^{\nu}{}_{\tau}\gamma^{\tau})\}\psi \\
&= a^{\mu}{}_{\rho}a^{\nu}{}_{\tau}\,\overline{\psi}\frac{i}{2}(\gamma^{\rho}\gamma^{\tau} - \gamma^{\tau}\gamma^{\rho})\psi \\
&= a^{\mu}{}_{\rho}a^{\nu}{}_{\tau}\,\overline{\psi}\hat{\sigma}^{\rho\tau}\psi.
\end{aligned}
$$

□

D.2.14 Properties of Free Solutions

$$(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})\omega^{r}(\mathbf{p}) = 0 \tag{D.217}$$

$$
\begin{aligned}
[(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})\omega^{r}(\mathbf{p})]^{\dagger} &= 0 = \omega^{r\dagger}(\mathbf{p})(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})^{\dagger} \\
&= \omega^{r\dagger}(\mathbf{p})(p_{\mu}\gamma^{\mu\dagger} - \epsilon_{r}m_{0}) \\
&= \omega^{r\dagger}(\mathbf{p})(p_{0}\gamma^{0} - p_{k}\gamma^{k} - \epsilon_{r}m_{0})
\end{aligned}
$$

Multiplication from the right by γ^0 yields

$$
\begin{aligned}
\omega^{r\dagger}(\mathbf{p})(p_{\mu}\gamma^{\mu\dagger} - \epsilon_{r}m_{0})\gamma^{0} &= 0 = \omega^{r\dagger}(\mathbf{p})\gamma^{0}(p_{0}\gamma^{0} + p_{k}\gamma^{k} - \epsilon_{r}m_{0}) \\
&= \overline{\omega}^{r}(\mathbf{p})(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})
\end{aligned}
$$

The normalisation condition

The quantity $\overline{\omega}^{r}(\mathbf{p})\omega^{r'}(\mathbf{p})$ is a Lorentz scalar and hence

$$\overline{\omega}^{r}(\mathbf{p})\omega^{r'}(\mathbf{p}) = \overline{\omega}^{r}(0)\omega^{r'}(0) = \omega^{r\dagger}(0)\gamma^{0}\omega^{r'}(0) = \delta_{rr'}\epsilon_{r}. \tag{D.218}$$

The completenes relation

We have

$$\omega^{r\dagger}(\epsilon_r \mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \delta_{rr'}(E/m_0)$$

This is proved in section D.14.6.

The closure relation

In the rest frame of the electron we have

$$\sum_{r=1}^{4} \epsilon_r \omega^r(0)\overline{w}^r_\beta(0) = \delta_{\alpha\beta} \tag{D.219}$$

We know that

$$\omega^r(p) = \hat{S}\left(\frac{-\mathbf{p}}{E}\right) w^r(0)$$

and so

$$\begin{aligned} \overline{\omega}^r(p) &= \omega^{r\dagger}(p)\gamma^0 \\ &= \left(\hat{S}\left(\frac{-\mathbf{p}}{E}\right)\omega^r(0)\right)^\dagger \gamma^0 \\ &= w^{r\dagger}(0)\gamma^0\gamma^0\hat{S}^\dagger\left(\frac{-\mathbf{p}}{E}\right)\gamma^0 \\ &= \overline{\omega}^r(0)\hat{S}^{-1}\left(\frac{-\mathbf{p}}{E}\right). \end{aligned} \tag{D.220}$$

where we have used

$$\hat{S}^\dagger = \gamma^0\hat{S}^{-1}\gamma^0.$$

Using these we find

$$
\begin{aligned}
\sum_{r=1}^{4} \epsilon_r \omega_\alpha^r(p) \overline{\omega}_\beta^r(p) &= \sum_{r=1}^{4} \sum_{\gamma,\lambda=1}^{4} \epsilon_r \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E}\right) \omega_\gamma^r(0)\, \overline{\omega}_\lambda^r(0) \hat{S}_{\lambda\beta}^{-1} \left(\frac{-\mathbf{p}}{E}\right) \\
&= \sum_{\gamma,\lambda=1}^{4} \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E}\right) \hat{S}_{\lambda\beta} \left(\frac{-\mathbf{p}}{E}\right) \sum_{r=1}^{4} \epsilon_r \omega_\gamma^r(0) \overline{\omega}_\lambda^r(0) \\
&= \sum_{\gamma,\lambda=1}^{4} \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E}\right) \hat{S}_{\lambda\beta} \left(\frac{-\mathbf{p}}{E}\right) \delta_{\gamma\lambda} \\
&= \delta_{\alpha\beta}
\end{aligned}
\tag{D.221}
$$

Therefore we have the closure relation

$$
\sum_{r=1}^{4} \epsilon_r \omega_\alpha^r(p) \overline{\omega}_\beta^r(p) = \delta_{\alpha\beta}. \tag{D.222}
$$

D.2.15 Projection Operators for Energy and Spin

Projection Operators for Energy

Recall

$$
(p_\mu \gamma^\mu - \epsilon_r m_0 c) w^r(\mathbf{p}) \quad \text{which implies} \quad \epsilon_r p_\mu \gamma^\mu w^r(\mathbf{p}) = m_0 c w^r(\mathbf{p})
$$

We immediately see that the projection operator for eigenstates with positive or negative energy is given by

$$
\hat{\Lambda}_r(p) = \frac{\epsilon_r p_\mu \gamma^\mu + m_0}{2m_0} \tag{D.223}
$$

and that it is Lorentz covariant. We check that it has all the properties of a projection operator. Obviously

$$
\hat{\Lambda}_+(p) + \hat{\Lambda}_-(p) = \frac{+p_\mu \gamma^\mu + m_0}{2m_0} + \frac{-p_\mu \gamma^\mu + m_0}{2m_0 c} = 1
$$

We now establish $(\hat{\Lambda}_+)^2 = \hat{\Lambda}_+$, $(\hat{\Lambda}_-)^2 = \hat{\Lambda}_-$, and $\hat{\Lambda}_+ \hat{\Lambda}_- = 0$. This is done with the help of

$$\begin{aligned}
p^\mu \gamma_\mu p^\nu \gamma_\nu &= \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) p^\mu p^\nu \\
&= \eta_{\mu\nu} p^\mu p^\nu \\
&= E^2 - \mathbf{p}^2 \\
&= (m_0^2 + \mathbf{p}^2) - \mathbf{p}^2 = m_0^2. \qquad \text{(D.224)}
\end{aligned}$$

Now

$$\begin{aligned}
\hat{\Lambda}_r(p)\hat{\Lambda}_{r'}(p) &= \frac{(\epsilon_r p^\mu \gamma_\mu + m_0)(\epsilon_{r'} p^\mu \gamma_\mu + m_0)}{4m_0^2} \\
&= \frac{(\epsilon_r \epsilon_{r'} p^\mu p^\nu \gamma_\mu \gamma_\nu + m_0^2 + (\epsilon_r + \epsilon_{r'}) m_0 p^\mu \gamma_\mu}{4m_0^2} \\
&= \frac{m_0^2(1 + \epsilon_r \epsilon_{r'}) + m_0 p^\mu \gamma_\mu \epsilon_r (1 + \epsilon_r \epsilon_{r'})}{4m_0^2} \\
&= \frac{1 + \epsilon_r \epsilon_{r'}}{2} \frac{\epsilon_r p_\mu \gamma^\mu + m_0}{2m_0} = \frac{1 + \epsilon_r \epsilon_{r'}}{2} \hat{\Lambda}_r(p). \qquad \text{(D.225)}
\end{aligned}$$

Projection Operators for Spin

In the non-relativistic limit the operator for "spin up" or "spin down"

$$\hat{P}_\pm = \frac{1 \pm \hat{\sigma}_3}{2}$$

We can generalise this to a spin-projection operator in an arbitrary direction

$$\hat{P}(\mathbf{u}) = \frac{1 + \hat{\sigma} \cdot \mathbf{u}}{2} \qquad \text{(D.226)}$$

where $\mathbf{u}$ is a unit vector. We need the relativistic generalisation of this. To that end introduce the four-vector

$$u_z^\nu \qquad \text{(D.227)}$$

which in the rest system of the electron is

$$(u_z^\nu)_{R.S.} = (0, 0, 0, 1) = (0, 0, 0, \mathbf{u}_z) \qquad \text{(D.228)}$$

$$
\begin{aligned}
\gamma_5\gamma_3(u_z^3)_{R.S.} &= \gamma_5\gamma_3 \\
&= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma_3 \\
&= i\gamma^0\gamma^1\gamma^2 \\
&= i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \qquad \text{(D.229)}
\end{aligned}
$$

In the rest frame we have for positive-energy $\omega^{1,2}(0)$

$$
\begin{aligned}
\hat{\Sigma}(u_z^3)\omega^{1,2}(0) &= \frac{1+\gamma_5\gamma^3(u_z^3)_{R.S.}}{2}\omega^{1,2}(0) \\
&= \frac{1}{2}\begin{pmatrix} I+\sigma_3 & 0 \\ 0 & I-\sigma_3 \end{pmatrix}\omega^{1,2}(0) \\
&= \frac{1}{2}\left\{\begin{array}{l} 1\cdot\omega^1(0) \\ 0\cdot\omega^2(0) \end{array}\right. . \qquad \text{(D.230)}
\end{aligned}
$$

In the rest frame we have for negitive-energy

$$
\begin{aligned}
\hat{\Sigma}(u_z^3)\omega^{3,4}(0) &= \frac{1+\gamma_5\gamma^3(u_z^3)_{R.S.}}{2}\omega^{3,4}(0) \\
&= \frac{1}{2}\begin{pmatrix} I+\sigma_3 & 0 \\ 0 & I-\sigma_3 \end{pmatrix}\omega^{3,4}(0) \\
&= \frac{1}{2}\left\{\begin{array}{l} 0\cdot\omega^3(0) \\ 1\cdot\omega^4(0) \end{array}\right. \qquad \text{(D.231)}
\end{aligned}
$$

The projection of negative-energy states arre opposite to those of positive-energy states. The opposite occurs because the spin of the missing particle of spin $\uparrow$ corresponds to a particle of spin $\downarrow$.

We generalise the spin projection operator for an arbitrary spin vector s^μ with $s^\mu p_\mu = 0$:

$$
\hat{\Sigma}(s) = \frac{1}{2}(1+\gamma_5 s^\mu\gamma_\mu) \qquad \text{(D.232)}
$$

We show that it is a true projection operator. We have

$$
\hat{\Sigma}(s) + \hat{\Sigma}(-s) = 1 \qquad \text{(D.233)}
$$

$$
\begin{aligned}
\hat{\Sigma}^2(s) &= \frac{1}{4}(1+\gamma_5 s^\mu\gamma_\mu)(1+\gamma_5 s^\nu\gamma_\nu) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu + s^\mu s^\nu\gamma_5\gamma_\mu\gamma_5\gamma_\nu) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu - s^\mu s^\nu\gamma_5^2\gamma_\mu\gamma_\nu) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu - s^\mu s^\nu\gamma_5^2\frac{1}{2}(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu - s\cdot s) \\
&= \frac{1}{2}(1+\gamma_5 s^\mu\gamma_\mu) = \hat{\Sigma}(s)
\end{aligned} \tag{D.234}
$$

Similarly $\hat{\Sigma}^2(-s) = \hat{\Sigma}(-s)$.

$$
\begin{aligned}
\hat{\Sigma}(s)\hat{\Sigma}(-s) &= \frac{1}{4}(1+\gamma_5 s^\mu\gamma_\mu)(1-\gamma_5 s^\nu\gamma_\nu) \\
&= \frac{1}{4}(1+s\cdot s) = 0.
\end{aligned} \tag{D.235}
$$

Simjultaneous Projection Operators for Energy and Spin

$$
\begin{aligned}
p_\mu\gamma^\mu\gamma_5\gamma^\nu s_\nu &= p_\mu\gamma^\mu i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\nu s_\nu \\
&= -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^\nu s_\nu p_\mu \\
&= -\gamma_5(2\eta^{\mu\nu}-\gamma^\nu\gamma^\mu)s_\nu p_\mu \\
&= s^\mu p_\mu\gamma_5 + s_\nu p_\mu\gamma_5\gamma^\nu\gamma^\mu \\
&= \gamma_5 s_\nu\gamma^\nu p_\mu\gamma^\mu
\end{aligned} \tag{D.236}
$$

implies

$$
\left[\hat{\Sigma}(s), \hat{\Lambda}_\pm(p)\right] = 0 \tag{D.237}
$$

$$
\begin{aligned}
\hat{P}_1 &= \hat{\Lambda}_+(p)\hat{\Sigma}(u_z) \\
\hat{P}_2 &= \hat{\Lambda}_+(p)\hat{\Sigma}(-u_z) \\
\hat{P}_3 &= \hat{\Lambda}_-(p)\hat{\Sigma}(u_z) \\
\hat{P}_4 &= \hat{\Lambda}_-(p)\hat{\Sigma}(-u_z)
\end{aligned} \tag{D.238}
$$

D.2.16 Summary

Maxwell's equations with source are

$$\Box A^{\mu} - \partial^{\mu}(\partial_{\nu} A^{\nu}) = j^{\mu} \tag{D.239}$$

where we are free to perform gauge transformations

$$A^{\mu} \to A^{\mu'} = A^{\mu} + \partial^{\mu}\Lambda. \tag{D.240}$$

The free Dirac equation can be written

$$i\hbar\frac{\partial\psi(x)}{\partial t} = [\alpha\cdot(-i\hbar\nabla) + \beta m_0]\psi(x) \tag{D.241}$$

where $= (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ and $\hat{\beta}$ are 4×4 Hermitian matrices satisfying

$$\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = 2\delta_{ij}, \quad \hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i = 0, \quad \beta^2 = 1, \quad i = 1, 2, 3. \tag{D.242}$$

With

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i \tag{D.243}$$

Dirac's equation becomes

$$i\hbar\gamma^{\mu}\frac{\partial\psi(x)}{\partial x^{\mu}} - m_0\psi(x) = 0 \tag{D.244}$$

with the 4×4 matrices γ^{μ}, $\mu = 0, \ldots, 3$, satifying the anticommutation relations

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} \tag{D.245}$$

and Hermiticity conditions

$$\gamma^{\mu\dagger} = \gamma^0\gamma^{\mu}\gamma^0. \tag{D.246}$$

Coupling of a Spinor to the electromagnetic field is given by

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu(x) - m_0)\psi(x) = 0. \tag{D.247}$$

The four current density of the dirac field is given by

$$j^\mu = \overline{\psi}\gamma^\mu \psi \tag{D.248}$$

The plane wave for a photon is

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{D.249}$$

Plane waves of electrons: incoming

$$\psi(x) = \sqrt{\frac{m_0}{EV}}\, u(p,s)e^{-ip\cdot x} \tag{D.250}$$

and outgoing

$$\overline{\psi}(x) = \sqrt{\frac{m_0}{EV}}\, \overline{u}(p,s)e^{ip\cdot x}. \tag{D.251}$$

The general free solution has the form

$$\psi^r(x) = \omega^r(p)e^{-i\epsilon_r p_\mu x^\mu} \tag{D.252}$$

where

$$[\omega^1(p), \omega^2(p), \omega^3(p), \omega^4(p)] = \sqrt{\frac{E+m_0}{2m_0}} \begin{bmatrix} 1 & 0 & \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} \\ 0 & 1 & \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} \\ \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} & 1 & 0 \\ \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} & 0 & 1 \end{bmatrix} \tag{D.253}$$

The orthogonality condition for spinors

$$\omega^{r\dagger}(\epsilon_r p)\omega^{r'}(\epsilon_r{}' p) = \frac{E_p}{m_0}\delta_{rr'} \tag{D.254}$$

The completeness relation for spinors

$$\sum_{r=1}^{4} \epsilon_r \omega^r(p) \overline{\omega}^r_\alpha(p) = \delta_{\alpha\beta} \tag{D.255}$$

There are projection operators for Energy

$$\hat{\Lambda}_{\pm}(p) = \frac{\pm p_\mu \gamma^\mu + m_0}{2m_0} \tag{D.256}$$

and spin

$$\hat{\Sigma} = \frac{1}{2}(1 + \gamma_5 s_\mu \gamma^\mu) \tag{D.257}$$

such that

$$\hat{\Sigma} u(p, +s) = u(p, +s), \quad \hat{\Sigma} u(p, -s) = 0. \tag{D.258}$$

The basic bilinear covariants of Dirac theory are

$$\begin{array}{rl}
\overline{\psi}\psi & \text{scalar} \\
\overline{\psi}\gamma^\mu\psi & \text{vector} \\
\overline{\psi}\sigma^{\mu\nu}\psi & \text{antisymmetric second-rank tensor} \\
\overline{\psi}\gamma^5\gamma^\mu\psi & \text{pseudo-vector} \\
\overline{\psi}\gamma^5\psi & \text{pseudo-scalar}
\end{array} \tag{D.259}$$

D.3 Perturbation Theory

D.3.1 Non-Relativistic Green's Function

Given Schrodinger's equation

$$\left(i\hbar \frac{\partial}{\partial t} - \hat{H}_0(x) - V(x) \right) \psi(x) = 0 \tag{D.260}$$

The retarded Green's function is defined by the differential equation

$$\left(i\hbar\frac{\partial}{\partial t'}-\hat{H}_0(x')-V(x')\right)G^+(x';x)=\delta^4(x'-x) \tag{D.261}$$

and the boundary condition

$$G^+(x';x)=0 \quad \text{for } t'<t. \tag{D.262}$$

Free Green's function in momentum space

$$\left(i\hbar\frac{\partial}{\partial t'}-\hat{H}_0(x')\right)G^+(x';x)=\delta^4(x'-x) \tag{D.263}$$

where

$$\hat{H}_0=-\frac{\hbar^2}{2m}\nabla'^2 \tag{D.264}$$

As the above differential equation can be turned into an algebraic equation in energy-momentum space, we write

$$G_0^+(x'-x)=\int\frac{d^3p dE}{(2\pi\hbar)^4}\exp\left[-\frac{i}{\hbar}E(t'-t)\right]\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right]\ G_0^+(p;E) \tag{D.265}$$

and apply

$$\begin{aligned}
&\left(i\hbar\frac{\partial}{\partial t'}+\frac{\hbar^2}{2m}\nabla'^2\right)G_0^+(x'-x)\\
=&\int\frac{d^3p dE}{(2\pi\hbar)^4}\left\{\left(E-\frac{\mathbf{p}^2}{2m}\right)G_0^+(p;E)\right\}\exp\left[-\frac{i}{\hbar}E(t'-t)\right]\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right]\\
=&\ \hbar\delta^4(x'-x)
\end{aligned} \tag{D.266}$$

We will recall the δ-function integral representation

$$\int\frac{d^3p dE}{(2\pi\hbar)^4}\exp\left[-\frac{i}{\hbar}E(t'-t)\right]\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right]=\delta^4(x'-x)$$

Therefore for $E\neq\mathbf{p}^2/2m$ we obtain

$$G_0^+(\mathbf{p}, E) = \frac{\hbar}{E - \frac{\mathbf{p}^2}{2m}} \tag{D.267}$$

How do we deal with the singularity when we do the inverse Fourier transformation? The clue how to proceed comes from the integral representation of the step function:

$$\Theta(\tau) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon}. \tag{D.268}$$

By adding a small imaginary part $i\epsilon$ to the energy one will obtain the retardation condition (D.262), while the resulting Green's function still satisfies the Green's function differential equation (D.263) in the limit $\epsilon \to 0$. Write

$$G_0^+(x' - x) = \hbar \int \frac{d^3p}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar}\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})\right] \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \frac{\exp\left[-iE(t' - t)/\hbar\right]}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon} \tag{D.269}$$

With the substitution $E' = E - \mathbf{p}^2/2m$ the last integral becomes

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{dE'}{2\pi\hbar} \frac{\exp[-i(E' + \mathbf{p}^2/2m)(t' - t)/\hbar]}{E' + i\epsilon} \\
= \ & \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \int_{-\infty}^{\infty} \frac{dE'}{2\pi\hbar} \frac{\exp[-iE'(t' - t)/\hbar]}{E' + i\epsilon} \\
= \ & \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \frac{-i}{\hbar} \cdot \frac{-1}{2\pi i} \int_{-\infty}^{\infty} dE' \frac{\exp[-iE'(t' - t)/\hbar]}{E' + i\epsilon} \\
= \ & \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \left[-\frac{i}{\hbar}\Theta\left(\frac{t' - t}{\hbar}\right)\right] \\
= \ & -\frac{i}{\hbar} \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \Theta(t' - t)
\end{aligned} \tag{D.270}$$

We do indeed recover the retardation condition. Now (D.269) becomes

$$G_0^+(x' - x) = -i\Theta(t' - t) \int \frac{d^3p}{(2\pi\hbar)^3} \exp\left\{\frac{i}{\hbar}\left[\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x}) - \frac{\mathbf{p}^2}{2m}(t' - t)\right]\right\} \tag{D.271}$$

This can be expressed in terms of plane waves of the free Schrodinger's equation. The δ-function normalised plane waves are

$$
\begin{aligned}
\phi_p(\mathbf{x},t) &= \frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{i}{\hbar}\left((\mathbf{p}\cdot\mathbf{x}-\frac{\mathbf{p}^2}{2m}t\right)\right] \\
&= \frac{1}{\sqrt{2\pi\hbar}}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]
\end{aligned} \tag{D.272}
$$

where

$$
\hbar\omega = \frac{\mathbf{p}^2}{2m}, \quad \hbar\mathbf{k} = \mathbf{p}.
$$

$$
G_0^+(x'-x) = -i\Theta(t'-t)\int d^3p\, \phi_p(\mathbf{x}',t')\phi_p^*(\mathbf{x},t) \tag{D.273}
$$

Full Green's function in terms of plane waves

We give a proof that the Green's function can be written as

$$
G^+(x';x) = -i\Theta(t'-t)\sum_n \psi_n^*(x)\psi_n(x'). \tag{D.274}
$$

where $\psi_n(\mathbf{x},t)$ are a complete set of eigenfunctions of Schrodinger's equation. We do this using the closure relation

$$
\sum_n \psi_n^*(\mathbf{x},t)\psi_n(\mathbf{x}',t) = \delta^3(\mathbf{x}'-\mathbf{x}). \tag{D.275}
$$

for the eigenfunctions of Schrodinger's equation

$$
\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)\psi_n(x') = 0. \tag{D.276}
$$

Using this we have

$$
\begin{aligned}
\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)G^+(x';x) &= \hbar\delta(t'-t)\sum_n \psi_n^*(\mathbf{x},t)\psi_n(\mathbf{x}',t) \\
&\quad -i\Theta(t'-t)\sum_n\left[\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)\psi_n(x')\right]\psi_n^*(x) \\
&= \hbar\delta(t'-t)\delta^3(\mathbf{x}'-\mathbf{x}) \\
&= \hbar\delta^4(x'-x).
\end{aligned} \tag{D.277}
$$

We have the relations describing the evolution of solutions of Schrodinger's equation:

$$
\begin{aligned}
i\int d^3x G^+(x';x)\psi_n(x) &= \Theta(t'-t)\sum_m \psi_m(x')\int d^3x\psi_m^*(x)\psi_n(x) \\
&= \Theta(t'-t)\psi_n(x')
\end{aligned} \tag{D.278}
$$

and on the other hand

$$
\begin{aligned}
i\int d^3x\psi_n^*(x')G^+(x';x) &= \Theta(t'-t)\sum_m \int d^3x'\psi_m(x')\psi_n^*(x')\ \psi_m^*(x) \\
&= \Theta(t'-t)\psi_n^*(x)
\end{aligned} \tag{D.279}
$$

The first of these relations expresses the propagation of $\psi_n(x)$ forward in time and the second corresponding backward propagation of $\psi_n^*(x')$.

Perturbation theory

$$
\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0\right)G^+(x';x) = \delta^4(x'-x) + V(x')G^+(x';x) \tag{D.280}
$$

The RHS can be interpreted as the source term in an inhomogeneous Schrodinger equation:

$$
\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0(x')\right)G^+(x';x) = \rho(x';x) \tag{D.281}
$$

Using the free Green's function G_0 the solution is

$$
G^+(x';x) = \int d^4x_1 G_0^+(x';x_1)\rho(x_1;x) \tag{D.282}
$$

This leads to the following integral equation for the integrating Green's function

$$
\begin{aligned}
G^+(x';x) &= \int d^4x_1 G_0^+(x';x_1)\big(\delta^4(x_1-x) + V(x_1)G^+(x_1;x)\big) \\
&= G_0^+(x';x) + \int d^4x_1 G_0^+(x';x_1)V(x_1)G^+(x_1;x)
\end{aligned} \tag{D.283}
$$

Repeatedly substituting this equation into itself we obtain perturbative series

$$
\begin{aligned}
G^{+}(x';x) &= G_0^{+}(x';x) + \int d^4x_1 G_0^{+}(x';x_1)V(x_1)G^{+}(x_1;x) \\
&= G_0^{+}(x';x) + \int d^4x_1 G_0^{+}(x';x_1)V(x_1)G_0^{+}(x_1;x) \\
&\quad + \int d^4x_1 d^4x_2 G_0^{+}(x';x_1)V(x_1)G_0^{+}(x_1;x_2)V(x_2)G_0^{+}(x_2;x) \\
&\quad + \dots \quad .
\end{aligned}
\tag{D.284}
$$

Boundary condition

$$
\begin{aligned}
\psi(x') &= \lim_{t\to-\infty} i \int d^3x G^{+}(x';x)\phi(x) \\
&= \lim_{t\to-\infty} i \int d^3x \left(G_0^{+}(x';x) + \int d^4x_1 G_0^{+}(x';x_1)V(x_1)G^{+}(x_1;x) \right) \phi(x) \\
&= \phi(x') + \lim_{t\to-\infty} \int d^4x_1 G_0^{+}(x';x_1)V(x_1) i \int d^3x G^{+}(x_1;x)\phi(x) \\
&= \phi(x') + \lim_{t\to-\infty} \int d^4x_1 G_0^{+}(x';x_1)V(x_1)\psi(x_1)
\end{aligned}
\tag{D.285}
$$

The second the on the RHS is the scattered wave.

We consider a scattering problem where no interaction occurs in the distant past and future:

$$
V(\mathbf{x},t) \to 0 \quad \text{for} \quad t \to \mp\infty \tag{D.286}
$$

The initial wave ϕ is therefore a solution of the Schrodinger equation for free particles, which fulfills the initial conditions of the experiment. The exact wavefunction $\psi(\mathbf{x},t)$ then approaches the incoming wave $\phi(\mathbf{x},t)$ in the limit $t \to -\infty$:

$$
\psi(\mathbf{x},t) \to \phi(\mathbf{x},t). \tag{D.287}
$$

The scattering matrix

Let $\phi_i(x)$ and $\phi_f(x)$ denote the intial and final free wave with quantum numbers i and f that are emitted, observed at the begining, end of the scattering process respectively. The full wavefunction $\psi_i(x)$ is given in terms the integral equation,

$$\psi_i(x) = \phi_i(x) + \int d^4x_1 G_0^+(x, x_1)V(x_1)\psi_i(x_1) \tag{D.288}$$

The wavefunction $\psi_i(x)$ satifies the boundary condition $\psi_i(\mathbf{x}, t) \to \phi_i(\mathbf{x}, t)$ for $t \to -\infty$. The scattering matrix results from the projection of $\psi_i(x)$ on the final state $\phi_f(\mathbf{x}, t)$

$$S_{fi} = \lim_{t\to+\infty} \left\langle \phi_f(x) \middle| \psi_i(x) \right\rangle \tag{D.289}$$

$$\begin{aligned} S_{fi} &= \lim_{t\to+\infty} \left\langle \phi_f(x) \middle| \phi_i(x) + \int d^4x_1 G_0^+(x, x_1)V(x_1)\psi(x_1) \right\rangle \\ &= \delta_{fi} + \lim_{t\to+\infty} \int d^3x \phi_f^*(x) \int d^4x_1 G_0^+(x, x_1)V(x_1)\psi(x_1) \\ &= \delta_{fi} + \lim_{t\to+\infty} \int d^4x_1 \left(\int d^3x \phi_f^*(x) G_0^+(x, x_1) \right) V(x_1)\psi(x_1) \end{aligned} \tag{D.290}$$

Using the equations (D.279) for free particles we obtain

$$\int d^3x \phi_f^*(x) G_0^+(x'; x_1) = -i\phi_f(x_1) \quad \text{for } t' > t_1, \tag{D.291}$$

so the x integral can be carried out resulting in

$$S_{fi} = \delta_{fi} - i \lim_{t\to+\infty} \int d^4x_1 \phi_f^*(x_1)V(x_1)\psi(x_1) \tag{D.292}$$

Now repeatedly substituting (D.288) into this we obtain

$$\begin{aligned} S_{fi} &= \delta_{fi} - i \int d^4x_1 \phi_f^*(x_1)V(x_1)\phi_i(x_1) \\ &\quad -i \int d^4x_1 d^4x_2 \phi_f^*(x_1)V(x_1)G_0^+(x_1; x_2)V(x_2)\phi_i(x_2) \\ &\quad -i \int d^4x_1 d^4x_2 \phi_f^*(x_1)V(x_1)G_0^+(x_1; x_2)V(x_2)G_0^+(x_2; x_3)V(x_3)\phi_i(x_3) \\ &\quad + \ldots \quad . \end{aligned} \tag{D.293}$$

Each line represents a free Green's function $G_0^+(x_i; x_{i-1})$, i.e. the amplitude that a particle wave originating at the spacetime point x_{i-1} and propagates freely to the spacetime point x_i. At the point x_i the particle wave is scattered with probability amplitude $V(x_i)$ per

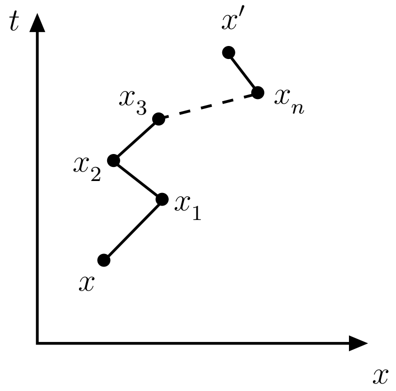


Figure D.1: nth-order Green's function as the probability amplitude for multiple scattering.

unit spacetime volume. Such points are called interaction vertices and are denoted by filled-in circles. The resulting scattered wave then again propagates freely foward in time (recall $G_0^+(x_{i+1}; x_i) = 0$ for $t_{i+1} < t_i$) from the spacetime point x_i towards the point x_{i+1} with the amplitude $G_0^+(x_{i+1}; x_i)$ where the next interaction takes place, and so on.

D.3.2 The Electron and Positron Propagator

Differential equation for relativistic propagator

Let us introduce the relativistic propagator

$$S_F(x', x; A) \tag{D.294}$$

in analogy to the nonrelativistic propagator, by satisfying the following differential equation

$$\left[\gamma_\mu \left(i\hbar \frac{\partial}{\partial x'_\mu} - \frac{e}{c} A^\mu(x')\right) - m_0\right] S_F(x', x; A) = \hbar \delta^4(x' - x) \mathbf{1}. \tag{D.295}$$

We from now on use natural units

$$\frac{e}{\hbar c} \to e, \quad \frac{m_0 c}{\hbar} \to m_0. \tag{D.296}$$

Thus we write

$$\left[\gamma_\mu\left(i\partial'^\mu - eA^\mu(x')\right) - m_0\right] S_F(x', x; A) = \delta^4(x' - x). \tag{D.297}$$

where we have suppressed the unit matrix, however, it must be kept in mind that we are dealing with a matrix equation.

The free-particle propagator satisfies (D.297) with the interaction term $\gamma_\mu A^\mu(x')$ absent, i.e.

$$(i\gamma_\mu\partial'^\mu - m_0)S_F(x', x) = \delta^4(x' - x). \tag{D.298}$$

As in the non-relativistic case we calculate $S_F(x', x)$ in momentum space.

Non-interacting propagator in momentum space

$$S_F(x', x) = S_F(x' - x) = \int \frac{d^4p}{(2\pi)^4} \exp[-ip\cdot(x' - x)]\; S_F(p) \tag{D.299}$$

which implies that

$$(p^\mu\gamma_\mu + m_0)S_F(p) = \mathbf{1} \tag{D.300}$$

This can be solved for $S_F(p)$ by multiplying by $(p^\mu\gamma_\mu + m_0)$ from the left

$$(p^\mu\gamma_\mu + m_0)(p^\nu\gamma_\nu - m_0)S_F(p) = (p^\mu\gamma_\mu + m_0) \tag{D.301}$$

Since

$$p^\mu\gamma_\mu p^\nu\gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)p^\mu p^\nu = \eta_{\mu\nu}p^\mu p^\nu = p_\mu p^\mu = p^2$$

we then have

$$(p^2 - m_0^2)S_F(p) = p^\mu\gamma_\mu + m_0$$

so

$$S_F(p) = \frac{p^\mu\gamma_\mu + m_0}{p^2 - m_0^2} \quad \text{for } p^2 \neq m_0^2 \tag{D.302}$$

Let us consider the inverse Fourier transformation.

$$\begin{aligned}
S_F(x'-x) &= \int \frac{d^4}{(2\pi)^4} S_F(p) \exp[-ip\cdot(x'-x)] \\
&= \int \frac{d^4}{(2\pi)^4} S_F(p) \exp\{[-ip_0\cdot(t'-t) - \mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\} \\
&= \int \frac{d^3}{(2\pi)^3} S_F(p) \exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})] \times \int_C \frac{dp_0}{2\pi} \frac{\exp[-ip_0\cdot(t'-t)]}{p^2-m_0^2}
\end{aligned} \tag{D.303}$$

where C is contour of integration choosen to avoid the singularities of $S_F(p)$. As we know from the nonrelativistic case the choice of contour encodes the boundary conditions imposed on $S_F(x'-x)$.

Propagator describing positive-energy particle waves

Considering the particle's propagation forward in time implies that $t'-t$ is positive so that the p_0 integration must be performed along the contour closed in the lower half plane as this gives zanishing contribution. Then the only pole is at

$$p_0 = +E_p = +\sqrt{\mathbf{p}^2+m_0^2}.$$

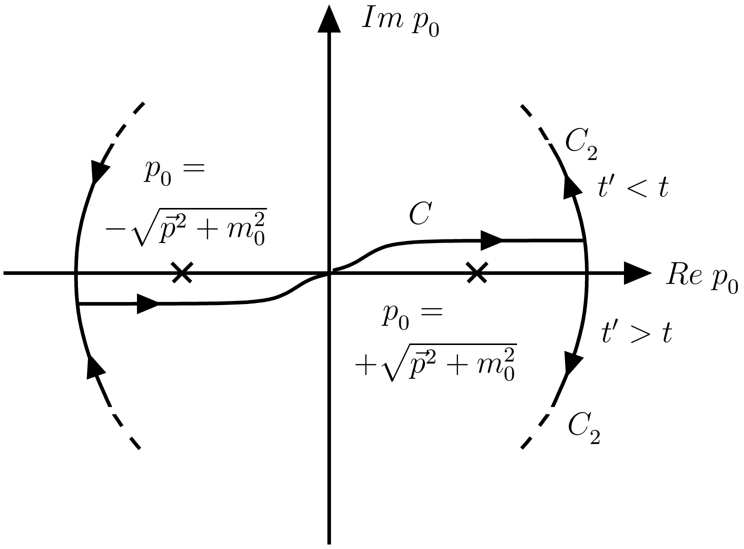


Figure D.2:

The propagator is then

$$
\begin{aligned}
S_F^{(t'>t)}(x'-x) &= -i\int \frac{d^3p}{(2\pi)^3}\exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\exp[-iE_p(t'-t)] \\
&\quad \times\frac{(E_p\gamma^0-\mathbf{p}\cdot\gamma+m_0)}{2E_p} \qquad \text{for} \quad t'>t. \qquad \text{(D.304)}
\end{aligned}
$$

Instead of deforming the contour as in fig (D.3.2), we can move the poles an infintesimal distance η off the real axis, as shown in fig (), and perform the p_0 integration alng the whole real axis

Propagator describing negative-energy particle waves

On the other hand, considering the particle's propagation backward in time implies that $t'-t$ is negative so that the p_0 integration must be performed along the contour closed in the upper half plane as this gives zanishing contribution. Then the only pole is at

$$p_0 = -E_p = -\sqrt{\mathbf{p}^2+m_0^2}.$$

$$
\begin{aligned}
S_F^{(t>t')}(x'-x) &= -i\int \frac{d^3p}{(2\pi)^3}\exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\exp[+iE_p(t'-t)] \\
&\quad \times\frac{(-E_p\gamma^0-\mathbf{p}\cdot\gamma+m_0)}{2E_p} \qquad \text{for} \quad t'<t. \qquad \text{(D.305)}
\end{aligned}
$$

D.3.3 Propagating Positive and Negative Particles

We combine the two propagators describing positive-energy particle waves and negative-energy particle waves moving forward and backward in time, respectively.

$$S_F(x'-x) = S_F^{(t'>t)}(x'-x) + S_F^{(t>t')}(x'-x) \qquad \text{(D.306)}$$

$$
\begin{aligned}
S_F(x'-x) &= -i\int \frac{d^3p}{(2\pi)^3} \\
&\quad \left\{ \exp[-i(+E_p)(t'-t)]\exp[+i\vec{p}\cdot(\vec{x}'-\vec{x})]\frac{(+E_p\gamma^0+p_i\gamma^i+m_0)}{2E_p}\,\Theta(t'-t) \right. \\
&\quad \left. +\exp[-i(-E_p)(t'-t)]\exp[-i\vec{p}\cdot(\vec{x}'-\vec{x})]\frac{(-E_p\gamma^0-p_i\gamma^i+m_0)}{2E_p}\Theta(t-t') \right\} \\
&= -i\int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\left\{\frac{p_\mu\gamma^\mu+m_0}{2m_0}\exp[-ip\cdot(x'-x)]\Theta(t'-t)\right. \\
&\qquad\qquad + \frac{-p_\mu\gamma^\mu+m_0}{2m_0}\exp[ip\cdot(x'-x)]\Theta(t'-t)) \\
&= -i\int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}(\hat{\Lambda}_+(p)\exp[-ip\cdot(x'-x)]\Theta(t-t') \\
&\qquad\qquad +\hat{\Lambda}_-(p)\exp[ip\cdot(x'-x)]\Theta(t'-t))
\end{aligned}
\tag{D.307}
$$

Free propagator in terms of plane waves

This can also by written in terms of the normalised Dirac plane waves

$$
\begin{aligned}
S_F(x'-x) &= -i\Theta(t'-t)\int d^3p\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x) \\
&\quad +i\Theta(t-t')\int d^3p\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x)
\end{aligned}
\tag{D.308}
$$

Proof:

$$
\begin{aligned}
\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x) &= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\sum_{r=1}^{2}\omega^r(p)\overline{\omega}^r(p) \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\underbrace{\sum_{r=1}^{4}\epsilon_r\omega^r(p)\overline{\omega}^r(p)}_{=1}\frac{p_\mu\gamma^\mu+m_0}{2m_0} \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\ \frac{p_\mu\gamma^\mu+m_0}{2m_0} \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\ \hat{\Lambda}_+(p)
\end{aligned}
$$

A similary calulation for the secod part gives

$$\sum_{r=3}^{4} \psi_p^r(x')\overline{\psi}_p^r(x) = -\frac{1}{(2\pi)^3}\frac{m_0}{E_p} \exp[ip\cdot(x'-x)]\ \hat{\Lambda}_-(p) \tag{D.309}$$

$\square$

Using this we easily verify:

$$\Theta(t'-t)\psi^{(+E)}(x') = i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x), \tag{D.310}$$

$$\Theta(t-t')\psi^{(-E)}(x') = -i\int d^3x S_F(x'-x)\gamma_0\psi^{(-E)}(x), \tag{D.311}$$

(see section D.14.9). Equation (D.310) explicitly expresses the interpretation of electrons in terms of positive-energy solutions propagating forward in time and equation (D.311) the interpretation of positrons in terms of negative-energy solutions moving backward in time.

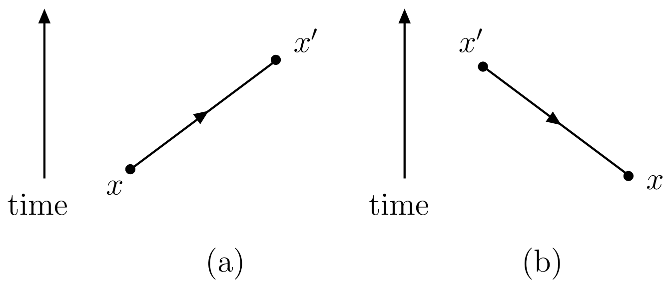


Figure D.3: (a) $t < t'$: an electron propagated from x to x'. (b) $t > t'$: a positron propagated from x' to x.

The reader should be warnerd no to take this pictorial description of the mathematics as a literal process in space and time. For example, for x and x' with space-like separation, our naive interptetation of propagation would imply the electron/positron travels between the two points with speed greater than the speed of light.

D.3.4 Perturbation Expansion for the Stuckelberg-Feynmann Propagator

Equations () or () determine the free-particle propagator of the electron-positron theory. Here we develop a perturbative expansion for how this is modified to the exact propagator in the presence of an electromagnetic potential, the so-called Stuckelberg-Feynmann propagator, $S_F(x', x; A)$.

$$(i\gamma_\mu \partial^\mu - m_0) S_F(x', x; A) = \delta^4(x' - x) + eA_\mu(x')\gamma^\mu S_F(x', x; A) \tag{D.312}$$

This can be viewed as in inhomogeneous Dirac equation of the form

$$(i\gamma_\mu \partial^\mu - m_0)\Psi(x) = \rho(x) \tag{D.313}$$

which is solved by

$$\Psi(x) = \Psi_0(x) + \int d^4y S_F(x - y)\rho(y) \tag{D.314}$$

As is easily seen:

$$\begin{aligned}
(i\gamma_\mu \partial_x^\mu - m_0)\Psi(x) &= (i\gamma_\mu \partial_x^\mu - m_0)\Psi_0(x) + \int d^4y (i\gamma_\mu \partial_x^\mu - m_0) S_F(x - y)\rho(y) \\
&= \int d^4y \delta^4(x - y)\rho(y) \\
&= \rho(x)
\end{aligned} \tag{D.315}$$

In this way we obtain an integral equation for $S_F(x', x; A)$

$$\begin{aligned}
S_F(x', x; A) &= \int d^4y S_F(x' - y) \left[\delta^4(y - x) + eA_\mu(x')\gamma^\mu S_F(y, x; A) \right] \\
&= S_F(x' - y) + e \int d^4y S_F(x' - y) A_\mu(x')\gamma^\mu S_F(y, x; A).
\end{aligned} \tag{D.316}$$

Repeatedly substituting this equation into itself we obtain

$$\begin{aligned}
S_F(x', x; A) &= \int d^4y S_F(x' - y) + e \int d^4x_1 S_F(x' - x_1) A_\mu(x_1)\gamma^\mu S_F(x_1 - x) \\
&\quad + e^2 \int d^4x_1 d^4x_2 S_F(x' - x_1) A_\mu(x_1)\gamma^\mu S_F(x_1 - x_2) A_\nu(x_2)\gamma^\nu S_F(x_2 - x) \\
&\quad + \ldots \ .
\end{aligned} \tag{D.317}$$

Boundary condition of Feynman and Stuckelberg

$$\Psi(x) = \psi(x) + \int S_F(x-y) e\gamma_\mu A^\mu(y)\Psi(y) \tag{D.318}$$

The second term on the RHS represents the scattered wave.

Now by () $t \equiv x^0 \to +\infty$

$$S_F(x-y) \to -i \int d^3p \sum_{r=1}^{2} \psi_p^r(x)\overline{\psi}_p^r(y)$$

and $t \equiv x^0 \to -\infty$

$$S_F(x-y) \to +i \int d^3p \sum_{r=3}^{4} \psi_p^r(x)\overline{\psi}_p^r(y)$$

So that

$$\Psi(x) - \psi(x) \to \int d^3p \sum_{r=1}^{2} \psi_p^r(x) \left(-ie \int d^4y\, \overline{\psi}_p^r(x) A_\mu(y)\gamma^\mu \Psi(y) \right) \quad \text{for} \quad t \to +\infty \tag{D.319}$$

and

$$\Psi(x) - \psi(x) \to \int d^3p \sum_{r=3}^{4} \psi_p^r(x) \left(+ie \int d^4y\, \overline{\psi}_p^r(x) A_\mu(y)\gamma^\mu \Psi(y) \right) \quad \text{for} \quad t \to -\infty \tag{D.320}$$

Therefore the scattered wave contains only positive frequencies

D.3.5 The $S-$Matrix Elements

The S-matrix elements are defined in the same manner as in the nonrelativistic case.

Let $\psi_f(x)$ denote the final free wave with quantum numbers f that is observed at the end of the scattering process.

$$
\begin{aligned}
S_{fi} &= \lim_{t\to\pm\infty} < \psi_f(x)|\Psi_i(x) > \\
&= \lim_{t\to\pm\infty} \left\langle \psi_f(x) \middle| \psi_i(x) + \int d^4y S_F(x-y) e A_\mu(y)\gamma^\mu \Psi_i(x) \right\rangle
\end{aligned} \tag{D.321}
$$

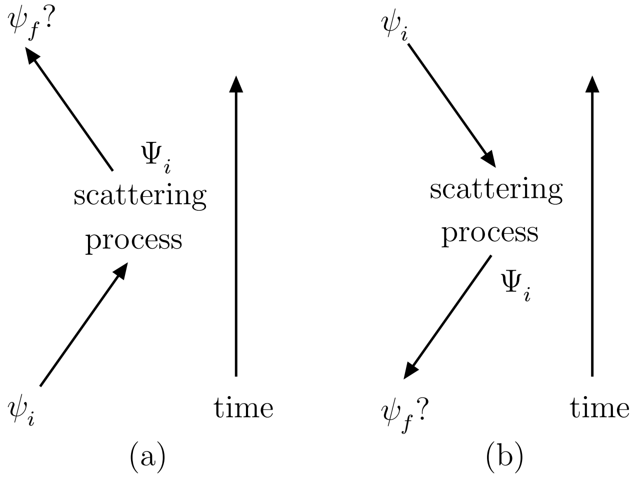


Figure D.4: $\Psi_i(x)$ stands for the incoming wave, which either reduces at $y_0 \to -\infty$ to an incident positive energy wave $\psi_i(x)$ or at $y_0 \to +\infty$ to an incident negative energy wave $\psi_i(x)$. (a) ψ_f describes an electron in the limit $t \to +\infty$. (b) ψ_f describes a positron in the limit $t \to -\infty$.

There are four basic processes to consider: (a) electron scattering; (b) positron scattering; (c) electron-positron pair creation; (d) pair annihilation.

We will need the following relations for adjoint spinors (proven in section D.14.9).

$$
\Theta(t-t')\overline{\psi}^{(+E)}(x') = i\int d^3x \overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \tag{D.322}
$$

$$
\Theta(t'-t)\overline{\psi}^{(-E)}(x') = -i\int d^3x \overline{\psi}^{(-E)}(x)\gamma_0 S_F(x'-x). \tag{D.323}
$$

These are the adjoint spinor versions of equations (D.310) and (D.311).

Using (D.322), for electron scattering we have

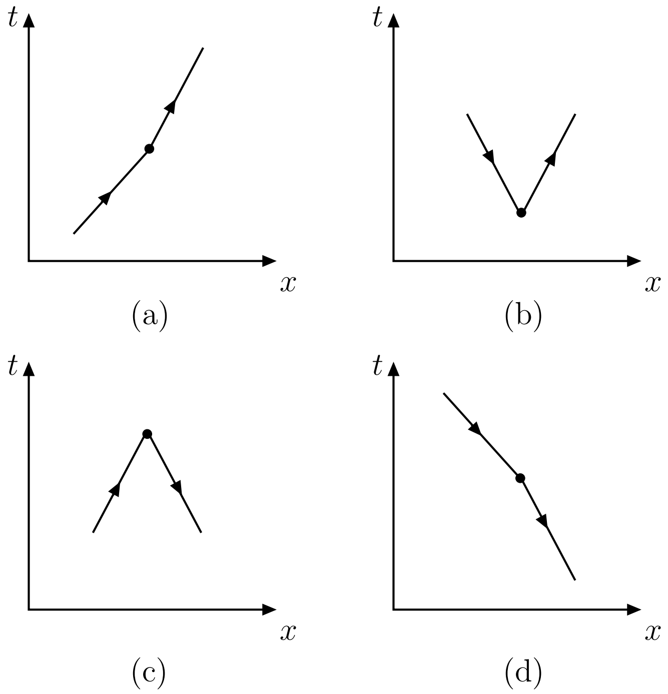


Figure D.5: (a) electron scattering; (b) electron-positron pair creation; (c) pair annihilation (d) positron scattering.

$$
\begin{aligned}
S_{fi} &= \lim_{t\to+\infty} \left\langle \psi_f(x) \middle| \psi_i(x) + \int d^4y S_F(x-y) e A_\mu(y)\gamma^\mu \Psi_i(x) \right\rangle \\
&= \delta_{fi} + e \lim_{t\to+\infty} \int d^3x \psi_f^*(x) \int d^4y S_F(x-y) A_\mu(y)\gamma^\mu \Psi_i(x) \\
&= \delta_{fi} - ie \lim_{t\to+\infty} \int d^4y \left(i \int d^3x \overline{\psi}_f(x)\gamma^0 S_F(x-y) \right) A_\mu(y)\gamma^\mu \Psi_i(x) \\
&= \delta_{fi} - ie \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x)
\end{aligned}
$$

while, using (D.323) similarly, positron scattering is described by

$$
S_{fi} = \delta_{fi} + ie \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x)
$$

Both results can be combined by writing

$$
S_{fi} = \delta_{fi} - ie\varepsilon_f \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x) \tag{D.324}
$$

where $\varepsilon_f = +1$ for positive energy waves in the future and $\varepsilon_f = -1$ for energy waves in the past. $\Psi_i(x)$ stands for the incoming wave

Repeated substitution of (D.318)

$$
\begin{aligned}
S_{fi} &= \delta_{fi} - e\varepsilon_f \int d^4y \overline{\psi}_f(y) A_\mu \gamma^\mu(y) \Psi_i(y) \\
&= \delta_{fi} - e\varepsilon_f \Big[\int d^4y_1 \overline{\psi}_f(y_1) A_\mu(y_1)\gamma^\mu \psi_i(y_1) \\
&\quad + \int d^4y_1 \int d^4y_2 \overline{\psi}_f(y_2) A_{\mu_2}(y_2)\gamma^{\mu_2} S_F(y_2 - y_1) A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i(y_1) \Big] \\
&\quad + \dots \quad . \\
&= \delta_{fi} + \sum_{n=1}^{\infty} S_{fi}^{(n)} \qquad \text{(D.325)}
\end{aligned}
$$

where

$$
\begin{aligned}
S_{fi}^{(n)} &= -ie^n \varepsilon_f \int d^4y_1 \dots \int d^4y_n \overline{\psi}_f(y_n) A_{\mu_n}(y_n)\gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}} \dots \\
&\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i(y_1) \qquad \text{(D.326)}
\end{aligned}
$$

"Ordinary" scattering of electrons

- $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with positive energy.

In this case

$$
\Psi_i(y) \to \psi_i^{(+E)}(y) = \sqrt{\frac{m_0}{E_-}} \frac{1}{(2\pi)^{3/2}} u(p_-, s_-) \exp(-ip_- \cdot x) \quad \text{as} \quad y_0 \to -\infty \qquad \text{(D.327)}
$$

an incoming electron with positive energy E_- and momentum p_- and spin s_-

$$
\begin{aligned}
S_{fi}^{(n)} &= -ie^n \int d^4y_1 \dots \int d^4y_n \overline{\psi}_f^{(+E)}(y_n) A_{\mu_n}(y_n)\gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}} \dots \\
&\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i^{(+E)}(y_1) \qquad \text{(D.328)}
\end{aligned}
$$

In addition to ordinary scattering intermediate pair creation and pair annihilation are included in the series.

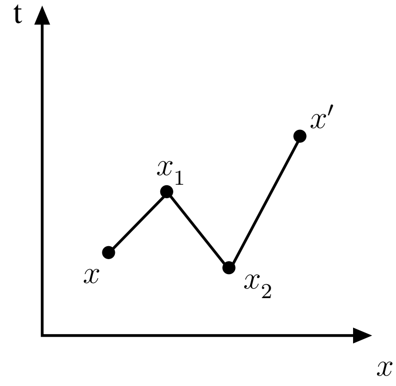


Figure D.6: The electron at x_1 propagates backward in time from x_1 to x_2. Physically a positron-electron pair is created at x_2, the positron propagates forward in time where it anihilates with the intial electron at x_1.

Pair production processes

- $\Psi_i(y)$ in this case at $y_0 \to +\infty$ reduces to a plane wave with negative energy.

The positron state at $t \to +\infty$ is described by a plane wave of negative energy. We use the notation

By hole theory a positron is an electron with negative energy, negative momentum and negative spin.

Now we need the plane wave propagating backward in time. There will be an exponential with a positive sign in the exponent

$$e^{(i+p_+ \cdot y)}$$

expressing the property that it has negative energy and momentum. It also will involve

$$v(p_+, +1/2) = \omega^4(p_+) \quad \text{and} \quad v(p_+, -1/2) = \omega^3(p_+)$$

where ω^4 is the spinor corresponding to a negative energy electron with spin up and ω^3 a negative energy electron with spin down. By using the spinors $v(p, s)$ we take care of the fact that the spin of electrons withe negative enery is $-s$. Here s is the spin of the positron.

$$\psi_i^{(electron)}(-p_f, -s_f) = Const.\ v(p_f, s_f) \exp(+ip_f \cdot x) \tag{D.329}$$

$$\Psi_i(x) \to \sqrt{\frac{m_0}{E_-}} \frac{1}{(2\pi)^{3/2}} v(p_+, s_+) e^{i+p_+\cdot y)} \quad \text{as} \quad y_0 \to \infty \tag{D.330}$$

$$\begin{aligned} S_{fi}^{(n)} &= -ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(+E)}(y_n) A_{\mu_n}(y_n) \gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1}) \gamma^{\mu_{n-1}} \ldots \\ &\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i(-E)(y_1) \end{aligned} \tag{D.331}$$

Pair annihilation processes

- $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with negative energy.

$$\begin{aligned} S_{fi}^{(n)} &= +ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(-E)}(y_n) A_{\mu_n}(y_n) \gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1}) \gamma^{\mu_{n-1}} \ldots \\ &\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i^{(+E)}(y_1) \end{aligned} \tag{D.332}$$

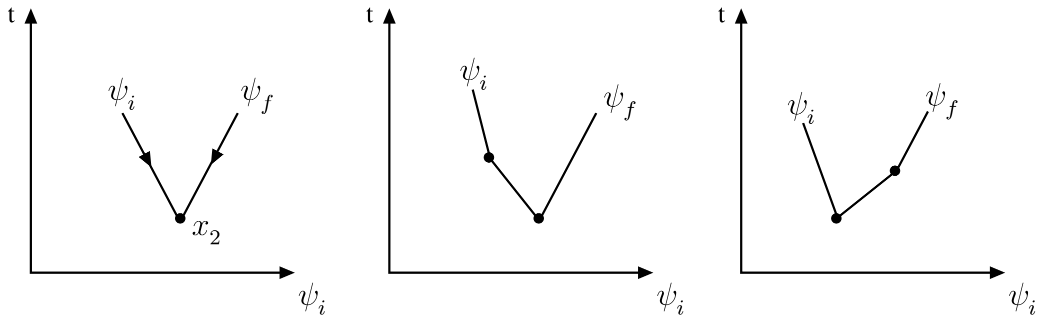
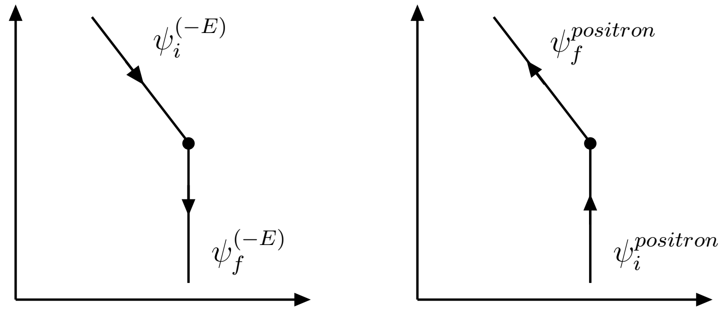


Figure D.7:

Scattering of positrons

- $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with negative energy.

$$\begin{aligned} S_{fi}^{(n)} &= +ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(-E)}(y_n) A_{\mu_n}(y_n) \gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1}) \gamma^{\mu_{n-1}} \ldots \\ &\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i^{(-E)}(y_1) \end{aligned} \tag{D.333}$$



(a) (b)

Figure D.8: Lowest order positron scattering. (a) incoming negative energy electron $\psi_i^{(-E)}$ is scattered into an outgoing negative energy electron $\psi_f^{(-E)}$. (b) This corresponds to an incident positron $\psi_f^{positron}$ and emerging positron $\psi_f^{positron}$. This is the link between the calculational technique and the real physical picture of positron scattering.

D.4 Scattering of an Electron off a Coulomb Potential

D.4.1 The Scattering Amplitude

We calculate the Rutherford scattering of an electron at a fixed Coulomb potential to lowest order of perturbation theory. The appropriate S-Matrix element is the first order term of (D.328)

$$S_{fi} = -ie \int d^4x \; \overline{\psi}_f(x) A_\mu \gamma^\mu(x) \psi_i(x) \tag{D.334}$$

$\psi_i(x)$ is given by the incoming plane wave of an electron with momentum p_i and s_i:

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}}\; u(p_i, s_i) e^{-ip_i \cdot x} \tag{D.335}$$

$\overline{\psi}_f(x)$ is given by

$$\overline{\psi}_f(x) = \sqrt{\frac{m_0}{E_f V}}\; \overline{u}(p_f, s_f) e^{ip_f \cdot x}. \tag{D.336}$$

Recall

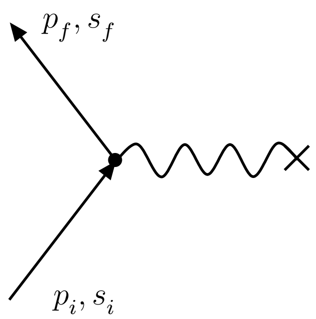


Figure D.9:

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}, \quad \mathbf{B} = -\nabla \times \mathbf{A}$$

Choosing

$$A_0(x) = A_0(\mathbf{x}) = -\frac{Ze}{|\mathbf{x}|}, \quad \mathbf{A}(x) = 0. \tag{D.337}$$

corresponds to a Coulomb force generated by a static charge $-Ze$. With these assumptions the S-Matrix element becomes

$$S_{fi} = iZe^2 \frac{1}{V} \sqrt{\frac{m_0^2}{E_f E_i}} \overline{u}(p_f, s_f) \gamma^0 u(p_i, s_i) \int d^4x \; e^{i(p_f - p_i)\cdot x} \frac{1}{|\mathbf{x}|}. \tag{D.338}$$

The integral over the time coordinate can be integrated

$$\int_{-\infty}^{\infty} dx_0 \; e^{i(E_f - E_i)t} = 2\pi\delta(E_f - E_i) \tag{D.339}$$

The remaining integral is

$$A_0(\mathbf{x}) = -Ze \int d^3x \frac{1}{|x|} e^{-i\mathbf{q}\cdot\mathbf{x}}$$

where $\mathbf{q}$ is the momentum transfer i.e. $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$. This can be evaluated using integration by parts of Poisson's formula $\Delta(1/|\mathbf{x}|) = -4\pi\delta^3(\mathbf{x})$:

$$\begin{aligned}
\int d^3x \frac{1}{|\mathbf{x}|} e^{-i\mathbf{q}\cdot\mathbf{x}} &= -\frac{1}{\mathbf{q}^2} \int d^3x \frac{1}{|\mathbf{x}|} \Delta e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= -\frac{1}{\mathbf{q}^2} \int d^3x \Delta \left(\frac{1}{|\mathbf{x}|}\right) e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= -\frac{1}{\mathbf{q}^2} \int d^3x (-4\pi\delta^3(\mathbf{x})) e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= \frac{4\pi}{\mathbf{q}^2} \qquad \text{(D.340)}
\end{aligned}$$

Thus the $S-$matrix element becomes

$$S_{fi} = iZe^2 \frac{1}{V} \sqrt{\frac{m_0^2}{E_f E_i}}\, \overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\, \frac{4\pi}{q^2}\, 2\pi\delta(E_f - E_i). \qquad \text{(D.341)}$$

D.4.2 The Cross Section

A differential cross section σ is defined by the effective area of target particles.

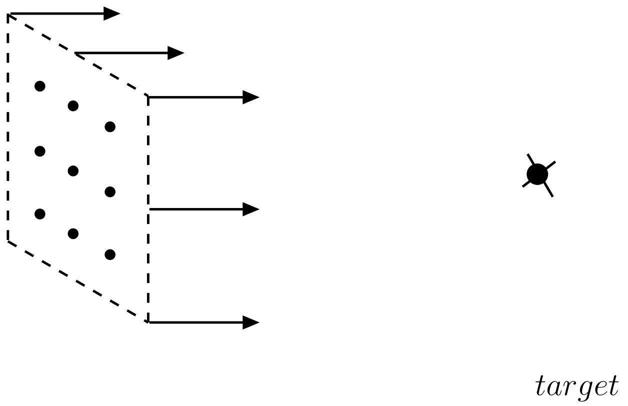


Figure D.10: .

$$\text{Chance of hitting a Coulomb potential} = \frac{N_T \sigma}{A} \qquad \text{(D.342)}$$

Let us say that there are N_I incoming particles. The number of scattering events is then

$$\text{number of events} = N_I \frac{N_T \sigma}{A} \qquad \text{(D.343)}$$

so that the cross section is then expressed as

$$\sigma = \left(\frac{\text{number of events}}{N_I N_T} \right) A. \tag{D.344}$$

We wish the express the cross section in terms of the flux of the incoming beam,

$$\text{flux} = \rho v$$

where v is the velocity of the beam moving toward the stationary target. The number of particles in the beam N_I is equal to the density of the beam ρ times the volume of the beam, vtA. The cross section can therefore be written as

$$\begin{aligned} \sigma &= \frac{\text{number of events}/t}{(\rho v t A) N_T / t} A \\ &= \frac{\text{number of events}/t}{\rho v} \cdot \frac{1}{N_T} \\ &= \frac{\text{transition rate}}{\text{flux}} \cdot \frac{1}{N_T} \end{aligned} \tag{D.345}$$

$$dW = \frac{|S_{fi}|^2 dN_f}{T} \frac{1}{J} \tag{D.346}$$

dN_f is now determined.

D.4.3 Transition Probability Per Particle into Final States

Standing waves in a cubical box of volume $V = L^3$ require

$$\begin{aligned} p_x L &= n_x 2\pi, \\ p_y L &= n_y 2\pi, \\ p_z L &= n_z 2\pi, \end{aligned} \tag{D.347}$$

with integer number n_x, n_y, n_z. For large L the discrete set of $\mathbf{p}-$values approaches a continuum. The number of states is

$$\begin{aligned} dN &= dn_x dn_y dn_z \\ &= \frac{1}{(2\pi)^3} L^3 dp_x dp_y dp_z \\ &= \frac{V}{(2\pi)^3} d^3p. \end{aligned} \tag{D.348}$$

The transition probability per particle into these final states is

$$\begin{aligned} dW &= |S_{fi}|^2 \frac{V d^3 p_f}{(2\pi)^3} \\ &= \frac{Z^2(4\pi\alpha)^2 m_0^2}{E_i V} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{d^3 p_f}{(2\pi)^3 E_f} (2\pi\delta(E_f - E_i))^2 \end{aligned} \tag{D.349}$$

D.4.4 Transition Probability Per Particle, Per Unit Time

We smear out the $\delta-$function $2\pi\delta(E_f - E_i)$:

$$\begin{aligned} \int_{-T/2}^{T/2} dx_0 e^{i(E_f - E_i)x_0} &= \left[\frac{1}{i(E_f - E_i)} e^{i(E_f - E_i)x_0} \right]_{-T/2}^{T/2} \\ &= \frac{2\sin(E_f - E_i)T/2}{E_f - E_i} \end{aligned} \tag{D.350}$$

Thus we replace the square of the $\delta-$function is replaced by

$$(2\pi\delta(E_f - E_i))^2 \Rightarrow 4\frac{\sin^2(E_f - E_i)T/2}{(E_f - E_i)^2} \tag{D.351}$$

The area of this function is

$$\int_{-\infty}^{\infty} 4\frac{\sin^2(E_f - E_i)T/2}{(E_f - E_i)^2} dE_f = 2\pi T. \tag{D.352}$$

Knowing that the "area" under the square of the δ is $\lim_{T\to\infty} 2\pi T$, we make the replacement

$$(2\pi\delta(E_f - E_i))^2 \rightarrow 2\pi T\delta(E_f - E_i). \tag{D.353}$$

Denote the rate R

$$dR = \frac{dW}{T} = \frac{Z^2\alpha^2 m_0^2}{E_i V}\frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4}\frac{d^3p_f}{E_f}\delta(E_f - E_i) \tag{D.354}$$

D.4.5 Formula for Differential Cross Section

The scattering cross section can be defined as the transition probabilty per particle and per unit time divided by the incoming current of particles

$$J^a_{inc}(x) = \overline{\psi}_i(x)\gamma^a\psi_i(x) \tag{D.355}$$

Taking the spinors with spin polarisation in the z-direction we determine the current

$$\begin{aligned}
J^a_{inc} &= \overline{\psi}_i(x)\gamma^a\psi_i(x) \\
&= \frac{m_0}{E_i V}\,\overline{u}(p_i, s_i)\gamma^3 u(p_i, s_i) \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\begin{pmatrix} 1 \; 0 \; \frac{p_i}{E_i + m_0} \; 0 \end{pmatrix}\,\gamma^0\gamma^3\begin{pmatrix} 1 \\ 0 \\ \frac{p_i}{E_i+m_0} \\ 0 \end{pmatrix} \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\begin{pmatrix} 1 \; 0 \; \frac{p_i}{E_i + m_0} \; 0 \end{pmatrix}\begin{pmatrix} 1 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{p_i}{E_i+m_0} \\ 0 \end{pmatrix} \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\begin{pmatrix} 1 \; 0 \; \frac{p_i}{E_i + m_0} \; 0 \end{pmatrix}\begin{pmatrix} \frac{p_i}{E_i+m_0} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{p_i}{E_i}\frac{1}{V}.
\end{aligned} \tag{D.356}$$

$$|\mathbf{J}_{inc}| = \frac{|\mathbf{v}_i|}{V}. \tag{D.357}$$

The differential cross section can now be determined

$$d\sigma = \frac{dR}{|\mathbf{J}_{inc}|} = \frac{4Z^2\alpha^2 m_0^2}{E_i V \frac{|v_i|}{V}} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{d\mathbf{p}_f^3}{E_f} \delta(E_f - E_i) \tag{D.358}$$

Use

$$d^3 p_f = \mathbf{p}_f^2 d|\mathbf{p}_f| d\Omega_f \tag{D.359}$$

Then the differential cross section becomes

$$d\sigma = \frac{4Z^2\alpha^2 m_0^2}{E_i V \frac{|v_i|}{V}} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{\mathbf{p}_f^2 d|\mathbf{p}_f|}{E_f} d\Omega_f \ \delta(E_f - E_i) \tag{D.360}$$

D.4.6 Averaging Over Spin

The differential cross section above can be applied to calculate the scattering of a particle with initial polarisation (s_i) to final polarisation (s_f).

First we give a simple example. From the relation

$$\overline{w}^r(p_\mu \gamma^\mu - \epsilon_r m_0) = 0.$$

we see that

$$\overline{w}^r_\gamma(p_i)\Lambda_{\gamma\delta}(p) = 0$$

for $r = 3, 4$, where

$$\Lambda_{\gamma\delta}(p) = \frac{-p_\mu \gamma^\mu + m_0}{2m_0}$$

$$\begin{aligned}
\sum_{s_i} u_\beta(p_i,s_i)\overline{u}_\beta(p_i,s_i) &= u_\beta(p_i,\uparrow)\overline{u}_\beta(p_i,\uparrow) + u_\beta(p_i,\downarrow)\overline{u}_\beta(p_i,\downarrow) \\
&= \sum_{r=1}^{2} w^r_\beta(p_i)\overline{w}^r_\delta(p_i) \\
&= \sum_{r=1}^{2} w^r_\beta(p_i)\sum_{\gamma=1}^{4}\overline{w}^r_\gamma(p_i)\Lambda_{\gamma\delta}(p) \\
&= \sum_{\gamma,r=1}^{4} \epsilon_r w^r_\beta(p_i)\overline{w}^r_\gamma(p_i)(\Lambda(p))_{\gamma\delta} \\
&= (\Lambda(p))_{\beta\delta} \qquad \text{(D.361)}
\end{aligned}$$

where we used the completeness relation

$$\sum_{r=1}^{4} \epsilon_r w^r_\beta(p_i)\overline{w}^r_\gamma(p_i) = \delta_{\beta\gamma}.$$

Using this result and similar considerations we now calculate the spin sum.

$$\begin{aligned}
&\sum_{\alpha,\sigma,\beta,\delta}\sum_{s_f}\overline{u}_\alpha(p_f,s_f)\gamma^0_{\alpha\beta}\left(\sum_{s_f} u_\alpha(p_i,s_i)\overline{u}_\delta(p_i,s_i)\right)\gamma^0_{\delta\sigma}u_\sigma(p_f,s_f) \\
&= \sum_{\alpha,\sigma}\sum_{s_f}\overline{u}_\alpha(p_f,s_f)\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu+m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}u_\sigma(p_f,s_f) \\
&= \sum_{\alpha,\sigma}\sum_{r=1}^{2}\overline{w}^r_\alpha(p_f)\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu+m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}w^r_\sigma(p_f) \\
&= \sum_{\alpha,\sigma}\sum_{r=1}^{4}\epsilon_r\overline{w}^r_\alpha(p_f)\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu+m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}\sum_{\tau=1}^{4}\left(\frac{(p_f)_\nu\gamma^\nu+m_0}{2m_0}\right)_{\sigma\tau}w^r_\tau(p_f) \\
&= \sum_{\alpha,\sigma}\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu+m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}\left(\frac{(p_f)_\nu\gamma^\nu+m_0}{2m_0}\right)_{\sigma\alpha} \\
&= Tr\left[\gamma^0\frac{(p_i)_\mu\gamma^\mu+m_0}{2m_0}\gamma^0\frac{(p_f)_\nu\gamma^\nu+m_0}{2m_0}\right] \qquad \text{(D.362)}
\end{aligned}$$

Using this the differential cross section can be written

$$\frac{d\overline{\sigma}}{d\Omega_f} = \frac{4Z^2\alpha^2m_0^2}{2|\mathbf{q}|^4}Tr\left[\gamma^0\frac{(p_i)_\mu\gamma^\mu+m_0}{2m_0}\gamma^0\frac{(p_f)_\nu\gamma^\nu+m_0}{2m_0}\right]. \qquad \text{(D.363)}$$

D.4.7 Taking the Trace of the Product of Gamma Matrices in the Differential Cross Section

We first prove that the trace of an odd number of $\gamma-$matrices vanishes. To do this we use $(\gamma^5)^2 = I$ and $\gamma_\mu \gamma^5 + \gamma^5 \gamma_\mu = 0$.

$$
\begin{aligned}
Tr\gamma_\mu \ldots \gamma_\nu &= Tr\gamma_\mu \ldots \gamma_\nu \gamma^5 \gamma^5 \\
&= (-1)^n Tr\gamma^5 \gamma_\mu \ldots \gamma_\nu \gamma^5 \\
&= (-1)^n Tr\gamma_\mu \ldots \gamma_\nu \gamma^5 \gamma^5 \\
&= (-1)^n Tr\gamma_\mu \ldots \gamma_\nu
\end{aligned}
\tag{D.364}
$$

where in the third line we used the cyclic permutation of the trace. With this () reduces to

$$
\frac{d\overline{\sigma}}{d\Omega_f} = \frac{4Z^2\alpha^2 m_0^2}{2|\mathbf{q}|^4} \left[Tr(\gamma^0 (p_i)_\mu \gamma^\mu \gamma^0 (p_f)_\nu \gamma^\nu) + m_0^2 Tr(\gamma^0)^2 \right]. \tag{D.365}
$$

We have $Tr(\gamma^0)^2 = TrI = 4$. To evaluate the first trace we derive a couple of results: Firstly

$$
\begin{aligned}
a_\mu b_\nu Tr\gamma^\mu \gamma^\nu &= a_\mu b_\nu \frac{1}{2} Tr(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \\
&= a_\mu b_\nu \eta^{\mu\nu} TrI \\
&= 4a \cdot b.
\end{aligned}
\tag{D.366}
$$

where we have used μ and ν are dummy variables in the first line. Secondly, starting with

$$
\begin{aligned}
a_\mu b_\nu c_\gamma d_\delta \, Tr\gamma^\mu \gamma^\nu \gamma^\gamma \gamma^\delta &= 2a \cdot b \, c_\gamma d_\delta \, Tr\gamma^\gamma \gamma^\delta - b_\mu a_\nu c_\gamma d_\delta \, Tr\gamma^\mu \gamma^\nu \gamma^\gamma \gamma^\delta \\
&= 2a \cdot b \, c_\gamma d_\delta \, Tr\gamma^\gamma \gamma^\delta - 2a \cdot c b_\mu d_\delta \, Tr\gamma^\mu \gamma^\delta \\
&\quad + b_\mu c_\nu a_\gamma d_\delta \, Tr\gamma^\mu \gamma^\nu \gamma^\gamma \gamma^\delta \\
&= 2a \cdot b \, c_\gamma d_\delta \, Tr\gamma^\gamma \gamma^\delta - 2a \cdot c \, b_\mu d_\delta \, Tr\gamma^\mu \gamma^\delta \\
&\quad + 2a \cdot d \, b_\mu c_\nu \, Tr\gamma^\mu \gamma^\nu - b_\mu c_\nu d_\gamma a_\delta \, Tr\gamma^\mu \gamma^\nu \gamma^\gamma \gamma^\delta
\end{aligned}
$$

then using the cyclic property of the trace we find

$$
\begin{aligned}
a_\mu b_\nu c_\gamma d_\delta \, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta &= a\cdot b\; c_\mu d_\nu \, Tr\gamma^\mu\gamma^\nu - a\cdot c\; b_\mu d_\nu \, Tr\gamma^\mu\gamma^\nu \\
&\quad + a\cdot d\; b_\mu c_\nu \, Tr\gamma^\mu\gamma^\nu
\end{aligned} \tag{D.367}
$$

Using the second result first with $a = c = (1, 0, 0, 0)$ we get

$$
(p_i)_\mu (p_f)_\nu \, Tr(\gamma^0\gamma^\mu\gamma^0\gamma^\nu) = a\cdot p_i \, Tr - a\cdot a \, Tr + a\cdot p_f \, Tr
$$

Now using the first result we have

$$
\begin{aligned}
(p_i)_\mu (p_f)_\nu \, Tr(\gamma^0\gamma^\mu\gamma^0\gamma^\nu) &= 4(a\cdot p_i)(a\cdot p_f) - (a\cdot a)4(p_i\cdot p_f) + 4(a\cdot p_f)(a\cdot p_i) \\
&= 4E_iE_f - 4(E_iE_f - \vec{p}_i\cdot\vec{p}_f) + 4E_iE_f
\end{aligned} \tag{D.368}
$$

D.4.8 Mott Scattering Formula

The $\delta(E_i - E_f)$ function of the cross section ensures energy conservation $E_i = E_f$, thus $E_i^2 = E_f^2$:

$$
m_0^2 + \vec{p}_i^{\,2} = m_0^2 + \vec{p}_f^{\,2}
$$

implying

$$
|\vec{p}_i| = |\vec{p}_f| = |\vec{p}|
$$

The scalar product of initial and final momentum is the following function of the scattering angle θ

$$
\begin{aligned}
p_i\cdot p_f &= |\mathbf{p}|^2\cos\theta \\
&= |\mathbf{p}|^2\left(1 - 2\sin^2\frac{\theta}{2}\right) \\
&= \beta^2E^2\left(1 - 2\sin^2\frac{\theta}{2}\right)
\end{aligned} \tag{D.369}
$$

From this we have for the momentum transfer

$$
\begin{aligned}
|q| &= |p_f - p_i| \\
&= \sqrt{|p_f|^2 + |p_i|^2 - p_f \cdot p_i} \\
&= \sqrt{2p^2 - |p|\cos\theta} \\
&= 2|p|\sin\frac{\theta}{2}
\end{aligned}
\tag{D.370}
$$

A simple exercise, left to the reader, gives us the well known Mott scattering formula

$$
\frac{d\overline{\sigma}}{d\Omega_f} = \frac{Z^2\alpha^2(1-\beta^2\sin^2\frac{\theta}{2})}{4\beta^2|\mathbf{p}|^2\sin^4\frac{\theta}{2}} \tag{D.371}
$$

In the limit $\beta \to 0$ (small velocities) reduces to Rutherford's scattering formula

$$
\frac{d\overline{\sigma}}{d\Omega_f} = \frac{Z^2\alpha^2}{4\beta^2|\mathbf{p}|^2\sin^4\frac{\theta}{2}} \tag{D.372}
$$

D.5 Scattering of an Electron off a Free Proton

In the last example we considered scattering off a central potential. Now we make our first step toward the derivation of Feynmann's rules by considering scattering off two free particles.

D.5.1 Inhomogeneous Wave Equation and Photon Proporgator

In electromagnetism the invariance in A_μ comes about because the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is left unchanged by the gauge transformations

$$
A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)
$$

We wish to calculate the four-potential $A_\mu(x)$ produced by a source current $J^\nu(x)$ term,

$$
\Box A^\nu(x) - \partial^\mu\partial_\mu A^\nu(x) = 4\pi J^\nu(x) \tag{D.373}
$$

We are free to choose the most convenient gauge for the calculation intened to make.

We will choose the Lorentz gauge

$$\partial_\mu A^\mu(x) = 0$$

In momentum space this reads

$$k^\mu A^\mu(k) = 0.$$

$$\Box A^\nu(x) = 4\pi J^\nu(x) \tag{D.374}$$

The solution of the above equation may be systematically formulated using the appropriate Green's function which we call $D_F(x-y)$, the propogator for electromagnetism.

$$\Box D_F(x-y)(x) = 4\pi\delta^4(x-y). \tag{D.375}$$

The Fourier-transformed proporgator is defined by

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp[-iq\cdot(x-y)]D_F(q) \tag{D.376}$$

Using

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp[-iq\cdot(x-y)] \tag{D.377}$$

and making comparison we get

$$D_F(q) = -\frac{4\pi}{q^2} \tag{D.378}$$

The four-potential $A^\mu(x)$ solving (D.374) is

$$A^\mu(x) = \int d^4y D_F(x-y)J^\mu(y). \tag{D.379}$$

D.5.2 Potential of Proton Current

$$S_{fi} = -ie \int d^4x \overline{\psi}_f(x)\gamma_\mu A^\mu(x)\Psi_i(x) \tag{D.380}$$

At first order the four-potential $A^\mu(x)$ is the field produced by the proton to lowest order in α.

$$S_{fi} = -i \int d^4x d^4y \left[e\overline{\psi}_f(x)\gamma_\mu\psi_i(x)\right] D_F(x-y)J^\mu(y). \tag{D.381}$$

The term in the brackets represents the current of the electron. As the electron and proton play equivalent roles in the scattering process, the proton's current should be of the same form as the electronic current. Therefore we make the replacement

$$J^\mu(y) \to e_p \overline{\psi}^p_f(y)\gamma^\mu\psi^p_i(y) \tag{D.382}$$

where $\overline{\psi}^p_f(y)$ and $\psi^p_i(y)$ have the same for as the electron wavefunctions

$$\begin{aligned} \psi^p_i(y) &= \sqrt{\frac{M_0}{E^p_i V}} u(P_i, S_i)\exp(-iP_i \cdot y) \\ \psi^p_f(y) &= \sqrt{\frac{M_0}{E^p_f V}} u(P_f, S_f)\exp(-iP_f \cdot y) \end{aligned} \tag{D.383}$$

where P_i and P_f denote the four-momentum of the proton, S_i, S_f and E^p_i, E^p_f denote its spin and energy respectively. M_0 is the proton's rest mass. The proton's current is then

$$J^\mu_{fi}(y) = -\sqrt{\frac{M_0^2}{E^p_f E^p_i}} \frac{e}{V} \exp[i(P_f - P_i)\cdot y]\, \overline{u}(P_f, S_f)\gamma^\mu u(P_i, S_i). \tag{D.384}$$

Inserting this into the expression for the S-matrix gives

$$\begin{aligned} S_{fi} &= +i\frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E^p_f E^p_i}}\, [\overline{u}(p_f, s_f)\gamma_\mu u(p_i, s_i)] \\ &\times \int d^4x d^4y \frac{d^4q}{(2\pi)^4} \exp[-iq\cdot(x-y)]\exp[i(p_f - p_i)\cdot x]\exp[i(P_f - P_i)\cdot x] \\ &\times \left(\frac{-4\pi}{q^2 + i\epsilon}\right)\, [\overline{u}(P_f, S_f)\gamma_\mu u(P_i, S_i)] \end{aligned} \tag{D.385}$$

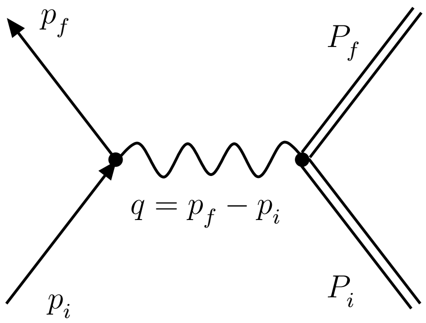


Figure D.11: Lowest order electron-proton scattering.

D.5.3 Conservation of Four-Momentum

The $x-$ and $y-$integrations give

$$\begin{aligned}\int d^4x \exp(i(p_f - p_i - q)\cdot x) &= (2\pi)^4\delta^4(p_f - p_i - q)\\ \int d^4y \exp(i(p_f - p_i - q)\cdot y) &= (2\pi)^4\delta^4(P_f - P_i - q)\end{aligned} \tag{D.386}$$

The integration over q is then readily done:

$$\begin{aligned}&\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^4(p_f - p_i - q)(2\pi)^4\delta^4(P_f - P_i - q)\left[-\frac{4\pi}{q^2 + i\epsilon}\right]\\ &= (2\pi)^4\delta^4(P_f - P_i + p_f - p_i)\left[-\frac{4\pi}{(p_f - p_i)^2 + i\epsilon}\right]\end{aligned} \tag{D.387}$$

D.5.4 Remarks on the Form of the S-matrix Element

Here we display properties of S-matrix element that are a first step toward “deriving” the Feynmann rules for QED. The total S-matrix element the reads

$$S_{fi} = i(2\pi)^4\delta^4(P_f - P_i + p_f - p_i)\ M_{fi}\ \frac{1}{V^2}\sqrt{\frac{m_0^2}{E_fE_i}}\sqrt{\frac{M_0^2}{E_f^pE_i^p}} \tag{D.388}$$

where

$$M_{fi} = [\overline{u}(p_f, s_f)(-ie\gamma_\mu)u(p_i, s_i)]\frac{-4\pi\ \eta^{\mu\nu}}{(p_f - p_i)^2 + i\epsilon}[\overline{u}(P_f, S_f)(-ie_p\gamma_\mu)u(P_i, S_i)] \qquad \text{(D.389)}$$

This describes the lowest order contribution. This is put in diagrammatic form in fig (E.7). The wavy line represents the virtual photon being exchanged between the electron and proton. The four momentum of the photon is

$$q = p_f - p_i = P_f - P_i \qquad \text{(D.390)}$$

• **The following factor represents the amplitude for the propagation of a photon with momentum q:**

$$\frac{-4\pi\ \eta^{\mu\nu}}{q^2 + i\epsilon} \qquad \text{(D.391)}$$

• **There is a factor of $-ie\gamma_\mu$ for every vertex.**

• **These act between spinors $u(p, s)$ describing the free ingoing and outgoing Dirac particles.**

• **There is a four dimensional $\delta-$function, ensuring conservation of total energy and momentum in the scattering process.**

D.5.5 The Scattering Cross Section

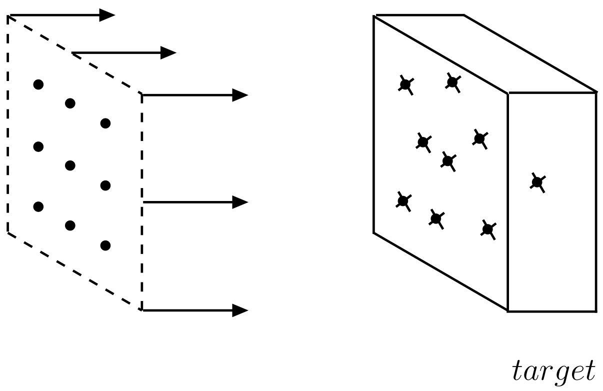


Figure D.12: .

We divide $|S_{fi}|^2$ by the time interval and the space volume of the reaction (Dirac waves normalised so that there is one particle per unit volume)

$$W_{fi} = \frac{|S_{fi}|^2}{VT} \tag{D.392}$$

Now we come to calculating the cross section. As in section we have to consider the square of the δ^4-function

$$(2\pi)^4 \delta^4 \left(p_f + P_f - p_i - P_i\right) = \lim_{T\to\infty, V\to\infty} \int_{-T/2}^{T/2} \int_V d^3\mathbf{x} \exp\left[ix \cdot \left(p_f + P_f - p_i - P_i\right)\right] \tag{D.393}$$

$$\left[(2\pi)^4 \delta^4 \left(p_f + P_f - p_i - P_i\right)\right]^2 \to TV(2\pi)^4 \delta^4 \left(p_f + P_f - p_i - P_i\right) \tag{D.394}$$

To obtain the transition rate to a group of final states with momenta in the intervals $f = 1, 2$, we multiply by the number of these states which is

$$\frac{V d^3\mathbf{P}_1}{(2\pi)^3} \frac{V d^3\mathbf{P}_2}{(2\pi)^3} \tag{D.395}$$

number of target particles per unit volume $= 1/V$

Combining these results with (), we obtain the required formula for the differential cross section

$$\begin{aligned} d\sigma &= V^2 \frac{d^3\mathbf{p}_f}{(2\pi)^3} \frac{d^3\mathbf{P}_f}{(2\pi)^3} \frac{1}{|\mathbf{J_{inc}}|} \frac{1}{1/V} W_{fi} \\ &= \end{aligned} \tag{D.396}$$

D.5.6 Lorentz Invariance

Each particle leaving the scattering process contributes a factor

$$\frac{m_0}{E_f} \frac{d^3p_f}{(2\pi)^3} \tag{D.397}$$

to the cross section. Consider

$$
\begin{aligned}
\int_{-\infty}^{\infty} d^4p\ \delta(p^2-m_0^2)\Theta(p_0) &= \int_0^{\infty} dp_0\ \delta(p_0^2-\mathbf{p}^2-m_0^2)\ d^3p \\
&= \int_0^{\infty} dp_0\ \delta(p_0^2-E^2)\ d^3p \\
&= \int_0^{\infty} dp_0\ \delta[(p_0-E)(p_0+E)]\ d^3p \\
&= \int_0^{\infty} dp_0\ \delta[2E(p_0-E)]\ d^3p \\
&= \frac{d^3p}{2E}
\end{aligned}
$$

where

$$
\Theta(p_0) = \begin{cases} 1 & \text{for } p_0 > 0 \\ 0 & \text{for } p_0 < 0 \end{cases}
$$

This step function is obviously Lorentz invariant since Lorentz transformations always transform timelike four vectors into timelike four vectors. Thus we have established that $d^3p/2E$ is a Lorentz-invariant factor.

Now we consider the factor

$$
\frac{m_0}{E_i}\frac{M_0}{E_i^P}\frac{1}{|\mathbf{J}_{inc}|V}.
$$

$$
|\mathbf{J}_{inc}| = \frac{1}{V}|\mathbf{v}_i - \mathbf{V}_i| \tag{D.398}
$$

$$
\mathbf{v}_i = \frac{\mathbf{p}_i}{E_i}, \quad \mathbf{V}_i = \frac{\mathbf{P}_i}{E_i^P}
$$

This gives

$$
\begin{aligned}
\frac{m_0}{E_i}\frac{M_0}{E_i^P}\frac{1}{|\mathbf{J}_{inc}|V} &= \frac{m_0 M_0}{E_i E_i^P|\mathbf{v}_i - \mathbf{V}_i|} \\
&= \frac{m_0 M_0}{E_i E_i^P\sqrt{\mathbf{v}_i^2 + \mathbf{V}_i - 2\mathbf{v}_i\cdot\mathbf{V}_i}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + \mathbf{P}_i^2 E_i^2 - 2\mathbf{p}_i\cdot\mathbf{P}_i E_i E_i^P}}
\end{aligned} \tag{D.399}
$$

We prove that for collinear collisions that this is equivalent to the Lorentz invariant scalar

$$\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}},$$

because

$$\begin{aligned}
\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} &= \frac{m_0 M_0}{\sqrt{(E_i E_i^P - \mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{E_i^2 E_i^{P2} - 2E_i E_i^P\ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{(m_0^2 + \mathbf{p}_i^2)(M_0^2 + \mathbf{P}_i^2) - 2E_i E_i^P\ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + m_0^2 \mathbf{P}_i^2 - 2E_i E_i^P\ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2}}
\end{aligned} \tag{D.400}$$

As the velocity vectors are collinear we have that $(\mathbf{p}_i \cdot \mathbf{P}_i)^2 = \mathbf{p}_i^2 \mathbf{P}_i^2$.

$$\begin{aligned}
\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} &= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + m_0^2 \mathbf{P}_i^2 - 2E_i E_i^P\ \mathbf{p}_i \cdot \mathbf{P}_i + \mathbf{p}_i^2 \mathbf{P}_i^2}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + \mathbf{P}_i^2 E_i^2 - 2E_i E_i^P\ \mathbf{p}_i \cdot \mathbf{P}_i}}
\end{aligned} \tag{D.401}$$

We can use this Lorentz-invariant flux factor to write the cross section in a invariant form

$$d\sigma = \frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}}\ |M_{fi}|^2\ (2\pi)^4 \delta^4(P_f - P_i + p_f - p_i)\ \frac{m_0 d^3 p_f}{(2\pi)^3 E_f}\ \frac{M_0 d^3 p_f}{(2\pi)^3 E_f^P}. \tag{D.402}$$

D.5.7 Averaging over Spin

The squared invariant matrix element averaged over initial and final spin is

$$\overline{|M_{fi}|^2} = \frac{1}{4} \sum_{S_f, S_i, s_f, s_i} \left| \overline{u}(p_f, s_f) \gamma^\mu u(p_i, s_i)\ \frac{e e_p (4\pi)}{q^2 + i\epsilon}\ \overline{u}(P_f, S_f) \gamma_\mu u(P_i, S_i) \right|^2 \tag{D.403}$$

$$\begin{aligned}
& \sum_{S_f,S_i,s_f,s_i} \left| [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(P_f,S_f)\gamma_\mu u(P_i,S_i)] \right|^2 \\
= & \sum_{S_f,S_i,s_f,s_i} [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(P_f,S_f)\gamma_\mu u(P_i,S_i)] \\
& [\overline{u}(p_f,s_f)\gamma^\nu u(p_i,s_i)]^*\ [\overline{u}(P_f,S_f)\gamma_\nu u(P_i,S_i)]^* \\
= & \sum_{s_f,s_i} [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(p_f,s_f)\gamma^\nu u(p_i,s_i)]^* \\
& \sum_{S_f,S_i} [\overline{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)]\ [\overline{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)]^*
\end{aligned} \tag{D.404}$$

At this point the reader should go throught section D.14.8. The answer according to (D.588) is

$$Tr\left[\frac{p_{f\alpha}\gamma^\alpha + m_0}{2m_0}\ \gamma^\mu\ \frac{p_{i\beta}\gamma^\beta + m_0}{2m_0}\ \gamma^\nu\right]\ Tr\left[\frac{P_{f\gamma}\gamma^\gamma + M_0}{2M_0}\ \gamma_\mu\ \frac{P_{i\delta}\gamma^\delta + M_0}{2M_0}\ \gamma_\nu\right] \tag{D.405}$$

$$\overline{|M_{fi}|^2} = \frac{e^2 e_p^2 (4\pi)^2}{q^4} L^{\mu\nu} H_{\mu\nu} \tag{D.406}$$

where we have introduced the lepton tensor $L^{\mu\nu}$ and the hadron tensor $H_{\mu\nu}$, defined as

$$L^{\mu\nu} = Tr\left[\frac{p_{f\alpha}\gamma^\alpha + m_0}{2m_0}\ \gamma^\mu\ \frac{p_{i\beta}\gamma^\beta + m_0}{2m_0}\ \gamma^\nu\right] \tag{D.407}$$

and

$$H_{\mu\nu} = Tr\left[\frac{P_{f\gamma}\gamma^\gamma + M_0}{2M_0}\ \gamma_\mu\ \frac{P_{i\delta}\gamma^\delta + M_0}{2M_0}\ \gamma_\nu\right] \tag{D.408}$$

Using methods already introduced in the previous section on Coulomb scattering, we can easily evaluate the trace in the lepton tensor $L^{\mu\nu}$ to obtain:

$$L^{\mu\nu} = \frac{1}{2}\frac{1}{m_0^2}\left[p_f^\mu p_i^\nu + p_i^\mu p_f^\nu - \eta^{\mu\nu}(p_f \cdot p_i - m_0^2)\right] \tag{D.409}$$

The Hadron trace has the same structure, we just replace small letters by capitals and lower the spacetime indices.

D.5.8 Differential Cross Section in Rest Frame of Proton

The calculation of $\overline{|M_{fi}|^2}$ which we leave to the reader results in

$$\begin{aligned}\overline{|M_{fi}|^2} &= \frac{e^2 e_p^2 (4\pi)^2}{4m_0^2 M_0^2 (q^2)^2} \Big[(p_i \cdot P_i)(p_f \cdot P_f) + (p_i \cdot P_f)(p_f \cdot P_i) \\ &\quad -(p_i \cdot p_f) M_0^2 - (P_i \cdot P_f) m_0^2 + 2m_0^2 M_0^2\Big] . \end{aligned} \tag{D.410}$$

Let us work in the rest frame of the proton. We define

$$\begin{aligned} p_f &= (E', \mathbf{p}') =: p' \\ p_f &= (E, \mathbf{p}) =: p \\ P_i &= (M_0, 0) \end{aligned} \tag{D.411}$$

We calculate the differential cross section for electron scattering into a solid angle $d\Omega'$ centered around the scattering angle θ. Thus we will integrate the differential cross section over all momentum variable except for the direction of $\mathbf{p_f}$. First we will want to write down the spin averaged differential cross section in the proton rest system. The invariant flux factor reduces to

$$\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i) - m_0^2 M_0^2}} = \frac{m_0 M_0}{\sqrt{E^2 M_0^2 - m_0^2 M_0^2}} = \frac{m_0}{|\mathbf{p}|} \tag{D.412}$$

We will use

$$\frac{M_0 d^3 P_f}{(2\pi)^3 E_f^p} = \frac{2M_0}{(2\pi)^3} \int_{-\infty}^{\infty} d^4 P_f \; \delta(P_f^2 - M_0^2) \Theta(P_f^0).$$

The spin averaged differential cross section $d\overline{\sigma}$ is then

$$\begin{aligned} d\overline{\sigma} &= \frac{m_0}{|\mathbf{p}|} \overline{|M_{fi}|^2} (2\pi)^4 \delta^4 (P_f + p' - P_i - p) \\ &\quad \times \frac{m_0}{(2\pi)^3} |\mathbf{p}'| dE' d\Omega' \times \frac{2M_0}{(2\pi)^3} \int_{-\infty}^{\infty} d^4 P_f \; \delta(P_f^2 - M_0^2) \Theta(P_f^0) \end{aligned} \tag{D.413}$$

Now integrating over dE' and $d^3 P_f$ we obtain

$$
\begin{aligned}
\frac{d\overline{\sigma}}{d\Omega'} &= \int dE'|\mathbf{p}'|E' \; (d\overline{\sigma}) \\
&= \frac{m_0^2 M_0}{|\mathbf{p}|2\pi^2} \int dE'|\mathbf{p}'| \; \overline{|M_{fi}|^2}\delta\left((p' - P_i - p)^2 - M_0^2\right) \Theta(P_i^0 + E - E') \\
&= \frac{m_0^2 M_0}{|\mathbf{p}|2\pi^2} \int_{m_0}^{M_0+E} dE'|\mathbf{p}'| \; \overline{|M_{fi}|^2}\delta(2m_0^2 - 2(E' - E) - 2E'E2|\mathbf{p}||\mathbf{p}'|\cos\theta)
\end{aligned}
\tag{D.414}
$$

where the upper limit in the integral comes from the step function and the lower from the fact that E' cannot be less than m_0. Furthermore, the argument of the delta function has been expressed in terms of the kinematical variables of the laboratory frame. The remaining integral over E' can be performed using

$$
\delta\left(f(x)\right) = \sum_k \frac{\delta(x - x_k)}{|\frac{df}{dx}|_{x_k}}
$$

x_i being the roots of $f(x)$ inside the range of integration. We get

$$
\frac{d\overline{\sigma}}{d\Omega'} = \frac{m_0^2 M_0}{4\pi^2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \frac{\overline{|M_{fi}|^2}}{M_0 + E - |\mathbf{p}|(E'/|\mathbf{p}'|)\cos\theta} \tag{D.415}
$$

where we have used $d|\mathbf{p}'|/dE' = E'/|\mathbf{p}'|$ and we have for E'

$$
E'(M_0 + E) - |\mathbf{p}||\mathbf{p}'|\cos\theta = EM_0 + m_0^2 \tag{D.416}
$$

For given scattering angle θ the final energy E' of the electrons can be determined as a function of E and θ. The resulting E' and the corresponding $|\mathbf{p}'| = E'^2 - m_0^2$ have to be inserted into (D.415).

D.6 Scattering of Identical Fermions

We can take over many aspects of electron-proton scattering. But now because the two particles are of the same type there is no way to tell which of the two emerging electrons was the "incident" and which was the "target" particle. This is taken into account by adding the amplitudes for both processes but incuding a change of sign since we are exchanging two identical fermions. The resulting total amplitude is then

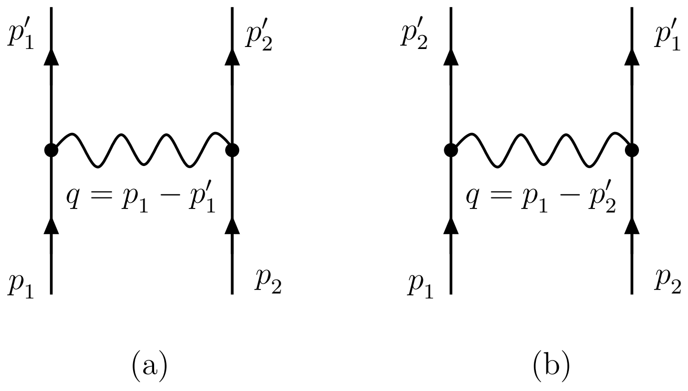


Figure D.13:

$$
\begin{aligned}
S_{fi} \;=\; & -\frac{1}{2}\sqrt{\frac{m_0^2}{E_1 E_2}}\sqrt{\frac{m_0^2}{E_1' E_2'}}(2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2') \\
& \times \Bigg\{ + \left[\overline{u}(p_1', s_1')(-ie\gamma_\mu)u(p_1, s_1)\right]\frac{i4\pi}{(p_1 - p_1')^2 + i\epsilon}\left[\overline{u}(p_2', s_2')(-ie\gamma_\mu)u(p_2, s_2)\right] \\
& - \left[\overline{u}(p_2', s_2')(-ie\gamma_\mu)u(p_1, s_1)\right]\frac{i4\pi}{(p_1 - p_2')^2 + i\epsilon}\left[\overline{u}(p_1', s_1')(-ie\gamma_\mu)u(p_2, s_2)\right]\Bigg\}
\end{aligned}
\tag{D.417}
$$

D.6.1 Averaging over Spin

The squared invariant matrix element averaged over initial and final spin is

$$
\begin{aligned}
& \overline{|M_{fi}|^2} \\
= \;& e^4(4\pi)^2\frac{1}{4}\sum_{s_1', s_1}\sum_{s_2', s_2}\Bigg|\overline{u}(p_1', s_1')\gamma_\mu u(p_1, s_1)\frac{1}{(p_1 - p_1')^2}\overline{u}(p_2', s_2')\gamma^\mu u(p_2, s_2) \\
& - \overline{u}(p_2', s_2')\gamma_\mu u(p_1, s_1)\frac{1}{(p_1 - p_2')^2}\overline{u}(p_1', s_1')\gamma^\mu u(p_2, s_2)\Bigg|^2
\end{aligned}
\tag{D.418}
$$

We gain familiarity with calculating spin averaging, the reader will be left to put together the results to obtain the final answer. Consider the mod-squared terms in the square $|\cdots|^2$. Take the first such term:

$$
\begin{aligned}
&\sum_{s_1',s_1}\sum_{s_2',s_2} \left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\left(\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right) \\
&\quad \times \left(\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\right)^* \left(\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right)^* \\
= &\left[\sum_{s_1',s_1} \left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\left(\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\right)^*\right] \\
&\quad \times \left[\sum_{s_2',s_2} \left(\overline{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\right)\left(\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right)^*\right]
\end{aligned}
\tag{D.419}
$$

It is sufficient to consider just the first sum over s_1', s_1, since the second term s_2', s_2 has the same structure. The reader should see section D.14.8 on how to turn this into the following trace

$$
Tr\left[\gamma_\mu \frac{p_{1\alpha}\gamma^\alpha + m_0}{2m_0}\gamma_\nu \frac{p_{1\beta}'\gamma^\beta + m_0}{2m_0}\right]. \tag{D.420}
$$

Now we consider the more complicated mixed terms in the square $|\cdots|^2$. The first of these is

$$
\begin{aligned}
&\sum_{s_1',s_1}\sum_{s_2',s_2} \left[\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\left(\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right)\right] \\
&\quad \times \left[\left(\overline{u}(p_2',s_2')\gamma_\nu u(p_1,s_1)\right)\left(\overline{u}(p_1',s_1')\gamma^\nu u(p_2,s_2)\right)\right]^* \\
&\sum_{s_1',s_1}\sum_{s_2',s_2} \left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\left(\overline{u}(p_1,s_1)\gamma_\nu u(p_2',s_2')\right) \\
&\quad \times \left(\overline{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\right)\left(\overline{u}(p_2,s_2)\gamma_\nu u(p_1',s_1')\right) \\
= &\sum_{s_1'}\sum_{s_2'} \left(\overline{u}(p_1',s_1')\gamma_\mu \frac{p_{1\alpha}\gamma^\alpha + m_0}{2m_0}\gamma_\nu u(p_2',s_2')\right) \\
&\quad \times \left(\overline{u}(p_2',s_2')\gamma_\mu \frac{p_{2\alpha}\gamma^\alpha + m_0}{2m_0}\gamma_\nu u(p_1',s_1')\right)
\end{aligned}
\tag{D.421}
$$

where we have used the identity

$$
\sum_{s_1} u_\alpha(p_1,s_1)\overline{u}_\beta(p_1,s_1) = \left(\frac{p_{1\alpha}\gamma^\alpha + m_0}{2m_0}\right)_{\alpha\beta}
$$

We use this identity another two times to obtain the final result

$$Tr\left[\gamma_\mu \frac{p_{1\alpha}\gamma^\alpha + m_0}{2m_0}\gamma_\nu \frac{p'_{2\beta}\gamma^\beta + m_0}{2m_0}\gamma^\mu \frac{p_{2\gamma}\gamma^\gamma + m_0}{2m_0}\gamma^\nu \frac{p'_{1\delta}\gamma^\delta + m_0}{2m_0}\right]. \tag{D.422}$$

In section D.14.1 we provide theorems on the trace of $\gamma-$matrices sufficient for the reader to evaluate the above expresions.

D.7 Electron-Positron Scattering

D.7.1 Scattering of an Positron off a Coulomb Potential

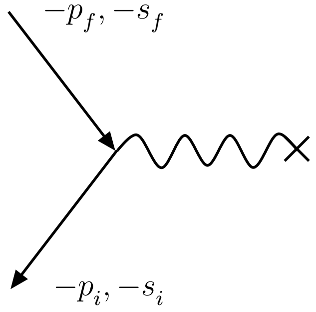


Figure D.14: The incoming positron with momentum p_i and spin s_i is described by an outgoing electron with negative energy, with momentum $-p_i$ and spin $-s_i$. Similarly for the outgoing positron.

$$S_{fi} = -iZe^2 \frac{1}{V}\sqrt{\frac{m_0^2}{E_f E_i}}\overline{v}(p_i, s_i)\gamma^0 v(p_f, s_f) \int d^4x\, e^{i(p_f - p_i)\cdot x}\frac{1}{|x|}. \tag{D.423}$$

D.7.2 Electron-Positron Scattering Amplitude

Make the replacements

an incoming electron spinor $u(p_i, s_i) \rightarrow$ an outgoing positron spinor $\overline{v}(p_f, s_f)$

$$S_{fi}(dir.) = -\frac{1}{2}\sqrt{\frac{m_0^2}{E_1\overline{E}_2'}}\sqrt{\frac{m_0^2}{E_1'\overline{E}_2}}(2\pi)^4\delta^4(p_1+\overline{p}_2'-p_1'-\overline{p}_2)$$
$$\times\,[\overline{u}(p_1',s_1')(-ie\gamma_\mu)u(p_1,s_1)]\frac{i4\pi}{(p_1-p_1')^2+i\epsilon}[\overline{v}(\overline{p}_2,\overline{s}_2)(-i(-e)\gamma_\mu)v(p_2',s_2')] \tag{D.424}$$

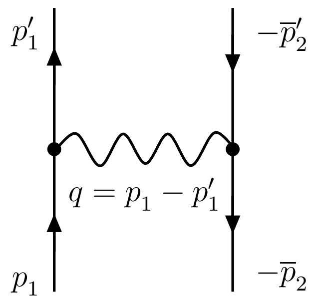


Figure D.15:

The exchange amplitude

$$S_{fi}(exch.) = \frac{1}{2}\sqrt{\frac{m_0^2}{E_1E_2}}\sqrt{\frac{m_0^2}{E_1'E_2'}}(2\pi)^4\delta^4(p_1+\overline{p}_2'-p_1'-\overline{p}_2)$$
$$\times\,[\overline{v}(\overline{p}_2,\overline{s}_2)(-i(-e)\gamma_\mu)u(p_1,s_1)]\frac{i4\pi}{(p_1-p_1')^2+i\epsilon}[\overline{u}(p_1',s_1')(-ie\gamma_\mu)u(\overline{p}_2',\overline{s}_2')] \tag{D.425}$$

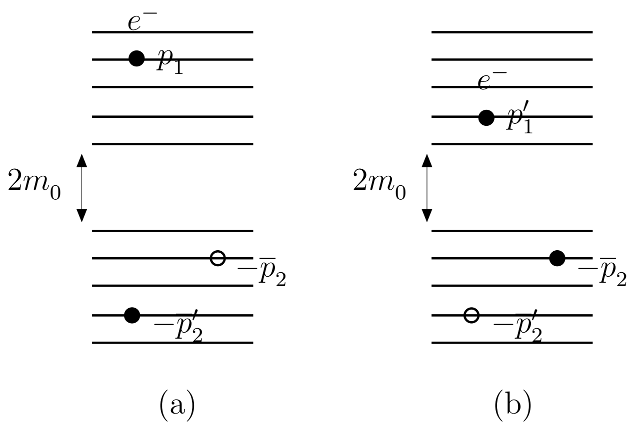


Figure D.16: (a) The intial state. (b) The final state.

The exclusion principle requires that antisymmetric combinations of amplitudes be chosen for processes which differ only by an exchange of particles. In the final state, fig (**??**), has to be antisymmetric with respect to the exchange $p_1' \leftrightarrow -\overline{p}_2$

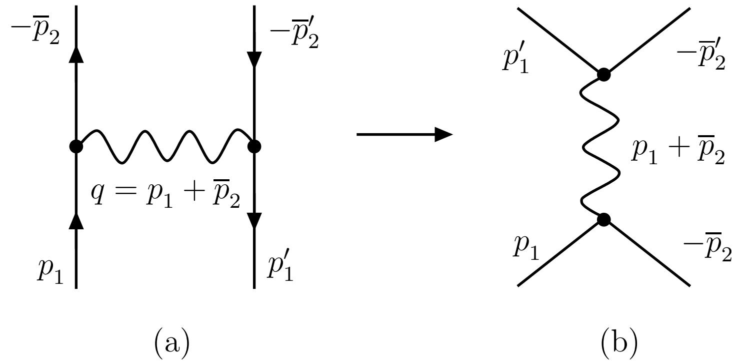


Figure D.17: (a) . (b) The exchange graph is usually written this way.

D.7.3 Remarks on the Form of the $S-$Matrix element

D.7.4 Crossing Symmetry

The squared invariant matrix element for electron-positron scattering can be obtained from the squared invariant matrix element for electron-electron scattering by making the following substitutions of four-momenta

$$\begin{aligned} p_1 &\rightarrow p_1 \\ p_1' &\rightarrow p_1' \\ p_2 &\rightarrow -\overline{p}_2' \\ p_2' &\rightarrow -\overline{p}_2. \end{aligned} \tag{D.426}$$

D.8 Scattering of Polarised Dirac Particles

s^μ is a Lorentz vector which is properly defined in the rest system of the particle where it reduces to a spacial unit vector

$$(s^\mu)_{RS} = (0, \mathbf{s}'). \tag{D.427}$$

We wish to obtain the components of s^μ in a frame in which the particle moves with momentum $\mathbf{p}$. What is the Lorentz transformation formula os an aritrary four-vecto a^μ for when $\mathbf{v}$ is not parallel to the x-axis? Consider

$$a^{0'} = \gamma\left(a^0 - \mathbf{v}\cdot\mathbf{a}\right), \qquad \mathbf{a}' = \mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{v^2}(\gamma-1) - \gamma a^0\right)\mathbf{v} \tag{D.428}$$

where

$$\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}.$$

Specialising to $\mathbf{v} = (v, 0, 0)$ we find

$$a^{0'} = \gamma\left(a^0 - v a_x\right), \tag{D.429}$$

$$(a'_x, a'_y, a'_z) = (a_x, a_y, a_z) - \left(\frac{\mathbf{v}\cdot a}{v^2}(1-\gamma) - \gamma a_t\right)(v, 0, 0) \tag{D.430}$$

which reads separately

$$a'_t = \gamma(a_t - v a_x), \quad a'_x = \gamma(a_x - v a_t), \quad a'_y = a_y, \quad a'_z = a_z \tag{D.431}$$

We want the inverse of this transformation which is easily obtained by making the replacement $\mathbf{v} \to -\mathbf{v}$. We have that $s^{0'} = 0$. We obtain for s^μ,

$$s^\mu = \left[\gamma\mathbf{v}\cdot\mathbf{s}'\ ,\ \mathbf{s}' + \frac{\mathbf{v}\cdot\mathbf{s}'}{v^2}(\gamma-1)\mathbf{v}\right] \tag{D.432}$$

We use the following:

$$\mathbf{v} = \frac{\mathbf{p}}{E}, \tag{D.433}$$

$$E/\gamma = m_0, \tag{D.434}$$

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\mathbf{p}^2/E^2}} \\ &= \frac{E}{\sqrt{E^2-\mathbf{p}^2}} = \frac{E}{m_0}, \end{aligned} \tag{D.435}$$

$$\begin{aligned} \frac{\gamma - 1}{E^2 - \mathbf{p}^2} &= \frac{E/m_0 - 1}{(E + m_0)(E - m_0)} \\ &= \frac{1}{m_0(E + m_0)}, \end{aligned} \tag{D.436}$$

so that () finally becomes

$$s^\mu = \left[\frac{\mathbf{p} \cdot \mathbf{s}'}{m_0} \ , \ \mathbf{s}' + \frac{\mathbf{p} \cdot \mathbf{s}'}{m_0(E + m_0)} \mathbf{p} \right] \tag{D.437}$$

Because of the Lorentz invariance of the four-diemnsional scalar product it follows

$$s_\mu s^\mu = (s_\mu)_{RS} (s^\mu)_{RS} = -\mathbf{s} \cdot \mathbf{s} = -1 \tag{D.438}$$

and

$$p^\mu s_\mu = (p^\mu)_{RS} (s_\mu)_{RS} = (m_0, 0, 0, 0) \begin{pmatrix} 0 \\ -s_x \\ -s_y \\ -s_z \end{pmatrix} = 0. \tag{D.439}$$

So we have the normalisation and orthogonality relations

$$s^2 = -1, \quad p \cdot s = 0. \tag{D.440}$$

We will specialise to helicity states, namely states where the spin points in the direction (or opposite direction) of the momentum:

$$s'_\lambda = \lambda \frac{\mathbf{p}}{|\mathbf{p}|}, \quad \text{where} \quad \lambda = \pm 1. \tag{D.441}$$

Substituting this into (D.437) leads to the spin four-vector

$$s^\mu = \lambda \left(\frac{|\mathbf{p}|}{m_0} \ , \ \frac{\mathbf{p}}{|\mathbf{p}|} + \frac{\mathbf{p}^2}{m_0(E + m_0)} \frac{\mathbf{p}}{|\mathbf{p}|} \right) = \lambda \left(\frac{|\mathbf{p}|}{m_0} \ , \ \frac{E}{m_0} \frac{\mathbf{p}}{|\mathbf{p}|} \right) \tag{D.442}$$

After this preliminar work we now look the cross section for Coulomb scattering.

D.8.1 Polarised Electron Scattering of a Coulomb Potential

$$\frac{d\sigma}{d\Omega}(s_i, s_f) = \frac{4Z^4\alpha^2 m_0^2}{|q|^4}|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2 \tag{D.443}$$

We introduce auxialiary summations over the spin orientations s_i and s_f using the spin projection operator $\mathbf{\Sigma}(s)$ which suppresses the "wrong" spin state $u(p, -s)$.

$$\begin{aligned}\frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\Big(\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\Big)\ \Big(u^\dagger(p_i, s_i)\gamma_0^\dagger\gamma_0^\dagger u(p_f, s_f)\Big) \\ &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\sum_{s_i', s_f'}\Big(\overline{u}(p_f, s_f')\gamma^0\hat{\Sigma}(s_i)u(p_i, s_i')\Big)\ \Big(\overline{u}(p_i, s_i')\gamma^0\hat{\Sigma}(s_f)u(p_f, s_f')\Big)\end{aligned} \tag{D.444}$$

The same calculation as in (D.362) but with the replacement $\gamma^0_{\alpha\beta} \rightarrow \sum_{\delta=1}^4 \gamma^0_{\alpha\delta}(\hat{\Sigma})_{\delta\beta}$

$$\begin{aligned}\frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0\hat{\Sigma}(s_i)\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\hat{\Sigma}(s_f)\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right] \\ &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0\frac{1+\gamma_5 s_i^\sigma\gamma_\sigma}{2}\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\frac{1+\gamma_5 s_f^\delta\gamma_\delta}{2}\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right]\end{aligned} \tag{D.445}$$

D.8.2 When the Incoming Beam is Unpolarised

Before examinig the above we warm up by looking at when the scattering process in which the incoming beam is unpolarised, and ask if polarisation of the scaterred particle takes place. The above cross section is then replaced by

$$\frac{d\sigma}{d\Omega}(s_i, s_f) = \frac{1}{2}\frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0\frac{1+\gamma_5 s_i^\sigma\gamma_\sigma}{2}\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right] \tag{D.446}$$

The factor 1/2 comes from averaging over the initial spins.

Expanding (D.446) we find the traces

$$Tr\gamma_0\gamma_5\gamma_\sigma\gamma_0, \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_\mu\gamma_0\gamma_\nu, \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_\mu\gamma_0, \quad \text{and} \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_0\gamma_\nu.$$

It is easily seen this reduces to evaluating

$$Tr\gamma_5\gamma_\sigma, \quad Tr\gamma_5\gamma_\sigma\gamma_\mu\gamma_\nu, \quad \text{and} \quad Tr\gamma_5\gamma_\sigma\gamma_\mu.$$

The first two obviously vanish as the trace of the product of γ_5 and an odd number of γ matrices is zero

$$Tr\gamma_5\gamma_\mu\dots\gamma_\nu = (-1)^n Tr\gamma_\mu\dots\gamma_\nu\gamma_5 = (-1)^n Tr\gamma_5\gamma_\mu\dots\gamma_\nu$$

where we first used $\gamma_5\gamma_\mu = \gamma_\mu\gamma_5$ and then the cyclic property of the trace. We consider the last trace. Say first that $\mu = \nu$ then

$$Tr\gamma_5\gamma_\mu\gamma_\nu = Tr\gamma_5(\gamma_\mu)^2 = \eta^{\mu\mu}Tr\gamma_5.$$

As $\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$ and thus inparticular $\gamma^0\gamma^5 = -\gamma^5\gamma^0$ we get

$$\begin{aligned} Tr\gamma_5 &= Tr\gamma_5(\gamma^0)^2 \\ &= -Tr\gamma^0\gamma_5\gamma^0 \\ &= -Tr\gamma_5(\gamma^0)^2 \\ &= -Tr\gamma_5 = 0. \end{aligned}$$

Now if $\mu \neq \nu$ we choose λ that differs from μ and ν and use $\gamma_\lambda\gamma_5\gamma_\mu\gamma_\nu = (-1)^3\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda$

$$\begin{aligned} Tr\gamma_5\gamma_\mu\gamma_\nu &= Tr\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1}\gamma_\lambda \\ &= Tr\gamma_\lambda\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1} \\ &= (-1)^3 Tr\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\lambda^{-1} \\ &= -Tr\gamma_5\gamma_\mu\gamma_\nu = 0. \end{aligned}$$

Thus the cross section is independent of the final spin and agrees with half the unpolarised Mott scattering cross section

$$\frac{d\sigma}{d\Omega}(s_f) = \frac{1}{2}\frac{d\sigma_{Mott}}{d\Omega} \tag{D.447}$$

Thus at first order in perturbatin theory Coulomb scattering of electrons does not lead to polarisation of the incoming beam.

D.8.3 Polarised Scattering

We assume that the spin of the incoming electron is parallel to its direction of motion, i.e. it has well defined helicity $\lambda_i = +1$

$$
\begin{aligned}
s_{i_{\lambda_i}} &= \lambda_i \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0} \frac{\mathbf{p}_i}{|\mathbf{p}|} \right) \equiv \lambda_i s_i \\
s_{f_{\lambda_f}} &= \lambda_i \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0} \frac{\mathbf{p}_f}{|\mathbf{p}|} \right) \equiv \lambda_f s_f
\end{aligned} \tag{D.448}
$$

Dropping terms we know to zanish from the previous example, the polarised scattering cross section becomes

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0 \frac{1+\gamma_5 s_i^\sigma \gamma_\sigma}{2} \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma_0 \frac{1+\gamma_5 s_f^\mu \gamma_\mu}{2} \frac{(p_f)_\nu \gamma^\nu + m_0}{2m_0}\right] \\
&= \frac{4Z^4\alpha^2 m_0^2}{|q|^4} \frac{1}{4} \frac{1}{(2m_0)^2} \Big(Tr[\gamma_0((p_i)_\mu \gamma^\mu + m_0)\gamma_0((p_f)_\nu \gamma^\nu + m_0)] \\
&\quad + \lambda_i \lambda_f\, Tr[\gamma_0 \gamma_5 s_i^\sigma \gamma_\sigma ((p_i)_\mu \gamma^\mu + m_0)\gamma_0 \gamma_5 s_f^\delta \gamma_\delta ((p_f)_\nu \gamma^\nu + m_0)] \Big)
\end{aligned} \tag{D.449}
$$

Here we define the degree of polarisation P of the scattered particles by as the difference between counting rates for the positive and negative helicities, normalised by the total counting rate:

$$
P = \frac{d\sigma(\lambda_f = +1) - d\sigma(\lambda_f = -1)}{d\sigma(\lambda_f = +1) + d\sigma(\lambda_f = -1)} \tag{D.450}
$$

If the initial state is fully polarised, e.g. $\lambda_i = +1$, the final degree of polarisation becomes, using ()

$$
P = \frac{Tr[\gamma_0 \gamma_5 s_i^\sigma \gamma_\sigma ((p_i)_\mu \gamma^\mu + m_0)\gamma_0 \gamma_5 s_f^\mu \gamma_\mu ((p_f)_\nu \gamma^\nu + m_0)]}{Tr[\gamma_0((p_i)_\mu \gamma^\mu + m_0)\gamma_0((p_f)_\nu \gamma^\nu + m_0)]} \tag{D.451}
$$

The evaluation of the trace in the denominator is done along the same lines as early calculations. Expand the numerator, using that the trace of an odd number of γ matrices vanishes,

$$
\begin{aligned}
& Tr[\gamma_0\gamma_5 s_i^\sigma \gamma_\sigma((p_i)^\mu\gamma^\mu + m_0)\gamma_0\gamma_5 s_f^\mu \gamma_\mu((p_f)^\nu\gamma^\nu + m_0) \\
= \;& s_i^\sigma p_i^\mu s_f^\delta p_f^\nu \; Tr[\gamma_0\gamma_\sigma\gamma_\mu\gamma_0\gamma_\delta\gamma_\nu] + m_0^2 s_i^\sigma s_f^\delta \; Tr[\gamma_0\gamma_\sigma\gamma_0\gamma_\delta]
\end{aligned}
\tag{D.452}
$$

To evaluate the first trace we generalise the result of () to arbitrary even number of γ matrices. We know

$$
Tr\gamma^{\mu_1}\ldots\gamma^{\mu_n} = 2\eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\ldots\gamma^{\mu_n} - Tr\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\ldots\gamma^{\mu_n}
$$

Repeating this procedure we get

$$
\begin{aligned}
Tr\gamma^{\mu_1}\ldots\gamma^{\mu_n} &= 2\eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\ldots\gamma^{\mu_n} - \\
&\quad +2\eta^{\mu_1\mu_n}Tr\gamma^{\mu_2}\ldots\gamma^{\mu_n} \\
&\quad -Tr\gamma^{\mu_2}\ldots\gamma^{\mu_n}\gamma^{\mu_1}
\end{aligned}
$$

Using the cyclic property of traces we get

$$
Tr\gamma^{\mu_1}\ldots\gamma^{\mu_n} = \eta^{\mu_1\mu_2}\; Tr\gamma^{\mu_3}\ldots\gamma^{\mu_n} - \cdots + \eta^{\mu_1\mu_n}\; Tr\gamma^{\mu_2}\ldots\gamma^{\mu_n}
\tag{D.453}
$$

To do the calculation we need the following scalar products:

It satifies the orthogonality relations. Firstly

$$
\begin{aligned}
p_i \cdot s_i &= (E, \mathbf{p}_i)\cdot\left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_i}{|\mathbf{p}|}\right) \\
&= \frac{E}{m_0}(|\mathbf{p}| - \frac{\mathbf{p}_i\cdot\mathbf{p}_i}{|\mathbf{p}|}) = 0.
\end{aligned}
\tag{D.454}
$$

Similarly we have

$$
p_f \cdot s_f = 0. \tag{D.455}
$$

$$
\begin{aligned}
p_i \cdot s_f &= (E, \mathbf{p}_i)\cdot\left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_f}{|\mathbf{p}|}\right) \\
&= \frac{E}{m_0}(|\mathbf{p}| - \frac{\mathbf{p}_i\cdot\mathbf{p}_f}{|\mathbf{p}|}) \\
&= \frac{E|\mathbf{p}|}{m_0}(1 - \cos\theta)
\end{aligned}
\tag{D.456}
$$

$$p_f \cdot s_i = \frac{E|\mathbf{p}|}{m_0}(1 - \cos\theta) \tag{D.457}$$

Similarly we have

$$\begin{aligned} s_i \cdot s_f &= \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_i}{|\mathbf{p}|}\right) \cdot \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_f}{|\mathbf{p}|}\right) \\ &= \frac{1}{m_0^2}(\mathbf{p}^2 - \frac{E^2}{\mathbf{p}^2}\mathbf{p}_i \cdot \mathbf{p}_f) \\ &= \frac{1}{m_0^2}(\mathbf{p}^2 - E^2 \cos\theta) \end{aligned} \tag{D.458}$$

$$p_i \cdot p_f = E^2 - \mathbf{p}^2 \cos\theta \tag{D.459}$$

We leave the details of the calculation to the reader. The result leads to

$$P = 1 - \frac{2\sin\frac{\theta}{2}}{\left(\frac{E}{m_0}\right)^2 \cos\frac{\theta}{2} + \sin\frac{\theta}{2}} \tag{D.460}$$

In the nonrelativistic limit $E \to m_0$ this reduces to

$$P \simeq 1 - 2\sin\frac{\theta}{2} = \cos\theta. \tag{D.461}$$

D.9 Bremsstrahlung

When electrons scatter at protons or in the field of a nucleus, they can emit real photons. Bremsstrahlung is a second order process

$$S_{fi}^{(2)} = -ie^2 \int d^4y d^4x \overline{\psi}_f(x) A_\mu(x) \gamma^\mu S_F(x-y) A_\nu(y) \gamma^\nu \psi_i(y) \tag{D.462}$$

the outgoing photon by

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\, \epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{D.463}$$

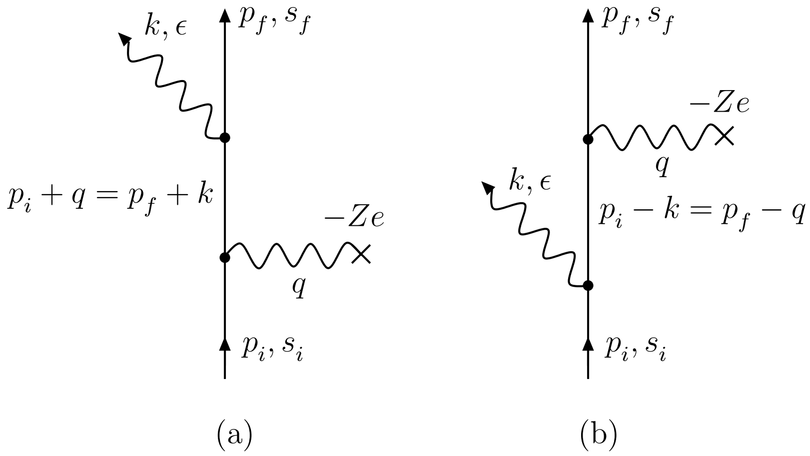


Figure D.18: (a) . (b) .

the incoming electron

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot x} \tag{D.464}$$

the outgoing electron

$$\psi_f(x) = \sqrt{\frac{m_0}{E_f V}} u(p_f, s_f) e^{ip_f \cdot x}. \tag{D.465}$$

and the Coulomb potetial is

$$A_0^{coul}(x) = -\frac{Ze}{|\mathbf{x}|} \tag{D.466}$$

$$\begin{aligned} S_{fi} &= e^2 \int d^4y d^4x \overline{\psi}_f(x) \Big[(-iA_\mu(x,k)\gamma^\mu) iS_F(x-y)(-i\gamma^0) A_0^{coul}(y) \\ &\quad + (-i\gamma^0) A_0^{coul}(x) iS_F(x-y)(-iA_\mu(y,k)\gamma^\mu) \Big] \psi_i(x) \end{aligned} \tag{D.467}$$

Again we transform to momentum space. The Fourier transformation of the Coulomb potential

$$-\frac{Ze}{|\mathbf{x}|} = -Ze4\pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{|\mathbf{q}|^2} e^{-iq\cdot x} \tag{D.468}$$

Substituting all of the above into (D.467)

$$\begin{aligned}
S_{fi} &= e^2 \int d^4y d^4x \left(\sqrt{\frac{m_0}{E_f V}} \overline{u}(p_f, s_f) e^{ip_f \cdot x} \right) \left[-i \left(\sqrt{\frac{4\pi}{2\omega V}}\, \epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \right) \gamma^\mu \times \right. \\
&\quad \times \left(\int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p_\alpha \gamma^\alpha - m_0 + i\epsilon} \right) \times (-i\gamma^0) \left. \left(-Ze4\pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{|\mathbf{q}|^2} e^{-iq\cdot y} \right) \right] \\
&\quad \times \left(\sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot y} \right) + exch. \qquad \text{(D.469)}
\end{aligned}$$

rearanging

$$\begin{aligned}
S_{fi} &= -\frac{Ze^3 4\pi}{V^{3/2}} \sqrt{\frac{4\pi}{2\omega}} \sqrt{\frac{m_0^2}{E_f E_i}} \int d^4y d^4x \frac{d^3q}{(2\pi)^3} \frac{d^4p}{(2\pi)^4} \\
&\quad \times \overline{u}(p_f, s_f) e^{ip_f\cdot x} \left[-i\epsilon_\mu \gamma^\mu (e^{-ik\cdot x} + e^{ik\cdot x})\, \frac{ie^{-ip\cdot(x-y)}}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}\, (-i\gamma^0) \frac{e^{-iq\cdot y}}{|\mathbf{q}|^2} \right. \\
&\quad \left. (-i\gamma^0) \frac{e^{-iq\cdot y}}{|\mathbf{q}|^2}\, \frac{ie^{-ip\cdot(x-y)}}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}\, (-i\epsilon_\mu \gamma^\mu)(e^{-ik\cdot x} + e^{ik\cdot x}) \right] u(p_i, s_i) e^{-ip_i\cdot y}
\end{aligned}$$
(D.470)

Peforming the integrations

$$\begin{aligned}
&\int d^4x (e^{ix\cdot(p_f - k - p)} + e^{ix\cdot(p_f + k - p)}) \int d^4y e^{iy\cdot(-q+p-p_i)} \\
&= [(2\pi)^4 \delta^4(p_f - k - p) + (2\pi)^4 \delta^4(p_f + k - p)](2\pi)^4 \delta^4(q + p - p_i) \\
&\int d^4y (e^{iy\cdot(p-k-p_i)} + e^{iy\cdot(p+k-p_i)}) \int d^4x e^{ix\cdot(p_f + q - p)} \\
&= [(2\pi)^4 \delta^4(p - k - p_i) + (2\pi)^4 \delta^4(p + k - p_i)](2\pi)^4 \delta^4(-q + p - p_i)
\end{aligned}$$

The $S-$matrix becomes

$$
\begin{aligned}
S_{fi} &= -\frac{Ze^3 4\pi}{V^{3/2}}\sqrt{\frac{4\pi}{2\omega}}\sqrt{\frac{m_0^2}{E_f E_i}}\int d^4y d^4x \frac{d^3q}{(2\pi)^3}\frac{d^4p}{(2\pi)^4} \\
&\times\Big\{[(2\pi)^4\delta^4(p_f-k-p)+(2\pi)^4\delta^4(p_f+k-p)](2\pi)^4\delta^4(q+p-p_i) \\
&\times\overline{u}(p_f,s_f)(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_\alpha\gamma^\alpha-m_0+i\epsilon}(-i\gamma^0)\frac{1}{|\mathbf{q}|^2}u(p_i,s_i) \\
&+[(2\pi)^4\delta^4(p-k-p_i)+(2\pi)^4\delta^4(p+k-p_i)](2\pi)^4\delta^4(-q+p-p_i) \\
&\times\overline{u}(p_f,s_f)(-i\gamma^0)\frac{1}{|\mathbf{q}|^2}\frac{i}{p_\alpha\gamma^\alpha-m_0+i\epsilon}(-i\epsilon_\mu\gamma^\mu)u(p_i,s_i)\Big\} \qquad \text{(D.471)}
\end{aligned}
$$

In the following we will need the formula

$$
\int dx\delta(x-a)\delta(x-b)=\delta(a-b).
$$

Consider the momentum integrals coming from the direct graph, we find

$$
\begin{aligned}
&\int\frac{d^3q}{(2\pi)^3}\int\frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^4(p_f\pm k-p)(2\pi)^4\delta^4(p-q-p_i)f(p,|\mathbf{q}|) \\
&= \int\frac{d^3q}{(2\pi)^3}(2\pi)^4\delta^4(p_f\pm k-q-p_i)f(p,|\mathbf{q}|) \\
&= 2\pi\delta(E_f-E_i\pm\omega)f(p,|\mathbf{q}|) \qquad \text{(D.472)}
\end{aligned}
$$

where $q=p_f\pm k-p_i$ and $p=p_f\pm k$. There is something similary for the exchange graph.

Since we want to describe photon emmission the electron loses energy, $E_f<E_i$, which correspnds to $E_f=E_i-\omega$ - this is the arrangement measured experimentally. The $S-$matrix we require to describe emmision of a photon is then

$$
\begin{aligned}
S_{fi} &= -Ze^3 2\pi\delta(E_f+\omega-E_i)\sqrt{\frac{4\pi}{2\omega V}}\sqrt{\frac{m_0^2}{E_fE_iV^2}}\frac{4\pi}{|\mathbf{q}|^2} \\
&\times\ \overline{u}(p_f,s_f)\Bigg[(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu+k_\nu\gamma^\nu-m_0}(-i\gamma_0) \\
&\qquad +(-i\gamma_0)\frac{i}{p_{f\nu}\gamma^\nu+k_\nu\gamma^\nu-m_0}(-i\epsilon_\mu\gamma^\mu)\Bigg]u(p_i,s_i) \qquad \text{(D.473)}
\end{aligned}
$$

Using the relations $p_i^2=p_f^2=m_0^2$, $k^2=0$, we have

$$
\begin{aligned}
\frac{1}{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu - m_0 + i\epsilon} &= \frac{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu + m_0}{(p_f+k)^2 - m_0^2 + i\epsilon} \\
&= \frac{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu + m_0}{2p_f\cdot k + i\epsilon}.
\end{aligned} \tag{D.474}
$$

D.9.1 Remarks on the Form of the $S-$Matrix element

• At the free vertex, where a free photon with polarisation vector ϵ_μ is emitted, a factor

i) $(-i\epsilon_\mu\gamma^\mu)$ occurs

ii) and the normalisation factor of the photon $\sqrt{4\pi/2\omega V}$ enters.

D.9.2 Bremsstrahlung Cross Section

We can simplify the notation by writing

$$
S_{fi} = iZe^3 2\pi\delta(E_f + \omega - E_i)\sqrt{\frac{4\pi}{2\omega V}}\sqrt{\frac{m_0^2}{E_f E_i V^2}}\frac{4\pi}{|\mathbf{q}|^2}\,\epsilon^\mu M_\mu(k), \tag{D.475}
$$

where

$$
M_\mu(k) = \overline{u}(p_f,s_f)\left[\gamma_\mu\frac{p_{f\alpha}\gamma^\alpha + k_\alpha\gamma^\alpha + m_0}{2p_f\cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{p_{i\alpha}\gamma^\alpha - k_\alpha\gamma^\alpha + m_0}{-2p_i\cdot k + i\epsilon}\gamma_\mu\right]u(p_i,s_i) \tag{D.476}
$$

The bremsstrahlung cross section is given by

$$
\begin{aligned}
d\sigma &= \frac{1}{\frac{|\mathbf{v_i}|}{V}T}\,|S_{fi}|^2\,\frac{V d^3k}{(2\pi)^3}\frac{V d^3p_f}{(2\pi)^3} \\
&= \frac{Z^2e^6}{|\mathbf{v_i}|}\frac{4\pi}{2\omega}\frac{m_0^2}{E_fE_i}\,\frac{(4\pi)^2}{|\mathbf{q}|^4}\,|\epsilon^\mu M_\mu(k)|^2\,2\pi\delta(E_f+\omega-E_i)\frac{d^3k}{(2\pi)^3}\frac{d^3p_f}{(2\pi)^3}
\end{aligned} \tag{D.477}
$$

D.9.3 Sum Over Polarisations of Photon

We know that gauge invariance implies the condition of conservered current

$$\frac{\partial J_\mu(x)}{\partial x_\mu} = 0, \tag{D.478}$$

where $J_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$. In momentum space the conservation condition reads

$$k^\mu J_\mu(k) = 0 \tag{D.479}$$

Now the matrix element $M_\mu(k)$ given in () is a quantum mchanical transition current for Bremsstrahlung in lowest order perurbation theory and also satifies

$$k^\mu M_\mu(k) = 0 \tag{D.480}$$

This condition is easily verified using $k_\mu\gamma^\mu p_\nu\gamma^\nu = -p_\mu\gamma^\mu k_\nu\gamma^\nu + 2p\cdot k$ and the dirac equation:

$$\overline{u}(p_f, s_f)(p_{f\mu}\gamma^\mu - m_0) = 0, \quad (p_{i\mu}\gamma^\mu - m_0)u(p_i, s_i) = 0$$

$$\begin{aligned}
k^\mu M_\mu(k) &= \overline{u}(p_f, s_f)\left[k_\mu\gamma^\mu\,\frac{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu + m_0}{2p_f\cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{p_{i\mu}\gamma^\mu - k_\mu\gamma^\mu + m_0}{-2p_i\cdot k + i\epsilon}\,k_\nu\gamma^\nu\right]u(p_i, s_i) \\
&= \overline{u}(p_f, s_f)\left[\frac{-(p_{f\nu}\gamma^\nu - m_0)k_\mu\gamma^\mu + 2p_f\cdot k + k^2}{2p_f\cdot k + i\epsilon}\gamma_0\right. \\
&\quad \left. +\gamma_0\frac{-k_\nu\gamma^\nu(p_{i\mu}\gamma^\mu - m_0) + 2p_i\cdot k - k^2}{-2p_i\cdot k + i\epsilon}\right]u(p_i, s_i) \\
&= \overline{u}(p_f, s_f)\left[\frac{2p_f\cdot k}{2p_f\cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{2p_i\cdot k}{-2p_i\cdot k + i\epsilon}\right]u(p_i, s_i) \\
&= 0.
\end{aligned} \tag{D.481}$$

We now perform the summation over the photon polarisation vectors $\epsilon_\mu(\mathbf{k}, \lambda)$ with $\lambda = 1, 2$. The quantity of interest is

$$\overline{|\epsilon\cdot M_\mu(k)|^2} = \sum_{\lambda=1,2}|\epsilon_\mu(\mathbf{k},\lambda)M^\mu(k)|^2 = \sum_{\lambda=1,2}\epsilon_\mu(\mathbf{k},\lambda)\epsilon^*_\nu(\mathbf{k},\lambda)M^\mu(k)M^{*\nu}(k) \tag{D.482}$$

We work in the radiation gauge and choose a particular coordinate which simplifies the calculation. Consider the coordinate system such that the momentum vector $\mathbf{k}$ points in the z-direction

$$k^\mu = \omega(1,0,0,1) \tag{D.483}$$

We choose the two transverse polarisation vectors

$$\begin{aligned}\epsilon(\mathbf{k},1) &= (0,1,0,0),\\ \epsilon(\mathbf{k},2) &= (0,0,1,0).\end{aligned} \tag{D.484}$$

Now we use the condition of current conservation

$$k^\mu M^\mu = \omega(M^0 - M^3) = 0, \tag{D.485}$$

which implies $M^0 = M^3$. Therefore we can write

$$\overline{|\epsilon \cdot M|^2} = M^1 M^{*1} + M^2 M^{*2} + M^3 M^{*3} - M^0 M^{*0} = -M_\mu M^{*\mu} \tag{D.486}$$

Obviously this is Lorentz covariant. In general we have

$$\sum_{\lambda=1,2} \epsilon_\mu(\mathbf{k},1)\epsilon_\nu(\mathbf{k},2) = -\eta_{\mu\nu} + \text{gauge terms}. \tag{D.487}$$

The additional terms are proportional to k_μ or k_ν and thus do not contribute to any observable quantitiy since the sum is multiplied by conserved currents which satisfy $k \cdot J$.

D.9.4 The Infrared Catastrophe

A photon may be emmited which is too soft to be detected because of the energy resolution ΔE of the apparatus. Consequently, the experimental cross section is the sum the cross section for bremsstrahlung of energy less than ΔE and second order (raidiative corrected) elastic cross section, i.e.,

$$\left(\frac{d\sigma}{d\Omega'}\right)_{Exp} = \left(\frac{d\sigma}{d\Omega'}\right)_B + \left(\frac{d\sigma}{d\Omega'}\right)_{El}. \tag{D.488}$$

Here $(d\sigma/\Omega')_B$ is the soft bremsstrahlung cross section integrated over the range of photon energy $0 \ll \omega \ll \Delta E$ and $(d\sigma/\Omega')_{El}$ is the cross section for raidiative corrected elastic electron off the Coulomb potential.

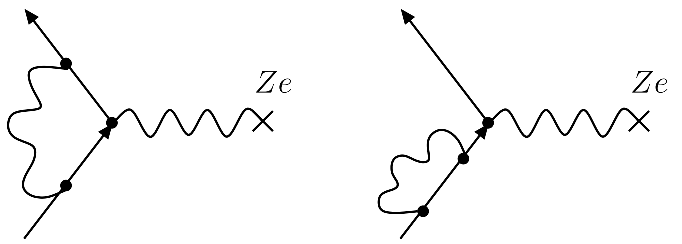


Figure D.19: The two types of lowest order radiative corrections to elastic scattering of an electron of a Coulomb potential.

D.10 Compton Scattering

We describe the incoming photon as a plane wave:

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\,\epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{D.489}$$

and the outgoing (scattered) photon by

$$A'_\mu(x',k') = \sqrt{\frac{4\pi}{2\omega' V}}\,\epsilon_\mu(k,\lambda)(e^{-ik'\cdot x'} + e^{ik'\cdot x'}) \tag{D.490}$$

the incoming electron

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i\cdot x} \tag{D.491}$$

the outgoing electron

$$\psi_f(x) = \sqrt{\frac{m_0}{E_f V}} u(p_f, s_f) e^{ip_f\cdot x}. \tag{D.492}$$

$$\begin{aligned} S_{fi} \;=\; & e^2 \int d^4x d^4y\, \overline{\psi}_f(x) \Big[(-iA_\mu(y,k')\gamma^\mu)\; iS_F(x-y)(-iA_\nu(y,k)\gamma^\nu) \\ & + (-iA_\mu(y,k)\gamma^\mu)\; iS_F(x-y)(-iA_\nu(y,k')\gamma^\nu)\Big]\,\psi_i(y). \end{aligned} \tag{D.493}$$

From previous experience we know to write this in momentum space to be

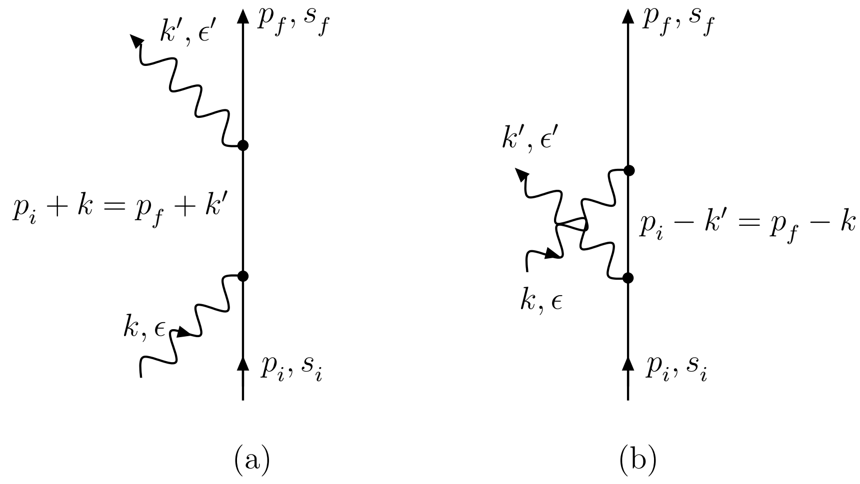


Figure D.20: The direct and exchange diagrams describing Compton scattering.

$$
\begin{aligned}
S_{fi} &= \frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_i E_f}}\sqrt{\frac{(4\pi)^2}{2\omega 2\omega'}}\,(2\pi)^4\delta^4(p_f + k' - p_i - k) \\
&\times\ \overline{u}(p_f, s_f)\left[(-i\epsilon'_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu - m_0}(-i\epsilon_\sigma\gamma^\sigma)\right. \\
&\qquad \left. +(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu - k'_\nu\gamma^\nu - m_0}(-i\epsilon'_\sigma\gamma^\sigma)\right] u(p_i, s_i) \qquad \text{(D.494)}
\end{aligned}
$$

In going from (D.493) to (D.494) we integrated over plane waves which gave delta functions which impose energy-momentum conservation at the vertices. It results in four different processes, two of which are not allowed kinematically: the emission or absorption of two photons by a free electron. A third process is not compatible with the kinematic conditions fixed by the experiment. The process describing Compton scattering corresponds to the followings constraints on four momentum:

$$
+k + p_i = +k' + p_f \qquad \text{(D.495)}
$$

The situation is similarly to what we encoutered in Bremsstrahlung in that not every term is physically relevant for the process considered. The term in the Compton scattering amplitude stems from the part $\exp(-ik \cdot x)$ of the photon field in (D.489) describing the absorption of a photon with four-momentum k^μ by the electron and from the part $\exp(-ik' \cdot x')$ of the photon field in (D.490) describing a photon emitted by the electron with four-momentum k'^μ.

D.10.1 Compton Scattering Cross Section

We split the S-matrix into two parts:

$$S_{fi} = -i\frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_i E_f}}\sqrt{\frac{(4\pi)^2}{2\omega 2\omega'}}(2\pi)^4\delta^4(p_f + k' - p_i - k)\ \epsilon^\mu(\mathbf{k}', \lambda')\epsilon^\nu(\mathbf{k}, \lambda)\ M_{\mu\nu} \tag{D.496}$$

Here $M_{\mu\nu}$ is the Compton tensor

$$M_{\mu\nu} = \overline{u}(p_f, s_f)\left[\gamma_\mu \frac{p_{i\alpha}\gamma^\alpha + k_\alpha\gamma^\alpha + m_0}{2p_i\cdot k + i\epsilon}\gamma_\nu + \gamma_\nu \frac{p_{i\alpha}\gamma^\alpha - k'_\alpha\gamma^\alpha + m_0}{-2p_i\cdot k' + i\epsilon}\gamma_\mu\right] u(p_i, s_i) \tag{D.497}$$

we have

$$k'^\mu M_{\mu\nu} = k^\nu M_{\mu\nu} = 0. \tag{D.498}$$

The proof is analogous to the bremsstrahlung case.

The cross section starts as

$$d\sigma = \int \frac{|S_{fi}|^2}{T|\mathbf{v}_{rel}|/V}\ \frac{Vd^3p_f}{(2\pi)^3}\frac{Vd^3k'}{(2\pi)^3} \tag{D.499}$$

with

$$\frac{|S_{fi}|^2}{T} = \frac{|S_{fi}|^2}{VT/V}$$

being the transition rate per unit volume and normalised to one electron per volume. $|\mathbf{v}_{\mathbf{rel}}|/V$ is the incoming photon flux.

$$d\sigma \quad = \quad \frac{e^4}{V^4}\frac{m_0^2}{E_i E_f} \tag{D.500}$$

We calculate the cross section in the laboratory frame.

$$p_i = (m_0, 0)$$

Also

$$|\mathbf{v}_{rel}| = |\mathbf{c} - \mathbf{v_e}| = |\mathbf{c}| = c$$

We use the covariant expression for the density of final states:

$$\frac{d^3p}{2E} = \int_{-\infty}^{\infty} d^4p \; \delta(p^2 - m_0^2)\Theta(p_0) \tag{D.501}$$

Averaging over initial and final electron spins and polarisations

$$\frac{1}{4}\sum_{pol}\sum_{spin} |\epsilon^{\mu}(\mathbf{k}', \lambda')\epsilon^{\nu}(\mathbf{k}, \lambda) \; M_{\mu\nu}|^2 = \frac{1}{4}\sum_{spin} M^{\mu\nu}M^*_{\mu\nu} \tag{D.502}$$

(on account of (D.498) and (D.487)).

the unpolarised differential cross section for Compton scattering is then

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m}\left(\frac{\omega'}{\omega}\right)^2 \left\{\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right\}. \tag{D.503}$$

D.11 Annihilation of Particle and Antiparticle

$$\begin{aligned} S_{fi} &= e^2 \int d^4x d^4y \; \overline{\psi}_+(x) \Big[(-iA_\mu(y,k')\gamma^\mu) \; iS_F(x-y)(-iA_\nu(y,k)\gamma^\nu) \\ &\quad + (-iA_\mu(y,k)\gamma^\mu) \; iS_F(x-y)(-iA_\nu(y,k')\gamma^\nu)\Big] \; \psi_-(y). \end{aligned} \tag{D.504}$$

To fit the experimental situation both photon outgoing plane waves should be used. We are lead to the following expression in momentum space:

$$\begin{aligned} S_{fi} &= \frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_+E_-}}\sqrt{\frac{(4\pi)^2}{\omega_1\omega_2}}(2\pi)^4\delta^4(k_1 + k_2 - p_+ - p_-) \\ &\times \; \overline{v}(p_+, s_+)\left[(-i\epsilon_{2\mu}\gamma^\mu)\frac{i}{p_{-\alpha}\gamma^\alpha - k_{1\alpha}\gamma^\alpha - m_0}(-i\epsilon_{1\nu}\gamma^\nu)\right. \\ &\quad \left. +(-i\epsilon_{1\mu}\gamma^\mu)\frac{i}{p_{-\alpha}\gamma^\alpha - k_{2\alpha}\gamma^\alpha - m_0}(-i\epsilon_{2\nu}\gamma^\nu)\right] u(p_-, s_-) \end{aligned} \tag{D.505}$$

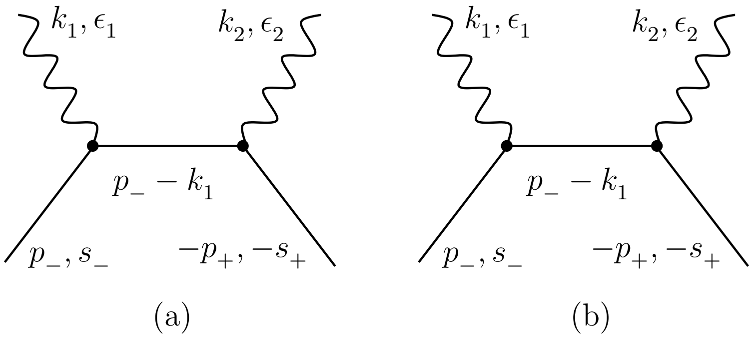


Figure D.21: Direct and exchange graph of pair annihilation into two photons.

D.12 Second Order Electron-Proton Scattering

Before we state Feynman's rules for QED we wish to examine how to formulate the amplitude for a second order scattering problem, namely the secon order S-matrix for electron-proton scattering.

The amplitude for second order electron proton scattering

$$S_{fi}^{(2)} = -ie^2 \int d^4x d^4y \overline{\psi}_f(x) A_\mu \gamma^\mu S_F(x-y) A_\nu \gamma^\mu \psi_i(y) \tag{D.506}$$

The second order electron current is given by

$$J_{\mu\nu}^{(2)}(x,y) = ie^2 \overline{\psi}_f(x)\ \gamma_\mu S_F(x-y) \gamma_\nu\ \psi_i(y) \tag{D.507}$$

We generalise the relation

$$A^\mu(x) = \int d^4y\ D_F(x-y)\ J^\mu(y).$$

by conjecturing

$$A_\mu(x) A_\nu(y) = \int d^4X d^4Y\ D_F(x-X) D_F(y-Y) J_{\mu\nu}^{p(2)}(X,Y). \tag{D.508}$$

By symmetry the proton current $J_{\mu\nu}^{p(2)}(X,Y)$ should be

$$J_{\mu\nu}^{p(2)}(X,Y) = ie^2\ \overline{\psi}_f^p(X)\ \gamma_\mu S_F(X-Y) \gamma_\nu\ \psi_i^p(Y) \tag{D.509}$$

Substituting () and (D.508) into

$$\begin{aligned} S_{fi}^{(2)}(dir.) &= e^2 e_p^2 \int d^4x d^y d^4X d^4Y \\ &\quad \times[\overline{\psi}_f(x)\gamma^\mu S_F(x-y)\gamma^\nu\ \psi_i(y)] \\ &\quad \times D_F(x-X)D_F(y-Y) \\ &\quad \times[\overline{\psi}_f^p(X)\gamma_\mu S_F(X-Y)\gamma_\nu\ \psi_i^p(Y)] \end{aligned} \tag{D.510}$$

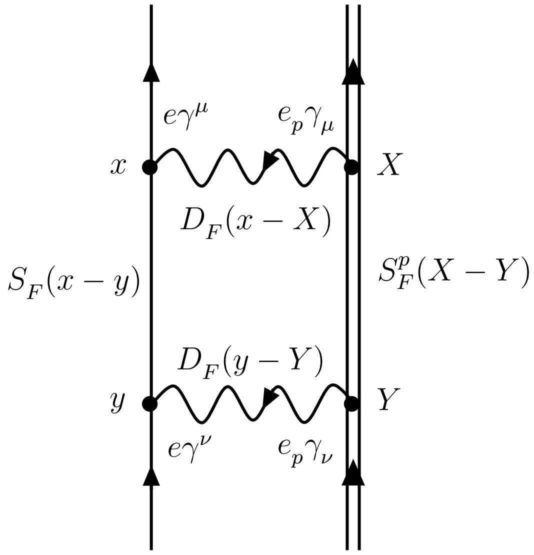


Figure D.22:

Since the two photons emitted by the proton current are indistiguishable the electron at x does not know whether the photon absorbed there has been emitted at X or Y. According to quantum mechanics we must coherently add the contribution coming from the corresponding exchange graph in (D.12).

$$\begin{aligned} S_{fi}^{(2)}(exch.) &= e^2 e_p^2 \int d^4x d^y d^4X d^4Y \\ &\quad \times[\overline{\psi}_f(x)\gamma^\mu S_F(x-y)\gamma^\nu\ \psi_i(y)] \\ &\quad \times D_F(x-Y)D_F(y-X) \\ &\quad \times[\overline{\psi}_f^p(X)\gamma_\nu S_F(X-Y)\gamma_\mu\ \psi_i^p(Y)] \end{aligned} \tag{D.511}$$

Notice how the indicies μ and ν in the 'proton current' term have been exchanged with respect to the direct term.

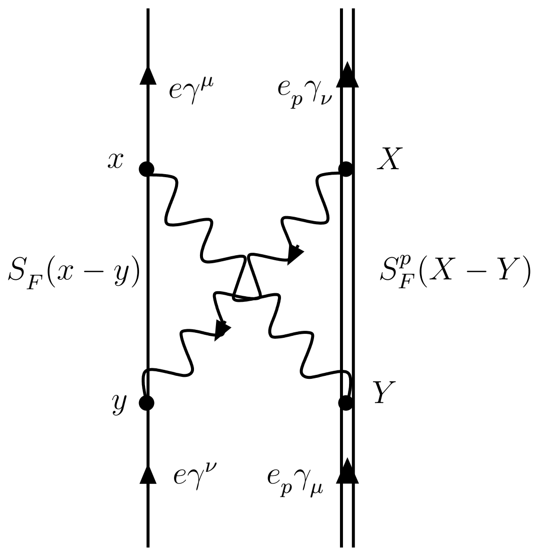


Figure D.23:

D.12.1 Feynman Diagram in Momentum Space

As always, the external particles (incoming and outgoing electron and proton) are described by plane waves. The direct term becomes

$$\begin{aligned} S_{fi}^{(2)}(dir.) &= \frac{(4\pi)^2 e^4}{V^2} \int d^4x d^4y d^4X d^4Y \sqrt{\frac{m_0^2}{E_f E_i}} \sqrt{\frac{M_0^2}{E_f^p E_i^p}} \\ & \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4P}{(2\pi)^4} \end{aligned} \tag{D.512}$$

It is easy to perform the integration over spacetime coordinates. It results in the product δ^4-functions:

$$\begin{aligned} &(2\pi)^4\delta^4(q_1+p-p_f)(2\pi)^4\delta^4(q_2-p+p_i) \\ &\times(2\pi)^4\delta^4(-q_2-P+P_i)(2\pi)^4\delta^4(-q_1+P-P_f). \end{aligned} \tag{D.513}$$

Each δ^4-function expresses the energy-momentum conservation at one of the four vertices. Now we can integrate over

$$
\begin{aligned}
S_{fi}^{(2)}(dir.) \quad = \quad & \frac{(4\pi)^2 e^4}{V^2} \sqrt{\frac{m_0^2}{E_f E_i}} \sqrt{\frac{M_0^2}{E_f^p E_i^p}} \, (2\pi)^4 \delta^4(P_f + p_f - P_i - p_i) \\
& \times \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{q_1^2 + i\epsilon} \frac{1}{(q-q_1)^2 + i\epsilon} \\
& \times \left[\overline{u}(p_f, s_f) \gamma^\mu \frac{1}{p_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - m_0 + i\epsilon} \gamma^\nu u(p_i, s_i) \right] \\
& \times \left[\overline{u}(P_f, S_f) \gamma_\mu \frac{1}{P_{f\alpha}\gamma^\alpha + q_{1\alpha}\gamma^\alpha - M_0 + i\epsilon} \gamma_\nu u(P_i, S_i) \right] \qquad \text{(D.514)}
\end{aligned}
$$

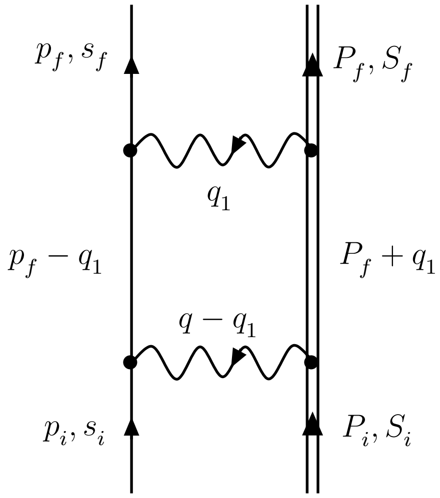


Figure D.24:

$$
\begin{aligned}
S_{fi}^{(2)}(exch.) \quad = \quad & \frac{(4\pi)^2 e^4}{V^2} \sqrt{\frac{m_0^2}{E_f E_i}} \sqrt{\frac{M_0^2}{E_f^p E_i^p}} \, (2\pi)^4 \delta^4(P_f + p_f - P_i - p_i) \\
& \times \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{q_1^2 + i\epsilon} \frac{1}{(q-q_1)^2 + i\epsilon} \\
& \times \left[\overline{u}(p_f, s_f) \gamma^\mu \frac{1}{p_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - m_0 + i\epsilon} \gamma^\nu u(p_i, s_i) \right] \\
& \times \left[\overline{u}(P_f, S_f) \gamma_\nu \frac{1}{P_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - M_0 + i\epsilon} \gamma_\mu u(P_i, S_i) \right] \qquad \text{(D.515)}
\end{aligned}
$$

D.12.2 Remarks on form of scattering Matrix

• each vertex contributes a factor of the form $-ie\gamma_\mu ...$

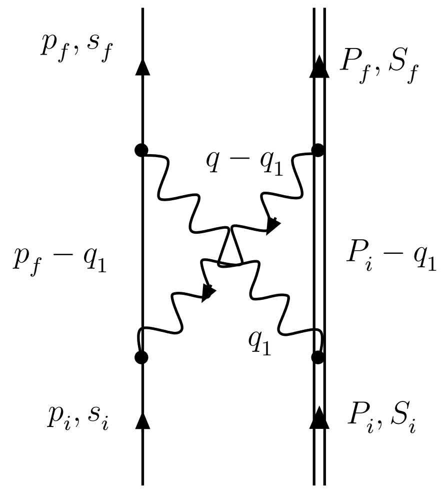


Figure D.25:

- each external particle yields a facttor $\sqrt{m_0/E}$.

D.13 Feynmann Rules of QED

There origins should be clear to the reader.

D.13.1 Scattering Amplitudes

We consider a scattering process in which two particles, they may be electrons, positrons or photons, with four-momenta

$$p_i = (E_i, \mathbf{p}_i), \quad i = 1, 2,$$

collide and produce N final particles with momenta

$$p_f = (E_f, \mathbf{p}_f), \quad f = 1, \ldots, N.$$

Individual energy-momentum conservation at each verrtex leads to conservation of total energy-momentum, represented by the delta function

$$\delta^4 \left(p_1 + p_2 - \sum_{i=1}^{n} p'_i \right).$$

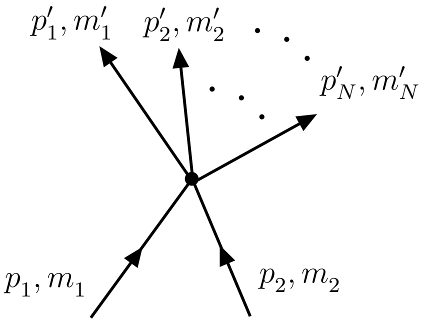


Figure D.26: We are considering reactions in which there are two particles in the initial state and n particles in the final state.

The scattering matrix element S_{fi} is given by

$$S_{fi} = i(2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{i=1}^{n} p'_i \right) \; M_{fi} \prod_{i=1}^{2} \sqrt{\frac{N_i}{2E_i V}} \prod_{i=1}^{n} \sqrt{\frac{N'_i}{2E'_i V}} \tag{D.516}$$

The normalisation factors N_i:

$$N_i = \begin{cases} 4\pi & photons \\ 2m_0 & spin - \frac{1}{2} particles \end{cases} \tag{D.517}$$

After drawing any Feynman diagram in momentummm space we see clearly how to translate the various lines in the graph directly into mathematical expressions.

The Feynman rules concern the calculation of the reduced scattering matrix element M_{fi}. A Feynman graph describing a scattering process consists of three parts:

(1) the external lines representing the wave functions of incomingand outgoing particles,

(2) the internal lines described by propagators, and

(3) the vertices representing the interactions between the particles.

With each external line one associates the following factors:

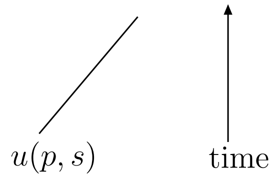


Figure D.27: An electron entering an interaction.

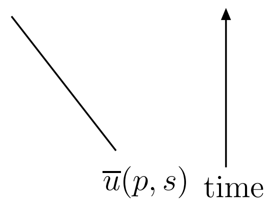


Figure D.28: An electron leaving an interaction.

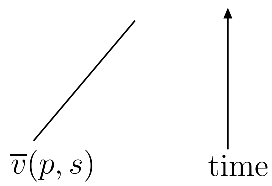


Figure D.29: .

electron:

$$iS_F = \frac{i(p_\mu \gamma^\mu + m_0)}{p^2 - m_0^2 + i\epsilon} \tag{D.518}$$

photon:

$$D_F^{\mu\nu}(k) = \frac{-i4\pi\ \eta^{\mu\nu}}{k^2 + i\epsilon} \tag{D.519}$$

Each vertex is associated with a factor

$$-ie\gamma_\mu . \tag{D.520}$$

a) a factor of -1 for each incoming positron (outgoing electtron with negative energy)

b) a factror of -1 in the case that two graphs which differ only by the exchange of two fermion lines.

c) a factor of -1 for each closed fermion loop.

For each internal loop, integrate over

$$\int \frac{d^4 q}{(2\pi)^4}$$

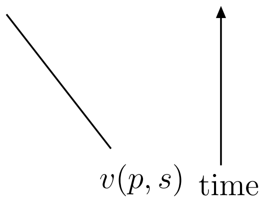


Figure D.30: .

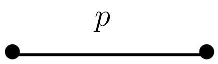


Figure D.31: Electron propagator.

D.13.2 Differential Cross Section

To obtain the transition rate to a group of final states with momenta in the intervals $f = 1, \ldots, N$, we multiply by the number of these states which is

$$\prod_f \frac{V d^3\mathbf{p}'_f}{(2\pi)^3} \tag{D.521}$$

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} N_1 N_2 (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{i=1}^{N} p'_i) S |M_{fi}|^2 \prod_f \frac{d^3\mathbf{p}'_f}{2E'_f (2\pi)^3} \tag{D.522}$$

The degeneracy factor S exists when the final state contains identical particles. Its taken into account by

$$S = \prod_k \frac{1}{g_k!}, \tag{D.523}$$

where g_k particles of the kind k in the final state.

External static electromagnetic fields

D.14 Details

D.14.1 Traces of Products of $\gamma-$Matrices

1. The trace of an odd number of $\gamma-$matrices vanishes.

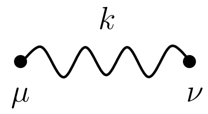


Figure D.32: Photon propagator.

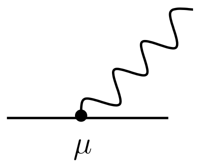


Figure D.33: Vertex

2. $a_\mu b_\nu \, Tr\gamma^\mu\gamma^\nu = 4a\cdot b.$

3.

$$\begin{aligned} Tr\gamma^{\mu_1}\ldots\gamma^{\mu_n} &= \eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\ldots\gamma^{\mu_n} \\ &\quad -\eta^{\mu_1\mu_3}Tr\gamma^{\mu_2}\gamma^{\mu_4}\ldots\gamma^{\mu_n} \\ &\quad +\eta^{\mu_1\mu_n}Tr\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}} \end{aligned} \tag{D.524}$$

4. $Tr\gamma^5 = 0.$

5. $Tr\gamma^5\gamma^\mu\gamma^\nu = 0.$

6. $a_\mu b_\nu c_\sigma d_\gamma Tr\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\gamma = -4i\epsilon^{\mu\nu\sigma\gamma}a_\mu b_\nu c_\sigma d_\gamma.$

7. $Tr\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_{2n}} = Tr\gamma_{\mu_{2n}}\ldots\gamma_{\mu_1}$.

8.

i) $\gamma_\mu\gamma^\mu = 4\mathbf{1}$

ii) $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$

iii) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu = 4\eta^{\nu\sigma}\mathbf{1}$

iv) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\mu = -2\gamma^\gamma\gamma^\sigma\gamma^\nu$

v) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\delta\gamma^\mu = 2\gamma^\delta\gamma^\nu\gamma^\sigma\gamma^\gamma + 2\gamma^\gamma\gamma^\sigma\gamma^\nu\gamma^\delta$

Proof.

1. See ()-()

2. See ()-()

3. See ()-()

4. See ()-()

5. See ()-()

6. See ()-()

7. We make use of the matrix $\hat{C}$, involved in charge conjugation. $\hat{C}$ has the property $\hat{C}\gamma_\mu\hat{C}^{-1} = -\gamma_\mu^T$. It follows that

$$\begin{aligned} Tr\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_{2n}} &= Tr(\hat{C}\gamma_{\mu_1}\hat{C}^{-1})(\hat{C}\gamma_{\mu_2}\hat{C}^{-1})\ldots(\hat{C}\gamma_{\mu_{2n}}\hat{C}^{-1}) \\ &= (-1)^{2n}\, Tr\gamma_{\mu_1}^T\gamma_{\mu_2}^T\ldots\gamma_{\mu_{2n}}^T \\ &= Tr[\gamma_{\mu_{2n}}\ldots\gamma_{\mu_1}]^T \\ &= Tr\gamma_{\mu_{2n}}\ldots\gamma_{\mu_1}. \end{aligned}$$

8.

i) $\gamma_\mu\gamma^\mu = \frac{1}{2}(\gamma_\mu\gamma^\mu + \gamma^\mu\gamma_\mu) = \frac{1}{2}2g^\mu{}_\mu 1 = 41$

ii)

$$\begin{aligned} \gamma_\mu\gamma^\nu\gamma^\mu &= \gamma_\mu(2\eta^{\mu\nu} - \gamma^\mu\gamma^\nu) \\ &= 2\gamma^\nu - 4\gamma^\nu \\ &= -2\gamma^\nu. \end{aligned}$$

iii)

$$\begin{aligned} \gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu &= \gamma_\mu\gamma^\nu(2\eta^{\mu\sigma} - \gamma^\mu\gamma^\sigma) \\ &= 2\gamma^\sigma\gamma^\nu - \gamma_\mu\gamma^\nu\gamma^\mu\gamma^\sigma \\ &= 2\gamma^\sigma\gamma^\nu + 2\gamma^\nu\gamma^\sigma \\ &= 2(2\eta^{\nu\sigma} - \gamma^\nu\gamma^\sigma) + 2\gamma^\nu\gamma^\sigma \\ &= 4\eta^{\nu\sigma}. \end{aligned}$$

□

iv)

$$\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\sigma (2\eta^{\mu\gamma} - \gamma^\mu \gamma^\gamma) \\
&= 2\gamma^\gamma \gamma^\nu \gamma^\sigma - (\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\mu)\gamma^\gamma \\
&= 2\gamma^\gamma \gamma^\nu \gamma^\sigma - 4\eta^{\nu\sigma}\gamma^\gamma \\
&= 2\gamma^\gamma (2\eta^{\nu\sigma} - \gamma^\sigma \gamma^\nu) - 4\eta^{\nu\sigma}\gamma^\gamma \\
&= -2\gamma^\gamma \gamma^\sigma \gamma^\nu.
\end{aligned}$$

□

v)

$$\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\delta \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma (2\eta^{\mu\delta} - \gamma^\mu \gamma^\delta) \\
&= 2\gamma^\delta \gamma^\nu \gamma^\sigma \gamma^\gamma - (\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\mu)\gamma^\delta \\
&= 2\gamma^\delta \gamma^\nu \gamma^\sigma \gamma^\gamma + 2\gamma^\gamma \gamma^\sigma \gamma^\nu \gamma^\delta.
\end{aligned}$$

□

D.14.2 Complete Set of 4×4 Matrices

Any 4×4 matrix can be written as

$$\sum_{A=1}^{16} a_A \hat{\Gamma}_A \tag{D.525}$$

where

$$\begin{aligned}
\hat{\Gamma}_A = \; & \mathbf{I}, \\
& \gamma_0,\ i\gamma_1,\ i\gamma_2,\ i\gamma_3, \\
& i\gamma_2\gamma_3,\ i\gamma_3\gamma_1,\ i\gamma_1\gamma_2,\ \gamma_1\gamma_0,\ \gamma_2\gamma_0,\ \gamma_3\gamma_0, \\
& \gamma_1\gamma_2\gamma_3,\ i\gamma_1\gamma_2\gamma_0,\ i\gamma_3\gamma_1\gamma_0,\ i\gamma_2\gamma_3\gamma_0 \\
& i\gamma_1\gamma_2\gamma_3\gamma_0
\end{aligned} \tag{D.526}$$

Proof:

Note

$$\hat{\Gamma}_A^2 = \mathbf{I} \quad (A = 1, \ldots, 16) \tag{D.527}$$

For all $\hat{\Gamma}_A$ but $\mathbf{I}$ there exists a $\hat{\Gamma}_B$ with

$$\hat{\Gamma}_B \hat{\Gamma}_A \hat{\Gamma}_B = -\hat{\Gamma}_A \tag{D.528}$$

The trace of all $\hat{\Gamma}_A$ $(A = 2, \ldots, 16)$ are zero,

$$Tr(\hat{\Gamma}_A) = -Tr(\hat{\Gamma}_B \hat{\Gamma}_A \hat{\Gamma}_B) = -Tr(\hat{\Gamma}_B^2 \hat{\Gamma}_A) = -Tr(\hat{\Gamma}_A).$$

Linear independence

Say

$$\sum_{A=1}^{16} a_A \hat{\Gamma}_A = 0.$$

Multiply this sum from the right by $\hat{\Gamma}_B$,

$$a_B \mathbf{I} + \sum_{A \neq B} a_A \hat{\Gamma}_A \hat{\Gamma}_B = 0. \tag{D.529}$$

Then take the trace

$$4a_B + \sum_{A \neq B} a_A Tr(\hat{\Gamma}_A \hat{\Gamma}_B) = 0. \tag{D.530}$$

Now from (D.526) we see that $\hat{\Gamma}_A \hat{\Gamma}_B = Const.\hat{\Gamma}_C$. In the case where $A \neq B$, $\hat{\Gamma}_C \neq \mathbf{I}$. This implies in (D.530) that $a_B = 0$.

Expansion of 4×4

Each 4×4 matrix can be expanded as

$$\hat{X} = \sum_{A=1}^{16} x_A \hat{\Gamma}_A. \tag{D.531}$$

This is evident since 4×4 matrices represents a 16-dimension space and the $\hat{\Gamma}_A$ are linearly independent. The coefficents are then given by

$$x_B = \frac{1}{4} Tr(\hat{\Gamma}_A \hat{X}).$$

□

D.14.3 Unitary Equivalence of Representations of the Dirac Algebra

All representations of the Dirac algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{I}$ which satify $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$ are unitary equivalent.

Proof: The proof is split into five parts.

i) First we prove that each 4×4 matrix which commutes with all $\hat{\Gamma}_A$ is a multiple of $\mathbf{I}$. Consider such a the matrix which write

$$\hat{X} = x_B\hat{\Gamma}_B + \sum_{A\neq B} x_A\hat{\Gamma}_A \tag{D.532}$$

where we have picked out a particular matrix which is not $\mathbf{I}$. We choose $\hat{\Gamma}_C$ such that

$$\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C = -\hat{\Gamma}_B. \tag{D.533}$$

Since $\hat{X}$ comutes wth all $\hat{\Gamma}_A$,

$$\hat{X} = \hat{\Gamma}_C\hat{X}\hat{\Gamma}_C$$

we have

$$\begin{aligned} x_B\hat{\Gamma}_B + \sum_{A\neq B} x_A\hat{\Gamma}_A &= x_B\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C + \sum_{A\neq B} x_A\hat{\Gamma}_C\hat{\Gamma}_A\hat{\Gamma}_C \\ &= x_B\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C + \sum_{A\neq B} x_A(\pm\hat{\Gamma}_A\hat{\Gamma}_C)\hat{\Gamma}_C \\ &= -x_B\hat{\Gamma}_B + \sum_{A\neq B}(\pm)x_A\hat{\Gamma}_A \end{aligned} \tag{D.534}$$

where we have used $\hat{\Gamma}_C\hat{\Gamma}_A = (\pm)\hat{\Gamma}_A\hat{\Gamma}_C$ (this established by inspection (D.526)). Next multiply by $\hat{\Gamma}_B$ and take the trace

$$4x_B + \sum_{A\neq B} x_A Tr(\hat{\Gamma}_A\hat{\Gamma}_B) = -4x_B + \sum_{A\neq B}(\pm)x_A Tr(\hat{\Gamma}_A\hat{\Gamma}_B) \tag{D.535}$$

implying

$$x_B = -x_B = 0.$$

So we conclude that

$$\hat{X} = x_1\mathbf{I}. \tag{D.536}$$

This result is actually just a special case of Schur's lemma which states that every matrix which commutes with every element of and irreducible representation must be a muliply of the identity matrix; we know 4×4 matrix reprersentations of the Dirac algebra is irreducible as there are no lower dimensional representations.

ii) Let γ_μ and γ'_μ be two representations of the Dirac algebra and $\hat{\Gamma}_A, \hat{\Gamma}'_A$ are respectively their basis. We wish to prove that

$$\hat{\Gamma}'_A\hat{S} = \hat{S}\hat{\Gamma}_A \tag{D.537}$$

where

$$\hat{S} = \sum_{B=1}^{16}\hat{\Gamma}'_B\hat{F}\hat{\Gamma}_B \tag{D.538}$$

and $\hat{F}$ is an aritrary 4×4 matrix. To this end, consider the matrix

$$\hat{\Gamma}'_A\hat{S}\hat{\Gamma}_A = \sum_{B=1}^{16}\hat{\Gamma}'_A\hat{\Gamma}'_B\hat{F}\hat{\Gamma}_B\hat{\Gamma}_A. \tag{D.539}$$

By inspection, from (D.526) we have $\hat{\Gamma}_B\hat{\Gamma}_A = \alpha_C\hat{\Gamma}_C$ where $\alpha_C \in \{\pm 1, \pm i\}$.

$$\hat{\Gamma}'_B\hat{\Gamma}'_A = \alpha_C\hat{\Gamma}'_C. \tag{D.540}$$

Multiply from the right by $\hat{\Gamma}'_A$

$$\hat{\Gamma}'_B = \alpha_C \hat{\Gamma}'_C \hat{\Gamma}'_A$$

then the right by $\hat{\Gamma}'_C$

$$\hat{\Gamma}'_C \hat{\Gamma}'_B = \alpha_C \hat{\Gamma}'_A$$

and from the left by $\hat{\Gamma}'_B$ gives

$$\hat{\Gamma}'_A \hat{\Gamma}'_B = \frac{1}{\alpha_C} \hat{\Gamma}'_C. \tag{D.541}$$

Substituting this into (D.539) then gives

$$\hat{\Gamma}'_A \hat{S} \hat{\Gamma}_A = \sum_{C=1}^{16} \left(\frac{1}{\alpha_C} \hat{\Gamma}'_C \right) \hat{F} \, (\alpha_C \hat{\Gamma}'_C) = \hat{S} \tag{D.542}$$

proving (D.537).

Suppose we could choose $\hat{F}$ (recall $\hat{F}$ is completely arbitrary) so that $\hat{S}$ is non-singular, we would then have in particular

$$\hat{\Gamma}'_2 = \hat{S}\hat{\Gamma}_2 \hat{S}^{-1}, \quad \hat{\Gamma}'_3 = \hat{S}\hat{\Gamma}_3 \hat{S}^{-1}, \quad \hat{\Gamma}'_4 = \hat{S}\hat{\Gamma}_4 \hat{S}^{-1}, \quad \hat{\Gamma}'_5 = \hat{S}\hat{\Gamma}_5 \hat{S}^{-1}$$

equivalently

$$\gamma'_\mu = \hat{S} \gamma_\mu \hat{S}^{-1}.$$

The next two steps are to prove we can choose $\hat{F}$ so the above conditions for $\hat{S}$ are fulfilled. In the final step we show that the additional conditions $\gamma_0 = \gamma_0^\dagger$, $\gamma_i = -\gamma_i^\dagger$, $\gamma'_0 = \gamma_0'^\dagger$, $\gamma'_i = -\gamma_i'^\dagger$ imply that $\hat{S}$ can be choosen to be unitary, proving the entire result.

iii) The matrix $\hat{F}$ can be choosen so that $\hat{S}$ does not vanish. We prove this by contraction. Say $\hat{S} = 0$ held for all choices of $\hat{F}$, then

$$0 = (\hat{S})_{\mu\rho} = \sum_{B=1}^{16} \sum_{\alpha,\beta=1}^{4} (\hat{\Gamma}'_B)_{\mu\alpha} (\hat{F})_{\alpha\beta} (\hat{\Gamma}_B)_{\beta\rho}. \tag{D.543}$$

for all μ and ρ. Now let us choose $\hat{F}$ such that a single element has the value 1 with all other elements being zero. Say it is the element $(\hat{F})_{\nu\sigma}$ that is equal to 1, then (D.543) reads

$$\sum_{B=1}^{16} (\hat{\Gamma}'_B)_{\mu\nu} (\hat{\Gamma}_B)_{\sigma\rho} = 0 \tag{D.544}$$

This equation can be written for all possible choices of ν and σ, so we can infer

$$\sum_{B=1}^{16} (\hat{\Gamma}'_B)_{\mu\nu}\, \hat{\Gamma}_B = 0 \tag{D.545}$$

holds for all μ and ν. Since $\hat{\Gamma}_B^2 = \mathbf{I}$ this would imply

$$\sum_{B=1}^{16} (\hat{\Gamma}'_B)_{\mu\nu} = 0 \quad \text{for all} \quad \mu, \nu. \tag{D.546}$$

As $(\hat{\Gamma}'_B)_{\mu\nu}$ cannot be equal to zero simultaneously, we have a contradiction to the linear independence of the $\hat{\Gamma}_B$.

iv) Now we prove that $\hat{S}$ is not singular with appropriate choice of $\hat{F}$. To this end construct

$$\hat{T} = \sum_{B=1}^{16} \hat{\Gamma}_B \hat{G} \hat{\Gamma}'_B \tag{D.547}$$

where $\hat{G}$ is arbitrary. Obviously we have

$$\hat{\Gamma}_A \hat{T} = \hat{T} \hat{\Gamma}'_A \tag{D.548}$$

(same argument as in iii) which together with (D.537) implies

$$(\hat{\Gamma}_A \hat{T})\hat{S} = (\hat{T}\hat{\Gamma}'_A)\hat{S} = \hat{T}(\hat{\Gamma}'_A \hat{S}) = \hat{T}(\hat{S}\hat{\Gamma}_A)$$

i.e.

$$\hat{\Gamma}_A(\hat{T}\hat{S}) = (\hat{T}\hat{S})\hat{\Gamma}_A, \tag{D.549}$$

accordingly $\hat{T}\hat{S}$ must be a multiple of the identity

$$\hat{T}\hat{S} = k\,\mathbf{I}. \tag{D.550}$$

Obviously we can choose $\hat{G}$ so that $\hat{T} \neq 0$ (same argument as iii). With the same kind of choice of $\hat{F}$ as in part iv) we will now show that we must have $k \neq 0$, and hence that $\hat{S}$ is not singular! We prove it by contradiction

$$\sum_{B=1}^{16} \hat{T}\hat{\Gamma}'_B \hat{F}\hat{\Gamma}_B = 0 \tag{D.551}$$

with the choice of $(\hat{F})_{\nu\rho} = 1$ with all other terms zero,

$$\sum_{B=1}^{16} (\hat{T}\hat{\Gamma}'_B)_{\mu\nu} (\hat{\Gamma}_B)_{\rho\sigma} = 0 \tag{D.552}$$

or

$$\sum_{B=1}^{16} (\hat{T}\hat{\Gamma}'_B)_{\mu\nu} \hat{\Gamma}_B = 0. \tag{D.553}$$

From $\hat{\Gamma}'^2_A = \mathbf{I}$ and the fact that $(\hat{T}\hat{\Gamma}'_B)_{\mu\nu}$ cannot all be simulateously zero as $\hat{\Gamma}'_1 = \mathbf{I}$ and $\hat{T} \neq 0$. This is in contradiction to the linear independence of the $\hat{\Gamma}_B$.

v) We now show that in the case of

$$\gamma_0 = \gamma_0^\dagger, \quad \gamma_i = -\gamma_i^\dagger, \quad \gamma'_0 = \gamma_0'^\dagger, \quad \gamma'_i = -\gamma_i'^\dagger,$$

equivalently

$$\gamma_\mu^\dagger = \eta_{\mu\mu}\gamma_\mu, \quad \gamma_\mu'^\dagger = \eta_{\mu\mu}\gamma'_\mu, \tag{D.554}$$

then $\hat{S}$ can be choosen as a unitary operator. To see this put $\hat{V} \equiv (det\,\hat{S})^{-1}\hat{S}$ then

$$\gamma'_\mu = \hat{V}\gamma_\mu\hat{V}^{-1}, \quad det\,\hat{V} = 1. \tag{D.555}$$

Let us see if there exist another choice for $\hat{V}$. We must have $det\,\hat{V}_1 = det\,\hat{V}_2 = 1$ and

$$\gamma'_\mu = \hat{V}_1\gamma_\mu\hat{V}_1^{-1} = \hat{V}_2\gamma_\mu\hat{V}_2^{-1}. \tag{D.556}$$

Eq (D.556) implies

$$\hat{V}_1\hat{\Gamma}_A\hat{V}_1^{-1} = \hat{V}_2\hat{\Gamma}_A\hat{V}_2^{-1}, \tag{D.557}$$

for example

$$\begin{aligned} \hat{V}_1(i\gamma_1\gamma_2)\hat{V}_1^{-1} &= i(\hat{V}_1\gamma_1\hat{V}_1^{-1})(\hat{V}_1\gamma_2\hat{V}_1^{-1}) \\ &= i(\hat{V}_2\gamma_1\hat{V}_2^{-1})(\hat{V}_2\gamma_2\hat{V}_2^{-1}) \\ &= \hat{V}_2(i\gamma_1\gamma_2)\hat{V}_2^{-1}. \end{aligned} \tag{D.558}$$

Eq (D.557) rearranged becomes

$$(\hat{V}_2^{-1}\hat{V}_1)\hat{\Gamma}_A = \hat{\Gamma}_A(\hat{V}_2^{-1}\hat{V}_1). \tag{D.559}$$

By result i) (Schur's lemma)

$$\hat{V}_2^{-1}\hat{V}_1 = k'\mathbf{I}$$

hence

$$\hat{V}_1 = k'\hat{V}_2. \tag{D.560}$$

As $det\ \hat{V}_2 = det\ \hat{V}_1 = k'^4 det\ \hat{V}_2$, we must have $k' \in \{\pm 1, \pm i\}$. Now take the Hermitian conjugate of (D.555),

$$\gamma_\mu{}'\dagger = (\hat{V}^{-1})^\dagger\gamma_\mu^\dagger\hat{V}^\dagger \tag{D.561}$$

then by means of (D.554),

$$\gamma_\mu' = (\hat{V}^\dagger)^{-1}\gamma_\mu\hat{V}^\dagger \tag{D.562}$$

We see that $(\hat{V}^\dagger)^{-1}$ fulfills (D.555) as does $\hat{V}$. From (D.560) ($k' \in \{\pm 1, \pm i\}$) it follows

$$\begin{aligned} (\hat{V}^\dagger)^{-1} &= k'\hat{V}, \quad \hat{V}^\dagger = k'^{-1}\hat{V}^{-1} \\ \hat{V}^\dagger\hat{V} &= k'^{-1}\mathbf{I}. \end{aligned} \tag{D.563}$$

Since

$$(\hat{V}^\dagger \hat{V})_{ii} = \sum_j (\hat{V}^\dagger)_{ij} (\hat{V})_{ji} = \sum_j |V_{ji}|^2 = k^{'-1} \tag{D.564}$$

Hence $k^{'-1}$ must be real and positive, i.e. $k^{'-1} = 1$. Hence,

$$\hat{V}^\dagger \hat{V} = \mathbf{I}. \tag{D.565}$$

□

D.14.4 Coefficients of Infintesimal Lorentz Transformation

We prove that the

$$\hat{\sigma}_{\alpha\beta} = \frac{i}{2}(\gamma_\alpha \gamma_\beta - \gamma_\alpha \gamma_\beta)$$

fulfill ()

Proof: Insert the above expression in the RHS of ()

$$\begin{aligned}
[\gamma^\nu, \hat{\sigma}_{\alpha\beta}] &= \frac{i}{2}\left[\gamma^\nu, [\gamma_\alpha, \gamma_\beta]\right] \\
&= \frac{i}{2}\left(\left[\gamma^\nu, \gamma_\alpha \gamma_\beta\right] - \left[\gamma^\nu, \gamma_\beta \gamma_\alpha\right]\right) \\
&= \frac{i}{2}\left(2\left[\gamma^\nu, \gamma_\alpha \gamma_\beta\right] - 2\left[\gamma^\nu, \eta_{\alpha\beta}\right]\right) \\
&= i[\gamma^\nu, \gamma_\alpha \gamma_\beta]
\end{aligned} \tag{D.566}$$

where we used $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\eta_{\alpha\beta}$. Furthermore we have

$$\begin{aligned}
i[\gamma^\nu, \gamma_\alpha \gamma_\beta] &= i(\gamma^\nu \gamma_\alpha \gamma_\beta - \gamma_\alpha \gamma_\beta \gamma^\nu) \\
&= i(\gamma^\nu \gamma_\alpha \gamma_\beta - 2\eta^\nu{}_\beta \gamma_\alpha + \gamma_\alpha \gamma^\nu \gamma_\beta) \\
&= i(\gamma^\nu \gamma_\alpha \gamma_\beta - 2\eta^\nu{}_\beta \gamma_\alpha + 2\eta^\nu{}_\alpha \gamma_\beta - \gamma^\nu \gamma_\alpha \gamma_\beta) \\
&= 2i(\eta^\nu{}_\alpha \gamma_\beta - \eta^\nu{}_\beta \gamma_\alpha).
\end{aligned} \tag{D.567}$$

□

D.14.5 Proof of Relation $\hat{S}^{-1} = \gamma_0 \hat{S}^\dagger \gamma_0$

We show that for

$$\hat{S} = \exp\left(-\frac{i}{4}\omega\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu}\right)$$

the inverse operator is given by

$$\hat{S}^{-1} = \gamma_0 \hat{S}^\dagger \gamma_0 \tag{D.568}$$

Proof:

(i) Rotations:

For spacial rotations we can write:

$$\hat{S} = \exp\left(-\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}\right). \tag{D.569}$$

The $\hat{\sigma}_{ij}$ are Hermitian because

$$\begin{aligned}
\hat{\sigma}_{ij}^\dagger &= -\frac{i}{2}\left\{(\gamma_i\gamma_j)^\dagger - (\gamma_j\gamma_i)^\dagger\right\} \\
&= -\frac{i}{2}\left\{\gamma_j\gamma_i - \gamma_i\gamma_j\right\} \\
&= \hat{\sigma}_{ij}.
\end{aligned} \tag{D.570}$$

This implies

$$\hat{S}^\dagger = \exp\left(\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}^\dagger\right) = \exp\left(\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}\right). \tag{D.571}$$

Obviously, γ_0 commutes with $\hat{\sigma}_{ij}$ and thus with $\hat{S}^\dagger$. Hence we have

$$\gamma_0\hat{S}^\dagger\gamma_0 = \hat{S}^\dagger = \hat{S}^{-1}. \tag{D.572}$$

(ii) Lorentz boosts:

Given that a general Loreentz transformation can be decomposed into first a rotation, a Lorentz boost along the x-direction and then undoing the rotation, and given that

$\gamma^i\gamma^0 + \gamma^0\gamma^i = 0$, it suffices to prove the result for a simple boost along the $x-$direction. For this transformation we have

$$\hat{S} = \exp\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)$$

$\hat{\sigma}_{01}$ is antihermitian because

$$\begin{aligned}
\hat{\sigma}_{01}^{\dagger} &= -\frac{i}{2}\left\{(\gamma_0\gamma_1)^{\dagger} - (\gamma_1\gamma_0)^{\dagger}\right\} \\
&= \frac{i}{2}\left\{\gamma_1\gamma_0 - \gamma_0\gamma_1\right\} \\
&= -\hat{\sigma}_{01}.
\end{aligned} \tag{D.573}$$

Therefore

$$\hat{S}^{\dagger} = \exp\left(\frac{i}{2}\omega\hat{\sigma}_{01}^{\dagger}\right) = \exp\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right) = \hat{S}. \tag{D.574}$$

From

$$\begin{aligned}
\gamma_0\hat{\sigma}_{01} &= \frac{i}{2}\left\{\gamma_0\gamma_0\gamma_1 - \gamma_0\gamma_1\gamma_0\right\} \\
&= \frac{i}{2}\left\{\gamma_1\gamma_0\gamma_0 - \gamma_0\gamma_1\gamma_0\right\} \\
&= \hat{\sigma}_{10}\gamma_0 = -\hat{\sigma}_{01}\gamma_0.
\end{aligned} \tag{D.575}$$

we get

$$\begin{aligned}
\gamma_0\hat{S}^{\dagger}\gamma_0 &= \gamma_0\left[\sum_{n=0}^{\infty}\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)^n\right]\gamma_0 \\
&= \sum_{n=0}^{\infty}\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)^n\gamma_0 \\
&= \sum_{n=0}^{\infty}\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)\gamma_0\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)\gamma_0\ldots\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)\gamma_0 \\
&= \sum_{n=0}^{\infty}\left(+\frac{i}{2}\omega\hat{\sigma}_{01}\right)^n \\
&= \exp\left(\frac{i}{2}\omega\hat{\sigma}_{01}\right) = \hat{S}^{-1}.
\end{aligned} \tag{D.576}$$

$\square$

D.14.6 Proof of the Completenes Relation for spinors

Proof of completenes relation: $\omega^{r\dagger}(\epsilon_r \mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \delta_{rr'}(E/m_0)$

Proof:

We calculate some examples

$r = 1, r' = 1$:

$$\begin{aligned}
& \frac{E+m_0}{2m_0}\left(1, 0, \frac{p_z}{E+m_0}, \frac{p_-}{E+m_0}\right)\left(\begin{array}{c} 1 \\ 0 \\ \frac{p_z}{E+m_0} \\ \frac{p_-}{E+m_0} \end{array}\right) \\
= & \frac{E+m_0}{2m_0}\left\{1 + \frac{\mathbf{p}^2}{(E+m_0)^2}\right\} \\
= & \frac{E+m_0}{2m_0}\left\{\frac{(E+m_0)^2 + \mathbf{p}^2}{(E+m_0)^2}\right\} \\
= & \left\{\frac{2E + 2m_0 E}{2m_0(E+m_0)}\right\} \\
= & \frac{E}{m_0}\delta_{11}
\end{aligned} \tag{D.577}$$

$r = 2, r' = 4$:

$$\begin{aligned}
& \frac{E+m_0}{2m_0}\left(0, 1, \frac{p_+}{E+m_0}, -\frac{p_z}{E+m_0}\right)\left(\begin{array}{c} -\frac{p_-}{E+m_0} \\ \frac{p_z}{E+m_0} \\ 0 \\ 1 \end{array}\right) \\
= & \frac{E+m_0}{2m_0}\left\{\frac{p_z}{E+m_0} - \frac{p_z}{E+m_0}\right\} = 0.
\end{aligned} \tag{D.578}$$

$r = 4, r' = 4$:

$$
\begin{aligned}
& \frac{E+m_0}{2m_0}\left(1,0,\frac{p_+}{E+m_0},-\frac{p_z}{E+m_0}\right)\begin{pmatrix} -\frac{p_-}{E+m_0} \\ \frac{p_z}{E+m_0} \\ 0 \\ 1 \end{pmatrix} \\
= \; & \frac{E}{m_0}\delta_{44} \qquad \text{(D.579)}
\end{aligned}
$$

The other combinations can be calculated similarly.

□

D.14.7 Integral representation for the step function

We show that

$$
\Theta(\tau) = -\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon} \qquad \text{(D.580)}
$$

Proof: We can evaluate the integral by means of complex integration in the complex $\omega-$plane. This can be done if we can show if the contribution from the upper (lower), infinitely distant half circle vanishes. Let I_R be the integral along the upper (lower) semicircle, then we can write

$$
\begin{aligned}
I_R &= \lim_{R\to\infty}\int_0^{\pm\pi}[zf(z)]\frac{dz}{z} \\
&\leq \lim_{R\to\infty} Max[zf(z)]\int_0^{\pm\pi}\frac{dz}{z} \\
&= \pm i\pi\lim_{R\to\infty} Max[zf(z)] \qquad \text{(D.581)}
\end{aligned}
$$

Hence, if we can show that $\lim_{R\to\infty} Max\,|[zf(z)]| \to 0$ then the integral over the semicircle can be ignored and the integral along the real line can be converted into a closed contour integral.

For $\tau < 0$ we show that the contribution from the upper, infinitely distant half circle vanishes

$$\begin{aligned} f(R,\theta) &= \frac{e^{-i\omega\tau}}{\omega} \\ &= \frac{e^{-iR\tau(\cos\theta+i\sin\theta)}}{re^{i\theta}} \\ &= e^{-iR\tau\cos\theta}\frac{e^{+R\tau\sin\theta}}{Re^{i\theta}} \end{aligned} \tag{D.582}$$

and

$$\lim_{R\to\infty}|Rf(R,\theta)| = e^{-R|\tau|\sin\theta} = 0$$

For $\tau < 0$ we close the contour in the upper half plane. There is only a first order pole at $-i\epsilon$. Therefore this integral will be zero.

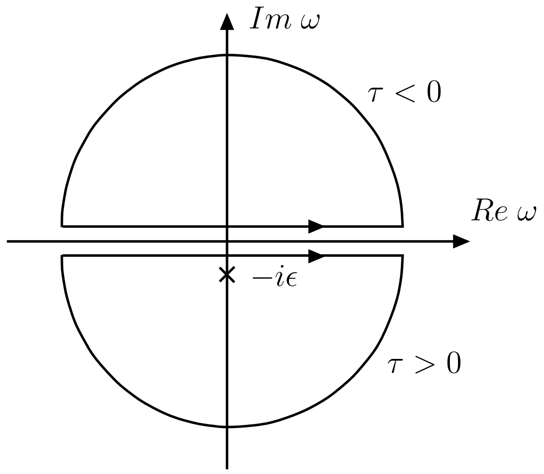


Figure D.34:

In the case $\tau > 0$ for similar reasons one can close the contour by means of an infinitely large half circle below the real axis. Cauchy's integral theorem says that the integrand at the pole

$$\begin{aligned} \Theta(\tau > 0) &= -\frac{1}{2\pi i}(-1)2\pi i \lim_{\epsilon\to 0} Res\left[\frac{e^{-i\omega\tau}}{\omega + i\epsilon}\right] \\ &= e^{-\epsilon\tau}\big|_{\epsilon=0} = 1. \end{aligned} \tag{D.583}$$

where we have a minus sign coming from the clockwise direction of the integration.

□

D.14.8 Averaging over Spin

We have

$$\left(\overline{u}(f)\ \hat{\Gamma}_1 u(i)\right)\ \left(\overline{u}(f)\ \hat{\Gamma}_2 u(i)\right)^* = (\overline{u}(f)\ \hat{\Gamma}_1 u(i))\ (\overline{u}(i)\ \hat{\overline{\Gamma}}_2 u(f)) \tag{D.584}$$

where

$$\hat{\overline{\Gamma}} = \gamma^0 \hat{\Gamma}^\dagger \gamma^0. \tag{D.585}$$

Proof:

First note what the complex conjugate $(\overline{u}(f)\hat{\Gamma}u(i))^*$ of the number $\overline{u}(f)\hat{\Gamma}u(i)$ is equal to

$$\begin{aligned}
(\overline{u}(f)\hat{\Gamma}u(i))^\dagger &= \left(\overline{u}^\dagger(f)\gamma^0\hat{\Gamma}u(i)\right)^\dagger \\
&= u(i)^\dagger \hat{\Gamma}^\dagger \gamma^{0\dagger}\ u^{\dagger\dagger}(f) \\
&= \overline{u}(i)(\gamma^0\hat{\Gamma}^\dagger\gamma^0)u(f) \\
&= \overline{u}(i)\hat{\overline{\Gamma}}u(f)
\end{aligned} \tag{D.586}$$

where we have used $\gamma^{0\dagger} = \gamma^0$ and $(\gamma^0)^2 = 1$.

$$\left(\overline{u}(f)\ \hat{\Gamma}_1 u(i)\right)\ \left(\overline{u}(f)\ \hat{\Gamma}_2 u(i)\right)^* = (\overline{u}(f)\ \hat{\Gamma}_1 u(i))\ (\overline{u}(i)\ \hat{\overline{\Gamma}}_2 u(f)) \tag{D.587}$$

$\square$

The barred matrices $\hat{\overline{\Gamma}}$ can be directly calculated for a number of operators:

(i) $\overline{\gamma}^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$

(ii) $\overline{i\gamma^5} = i\gamma^5$

(iii) $\overline{\gamma^\mu\gamma^5} = \gamma^\mu\gamma^5$

(iv) $\overline{\gamma^\mu\gamma^\nu \dots \gamma^\lambda} = \gamma^\lambda \dots \gamma^\nu\gamma^\mu$

Proof:

(i) First

$$\gamma^0\gamma^{0\dagger}\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$$

and secondly

$$\gamma^0\gamma^{i\dagger}\gamma^0 = -\gamma^0\gamma^i\gamma^0 = \gamma^i\gamma^0\gamma^0 = \gamma^i.$$

(ii) As $i\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3$ and

$$\overline{i\gamma^5} = -\gamma^0\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger}\gamma^0 = +\gamma^0\gamma^3\gamma^2\gamma^1 = -\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^5.$$

(iii) Similar to (ii).

(iv) Proved using (i).

$\square$

Spin summation of the general squared matrix element

$$\sum_{s_f\ s_i} \left(\overline{u}(p_f, s_f)\ \hat{\Gamma}_1\ u(p_i, s_i)\right)\ \left(\overline{u}(p_f, s_f)\ \hat{\Gamma}_2\ u(p_i, s_i)\right)^* = Tr\left[\hat{\Gamma}_1\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\ \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right] \tag{D.588}$$

A special case is

$$\sum_{s_f\ s_i} |\overline{u}(p_f, s_f)\ \hat{\Gamma}\ u(p_i, s_i)|^2 = Tr\left[\hat{\Gamma}\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}\ \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right] \tag{D.589}$$

Proof: We use Einstein's summation convention.

$$\begin{aligned}
&\sum_{s_f\ s_i} \left(\overline{u}_\alpha(p_f, s_f)\ (\hat{\Gamma}_1)_{\alpha\beta}\ u_\beta(p_i, s_i)\right)\ \left(\overline{u}_\gamma(p_i, s_i)\ (\hat{\overline{\Gamma}}_2)_{\gamma\delta}\ u_\delta(p_f, s_f)\right) \\
=\ &\sum_{s_f} \overline{u}_\alpha(p_f, s_f)\ (\hat{\Gamma}_1)_{\alpha\beta} \left(\sum_{s_i} u_\beta(p_i, s_i)\overline{u}_\gamma(p_i, s_i)\right) \hat{\overline{\Gamma}}_{\gamma\tau}\ u_\tau(p_f, s_f) \\
=\ &\sum_{s_f} \overline{u}_\alpha(p_f, s_f)\ \left(\hat{\Gamma}_1\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\right)_{\alpha\beta} u_\beta(p_f, s_f) \\
=\ &\sum_{r=1}^{4} \epsilon_r \overline{\omega}^r_\alpha(p_f)\ \left(\hat{\Gamma}_1 \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\right)_{\alpha\beta} \left(\frac{p_{f\mu}\gamma^\mu + m_0}{2m_0}\right)_{\beta\gamma} \omega^r_\gamma(p_f) \\
=\ &Tr\left[\hat{\Gamma}_1\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\ \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right]
\end{aligned} \tag{D.590}$$

$\square$

D.14.9 Proof of Equations (D.310) and (D.311)

$$\Theta(t-t')\psi^{(-E)}(x') = -i\int d^3x S_F(x'-x)\gamma_0\psi^{(-E)}(x)$$

We prove (D.310):

$$\Theta(t'-t)\psi^{(+E)}(x') = i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x)$$

Proof:

Any wave packet of positive energy may be expressed in terms of normalised plane waves

$$\psi^{(+E)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b(p,r)\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x) \tag{D.591}$$

where $E_p = \sqrt{\mathbf{p}^2+m_0^2}$ and $\epsilon_1 = \epsilon_2 = +1$.. We will need to make use of the orthogonality condition

$$\omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E_p}{m_0}\delta_{rr'}. \tag{D.592}$$

We start with the plane-wave representation of the Feynman propagator

$$S_F(x'-x) = -i\Theta(t'-t)\int d^3p\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x) + i\Theta(t-t')\int d^3p\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x) \tag{D.593}$$

where

$$\psi_p^r = \sqrt{\frac{m_0}{E_p}}\frac{1}{(2\pi)^{3/2}}\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x). \tag{D.594}$$

Inserting the above into the RHS of ()

$$
\begin{aligned}
& i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x) \\
= \;& \Theta(t'-t)\int d^3x\int d^3p\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x)\gamma_0\psi^{(+E)}(x) \\
& -\Theta(t'-t)\int d^3x\int d^3p\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x)\gamma_0\psi^{(+E)}(x) \\
= \;& \Theta(t'-t)\int d^3x\int\frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\sum_{r=1}^{2}\omega^r(p)\overline{\omega}^r(p)\gamma_0\exp[-i\epsilon_r p\cdot(x'-x)] \\
& \times\int\frac{d^3p'}{(2\pi)^3}\frac{m_0}{E_{p'}}\sum_{r=1}^{2}b(p',r')\omega^{r'}(p')\exp(-i\epsilon_{r'}p'\cdot x) \\
& -\Theta(t'-t)\int d^3x\int\frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\sum_{r=3}^{4}\omega^r(p)\overline{\omega}^r(p)\gamma_0\exp[-i\epsilon_r p\cdot(x'-x)] \\
& \times\int\frac{d^3p'}{(2\pi)^3}\frac{m_0}{E_{p'}}\sum_{r=1}^{2}b(p',r')\omega^{r'}(p')\exp(-i\epsilon_r p'\cdot x) \\
= \;& \Theta(t'-t)\int\frac{d^3pd^3p'}{(2\pi)^3}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_p}}\sum_{r=1,2;r'=1,2}\omega^r(p)\omega^{r\dagger}(p)\omega^{r'}(p')b(p',r')\exp(-i\epsilon_{r'}p'\cdot x) \\
& \quad\times\int\frac{d^3x}{(2\pi)^3}\exp[i(\epsilon_r p-\epsilon_{r'}p')\cdot x] \\
- \;& \Theta(t'-t)\int\frac{d^3pd^3p'}{(2\pi)^3}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_{p'}}}\sum_{r=3,4;r'=1,2}\omega^r(p)\omega^{r\dagger}(p)\omega^{r'}(p')b(p',r')\exp(-i\epsilon_r p\cdot x') \\
& \quad\times\int\frac{d^3x}{(2\pi)^3}\exp[i(\epsilon_r p-\epsilon_{r'}p')\cdot x] \qquad\text{(D.595)}
\end{aligned}
$$

Performing the x integration in the Θ term yields

$$
\exp[i(E_p-E_{p'})t]\delta^3(\mathbf{p}-\mathbf{p}')\to\delta^3(\mathbf{p}-\mathbf{p}') \qquad\text{(D.596)}
$$

Performing the x integration in the Θ term yields

$$
\exp[-i(E_p+E_{p'})t]\delta^3(\mathbf{p}+\mathbf{p}')\to\exp(-2iE_pt)\delta^3(\mathbf{p}+\mathbf{p}'). \qquad\text{(D.597)}
$$

Integrating over $\mathbf{p}$ and relabelling $\mathbf{p}'$ as $\mathbf{p}$ we find

$$
\begin{aligned}
& i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x) \\
= \; & \Theta(t'-t)\int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2}\sum_{r=1,2;r'=1,2}\omega^r(\mathbf{p})\omega^{r\dagger}(\mathbf{p})\omega^{r'}(\mathbf{p})b(p,r')\exp(i\epsilon_r p\cdot x') \\
& -\Theta(t-t')\int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2}\sum_{r=3,4;r'=1,2}\omega^r(-\mathbf{p})\omega^{r\dagger}(-\mathbf{p})\omega^{r'}(+\mathbf{p})b(p,r') \\
& \times\exp(i\epsilon_r p\cdot x')\exp(-2iE_p t) \qquad \text{(D.598)}
\end{aligned}
$$

Now we make use of the orthogonality relation. For $r, r' = 1, 2$

$$
\omega^{r\dagger}(\mathbf{p})\omega^{r'}(\mathbf{p}) = \omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E_p}{m_0}\delta_{rr'} \qquad \text{(D.599)}
$$

and for $r = 3, 4$ and $r' = 1, 2$,

$$
\omega^{r\dagger}(-\mathbf{p})\omega^{r'}(\mathbf{p}) = \omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = 0 \qquad \text{(D.600)}
$$

The second term vanishes. The remaining term gives

$$
\begin{aligned}
i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x) &= \Theta(t'-t)\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b(p,r)\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x') \\
&= \Theta(t'-t)\psi^{(+E)}(x') \qquad \text{(D.601)}
\end{aligned}
$$

□

Similar relations can be deduced the propagation of adjoint spinors $\overline{\psi}^{(+E)}(x), \overline{\psi}^{(-E)}(x)$:

$$
\Theta(t-t')\overline{\psi}^{(+E)}(x') = i\int d^3x\overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \qquad \text{(D.602)}
$$

and

$$
\Theta(t'-t)\overline{\psi}^{(-E)}(x') = -i\int d^3x\overline{\psi}^{(-E)}(x)\gamma_0 S_F(x'-x) \qquad \text{(D.603)}
$$

Proof:

Any adjoint wave packet of positive energy may be expressed in terms of normalised plane waves

$$\overline{\psi}^{(+E)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=1}^{2} b^*(p,r)\overline{\omega}^r(\mathbf{p}) \exp(+ip\cdot x) \tag{D.604}$$

Consider the integral

$$\begin{aligned}
& i\int d^3x \overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \\
= \; & i\int d^3x \int \frac{d^3p'}{(2\pi)^{3/2}} \frac{m_0}{E_p} \sqrt{\frac{m_0}{E_{p'}}} \int \frac{d^3p}{(2\pi)^3} \sum_{r'=1}^{2} \overline{\omega}^{r'}(p') \exp(ip'\cdot x)\gamma_0 \\
& \times \left\{ -i\Theta(t-t') \sum_{r=1}^{2} \omega^r(p)\overline{\omega}^r(p) \; \exp[-ip\cdot(x-x')] \right. \\
& \left. \quad + i\Theta(t-t') \sum_{r=3}^{4} \omega^r(p)\overline{\omega}^r(p) \; \exp[+ip\cdot(x-x')] \right\}
\end{aligned} \tag{D.605}$$

Again we do the x integration and use the orthogonality relations for spinors. We obtain

$$\int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=1}^{2} b^*(p,r)\overline{\omega}(\mathbf{p}) \exp(ip\cdot x)\Theta(t-t'). \tag{D.606}$$

This is just the expansion of the adjoint spinor $\overline{\psi}^{(+E)}$ multiplied by the step function $\Theta(t-t')$.

□

D.15 Scalar Quantum Electrodynamics

The Klein-Gordon equation results from substitution of quantum mechanical operators for energy and momentum into the Einstein relation $E^2 - \mathbf{p}^2 = m_0^2$ of special relativity. As we shall seee this leads to inconsistences such as negative probabilities and negative energy states. These inconsistencies come from interpreting the equation as a single particle wave equation.

Instead we will interpret ϕ as a field, much as the electron field, adopting a description in which particles are created and destroyed at vertices and negative energy solutions propagating backward in time as antiparticles propagating forward in time. We reintepret probability density as charge density, with the existence of particles and antiparticles.

D.15.1 Klein-Gordon Equation

Four current

We construct the four current j_μ for the Klein-Gordon equation. The current will satify a conservation equation. Take the complex conjugate of

$$(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi = 0,$$

i.e.

$$(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi^* = 0,$$

Multiplying the first equation from the left by ϕ^* and the second from the left by ψ and take the difference

$$\phi^*(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi - \phi(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi^* = 0,$$

or

$$-\phi^*(\nabla_\mu \nabla^\mu)\phi - \phi(\nabla_\mu \nabla^\mu)\phi^* = 0,$$

which implies

$$\nabla_\mu(\phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^*) = 0 \tag{D.607}$$

We define the four-current density

$$j_\mu = \frac{i\hbar}{2m_0}(\phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^*) \tag{D.608}$$

which by (D.607) satifies

$$\nabla_\mu j^\mu = 0 \tag{D.609}$$

The constant $i\hbar/2m_0$ was choosen so that j_0 has the dimentions of a probability density.

Charge density

$$\rho^Q = \frac{i\hbar e}{2m_0 c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \tag{D.610}$$

Minimal coupling

minimal coupling

$$[(\hat{p}^\mu - eA^\mu)(\hat{p}_\mu - eA_\mu) - m_0^2]\phi(x) \tag{D.611}$$

rearanging gives

$$[\partial^\mu\partial_\mu - m_0^2]\phi(x) = -\hat{V}\phi(x) \tag{D.612}$$

where

$$\hat{V}\phi(x) = ie(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi - e^2 A^\mu A_\mu \phi. \tag{D.613}$$

D.15.2 Current density in presence of electromagnetic potential

$$\begin{aligned}
0 &= \phi^*[\partial^\mu\partial_\mu\phi + ie(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi] - \\
&\qquad \phi[\partial^\mu\partial_\mu\phi - ie(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi^*] \\
&= \phi^*\partial^\mu\partial_\mu\phi - \phi\partial^\mu\partial_\mu\phi + 2ie\phi^*\phi\partial^\mu A^\mu + 2ieA_\mu\phi^*\partial^\mu\phi + 2ieA_\mu\phi\partial^\mu\phi^* \\
&= \partial^\mu[\phi^*\partial^\mu\phi - \phi\partial_\mu\phi^* + 2ie\phi^*\phi A^\mu]
\end{aligned} \tag{D.614}$$

$$j_\mu = ie\phi^* \overleftrightarrow{\partial}\, \phi - 2e^2 A_\mu \phi^*\phi \tag{D.615}$$

D.15.3 Feynman Propagator for Scalar Particles

$$(\Box + m_0)\Delta_F(x-y) = -\delta^4(x-y). \tag{D.616}$$

$$\Delta_F(p) = \frac{1}{p^2 - m_0^2 + i\epsilon}. \tag{D.617}$$

D.15.4 Perturbative Series

$$
\begin{aligned}
S_{fi} &= \lim_{t\to+\infty} \left(\varphi_{p_f}^{(+)}(x) \middle| \phi_{p_i}(x) \right) \\
&= \lim_{t\to+\infty} \int d^3x \; \varphi_{p_f}^{(+)*}(\mathbf{x},t) i \overleftrightarrow{\partial} \phi_{p_i}(\mathbf{x},t)
\end{aligned}
\tag{D.618}
$$

D.15.5 Scattering off a Coulomb Potential

Scattering off a Coulomb Potential

$$
S_{fi} = -ie \int d^4x \varphi_f^*(x)(\partial^\mu A_\mu(x) + A_\mu(x)\partial_\mu)\phi_i(x) \tag{D.619}
$$

To first order $\phi_i(x)$ is given by the incoming plane wave $\varphi_i(x)$ of a scalar particle with momentum p_i:

$$
\varphi_i(x) = \sqrt{\frac{1}{2E_pV}} e^{-ip_i\cdot x} \tag{D.620}
$$

$\varphi_f^*(x)$ is given by

$$
\varphi_f(x) = \sqrt{\frac{1}{2E_pV}} e^{ip_f\cdot x} \tag{D.621}
$$

$$
\begin{aligned}
S_{fi} &= e\frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}} \int d^4x e^{+ip_f\cdot x}(\partial^\mu A_\mu(x) + A_\mu(x)\partial_\mu)e^{-ip_i\cdot x} \\
&= e\frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}} \int d^4x [-(ip_f^\mu) + (-ip_i^\mu)] A_\mu(x) e^{i(p_f-p_i)\cdot x} \\
&= [(-ie)(p_f^\mu + p_i^\mu)] \frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}} A_\mu(p_f - p_i) \\
&= iZe^2 \frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}} \frac{4\pi}{q^2} \; 2\pi\delta(E_f - E_i).
\end{aligned}
\tag{D.622}
$$

Formula for Differential Cross Section

D.15.6 Scattering of Identical Bosons

$$A^{\mu}(x) = \int d^4y D_F(x-y) j^{\mu}_{fi}(y). \tag{D.623}$$

Appendix E

Quantum Field Theory: Functional Integral and Canonical Approach

E.1 Lagrangian Field Theory

$$
\begin{aligned}
0 &= \delta S \\
&= \delta \int d^4 \mathcal{L} \\
&= \int d^4 \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta (\partial_\mu \varphi) \right\}
\end{aligned} \tag{E.1}
$$

use

$$
\delta(\partial_\mu \varphi) = \partial_\mu (\delta \varphi)
$$

and apply integration by parts, the boundary terms vanish because the end points are fixed:

$$
\begin{aligned}
\int d^4 \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta (\partial_\mu \varphi) &= \int d^4 \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \partial_\mu (\delta \varphi) \\
&= - \int d^4 \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) \delta \varphi
\end{aligned} \tag{E.2}
$$

Altogether, the variation of the action is

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right\} \delta\varphi \tag{E.3}$$

Either the integrand takes positive and negative values or the integrand is zero over the domain of integration. As $\delta\varphi$ is arbitrary we know that the integrand must be zero. The terminside the braces vanishes. This gives the Euler-Lagrange equations for the field φ

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \tag{E.4}$$

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L} \tag{E.5}$$

$$H = \int \mathcal{H} d^3x \tag{E.6}$$

following

$$p_r(t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_r}$$

conjugate momentum defined in the usual way

$$\pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \tag{E.7}$$

E.2 Bosonic Integration

E.2.1 N Real variables

$$Z(j) = \int \prod_{i=1}^{N} dx_i \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_{i=1}^{N} j_i x_i \right) \tag{E.8}$$

where the matrix A_{ij} is symmetric and strictly positive.A more compact notation is to represent the column vectors $(x_i ... x_N)$ and $(j_1 ... j_N)$ as x and j, then the row vectors would be x^T and j^T where T=transpose. We then have:

$$\sum_{i,j=1}^{N} x_i A_{ij} x_j = x^T A x, \qquad \sum_{i=1}^{N} j_i x_i = j^T x \tag{E.9}$$

Change variables to x' given by

$$x = x' + A^{-1} j, \tag{E.10}$$

(the matrix A^{-1} exists because A is assumed positive) then

$$-\frac{1}{2} x^T A X + J^T x = -\frac{1}{2} x'^T A x' + \frac{1}{2} j^T A^{-1} j, \tag{E.11}$$

The integral then becomes

$$Z(j) = \exp(\frac{1}{2} j^T A^{-1} j) Z(0) \tag{E.12}$$

In many cases equation () is all one needs (for example in calculating correlation functions, from which Z(0) cancels). Z(0) reads

$$Z(0) = \int \prod_{i=1}^{N} dx'_i \exp(-\frac{1}{2} x^T A x), \tag{E.13}$$

Let R be an orthogonal transformation ($RR^T = I$) diagonalising A,

$$A = \mathbf{R}^T \mathbf{D} \mathbf{R}, \qquad \mathbf{D} = \begin{pmatrix} d_1 & & & \\ & d_2 & & 0 \\ 0 & & \ddots & \\ & & & d_N \end{pmatrix}, \qquad d_i > 0 \qquad \forall i. \tag{E.14}$$

Make the following change of variables with unit Jacobian:

$$x' = \mathbf{R} x \qquad (det \mathbf{R} = 1), \tag{E.15}$$

$$\int \prod_{i=1}^{N} dx_i \exp(\frac{1}{2} x^T \mathbf{A} x) = \int \prod_{i=1}^{N} dx'_i \exp(\frac{1}{2} x'^T \mathbf{D} x') \tag{E.16}$$

The last integral is the product of N independent gaussian integrals, and is given by

$$(2\pi)^{N/2}\prod_{i=1}^{N}(d_i)^{-1/2} = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{E.17}$$

$$Z(0) = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{E.18}$$

$$\int\prod_{i=1}^{N} dx_i \exp(-x^T\mathbf{A}x + j^T x+) = \frac{\pi^{N/2}}{det\mathbf{A}^{1/2}}\exp((1/2)j^T A^{-1}j), \tag{E.19}$$

E.2.2 Complex variables

First consider the case of a single complex variable z=x+iy with integral

$$I = \int d^2 z e^{-z^* Az + z^* j + zj^*} = \int dxdy e^{-A(x^2+y^2)+2j_1 x + 2j_2 y} \tag{E.20}$$

where $j = j_1 + ij_2$ and $A = a_1 + ia_2$, with $a_1 > 0$. Then I follows immediately:

$$I = \frac{\pi}{A}a^{j^* A^{-1} j} \tag{E.21}$$

Next, consider the case of N complex variables z_i,

$$I = \int\prod_{i=1}^{N} d^2 z_i e^{-z^\dagger \mathbf{A} z + z^\dagger j + j^\dagger z}, \tag{E.22}$$

where transposes have been replaced by Hermitian conjugates. Assume that $\mathbf{A}$ can be diagonalized by a unitary transformation U,

$$A = U^\dagger D U, \tag{E.23}$$

where D is a diagonal matrix with elements d_i whose real parts are positive. Write

$$U = R + iS, \tag{E.24}$$

where R and S are real matrices; the relation $U^\dagger U = I$ entails

$$RR^T + SS^T = I, \qquad RS^T - SR^T = 0. \tag{E.25}$$

The transformation $z' = Uz$ amounts to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} R & -S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{E.26}$$

and the matrix which transforms (x,y) into (x', y') is orthogonal so that the jacobian of the transformation is 1, which leads to the end result

$$\int \prod_{i=1}^{N} e^{-z^\dagger \mathbf{A} z + z^\dagger j + j^\dagger z} = \frac{\pi^N}{det\mathbf{A}} e^{j^\dagger \mathbf{A^{-1}} j} \tag{E.27}$$

E.3 Feynmann Rules for Scalar Quantum field theory

E.4 Perturbation Theory

E.4.1 Diagrammatic Perturbation Theory

In this section we investigate general rules for the perturbative calculation of correlation functions, rules designed to yield the result in the form of an expansion in powers of g,

$$G = G_0 + gG_1 + g^2 G_2 + g^3 G_3 + ... + g^n G_n + ... \tag{E.28}$$

Where G_0 is the correlation function of the Gaussian model, (non-interacting model). These rules are easily represented in diagrammatic form. These diagrams are the so-called *Feynman Diagrams*. As a simple example we examine the Ginzburg-Landau theory (see eq.(**??**)). It is impossible to find an exact closed formula for Z(0), but if g is small on can expand $\exp(-g \int d^d x \phi^4(x)/4!)$.

The calculation of $\mathbf{G^{(2)}}$ to order g

First we calculate the 2-point greens function to order g. One must evaluate the integral

$$I(x,y) = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H} = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H_0} \left[1 - \frac{g}{4!} \int d^d z \phi^4(z) + \cdots \right]. \tag{E.29}$$

The first term in the square brackets merely yields

$$\mathcal{N} \langle \phi(x)\phi(y) \rangle_0 = \mathcal{N} G_0(x-y) \quad \text{where } \mathcal{N} = Z_0(j=0). \tag{E.30}$$

To evaluate the integral in the second term,

$$\int \mathcal{D}\phi\phi(x)\phi(y)e^{-H_0}\int d^dz\phi^4(z), \tag{E.31}$$

we use Wick's theorem (**??**). There are two types of result from the contractions:

$$(a)\ \ \langle\phi(x)\phi(y)\rangle_0\left\langle\phi^4(z)\right\rangle \quad \text{and} \quad (b)\ \ \langle\phi(x)\phi(z)\rangle_0\left\langle\phi^2(z)\right\rangle_0\langle\phi(y)\phi(z)\rangle_0 \tag{E.32}$$

in the wick expansion there were $4\times = 12$ terms of type (a) and 3 terms of type (b). It is convenient to represent these contractions as diagrams, by drawing two "external" points x and y ("external" means that they refer to the arguments of the correlation function), and "internal" point or "vertex" z, which stems from the expansion of exp(-V), and over which we integrate. Every contraction is represented by a line joining arguments of ϕ. The two types of terms possible in (E.32) are drawn

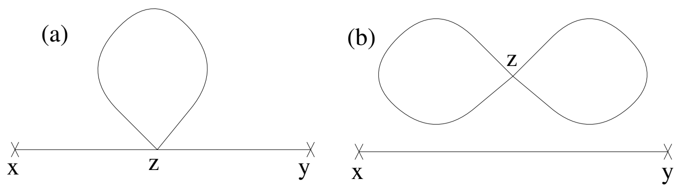


Figure E.1: The two diagrams of order g

These diagrams are called *Feynman diagrams (or graphs)*; one such diagram corresponds to every distinct group of terms of the perturbation expansion. The integral I reads

$$I(x,y) = \mathcal{N}\left[G_0(x-y) - \frac{1}{2}g\int d^dzG_0(x-z)G_0(0)G_0(z-y) - \frac{1}{8}gG_0(x-y)(G_0(0))^2\int d^dz\right] \tag{E.33}$$

In order to obtain the correlation function, we must divide by Z(0):

$$Z(0) = \int \mathcal{D}\phi e^{-H_0}\left(1 - \frac{g}{4!}\int d^dz\phi^4(z) + \cdots\right) = \mathcal{N}\left[1 - \frac{g}{8}(G_0(0))^2\int d^dz + \cdots\right]. \tag{E.34}$$

The second term in the square brackets is represented by the diagram.

Dividing (E.33) by (E.34) we obtain the correlation function to order g

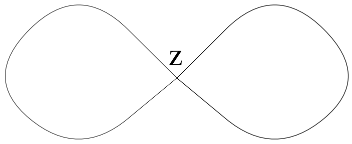


Figure E.2: The vacuum-fluctuation diagram

$$G^{(2)}(x-y) = \frac{I(x,y)}{Z(0)} = G_0(x-y) - \frac{1}{2}g \int d^d z G_0(x-z) G_0(0) G_0(z-y) + \mathcal{O}(g^2). \quad \text{(E.35)}$$

The graph(b) from fig.(E.1) does not feature in the perturbation expansion of G. Diagrams of this type are called "vacuum-fluctuation" (sub)diagrams, meaning a subgraph that is completely disconnected from the "external" points x and y. The sum of all vacuum-fluctuation diagrams is equal to $Z(0) = \mathcal{D}\phi e^{-H}$. **Division by $Z(0)$ cancels all graphs containing "vacuum-fluctuations" parts disconnected from the rest of the diagram**. A proof is given in citeBellac (p 160).

On taking the Fourier transform, eq.(E.35) becomes

$$G^{(2)}(k) = G_0(k) - \frac{1}{2}g G_0(k) \left[\int \frac{d^d q}{(2\pi)^d} G_0(q) \right] G_0(k). \quad \text{(E.36)}$$

The factor in front of the second term on the r.h.s. is called the *symmetry factor* of the diagram. To become familiar with the "Feynman rules", i.e. the rules for associating diagrams with the perturbation expansion, we move to the calculation of $G^{(2)}$ to order g^2.

The calculation of $\mathbf{G^{(2)}}$ to order $\mathbf{g^2}$

We use Wick's theorem to compute the expression

$$\left\langle \phi(x)\phi(y) \int d^d z d^d u \phi^4(z) \phi^4(u) \right\rangle_0 . \quad \text{(E.37)}$$

Eliminating the terms that contain vacuum-fluctuation parts, one finds three types of graphs shown in fig.(E.3), with their symmetry factors given in brackets:

The vertices z and u may be permuted, which yields a multiplicative factor 2!; however this is exactly cancelled by the factor 1/2! from the expansion of the exponential. This is the same kind of cancellation happens in the nth order.

We shall settle for examining the contribution $\bar{G}(x-y)$ to the correlation function from graph (a) in fig.(E.3). Thus

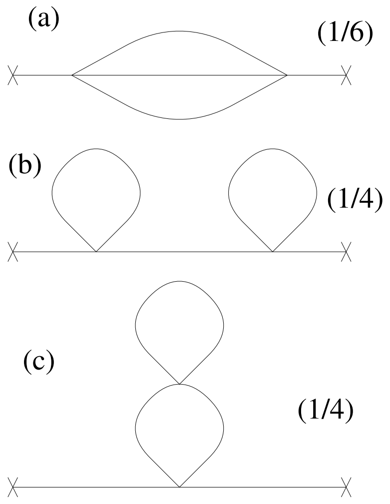


Figure E.3: The vacuum-fluctuation diagram

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^d z d^d u G_0(x-z)[G_0(z-u)]^3 G_0(u-y). \tag{E.38}$$

Let us write $\bar{G}(x-y)$ as a Fourier transform, by replacing every factor G_0 by its Fourier representation

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^d z d^d u \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d} \prod_{l=1}^{3} \left\{ \frac{d^d q_l}{(2\pi)^d} e^{i\sum_{l=1}^{3} q_l\cdot(z-u)} \right\}$$
$$\times e^{ik\cdot(x-z)} e^{ik'\cdot(u-y)} G_0(k) G_0(k') \prod_{l=1}^{3} G_0(q_l). \tag{E.39}$$

The integration over z and u yield a product of two delta functions

$$2\pi)^d \delta^d(k - q_1 - q_2 - q_3) \times (2\pi)^d \delta^d(k' - q_1 - q_2 - q_3) \tag{E.40}$$

which represent "momentum conservation" at the two vertices. Hence

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot(x-y)} [G_0(k)]^2 \times \int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d} G_0(q_1) G_0(q_2) G_0(k - q_1 - q_2). \tag{E.41}$$

The last expression shows that $\bar{G}(x-y)$ is the Fourier transform of the function $\bar{G}(k)$,

$$\bar{G}(k) = \frac{1}{6} g^2 G_0(k) \left[\int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} G_0(q_1) G_0(q_2) G_0(k - q_1 - q_2) \right] G_0(-k), \tag{E.42}$$

(where we have used $G_0(k) = G_0(-k)$). (E.42) can be represented diagrammatically in fig.(E.4). The graph shown there has two external propagators $G_0(k)$ and $G_0(-k)$, and three internal propagators; because of the delta-functions $\delta^d(\ldots)$ ("momentum conservation"), only two of the three internal lines are independent. By following the internal propagators one can describe three different closed loops, but because of "momentum conservation" only two of these are independent; i.e. there are only two integration variables in (E.42).

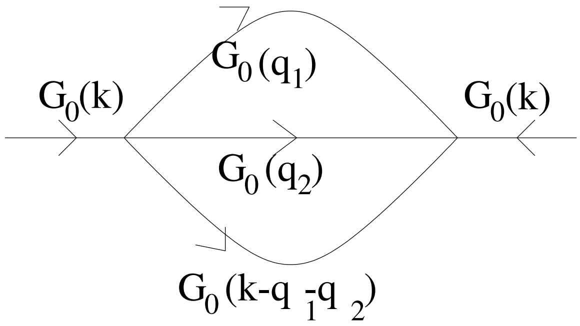


Figure E.4: Diagrammatic representation of (E.42)

Our experience with the previous examples suggest the following "Feynman rules" in k-space ("momentum space"):

1. We draw the Feynman diagram with a momentum assigned to each line. We must have overall momentum conservation and conservation at each vertex.

2. To every vertex we assign a factor -g

3. To every line we assign a factor $G_0(k)$

4. To every independent loop there corresponds an integration $\int d^d q/(2\pi)^d$.

5. Finally, every graph is multiplied by a symmetry factor.

E.4.2 The Generating Functional of Connected Diagrams

We start with an example, by investigating the correlation function $G^{(4)}$. It subdivides into one connected and three disconnected diagrams,

$$G^{(4)}(1,2,3,4) = G_c^{(4)}(1,2,3,4) + \{G_c^{(2)}(1,2)G_c^{(2)}(3,4) + permutations\}, \tag{E.43}$$

where G_c denotes a connected correlation function. (note $G_c^{(2)} = G^{(2)}$. In terms of graphs this is represented as in fig.(refbubble0)

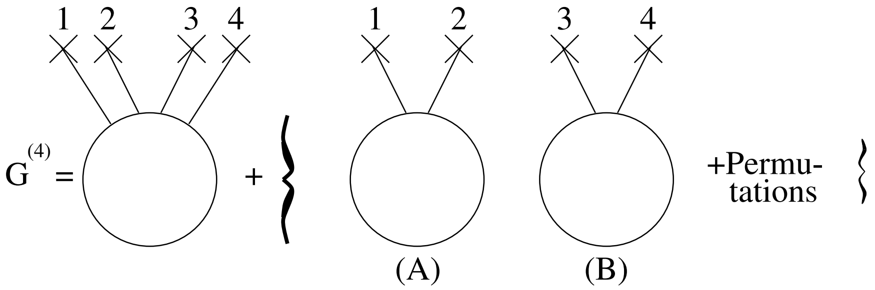


Figure E.5:

The number of disconnected terms is $3 = 4!/[(2!)^2 \times (2!)]$. 4! is the number of permutations of the external points (1,2,3,4); but the result is unaffected by permuting (1,2), or (3,4), or the two bubbles (A) and (B), hence a factor $(2!)^2 \times 2!$.

We have been only considering theories where the n-point correlation functions with n odd vanish: $G^{(2k+1)} = 0$. For more generality, we shall assume that the interaction contains terms inφ^{2p+1}. Consider a disconnected diagram of $G^{(N)}$ corresponding to the subdivision into connected diagrams (fig E.6):

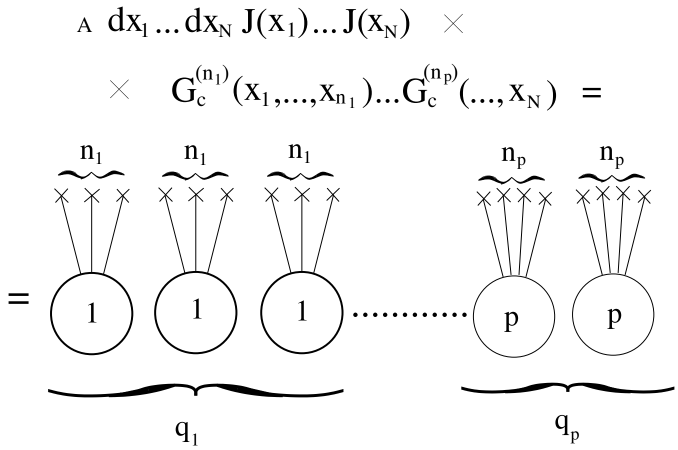


Figure E.6:

There are q_l bubbles connected to n_l external points,...., q_p bubbles connected to n_p external points, with

$$q_1 n_1 + \cdots + q_p n_p = N. \tag{E.44}$$

The number of independent terms is

$$\frac{N!}{[(n_1!)^{q_1}q_1!]\dots[(n_p!)^{q_p}q_p!]} \tag{E.45}$$

It is found that the Functional that generates just connected diagrams is the logarithm of the normalised Generating functional. Hence, consider the exponential of the generating functional of connected diagrams:

$$\exp\sum_{N=1}^{\infty}\frac{1}{N!}\int dx_1\dots dx_N j(x_1)\dots j(x_N)G_c^N(x_1\dots x_N) \tag{E.46}$$

This should give the expansion for the generating function of all possible diagrams. When the exponential is expanded it is obvious that the amplitude for every possible disconnected diagram will be produced. To complete the proof that this is the correct Generating Functional we need to check each diagram comes with the correct prefactor.(i.e. equation (E.45). So expanding equation (E.46)

$$\sum_{q=0}^{\infty}\frac{1}{q!}\left(\sum_{n=1}^{\infty}\int dx_1....dx_n j(x_1)....j(x_n)G_c^{(n)}(x_1....x_n)\right)^q \tag{E.47}$$

We convert this sum into a summation over N, the number of legs of the disconnected diagrams (figure).

$$\sum_{N=0}^{\infty}\sum_{q_1n_1+...+q_pn_p=N}\prod_{i=1}^{p}\frac{1}{q_i!}\left[\frac{\int dx_1....dx_{n_i}j(x_1)....j(x_{n_i})G_c^{n_i}(x_1....x_{n_i})}{n_i!}\right]^{q_i} \tag{E.48}$$

Now we use (E.45) and the symmetry of G_c with respect to its arguments to rewrite the above equation as

$$\sum_{N=0}^{\infty}\frac{1}{N!}\int dx_1....dx_N j(x_1)....j(x_N)\sum_{q_1n_1+...+q_pn_p=N}G_c^{n_i}(x_1....x_{n_i})....G_c^{n_p}(....x_N) \tag{E.49}$$

Which is the correct form for the generating functional. Thus we have found that the generating functional of connected diagrams W(j) is indeed ln[Z(j)/Z(0)],

$$W(j) = ln\frac{Z(j)}{Z(0)} = \sum_{N=1}^{\infty}\frac{1}{N!}\int dx_1... dx_N j(x_1)...j(x_N)G_c^N(x_1...x_N) \tag{E.50}$$

E.4.3 Generating functional of proper vertices

We have seen that to find $\log Z$ we only need consider the connected diagrams. The number of diagrams that need to be calculated can be reduced further. This is possible because some connected diagrams are made up of two or more connected diagrams joined together in a simple way. The following examples from ϕ^4 theory illustrates this redundancy connected diagrams can still have:

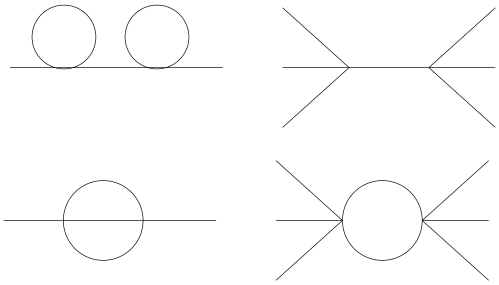


Figure E.7: Examples of reducible and irreducible diagrams from ϕ^4 theory.

The diagrams in the first row can be split into two disjoint parts by cutting a single line, thus they are called *reducible*. The examples on the second row are called *one-particle irreducible*, 1PI, since they cannot be dissected in this way. The irreducible diagrams will play an important role later because they are closely related to the parameters of the theory and also they play an important role for the systematic construction of diagrams in higher loop orders. We define an *irreducible vertex function* which is an n-point function $\Gamma^{(n)}(x_1, \ldots, x_n)$. It is possible to construct a generating functional from which the irreducible vertex functions can be obtained. This functional is defined by a Legendre transformation:

$$\Gamma[\langle\phi\rangle] = W[J] - \int d^d x J(x) \langle\phi(x)\rangle \tag{E.51}$$

where

$$\langle\phi(x)\rangle = \frac{\delta W}{\delta J(x)} \tag{E.52}$$

This functional Γ has no explicit dependence on $j(x)$. It is a functional of the expectation value of the operator $\phi(x)$ in the presence of the source J denoted $\langle\phi\rangle$. It is easy to find from Eq.(E.51) that:

$$J(x) = -\frac{\delta\Gamma}{\delta\langle\phi(x)\rangle} \tag{E.53}$$

If we functionally differentiate equation E.52 by $J(y)$ and use the chain rule we get the identity

$$\int d^d z \frac{\delta^2 W}{\delta j(x)\delta j(z)} \frac{\delta^2 \Gamma}{\delta \langle \phi(z) \rangle \, \delta \langle \phi(y) \rangle} = \int d^d z G^{(2)}(x-z)\Gamma^{(2)}(z-y) = -\delta^{(d)}(x-y) \quad \text{(E.54)}$$

Which shows that $-\Gamma^{(2)}(x-y)$ is the inverse of the connected Green's function $G^{(2)}(x-y)$

The self-energy $\Sigma(k)$ is defined as the sum of all two point 1PI diagrams shorn of their external lines. The correlation function $G^{(2)}(k)$ can be written in terms of $\Sigma(k)$ as

$$G^{(2)}(k) = G_0(k) + G_0(k)\Sigma(k)G_0(k) + \cdots = \left[G_0^{-1}(k) - \Sigma(k)\right]^{-1} \quad \text{(E.55)}$$

We have the relationships in ϕ^4 theory:

$$G^{(2)}(k) = \frac{1}{k^2 + m^2 - \Sigma(k)}, \qquad \Gamma^{(2)}(k) = -(k^2 + m^2 - \Sigma(k)) \quad \text{(E.56)}$$

We have a new effective mass $-\Gamma^{(2)}(0) = m^2 - \Sigma(0)$.

One can begin to see how the functions $\Gamma^{(n)}$ are related to the running parameters of the model.

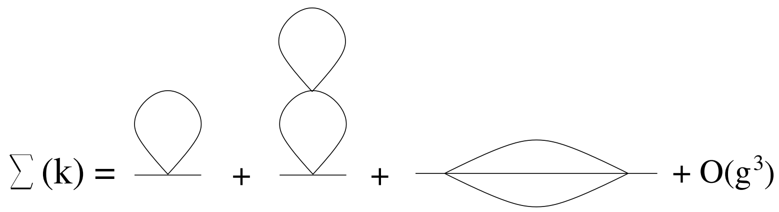


Figure E.8: The self-energy in ϕ^4 theory to order g^2

.

To ease the notation in the proof of what follows, we use

$$\frac{\delta}{\delta J(x)} \to \frac{\delta}{\delta j_i}, \qquad \int d^d x \to \sum_i \quad \text{(E.57)}$$

We have just met the identity

$$\sum_l \frac{\delta^2 W}{\delta j_i \delta j_l} \frac{\delta^2 \Gamma}{\delta \bar{\phi}_l \delta \bar{\phi}_k} = \sum_l G^{(2)}_{il} \Gamma^{(2)}_{lk} = -\delta_{ik} \quad \text{(E.58)}$$

which shows that $-\Gamma^{(2)}_{lk}$ is the inverse of the(connected) Green's function $G^{(2)}_{kl}$.

We now proceed to Green's functions of higher order, by differentiating the identity Eq.(E.58) with respect to j_m:

$$\sum_l \frac{\delta^3 W}{\delta j_i \delta j_l \delta j_m} \frac{\delta^2 \Gamma}{\delta \langle\phi\rangle_l \delta \langle\phi\rangle_k} + \sum_l \frac{\delta^2 W}{\delta j_i \delta j_l} \frac{\delta^3 \Gamma}{\delta j_m \delta \langle\phi\rangle_l \delta \langle\phi\rangle_k} = 0 \tag{E.59}$$

Since Γ is a function of the $\langle\phi\rangle_i$, we must transform the second derivative in Eq.(E.59); we do this for the general case ($\Gamma^{(N)}_{i_1 \dots i_N} = \delta^{(N)}\Gamma/\delta\langle\phi\rangle_{i_1} \dots \delta\langle\phi\rangle_{i_N}$):

$$\frac{\delta}{\delta j_m} \Gamma^{(N)}_{i_1 \dots i_N} = \sum_n \frac{\delta\langle\phi\rangle_n}{\delta j_m} \frac{\delta\Gamma^{(N)}_{i_1 \dots i_N}}{\delta\langle\phi\rangle_n} = \sum_n G^{(2)}_{mn} \Gamma^{(N+1)}_{n i_1 \dots i_N}. \tag{E.60}$$

Equations (E.59) and (E.60) can be put into a graphical form if we represent the $\Gamma^{(N)}$ by shaded bubbles Fig.(E.9) (Here we have used Eq.(E.60) in order to transform Eq.(E.59).)

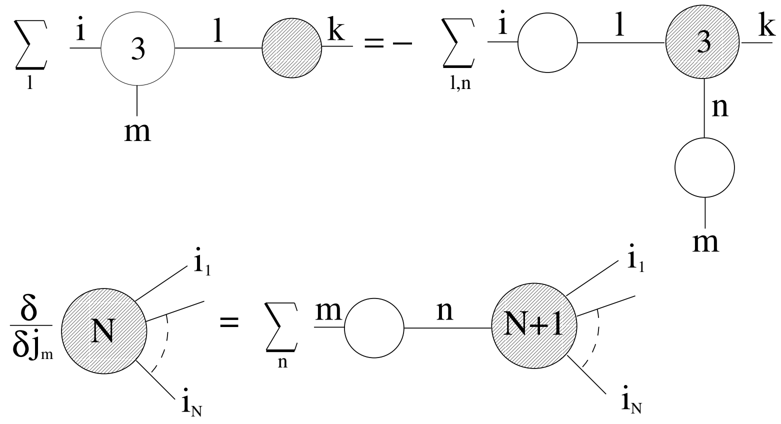


Figure E.9: Graphical representation of Eq.(E.59) and Eq.(E.60).

Multiplying the two terms in Eq.(E.59) from the right by G_{kp} and summing over k we find the relation between $G^{(3)}_c$ and $\Gamma^{(3)}$,

$$G^{(3)}_{ijk} = \sum_{l,m,n} G_{il} G_{jm} G_{kn} \Gamma^{(3)}_{lmn} \tag{E.61}$$

see Fig(E.10). Clearly, $\Gamma^{(3)}$ represents the connected, truncated 3-point function since $G^{(3)}$ is connected, and each $\Gamma^{(2)}$ above chops off one of the external legs. $\Gamma^{(3)}$ is automatically 1PI, because it is a 3-point function.

We continue the process by differentiating Eq.(E.61) once more with respect to j_l. Using Eq.(E.59) and Eq.(E.60) we obtain the relation between $G^{(4)}$ and $\Gamma^{(4)}$, Fig.(E.11)

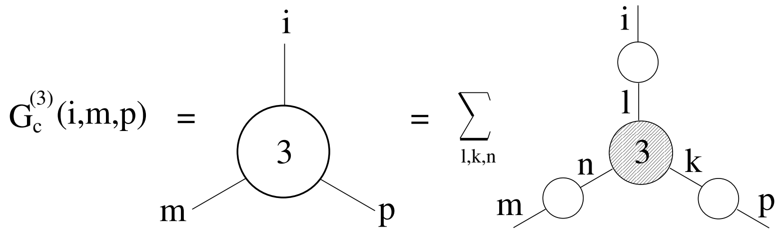


Figure E.10: 3-point Green's function, $G^{(3)}$, written in terms of the proper 3-point vertex and 2-point Green's functions $G^{(2)}$

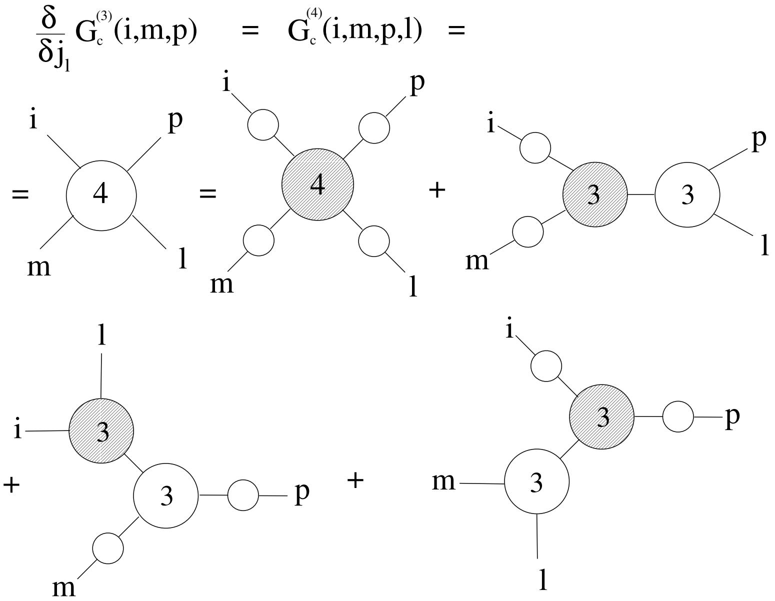


Figure E.11: Relation between $G^{(4)}$ and $\Gamma^{(4)}$, $\Gamma^{(3)}$ and $G^{(2)}$.

As we know that $\Gamma^{(3)}$ is a sum of all 3-point 1PI diagrams, the last three terms of Fig.(E.11) supply all the diagrams of the 4-point Green's functions that can rendered disconnected by cutting one internal line. So the first term in Fig.(E.11) must be the summation over those diagrams of $G^{(4)}$ that don't become disconnected by cutting an internal line. Hence, we can identify the fourth derivative of the Effective action, $\Gamma^{(4)}$, as the summation over all 4-point 1PI diagrams.

It transpires that the higher order derivatives of Γ are the summation over 1PI diagrams. These can be joined together with 2-point Green's functions to construct the four and higher point connected Green's functions. The special cases N=3 and N=4 which we have just studied enable us to see how one can set out to prove the above statement.

One can prove by induction, assuming that an equation like that of Fig(E.11) can be written to order N, and that the N-th order proper vertex can be identified with the Nth derivative of the generating functional. Differentiating this equation with respect to j_i,

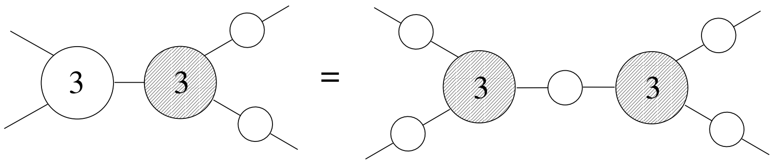


Figure E.12: We can replace each $G^{(3)}$ for $\Gamma^{(3)}$ in Fig.(E.11) using Fig.(E.10).

one finds: see Fig.(E.13)

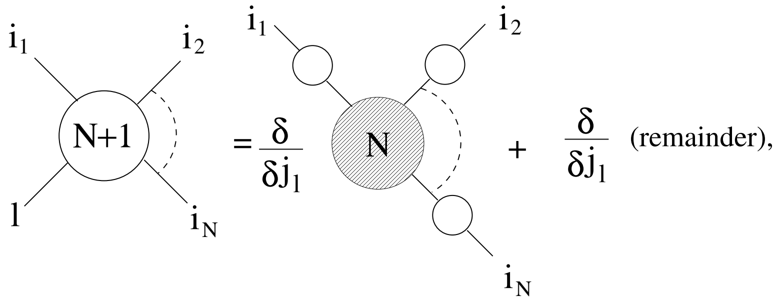


Figure E.13: Differentiating $G^{(N)}$ with respect to source j_i.

where the "remainder" does not contain $\Gamma^{(N)}$, but only $\Gamma^{(N)}$ $\Gamma^{(N-2)}$, etc. (see Fig.(E.11)). Hence we obtain

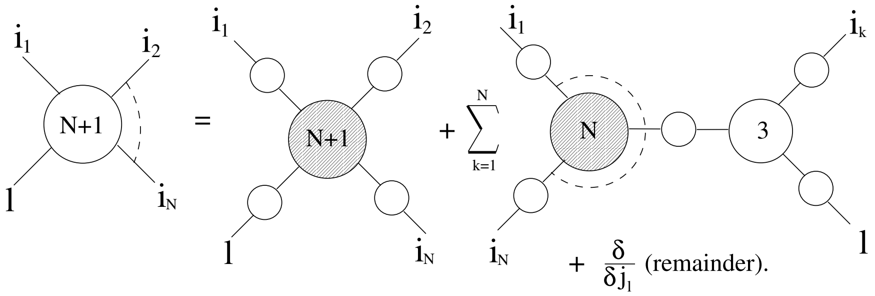


Figure E.14: All possible Green's functions with N external legs.

On removing the full external propagators from both sides, we can identify $\Gamma^{(N+1)}$ as the N+1-point 1PI diagram.

There is an algebraic proof that by cutting one line in all possible ways in a diagram contributing to $\Gamma[\langle\phi\rangle]$ does not produce disconnected diagrams that is given in [**?**] (p. 135). This proof is easily generalsied to models that are more complicated than ϕ^4. Their proof is presented in Appendix C.

E.5 Grassmann Integration

Grassmann Algebra

We consider a set of anticommutating Grassmann variables $\{\zeta_i\}_{i=1,...,n}$, with complex linear coefficients, where n is the dimension of the algebra. The decisive relation defining the structure of the algebra is the anticommutation relation

$$\zeta_i \zeta_j + \zeta_j \zeta_i = 0 \tag{E.62}$$

for all i and j. As a particular consequence of this condition the square and all higher powers of a variable vanish,

$$\zeta_i^2 = 0 \tag{E.63}$$

The Grassmann algebra generate a Grassmann algebra of functions which have the form

$$f(\zeta) = f^{(0)} + \sum_i f_i^{(1)} + \sum_{i_1<i_2} f_{i_1 i_2}^{(2)} \zeta_{i_1} \zeta_{i_2} + ... + f^{(n)} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_n} \tag{E.64}$$

where the coefficients $f^{(k)}$ are ordinary complex numbers.

On this algebra we will need to define the derivative. We first consider a simple Grassmann algebra of order $n = 2$ with the variables ζ_1 and ζ_2.

$$f(\zeta_1, \zeta_2) = f^{(0)} + f_1^{(1)} \zeta_1 + f_2^{(1)} \zeta_2 + f^{(2)} \zeta_1 \zeta_2$$

$$\frac{\partial f}{\partial \zeta_1} = f_1^{(1)} + f^{(2)} \zeta_2, \qquad \frac{\partial f}{\partial \zeta_2} = f_2^{(1)} - f^{(2)} \zeta_1. \tag{E.65}$$

Note the minus sign in the last equation of (E.65),

$$\frac{\partial}{\partial \zeta_j} \zeta_1 \zeta_2 = \delta_{j1} \zeta_2 - \delta_{j2} \zeta_1.$$

The general rule for differentiation of a product is given by

$$\frac{\partial}{\partial \zeta_j} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_m} = \delta_{ji_1} \zeta_{i_2} ... \zeta_{i_m} - \delta_{ji_2} \zeta_{i_1} \zeta_{i_3} ... \zeta_{i_m} + ... + (-1)^{m-1} \delta_{ji_2} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_{m-1}} \tag{E.66}$$

The respective factor ζ_{i_k} is anticommuted to the left until the derivative operator can be directly applied. We may prove the following properties of the derivatives

$$\frac{\partial}{\partial\zeta_i}\frac{\partial}{\partial\zeta_j}+\frac{\partial}{\partial\zeta_j}\frac{\partial}{\partial\zeta_i}=0 \tag{E.67}$$

$$\frac{\partial}{\partial\zeta_i}\zeta_j+\zeta_j\frac{\partial}{\partial\zeta_i}=0 \tag{E.68}$$

Grassmann integration

An attempt to to introduce an indefinite integral as the inverse of differentiation is bound to fail. This illustrated by the fact that according to (E.67) the second derivative of any Grassmann function vanishes, so that the inverse operation does not exist, for if there was an inverse to $\frac{\partial^2 F}{\partial\zeta^2}$ it should give

$$\int d\zeta\frac{\partial^2 F}{\partial\zeta^2}=\frac{\partial F}{\partial\zeta}$$

However as

$$0=\int d\zeta 0=\int d\zeta\frac{\partial^2 F}{\partial\zeta^2}$$

this would imply we always have

$$\frac{\partial F}{\partial\zeta}=0$$

which is not true in general.

We must be content with some formal definition. One way to arrive at it is to require that integration be translationally invariant. For an arbitrary function $g(\zeta)=g_1+g_2\zeta$ we have

$$\begin{aligned}\int d\zeta g(\zeta+\eta) &= \int d\zeta\,[g_1+g_2(\zeta+\eta)]=\int d\zeta\,[g_1+g_2\zeta]+\int d\zeta g_2\eta \\ &= \int d\zeta g(\zeta)+\left[\int d\zeta 1\right]g_2\eta=\int d\zeta g(\zeta)\end{aligned} \tag{E.69}$$

The translational invariance requires the integral of 1 is zero. The following postulates uniquely fix the value of any integral.

$$\int d\zeta 1 = 0, \tag{E.70}$$

$$\int d\zeta\zeta = 1. \tag{E.71}$$

Eq. (E.70) comes from the condition of translational invariance. The sole non-vanishing integral $\int d\zeta\zeta$ arbitrarily is assigned the value 1. This is a convenient normalisation condition.

We see that integration is equivalent to differentiation. Generalising integration rules to higher dimensions straightforward

$$\int d\zeta_i 1 = 0, \tag{E.72}$$

$$\int d\zeta_i\zeta_j = \delta_{ij}. \tag{E.73}$$

Note that the differentials $d\zeta_i$ must anticommute with all other Grassmann variables as integration is equivalent to differentiation. In order to obtain analog results of conventional integration we introduce complex Grassmann variables. Let us start with two disjoint sets of Grassmann variables $\zeta_1^*, ..., \zeta_n^*$ and $\zeta_1, ..., \zeta_n$, which are all mutually anticommutating

$$\{\zeta_i, \zeta_j\} = \{\zeta_i^*, \zeta_j^*\} = \{\zeta_i, \zeta_j^*\} = 0 \tag{E.74}$$

The two sets are related, using complex cunjugation, according to

$$\begin{aligned}
(\zeta_i)^* &= \zeta_i^*, \\
(\zeta_i^*)^* &= -\zeta_i, \\
(\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m})^* &= \zeta_{i_m}^*...\zeta_{i_2}^*\zeta_{i_1}^* \\
(\lambda\zeta_i)^* &= \lambda^*\zeta_i^*
\end{aligned} \tag{E.75}$$

where λ is a complex number.

In order to develop functional integral formalism for Grassmann fields we need to solve *Gaussian integrals.*

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp\left\{ -\sum_{k,l=1}^{N} \zeta_k^* M_{kl} \zeta_l \right\} \tag{E.76}$$

To simplify the notation, let us write this as

$$I = \int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} \tag{E.77}$$

The calculation in principle is very simply because grassmann functions can at worst be linear in each variable, causing the series expansion of the exponential function to terminate. On the other hand , according to the rules for Grassmann integration, the integrand must contain as many different Grassmann variables as there are integrals or else the overall integration vanishes.

Let us consider the case where we have two pairs of variables. The exponential then reads

$$e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2 - \frac{1}{3!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^3 + \ldots \tag{E.78}$$

Obviously this series terminates beyond second order, so we have

$$e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2. \tag{E.79}$$

Let us consider the integration of the first two terms, obviously we have

$$\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; 1 = 0$$
$$\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b) = 0$$

as the number of variables integrated over is greater than the number of variables appearing in the integrand. For the case of two pairs of variables one effectively has

$$\begin{aligned} e^{-\zeta^* M \zeta} &\rightarrow \frac{1}{2!} (\zeta^* M \zeta)^2 \\ &= \frac{1}{2!} (\zeta_1^* M_{11} \zeta_1 + \zeta_1^* M_{12} \zeta_2 + \zeta_2^* M_{21} \zeta_1 + \zeta_2^* M_{22} \zeta_2)^2 \\ &= (M_{11} M_{22} - M_{12} M_{21}) \zeta_1^* \zeta_1 \zeta_2^* \zeta_2 \end{aligned} \tag{E.80}$$

where the last line follows from the anticommutating character of the Grassmann numbers. The integration of $\zeta_1^*\zeta_1\zeta_2^*\zeta_2$, gives unity, and so for this case

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = detM \tag{E.81}$$

One should suspect that this result holds in general. For the case of N pairs of variables, only the term of order $(\zeta^* M\zeta)^N$ survives in the expansion of the exponential and contributes to the integral:

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = \frac{(-1)^N}{N!} \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k)(\zeta^* M\zeta)^N. \tag{E.82}$$

In view of the anticommutativity of the Grassmann variables, these terms contain the appropriately signed products of matrix elements which define the determinant. But rather than go through this combinatorial exercise we will follow the method given in (Brown QFT).(page83) which is presented in Appendix(B). The integral is some function $I(M)$ of the matrix M, let us derive a differential equation for this function. Since $\zeta^* M\zeta$ contains the product of two anti commutating variables and thus a commutating variable itself,

$$\begin{aligned} \delta_M(\zeta^* M\zeta)^N &= (\zeta^*(M+\delta M)\zeta)^N - (\zeta^* M\zeta)^N \\ &= n(\zeta^*\delta M\zeta)(\zeta^* M\zeta)^{n-1} \end{aligned} \tag{E.83}$$

$$\begin{aligned} \delta I &= \int [d\zeta^* d\zeta] \; (e^{-\zeta^*(M+\delta M)\zeta} - e^{-\zeta^* M\zeta}) \\ &= \int [d\zeta^* d\zeta] \sum_{n=0} \frac{(-1)^n}{n!}[(\zeta^*(M+\delta M)\zeta)^n - (\zeta^* M\zeta)^n] \\ &= -\int [d\zeta^* d\zeta] \; (\zeta^*\delta M\zeta) \sum_{n=0} \frac{(-1)^{n-1}}{(n-1)!}(\zeta^* M\zeta)^{n-1} \\ &= -\int [d\zeta^* d\zeta] \; \zeta^*\delta M\zeta \; e^{-\zeta^* M\zeta} \\ &= -\int [d\zeta^* d\zeta] \sum_{m,l=1}^{n} \zeta_m^*(\delta M)_{ml}\zeta_l \; e^{-\zeta^* M\zeta} \end{aligned} \tag{E.84}$$

Since $\zeta^* M\zeta$ commute, the derivative of $e^{-\zeta^* M\zeta}$ is given by

$$
\begin{aligned}
\frac{\partial}{\partial \zeta_k^*} e^{-\zeta^* M \zeta} &= \frac{\partial}{\partial \zeta_k^*} \sum_{n=0} \frac{(-1)^n}{n!} (\zeta^* M \zeta)^n \\
&= \sum_{n=1} \frac{(-1)^n}{n!} \Big([\frac{\partial}{\partial \zeta_k^*}(\zeta^* M \zeta)](\zeta^* M \zeta)^{n-1} + \\
&\quad + (\zeta^* M \zeta)[\frac{\partial}{\partial \zeta_k^*}(\zeta^* M \zeta)](\zeta^* M \zeta)^{n-2} + \cdots + (\zeta^* M \zeta)^{n-1}[\frac{\partial}{\partial \zeta_k^*}(\zeta^* M \zeta)] \Big) \\
&= -\sum_{l=1} M_{kl} \zeta_l \sum_{n=1} \frac{(-1)^{n-1}}{(n-1)!} (\zeta^* M \zeta)^{n-1} \\
&= -\sum_{l=1} M_{kl} \zeta_l e^{-\zeta^* M \zeta}. \qquad \text{(E.85)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\delta I &= -\int [d\zeta^* d\zeta] \sum_{m,l=1} \zeta_m^* (\delta M)_{ml} \zeta_l \, e^{-\zeta^* M \zeta} \\
&= -\int [d\zeta^* d\zeta] \sum_{m,k,l=1} \zeta_m^* (\delta M M^{-1})_{mk} M_{kl} \zeta_l e^{-\zeta^* M \zeta} \\
&= \int [d\zeta^* d\zeta] \sum_{m,k=1} \zeta_m^* (\delta M M^{-1})_{mk} \frac{\partial}{\partial \zeta_k^*} e^{-\zeta^* M \zeta}. \qquad \text{(E.86)}
\end{aligned}
$$

This integral may be evaluated by "integration by parts" by using the following relation, where F is an arbitrary function of Grassmann variables

$$
\frac{\partial}{\partial \zeta_k^*}(\zeta_m^* F) = \delta_{km} F - \zeta_m^* \frac{\partial}{\partial \zeta_k^*} F. \qquad \text{(E.87)}
$$

To prove the above equation we need only consider three cases

(i) $k = m$ and F is a function of ζ_k^* as well. The r.h.s. vanishes as it should.

(ii) $k = m$ and F is not a function of ζ_k^* as well. Then we should just be left with F on thee r.h.s.

(iii) $k \neq m$. For this case the equation gives the correct answer again. The minus sign is there as the derivative has to pass through the first Grassmann variable.

Therefore,

$$\delta I = \int [d\zeta^* d\zeta] \sum_{k,m} (\delta M M^{-1})_{mk} \left\{ \delta_{km} e^{-\zeta^* M \zeta} - \frac{\partial}{\partial \zeta_k^*} (\zeta_m^* e^{-\zeta^* M \zeta}) \right\} \tag{E.88}$$

Since the Grassmann integral of a derivative vanishes,

$$\delta I = (Tr \delta M M^{-1}) I, \tag{E.89}$$

which gives

$$\delta \ln I = \frac{\delta I}{I} = Tr(\delta M M^{-1}), \tag{E.90}$$

It turns out that

$$Tr(\delta M M^{-1}) = \delta(\ln \ detM) \tag{E.91}$$

which we digress to prove.

$$detM = \sum_n M_{kn} C_{nk} \tag{E.92}$$

Taking the derivative of this, noting that C_{mk} is independent of the element M_{km},

$$\frac{\partial}{\partial M_{km}} detM = C_{mk} \tag{E.93}$$

Now the well known formul for the inverse matrix M^{-1} is

$$(M^{-1})_{kl} = \frac{C_{kl}}{detM} \tag{E.94}$$

So that we have

$$\frac{\partial}{\partial M_{km}} detM = (M^{-1})_{km} detM \tag{E.95}$$

or

$$\frac{\partial}{\partial M_{km}} \ln detM = (M^{-1})_{km}. \tag{E.96}$$

Accodingly,

$$\begin{aligned}
\delta \ln detM &= \sum_{k,m} \delta M_{km} \frac{\partial \ln detM}{\partial M_{km}} \\
&= \sum_{k,m} \delta M_{km} (M^{-1})_{mk} \\
&= Tr(\delta M M^{-1}) \\
&= Tr(M^{-1} \delta M).
\end{aligned} \tag{E.97}$$

Comparing this with (E.91), we see the equation (E.90) for $I(M)$ becomes

$$\delta \ln I = \delta(\ln\ detM), \tag{E.98}$$

with the solution

$$I(M) = Const.\ detM. \tag{E.99}$$

Treating the problem as a differential equation for $I(M)$, we set $M = 1$ in order to determine the proportionalitiy constant,

$$\begin{aligned}
I(1) &= Const. = \left[\int d\zeta^* d\zeta e^{-\zeta^*\zeta} \right]^N \\
&= \prod_{k=1}^{N} \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^{N} \zeta_i^* \zeta_i)^N \\
&= \frac{1}{N!} \prod_{k=1}^{N} \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^{N} \zeta_i^* \zeta_i)^N \\
&= (-1)^N \int d\zeta_N^* d\zeta_N \ldots d\zeta_1^* d\zeta_1 (\zeta_N^* \zeta_N \ldots \zeta_1^* \zeta_1) \\
&= 1^N = 1.
\end{aligned} \tag{E.100}$$

Hence the constant is unity and we do obtain the expected result:

$$I = \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) e^{-\zeta^* M \zeta} = detM \tag{E.101}$$

This should be compared to the ordinary integration where the corresponding integral gives $\det M^{-1}$.

Grassmann generating Functional

It is not surprising that the Gaussian integral formula (E.101) can be generalised to the case of general bilinear forms in the exponent:

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp - \left(\zeta^* M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho\right) = \det M \exp(\rho^\dagger M^{-1} \rho). \tag{E.102}$$

Here ρ is an n-component vector of Grassmann variables. Equation (E.102) is obtained by translating the integration variable, $\zeta' = \zeta + M^{-1}\rho$,

$$\begin{aligned}
\det M &= \int \prod_{k=1}^{N} (d\zeta'^*_k d\zeta'_k) \exp - \zeta'^* M \zeta' \\
&= \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp - (\zeta^\dagger + \rho^\dagger M^{-1}) M (\zeta + M^{-1}\rho) \\
&= \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp - \left(\zeta^\dagger M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho + \rho^\dagger M^{-1} \rho\right) \\
&= \left[\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp - \left(\zeta^\dagger M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho\right)\right] \exp - (\rho^\dagger M^{-1} \rho)
\end{aligned} \tag{E.103}$$

The construction of functional integration in section (4.1.2) did not make use of any special properties of the integration over field variables which might restrict the validity to ordinary c-numbers.

$$\boxed{\begin{aligned}\int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\left[-\int d^d x'\, d^d x \bar{\chi}(x') A(x', x) \chi(x) + \int d^d x (\bar{\rho}(x)\chi(x) + \bar{\chi}(x)\rho(x))\right] \\ = detA \exp\left[\int d^d x'\, d^d x \bar{\rho}(x') A^{-1}(x', x) \rho(x)\right].\end{aligned}} \tag{E.104}$$

in which the measure is $\propto \prod_{\mathbf{r}} d\bar{\varphi}(r) d\varphi(r)$ and $Z(\rho = 0) = detA$. Note that to normalise the functional we divide by $detA$ as apposed to $det(A^{-1})$ in the bosonic case (**??**).

It is rather straightforward to extend the results of esction 4.1 to the fermionic case: The Grassmann functional derivative is defined

$$\frac{\delta G[\chi(y)]}{\delta \chi(x)} = \lim_{\Delta V_i \to 0} \frac{\partial G}{\partial \chi_i} \quad \text{where } \mathbf{x} \text{ is located in cell } \Delta V_i \tag{E.105}$$

The (2n)-point correlators

$$G^{(2n)}(y_1,\ldots,y_n;x_1,\ldots,x_n) = \langle\chi(y_n),\ldots,\chi(y_1);\bar\chi(x_1),\ldots,\bar\chi(x_n)\rangle \tag{E.106}$$

can now be obtained by forming derivatives of the generating functional [1]

$$G^{(2n)}(y_1,\ldots,y_n;x_1,\ldots,x_n) = \left.\frac{\delta^{2n}Z[\rho,\bar\rho]}{\delta\rho(x_n)\cdots\delta\rho(x_1)\delta\bar\rho(y_1)\cdots\delta\bar\rho(y_n)}\right|_{\rho=\bar\rho=0}. \tag{E.107}$$

E.6 QED from a Functional Integral

E.6.1 Photon Propagator in Different Gauges

The form of the propagator we used in our QED calculations was (in momentum space)

$$D_{F\mu\nu} = -\frac{4\pi}{k^2+i\varepsilon}\eta_{\mu\nu} \tag{E.108}$$

which is only one of many choices of defining the photon propagator.

When we construct S-matrix elemenet, e.g.

$$j^\mu_{43}(p_4,p_3)D_{F\mu\nu}(k)j^\nu_{21}(p_2,p_1) \tag{E.109}$$

where $p_2 = p_1 - k$ and $p_4 = p_3 + k$. Now the transition currents obey the equation of continuity. Their four divergencies vanish, i.e. in momentum space

$$\begin{aligned} k_\nu j^\nu_{21}(p_1-k,p_1) &= 0 \\ k_\nu j^\nu_{43}(p_3+k,p_3) &= 0. \end{aligned} \tag{E.110}$$

Therefore one can add to $D_{F\mu\nu}$ the expression $k_\mu f_\nu(k) + k_\nu g_\mu(k)$ with arbitrary functions $f_\nu(k)$ and $g_\mu(k)$ without changing the result of the calculation.

Keeping the symmetry between the transition currents, we generalise () to

$$D_{F\mu\nu} = -\frac{4\pi}{k^2+i\varepsilon}\eta_{\mu\nu} + k_\mu f_\nu(k) + k_\nu f_\mu(k). \tag{E.111}$$

[1] The order of the derivatives was chosen in such that we get agreement with the bosionic case. This is not a trivial matter as the Grassmann derivatives $\delta/\delta\rho(x)$ and $\delta/\delta\rho(x)$ anticommute with the field variables $\chi(x)$ and $\bar\chi(x)$. One can show, however, that there is an even number of commutations when we carry out the differentiations of (E.107) and write the result in the form (E.106).

The origin of this amniguity of the photon propagator is the gauge degrees of freedon of the electromagnetic field:

$$A_\mu(x) \to A_\mu(x) + \frac{\partial}{\partial x^\mu}\chi(x). \tag{E.112}$$

The Coulomb gauge

The propagator takes the form

$$\begin{aligned}
D_{Cij}(k) &= \frac{4\pi}{k^2 + i\varepsilon}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \\
D_{C0j}(k) &= D_{Ci0}(k) = 0 \\
D_{C00}(k) &= \frac{4\pi}{\mathbf{k}^2}
\end{aligned} \tag{E.113}$$

The component $D_{C00}(k)$ is just the fourier transform of the electrostatic potential $1/r$.

E.6.2 The Generating Functional Integral of QED

We include coupling of the electromagnetic field to the Dirac field. Since our previous work is valid in any sort of current coupling to the photon field

$$Z[J, \eta, \overline{\eta}] = \int [\mathcal{D}A][\mathcal{D}\psi][\mathcal{D}\overline{\psi}] \exp\left\{ i \int \mathcal{L}_{QED} \right\}, \tag{E.114}$$

where

$$\begin{aligned}
\mathcal{L}_{QED} = {} & -\frac{1}{4e^2}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \overline{\psi}\left[\gamma_\mu\left(\frac{1}{i}\partial^\mu - A^\mu\right) + m_0\right]\psi + \\
& J^\mu A_\mu + \overline{\psi}\eta_0 + \overline{\eta_0}\psi.
\end{aligned} \tag{E.115}$$

E.7 Canonical Quantisation of Scalar Field

We promote fields to operators, and impose equal time commutation relations on the fields and their conjugate momenta.

Position and momentum p are not operrators, instead they are just numbers.

The fields $\varphi(x,t)$ and their conjugate momentum fields $\pi(x,t)$ are operators

We have states as we do in ordinary quantum mechanics, but these are states of the field

$$\varphi = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\varphi(\vec{k})e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + \varphi^*(\vec{k})e^{i(\omega_k t - \vec{k}\cdot\vec{x})}\right] \tag{E.116}$$

We promote φ to an operator by promoting the coefficients $\varphi(\vec{k})$ and $\varphi^*(\vec{k})$:

$$\begin{aligned} \varphi(\vec{k}) &= \hat{a}(\vec{k}) \\ \varphi^*(\vec{k}) &= \hat{a}^\dagger(\vec{k}) \end{aligned} \tag{E.117}$$

Conjugate momentum

$$\begin{aligned} \partial_t\hat{\varphi} &= \partial_t \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger(\vec{k})e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})(-i\omega_k)e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger(\vec{k})(+i\omega_k)e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \\ &= -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}} \left[\hat{a}(\vec{k})e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} - \hat{a}^\dagger(\vec{k})e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \end{aligned} \tag{E.118}$$

E.7.1 Commutatin Relations

To quantise the scalar field we postulate the standard commutatin relations

$$\begin{aligned} [\hat{\varphi}(x), \hat{\pi}(y)] &= i\delta^3(\mathbf{x}-\mathbf{y}) \\ [\hat{\varphi}(x), \hat{\varphi}(y)] &= 0 \\ [\hat{\pi}(x), \hat{\pi}(y)] &= 0 \end{aligned} \tag{E.119}$$

E.7.2 Bose Statistics

$$\begin{aligned} |\vec{k}_1, \vec{k}_2 > &= \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)|0> \\ &= \hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0> \\ &= |\vec{k}_2, \vec{k}_1 > \end{aligned} \tag{E.120}$$

which implies

$$|\vec{k}_1, \vec{k}_2 >= |\vec{k}_2, \vec{k}_1 > \tag{E.121}$$

E.8 Perturbation Theory in Canonical Approach

Any physical process is a transition from an initial state $|i >= |A(t_0) >$ to a final state $|f >= |A(t) >$

E.8.1 Pictures

The Schrodinger picture

In the Schrodinger picture the time dependence is carried by the states according to Schrodinger's equation

$$i\hbar \frac{d}{dt}|A(t) >_S = \hat{H}|A(t) > \tag{E.122}$$

This can be formally solved in terms of the state of the system at an arbitrary initial time t_0

$$|A(t) >_S = \hat{U}|A(t_0) >_S \tag{E.123}$$

where $\hat{U}$ is the unitary operator:

$$\hat{U} := \hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar} \tag{E.124}$$

The Heisenberg picture

Using $\hat{U}$ we perform the following transformations, defining the state $|A(t) >_H$ and operator $O^H(t)$:

$$|A(t) >_H = \hat{U}^\dagger |A(t) >_S = |A(t_0) >_S \tag{E.125}$$

and

$$O^H(t) = \hat{U}^\dagger O^S \hat{U} \tag{E.126}$$

The H indicates that this is the Heisenberg picture. At $t = t_0$ states and operators in the two pictures are the same. We see that in the Heisenberg picture states are constant in time.

Differentiation of (E.126) gives the Heisenberg equation of motion

$$\begin{aligned} i\hbar\frac{d}{dt}O^H(t) &= (i\hbar\frac{d}{dt}\hat{U}^\dagger)O^S\hat{U} + \hat{U}^\dagger O^S(i\hbar\frac{d}{dt}\hat{U}) \\ &= \hat{U}^\dagger\hat{O}^S\hat{U}\hat{H} - \hat{H}\hat{U}^\dagger\hat{O}^S\hat{U} \\ &= [\hat{O}^H, \hat{H}] \end{aligned} \tag{E.127}$$

The Interaction picture

The Interaction picture arises if the Hamiltonian is split into two parts

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \tag{E.128}$$

The interaction picture is related to the Schrodinger picture by the unitary transformation

$$\hat{U}_0 = \hat{U}_0(t, t_0) = e^{-i\hat{H}_0(t-t_0)/\hbar} \tag{E.129}$$

i.e.

$$|A(t) >_I = \hat{U}_0^\dagger |A(t) >_S \tag{E.130}$$

and

$$O^I(t) = \hat{U}_0^\dagger O^S \hat{U}_0 \tag{E.131}$$

Differentiation of (E.131) gives the equation of motion in the interaction picture

E.8.2 Perturbation Theory

E.9 QED from Interaction Picture

$$\mathcal{H}_{int} = -ie\overline{\psi}\gamma^{\mu}\psi A_{\mu}. \tag{E.132}$$

There are various factrs of $\mathcal{H}_{int}$ in the expansion of thr interacting Lagrangian. We use Wick's theorem to pair off the various fermion photon lines to for propagators and vertices.

Appendix F

Electro-Weak Theory

F.1 Fermi Interactions

Recall because of crossing symmetry, processes which differ in the grouping of incoming and outgoing particles are related to each other. In particular the matrix element of three body decay can be derived from that of the two-body scattering process. For example fig (F.1)

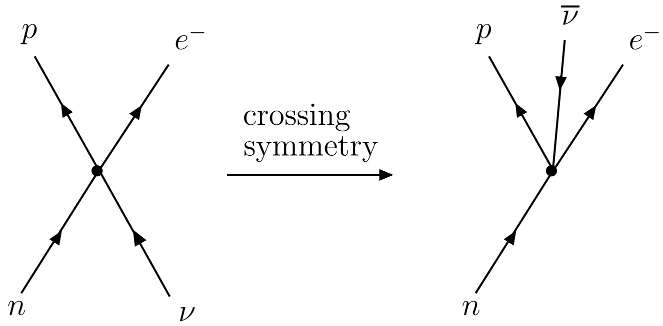


Figure F.1: .

From parity violatoin in nuclear β decay interaction was postulated to be of the form

$$H_{int} = \frac{G}{\sqrt{2}} \int d^3x [\overline{u}_p(x)\gamma_\mu(C_V + C_A)u_n(x)] \times [\overline{u}_e(x)\gamma^\mu(1-\gamma_5)u_\nu(x)] \tag{F.1}$$

where the leptonic contribution

$$\overline{u}_e(x)\gamma^\mu(1-\gamma_5)u_{\nu_e}(x) \tag{F.2}$$

resembles the electromagnetic transition current.

By analogy with the electromagnetic current, we therefore introduce the weak leptonic current:

$$\begin{aligned} J_\mu^{(L)}(x) &= J_\mu^{(e)}(x) + J_\mu^{(\mu)}(x) + J_\mu^{(\tau)}(x) \\ &= \overline{u}_e(x)\gamma_\mu(1-\gamma_5)u_{\nu_e}(x) + \overline{u}_\mu(x)\gamma_\mu(1-\gamma_5)u_{\nu_\mu}(x) \\ &\quad + \overline{u}_\tau(x)\gamma_\mu(1-\gamma_5)u_{\nu_\tau}(x) \end{aligned} \tag{F.3}$$

Motivated by (F.1)

$$H_{int} = \frac{G}{\sqrt{2}} \int d^3x J_\mu^{(L)\dagger}(x) J^\mu_{(L)}(x). \tag{F.4}$$

We consider purely leptonic processes.

F.2 Intermediate Vector Gauge Boson Theory

F.2.1 Free Massive Vector Boson

We construct wave functions which describe particles with spin 1 out of solutions of the Dirac equation and the wave equation it generates.

In the rest system we find the following linearly independent combinations

$$\begin{aligned} \omega^{(+)}_{\alpha\beta}(0, i=0) &= \delta_{\alpha 1}\delta_{\beta 1} \\ \omega^{(+)}_{\alpha\beta}(0, i=1) &= \delta_{\alpha 2}\delta_{\beta 1} + \delta_{\alpha 1}\delta_{\beta 2} \\ \omega^{(+)}_{\alpha\beta}(0, i=2) &= \delta_{\alpha 2}\delta_{\beta 2} \end{aligned} \tag{F.5}$$

Each of these spinors represents an eigenvector of the operator of total spin, which in the rest system is defined by

$$\frac{1}{2}\hbar\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'} = \frac{1}{2}\hbar\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\beta\beta'} + \frac{1}{2}\hbar\hat{\Sigma}^3_{\beta\beta'}\delta_{\alpha\alpha'} \tag{F.6}$$

where

$$\hat{\Sigma}^3_{\alpha\alpha'} = \begin{pmatrix} \hat{\sigma}_3 & 0 \\ 0 & \hat{\sigma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{F.7}$$

We verify that the multispinors $\omega^{(+)}(0,4)$ fulfill the eigenvector equation

$$\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i) = \hbar(s-i)\omega^{(+)}(0,i)$$

Obviously we have

$$\begin{aligned} \hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha' 1} &= +\delta_{\alpha 1} \\ \hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha' 2} &= -\delta_{\alpha 2} \end{aligned} \tag{F.8}$$

$$\begin{aligned} \left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=0)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\;\omega^{(+)}_{\alpha'\beta'}(0,i=0) \\ &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha' 1}\delta_{\beta 1} + \delta_{\alpha 1}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta' 1} \\ &= \frac{\hbar}{2}2(\delta_{\alpha 1}\delta_{\beta 1}) \\ &= \hbar(3-2)\omega^{(+)}_{\alpha\beta}(0,i=0) \end{aligned} \tag{F.9}$$

$$\begin{aligned} \left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=1)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\;\omega^{(+)}_{\alpha'\beta'}(0,i=1) \\ &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha' 2}\delta_{\beta 1} + \delta_{\alpha 2}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta' 1} \\ &+ \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha' 1}\delta_{\beta 2} + \delta_{\alpha 1}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta' 2} \\ &= -\frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 1} + \frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 1} \\ &+ \frac{\hbar}{2}\delta_{\alpha 1}\delta_{\beta 2} - \frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 2} = 0 \end{aligned} \tag{F.10}$$

$$
\begin{aligned}
\left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=2)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\,\omega^{(+)}_{\alpha'\beta'}(0,i=2) \\
&= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'2}\delta_{\beta 2}+\delta_{\alpha 2}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'2} \\
&= -\frac{\hbar}{2}2(\delta_{\alpha 2}\delta_{\beta 2}) \\
&= -\hbar(3-2)\omega^{(+)}_{\alpha\beta}(0,i=2) \qquad \text{(F.11)}
\end{aligned}
$$

Now we can transform these multispinors into an arbitrary frame of reference via the inverse Lorentz transform.

$$
\hat{S}_{\alpha\alpha'\beta\beta'}\left(\frac{\mathbf{p}}{E}\right)=\hat{S}_{\alpha\alpha'}\left(\frac{\mathbf{p}}{E}\right)\hat{S}_{\beta\beta'}\left(\frac{\mathbf{p}}{E}\right) \qquad \text{(F.12)}
$$

Applying this we get

$$
\omega^{(+)}(x;p,i)=\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^{(+)}(0,i), \qquad \omega^{(-)}(x;p,i)=\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^{(-)}(0,i) \qquad \text{(F.13)}
$$

Now every wave function can be written as a superposition of plane waves:

$$
\begin{aligned}
\Psi^{(+)}_{\alpha\beta}(x;p,i) &= \omega^{(+)}_{\alpha\beta}(x;p,i)e^{-ip\cdot x/\hbar} \\
\Psi^{(-)}_{\alpha\beta}(x;p,i) &= \omega^{(-)}_{\alpha\beta}(x;p,i)e^{+ip\cdot x/\hbar} \qquad \text{(F.14)}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\Psi_{\alpha\beta}(x) &= \sum_i\int c^{(+)}(p,i)\Psi^{(+)}_{\alpha\beta}(x;p,i)d^3p \\
&+ \sum_i\int c^{(-)}(p,i)\Psi^{(-)}_{\alpha\beta}(x;p,i)d^3p. \qquad \text{(F.15)}
\end{aligned}
$$

It is easily seen that the plane wave satify Diracs equation. We consider one particular example

$$
(i\hbar\gamma\cdot\partial-m_0c)(\omega^{(+)}(x;p,i=1)e^{-ip\cdot x/\hbar}).
$$

first recall that

$$(i\hbar\gamma\cdot\partial - m_0 c)\left[\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^1(0)e^{-ip\cdot x/\hbar}\right] = 0$$

where

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now

$$\begin{aligned}
& (i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}(\omega^{(+)}_{\alpha'\beta}(x;p,i=1)e^{-ip\cdot x/\hbar}) \\
= \;& (i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\left(\hat{S}_{\alpha'\gamma}\left(\frac{\mathbf{p}}{E}\right)\hat{S}_{\beta\delta}\left(\frac{\mathbf{p}}{E}\right)\omega^{(+)}_{\gamma\delta}(0,i=1)e^{-ip\cdot x/\hbar}\right) \\
= \;& (i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\left(\hat{S}_{\alpha'1}\left(\frac{\mathbf{p}}{E}\right)e^{-ip\cdot x/\hbar}\right)\hat{S}_{\beta 1}\left(\frac{\mathbf{p}}{E}\right) \\
= \;& \underbrace{\left[(i\hbar\gamma\cdot\partial - m_0 c)\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^1(0)e^{-ip\cdot x/\hbar}\right]_\alpha}_{=0} \times \hat{S}_{\beta 1}\left(\frac{\mathbf{p}}{E}\right)
\end{aligned} \tag{F.16}$$

Let us apply the Dirac equation to this first index α

$$\begin{aligned}
(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi_{\alpha'\beta}(x) &= \sum_i \int c^{(+)}(p,i)(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi^{(+)}_{\alpha'\beta}(x;p,i)d^3p \\
&+ \sum_i \int c^{(-)}(p,i)(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi^{(-)}_{\alpha'\beta}(x;p,i)d^3p
\end{aligned} \tag{F.17}$$

The multispinors therefore fulfill the following Dirac equations separately:

$$\begin{aligned}
(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi_{\alpha'\beta}(x) &= 0 \\
(i\hbar\gamma\cdot\partial - m_0 c)_{\beta\beta'}\Psi_{\alpha\beta'}(x) &= 0
\end{aligned} \tag{F.18}$$

These are the so-called Bargmann-Wigner equations. Each component is also a solution of the Klein-Gordon equation

$$\left(\Box + \frac{m_0^2 c^2}{\hbar^2}\right) \Psi_{\alpha\beta}(x) = 0. \tag{F.19}$$

The two Dirac equations for the symmetric matrix $\Psi_{\alpha\beta}(x)$ may be written as

Since the 4×4 spinor is symmetric, it may be expanded in terms of a complete set of symmetric elements of the Clifford algebra representation

$$\gamma^\mu \hat{C}, \quad \hat{\sigma}^{\mu\nu} \hat{C} \tag{F.20}$$

We define

$$\Psi(x) = m_0 \gamma_\mu \hat{C} W^\mu(x) + \frac{1}{2} \hat{\sigma}^{\mu\nu} \hat{C} G_{\mu\nu}(x) \tag{F.21}$$

where thye coefficients $W^\mu(x)$ and $G^{\mu\nu}(x)$ are generally complex and transform under Loretz transformations like a vector and an anti-symmetric tensor, respectively. The Bargmann-Winger equations now become

$$\begin{aligned}
(i\hbar \cdot \partial\gamma - m_0 c) \left(\frac{1}{\hbar} m_0 c \gamma_\mu W^\mu(x) + \frac{1}{2} \hat{\sigma}_{\mu\nu} G^{\mu\nu}(x)\right) \hat{C} &= 0 \\
\left(\frac{1}{\hbar} m_0 c \gamma_\mu W^\mu(x) + \frac{1}{2} \hat{\sigma}_{\mu\nu} G^{\mu\nu}(x)\right) \hat{C} (i\hbar \gamma^T \cdot \overleftarrow{\partial} - m_0 c) &= 0
\end{aligned} \tag{F.22}$$

Using that $\hat{C} \gamma_\mu^T = -\gamma_\mu \hat{C}$

$$\begin{aligned}
&\left(i m_0 c \partial_\alpha W_\mu(x) \gamma^\alpha \gamma^\mu - m_0^2 c^2 \gamma_\mu W^\mu(x) + \frac{1}{2} i\hbar \gamma_\alpha \hat{\sigma}^{\mu\nu} \partial^\alpha G_{\mu\nu}(x)\right. \\
&\quad \left. - \frac{1}{2} m_0 c \hat{\sigma}^{\mu\nu} G_{\mu\nu}(x)\right) \hat{C} = 0,
\end{aligned} \tag{F.23}$$

$$\begin{aligned}
&\left(i m_0 c \partial_\alpha W_\mu(x) \gamma^\mu \gamma^\alpha - m_0^2 c^2 \gamma_\mu W^\mu(x) + \frac{1}{2} i\hbar \gamma_\alpha \partial^\alpha \hat{\sigma}^{\mu\nu} G_{\mu\nu}(x)\right. \\
&\quad \left. + \frac{1}{2} m_0 c \hat{\sigma}^{\mu\nu} G_{\mu\nu}(x)\right) \hat{C} = 0,
\end{aligned} \tag{F.24}$$

Subtracting (F.24) from (F.23) gives

$$
\begin{aligned}
& i m_0 \partial_\alpha W_\mu(x)\{\gamma^\alpha\gamma^\mu - \gamma^\mu\gamma^\alpha\}\hat{C} - \frac{2m_0^2c^2}{\hbar}\gamma_\mu \hat{C} W_\mu(x) \\
& + \frac{1}{2} i\hbar\{\gamma_\alpha\hat{\sigma}^{\mu\nu} - \hat{\sigma}^{\mu\nu}\gamma_\alpha\}\partial_\alpha G_{\mu\nu}(x)\hat{C} - m_0 c\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x)\hat{C} = 0.
\end{aligned}
\tag{F.25}
$$

Using

$$
\begin{aligned}
i m_0 \partial_\alpha W_\mu(x)\{\gamma^\alpha\gamma^\mu - \gamma^\mu\gamma^\alpha\}\hat{C} &= 2m_0\partial_\alpha W_\mu(x)\hat{\sigma}^{\alpha\mu}\hat{C} \\
&= m_0(\partial^\alpha W^\mu(x) - \partial^\mu W^\alpha(x))\hat{\sigma}_{\alpha\mu}\hat{C}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2} i\hbar\{\gamma_\alpha\hat{\sigma}^{\mu\nu} - \hat{\sigma}^{\mu\nu}\gamma^\alpha\}\partial^\alpha G_{\mu\nu}(x)\hat{C} &= \frac{1}{2} i\hbar 2i(\eta^{\alpha\mu}\gamma^\nu - \eta^{\alpha\nu}\gamma^\mu)\hat{C}\partial_\alpha G_{\mu\nu}(x) \\
&= -2\hbar\gamma_\mu\hat{C}\partial_\alpha G^{\alpha\mu}(x)
\end{aligned}
$$

It follows that

$$
m_0 c(\partial^\alpha W^\mu(x) - \partial^\mu W^\alpha(x) - G^{\alpha\mu})\hat{\sigma}_{\alpha\mu}\hat{C} - 2\gamma_\mu\hat{C}\left(\hbar\partial_\alpha G^{\alpha\mu} + \frac{m_0^2c^2}{\hbar}W^\mu\right) = 0 \tag{F.26}
$$

The coefficients of the linearly independent matrices $\hat{C}$ must vanish separately. Hence, for $m_0 \neq 0$, this implies that

$$
G^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu, \tag{F.27}
$$

$$
\partial_\mu G^{\mu\nu} = -\frac{m_0^2c^2}{\hbar^2}W^\nu. \tag{F.28}
$$

Expressed in terms of the vector field W^μ

$$
\Box W^\mu(x) - \partial^\mu(\partial_\nu W^\nu(x)) + \frac{m_0^2c^2}{\hbar^2}\, W^\mu(x) = 0 \tag{F.29}
$$

On taking the divergence of this equation, one automatically obtains the Lorentz condition

$$\frac{m_0^2 c^2}{\hbar^2}\, \partial_\mu W^\mu(x) = 0 \tag{F.30}$$

This is in contrast to the photon case, where the Lorentz condition must be imposed as a subsidiary condition. The Proca equation then reduces to

$$\Box W^\mu(x) + \frac{m_0^2 c^2}{\hbar^2}\, W^\mu(x) = 0 \tag{F.31}$$

The Propagator

$$\eta^{\nu\lambda}\Box W_\lambda(x) - \partial^\nu(\partial^\lambda W_\lambda) + \frac{m_0^2 c^2}{\hbar^2}\eta^{\nu\lambda} W_\lambda(x) = J^\nu \tag{F.32}$$

In momentum space the left hand side reads

$$[\eta^{\nu\lambda}(-q^2 + M^2) + q^\nu q^\lambda] W_\lambda \tag{F.33}$$

The inverse operator $D_{\lambda\mu}(q)$ will have the structure

$$A(q^2)\eta_{\lambda\mu} + B(q^2) q_\lambda q_\mu \tag{F.34}$$

because there are only two second-rank tensors that can be formed.

$$\begin{aligned}
& [\eta^{\nu\lambda}(-q^2 + M^2) + q^\nu q^\lambda]\ [A(q^2)\eta_{\lambda\mu} + B(q^2) q_\lambda q_\mu] \\
= \ & A(q^2)\ [(-q^2 + M^2)\delta^\nu_\mu + q^\nu q_\mu] + B(q^2)\ [q^\nu q_\mu(-q^2 + M^2) + q^2 q^\nu q_\mu] \\
= \ & \delta^\nu_\mu .
\end{aligned}$$

Thus we get

$$A(q^2)\ (-q^2 + M^2) = 1$$

for $\mu = \nu$, and

$$A(q^2)\ [q^\nu q^\lambda] + B(q^2)\ [q^\nu q_\mu(-q^2 + M^2) + q^2 q^\nu q_\mu] = 0. \tag{F.35}$$

for $\mu \neq \nu$. Hence

$$A(q^2) = -\frac{1}{-q^2 + M^2} \tag{F.36}$$

and

$$B(q^2) = -\frac{1/M^2}{-q^2 + M^2} \tag{F.37}$$

The propagator

$$\begin{aligned} D^{\lambda\mu}(q) &= A(q^2)\eta^{\lambda\mu} + B(q^2)q^\lambda q^\mu \\ &= -\frac{\eta^{\lambda\mu}}{q^2 - M^2} + \frac{q^\lambda q^\mu/M^2}{q^2 - M^2} \\ &= \frac{-\eta^{\lambda\mu} + q^\lambda q^\mu/M^2}{q^2 - M^2} \end{aligned} \tag{F.38}$$

F.2.2 Interactions via Massive Bosons

vector meson exchange

$$\mathcal{L} = g_W^2(\overline{\Psi}_p\gamma^\mu\Psi_n)\frac{\eta_{\mu\nu} - q_\mu q_\nu/M_W^2}{q^2 - M_W^2 + i\epsilon}(\overline{\psi}_e\gamma^\nu\psi_\nu) \tag{F.39}$$

F.3 Lagrangian for Yang-Mills Theory

Of particular interest here are so called Yang-Mills theories which are a special example of gauge theory with a non-abelian group symmetry group. A nice introduction to the Yang-Mills field, with calculational details can be found in the book by Greiner (W. Greiner and B. Muler, "Gauge Theory of Weak Interactions", Springer-Verlag). For non-Abelian gauge theory a gauge transformation is given by

$$A_\mu^U \cdot \hat{T} = \hat{U}(A_\mu \cdot \hat{T})\hat{U}^{-1} + \frac{i}{g}\hat{U}\left(\partial_\mu\hat{U}^{-1}\right) \tag{F.40}$$

where

$$\hat{U} = \exp\left(i\theta^a(x)T^a\right). \tag{F.41}$$

with the generators of the Lie algebra corresponding to

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc}T^c \tag{F.42}$$

where the T^a are matrices, and as such they in general do not commute with each other. The curvature or field-strength tensor $\hat{F}_{\mu\nu} \equiv \sum_a F^a_{\mu\nu}T^a$. The index a is sometimes referred to a "colour" index. The Field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{F.43}$$

The Lagrangian is given by

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{Tr}(\hat{F}^2) = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} \tag{F.44}$$

The gauge field is called non-Abelian as we have

$$[A^a_\mu T^a, A^b_\nu T^b] = A^a_\mu A^b_\nu [T^a, T^b] = iA^a_\mu A^b_\nu f^{abc}T^c. \tag{F.45}$$

In fact, this is the origin of the quadratic term in the field strength tensor. This term results in the gauge bosons of the theory to interact with themselves, complicating the theory.

It can be shown that under a gauge transformation that the Field strength tensor transforms as:

$$\hat{T}\cdot\mathbf{F}'_{\mu\nu} = \hat{U}(\hat{T}\cdot\mathbf{F}_{\mu\nu})\hat{U}^{-1}, \tag{F.46}$$

This means that the Field strength tensor is not gauge invariant, however the quantity $Tr(\hat{F}^{\mu\nu}\cdot\hat{F}_{\mu\nu})$ is gauge invariant due to the cyclic property of the trace operation.

A method of quantizing the Yang-Mills theory is by functional methods, i.e. Path integral formulation. One introduces a generating functional for "n"-point functions as

$$Z[J] = \int \mathcal{D}[A]\exp\left[-\frac{i}{2}\int d^4x\,\mathrm{Tr}(\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}) + i\int d^4x\,J^a_\mu(x)A^{a\mu}(x)\right], \tag{F.47}$$

Quantum electrodynamics is the most famous example an Abelian Group—Abelian gauge theory. It relies on the symmetry group $U(1)$ and has one massless gauge field, the $U(1)$ gauge symmetry, dictating the form of the interactions involving the electromagnetic field, with the photon being the gauge boson. In this case $\hat{U} = \exp\{i\theta(x)\}$. The gauge field transforms as

$$A_\mu^U(x) = A_\mu(x) + \frac{1}{g}\partial_\mu\theta(x). \tag{F.48}$$

In QED there is no "colour" index, and no matrices, and the gauge group is $U(1)$ which corresponds to complex "numbers" of modulus (or Absolute value) 1, and hence the Abelian nature of QED:

$$[A_\mu, A_\nu] = A_\mu(x)A_\nu(x) - A_\nu(x)A_\mu(x) = 0. \tag{F.49}$$

Here the field strength tensor is simply given by,

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{F.50}$$

Note the absence of a term quadratic in the gauge fields. Also, note that here the field strength tensor is trivially gauge invariant.

Under the composition of two non-abelian gauge transformations we have

$$A_\mu^{U'U} \cdot \hat{T} = \hat{U}''(A_\mu \cdot \hat{T})\hat{U''}^{-1} + \frac{i}{g}\hat{U}''\left(\partial_\mu \hat{U''}^{-1}\right) \tag{F.51}$$

where

$$U'' = UU' \tag{F.52}$$

The infinitesimal form of the gauge transformation is

$$A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\theta^a + f^{abc}A^b_\mu\theta^c. \tag{F.53}$$

F.4 Fadeev-Popov Gauge Fixing

For many years the naive quantization of a Yang-Mills theory (Yang-Mills theories now a cornerstone of particle physics) was flawed, and it remained unclear how to resolve the problem. It was Ludvig Faddeev and Victor Popov who developed the correct procedure for properly defining the functional integral quantization, now known as Faddeev-Popov gauge fixing.

The Standard Model of particle physics is a theory concerning the electromagnetic, weak, and strong nuclear interactions, as well as classifying all the matter particles known, i.e. scalar fields (in particular the charged Higgs field of the Standard Model) and spinor fields describing fermionic matter, the most familiar examples of fermionic matter being electrons and up quark and down quarks (the up and down quarks are from which which protons and neutrons are comprised).

The calculation of probability amplitudes in theoretical particle physics as calculated as a perturbative series requires the use of rather large and complicated integrals over a large number of variables. These integrals do, however, have a regular structure, and may be represented graphically as Feynman diagrams. The Feynman diagrams are drawn according to the Feynman rules, which depend upon the interaction Lagrangian.

It is possible to read off the Feynman rules of quantum field theory from a functional integral - in particularly from the Lagrangian. Here we are concerned with gauge fields, which mediate the forces between material particles, the simplest example of which would be the electromagnetic gauge potential. It turns out that, because of our freedom to make gauge transformations, one encounters difficulties not present with the description of matter fields (this problem turns out to be trivial in the very first case of a quantum field theory, i.e. quantum electrodynamics, involving the electromagnetic gauge potential, which is known as an Abelian gauge field. Here we will show how to properly apply the functional integral method to gauge fields, a method essential to non-abelian gauge theory also known as Yang-Mills theory which is used in the description of the unification of the electromagnetic and weak forces (electroweak theory) as well as the strong force (quantum chromodynamics) which together form the interactions of the Standard Model of particle physics. The method is known as Faddeev-Popov gauge fixing.

Historically, for many years, however before the Faddeev-Popov method, the quantization of Yang-Mills theory was not clear for some time. In 1965 Feynman (Feynman, R. P. 1963, *Acta Physica Polonica* **24**, *697.*) showed that the naive quantization of the theory was not unitary (implying non conservation of probability). In order to to cancel the nonunitary terms from the theory, Feynman was lead to postulate the existence of a term that did not emerge from the naive quantization. The procedure for properly defining the functional integral was developed by Faddeev and Popov (Faddeev, L. D., and Popov, V. N., 1967, *Phys. Lett.* **25B** 29; Se also : Mandelstam, S., 1962, *Ann. Phys.* **19**, 1.), from which Feynman's postulated term naturally emerges, now know as the Faddeev-Popov ghost.

F.4.1 Gauge Fixing: Analogy in a simple context

The Faddeev-Popov procedure is just a change of coordinates, but since it is for a functional integral, it may at first sight seem unfamiliar. Therefore we begin with a simple example. Consider the integral

$$Z = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-(x-y)^2} \tag{F.54}$$

The integral is invariant of the transformation

$$x \quad \mapsto \quad x + a \tag{F.55}$$
$$y \quad \mapsto \quad y + a \tag{F.56}$$

This represents the gauge symmetry of this toy model. The action $-(x-y)^2$ is gauge invariant, with a an element of the gauge group.

To impose a gauge constraint,

$$g(x, y) = 0 \tag{F.57}$$

we cant just insert the delta function $\delta(g = 0)$. For example, if we have a function $g(x)$ that has a zero at $x = c$, we have that:

$$\delta(g(x)) = \frac{\delta(x - C)}{|g'(x)|}. \tag{F.58}$$

Instead, let us consider the inclusion of the integral

$$1 = \int da \; \delta(g(a)) \, \frac{dg}{da} \tag{F.59}$$

into the expression.

This expression generalizes to

$$1 = \left(\prod_{i}^{n} \int da_{i} \right) \delta^{n}(g(a)) \; det \left(\frac{\partial g_{i}}{\partial a_{j}} \right) \tag{F.60}$$

where $\frac{\partial g_i}{\partial a_j}$ is an $n-$dimensional matrix M

$$
\begin{aligned}
\Delta_g^{-1}(x,y) &= \int da\, \delta\left(g(x(a),y(a))\right) \\
&= \int dg\ \det\left|\frac{da}{dg}\right| \delta(g) \\
&= \det\left|\frac{da}{dg}\right|_{g=0}
\end{aligned}
\tag{F.61}
$$

First we prove that $\Delta_g(x,y)$ is gauge invariant. We do this by proving $\Delta_g(x(a'),y(a')) = \Delta_g(x,y)$ for constant a',

$$
\begin{aligned}
\Delta_g^{-1}(x(a'),y(a')) &= \int da\, \delta\left(g(x+a'+a, y+a'+a)\right) \\
&= \int d(a+a')\ \delta\left(g(x+a+a', y+a+a')\right) \\
&= \int da''\ \delta\left(g(x(a''),y(a''))\right) \\
&= \Delta_g^{-1}(x,y)
\end{aligned}
\tag{F.62}
$$

where we have used $da = da''$

We insert

$$
1 = \Delta_g(x,y) \int da\ \delta(g(x(a),y(a)) = 0) \tag{F.63}
$$

into Z, +

$$
Z = \int dxdy\ \left\{\Delta_g(x,y) \int da\ \delta(g(a) = 0)\right\} e^{-(x-y)^2} \tag{F.64}
$$

One then performs a gauge transformation taking $x(a)$ and $y(a)$ to $x = x(0)$ and $y = y(0)$ respectively, and using $dx(a) = dx, dy(a) = dy$, the action is gauge invariant, and so is $\Delta_g(x,y)$ we get

$$
Z = \int dxdy\ \left\{\Delta_g(x,y) \int da\ \delta(g(x(a),y(a)) = 0)\right\} e^{-(x-y)^2} \tag{F.65}
$$

$$
= \left(\int da\right) \int dxdy\ \Delta_g(x,y)\ \delta(g(x,y) = 0) e^{-(x-y)^2} \tag{F.66}
$$

where we have been able to take out the factor $\int da$ since the integrand is independent of a.

$$\Delta_g = \left|\frac{\partial g}{\partial a}\right|_{g=0} = M \tag{F.67}$$

where M is a constant.

We now have to deal with the $\delta(g = 0)$. Note that the result is independent of the choice of slice, so we could just as well have chosen $g = c$, where c is a real constant.

Here we would insert

$$1 = \left|\frac{\partial g}{\partial a}\right|_{g=c} \int da\ \delta(g(x(a), y(a)) - c = 0) \tag{F.68}$$

and obtain:

$$Z = \int da \int dxdy\ \left\{\delta(g(x,y) - c) \left|\frac{dg}{da}\right|_{g=c}\right\} e^{-(x-y)^2} \tag{F.69}$$

We take advantage of the fact that the integrand is independent of the c and integrate over this parameter with convenient weight. We take this weight to be the normalized Gaussian

$$1 = \frac{1}{\sqrt{\pi}} \int dc\ \exp(-c^2). \tag{F.70}$$

We obtain for Z,

$$\begin{aligned}
Z &= \int dxdy \int dc\ \left\{\delta(g - c = 0) \left|\frac{dg}{da}\right|_{g=c}\right\} \frac{e^{-c^2}}{\sqrt{\pi}} e^{-(x-y)^2} && \text{(F.71)}\\
&= \int dxdy\ \left\{\int dc\ \delta(g - c = 0) \frac{e^{-c^2}}{\sqrt{\pi}}\right\} \left|\frac{dg}{da}\right|_{a=0} e^{-(x-y)^2} && \text{(F.72)}\\
&= \int dxdy\ \frac{1}{\sqrt{\pi}} \left|\frac{dg}{da}\right|_{a=0} \exp\left(-(x-y)^2 + g^2\right) && \text{(F.73)}
\end{aligned}$$

where g^2 is the gauge fixing term.

F.4.2 Fadeev-Popov Gauge Fixing

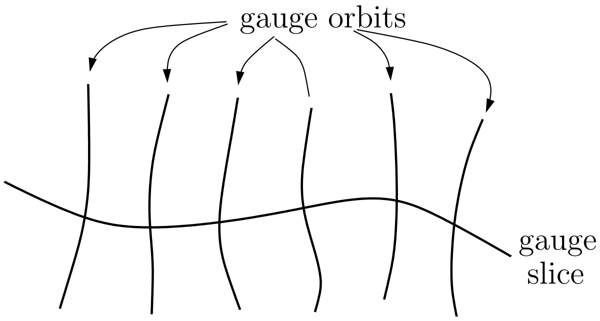


Figure F.2: Schematic representation of the vector potential gauge field configuration space.

The line running upward represents the gauge orbit of the vector potential A_μ^Λ as the gauge transformation function Λ varies. By gauge invariance, all point along these lines are physically equivalent. Functionally integrating over all configurations overcounts the integrand repeatedly an infinite number of times giving us the non-sensical result

$$\int \mathcal{D}A_\mu \exp(iS[A]) = \infty. \tag{F.74}$$

This surface, the gauge slice, is the surface in function space that is described a gauge-fixing constraint. One might be tempted to solve the problem by simply inserting a gauge fixing factor

$$\delta\left(F(A)\right).$$

into the functional integral, forcing it to respect the gauge choice. However, this is inconsistent. We know that the delta function changes when we make changes in it. For example, if we have a function $f(x)$ that has a zero at $x = a$, we recall that:

$$\delta(f(x)) = \frac{\delta(x-a)}{|f'(x)|}. \tag{F.75}$$

Consider the integral

$$1 = \int da\, \delta(g(a))\, \frac{dg}{da}. \tag{F.76}$$

$Z[J]$, in this form, is not particularly easy to calculate into the functional integral

$$1 = \int \mathcal{D}U(x)\ \delta(G(A^U))\ det\left(\frac{\delta G(A^U)}{\delta U}\right). \tag{F.77}$$

This is the continuum generalisation of

$$1 = \left(\prod_i \int da_i\right) \delta^n(g(a))\ det\left(\frac{\partial g_i}{\partial a_j}\right) \tag{F.78}$$

Define

$$\Delta_{FP}^{-1}(A_\mu) := \int \mathcal{D}U\ \delta\left[G(A_\mu^U)\right] = \left(det\left|\frac{\delta G(A^U)}{\delta U}\right|_{G=0}\right)^{-1} \tag{F.79}$$

This is the Fadeev-Popov determinant. First we prove that it is gauge independent. This is done by proving that $\Delta_{FP}^{-1}(A_\mu) = \Delta_{FP}^{-1}(A_\mu^{U'})$ where $A_\mu^{U'}$ is obtained from A_μ via a gauge transformation corresponding to U':

$$\begin{aligned}
\Delta_{FP}^{-1}(A_\mu^{U'}) &= \int \mathcal{D}U\ \delta\left[G(A_\mu^{U'U})\right] \\
&= \int \mathcal{D}[U'U]\ \delta\left[G(A_\mu^{U'U})\right] \\
&= \int \mathcal{D}[U'']\ \delta\left[G(A_\mu^{U''})\right] \\
&= \Delta_{FP}^{-1}(A_\mu),
\end{aligned} \tag{F.80}$$

where we have used that (for compact groups) the volume element in group space defines and invariant measure (See, for example: Hamermesh, M. (1962), "Group Theory and its Application to Physical Problems", Addison-Wesley or Talman, J.D. (1968), "Special functions: A group Theoretical Approach', Benjamin):

$$\mathcal{D}U = \mathcal{D}U''. \tag{F.81}$$

We have

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}U(x)\ \delta(G(A^U)) \tag{F.82}$$

Following Faddeev and Popov we insert this identity into the functional integral $Z[J]$,

$$Z[J] = \int \mathcal{D}A \, \left\{ \Delta_{FP}(A_\mu) \int \mathcal{D}U(x) \; \delta(G(A^U)) \right\} \exp(iS[A]) \tag{F.83}$$

One then performs a gauge transformation taking A_μ^Λ to A_μ and using $\mathcal{D}A^\Lambda = \mathcal{D}A$, the action S is gauge invariant, and so is $\Delta_{FP}(A)$ we get

$$\begin{aligned} Z &= \int \mathcal{D}A \, \left\{ \Delta_{FP}(A_\mu) \int \mathcal{D}U(x) \; \delta(G(A)) \right\} \exp(iS[A]) \\ &= \left(\int \mathcal{D}U(x) \right) \int \mathcal{D}A \; \Delta_{FP}(A_\mu) \delta(G(A)) \exp(iS[A]) \end{aligned} \tag{F.84}$$

where we have been able to take out the factor $\mathcal{D}U(x)$ since the integrand is independent of U. Therefore, summation over gauge equivalent configurations has been factored out so that the divergent integral $\int \mathcal{D}U(x)$ gives a simply multiplicative factor. The correct expression to use is therefore:

$$Z = \int \mathcal{D}A \; \Delta_{FP}(A_\mu) \delta(G(A)) \exp(iS[A]) \tag{F.85}$$

The essential point is that the factor Δ_{FP} gives the correct measure in the functional integral, the factor needed for so many years to cure previous attempts to quantize gauge theories.

and so in (**??**) summation over gauge equivalent configurations has been factored put so that the divergent integral $\int \mathcal{D}U(x)$ gives a simply multiplicative factor.

The essential point is that the factor Δ_{FP} gives the correct measure in the functional integral, the factor needed for so many years to cure previous attempts to quantise gauge theories.

We have deal with the delta function. Note that $Z[J]$ as originally defined, is completely independent of the arbitrary function $\chi(x)$. It is convenient computationally to add together the contribution from all slices labeled by $\chi(x)$, each slice weighted with a gaussian function centred on $\chi = 0$.

F.4.3 Faddeev-Popov ghost fields

We now need an expression for

$$\Delta_{FP} = \det \left| \frac{\delta G(A^\Lambda)}{\delta \Lambda} \right|_{G=0} = \det M \tag{F.86}$$

Following Faddeev-Popov we convert the determinant of the matrix M into a Gaussian integral over two fields η and $\eta^\dagger$;

$$\Delta_{FP} = \det M = \int \mathcal{D}\overline{\eta}\mathcal{D}\eta \exp\left(\int d^4x d^4y \overline{\eta}^a(x) M_{ab}(x,y)\eta^b(y)\right) \tag{F.87}$$

The determinant appears in the numerator of the functional integral, rather than the denominator, which means we must integrate over Grassmann variables, rather than bosonic variables. The fields η and $\overline{\eta}$ are called"ghost fields", that is, scalar fields obeying Fermi-Dirac statistics. This is the origin of the Faddeev-Popov ghosts.

The Faddeev-Popov ghosts violate the *spin-statistics theorem*, and they are regarded as "non-physical" and only occur as *virtual particles* in Feynman diagrams.

We have deal with the delta function. We can modify the the gauge fixing term $\delta(G(A))$ by making the exchange:

$$G^a \mapsto G^a - \chi^a(x) \tag{F.88}$$

where $\chi^a(x)$ is an arbitrary function. Note that $Z[J]$ as originally defined, is completely independent of the arbitrary function $\chi(x)$. It is convenient computationally to multiply Z by an overall factor

$$\exp\left(-\frac{i}{2\xi}\int \chi^2 dx\right) \tag{F.89}$$

and then integrating over χ,

$$\begin{aligned}
Z &= \int \mathcal{D}\chi \int \mathcal{D}A\ \Delta_{FP}(A_\mu)\delta(G(A)-\chi)\exp\left(-\frac{i}{2\xi}\int \chi^2 dx\right)\exp(iS[A]) \\
&= \int \mathcal{D}A\ \Delta_{FP}(A_\mu)\left\{\int \mathcal{D}\chi\ \delta(G(A)-\chi)\exp\left(-\frac{i}{2\xi}\int \chi^2 dx\right)\right\}\exp(iS[A]) \\
&= \int \mathcal{D}A \int \mathcal{D}\overline{\eta}\mathcal{D}\eta\ \exp\left(iS_{eff}[A]\right) \qquad \text{(F.90)}
\end{aligned}$$

where

$$S_{eff} = \int\left(-\frac{1}{2}Tr(F^{\mu\nu}F_{\mu\nu}) - \frac{1}{2\xi}G^2\right)dx + \int \overline{\eta}^a(x)M_{ab}(x,y)\eta^b(y)dxdy \tag{F.91}$$

As we obtain $M_{ab}(x,y)$ by taking a functional derivative, it will get put in the form: $\delta(x-y)\tilde{M}_{ab}$ in the integral. Therefore, we can write

$$Z = \int \mathcal{D}A \int \mathcal{D}\overline{\eta}\mathcal{D}\eta \, \exp\left(i \int \mathcal{L}_{eff}[A] dx \right) \tag{F.92}$$

where this effective Lagrangian, $\mathcal{L}_{eff}$ is given by

$$\begin{aligned} \mathcal{L}_{eff} &= \mathcal{L}[A] - \tfrac{1}{2\xi} G^2[A] + \mathcal{L}_{FPG}[\partial\overline{\eta}, \partial\eta, \overline{\eta}, \eta; A] & \text{(F.93)} \\ &= \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FPG} & \text{(F.94)} \end{aligned}$$

where $\mathcal{L}_{GF}$ is the gauge fixing term and $\mathcal{L}_{FPG}$ is the Faddeev-Popov ghost term. The naive quantization missed off the Faddeev-Popov ghost term.

The gauge fixing and this ghost Lagrangian modify the original theory in a compensating manner which allows one to define Feynman rules of the theory and carry out any perturbative calculation.

F.4.4 The R_ξ gauge

The exact form or formulation of ghosts is dependent on the particular gauge chosen, although the same physical results are obtained with all the gauges. The Gauge fixing R_ξ gauge is usually the simplest gauge for this purpose. It is a generalization of the Lorentz gauge, and obtained by putting

$$G^a = \partial^\mu A^a_\mu . \tag{F.95}$$

In evaluating the determinant, we start from the configuration satisfying the gauge constraint, and then perform an infinitesimal gauge transformation, given by

$$(A^\theta)^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \theta^a + f^{abc} A^b_\mu \theta^c . \tag{F.96}$$

then

$$\begin{aligned} G(A^\theta_\mu) - \chi^a(x) &= \partial^\mu A^a_\mu + \frac{1}{g}\left(\partial^\mu \partial_\mu \theta^a + g f^{abc} \partial^\mu (A^b_\mu \theta^c) \right) - \chi^a(x) \\ &= \frac{1}{g}\left(\partial^\mu \partial_\mu \theta^a + g f^{abc} \partial^\mu (A^b_\mu \theta^c) \right) & \text{(F.97)} \end{aligned}$$

and

$$\left| \frac{\delta G(A_\mu^\theta)}{\delta \theta^a} \right|_{\theta=0} = \frac{1}{g} \left(\delta^{ac} \partial^\mu \partial_\mu + g f^{abc} \partial^\mu A_\mu^b \right) \delta^4(x-y) \tag{F.98}$$

We convert this into a Grassmann integration

$$det\tilde{M} = \int \mathcal{D}\overline{\eta}\mathcal{D}\eta \, \exp\left(i \int -\overline{\eta}^a (\delta^{ac}\partial^\mu \partial_\mu + g f^{abc} \partial^\mu A_\mu^b) \eta^c \, dx \right) \tag{F.99}$$

(The factor of $1/g$ is absorbed into the normalization of the fields η and $\overline{\eta}$.) The effective Lagrangian, $\mathcal{L}_{eff}$, is then mde up of the parts:

$$\mathcal{L} = -\frac{1}{2} \operatorname{Tr}(F^{\mu\nu} F_{\mu\nu}) \tag{F.100}$$

for the field,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial \cdot A)^2 \tag{F.101}$$

for the gauge fixing and

$$\mathcal{L}_{FPG} = -(\bar{\eta}^a \partial_\mu \partial^\mu \eta^a + g \bar{\eta}^a f^{abc} \partial_\mu A^{b\mu} \eta^c) \tag{F.102}$$

for the ghost.

We obtain the quantization of Yang-Mills theory given in section.

F.4.5 Decoupling of ghost fields in QED

In particle physics, quantum electrodynamics (QED) is the relativistic quantum field theory of electrodynamics. In essence, it describes how the electromagnetic field and charged matter (electrons and protons as well as their anti-particles) interact and is the first theory where full agreement between quantum mechanics and special relativity is achieved. It is the simplest example of a gauge theory.

To obtain some familiarity with the Faddeev-Popov term, let us commute the Faddeev-Popov determinant for this simplest of gauge theory, Maxwell's theory. As it is an Abelian gauge theory we are lucky and the Faddeev-Popov ghost fields completely decouple from the theory and are not required; $\theta(x)$ has no "colour" index and the structure constants are zero. The infinitesimal form of the transformation is

$$(A^\theta)_\mu = A_\mu + \frac{1}{g}\partial_\mu \theta. \tag{F.103}$$

Then choosing

$$G = \partial^\mu A_\mu \tag{F.104}$$

we have

$$\begin{aligned} G(A^\theta) - \chi &= \partial^\mu A_\mu + \frac{1}{g}\partial^\mu \partial_\mu \theta - \chi \\ &= \frac{1}{g}\partial^\mu \partial_\mu \theta \end{aligned} \tag{F.105}$$

So in QED the determinant

$$det \left| \frac{\delta G(A^\theta)}{\delta \theta} \right|_{\theta=0} = \frac{1}{g}\partial^\mu \partial_\mu \delta^4(x-y) \tag{F.106}$$

is independent of A_μ and therefore just contributes to the overall normalization constant:

$$det\tilde{M} = \int \mathcal{D}\overline{\eta}\mathcal{D}\eta \ \exp\left(i\int -\overline{\eta}(\partial^\mu \partial_\mu)\eta \ dx\right). \tag{F.107}$$

F.5 Example: QED

F.5.1 Photon propagator

We have

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \\
&= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \\
&= -\frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \\
&= \frac{1}{2}[A^\nu \partial^\rho \partial_\rho A_\nu - \partial^\mu (A^\nu \partial_\mu A_\nu)] - \frac{1}{2}[A^\mu \partial_\mu \partial_\nu A^\nu - \partial_\mu (A^\nu \partial_\nu A^\mu)] \\
&\quad + \frac{1}{2\xi}[A^\mu \partial_\mu \partial_\nu A^\nu - \partial_\mu (A^\mu \partial_\nu A^\nu)] \\
&= \frac{1}{2}A^\mu \left[\Box \eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right] A^\nu - \frac{1}{2}\partial^\mu \left[A^\nu \partial_\mu A_\nu - A^\nu \partial_\nu A_\mu + \frac{1}{\xi}A_\mu \partial_\nu A^\nu\right]
\end{aligned}
\tag{F.108}
$$

We may write $\mathcal{L}$ as

$$
\mathcal{L} = \frac{1}{2}A^\mu \left[\Box \eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right] A^\nu = \frac{1}{2}A^\mu \tilde{P}_{\mu\nu} A^\nu \tag{F.109}
$$

In momentum space the operator $\tilde{P}_{\mu\nu}$ becomes

$$
(-q^2 \eta^{\nu\lambda} + (1 - \xi^{-1}) q^\nu q^\lambda) \tag{F.110}
$$

The inverse operator $D_{\lambda\mu}(q)$ will have the structure

$$
(A(q^2)\eta_{\lambda\mu} + B(q^2) q_\mu q_\lambda) \tag{F.111}
$$

$$
\begin{aligned}
& [(-q^2 \eta^{\nu\lambda} + (1 - \xi^{-1}) q^\nu q^\lambda)]\ [(A(q^2)\eta_{\lambda\mu} + B(q^2) q_\mu q_\lambda)] \\
=\ & A(q^2)\ [-q^2 \delta^\nu_\mu + (1 - \xi^{-1}) q^\nu q_\mu] + B(q^2)\ [-\xi^{-1} q^2 q^\nu q_\mu] \\
=\ & \delta^\nu_\mu.
\end{aligned}
\tag{F.112}
$$

Thus we get

$$
A(q^2) \cdot -q^2 = 1
$$

for $\mu = \nu$, and

$$A(q^2)\ [(1-\xi^{-1})q^\nu q_\mu] - \xi^{-1}B(q^2)\ [q^2 q^\nu q_\mu] = 0 \tag{F.113}$$

for $\mu \neq \nu$. Hence

$$A(q^2) = -\frac{1}{q^2} \tag{F.114}$$

and

$$B(q^2) = \frac{1-\xi}{q^4} \tag{F.115}$$

So that

$$(A(q^2)\eta_{\mu\nu} + B(q^2)q_\mu q_\nu) = \left(-\frac{\eta_{\mu\nu}}{q^2} + \frac{(1-\xi)q_\mu q_\nu}{q^4}\right). \tag{F.116}$$

The propagator is then

$$D_{\mu\nu}(q) = -\frac{1}{q^2+i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2+i\epsilon}\right). \tag{F.117}$$

F.6 Non-Abelian Case

In QED, the determinant was independent of A, so could be factorised out as another unimportant overall constant

In the non-Abelian case $Z[J]$ then becomes

$$\begin{aligned} Z[J] &= N(\xi)\int \mathcal{D}\chi \exp\left[-i\int d^4x \frac{Tr\chi^2(x)}{2\xi}\right]\int \mathcal{D}U(x) \\ &\quad \times \int \mathcal{D}A\, \delta(F(A)-\xi(x))\ det\left(\frac{\delta F^a(A)}{\delta\Lambda^b}\right)\exp(iS[A]) \\ &= N(\xi)\int \mathcal{D}U(x)\int \mathcal{D}A\ det\left(\frac{\delta F^a(A)}{\delta\Lambda^b}\right)\exp\left(iS[A] - i\int d^4x \frac{Tr(F[A])^2}{2\xi}\right) \end{aligned} \tag{F.118}$$

where $N(\xi)$ is an unimportant noralisation constant.

$$[T^a, T^b] = i\epsilon^{abc}T^c$$

$$A_\mu^\Lambda = UA_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger, \tag{F.119}$$

where

$$U(x) = \exp(ig\Lambda^a(x)T^a) \tag{F.120}$$

We have

$$\begin{aligned}
A_\mu^{c\Lambda}T^c(x) &= (\mathbf{1} + ig\Lambda^a(x)T^a)A_\mu^b(x)T^b(\mathbf{1} - ig\Lambda^a(x)T^a) + \frac{i}{g}(ig\partial_\mu\Lambda^c(x)T^c)(\mathbf{1} - ig\Lambda^b(x)T^b) \\
&= A_\mu^c(x)T^c - \partial_\mu\Lambda^c(x)T^c + ig\Lambda^a(x)A_\mu^b(x)[T^a, T^b] + \mathcal{O}(\Lambda^2) \\
&= A_\mu^c(x)T^c - \partial_\mu\Lambda^c(x)T^c - g\Lambda^a(x)A_\mu^b(x)\epsilon^{abc}T^c + \mathcal{O}(\Lambda^2)
\end{aligned} \tag{F.121}$$

or

$$A_\mu^{a\Lambda}(x) = A_\mu^a(x) - \partial_\mu\Lambda^a(x) \tag{F.122}$$

$$\begin{aligned}
M_{ab} = \frac{\delta F_a(x)}{\delta\Lambda^b(y)} &= \int d^4z \frac{\delta F_a(x)}{\delta A_c^\mu(z)} \frac{\delta A_c^\mu(z)}{\delta\Lambda^b(y)} \\
&= \int d^4z[-\partial_\mu\delta_{ac}\delta^4(x-z)][] \\
&= \\
&= \int d^4z\partial_\mu\delta_{ac}\delta^4(x-z)(D^\mu)_{cb}\delta^4(z-y) \\
&= (\partial_\mu D^\mu)_{ab}\delta^4(x-y)
\end{aligned} \tag{F.123}$$

where

$$(D^\mu)_{ab} = \partial_\mu\delta_{ab} + ig\epsilon_{abc}A^{\mu c}$$

The determinant in terms of a fermionic Gaussian integral over complex anticommutationg functions of the operator put in between the fields,

$$det\ M = \int \mathcal{D}\eta^* \mathcal{D}\eta \exp\left(\int d^4x d^4x' \eta^*(x) M(x-x')\eta(x')\right) \tag{F.124}$$

The Jacobian, Δ^F, may be represented as

$$\begin{aligned} & det\ |M_{ab}|_{F=0} \\ = & \int \mathcal{D}\eta^{*a} \mathcal{D}\eta^a \exp\left(\int d^4x d^4x' \eta^{*a}(x) M_{ab}(x-x')\eta^b(x')\right) \end{aligned} \tag{F.125}$$

So far we have worked only with the integral $\int \mathcal{D}A \exp(iS[A])$. Say we wanted to calculate the quantity

$$\frac{\int \mathcal{D}A\mathcal{O}(A)\exp(iS[A])}{\int \mathcal{D}A \exp(iS[A])} \tag{F.126}$$

where the operator is gauge invariant. The same procedure goes through with the numerator as the replacement of A by A^Λ works. We find for the correlation function

$$\frac{\int \mathcal{D}A \int \mathcal{D}\eta^* \mathcal{D}\eta\ \mathcal{O}(A)\exp(iS[A] - \frac{1}{2\xi}(F[A])^2 + iS_g[\eta,\eta^*;A])}{\int \mathcal{D}A \int \mathcal{D}\eta^* \mathcal{D}\eta \exp(iS[A] - \frac{1}{2\xi}(F[A])^2 + iS_g[\eta,\eta^*;A])} \tag{F.127}$$

where the awkard constant factors have canceled.

F.6.1 Final Lagrangian for Yang-Mills

The quadratic in the fields part of the actio, the part that gives the propagators, is given by

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \eta^\dagger_a \partial^2 \eta_a. \tag{F.128}$$

The interaction part of the Lagrangian is given by,

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{1}{2}g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\epsilon^{abc}A^{b\mu}A^{c\nu} \\ &\quad +\frac{1}{4}g^2\epsilon^{abc}\epsilon^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} \\ &\quad -ig\eta^{a\dagger}\epsilon^{abc}\partial^\mu A^c_\mu \eta^b. \end{aligned} \tag{F.129}$$

Boson propagator

The boson lagrangian is given by

$$\mathcal{L}_B = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \tag{F.130}$$

In analogy to the calculation (F.108), we may write $\mathcal{L}_B$ as

$$\mathcal{L}_B = \frac{1}{2}A^{a\mu}\left[\Box\eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^{a\nu} = \frac{1}{2}A^{a\mu}\tilde{P}^{ab}_{\mu\nu}A^{b\nu}. \tag{F.131}$$

In momentum space the operator $\tilde{P}^{ac}_{\mu\nu}$ becomes

$$(-q^2\eta^{\nu\lambda} + (1 - \xi^{-1})q^\nu q^\lambda)\delta^{ac}. \tag{F.132}$$

The inverse operator $D^{cb}_{\lambda\mu}(q)$ will have the structure

$$(A(q^2)\eta_{\lambda\mu} + B(q^2)q_\mu q_\lambda)\delta^{cb} \tag{F.133}$$

Performing the analogous calculation as in section F.5.1, the propagator is

$$D^{ab}_{\mu\nu}(q) = -\frac{1}{q^2 + i\epsilon}\left(\eta_{\mu\nu} - (1 - \xi)\frac{q_\mu q_\nu}{q^2 + i\epsilon}\right)\delta^{ab}. \tag{F.134}$$

Ghost Propagator

The free ghost lagrangian is given by

$$\mathcal{L}_{FP} = -\eta^\dagger_a\Box\eta_a \tag{F.135}$$

The ghost field propagates like a masslees spin-zero field:

$$\Delta^{ab}(p) = \frac{\delta^{ab}}{p^2 + i\epsilon}. \tag{F.136}$$

As we saw the functional integral method allows us to read off the Feynmann rules for vertices directly from the interacting field theory.

F.6.2 Interaction Vertices of Gauge Fields

The fermion and Yang-Mills vertex

In QED the interaction term of the Lagrangian is given by

$$\mathcal{L}_{int}^{EM} = -e\overline{\Psi}\gamma^{\mu}\Psi A_{\mu} \tag{F.137}$$

The vertex function is just

$$\Gamma^{\mu}_{EM} = -e\gamma^{\mu}. \tag{F.138}$$

In the same way we can read off from the interaction Lagrangian the vertex function for the coupling of fermions and Yang-Mills field

In general the Feynman rules are obtained by varying the corresponding action integral in momentum space.

Triple vertex

$$S_{int}^{trip} = \int d^4x \mathcal{L}_{int}^{trip} \tag{F.139}$$

$$\begin{aligned} \mathcal{L}_{int}^{quad} &= -\frac{1}{2}g(\partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a})\epsilon_{abc}A_{b}^{\mu}A_{c}^{\nu} \\ &= -g\partial_{\mu}A_{\nu}^{a}\epsilon_{abc}A_{b}^{\mu}A_{c}^{\nu} \end{aligned} \tag{F.140}$$

$$A_{\mu}^{a}(x) = \int \frac{d^4p}{(2\pi)^4} A_{\mu}^{a}(p)e^{-ip\cdot x} \tag{F.141}$$

in momentum space

$$
\begin{aligned}
S_{int}^{trip} &= \int d^4x \mathcal{L}_{int}^{trip} \\
&= -g\epsilon_{abc} \int d^4x \partial_\mu A_\nu^a A_b^\mu A_c^\nu \\
&= -g\epsilon_{abc} \int d^4x \frac{d^4pd^4kd^4q}{(2\pi)^{12}} (-ip_\mu) A_\nu^a(p) A_b^\mu(q) A_c^\nu(k) e^{-i(p+q+k)\cdot x} \\
&= ig\epsilon_{abc} \int \frac{d^4p\ d^4k\ d^4q}{(2\pi)^8} p_\mu A_\nu^a(p) A_b^\mu(q) A_c^\nu(k) \delta(p+q+k) \\
&= ig\epsilon_{abc} \int \frac{d^4p\ d^4k\ d^4q}{(2\pi)^8}\ \delta(p+q+k) \\
&\quad \frac{1}{3!}(p_\mu A_\nu^a(p) A_b^\mu(q) A_c^\nu(k) + \\
&\qquad p_\mu A_\nu^a(p) A_b^\mu(k) A_c^\nu(q) + \\
&\qquad q_\mu A_\nu^a(q) A_b^\mu(k) A_c^\nu(p) + \\
&\qquad q_\mu A_\nu^a(q) A_b^\mu(p) A_c^\nu(k) + \\
&\qquad k_\mu A_\nu^a(k) A_b^\mu(p) A_c^\nu(q) + \\
&\qquad k_\mu A_\nu^a(k) A_b^\mu(q) A_c^\nu(p))
\end{aligned} \tag{F.142}
$$

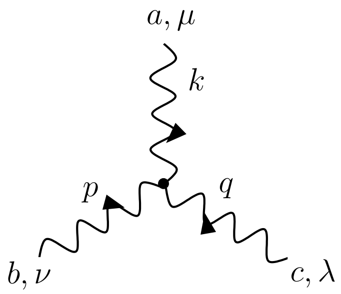


Figure F.3: .

Variation yields

$$
\frac{\delta A_\mu^a}{\delta A_\nu^b} = \eta_\beta^\alpha\ \delta_{ab} \tag{F.143}
$$

in momentum space

Quadruple vertex

$$
S_{int}^{quad} = \int d^4x \mathcal{L}_{int}^{quad} \tag{F.144}
$$

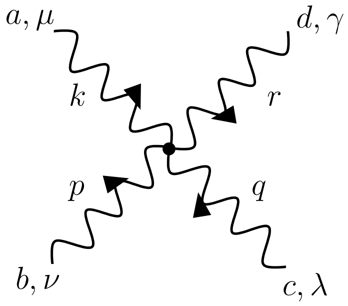


Figure F.4: .

F.6.3 Ghosts and Coupling to the Gauge Field

$$\mathcal{L}_g = \eta_a^\dagger \partial^2 \eta_a + g\eta^{a\dagger}\epsilon^{abc}\partial^\mu A_\mu^c \eta^b \tag{F.145}$$

F.6.4 Feynmann Rules for Yang-Mills Theory

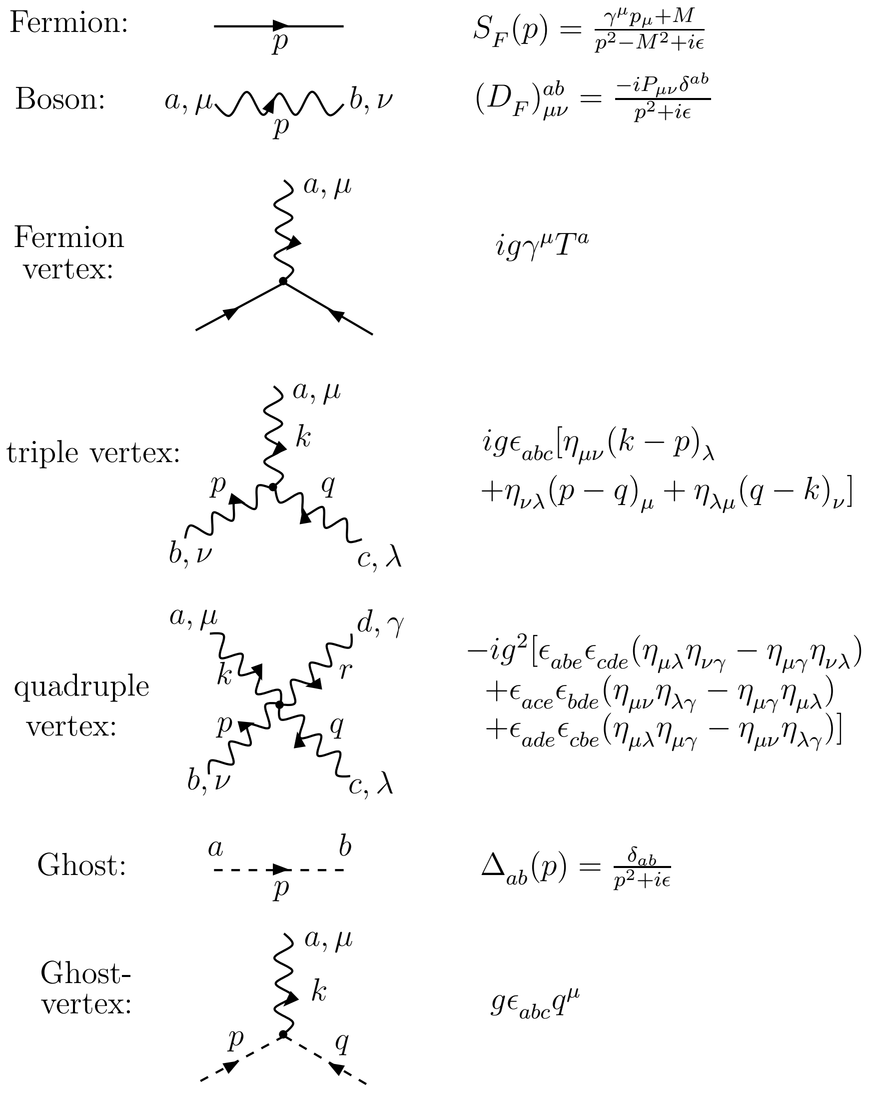


Figure F.5: .

F.6.5 The Axial Gauge and the Temporal Gauge

The axial gauge

The axial gauge is defined by the condition

$$n^\mu A^a_\mu = 0, \quad n^\mu n_\mu = 1 \tag{F.146}$$

where n^μ is a spacelike vector. The gauge constraint is then

$$G = n^\mu A^a_\mu \tag{F.147}$$

and

$$
\begin{aligned}
G^a(A_\mu^\theta) - \chi^a &= n^\mu A_\mu^a + \frac{1}{g} n^\mu \partial_\mu \theta^a + f^{abc} n^\mu A_\mu^b \theta^c - \chi^a \\
&= \frac{1}{g} n^\mu \partial_\mu \theta^a + f^{abc} \chi^b \theta^c
\end{aligned}
\tag{F.148}
$$

In this gauge the ghost ffield decouples from the gauge field.

We have

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a - \frac{1}{2\xi}(n^\mu A_\mu^a)^2 \\
&= \frac{1}{2}[A^{a\nu}\partial^\rho \partial_\rho A_\nu^a - \partial^\mu (A^{a\nu}\partial_\mu A_\nu^a)] - \frac{1}{2}[A^{a\mu}\partial_\mu \partial_\nu A^{a\nu} - \partial_\mu (A^{a\nu}\partial_\nu A^{a\mu})] - \frac{1}{2\xi}(n^\mu A_\mu^a)^2 \\
&= \frac{1}{2} A^{a\mu}\left[\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu - \frac{1}{\xi} n_\mu n_\nu\right] A^{a\nu} - \frac{1}{2}\partial^\mu \left[A^{a\nu}\partial_\mu A_\nu^a - A^{a\nu}\partial_\nu A_\mu^a\right]
\end{aligned}
\tag{F.149}
$$

We may write $\mathcal{L}$ as

$$
\mathcal{L} = \frac{1}{2} A^{a\mu}\left[\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu - \frac{1}{\xi} n_\mu n_\nu\right] A^{a\nu} = \frac{1}{2} A^{a\mu} \tilde{P}_{\mu\nu}^{ab} A^{b\nu}. \tag{F.150}
$$

The operator $\tilde{P}^{ab}_{\mu\nu}$ in momentum space,

$$(-q^2\eta_{\mu\nu} + q_\mu q_\nu - \frac{1}{\xi} n_\mu n_\nu)\delta^{ab}. \tag{F.151}$$

It can be demonstrated that the inverse is

$$-\frac{1}{q^2}\left(\eta^{\mu\nu} + \frac{(n^2+\xi q^2)q^\mu q^\nu}{(q\cdot n)^2} - \frac{q^\mu n^\nu + n^\mu q^\nu}{(q\cdot n)}\right)\delta^{ab}. \tag{F.152}$$

i.e.

$$\begin{aligned}
& [(-q^2\eta_{\mu\lambda} + q_\mu q_\lambda - \frac{1}{\xi} n_\mu n_\lambda)\delta^{ac}] \cdot - \left[\frac{1}{q^2}\left(\eta^{\lambda\nu} + \frac{(n^2+\xi q^2)q^\lambda q^\nu}{(q\cdot n)^2} - \frac{q^\lambda n^\nu + n^\lambda q^\nu}{(q\cdot n)}\right)\delta^{cb}\right] \\
= & \left(\delta^\nu_\mu + \cancel{\frac{(n^2+\xi q^2)q_\mu q^\nu}{(q\cdot n)^2}} - \frac{q_\mu n^\nu + n_\mu q^\nu}{(q\cdot n)} - \frac{q_\mu q^\nu}{q^2} - \cancel{\frac{(n^2+\xi q^2)q_\mu q^\nu}{(q\cdot n)^2}} + \frac{q_\mu n^\nu + (q\cdot n)q_\mu q^\nu/q^2}{(q\cdot n)}\right. \\
& \left. + \cancel{\frac{n_\mu n^\nu}{\xi q^2}} + \frac{(\cancel{n^2} + \xi q^2)n_\mu q^\nu}{\xi q^2 (q\cdot n)} - \frac{(q\cdot n)\cancel{n_\mu n^\nu} + \cancel{n^2} n_\mu q^\nu}{\xi q^2(q\cdot n)}\right)\delta^{ab} \\
= & \left(\delta^\nu_\mu - \frac{q_\mu n^\nu + n_\mu q^\nu}{(q\cdot n)} - \frac{q_\mu q^\nu}{q^2} + \frac{q_\mu n^\nu + (q\cdot n)q_\mu q^\nu/q^2}{(q\cdot n)} + \frac{n_\mu q^\nu}{(q\cdot n)}\right)\delta^{ab} \\
= & \ \delta^\nu_\mu \delta^{ab}.
\end{aligned} \tag{F.153}$$

The Temporal Gauge

$$n^\mu A^a_\mu = 0, \quad n^\mu n_\mu = -1 \tag{F.154}$$

where n^μ is a timelike vector.

In particualar, with $n_\mu = (1,0,0,0)$,

$$A^a_0 = 0. \tag{F.155}$$

F.7 Electro-Weak Theory

Veltman: I do not care what or how, but what we must have is at least one renormalisable theory with massive charged bosons, and whether that looks like Nature is of no concern, those are details that will be fixed later by some model freak...

't Hooft: I can do that.

Veltman: What do you say?

't Hooft: I can do that.

F.7.1 Introduction

The Higgs field is introduced into the model causing spontaneous symmetry breaking. This leads to the electron gaining mass.

F.7.2 Massless Dirac Lagrangian

The Dirac Lagrangian with zero mas is given by

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi \tag{F.156}$$

$$\begin{aligned}\mathcal{L} &= i\overline{\psi}\gamma^\mu\partial_\mu\psi \\ &= i(\overline{\psi}_L + \overline{\psi}_R)\gamma^\mu\partial_\mu(\psi_L + \psi_R) \\ &= i\overline{\psi}_L\gamma^\mu\partial_\mu\psi_L + i\overline{\psi}_R\gamma^\mu\partial_\mu\psi_R + i(\overline{\psi}_L\gamma^\mu\partial_\mu\psi_R + \overline{\psi}_R\gamma^\mu\partial_\mu\psi_L)\end{aligned} \tag{F.157}$$

The last term vanishes as

$$\begin{aligned}\overline{\psi}_L\gamma^\mu\partial_\mu\psi_R &= \left(\frac{1-\gamma_5}{2}\right)\overline{\psi}\gamma^\mu\partial_\mu\left(\frac{1+\gamma_5}{2}\right)\psi \\ &= \frac{1}{4}(1-\gamma_5+\gamma_5-\gamma_5^2)\,\overline{\psi}\gamma^\mu\partial_\mu\psi \\ &= 0\end{aligned} \tag{F.158}$$

so the mixed terms vanish and we are left

$$\mathcal{L} = i\overline{\psi}_L\gamma^\mu\partial_\mu\psi_L + i\overline{\psi}_R\gamma^\mu\partial_\mu\psi_R. \tag{F.159}$$

We see that the Lagrangian splits up into left- and right-handed parts.

F.7.3 Leptonic Fields in Electro-Weak Theory

$$\Psi_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \tag{F.160}$$

where ν_e is the electron neutrino and e_L is the left-handed electron field.

$$e_L = \left(\frac{1-\gamma_5}{2}\right) e, \qquad e_R = \left(\frac{1+\gamma_5}{2}\right) e \tag{F.161}$$

If we take the nuetrino be massless, the there is only the left-handed component of the neutrino field. Since the field is entirely left-handed, it satifies the equation

$$\left(\frac{1-\gamma_5}{2}\right) \nu_e = \nu_e. \tag{F.162}$$

With no right-handed component of the neutrino field, we define

$$\Psi_R = \begin{pmatrix} 0 \\ e_R \end{pmatrix} \tag{F.163}$$

F.7.4 Charges of the Electroweak Interaction

F.7.5 Higgs Field

In the standard model of particle physics, which obviusly contains electroweak theory, the massess of all the particles are zero. an extra field, the so-called Higgs field, is inserted by hand to give the particles mass.

F.7.6 Feynmann Rules for Electroweak Theory

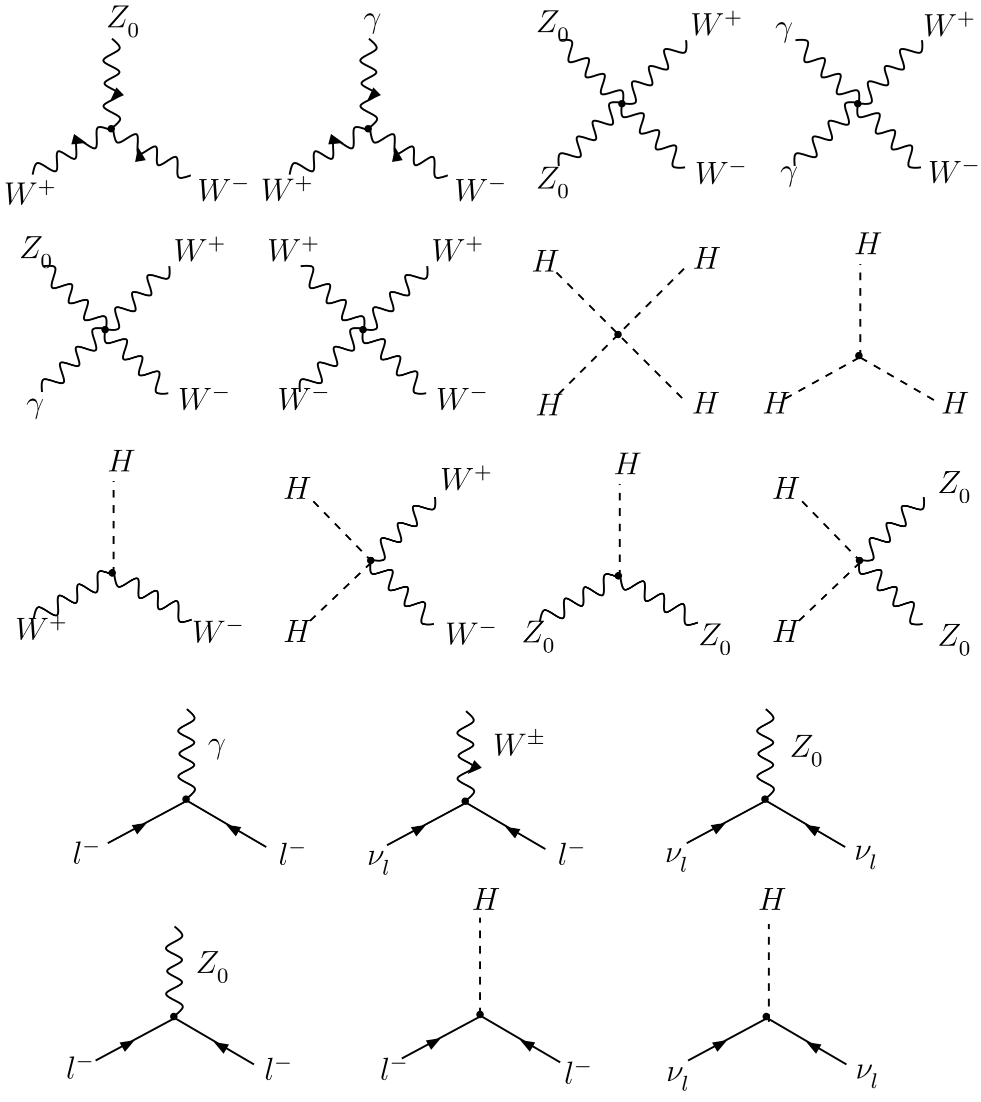


Figure F.6: The 18 vertices of standard electroweak theory.

Appendix G

The Standard Model of Particle Physics

G.0.7 The Weak Force

The gauge group of the weak force is $SU(2)$. The gauge bosons which mediate the weak force are denoted W^+, W^-, and Z. As the range of the weak force is short, these gauge bosons are massive.

G.0.8 The Strong Force

The gauge group of the strong force is $SU(3)$. The gauge bosons which mediate this force are called gluons.

G.0.9 Leptons

Leptons interact via the eletromagnetic and weak interaction, but do not participate in the sttrng interaction.

G.0.10 Higgs Field

In the standard model of particle physics the massess of all the particles are zero. An extra field, the so-called Higgs field, is inserted by hand to give the particles mass.

Appendix H

Perturbative Quantum Gravity

An open problem in quantum gravity is to compute particle scattering amplitudes from the background-independent theory and recover the low energy physics.

Calculations should agree with low energy conventional field theory. Here we introduce conventional scattering theory.

H.0.11 Renormalisability

In higher orders, when we allow bubbles and loops in the diagrams we infinities which can be accounted for by any kind of physical argument.

H.1 Coupling of Scalar Matter and Gravity

$$-\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R+\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-m^2\phi^2\right) \tag{H.1}$$

H.2 Expansion

Let $\hat{A}$ and $\hat{B}$ be matrices

$$\frac{1}{\hat{A}+\lambda\hat{B}}=\frac{1}{\hat{A}}-\lambda\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}+\lambda^2\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}-\dots \tag{H.2}$$

$$
\begin{aligned}
g^{\mu\nu} &= (\eta_{\mu\nu} + \lambda h_{\mu\nu})^{-1} \\
&= \eta^{\mu\nu} - \lambda \eta^{\mu\alpha} h_{\alpha\beta} \eta^{\beta\nu} + \lambda^2 \eta^{\mu\alpha} h_{\alpha\beta} \eta^{\beta\gamma} h_{\gamma\delta} \eta^{\delta\nu} - \dots \\
&= \eta^{\mu\nu} - \lambda h^{\mu\nu} + \lambda^2 h^{\mu}{}_{\alpha} h^{\alpha\nu} - \lambda^3 h^{\mu}{}_{\alpha} h_{\alpha\beta} h^{\beta\nu} + \dots
\end{aligned}
\tag{H.3}
$$

$$
\begin{aligned}
&\sqrt{-det\ g_{\mu\nu}} \\
&= \sqrt{-det\ \eta_{\mu\alpha}} \exp\left[\frac{1}{2} Tr\ \ln(\delta^{\alpha}_{\nu} + \lambda h^{\alpha}{}_{\nu})\right] \\
&= \exp\left[\frac{1}{2} Tr\left(\lambda h^{\alpha}{}_{\nu} - \frac{1}{2}\lambda^2 h^{\alpha}{}_{\tau} h^{\tau}{}_{\nu} + \frac{1}{3}\lambda^3 h^{\alpha}{}_{\tau} h^{\tau}{}_{\sigma} h^{\sigma}{}_{\nu} - \dots\right)\right] \\
&= 1 + \frac{1}{2}\lambda h^{\alpha}{}_{\alpha} - \frac{\lambda^2}{4} h^{\alpha}{}_{\rho}(h^{\rho}{}_{\alpha} - \frac{1}{2}\delta^{\rho}{}_{\alpha} h^{\sigma}{}_{\sigma}) + \dots
\end{aligned}
\tag{H.4}
$$

$$
\begin{aligned}
& S_m \\
&= \frac{1}{2}\int d^4x \sqrt{-g}\left(g^{\mu\nu}\phi,_{\mu}\phi,_{\nu} - m^2\phi^2\right) \\
&= \frac{1}{2}\int d^4x (1 + \lambda h^{\rho}{}_{\rho} - \lambda^2 h^{\sigma}{}_{\rho}(h^{\rho}{}_{\alpha} - \frac{1}{2}\delta^{\rho}{}_{\alpha} h^{\sigma}{}_{\sigma}) + \dots) \\
&\quad \times \left[(\eta^{\mu\nu} - \lambda h^{\mu\nu} + \lambda^2 h^{\mu}{}_{\alpha} h^{\alpha\nu} + \dots)\phi,_{\mu}\phi,_{\nu} - m^2\phi^2\right] \\
&= \frac{1}{2}\int d^4x (\phi,_{\mu}\phi,_{\nu} - m^2\phi^2) \\
&\quad - \frac{\lambda}{2}\int d^4x h^{\mu\nu}\left[\phi,_{\mu}\phi,_{\nu} - \frac{1}{2}\eta_{\mu\nu}\phi,^{\sigma}\phi,_{\sigma} + \frac{1}{2}m^2\phi^2\eta_{\mu\nu}\right] \\
&\quad - \frac{\lambda^2}{4}\int d^4x \left[\frac{1}{2} h^{\lambda}{}_{\rho}(h^{\rho}{}_{\lambda} - \frac{1}{2}\delta^{\rho}{}_{\lambda} h^{\sigma}{}_{\sigma})(\phi,^{\mu}\phi,_{\mu} - m^2\phi^2) - 2h^{\mu\rho}(h_{\rho}{}^{\nu} - \frac{1}{2}\delta_{\rho}{}^{\nu} h^{\sigma}{}_{\sigma})\phi,_{\mu}\phi,_{\nu}\right]
\end{aligned}
\tag{H.5}
$$

H.3 Graviton Propagator

The graviton propagator is obtained from the quadratic term in the field $h_{\mu\nu}$.

The Lagrangian, in this approximation, is

$$
\mathcal{L}_0 = \frac{1}{4}[-(\partial_{\nu} h_{\beta\gamma})^2 + (\partial_{\mu} h^{\beta}_{\beta})^2 - 2\partial_{\gamma} h^{\beta}_{\beta}\partial_{\mu} h^{\gamma\mu} + 2\partial_{\beta} h_{\nu\gamma}\partial^{\nu} h^{\beta\gamma}]
\tag{H.6}
$$

We can make a gauge chooice by adding the term

$$-\frac{1}{2}C_\mu^2; \qquad C_\mu = \partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h^\nu_\nu \tag{H.7}$$

to the action.

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\beta\gamma} V^{\beta\gamma\mu\nu}\partial^\lambda h_{\mu\nu} \tag{H.8}$$

where

$$V_{\alpha\beta\mu\nu} = \frac{1}{2}\delta_{\alpha\mu}\delta_{\beta\nu} - \frac{1}{4}\delta_{\alpha\beta}\delta_{\mu\nu} \tag{H.9}$$

We derive the source equations

$$-V^{(\alpha\beta)\mu\nu}\Box h_{\mu\nu}(x) = S^{\alpha\beta}(x) \tag{H.10}$$

$$h_{\mu\nu}(x) = \int d^4y D_{\mu\nu\lambda\gamma}(x-y)S^{\lambda\gamma}(y) \tag{H.11}$$

We must have

$$\begin{aligned} -V^{(\alpha\beta)\mu\nu}\Box h_{\mu\nu}(x) &= -\int d^4y V^{(\alpha\beta)\mu\nu}\Box D_{\mu\nu\lambda\gamma}(x-y)S^{\lambda\gamma}(y) \\ &= S^{\alpha\beta}(x) \end{aligned} \tag{H.12}$$

implying

$$-V^{(\alpha\beta)\mu\nu}\Box D_{\mu\nu\lambda\gamma}(x-y) = \delta^4(x-y)\delta_\lambda^{(\alpha}\delta_\gamma^{\beta)} \tag{H.13}$$

from

$$D_{\mu\nu\lambda\gamma}(x-y) = \int \frac{d^4q}{(2\pi)^4}\exp[-iq\cdot(x-y)]D_{\mu\nu\lambda\gamma}(q^2) \tag{H.14}$$

in momentum space

$$q^2 V^{(\alpha\beta)\mu\nu} D_{\mu\nu\lambda\gamma}(q) = \delta_\lambda^{(\alpha}\delta_\gamma^{\beta)} \tag{H.15}$$

The propagator should have the form

$$D_{\mu\nu\lambda\gamma}(q) \quad = \quad A(q^2)\delta_{\mu\nu}\delta_{\lambda\gamma} + B(q^2)\delta_{\mu\lambda}\delta_{\nu\gamma} + C(q^2)\delta_{\mu\gamma}\delta_{\nu\lambda} \tag{H.16}$$

Now

$$V^{(\alpha\beta)\mu\nu} = \frac{1}{4}\delta^{\alpha\mu}\delta^{\beta\nu} + \frac{1}{4}\delta^{\beta\mu}\delta^{\alpha\nu} - \frac{1}{4}\delta^{\alpha\beta}\delta^{\mu\nu} \tag{H.17}$$

so

$$\begin{aligned} q^2 V^{(\alpha\beta)\mu\nu} D_{\mu\nu\lambda\gamma}(q) &= q^2 A(q^2) V^{(\alpha\beta)\mu\nu}\delta_{\mu\nu}\delta_{\lambda\gamma} q^2 B(q^2) V^{(\alpha\beta)\mu\nu}\delta_{\mu\lambda}\delta_{\nu\gamma} \\ &\quad q^2 C(q^2) V^{(\alpha\beta)\mu\nu}\delta_{\mu\gamma}\delta_{\nu\lambda} + \ldots \\ &= q^2 A(q^2)\frac{1}{4}(\delta^{\alpha\beta}\delta_{\lambda\gamma} + \delta^{\beta\alpha}\delta_{\lambda\gamma} - 4\delta^{\alpha\beta}\delta_{\lambda\gamma}) \\ &\quad q^2 B(q^2)\frac{1}{4}(\delta_\lambda^\alpha\delta_\gamma^\beta + \delta_\lambda^\beta\delta_\gamma^\alpha - \delta^{\alpha\beta}\delta_{\lambda\gamma}) \\ &\quad q^2 C(q^2)\frac{1}{4}(\delta_\gamma^\alpha\delta_\lambda^\beta + \delta_\gamma^\beta\delta_\lambda^\alpha - \delta^{\alpha\beta}\delta_{\gamma\lambda}) + \ldots \\ &= \delta_\lambda^{(\alpha}\delta_\gamma^{\beta)} \end{aligned} \tag{H.18}$$

Implying

$$A(q^2) = -\frac{1}{q^2}, \quad B(q^2) = \frac{1}{q^2}, \quad C(q^2) = \frac{1}{q^2}. \tag{H.19}$$

The propagator is then

$$D_{\mu\nu\lambda\gamma}(q) = \frac{\delta_{\mu\lambda}\delta_{\nu\gamma} + \delta_{\mu\gamma}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\gamma}}{q^2 + i\epsilon} \tag{H.20}$$

H.3.1 Lowest Order Vertex

$$S_m = \frac{1}{2}\int d^4x(\phi^{,\mu}\phi_{,\mu} - m^2\phi^2) - \frac{\lambda}{2}\int h^{\mu\nu}[\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\eta_{\mu\nu}\phi^{,\rho}\phi_{,\rho} + \frac{1}{2}m^2\phi^2\eta_{\mu\nu}] + \mathcal{O}(\lambda^2) \tag{H.21}$$

This leads to the equations of motion

$$\begin{aligned}(\partial^\mu\partial_\mu + m_0^2)\phi(x) &= \lambda\partial_\mu[h^{\mu\nu}(x)\partial_\nu\phi(x)] \\ &\quad -\lambda\frac{1}{2}\partial_\mu[h^\rho{}_\rho(x)\eta^{\mu\nu}\partial_\nu\phi(x)] \\ &\quad -\lambda\frac{1}{2}m_0^2 h^\rho{}_\rho(x)\phi(x) + \mathcal{O}(\lambda^2) \\ &= -\hat{V}\phi(x) \end{aligned} \tag{H.22}$$

$$\hat{V}\phi(x) = -\lambda\left\{\partial_\mu[h^{\mu\nu}(x)\partial_\nu - \frac{1}{2}h^\rho{}_\rho(x)\eta^{\mu\nu}\partial_\nu] - \frac{1}{2}m_0^2 h^\rho{}_\rho(x)\right\}\phi(x)$$

$$\begin{aligned} S_{fi} &= \lambda Const. \int d^4x e^{+ip_f\cdot x}\left\{\partial_\mu[h^{\mu\nu}(x)\partial_\nu - \frac{1}{2}h^\rho{}_\rho(x)\eta^{\mu\nu}\partial_\nu] - \frac{1}{2}m_0^2 h^\rho{}_\rho(x)\right\}e^{-ip_i\cdot x} \\ &= \lambda Const. \int d^4x e^{+ip_f\cdot x}\left\{[-(ip_{f\mu})][h^{\mu\nu}(x) - \frac{1}{2}h^\rho{}_\rho(x)\eta^{\mu\nu}](-ip_{i\nu}) - \frac{1}{2}m_0^2 h^\rho{}_\rho(x)\right\}e^{-ip_i\cdot x} \\ &= -\lambda Const. \int d^4x \left[h^{\mu\nu}(x)p_{f\mu}p_{i\nu} - \frac{1}{2}h^\rho{}_\rho(x)(p_{f\tau}p_i^{\ \tau} - m_0^2)\right]e^{-i(p_f-p_i)\cdot x} \\ &= \lambda Const. \left[h^{\mu\nu}(p_f - p_i)p_{f\mu}p_{i\nu} - \frac{1}{2}h^\rho{}_\rho(p_f - p_i)(p_{f\tau}p_i^{\ \tau} - m_0^2)\right] \end{aligned} \tag{H.23}$$

If

$$h^{\mu\nu}(x) = e_{\mu\nu}e^{iq\cdot x}$$

then

$$h^{\mu\nu}(p_f - p_i) = (2\pi)^4\delta^4(q - p_i + p_f)$$

H.3.2 Harmonic Gauge

We choose the Harmonic gauge, where:

$$\frac{1}{2\alpha}(\partial_\mu h^{\mu\nu})^2 \tag{H.24}$$

is added to the action for arbitrary α. We calculate the graviton propagator, and the Fadeev-Popov ghost term.

$$-\lambda[p_{f\mu}p_{i\nu} - \tfrac{1}{2}\eta_{\mu\nu}(p_{f\tau}p_i^{\tau} - m_0^2)]$$

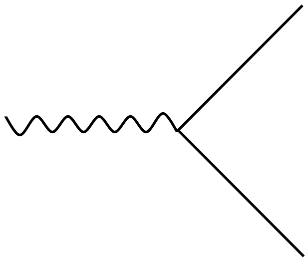


Figure H.1: interaction.

Appendix I

Algebraic Structures

overwhelmingly tested regimes of quantum theory and general relativity.

In loop quantum gravity, the geometry of space is built from qubits. Spacetime is like a parallel-processing quantum computer that constantly modifies its own topology.

Figure put how to simulate such systems more efficiently.

I.1 Introduction

Lie algebras,
Tensor product algebras,
Universal enveloping algebras,
Hopf algebras,
Quasi-triangular Hopf algebras,
Quantum groups.

Boolean algebras,
Heyting algebras.

Braid group algebras,
Categories,
Topos.

C*-algebras,
von Neumann algebras.

Physical applications we simplify the situation considerably, leaving out significant details.

I.1.1 Quantum Groups in Quantum Gravity

In three dimansions, Freidel and Livine have shown that it is possible to rigorously derive the effective quantum fields theory of a scalar self-gravitating field, starting from a spin foam model. Remarkably, the quantum properties of the graviatational field manifest itself in the effective QFT by a spaacetime non-commutativity. More precisely, there is a limit of the full theory leading to a QFT over a non-commutative spacetime. In the construction, a quantum group appears as a symmetry of an auxiliary formulation of the theory.

The situation is described by the partition function

$$\mathcal{Z} = \int \mathcal{D}g \int \mathcal{D}\phi \exp i(S[\phi, g] + S[g])$$

is the action for scalar filed ϕ (with non-derivative interactions) on a 3-dimensional spacetime with metric g_{ab} and $S[g]$ is the Einstein-Hilbert action. We wish to integrate out the gravitational degrees of freedom to obtain an effective action $S_{eff}[\phi]$ for the matter field ϕ:

$$\mathcal{Z} = \int \mathcal{D}\phi \exp iS_{eff}[\phi, g_0],$$

where g_0 is the flat metric. The standard Ableian products of fields in the original action $S[\phi, g]$ are just replaced by a non-Abelian $\star-$product in $S_{eff}[\phi, g_0]$.

While the action $S[\phi, g_0]$ is invariant under the six dimensional Poincare group, $S_{eff}[\phi, g_0]$ is invariant under the so called κ deformation of Poincare group.

These particles can be regrded as moving in a 'dual' non-commutative position space coordiatized by X_i satisfying the relations S

$$[X_i, X_j] = i\kappa\hbar\epsilon_{ijk}X_k. \tag{I.1}$$

The $\star-$algebra generated by the three operators X_i subject to these commutation relations is referred to as the $\kappa-$Minkowski space.

Role of quantum groups

Quantizing a system of self-gravitating masive particles on $\mathbb{E}^3$ is equivalent to quantizing a system of non-gravitating massive point particles whose symmetry group is $DSU(2)$ instead of $ISU(2)$.

Spacetime non-commutativity is associated with the fact that the symmetry group of the theory is in fact a quantum group. It is $DSU(2)$, Drinfeld double of the classical group

$SU(2)$. It appears rather clearly that quantum groups play a role in LQG, although the precise meaning of this appearance is not yet clear.

I.1.2 Quantum Deformation of Quantum Gravity

arXiv:gr-qc/9512020:

ABSTRACT

We describe a deformation of the observable algebra of quantum gravity in which the loop algebra is extended to framed loops. This allows an alternative nonperturbative quantization which is suitable for describing a phase of quantum gravity characterized by states which are normalizable in the measure of Chern-Simons theory. The deformation parameter, q, is

$$e^{i\hbar^2 G^2 \Lambda/6},$$

where Λ is the cosmological constant. The Mandelstam identities are extended to a set of relations which are governed by the Kau man bracket so that the spin network basis is deformed to a basis of $SU(2)_q$ spin networks. Corrections to the actions of operators in non-perturbative quantum gravity may be readily computed using recoupling theory; the example of the area observable is treated here. Finally, eigenstates of the $q-$deformed Wilson loops are constructed, which may make possible the construction of a $q-$deformed connection representation through an inverse transform.

I.2 Lie and Hopf Algebras

As an example, the reader may wish to keep ... in mind in what follows.

I.2.1 Lie Algebra

The Pauli matrices and

Their algebra can be written

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y \tag{I.2}$$

We say this is an algebra over the complex numbers a vector space with commutator relations (I.2).

A field is a generalization or abstraction of scalars to more general mathematical structures "number systems of mathematics".

$X \otimes Y$ spanned by elements $x \otimes y$

$$
\begin{aligned}
(x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\
x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\
\alpha(x \otimes y) &= (\alpha x) \otimes y = x \otimes (\alpha y)
\end{aligned} \tag{I.3}
$$

A vector basis of X is x_i and of Y is y_i provide a basis for the tensor product space $X \otimes Y$. An arbitrary element of $X \otimes Y$ can be written

$$\sum_{i,j} a_{ij} x_i \otimes y_j \tag{I.4}$$

where a_{ij} are scalars. If X has dimension m and Y has dimension n then $X \otimes Y$ has dimension mn.

We now introduce some formal mathematical notation used in the literature:

$$m \circ (a \otimes b) = a \cdot b \tag{I.5}$$

$\imath$ is an operation which takes a given value in k and gives you an element of A, we write $\imath : k \to A$ and say $\imath(k) \mapsto a$. The relationship between the identity 1_A and the unit 1 is $\imath(1) = 1_A$.

(a) m is associative, i.e., $(ab)c = a(bc)$

(b) $1_A \cdot a = a \cdot 1_A = a$, for all $a \in A$.

Equivalently, an algebra over k

(a) m is associative,

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (id_A \otimes m) \tag{I.6}$$

(b) unit condition

$$m \circ (\imath \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes \imath) = \mathrm{id}_A \tag{I.7}$$

An algebra over k is a vector space A over k with multiplication

Modules Let G be a group and let the field k (e.g. $\mathbb{R}$, $\mathbb{C}$). suppose is a representation of G. Write

Modules are also referred to as representations: for instance, representations of a group are essentially the same as modules over the group algebra. the term "module" you surely have come across some examples. Vector spaces are (rather simple) examples, as are abelian groups. Representations of algebras and of groups all are modules, as are representations of Lie algebras.

Example The identity element the over the complex numbers is an (unital) algebra A. Complex linear combinations of the 2×2 identity matrix together with the Pauli matrices

$$1_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; a_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; a_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{I.8}$$

acting (or "with action") on a 2-dimensional vector space V with elements

$$v = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x, y \in \mathbb{C} \tag{I.9}$$

is an example of a A-module.

A matrix acts on any 2-component vector and gives back some vector in V, we write $A \otimes V \to V$ and $a \cdot v = av$.

From the properties of matrices we have that

$$(a_i a_j)v = a_i(a_j v), \quad \text{for all } a_i, a_j \in A \text{ and for all } v \in V. \tag{I.10}$$

and that

$$1_A \cdot v = v, \quad \text{for all } v \in V. \tag{I.11}$$

Motivated by the above observations, we now define a $k-$module.

Definition modules

Let A be an algebra over k. An *A-module* is a vector space V over k with an A-action

$$\begin{aligned} A \otimes V &\to V \\ a \otimes v &\mapsto a \cdot v = av \end{aligned} \tag{I.12}$$

such that

(a) $(ab)v = a(bv)$, for all $a, b \in A$ and $v \in V$, and
(b) $1_A v = v$, for all $v \in V$.

Non-associativity of Lie Algebras

Definition Lie bracket multiplication

$$a \circ b := [a, b] \tag{I.13}$$

Lie bracket multiplication rule obviously satisfies the distributive law

$$\begin{aligned} [(\alpha a + \beta b)c] &= \alpha[a, c] + \beta[b, c] \\ (\alpha a + \beta b) \circ c &= \alpha a \circ \beta b \circ c \end{aligned} \tag{I.14}$$

$$\begin{aligned} [a, [b, c]] + [b, [c, a]] + [c, [a, b]] &= 0 \\ a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b) &= 0 \end{aligned} \tag{I.15}$$

$$a \circ (b \circ c) = (a \circ b) \circ c + b \circ (a \circ c) \tag{I.16}$$

$$a \circ (b \circ c) \neq (a \circ b) \circ c \tag{I.17}$$

Instead of associativity we have the Jacobi identity

associative algebras are easier to handle. the associative quotient algebra

I.2.2 Tensor Product Algebra

Let k be the complex numbers $\mathbb{C}$ or the reals $\mathbb{R}$, and let A, B be vector spaces over k. It is the direct product $\otimes$ that is the binary operation or product that make the tensor algebra an algebra,

(i) Distributive. left distributive

$$\begin{aligned} (x_1 \otimes \cdots \otimes x_r) \quad \otimes \quad & (y_1 \otimes \cdots \otimes y_s + z_1 \otimes \cdots \otimes z_s) \\ = \quad & x_1 \otimes \cdots \otimes x_r \otimes y_1 \otimes \cdots \otimes y_s \\ & + x_1 \otimes \cdots \otimes x_r \otimes z_1 \otimes \cdots \otimes z_s \end{aligned} \tag{I.18}$$

right distributive property demonstrated similarly.

Property involving scalar multiplication is easy to see,

$$\alpha(x_1 \otimes x_2 \otimes \cdots \otimes x_r) \tag{I.19}$$

It is an associative algebra

$$\begin{aligned} & x_1 \otimes \cdots \otimes x_r \otimes (y_1 \otimes \cdots \otimes y_s \otimes z_1 \otimes \cdots \otimes z_s) \\ = \ & x_1 \otimes \cdots \otimes x_r \otimes y_1 \otimes \cdots \otimes y_s \otimes z_1 \otimes \cdots \otimes z_s \\ = \ & (x_1 \otimes \cdots \otimes x_r \otimes y_1 \otimes \cdots \otimes y_s) \otimes z_1 \otimes \cdots \otimes z_s \end{aligned} \tag{I.20}$$

The tensor product of any number of elements can be taken as many times as one likes The tensor algebra $T(\mathcal{L})$ is the collection of (including up to an infinite number) collectively as direct summation:

$$T(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \ldots . \tag{I.21}$$

I.2.3 Universal Enveloping Algebra

we must take note of the commutators, or Lie brackets $[A, B] = AB - BA$. The Lie algebra is then constructed by means of the repeated application of the operations of $+$, its inverse $-$ and the bracket operation $[\ ,\ ]$, where we also allow the multiplication by real or complex numbers.

http://www.math.uni-bielefeld.de/ philfahr/enveloping/node1.html

$\mathfrak{A}$ $\mathfrak{su}(2, \mathbb{C})$

consider the set of elements generated by

$$u_{x,y} = (x \otimes y - y \otimes x) - [x, y] \in T. \tag{I.22}$$

that is, any element of $T(\mathcal{L})$ that can be written can be expressed in the form

$$v \otimes (x \otimes y - y \otimes x) \otimes w - v \otimes [x, y] \otimes w \tag{I.23}$$

we say the set of elements generated by $u_{x,y}$.Let $\mathcal{I}$ be a ideal in T generated by the set of all elements $u_{x,y}$.

Any element of $T(\mathcal{L})$ that can be written can be expressed in the form

$$v \otimes (x \otimes y - y \otimes x) \otimes w - v \otimes [x, y] \otimes w \tag{I.24}$$

Recall the notion of an ideal, but now in the context of tensor algebra: say $\mathcal{I}$ is a linear subspace of $T(\mathcal{L})$. If $\mathcal{I}$ satisfies the conditions $x \otimes i \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ and if $i \otimes x \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ then $\mathcal{I}$ is a two-sided ideal (or simply an ideal). It is trivial to see that the set of elements generated by (I.22) form an ideal of $T(\mathcal{L})$:

$$\frac{T(\mathcal{L})}{\mathcal{I}} =: U(\mathcal{L}) \tag{I.25}$$

is the *universal enveloping algebra*

$$x_{i_j} x_{i_j+1} - x_{i_j+1} x_{i_j} - [x_{i_j}, x_{i_j+1}] \in \mathcal{I} \tag{I.26}$$

It may be helpful to have to hand the flow diagram fig (I.4)

Recall the definition of a two-sided ideal, (or simply an ideal): say that $\mathcal{I}$ is a linear subspace of $\mathcal{A}$. If $\mathcal{I}$ satisfies the conditions $ai \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ and if $ia \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ then $\mathcal{I}$ is a two-sided ideal.

$[\mathcal{G}, \mathcal{G}]$ form an ideal of a Lie algebra, with ordinary multiplication $a \cdot b$.

$$[[a, b], c] = [d, c] \in \mathcal{I} \equiv [\mathcal{G}, \mathcal{G}] \tag{I.27}$$

$$T(\mathcal{L}) = \mathcal{F} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \ldots \tag{I.28}$$

$$x \in T(\mathcal{L}) \tag{I.29}$$

Say $\mathcal{I}$ is a linear subspace of $T(\mathcal{L})$. If $\mathcal{I}$ satisfies the conditions $x \otimes i \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ and if $i \otimes x \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ then $\mathcal{I}$ is a two-sided ideal.

$$[a, b] - (a \otimes b - b \otimes a) \tag{I.30}$$

so a two-sided ideal $\mathcal{I}$.

$$\frac{T(\mathcal{L})}{\mathcal{I}} =: U(\mathcal{L}) \tag{I.31}$$

is the *universal enveloping algebra*

The quotient

$$v \otimes (x \otimes y - y \otimes x - [x,y]) \otimes w \sim v \otimes w \tag{I.32}$$

universal enveloping algebra of $sl(2)$

I.2.4 Hopf Algebras

nice introduction to Hopf algebras [356] (in quantum field theory)

an algebra whose multiplication is associative, $(ab)c = a(bc)$. $m(a \otimes b) = ab$.

1. Algebra

multiplication $\mu : A \otimes A \to A$ $m(a \otimes b) = ab$. m is **associative**, $(ab)c = a(bc)$.

$$m(m \otimes \mathbf{1})(a \otimes b \otimes c) = m(\mathbf{1} \otimes m)(a \otimes b \otimes c) \tag{I.33}$$

mathematicians write this in the short-hand notation

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m) \tag{I.34}$$

and depict it as a diagram (I.2.4(a)).

unit η sends every number to the same number times the identity element, i.e.

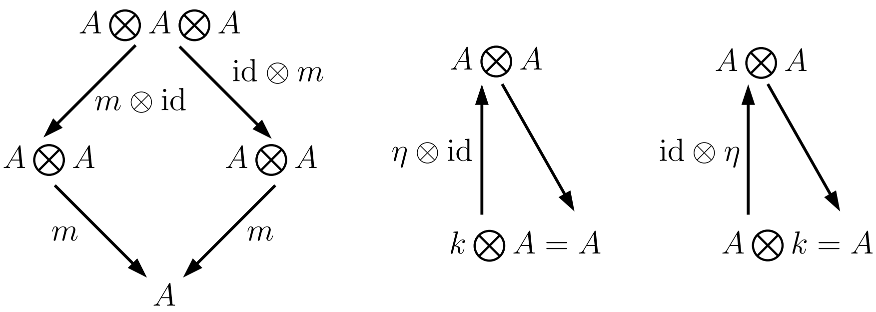
$$\eta(k) := \mathbf{1}k, \quad \text{for all } k \in \mathbb{C}. \tag{I.35}$$

$$m(a \otimes \eta(k)) = ak = ka = m(\eta(k) \otimes a), \quad \text{for all } a \in A, \text{ and for all } k \in \mathbb{C}. \tag{I.36}$$

$$m \circ (\eta \otimes \mathrm{id})(a) = a = m \circ (\mathrm{id} \otimes \eta)(a) \tag{I.37}$$

2. Coalgebra

A coalgebra is a vector space with the structure obtained by reversing the arrows in the diagrams Fig.(I.2.4) characterizing an algebra. A comultiplication $\Delta : C \to C \otimes C$. Δ is **coassociative**.



(a) (b)

Figure I.1: (a) Assiciativity condition diagram commutes. (b) Unit condition.

$$(\mathrm{id}_C \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_C)\Delta \tag{I.38}$$

seems a little too abstract can put in matrix form (exercise)

Sweedler notation for comultiplication

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)} \tag{I.39}$$

$$(\mathrm{id}_C \otimes \Delta)\Delta(a) = (\mathrm{id} \otimes \Delta) \sum_a a_{(1)} \otimes a_{(2)} = \sum_a a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \tag{I.40}$$

$$\sum_a a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum_a a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \tag{I.41}$$

counit condition, $\epsilon : C \to k$

$$m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta(a) = m \circ (\mathrm{id}_A \otimes \epsilon) \circ \sum_a a_{(1)} \otimes a_{(2)} = \sum_a a_{(1)} \epsilon(a_{(2)}) \tag{I.42}$$

$$m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta(a) = m \circ (\epsilon \otimes \mathrm{id}_A) \circ \Delta(a) = \mathrm{id}_A(a) = a, \tag{I.43}$$

$$\sum_a a_{(1)} \epsilon(a_{(2)}) = a \tag{I.44}$$

$$\sum_a \epsilon(a_{(1)}) a_{(2)} = a \tag{I.45}$$

$$
\begin{tikzcd}
 & C \bigotimes C \otimes C \arrow[dl, "\Delta \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes \Delta"] & \\
C \bigotimes C \arrow[dr, "\Delta"'] & & C \bigotimes C \arrow[dl, "\Delta"] \\
 & C &
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
C \bigotimes C \arrow[d, "\epsilon \otimes \mathrm{id}"'] \\
k \bigotimes C = C \arrow[u, bend right, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \bigotimes C \arrow[d, "\mathrm{id} \otimes \epsilon"'] \\
C \bigotimes k = C \arrow[u, bend right, "\Delta"']
\end{tikzcd}
$$

(b)

Figure I.2: A coalgebra has the structure obtained by reversing the arrows in the diagrams Fig.(I.2.4) characterizing an algebra. (a) Coassiciativity condition diagram commutes. (b) Counit condition.

3. Bialgebra

A bialgebra is a vector space A with the structure of an algebra (m, η) and the structure of a coalgebra (Δ, ϵ) which are compatible with each other, i.e.,

$$
\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b), & \Delta(\mathrm{id}_A) &= \mathrm{id}_A \otimes \mathrm{id}_A, & \text{(I.46)} \\
\epsilon(ab) &= \epsilon(a)\epsilon(b), & \epsilon(\mathrm{id}_A) &= 1_k, & \text{(I.47)}
\end{aligned}
$$

for all $g, h \in A$. These ensure that the comultplication Δ and counit ϵ are consistent with the multiplication m:

3.1. Algebra homomorphism condition for Δ:

$\Delta(ab) = \Delta(a)\Delta(b)$

Firstly $\Delta(ab)$

$$
\Delta(ab) = \Delta \circ m \circ (a \otimes b) \tag{I.48}
$$

Next in Sweedler notation $\Delta(a)\Delta(b) = \sum_{a,b} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}$

$$
(\Delta \otimes \Delta) \circ (a \otimes b) = (\sum_a a_{(1)} \otimes a_{(2)}) \otimes (\sum_b b_{(1)} \otimes b_{(2)}) \tag{I.49}
$$

$$
\begin{aligned}
(\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \quad \circ & \quad (\sum_a a_{(1)} \otimes a_{(2)}) \otimes (\sum_b b_{(1)} \otimes b_{(2)}) \\
= & \quad \sum_a \sum_b (a_{(1)} \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}) \qquad \text{(I.50)}
\end{aligned}
$$

$$(m \otimes m) \circ \sum_a \sum_b (a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}) = \sum_a \sum_b (a_{(1)} a_{(2)}) \otimes (a_{(2)} b_{(2)})) \tag{I.51}$$

$$\Delta(a)\Delta(b) = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta) \circ (a \otimes b) \tag{I.52}$$

The algebra homomorphism condition (I.46) in terms of m and Δ is

$$\Delta \circ m \circ (a \otimes b) = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta) \circ (a \otimes b). \tag{I.53}$$

3.2. Algebra homomorphism condition for ϵ:

The algebra homomorphism condition (I.47) $\epsilon(ab) = \epsilon(a)\epsilon(b)$ in terms of m and ϵ is straightforward

$$\epsilon \circ m \circ (a \otimes b) = (\epsilon \otimes \epsilon) \circ (a \otimes b). \tag{I.54}$$

Summerizing, a bialgebra is a quadraple of objects $(m, \eta, \Delta, \epsilon)$ which satisfy all of the commutative diagrams as well as the compatibility conditions (I.46) and (I.47).

4. Hopf algebra

There is no inverse operation

$$\mathrm{Inv}(a) \cdot a = a \cdot \mathrm{Inv}(a) = 1 \tag{I.55}$$

Replace $a \otimes a$ with the coproduct on a, and 1 with the counit, $\eta(a)$. We name S as the generalization of the inverse, Inv. So we have

$$m(S \otimes \mathrm{id})\Delta(a) = \eta(a). \tag{I.56}$$

An operation $S : A \to A$ satisfying the above equation, and also

$$m(\mathrm{id} \otimes S)\Delta(a) = \eta(a), \tag{I.57}$$

is called an *antipode*.

• **Hopf algebra.** Hopf algebra has an antipode S.

linear antipode $S : H \to H$

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{I.58}$$

the antipode is a kind of inverse.

In terms of its elements

$$\begin{aligned} \sum a_{(1)} S a_{(2)} &= \eta(a) \text{ and} & \text{(I.59)} \\ \sum S a_{(1)} a_{(2)} &= \eta(a). & \text{(I.60)} \end{aligned}$$

$$\begin{tikzcd} H \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & \mathbf{M} \arrow[r, "\eta"] & H \\ H \bigotimes H \arrow[rr, "\mathrm{id} \otimes S,\ S \otimes \mathrm{id}"] & & H \bigotimes H \arrow[u, "m"'] \end{tikzcd}$$

Figure I.3: links.

Uniqueness of the antipode. Let S, S' be two antipodes on a bialgebra, we show the antipode is unique by demonstrating that $S'a = Sa$ for all $a \in A$.

$$\begin{aligned} S'a &= S'a_{(1)} \eta(a_{(2)}) && \text{by counity } (I.44) \\ &= S'a_{(1)} a_{(2)(1)} S a_{(2)(2)} && \text{by antipode } (I.59) \\ &= S'a_{(1)(1)} a_{(2)(1)} S a_{(2)} && \text{by } m(m \otimes S) \text{ applied to both sides of } (I.41) \\ &= \eta(a_{(1)}) S a_{(2)} && \text{by antipode } (I.60) \text{ used on } S' \\ &= Sa && \text{by counity } (I.45). & \text{(I.61)} \end{aligned}$$

We now collect all the conditions (I.6), (I.7), (I.53)

Definition of Hopf algebras

A Hopf algebra is a vector space A over k with antipode S

a multiplication,	$m : A \otimes A \to A,$
a comultiplication,	$\Delta : A \to A \otimes A,$
a unit,	$\imath : k \to A,$
a counit,	$\epsilon : A \to k,$
an antipode,	$S : A \to A,$

such that

(1) m is associative,
$$m \circ (\mathrm{id}_A \otimes m) = m \circ (m \otimes \mathrm{id}_A), \tag{I.62}$$
(2) Δ is coassociative,
$$(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta, \tag{I.63}$$
(3) unit condition,
$$m \circ (\mathrm{id}_A \otimes \imath) = m \circ (\imath \otimes \mathrm{id}_A) = \mathrm{id}_A, \tag{I.64}$$
(4) counit condition,
$$m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta = m \circ (\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A, \tag{I.65}$$
(5) Δ is an algebra homomorphism,
$$\Delta \circ m = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta), \tag{I.66}$$
(6) ϵ is an algebra homomorphism,
$$\epsilon \circ m = \epsilon \otimes \epsilon, \tag{I.67}$$
(7) antipode condition
$$m \circ (\mathrm{id}_A \otimes S) \circ \Delta = m(S \otimes \mathrm{id}_A) \circ \Delta = \imath \circ \epsilon. \tag{I.68}$$

I.2.5 Hopf Algebras from Groups

Let G be a finite group, and $A = Fun(G)$ be a set of functions from G to the complex numbers $\mathbb{C}$. $A = Fun(G)$ is an algebra over $\mathbb{C}$ with the usual sum and product

$$(f+h)(g) = f(g) + h(g), \quad (f \cdot h)(g) = f(g)h(g), \quad (\lambda f)(g) = \lambda f(g),$$

for $f, h \in Fun(G)$, $g \in G$, $\lambda \in \mathbb{C}$. The unit of this algebra is I, defined by $I(g) = 1$, for all $g \in G$. Using the group structure of G (multiplication map, existence of the unit element and inverse element), we can introduce on $Fun(G)$ three other linear maps, the coproduct Δ, the counit ϵ, and the coinverse (antipode) S:

$$\begin{aligned}
\Delta(f)(g, g') &\equiv f(gg'), \quad \Delta : Fun(G) \to Fun(G) \otimes Fun(G) && \text{(I.69)}\\
\epsilon(f) &\equiv f(1_G), \quad \epsilon : Fun(G) \to \mathbb{C} && \text{(I.70)}\\
(Sf)(g) &\equiv f(g^{-1}), \quad S : Fun(G) \to Fun(G) && \text{(I.71)}
\end{aligned}$$

where 1_G is the unit of G.

In general a coproduct can be expanded on $Fun(G) \to Fun(G) \otimes Fun(G)$ as:

$$\Delta(f) = \sum_i f_1^i \otimes f_2^i \equiv f_1 \otimes f_2, \tag{I.72}$$

where $f_1^i, f_2^i \in A = Fun(G)$ and $f_1 \otimes f_2$ is shorthand notation. Thus we have:

$$\Delta(f)(g, g') = (f_1 \otimes f_2)(g, g') = f_1(g) f_2(g') = f(gg'). \tag{I.73}$$

It is not difficult to verify the following properties of the co-structures:

$$\begin{aligned}
(id \otimes \Delta)\Delta &= (\Delta \otimes id)\Delta & \text{(I.74)}\\
(id \otimes \epsilon)\Delta(a) &= (\epsilon \otimes id)\Delta(a) = a & \text{(I.75)}\\
m(S \otimes id)\Delta(a) &= m(id \otimes S)\Delta(a) = m(id \otimes S) = \epsilon(a)I & \text{(I.76)}
\end{aligned}$$

and

$$\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b), \quad \Delta(I) = I \otimes I, & \text{(I.77)}\\
\epsilon(ab) &= \epsilon(a)\epsilon(b), \quad \epsilon(I) = 1, & \text{(I.78)}\\
S(ab) &= S(b)S(a), \quad S(I) = I & \text{(I.79)}
\end{aligned}$$

where $a, b \in A = Fun(G)$ and m is the multiplication map $m(a \otimes b) \equiv ab$. The product in $\Delta(a)\Delta(b)$ is the product in $A \otimes A : (a \otimes b)(c \otimes d) = ab \otimes cd$.

Proof:

(i)

$$\begin{aligned}
(id \otimes \Delta)\Delta(f)(g, g', g'') &= f(g) \otimes \Delta(f)(g', g'')\\
&= f(g) \otimes f(g'g'')\\
&= f(gg'g'')\\
&= f(gg') \otimes f(g'')\\
&= (\Delta \otimes id)\Delta(f)(g, g', g'') & \text{(I.80)}
\end{aligned}$$

$f_1 \otimes f_{21} \otimes f_{22} = f_{11} \otimes f_{12} \otimes f_2$

(ii)

$$
\begin{aligned}
(id \otimes \epsilon)\Delta(f)(g) &= f_1(g)\epsilon(f_2) \\
&= f_1(g)f_2(1_G) \\
&= f(g1_G) = f(g) \\
&= f(1_G g) \\
&= \epsilon(f_1)f_2(g) \\
&= (\epsilon \otimes id)\Delta(f)(g)
\end{aligned} \tag{I.81}
$$

(iii)

$$
\begin{aligned}
m(S \otimes id)\Delta(f)(g) &= m(Sf_1 \otimes f_2) \\
&= f_1(g^{-1})f_2(g) \\
&= f(g^{-1}g) \\
&= f(1_G) = \epsilon(f) \\
&= f(gg^{-1}) \\
&= m(id \otimes S)\Delta(f)(g)
\end{aligned} \tag{I.82}
$$

(iv)

$$
\begin{aligned}
\Delta(fh)(g, g') &= (fh)(gg') \\
&= f(gg')h(gg') \\
&= \Delta(f)(g, g')\Delta(h)(g, g')
\end{aligned} \tag{I.83}
$$

(v)

$$
\begin{aligned}
\epsilon(fh) &= (fh)(1_G) \\
&= f(1_G)h(1_G) \\
&= \epsilon(f)\epsilon(h)
\end{aligned} \tag{I.84}
$$

(vi)

$$
\begin{aligned}
(Sfh)(g) &= (fh)(g^{-1}) \\
&= h(g^{-1})f(g^{-1}) \\
&= (Sh)(Sf)
\end{aligned} \tag{I.85}
$$

I.2.6 Example Where G is a Group of Matrices

We define $A = Fun(G)$ to be the algerba of polynomials in the matrix elements ${T^a}_b$ of this representation of G and $\det T^{-1}$.

$$\Delta({T^a}_b)(g, g') = {T^a}_b(gg') = {T^a}_b(gg') = {T^a}_c(g){T^c}_b(g'), \tag{I.86}$$

Therefore

$$\Delta({T^a}_b) = {T^a}_c \otimes {T^c}_b. \tag{I.87}$$

using (I.70) and (I.71), one finds:

$$\begin{aligned} \epsilon({T^a}_b) &= \delta^a_b & \text{(I.88)} \\ S({T^a}_b) &= {(T^{-1})^a}_b. & \text{(I.89)} \end{aligned}$$

Thus the algebra $A = Fun(G)$ of polynomials in the elements ${T^a}_b$ and $\det T^{-1}$ is a Hopf algebra with co-structures defined by (I.87)-(I.89) and (I.77)-(I.79).

Equivalence of the information contained in the group and the Hopf algebra

Noncommutative deformation, $Fun_q(G)$, of $Fun(G)$

Noncommutative deformation $Fun_q(G)$ of $Fun(G)$. Since G is a group, $Fun(G)$ is a Hopf algerba; the noncommutative deformation of $Fun(G)$ is usually calleda Quantum group. The term stems from the fact that the deformation is obtained by quantizing a Poisson structure of the algebra ([??]).

I.3 Duality

for all $\psi, \phi \in A'$ and $a, b \in A$

$$\begin{aligned} < \psi\phi, a > = < \psi \otimes \phi, \Delta(a) >, &\qquad < I, a >= \epsilon(a) & \text{(I.90)} \\ < \Delta(\psi), a \otimes b > = < \psi, ab >, &\qquad \epsilon(\psi) =< \psi, I > & \text{(I.91)} \\ < S(\psi), a > &= < \psi, S(a) > & \text{(I.92)} \end{aligned}$$

where $< \psi \otimes \phi, a \otimes b > \equiv < \psi, a >< \phi, b >$.

Example:

generators of the group vector space e_i and the generators of the dual space e^i.

$$m(g_1 \otimes g_2) := g_1 \cdot g_2. \tag{I.93}$$

If we consider its action on the generator of the group

$$m(e_i \otimes e_j) = m_{ij}^k e_k. \tag{I.94}$$

η sends every number to the same number times the identity element, i.e., $\mathbf{e}$ $\eta : \mathbb{R} \to \mathbb{R}(G)$

$$\eta(\lambda) = \lambda \mathbf{e} \tag{I.95}$$

The **antipode** - $S : \mathbb{R}(G) \to \mathbb{R}(G)$, $g \mapsto S(g)g^{-1}$

$$\begin{aligned} < m(g_1 \otimes g_2), f > &= =< g_1 \otimes g_2, \Delta(f) > \\ &= < g_1 \otimes g_2, f_1 \otimes f_2 > \\ &= < g_1 f_{(1)} >< g_2 f_{(2)} > . \end{aligned} \tag{I.96}$$

the coproduct map: Δ

$$(\lambda_1 g_1 + \lambda_2 g_2) \cdot g_3 = \lambda_1 g_1 \cdot g_3 + \lambda_2 g_2 \cdot g_3 \tag{I.97}$$

group algebra $\mathbb{R}(G)$.

As this has the structure of a vector space we may form a dual vector space via some inner product Fun(G) of functions on G. This bilinear inner product will take values from $< ., . >$: $\mathbb{R}(G) \otimes$Fun$(G) \to \mathbb{R}$. A natural choice is simply as functions on the group space $< g, f > := f(g)$.

given the inner product between these two vector spaces and given an operator on one of them we can define its dual action, that is its adjoint, acting on the other.

generators of the group vector space e_i and the generators of the dual space e^i.

$$m(g_1 \otimes g_2) := g_1 \cdot g_2. \tag{I.98}$$

Since these identities hold for arbitrary elements a, b, we will conclude that A' is a Hopf algerba.

coassociativity of Δ corresponds to associativity of the multiplication in the dual and similarly associativity to coassociativity in the dual.

coalgebra

$$\begin{aligned}
< (id \otimes \Delta)\Delta(\psi), a \otimes b \otimes c > &= < \psi_1 \otimes \Delta(\psi_2), a \otimes b \otimes c > \\
&= < \psi_1 \otimes \psi_2, a \otimes bc > \\
&= < \psi_1 \psi_2, a(bc) > = < \psi_1 \psi_2, (ab)c > \\
&= < \psi_1 \otimes \psi_2, ab \otimes c > \\
&= < \Delta(\psi_1) \otimes \psi_2, a \otimes b \otimes c > \\
&= < (\Delta \otimes id)\Delta(\psi), a \otimes b \otimes c > \qquad \text{(I.99)}
\end{aligned}$$

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$$

algebra

$$\begin{aligned}
< \psi(\phi\varphi), a > &= < \psi \otimes \phi\varphi, \Delta(a) > \\
&= < \psi \otimes \phi\varphi, a_1 \otimes a_2 > \\
&= < \psi \otimes \phi \otimes \varphi, a_1 \otimes \Delta(a_2) > \\
&= < \psi \otimes \phi \otimes \varphi, \Delta(a_1) \otimes a_2 > \\
&= < \psi\phi \otimes \varphi, a_1 \otimes a_2 > \\
&= < (\psi\phi)\varphi, a > \qquad \text{(I.100)}
\end{aligned}$$

$$\psi(\phi\varphi) = (\psi\phi)\varphi$$

Check:

(i)

$$\begin{aligned}
< \Delta(\psi\phi), a \otimes b > &= < \psi\phi, ab > \\
&= < \psi \otimes \phi, \Delta(ab) > \\
&= < \psi \otimes \phi, \Delta(a)\Delta(b) > \\
&= < \psi_1 \otimes \phi_1 \otimes \psi_2 \otimes \phi_2, \Delta(a) \otimes \Delta(b) > \\
&= < \Delta(\psi)\Delta(\phi), a \otimes b > \qquad \text{(I.101)}
\end{aligned}$$

as required. .... Implying

$$\Delta(\psi\phi) = \Delta(\psi)\Delta(\phi)$$

(ii)

$$\begin{aligned}
< (S\psi_1)\psi_2, a > &= < S\psi_1 \otimes \psi_2, a_1 \otimes a_2 > \\
&= < \psi_1 \otimes \psi_2, Sa_1 \otimes a_2 > \\
&= < \psi, (Sa_1)a_2 > \\
&= < \psi, I > \epsilon(a) \\
&= \epsilon(\psi)\epsilon(a) \\
&= < I\epsilon(\psi), a >
\end{aligned} \tag{I.102}$$

Implying

$$(S\psi_1)\psi_2 = I\epsilon(\psi)$$

(iii)

$$\begin{aligned}
< \Delta(I), a \otimes b > &= < I, ab > \\
&= < I, a >< I, b > \\
&= < I \otimes I, a \otimes b >
\end{aligned} \tag{I.103}$$

$$\Delta(I) = I \otimes I.$$

(iv)

$$\begin{aligned}
\epsilon(\psi\phi) &= < \psi\phi, I > \\
&= < \psi \otimes \phi, I \otimes I > \\
&= < \psi, I >< \phi, I > \\
&= \epsilon(\psi)\epsilon(\phi)
\end{aligned} \tag{I.104}$$

□

A subspace $I \subseteq C$ is a **coideal** if $\Delta \subseteq I \otimes C + C \otimes I$ and if $\epsilon(I) = 0$.

if I is a coideal then, the quotient C/I is a coalgebra with comultiplication induced from Δ.

I.3.1 Hopf algebra from a Lie group: Universal envelopping algebra

First account

The universal enveloping algebra $U(g)$ of a Lie algebra g, i.e. the algebra, with unit I, of polynomials in the generators χ_i modulo the commutation relations

$$[\chi_i, \chi_j] = C_{ij}^{\ \ k}\chi_k. \tag{I.105}$$

We define the co-structures on the generators as:

$$\begin{aligned}
\Delta(\chi_i) &= \chi_i \otimes I + I \otimes \chi_i, \quad && \Delta(I) = I \otimes I && \text{(I.106)}\\
\epsilon(\chi_i) &= 0, && \epsilon(I) = 1 && \text{(I.107)}\\
S(\chi_i) &= -\chi_i, && S(I) = I && \text{(I.108)}
\end{aligned}$$

and extend them to all $U(g)$ by requiring Δ and ϵ to be linear and multiplicative, $\Delta(\chi\chi') = \Delta(\chi)\Delta(\chi')$, $\epsilon(\chi\chi') = \epsilon(\chi)\epsilon(\chi')$, while S is linear and antimultiplicative. We have to check that the maps Δ, ϵ, S are well defined. Since $[\chi, \chi']$ is linear in the generators we have

$$\Delta[\chi, \chi'] = [\chi, \chi'] \otimes I + I \otimes [\chi, \chi'],$$

at the same time, using that Δ is multiplicative we have

$$\begin{aligned}
\Delta[\chi, \chi'] &= \Delta(\chi)\Delta(\chi') - \Delta(\chi')\Delta(\chi)\\
&= \chi\chi' \otimes I + \chi \otimes \chi' + \chi' \otimes \chi + I \otimes \chi\chi' - (\chi \leftrightarrow \chi')\\
&= [\chi, \chi'] \otimes I + I \otimes [\chi, \chi'] && \text{(I.109)}
\end{aligned}$$

Second account

The coproduct, counit, and antipode are

$$\Delta a = a \otimes 1 + 1 \otimes a, \qquad \epsilon a = 0, \qquad Sa = -a \tag{I.110}$$

Δ algebra map i.e., $\Delta(ab) = \Delta(a)\Delta(b)$, S antialgebra map i.e., $S(ab) = S(b)S(a)$.

$[a, b]$ is a member of the algebra, for Δ to be consitant we must have $\Delta[a, b] = [a, b] \otimes 1 + 1 \otimes [a, b]$. We can easily check this:

$$\begin{aligned}\Delta[a,b] &= \Delta(ab)-\Delta(ba)\\ &= (a\otimes 1+1\otimes a)(b\otimes 1+1\otimes b)-a\leftrightarrow b\\ &= (ab\otimes 1+1\otimes ba+a\otimes b+b\otimes a)-a\leftrightarrow b\\ &= [a,b]\otimes 1+1\otimes[b,a].\end{aligned} \tag{I.111}$$

If S is to be a consistent we must have $S([a,b])=-[a,b]$, we check this:

$$S([a,b])=S(ab)-S(ba)=(-b)(-a)-(-a)(-b)=-[a,b] \tag{I.112}$$

An element of a general Hopf algebra which has this linear form

$$\Delta a=a\otimes 1+1\otimes a \tag{I.113}$$

is called **primative**.

The Hopf algerbra $U(g)$ is noncommutative, $\xi\xi'\neq\xi'\xi$ but is cocommutative, i.e. for all $\xi\in U(g)$, $\xi_1\otimes\xi_2=\xi_2\otimes\xi_1$. This follows from the modulo commutation property (I.105):

$$\begin{aligned}\xi_1\otimes\xi_2 &= (\xi_1+[\xi_2,\xi_1])\otimes(\xi_2+[\xi_1,\xi_2])\\ &= \xi_2\otimes\xi_1.\end{aligned} \tag{I.114}$$

Tensor product universal enveloping algebra

tensor Hopf algebra

$$\mathcal{T}(V)=k\oplus V\oplus(V\otimes V)\oplus\dots \tag{I.115}$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the quotient of $\mathcal{T}(\mathfrak{g})$ modulo the ideal generated by $\xi\otimes\eta-\eta\otimes\xi-[\xi,\eta]$.

I.3.2 Quasi-Triangular Hopf Algebras

the map τ

$$\begin{aligned}\tau:\quad & A\otimes A\to A\otimes A\\ & a\otimes b\mapsto b\otimes b.\end{aligned} \tag{I.116}$$

$$\tau(a \otimes b) = b \otimes a. \tag{I.117}$$

Let $\Delta^{op} = \tau \circ \Delta^{op}$ so that, if $a \in A$ and

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}, \quad \text{then} \quad \Delta^{op}(a) = \sum_a a_{(2)} \otimes a_{(1)}. \tag{I.118}$$

Then $(A, m, \Delta^{op}, \epsilon, \imath, S^{-1})$ forms a Hopf algebra (verify).

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \tag{I.119}$$

$$\begin{aligned}(\mathbf{1} \otimes \Delta)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12} = \sum_i A_i A_j \otimes B_j \otimes B_i \\ (\Delta \otimes \mathbf{1})\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23} = \sum_i A_i \otimes A_j \otimes B_i B_j\end{aligned} \tag{I.120}$$

$$(S \otimes \mathbf{1}) = (\mathbf{1} \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1} \tag{I.121}$$

$$\begin{aligned}\mathcal{R}_{12} &= \sum_i A_i \otimes B_i \otimes \mathbf{1} \\ \mathcal{R}_{13} &= \sum_i A_i \otimes \mathbf{1} \otimes B_i \\ \mathcal{R}_{23} &= \sum_i \mathbf{1} \otimes A_i \otimes B_i\end{aligned} \tag{I.122}$$

I.3.3 Drinfeld's Quantum Double

As this has the structure of a vector space we may form a dual vector space via some inner product A of functions on A. This bilinear inner product will take values from $< .,. >$: $A^* \otimes A \to k$. A natural choice is simply as functions on the vector space $\alpha(a)$. Let $A^* = Hom_k(A, k)$ to be the dual of A.

$$D(A) = A \otimes A^{*coop} \tag{I.123}$$

elements $a \otimes \alpha$

$$< \alpha, a >:= \alpha(a) \tag{I.124}$$

$$< \alpha_1 \otimes \alpha_2, a_1 \otimes a_2 >=< \alpha_1, a_1 >< \alpha_2, a_2 > . \tag{I.125}$$

defining multiplication and comultiplication

$$< m^{op}(\alpha_1 \otimes \alpha_2), a >=< \alpha_1 \otimes \alpha_2, \Delta(a) > \quad \text{and} \quad < \Delta^{op}(\alpha), a_1 \otimes a_2 >=< \alpha, m(a_1 \otimes a_2) > \tag{I.126}$$

in components

$$\begin{aligned} e_i e_j &= m_{ij}^k e_k, & \Delta(e_k) &= \mu_k^{ij} e_i \otimes e_j \\ e^i e^j &= \mu_k^{ij} e^k, & \Delta(e^k) &= m_{ij}^k e^i \otimes e^j \end{aligned} \tag{I.127}$$

lower and upper indexed algebras are dual as bialgebras, the comultiplication of one being the multiplication of the other.

$\mathcal{R}-$matrix

$$\mathcal{R} = \sum_i b_i \otimes b^i, \tag{I.128}$$

comultiplication in $D(A)$,

$$\Delta(\alpha a) = \Delta(\alpha)\Delta(a) = \sum_{a,\alpha} a_{(1)}\alpha_{(1)} \otimes a_{(2)}\alpha_{(2)} \tag{I.129}$$

Theorem I.3.1 *Let there be a finite dimensional Hopf algebra over k and let A'^{coop} be the Hopf algerba $A^* = Hom_k(A, k)$ with opposite comultiplication. Then there exists a unique quasi-triangular Hopf algebra $(D(A), \mathcal{R})$ given by*

The $k-$linear map

$$A \otimes A^{\star coop} \to D(A)$$

$$a \otimes \alpha \mapsto a\alpha$$

is bijective.

$D(A)$ contains A and $A^{\star coop}$ as Hopf subalgebras.

The element $\mathcal{R} \in D(A) \otimes D(A)$ is given by

$$\mathcal{R} = \sum_i b_i \otimes b^i,$$

where $\{b_i\}$ is a basis of A and $\{b^i\}$ is dual basis in A'^{coop}.

Example with a finite group

Let G be a finite group and we assume that A is the group algebra of G i.e. $A = \mathbb{C}[G]$. As a result $A^* = F(G)$, the algebra of functions on G, and the quantum double

$$D(\mathbb{C}[G]) = \mathbb{C}[G] \otimes F(G)^{op}$$

is usually denoted $D(G)$. A basis of $D(G)$ is $(x \otimes \delta_g)_{x,g \in G}$ and the Hopf algerba and coalgebra structures are respectively given by:

$$\begin{aligned} (x \otimes \delta_g) \cdot (y \otimes \delta_h) &= xy \otimes \delta_h(y^{-1}gy)\delta_h && \text{(I.130)} \\ \Delta(x \otimes \delta_g) &= \sum_{g_1, g_2 | g_2 g_2 = H} (x \otimes g_2) \otimes (x \otimes g_1). && \text{(I.131)} \end{aligned}$$

The action of the antipode on the previous basis is givem by

$$S(x \otimes \delta_g) = x^{-1} \otimes \delta_{x^{-1}g^{-1}x}$$

and the counit is defined by

$$\epsilon(x \otimes \delta g) = \delta(g).$$

The R matrix is given by

$$R = \sum_{x,g} (x \otimes 1) \otimes (1 \otimes \delta_g).$$

Its elements are denoted $(u, \vec{a}) \in SU(2) \times \mathbb{R}^3$

$$(u_1, \vec{a}_1) \cdot (u_2, \vec{a}_2) = (u_1 u_2, u_1 \vec{a}_2 + \vec{a}_1), \tag{I.132}$$

The notation $u\vec{a}$ holds for the action of the vectorial representation of u on the vector $\vec{a}$.

$$ISU(2) = SU(2) \otimes_S \mathbb{R}^3 \tag{I.133}$$

The Drinfeld double $DSU(2)$

1st defifintion:

The Drinfeld double of a Hopf algebra A, for the case of the compact group $SU(2)$, i.e. where $A = \mathbb{C}[SU(2)]$.The Drinfeld double $DSU(2) = \mathbb{C}[SU(2)] \otimes F(SU(2))^{op}$ is a Hopf algebra whose definition given below. In particular, it is a quasi-triangular and admits the group algerba $\mathbb{C}[SU(2)]$ and the algerba of functions $F(SU(2))^{op}$ (with opposite co-product) as sub-Hopf algera.

Hopf algebra structure of $\mathbb{C}[SU(2)]$ is defined by the group law and the following co-algerba relations:

$$\Delta(x) = x \otimes x, \quad S(x) = x^{-1}, \quad \epsilon(x) = \delta(x), \quad \text{for all } x \in SU(2). \tag{I.134}$$

where δ is the delta distribution on the group $SU(2)$ with respect to the $SU(2)$ Haar measure.

The algebra of functions $F(SU(2))$ is a commutaive and its coalgerba structure is given by:

$$\Delta(f)(x, y) = f(xy), \quad S(f)(x) = f(x^{-1}), \quad \epsilon(f) = f(1),$$

where f is a function and $x, y \in G$.

2nd defintion:

The quantum double $DSU(2)$ is defined as a coalgerba as the tensor product of $SU(2)$ with $Fun(SU(2))$ where $Fun(SU(2))$ is a suitable set of functions on $SU(2)$. Its elements are usually denoted $(f, u) \in Fun(SU(2)) \times SU(2)$ in terms of which the algerba is given by

$$(f_1, u_1) \cdot (f_2, u_2) = (f_1 f_2 \circ ad_{u_1}, u_1 u_2), \tag{I.135}$$

and co-algebra by

$$\Delta(f,u) = \sum_{(f)} (f_{(1)}, u) \otimes (f_{(2)}, u). \tag{I.136}$$

where $f, g \in F(SU(2))$, $x, y, a, b \in SU(2)$. The action of the antipode reads

$$S(x \otimes f)(a) = x^{-1} \otimes f(x^{-1}a^{-1}x) \tag{I.137}$$

for any $a \in G$ and the co-unit is simply given by

$$\epsilon(x \otimes f) = f(1).$$

I.4 Introduction to Quantum Groups

An effective field theory space-time coordinates themselves are noncommutative.

q deformation of 2-plane.

$$xy = yx \tag{I.138}$$

$$\begin{aligned} [x, y] &= 0, \quad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0, \quad \left[\frac{\partial}{\partial x}, y\right] = 0, \quad \left[\frac{\partial}{\partial y}, x\right] = 0 \\ &\left[\frac{\partial}{\partial x}, x\right] = 1, \quad \left[\frac{\partial}{\partial y}, y\right] = 1. \end{aligned} \tag{I.139}$$

$$\begin{aligned} XY &= qYX, \quad \frac{\partial}{\partial X}\frac{\partial}{\partial Y} = q^{-1}\frac{\partial}{\partial X}\frac{\partial}{\partial Y}, \\ \frac{\partial}{\partial X}Y &= qY\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial Y}X = qX\frac{\partial}{\partial Y}, \\ \frac{\partial}{\partial X}X &- q^2X\frac{\partial}{\partial X} = 1 + (q^2-1)Y\frac{\partial}{\partial Y} \end{aligned} \tag{I.140}$$

$$\begin{aligned} AB = qBA, \quad & CD = qDC, \quad AC = qCA, \quad BD = qDB, \\ BC = CB, \quad & AD - DA = (q - q^{-1})BC, \end{aligned} \tag{I.141}$$

and

$$AD - qBC = \det_q T = 1. \tag{I.142}$$

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q^{-1}} \\ [n]_q! &= [n]_q[n-1]_q[n-2]_q \ldots [2]_q[1]_q, \quad n = 1, 2, \ldots, \quad [0]_q! := 1 \end{aligned} \tag{I.143}$$

$$[[n]]_q! = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{I.144}$$

$$\begin{aligned} \Delta(A) &= A \otimes A + B \otimes C, \quad \Delta(B) = A \otimes B + B \otimes D, \\ \Delta(C) &= C \otimes A + D \otimes C, \quad \Delta(D) = C \otimes B + D \otimes D. \end{aligned} \tag{I.145}$$

I.4.1 Q-Deformation of any Classical Lie Algebra

deform any classical Lie algebra. The initial Hopf algebra is the universal enveloping algebra of the Borel subalgebra of a Lie algebra.

$U(sl(2))$, the universal enveloing algebra of $sl(2)$

The classical Lie algebra is generated by X_+, X_- and H, with commutators

$$[X_+, X_-] = H, \quad [H, X_\pm] = \pm 2X_\pm \tag{I.146}$$

$$\Delta(J) = J \otimes \mathbf{1} + \mathbf{1} \otimes J, \quad J = X_\mp, H. \tag{I.147}$$

The antipode

$$\begin{gathered} \gamma(X_\pm) = -X_\pm, \quad \gamma(H) = -H, \quad \gamma(\mathbf{1}) = \mathbf{1}, \\ \epsilon(X_\pm) = \epsilon(H) = 0, \quad \epsilon(\mathbf{1}) = 1. \end{gathered} \tag{I.148}$$

$$\begin{gathered} e_a e_b = m^c_{ab}\, e_c, \quad \Delta(e^a) = m^a_{cb}\, e^b \otimes e^c \\ e^a e^b = \mu^{ab}_c\, e^c, \quad \Delta(e_a) = \mu^{bc}_a\, e_b \otimes e_c. \end{gathered} \tag{I.149}$$

the antipode γ

$$\gamma(e_a) = \gamma_a^b e_b, \quad e_a \text{ basis of } A, \tag{I.150}$$

$$\gamma(e^a) = (\gamma^{-1})_b^a e^b, \quad e^b \text{ basis of } A^*. \tag{I.151}$$

$$[X_+, X_-] = [H]_q = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad [H, X_\pm] = \pm 2x_\pm \tag{I.152}$$

Example. Let G be a compact group Lie group. Denote the set of algebraic functions (These are functions that are obtained as matrix elements of finite dimensional representations) on G by $\mathcal{C}(G)$. Then $\mathcal{C}(G)$ forms a commutative Hopf algebra as follows:

$$\begin{array}{ccc} & \mathbf{1}(g) = \mathbf{1}, & (f \cdot g)(g) = f(g)h(g), \\ \epsilon\, f = f(e), & \Delta f(g, g') = f(gg'), & (Sf)(g) = f(g^{-1}) \end{array}$$

where $f, h \in \mathcal{C}(G)$ and $g, g' \in G$ and e denotes the unit element in G. For the coproduct observe the implicit isomorphism $\mathcal{C}(G) \otimes \mathcal{C}(G) \cong \mathcal{C}(G \times G)$ which holds due to the Peter-Weyl theorem:

$$G \times G = \oplus$$

given $f, h \in \mathcal{C}(G)$ there exist a unique $F \in \mathcal{C}(G \times G)$

$$f(g) \times h(g') =$$

I.4.2 Quantum Poincare Lie Algebra

quite general method to deform the Hopf algebra $U(g)$, the universal enveloping algebra of a given Lie algerba g. It is based on the twist procedure.

the usual Poincare Lie algerba $iso(3, 1)$:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), && \text{(I.153)} \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}), && \text{(I.154)} \end{aligned}$$

I.4.3 Summary of Quantum Groups

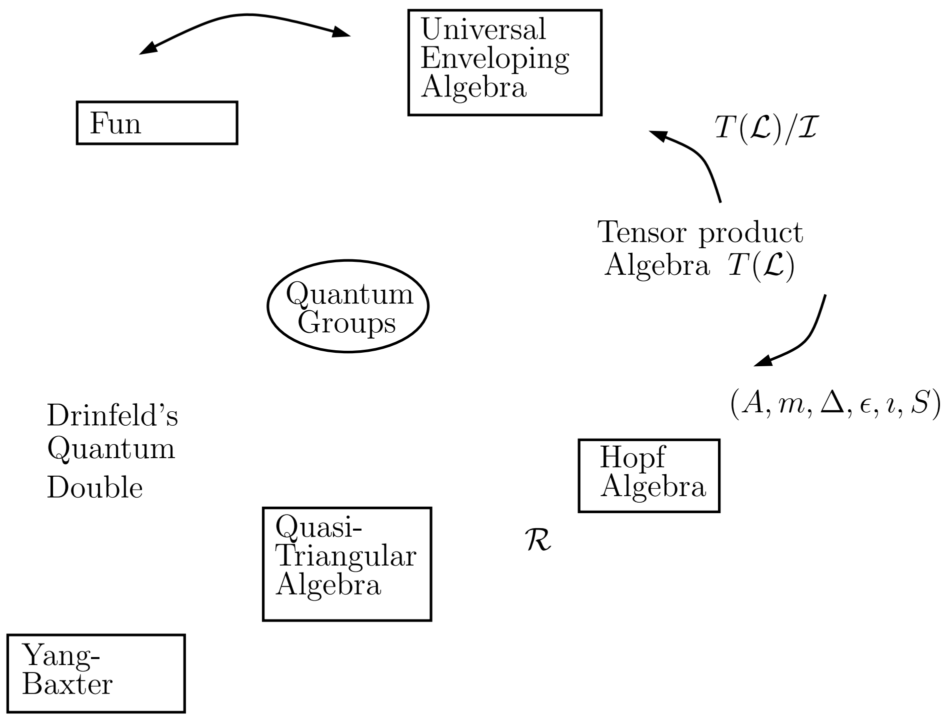


Figure I.4: $\mathcal{I}$ being the ideal generated by . (i) Deformations of commutative Hopf algerbas, of the kind $A = Fun(G)$, are related to deformations of cocommuttative Hopf algebras of the kind $U(g)$ where g is the Lie algebra of G. (ii) A non-cocommutative quasi-triangular Hopf algebra is called a quantum group. (iii)

I.5 WZW Conformal Field Theory

This section is in very early stage of development. Not really of priority.

Using: Introduction to conformal field theory by A.N. Schellekens and Quantum Groups in Two-dimensional Physics by C. Gomez et al.

Takes up the subject of knot and three-manifold invariants from Chern-Simons theory using the duality with WZW conformal field theory on a boundary, Σ, of $\mathcal{M}$.

primary fields are fields that transform as

$$\phi = \left[\frac{df}{dz}\right]^h \left[\frac{d\overline{f}}{d\overline{z}}\right]^{\overline{h}} \phi(z, \overline{z}) \tag{I.155}$$

The infinitesimal form of this transformation is

$$\delta\Phi(w,\overline{w}) = h\partial_w\epsilon(w)\Phi(w,\overline{w}) + \epsilon(w)\partial_w\Phi(w,\overline{w}) \tag{I.156}$$

where $f(z) = z + \epsilon(z)$.

Now we consider the quantum version of this transformation.

$$\delta\phi(w,\overline{w}) = [Q_\epsilon, \phi(w,\overline{w})]. \tag{I.157}$$

we use different contours in the two terms

$$[Q_\epsilon, \phi(w,\overline{w})] = \frac{1}{2\pi i}\oint_{|z|>|w|} dz\ \epsilon(z)T(z)\phi(w,\overline{w}) - \oint_{|z|<|w|} dz\ \epsilon(z)\phi(w,\overline{w})T(z) \tag{I.158}$$

$$[Q_\epsilon, \phi(w,\overline{w})] = \frac{1}{2\pi i}\left[\oint_{|z|>|w|} - \oint_{|z|<|w|}\right] dz\ \epsilon(z)R\big(T(z)\phi(w,\overline{w})\big) \tag{I.159}$$

$$[Q_\epsilon, \phi(w,\overline{w})] = \frac{1}{2\pi i}\oint_{C_w} dz\ \epsilon(z)R\big(T(z)\phi(w,\overline{w})\big) \tag{I.160}$$

where C_w is a closed integration contour encircling the point w.

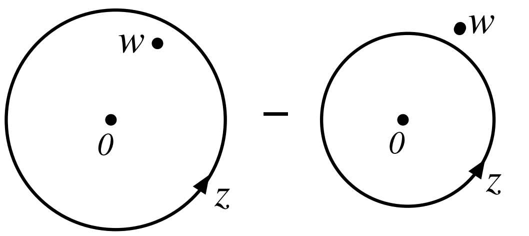


Figure I.5: contourInt. .

Recalling residue theorem:

$$\oint_C f(z)dz = 2\pi i\sum_{i=1}^{n} \mathrm{R}es[f(z_i)] \tag{I.161}$$

if $f(z)$ has an mth-order pole at z_0 it can be picked out with

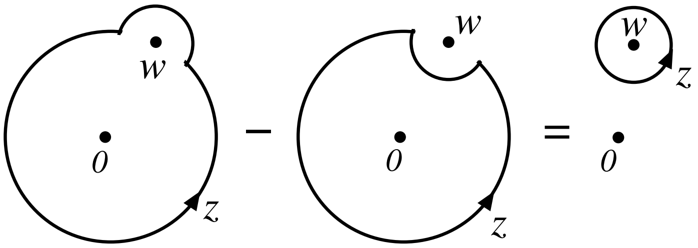


Figure I.6: contourInt2. .

$$
\mathrm{R}es[f(z_0)] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \tag{I.162}
$$

$$
R[T(z)\phi(w,\overline{w})] = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w \phi(w,\overline{w}) + \dots \tag{I.163}
$$

the other terms being regular so have no poles - these don't contribute to the integral.

I.5.1 Representations of the Virasoro Algebra

Descendants

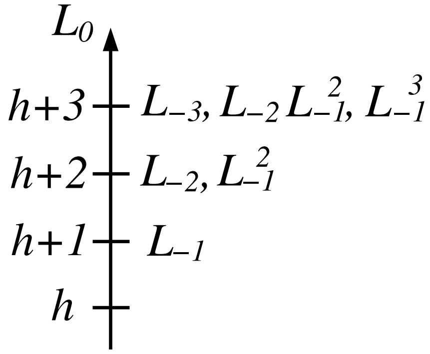


Figure I.7: decsenfig.

Irreducible representations of the Virasoro algebra is called the highest weight representations because they are correspond to a highest weight state $|h>$

$$
L_n|h> = 0 \quad n > 0, \qquad L_0|h> = h|h> \tag{I.164}
$$

The representation with highest weight $|h>$ consists of the vector space generated by $|h>$ and all descendant fields obtained from L_{-n} $n>0$ on the highest weight.

associated with each irreducible representation $(h,\overline{h})$, we can find a local quantum field $\phi(z,\overline{z})$ such that

$$|h,\overline{h}>=\phi(0,0)|0> \tag{I.165}$$

$$\begin{aligned}[L_n,\phi(w,\overline{w})] &= \frac{1}{2\pi i}\oint z^{n+1}T(z)\phi(w,\overline{w}) \\ &= h(n+1)w^n\phi(w,\overline{w})+w^{n+1}\partial_w\phi(w,\overline{w}),\end{aligned} \tag{I.166}$$

which vanishes if $w=0$ and $n>0$.

I.5.2 Correlation Functions

Two-point functions

$$G(z_1,z_2)=<\phi(z_1)\phi(z_2)>$$

This satisfies the differential equation

$$[\epsilon(z_1)\partial_1+h_1\partial\epsilon(z_1)+\epsilon(z_2)\partial_2+h_2\partial\epsilon(z_1)]G(z_1,z_2)=0,$$

for $\epsilon=1$ this gives

$$(\partial_1+\partial_2)G(z_1,z_2)=0$$

which implies $G(z_1,z_2)=G(z_1-z_2)$. For $\epsilon=z$ and using $\partial_2G(z_1,z_2)=-\partial_1G(z_1,z_2)$

$$[(z_1-z_2)\partial_1+h_1+h_2]G(z_1,z_2)=0.$$

this has the solution

$$G(z_1,z_2)=(z_1-z_2)^{-h_1-h_2}.$$

for the $\epsilon=z^2$ case

$$[z_1^2\partial_1 - z_2^2\partial_2 + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) = 0$$

we find

$$(h_1 - h_2)(z_1 - z_2)G(z_1, z_2) = 0,$$

(exercise) so that h_1 must be equal to h_2, or else the propagator vanishes. So finally we have

$$G(z_1, z_2) = \begin{cases} C(z_1 - z_2)^{-2h}, & \text{when } h_1 = h_2 = h \\ 0, & \text{otherwise} \end{cases} . \tag{I.167}$$

Three-point functions

for $\epsilon = 1$ this gives

$$(\partial_1 + \partial_2 + \partial_3)G(z_1, z_2, z_3) = 0$$

and implies that G is a fucntion of $z_{12} = z_1 - z_2$, $z_{23} = z_2 - z_3$ and $z_{31} = z_3 - z_1$

$$G = G(z_{12} = z_1 - z_2, z_2 - z_3, z_3 - z_1)$$

There are really two independent variables since $z_{31} = z_{23} - z_{12}$. We retain the redundent variable to maintain symmetry. For $\epsilon = z$,

$$z_{12}\frac{\partial G}{\partial z_{12}} + z_{23}\frac{\partial G}{\partial z_{23}} + z_{31}\frac{\partial G}{\partial z_{31}} = -(h_1 + h_2 + h_3)G.$$

We can solve this using separation of variables. Set G equal to

$$G = A(z_{12})B(z_{23})C(z_{13})$$

from which

$$\frac{z_{12}}{A}\frac{\partial A}{\partial z_{12}} = a, \quad \frac{z_{23}}{B}\frac{\partial B}{\partial z_{23}} = b, \quad \frac{z_{31}}{C}\frac{\partial C}{\partial z_{31}} = c,$$

where

$$a + b + c = -(h_1 + h_2 + h_3)$$

we have then

$$G_{ijk}(z_{12}, z_{23}, z_{13}) = C_{ijk} z_{12}^a z_{23}^b z_{31}^c$$

$\epsilon = z^2$ gives

$$(a + c + 2h_1)z_1 + (a + b + 2h_2)z_2 + (b + c + 2h_3)z_3 = 0.$$

As z_1, z_2 and z_3 are independent we may put $z_1 = 1, z_2 = z_3 = 0$, $z_1 = z_3 = 0, z_2 = 1$ and $z_1 = z_2 = 0, z_3 = 1$ in turn, and obtain

$$a + c + 2h_1 = a + b + 2h_2 = b + c + 2h_3 = 0$$

$$a = h_3 - h_1 - h_2, \quad b = h_1 - h_2 - h_3, \quad c = h_3 - h_1 - h_2.$$

$$G^{(3)}_{ijk}(z_{12}, z_{23}, z_{13}) = C_{ijk} z_{12}^{h_3 - h_1 - h_2} z_{23}^{h_1 - h_2 - h_3} z_{31}^{h_2 - h_3 - h_1}.$$

I.5.3 Conformal Ward Identities

$$< 0| \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi(w_1) \ldots \phi(w_n)|0 > \tag{I.168}$$

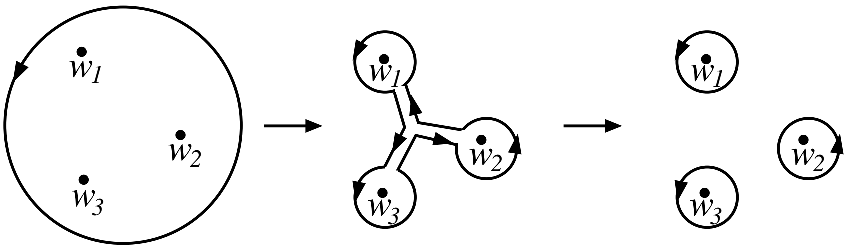


Figure I.8: ConWardfig.

$$\begin{aligned} & \sum_i < 0|\phi(w_1) \ldots \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi(w_i) \ldots \phi(w_n)|0 > \\ = \; & \sum_i < 0|\phi(w_1) \ldots \delta_\epsilon \phi(w_i) \ldots \phi(w_n)|0 > \end{aligned} \tag{I.169}$$

we get

$$\begin{aligned} &< 0|T(z)\phi(w_1)\dots\phi(w_n)|0 > \\ = &\sum_i \left[\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right] < 0|\phi(w_1)\dots\phi(w_n)|0 > \end{aligned} \tag{I.170}$$

I.5.4 Correlators of Descendants

$$< 0|\phi_1^{(-k)}(w_1)\phi_2(w_2)\dots\phi_n(w_n)|0 > \tag{I.171}$$

where $\phi_1^{(-k)}$ is the k-th descendant of ϕ.

$$\mathcal{L}^i_{-k} = -\frac{(1-k)h_i}{(w_i-w_1)^k} + \frac{1}{(w_i-w_1)^{k-1}}\frac{\partial}{\partial w_i} \tag{I.172}$$

we put

$$\mathcal{L}_{-k} := \sum_{i=2}^{n} \mathcal{L}^i_{-k}$$

we can write the correlation function as

$$< 0|\phi_1^{(-k)}(w_1)\phi_2(w_2)\dots\phi_n(w_n)|0 > = \mathcal{L}_{-k} < 0|\phi_1(w_1)\phi_2(w_2)\dots\phi_n(w_n)|0 > \tag{I.173}$$

I.5.5 Null State Decoupling

We wish to know if $\|\alpha L_{-2}|h> + \beta L_{-1}L_{-1}|h> \|$ is non positive. To this end we consider the Hermitian matrix

$$K_2 = \begin{pmatrix} < h|\, L^\dagger_{-2}L_{-2}\, |h > & < h|\, L^\dagger_{-2}L_{-1}L_{-1}\, |h > \\ < h|\, (L_{-1}L_{-1})^\dagger L_{-2}\, |h > & < h|\, (L_{-1}L_{-1})^\dagger L_{-1}L_{-1}\, |h > \end{pmatrix}, \tag{I.174}$$

noting that

$$vK_2v^T = \|\alpha L_{-2}|h> + \beta L_{-1}L_{-1}|h> \|$$

where $\vec{v} = (\alpha, \beta)$. If this matrix were to have negative or zero determinant then there must exist an eigenvector $\vec{v} = (\alpha, \beta)$ with zero or negative eigenvalue. If this were the case the quantity $\|\alpha L_{-2}|h> + \beta L_{-1}L_{-1}|h>\|$ would not be posive.

There are analogous matrices K_n for the n-*th* level. We cannot tell how many positive, zero and negative eigenvalues there are as if $\det K > 0$ there could be an even number of negative eigenvalues.

We are interested in the case of null vectors, the eigenvectors of zero norm.

I.5.6 Conformal Blocks

A two-dimensional chiral conformal field theory is a quantum field theory defined over cylindrical space time $\mathbb{R} \times S^1_R$ of radius R, with coordinates $(\tau,)$. The quantum mechanical degrees of freedom are chiral, which means that the dynamical modes of such a theory purely left-moving right-moving.

Given two representations, λ and μ, one can define their *fusion*, namely a *tensor product representation*, $\lambda \otimes \mu$, which is again a unitary representation of $\mathcal{A}$. A chiral algebra is called *rational* if and only if the number of inequivalent, irreducible unitary representations is finite. For a rational chiral algebra, the tensor product of two representations can be decomposed into a direct sum of irreduciable unitary representations.

Let $N^{\lambda_3}_{\lambda_1,\lambda_2}$ denote the multiplicity of λ_3 as a subrepresentation in the tensor product $\lambda_1 \otimes \lambda_2$. The multiplicities$N^{\lambda_3}_{\lambda_1,\lambda_2}$ are the structure constants of the ring and are called *fusion rules*; for a rational chiral algebra, they are finite non-negative integers.

Given number n of irreducible representations, $\lambda_1, \ldots, \lambda_n$, we define the linear space of *conformal blocks* as the space of invariant tensors, i.e., of invariant linear functionals, on the representation space of the tensor product representation $\lambda_1 \otimes \cdots \otimes \lambda_n$.

four-point functions

$$F_{ijkl}(z, \overline{z}) = \sum_m C_{ijm} C_{mkl} \mathcal{F}^m_{ijkl}(z) \overline{\mathcal{F}}^m_{ijkl}(\overline{z})$$

Decoupling

Conformal blocks form a basis to the space of solutions to the decoupling equations.

I.6 Classical and Quantum Logic

This difference is the following. In the classical propositional calculus the logical connectives between propositionals obey the distributive law,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{I.175}$$

while the prositional calculus for quantum mechanics $\wedge$ and $\vee$ do not obey this law but a weaker version of it. This is the modular law,

$$\text{if } a \leq c \text{ then } \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{I.176}$$

Classical logic (absolute):

'the value of A is a'

Classical logic (contextual):

'at a certain 'stage' the value of A is a'

Quantum logic:

'the state of the system is an eigenvector of the operator $\hat{\mathcal{O}}$ with eigenvalue λ'

The non commutativity of the observable operators results in a non Boolean logic for the propositions.

This is mathematically described by the orthomodular lattice structure of projectors on Hilbert space.

Quantum logic (relational or contextual):

'the state of the system S relative to another system, O, is an eigenvector of the operator $\hat{\mathcal{O}}$ with eigenvalue λ_O'

there is no meaning to the state of an isolated system.

quantum mechanical aspect and a distributive causal aspect.

I.7 Quantum Logic

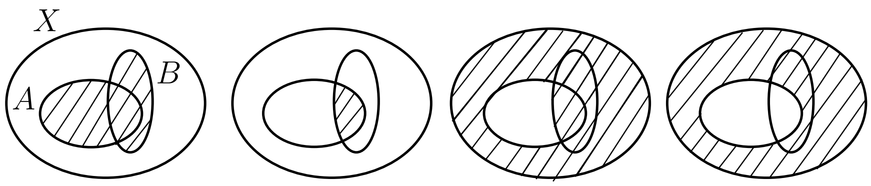
A positive answer to S implies a negative one to $\neg S$.

where the relation of inclusion and the operation of intersection between linear subspaces are the set inclusion and intersection, but where the set complement is replaced by the operation of taking the orthogonal subspace, and the union of two sets by the sum of subspaces.

propositional logic can express relationships that would be far too complicated to express in ordinary language.

x	y	$x \vee y$	$x \wedge y$	$x \Rightarrow y$	$\neg x$
1	1	1	0	1	0
1	0	1	0	0	0
0	1	1	0	1	1
0	0	0	1	1	1

Example: We define a partial relation between subsets of X: if $A \subseteq B$ then $A \leq B$. every two subsets $A, B \in X$ have their union $A \cap B$ is the least upper bound and their intersection $A \cup B$ as the greatest lower bound. Consider any other subset C which is smaller than A and B, i.e., $C \subset A$ and $C \subset B$ (i.e., $C \leq A$ and $C \leq B$).



(a) (b) (c) (d)

Figure I.9: If $A \subseteq B$ then $A \leq B$. partially ordered set of subsets of X is a lattice. (a) Their union $A \cap B$ is the l.u.b. (b) $A \cup B$ is the g.l.b.

Definition a ***lattice*** is a partially ordered set where the meet and join of any two elements always exists.

Definition a lattice is called ***complete*** if and only if the meet and join exist for arbitrary families of elements.

Lemma I.7.1 *Complete lattice always contains elements 0 and 1.*

Proof. Element 0 can be defined as the infimum of all elements of $\mathcal{L}$ and element 1 can be defined as their supremum. Both exist because of the completeness of the lattice.

□

Definition Element of the lattice is ***atomic*** if $0 \leq x$ such that there is no y that satisfies $0 \leq y \leq x$.

a join $y = \bigvee x$

The operations $\wedge$ and $\vee$ have the following properties:

$$\begin{aligned}
&\text{(a)}\ \ x \wedge x = x && \text{(b)}\ \ x \vee x = x && \text{(I.177)}\\
&\text{(a)}\ \ x \wedge y = y \wedge x && \text{(b)}\ \ x \vee y = y \vee x && \text{(I.178)}\\
&\text{(a)}\ \ x \wedge (y \wedge z) = (x \wedge y) \wedge z && \text{(b)}\ \ x \vee (y \vee z) = (x \vee y) \vee z && \text{(I.179)}\\
&\text{(a)}\ \ (x \wedge y) \vee x = x && \text{(b)}\ \ (x \vee y) \wedge x = x. && \text{(I.180)}
\end{aligned}$$

$$x \leq y \ \Leftrightarrow\ x \wedge y = x. \tag{I.181}$$

$x \wedge y = x$ and $x \vee y = y$ are equivalent. $x \wedge y = x$, then

$$\begin{aligned}
x \vee y &= (x \wedge y) \vee y\\
&= (y \wedge x) \vee y\\
&= y, \qquad\qquad \text{(I.182)}
\end{aligned}$$

and similarly $x \vee y = y$ implies $x \wedge y = x$.

In this way we see that characterizing lattices where there were relations are now operations, brings them closer to abstract algebra.

The (I.179 (a)) and (I.178 (a)) imply $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ can be given unambiguous meaning, for example

$$\begin{aligned}
x_1 \wedge x_2 \wedge x_3 \wedge x_4 &= (x_1 \wedge x_2) \wedge (x_3 \wedge x_4)\\
&= ((x_1 \wedge x_2) \wedge x_3) \wedge x_4\\
&= (x_1 \wedge (x_2 \wedge x_3)) \wedge x_4\\
&= ((x_1 \wedge x_3) \wedge x_2) \wedge x_4\\
&= (x_1 \wedge x_3) \wedge (x_2 \wedge x_4)\\
&= \ldots \qquad\qquad \text{(I.183)}
\end{aligned}$$

Similarly for $x_1 \vee x_2 \vee \cdots \vee x_n$.

A lattice is distributive if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{I.184}$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{I.185}$$

A *Boolean algebra* is a complemented distributive lattice.

lattice operation	propositional calculus
order relation $\leq$	implication $\rightarrow$
“meet” $\wedge$	disjunction “and” $\wedge$
“join” $\vee$	disjunction “or” $\vee$
“complement” ’	negation “not” $\neg$

To a given proposition $\mathcal{P}$, we assign a subset A with the interpretation that an element belongs to A if and only if it satisfies the proposition $\mathcal{P}$.

Orthomodular lattice

$$(a \vee (a^{\perp} \wedge (a \vee b))) = (a \vee b) \tag{I.186}$$

One can weaken the distributaty condition by requiring that (I.190) only hold if $x \leq z$. A lattice is called modular if for all y

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z \tag{I.187}$$

In an orthocomplemented lattice hold de Morgan laws

$$1^{\perp} = 0, \quad 0^{\perp} \quad (x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp} \quad (x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}. \tag{I.188}$$

Definition Orthocomplemented lattice $\mathcal{L}$ is called **orthomodular** if condition (I.187) holds for $y = x^{\perp}$, that is

$$x \leq z \Rightarrow x \vee (x^{\perp} \wedge z) = z. \tag{I.189}$$

Logic of Projectors on a Hilbert Space

A projection operator is characterized by *“if you measure again, you will get the same answer”*.

Closed subspaces of a Hilbert space, ordered by inclusion, form a complete lattice, (in a same way to how subsets of a set forms a lattice by inclusion). The greatest lower bound of the subspaces is their intersection, while their least upper bound is the closed span of their union.

projections, we may impose upon the set $L(H)$ the structure of a complete orthocomplemented lattice, defining $P \leq Q$, where $ran(P) \subset ran(Q)$ and $P' = 1 - P$ (so that $ran(P') = ran(P)^{\perp}$). It is straightforward that $P \leq Q$ just in case $PQ = QP = P$. More generally, if $PQ = QP$, then $PQ = PQ$, the meet of P and Q in $L(H)$; also in this case their join is given by $PQ = P + Q - PQ$.

from [367]:

This is still a distributional algebra. Namely, for propositions P, Q and R, if $P \vee Q$ denotes "P or Q", and $P \wedge Q$ means "P and Q", then $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$.

the observabls algebra is modified even at the classical level Quantum mechanics is linear, and as a result of the superposition principle, quantum mechanical propositions are not distributive. If P, Q and R are projection operators, $P \vee (Q \wedge R)$ is not equal to $(P \vee Q) \wedge (P \vee R)$.

$\rightarrow$ The logical propositions that represent physical questions in quantum mechanics are of the form the state of the system is an eigenvector of the operator $\hat{\mathcal{O}}$ with eigenvalue λ , for an observable $\mathcal{O}$. The non-commutativity of the observable operators gives place to a non-Boolean logic for the propositions, in contrast to the case of the propositions about the phase space of a system in classical physics.This is mathematically described by the orthomodular lattice structure of projectors in Hilbert space, which is often called quantum logic [357]. $\leftarrow$

$- - - - - - - - - - -$

Physical quantities or observables of a quantum entity are described as self-adjoint operators on a complex Hilbert space, where the outcomes of the observable are the numbers of the spectrum of the self-adjoint operator.

The most simple of all observables, an observable with only two possible outcomes, labelled 'yes' and 'no', is described by a self-adjoint operator with only two elements in its spectrum, and this is an orthogonal projection operator.

yes-no-experiments' can be taken as basic in quantum mechanics because a general experiment with more than two outcomes can always be subdivided in different yes-no-experiments, considering each time again a subset and its complement of the outcome set. This means that if we develop a framework for the general description of yes-no-experiments we can reconstruct all the experiments by considering them as compositions of their underlying yes-no-experiments. Much in the same way as a self-adjoint operator is completely defined by means of its spectral resolution, which is the set of orthogonal projections corresponding to the yes-no-experiments that are sub experiments of the observable described by this self-adjoint operator.

In Rovelli's axiomatic approach he postulates that 'yes-no-experiments' are the basic objects.

$- - - - - - - - - - -$

Gleason's Theorem

Quantum Logic

modern quantum mechanics is based on following assumption

The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of separable, infinite dimensional Hilbert space.

Quantum Logic

An atom of a lattice $\mathcal{L}$ is an element a for which $0 \leq x \leq a$ implies that $x = 0$ or $x = a$. A lattice with 0 is called atomic if for every $x \neq 0$ in $\mathcal{L}$ there is an atom $a \neq 0$ such that $a \leq x$.

Recall that a lattice is called distributive if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{I.190}$$

One can weaken the distributaty condition by requiring that (I.190) only hold if $x \leq z$. A lattice is called modular if for all y

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{I.191}$$

Boolean algebras - algebras for classical logic

I.7.1 Relational Models

Quantum mechanics has survived all tests and there is no reason to believe that there is any flaw in it.

It does not try to append a reasonable interpretation to the quantum mechanics formalism, nor to postulate as yet unknown new physics, rather it attempts to uncover the physical meaning of quantum mechanics.

nostalgia for the absolute

act of bringing about information

$$\underbrace{\alpha_1 \phi_{q_1} + \alpha_2 \phi_{q_2}}_{t_1} \Longrightarrow \underbrace{\alpha_2 \phi_{q_2}}_{t_2} \tag{I.192}$$

$$\psi_{init} \otimes (\alpha_1 \phi_{q_1} + \alpha_2 \phi_{q_2}) \tag{I.193}$$

if O' performs no measurement,

$$\alpha_1 \psi_{q_1} \otimes \phi_{q_1} + \alpha_2 \psi_{q_2} \otimes \phi_{q_2}. \tag{I.194}$$

A notation will not appear: the state of the system. The absence of this notion is a prime feature of the relational interpretation here. In place of the notion of state, which refers solely to the system, the notion of the information that a system has about another system is introduced.

With the conceptual foundations of the approach laid out. We describe the program of information-theoretic derivation of quantum theory.

I.8 Information-Theoretical Derivation of Quantum Theory via Quantum Logic

I.8.1 Introduction

Rovelli:

> "Quantum mechanics will cease to look puzzling only when we will be able to derive the formalism of the theory from a set of simple physical assertions ("postulates", "principles") about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanical formalism, but rather to derive the formalism from a set of experimentally motivated postulates."

From a system of information-theoretic axioms, we use quantum logical techniques for deriving the of quantum theory (Hilbert space structure). Inspired by the ideas of Rovelli's Relational Quantum Mechanics, we then propose a system of information-theoretic axioms and use quantum logical techniques for deriving lattice orthomodularity. Lattice orthomodularity follows from the postulate of finite amount of relevant information, while quantumness is due to the possibility to always obtain new information. To complete the information-theoretic reconstruction of the formalism, additional axioms are proposed from which follow the state space structure and unitary time evolution. Subsequently, Hilbert space, Born rule and unitary dynamics are all formally obtained.

In conclusion, it is shown how a twofold role of the observer as physical system and informational agent leads to a description of measurement by POVM, a necessary ingredient of quantum computation.

Axioms

Postulate 1 (Limited information). There is a maximum amount of relevant information that can be extracted from a system.

Postulate 2 (Unlimited information). It is always possible to acquire new information about a system.

The motivation for the second postulate is fully experimental. We know that all quantum systems (and all systems are quantum systems) have the property that even if we know their quantum state $|\psi>$ exactly, we can still learn "learn" more about them by performing a measurement of a quantity O such that $|\psi>$ is not an eigenstate of O. This is an *experimental* result about the world, coded in quantum mechanics. Postulate 2 expresses this result.

Since the amount of information that O can have about S is limited by postulate 1, when new information is acquired, part of the old relevant-information becomes irrelevant.

The physical content of the general formulism of quantum mechanics is (almost) nothing but a sequence of consequences of two physical facts expressed in postulates 1,2.

What do we need to reconstruct?

1) Obtain Hilbert space and prove quantumness.

2) Obtain Born rule with the state space.

3) Obtain unitary dynamics.

Quantum logical reconstruction of the Hilbert space

1. Definition of the lattice of yes-no questions.
2. Definition of orthogonal complement.
3. Definition of relevance and proof of orthomodularity.
4. Introduction of the space structure.
5. Lemmas about properties of the space.
6. Definition of the numeric field.
7. Construction of the Hilbert space.

A few notes.

Number 3 - Necessary (but not sufficient) condition to obtain a Hilbert space.

Number 6 - Fields are the "number systems" in maths.

I.8.2 Reconstruction Attempts

supply I-observer with information. The number of questions in I, though, can be much larger than N, as some of these questions are not independent. In particular, they may be related by implication ($Q_1 \Rightarrow Q_2$), union ($Q_3 = Q_1 \vee Q_2$), and intersection ($Q_3 = Q_1 \wedge Q_2$). One can define an always false question (Q_0) and an always true question (Q_∞), negation of a question ($\neg Q$), and a notion of orthogonality as follows: if $Q_1 \Rightarrow \neg Q_2$, then Q_1 and Q_2 are orthogonal ($Q_1 \perp Q_2$).

More Axioms

We will come to Axiom III in the section I.8.10.

Supplementary assumptions:

IV. The logical *or.* $\wedge$ is defined for every pair of questions.

V. The logical *and.* $\vee$ is defined for every pair of questions.

VI. The lattice of questions is complete.

VII. The underlying field of the space of the theory is one of the numeric fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{D}$ and the involutory anti-automorphism in this field is continuous.

Strings of Answers and Complete Questions

asks the N questions in the family c then the obtained answers form a string

$$s_c = [e_i, \ldots, e_N]_c. \tag{I.195}$$

The string s_c can take $2^N = K$ values. We denote them as $s_c^{(1)}$, $s_c^{(2)}, \ldots, s_c^{(K)}$ so that

$$\begin{aligned} s_c^{(1)} &= [0, 0, \ldots, 0]_c \\ s_c^{(2)} &= [0, 0, \ldots, 1]_c \\ &\ldots \\ s_c^{(K)} &= [1, 1, \ldots, 1]_c \end{aligned} \tag{I.196}$$

$$
\begin{aligned}
Q_c^{(1)} &= [(e_1 = 0) \wedge (e_1 = 0) \wedge \cdots \wedge (e_N = 0)]_c? = \neg Q_1 \wedge \neg Q_2 \wedge \cdots \wedge \neg Q_N \\
Q_c^{(2)} &= [(e_1 = 0) \wedge (e_1 = 0) \wedge \cdots \wedge (e_N = 1)]_c? = \neg Q_1 \wedge \neg Q_2 \wedge \cdots \wedge \ Q_N \\
&\qquad\qquad \cdots \\
Q_c^{(k)} &= [(e_1 = 1) \wedge (e_1 = 1) \wedge \cdots \wedge (e_N = 1)]_c? = \ Q_1 \wedge \ Q_2 \wedge \cdots \wedge \ Q_N
\end{aligned} \tag{I.197}
$$

the Questions of kind are referred to as complete questions. Here are some of its properties. It is obvious that $Q_c^{(i)} \wedge Q_c^{(i)} = Q_c^{(i)}$ and $Q_c^{(i)} \vee Q_c^{(i)} = Q_c^{(i)}$. It is also obvious that $Q_c^{(i)} \wedge Q_c^{(j)} = Q_c^{(j)} \wedge Q_c^{(i)}$ and $Q_c^{(i)} \vee Q_c^{(j)} = Q_c^{(j)} \vee Q_c^{(i)}$

$$
\begin{array}{lcl}
\text{“}Q_c^{(i)} \text{ and } Q_\infty\text{”} = Q_c^{(i)} & \qquad & \text{“}Q_c^{(i)} \text{ and } Q_0\text{”} = Q^0 \\
\text{“}Q_c^{(i)} \text{ or } Q_0\text{”} = Q_c^{(i)} & \qquad & \text{“}Q_c^{(i)} \text{ or } Q_0\text{”} = Q^\infty
\end{array} \tag{I.198}
$$

$$
\begin{array}{lcl}
Q_c^{(i)} \wedge Q_\infty = Q_c^{(i)} & \qquad & Q_c^{(i)} \wedge Q_0 = Q_0 \\
Q_c^{(i)} \vee Q_0 = Q_c^{(i)} & \qquad & Q_c^{(i)} \vee Q_0 = Q_\infty
\end{array} \tag{I.199}
$$

Lemma I.8.1 *Complete questions $Q_c^{(i)}$ are mutually exclusive, i.e.,*

$$
Q_c^{(i)} \wedge Q_c^{(j)} = Q_0 \quad \textit{for all } i \neq j. \tag{I.200}
$$

and for them holds the distributivity law:

$$
Q_c^{(i)} \vee (Q_c^{(j)} \wedge Q_c^{(k)}) = (Q_c^{(i)} \vee Q_c^{(j)}) \wedge (Q_c^{(i)} \vee Q_c^{(k)}). \tag{I.201}
$$

This implies, from (E.7), also that

$$
Q_c^{(i)} \wedge (Q_c^{(j)} \vee Q_c^{(k)}) = (Q_c^{(i)} \wedge Q_c^{(j)}) \vee (Q_c^{(i)} \wedge Q_c^{(k)}). \tag{I.202}
$$

Proof:

Equality to the always false question of the disjunction of any two different complete questions follows immediately from their definition (I.197). For example

$$
\begin{aligned}
& (Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \wedge (Q_1 \wedge \neg Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \\
= \; & (Q_1 \wedge Q_1) \wedge (Q_2 \wedge \neg Q_2) \wedge (\neg Q_3 \wedge \neg Q_3) \wedge (\neg Q_4 \wedge \neg Q_4) \\
= \; & Q_\infty \wedge Q_0 \wedge Q_\infty \wedge Q_\infty \\
= \; & Q_0.
\end{aligned} \tag{I.203}
$$

In words, the question "Q_2 and $\neg Q_2$" asks something that can never be true, so the question "$Q_c^{(i)}$ and $Q_c^{(j)}$" asks something that can never be true.

2. The case $i = j = k$ is trivial.

for $i \neq j = k$

$$
\text{"}(Q_c^{(i)} \text{ or } Q_c^{(j)}) \text{ and } (Q_c^{(i)} \text{ or } Q_c^{(j)})\text{"} \equiv \text{"}Q_c^{(i)} \text{ or } Q_c^{(j)}\text{"}.
$$

for $i \neq j \neq k$

$$
Q_c^{(i)} \vee (Q_c^{(j)} \wedge Q_c^{(k)}) = Q_c^{(i)} \vee Q_0 = Q_c^{(i)} \tag{I.204}
$$

for $i = j \neq k$

□

Definition (Material implication)

Definition Questions are **orthogonal** if

$$
Q_1 \to \neg Q_2. \tag{I.205}
$$

What would be the effect of applying negation to $Q_1 \wedge Q_2$?

$$
\neg(Q_1 \wedge Q_2) = \neg\text{"}Q_1 \text{ and } Q_2\text{"} = \text{"}\neg Q_1 \text{ or } \neg Q_2\text{"} = \neg Q_1 \vee \neg Q_2.
$$

Say $Q_1 \wedge Q_2 = Q_1$ then

$$
\neg Q_1 \wedge \neg Q_2 = \neg Q_2,
$$

that is, $\neg Q_2 \leq \neg Q_1$.

Definition Orthocomplement $Q^{\perp}$ is the union of all questions orthogonal to Q.

Lemma I.8.2 *The orthocomplementation as defined in $W(P)$ fulfills the requirements for a lattice orthocomplementation.*

Proof. A complemented lattice has a map $x \to x^{\perp}$, satisfying for all $x, y \in \mathcal{L}$ (i) $x^{\perp\perp} = x$, (ii) $x \leq y \Leftrightarrow y^{\perp} \leq x^{\perp}$, (iii) $x \wedge x^{\perp} = 0$, (iv) $x \vee x^{\perp} = 1$.

(i) The implication $Q_1 \to \neg Q_2$ is equivalent to $Q_2 \to \neg Q_1$. If we take Q_2 to be the orthocomplement to Q_1, i.e $Q_2 := Q_1^{\perp}$, this equivalence insures that $(Q_1^{\perp})^{\perp} = Q_2^{\perp} = Q_1$.

(iii) and **(iv)** It is trivial to verify that $Q \wedge Q^{\perp} = Q_0$ and $Q \vee Q^{\perp} = Q_{\infty}$ since $Q^{\perp}$ is greater or equal to $\neg Q$.

(ii) Assume that $Q_1 \leq Q_2$, i.e. $Q_1 \wedge Q_2 = Q_1$. We need to prove that $Q_2^{\perp} \leq Q_1^{\perp}$, i.e. $Q_2^{\perp} \wedge Q_1^{\perp} = Q_2^{\perp}$. The left-hand side of this last expression denotes such questions Q that $Q_1 \to \neg Q$ and $Q_2 \to \neg Q$ in all possible measurements.

In their turn, these two conditions holding separately in all measurements imply that it must not be the case that $[(Q_1 \vee Q_2) \wedge \neg Q]$. Note that the equality $Q_1 \wedge Q_2 = Q_1$ implies $\neg Q_1 \vee \neg Q_2 = \neg Q_1$. We get for the negative assumption

$$\begin{aligned} \neg[(Q_1 \vee Q_2) \wedge \neg Q] &= (\neg Q_1 \wedge \neg Q_2) \vee Q \\ &= [(\neg Q_1 \vee \neg Q_2) \wedge \neg Q_2] \vee Q \\ &= \neg Q_2 \vee Q. \end{aligned} \tag{I.206}$$

Recall that this must not be the case. Then negation of the last expression in the line entails that $\neg Q \wedge Q_2$. Since equivalence holds everywhere in (I.206) and we started with $Q_2^{\perp} \wedge Q_1^{\perp}$, we conclude that $Q_2^{\perp} \wedge Q_1^{\perp} = Q_2^{\perp}$.

□

Lemma I.8.3 *By taking all possible unions of sets of complete questions $Q_c^{(i)}$ of the same family c*

$$\begin{array}{l} Q_c^{(i)} \\ Q_c^{(i)} \vee Q_c^{(j)} \\ Q_c^{(i)} \vee Q_c^{(j)} \vee Q_c^{(k)} \\ \quad \cdots \\ \underbrace{Q_c^{(i)} \vee Q_c^{(j)} \vee \cdots \vee Q_c^{(k)}}_{2^N} \end{array} \tag{I.207}$$

one constructs a Boolean algebra that has $Q_c^{(i)}$ as atoms. Where Q_0 and Q_∞ serve as the lattice elements 0 and 1.

Proof: A Boolean algebra is a complemented distributive lattice.

Recall a lattice is a partially ordered set where the meet and join of any two elements always exists.

The partial ordering is defined as:

$$x \leq y \Leftrightarrow x \wedge y = x. \tag{I.208}$$

A complemented lattice has a map $x \to x^\perp$, satisfying for all $x, y \in \mathcal{L}$ (i) $x^{\perp\perp} = x$, (ii) $x \leq y \Leftrightarrow y^\perp \leq x^\perp$, (iii) $x \wedge x^\perp = 0$, (iv) $x \vee x^\perp = 1$.

Do all joins

$$(Q_c^{(i_1)} \vee Q_c^{(j_1)} \vee \cdots \vee Q_c^{(k_1)}) \vee (Q_c^{(i_2)} \vee Q_c^{(j_2)} \vee \cdots \vee Q_c^{(k_2)})$$

exist? That is, is (I.207) closed under the joins of any two of its members?

Does the meet of any two members of (I.207) exist?

$$(Q_c^{(i_1)} \vee Q_c^{(j_1)} \vee \cdots \vee Q_c^{(k_1)}) \wedge (Q_c^{(i_2)} \vee Q_c^{(j_2)} \vee \cdots \vee Q_c^{(k_2)})$$

exist?

$$\begin{aligned} Q_c^{(1)} &= \neg Q_1 \wedge \neg Q_2 \\ Q_c^{(2)} &= \neg Q_1 \wedge Q_2 \\ Q_c^{(3)} &= Q_1 \wedge Q_2 \end{aligned} \tag{I.209}$$

We construct elements:

$$v := Q_c^{(1)}, \quad w := Q_c^{(2)}, \quad y := Q_c^{(1)} \vee Q_c^{(2)} \tag{I.210}$$

1. First we must show that $v \wedge w$, $v \wedge x$ $v \wedge y$, $w \wedge x$, $w \wedge y$ and $x \wedge y$ exist. And similarly for $\wedge$ replaced by $\vee$.

$$
\begin{aligned}
v \vee w &= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y \\
v \vee y &= Q_c^{(1)} \vee (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y \\
w \vee y &= Q_c^{(2)} \vee (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y \\
y \vee y &= (Q_c^{(1)} \vee Q_c^{(2)}) \vee (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y
\end{aligned}
\tag{I.211}
$$

$$
\begin{aligned}
v \wedge w &= Q_c^{(1)} \wedge Q_c^{(2)} \\
&= Q_0 \\
v \wedge y &= Q_c^{(1)} \wedge (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \\
&= v \\
w \wedge y &= Q_c^{(2)} \wedge (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(2)} \\
&= w \\
y \wedge y &= (Q_c^{(1)} \vee Q_c^{(2)}) \wedge (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(2)} \vee Q_c^{(2)} \\
&= w
\end{aligned}
\tag{I.212}
$$

2. Second must show that there is no element q such that $Q_0 \leq q \leq Q_c^{(i)}$.

We need to show that the only element q in (I.207) such that $q \wedge Q_c^{(1)} = q$ is Q_0. By the distributive law (I.201) we have

$$
\begin{aligned}
(Q_c^{(i)} \wedge (Q_c^{(j)} \vee \cdots \vee Q_c^{(k)})) &= (Q_c^{(i)} \wedge Q_c^{(j)}) \vee \cdots \vee (Q_c^{(i)} \wedge Q_c^{(k)}) \\
&= Q_0 \vee \cdots \vee Q_0 \\
&= Q_0
\end{aligned}
\tag{I.213}
$$

Hence, the only element such that $Q_0 \leq q \leq Q_c^{(i)}$ is Q_0.

3. The existence of a complementation. Let us define the map $^\perp$ on (I.207) as

$$(\neg Q_1 \wedge Q_2 \wedge \cdots \wedge \neg Q_{N-1} \wedge Q_N)^\perp := (Q_1 \wedge \neg Q_2 \wedge \cdots \wedge Q_{N-1} \wedge \neg Q_N) \tag{I.214}$$

and the map $^\perp$ on $(Q_c^{(i)} \wedge \cdots \wedge Q_c^{(k)})$ as

$$(Q_c^{(i)} \wedge \cdots \wedge Q_c^{(k)})^\perp := \tag{I.215}$$

(i) Does $x^{\perp\perp} = x$? This is easy to see, for example

$$\begin{aligned}(\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4)^{\perp\perp} &= (Q_1 \wedge \neg Q_2 \wedge Q_3 \wedge Q_4)^\perp \\ &= \neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4\end{aligned} \tag{I.216}$$

So $Q_c^{(i)\perp\perp} = Q_c^{(i)}$. Now consider

$$(Q_c^{(i_1)} \wedge Q_c^{(j_1)} \wedge \cdots \wedge Q_c^{(k_1)})^{\perp\perp}$$

(ii) Does $x \leq y$ imply $y^\perp \leq x^\perp$? That is, does $x \wedge y = x$ imply $y^\perp \wedge x^\perp = y^\perp$. Note that only

$$Q_c^{(i)} \wedge Q_c^{(i)} = Q_c^{(i)}, \qquad Q_c^{(i)} \wedge Q_\infty = Q_c^{(i)} \tag{I.217}$$

$$Q_c^{(i)} \wedge Q_c^{(i)} = Q_c^{(i)}, \qquad Q_c^{(i)} \wedge Q_\infty = Q_c^{(i)} \tag{I.218}$$

(iii) Does $x \wedge x^\perp = Q_\infty$?

$$\begin{aligned}&(\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \wedge (\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4)^\perp \\ =\ &(\neg Q_1 \wedge Q_1) \wedge (Q_2 \wedge \neg Q_2) \wedge (\neg Q_3 \wedge Q_3) \wedge (\neg Q_4 \wedge Q_4) \\ =\ &Q_\infty \wedge \cdots \wedge Q_\infty = Q_\infty.\end{aligned} \tag{I.219}$$

(iv) Does $x \vee x^\perp = Q_0$?

$$
\begin{aligned}
& (\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \vee (\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4)^{\perp} \\
= \; & (\neg Q_1 \vee Q_1) \wedge (Q_2 \vee \neg Q_2) \wedge (\neg Q_3 \wedge Q_3) \wedge (\neg Q_4 \wedge Q_4) \\
= \; & Q_0 \wedge \cdots \wedge Q_0 = Q_0. \qquad \text{(I.220)}
\end{aligned}
$$

□

→ Alternatively, one can consider a different family d of N independent yes-no questions and obtain another Boolean algebra with different complete questions as atoms.

It follows, then, from Axiom I that the set of questions $W(P)$ that can be asked to P-observer is algebraically an orthomodular lattice containing subsets that form Boolean algebras.

$$
\begin{array}{lcl}
(Q_c^{(i)} \vee Q_c^{(j)} \vee \cdots \vee Q_c^{(k)}) & \vee & (Q_b^{(i)} \vee Q_b^{(j)} \vee \cdots \vee Q_b^{(k)}) \\
(Q_c^{(i)} \vee Q_c^{(j)} \vee \cdots \vee Q_c^{(k)}) & \wedge & (Q_b^{(i)} \vee Q_b^{(j)} \vee \cdots \vee Q_b^{(k)})
\end{array}
\qquad \text{(I.221)}
$$

As Rovelli says, “This is precisely the algebraic structure formed by the family of linear subsets of Hilbert space”.

This concludes his sketch. The sketch of the Hilbert space construction is not a rigorous derivation due to two key obstacles: First,

(i) orthomodularity of the lattice was not derived

and, strictly speaking, from Rovelli s construction one cannot derive it. Second, even if one admits that the lattice is orthomodular,

(ii) the fact that yes-no questions form an orthomodular lattice and that it contains as subsets Boolean algebras does not yet lead to emergence of the Hilbert space.

Both these claims were formalized by Gribaum in [376] where all the assumptions needed on the way to rigorous proof made explicit. ←

[20]

In fact, one may conjecture that this peculiar consistency between the observations of different observers is the missing ingredient for a reconstruction theorem of the Hilbert space formulism of quantum theory. Such a reconstruction theorem is still unavailable. On the basis of reasonable physical assumptions, one is able to derive the structure of an orthomodular lattice containing subsets that form Boolean algebras which “almost,” but *not quite, implies the existence of a Hilbert space and its projector's algebra. Perhaps an appropriate algebraic formulation of the condition of consistency between subsystems could provide the missing hypothesis to complete the reconstruction theorem.*

I.8.3 Relevance

Information is brought about in the answer to a yes-no question. . Consider b such that it entails the negation of $a : b \Rightarrow \neg a$. If the observer asks a and obtains an answer to $a \ldots$ but then asks a genuine new question b, it means that the observer expects either a positive or negative answer to b. This, is turn, is only possible if information a is no longer relevant; indeed, otherwise the observer would be bound to always obtain the negative answer to b. We say that, by asking b, the observer renders the information brought about by asking question a irrelevant.

Definition Relevance Question b is called irrelevant with respect to question a if $b \wedge a^{\perp} \neq 0$.

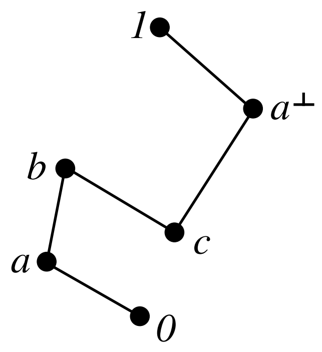


Figure I.10: relevance1.

Trivial in Hilbert spaces: $x \leq y$ are relevant with respect to y, all others irrelevant.

Non-trivial if used to derive what in Hilbert space lattices assumed.

I.8.4 Amount of Information

Two assumptions:

1. If relevance is not lost, the amount of information grows monotonously as new information comes in.

2. The lattice contains all possible information (yes-no questions). Thus, there are sufficiently many questions as to bring about any a priori allowed amount of information.

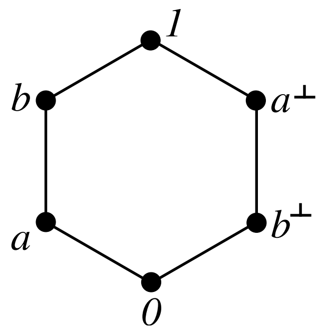


Figure I.11: relevance2.

I.8.5 Proof of Orthomodularity

By Axiom I there exists a finite upper bound of the amount of relevant information, call it N. Select an arbitrary question a and consider a question $\tilde{a}$ such that $\{a, \tilde{a}\}$ bring N bits of information. Then $a^\perp \wedge \tilde{a} = 0$.

Lemma I.8.4 *An orthocomplemented lattice is orthomodular if and only if* $a \leq b$ *and* $a^\perp \wedge b = 0$ *imply* $a = b$.

Question b is relevant with respect to a; and question $\tilde{a}$ is relevant with respect to b.

Consider $\{a, b, \tilde{a}\}$. If $b > a$, this sequence preserves relevance and brings about strictly more than N bits of relevant information.

From the contradiction follows $a = b$.

Proof of Lemma I.8.4

An orthocomplemented lattice $\mathcal{L}$ is orthomodular if and only if $x \leq z$ and $x^\perp \wedge z = 0$ imply $x = z$.

First prove the "only if" case. Say our orthocomplemented lattice $\mathcal{L}$ is orthomodular and that $x \leq z$ and $x^\perp \wedge z = 0$. By defintion (I.189), as $x \leq z$, one has

$$x \vee (x^\perp \wedge z) = x \vee 0 = z. \tag{I.222}$$

Now we turn to the "if" case. To prove this we assume the lattice is not orthomodular and show this conflicts with $x = z$ for all x and z when $x \leq z$ and $x^\perp \wedge z = 0$; as the lattice is either orthomodular or not orthomodular we can conclude that it is orthomodular.

Assume that the lattice is not orthomodular. We need to show is that there exist elements x and z such that

$$x \leq z, \quad x^\perp \wedge z = 0, \quad x \neq z. \tag{I.223}$$

Let us use the notation $x < z$ if $x \leq z$ and $x \neq z$. We can then rewrite (I.223) as

$$x < z, \quad x^\perp \wedge z = 0. \tag{I.224}$$

By the definition of an orthomodular latice (I.189), there exist elements y and z such that

$$y \leq z, \quad y \vee (y^\perp \wedge z) \neq z. \tag{I.225}$$

It can be shown, ([379] Chapter 2, Section 4), that in any lattice we have

$$a \leq b \quad \Rightarrow \quad (c \wedge b) \vee a \leq (c \vee a) \wedge b \quad \text{for all } c. \tag{I.226}$$

Put into this $a = y$, $b = z$ and $c = y^\perp$. It follows that

$$(y^\perp \wedge z) \vee y \leq (y^\perp \wedge y) \wedge z. \tag{I.227}$$

In the right-hand side replace $y^\perp \wedge y = 1$, and $1 \wedge z = z$, we then have

$$(y^\perp \wedge z) \vee y \leq z. \tag{I.228}$$

From this and (I.225) we obtain

$$(y^\perp \wedge z) \vee y < z. \tag{I.229}$$

On the other hand, form de Morgan laws (I.188) one has

$$\begin{aligned} z \wedge (y \vee (y^\perp \wedge z)^\perp) &= z \wedge (y^\perp \wedge (y^\perp \wedge z)^\perp) \\ &= z \wedge (y^\perp \wedge (y \vee z^\perp)) \\ &= (z \wedge y^\perp) \wedge (y \vee z^\perp) \\ &= (z \wedge y^\perp) \wedge (z \wedge y^\perp)^\perp = 0. \end{aligned} \tag{I.230}$$

Now put $x = y \vee (y^\perp \wedge z)$. Equations (I.229) and (I.230) can be rewritten as

$$x < z, \quad x^\perp \wedge z = 0. \tag{I.231}$$

□

I.8.6 Kalmbach's theorem

Infinite-dimensional Hilbert space characterization theorem:

Let H be an infinite-dimensional vector space over real or complex numbers or quaternions. Let L be a complete orthomodular lattice of subspaces of H which satisfies:

(i) Every finite-dimensional subspace of H belongs to L.

(ii) For every element U of L and for every finite-dimensional subspace V of H, linear sum $U + V$ belongs to L.

Then there exists an inner product f on H such that (H, f) is a Hilbert space with L as its lattice of closed subspaces.

I.8.7 Step 6: Definition of the numeric field

Axiom VII: The underlying numeric field of V is one of the real or complex numbers or quaternions, and the involutory anti-automorphism (conjugation) is continuous.

Substitutes: Solr's theorem assuming existence of an infinite orthonormal sequence of vectors. Also: Zieler, Holland, Landsman

I.8.8 Step 7: Construction of the Hilbert space

Theorem I.8.5 *Let $W(P)$ be an ensemble of yes-no questions that can be asked to a physical system and V a vector space over real or complex numbers or quaternions such that a lattice of its subspaces L is isomorphic to $W(P)$. Then there exists an inner product f on V such that V together with f form a Hilbert space.*

I.8.9 Quantumness

The Hilbert space was soley built using consequences of Axiom I (and supplementary axioms), and Axiom II remained unused through the whole discussion which preceded Theorem E.7.

Quantumness Axiom II: It is always possible to acquire new information about a system.

Criterion: Orthomodular lattice, in order to describe a quantum mechanical system, must be nondistributive.

Lemma I.8.6 *All Boolean subalgebras of $L(V)$ are proper.*

Corollary I.8.7 *$W(P)$ is non-Boolean.*

I.8.10 State space and the Born rule

$$p^{ij} = p(Q_b^{(i)}, Q_c^{(j)}). \tag{I.232}$$

From the way it is defined, the $2^N \times 2^N$ matrix cannot by fully arbitrary.

$$0 \geq p^{ij} \geq 1. \tag{I.233}$$

$$\sum_i p^{ij} \tag{I.234}$$

Rovelli would like to deduce the the existence of the states space and the Born rule from a third axiom [371],

tentative axiom III. Different observers hold information in a consistent way.

No one has completed the programme proposed by Rovelli. Grinbaumm chooses a different approach.

Axiom III Intratheoretic non-contextuality : If information is obtained by an observer, then it is obtained independently of how the measurement was eventually conducted, i.e. independent of the measurement context.

No metainformation : If information I about a system has been brought about, then it happened without bringing in information J about the fact of bringing about information I.

Gleason s theorem builds the state space.

[380]:

Theorem I.8.8 (Gleason). *Let f be any function from 1-diemensional projections on a Hilbert space of dimension $d > 2$ to the unit interval, such that for each resoltion of the identity in projections $\{P_k\}$, $k = 1 \dots d$*

$$\sum_{k=1}^{d} P_k = I, \quad \sum_{k=1}^{d} f(P_k) = 1. \tag{I.235}$$

Then there exists a unique density matrix ρ such that $f(P_k) = Tr(\rho P_k)$.

I.8.11 Time and unitary

Assume isomorphism between $W_t(P)$ at different time moments. In other words, time evolution commutes with orthogonal complementation, hence with relevance.

A symmetry, then, is a mapping of vectors in Hilbert space

$$|\psi > \rightarrow |\psi' >, \tag{I.236}$$

that preserves the absolute values of inner products

$$| < \varphi|\psi > | = | < \varphi'|\psi' > |, \tag{I.237}$$

for all $|\varphi >$ and $|\psi >$. According to Wigner's theorem, a mapping with this property can always be chosen to be either unitary or anti-unitary. The anti-unitary alternative, while important for discrete symmetries, can be excluded for continuous symmetries. Then the symmetry acts as

$$|\psi > \rightarrow |\psi' >= U|\psi >, \tag{I.238}$$

where U is unitary (and linear).

Applying Wigner's theorem, there is a unitary or anti-unitary transformation $U(t_1, t_2)$: $W_{t_1}(P) \rightarrow W_{t_2}(P)$. The unitary transformation is selected by virtue of the condition of continuity in the limit $t_2 \rightarrow t_1$.

$$Q(t_2) = U(t_2 - t_1)Q(t_1)U^{-1}(t_2 - t_1) \tag{I.239}$$

The Hamiltonian description follows by conventional arguments (Stone's theorem), these unitary operators form an abelean group and

$U(t_1 - t_2) = \exp[-i(t_1 - t_2)\mathcal{H}]$.

where $\mathcal{H}$ is a self-adjoint operator on the Hilbert space, the Hamiltonian. By defintion

$$U(t + \delta t, t_0) = (\mathbf{1} - i\delta t\mathcal{H}(t))U(t, t_0). \tag{I.240}$$

Hence

$$\mathcal{H}(t) = i\frac{\partial U(t)}{\partial t}U^{-1}(t). \tag{I.241}$$

The Schrödinger equation follows immediately if we transform from the Heisenberg to the Schrödinger picture.

I.8.12 POVM description

Twofold role of the observer: Observer is at the same time a physical system (P-observer) and an informational agent (I-observer). Information-based physical theory must give an account of P-observer, while I-observer must remain metatheoretic.

Starting with an orthogonal projector description of measurement and factoring out P-observer, one obtains the general POVM description of measurement.

I.8.13 List of axioms

Information-theoretic axioms:

I. There is a maximum amount of relevant information that can be extracted from a system.

II. It is always possible to acquire new information about a system.

III. If information I about a system has been brought about, then it happened independently of information J about the fact of bringing about information I.

Supplementary assumptions:

IV. The logical *or.* $\wedge$ is defined for every pair of questions. Or more formally:

For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to at least one of the initial question is positive. Thatis, for all $Q_1, Q_2 \in I$ there exists Q_3 such that $Q_3 = Q_1 \wedge Q_2$

V. The logical *and.* $\vee$ is defined for every pair of questions. Or more formally:

For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to both initial questions is positive. That is, all $Q_1, Q_2 \in I$ there exists Q_3 such that $Q_3 = Q_1 \vee Q_2$

VI. The lattice of questions is complete.

VII. The underlying field of the space of the theory is one of the numeric fields R, C or D and the involutory anti-automorphism in this field is continuous.

I.8.14 Open questions

1. Information-theoretic meaning of Axioms IV, V, and VI.
2. Information-theoretic meaning of Axiom VII or of Solr s theorem.
3. Meaning of the probability function f used in Gleason s theorem.
4. Origin of assumptions concerning time evolution.
5. Problem of dimension of the Hilbert space.
6. Superselection rules.

I.8.15 Example

maintained by the Copenhagen interpretation

is photon is absorbed at time t, the cat dies. At time t' the outside observer opens the box.

photon absorbed	cat alive	observer finds the cat alive
T	F	F
F	T	T

states Φ_1 Φ_2 interacts with system S with two states Ψ_1 Ψ_2. If the *up* (*down*) the led goes *on* (*off*).

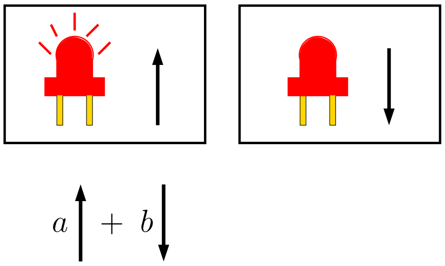


Figure I.12: If the *up* (*down*) the led goes *on* (*off*).

T for true , F for U for undecidable.

led (S) on	spin (O) up	observer (O')
T	T	T
F	F	F

T for true with respect to something, F for false with respect to something and U for undecidable with respect to something.

With respect to S before t'

led (S) on	spin (O) up	observer (O')
T	T	U
F	F	U

I.9 Heyting Algebra

Heyting algebras are algebraic structures that play a role in theories of intuitionistic logic analogous to that played by Boolean algebras in theories of classical logic.

The axioms for a Heyting algebra are the same as in (I.177), (I.178), (I.179), (I.180), with the following

$$\begin{aligned} (x \Rightarrow x) &= 1, \\ x \wedge (x \Rightarrow y) &= x \wedge y, \\ y \wedge (x \Rightarrow y) &= y, \\ (x \Rightarrow (y \wedge z)) &= (x \Rightarrow y) \wedge (x \Rightarrow z). \end{aligned} \tag{I.242}$$

Every Heyting algebra is distributive.

the observables algebra is modified even at the classical level. Internal observables satisfy a Heyting algebra, which is a weak version of the Boolean algebra of ordinary observables. This is still a distributional algebra. Namely, for propositions P, Q and R, if $P \vee Q$ denotes "P or Q", and $P \wedge Q$ means "P and Q", then $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$.

Having both quantum mechanics and internal observables in the same theory requires finding propositions that have a non-distributive quantum mechanical aspect and a distributive causal aspect. The aim of the present paper is to define the histories in which such observables may be encountered.

A boolean algebra is a Heyting algebra in which $x \vee \neg x = 1$.

negation is defined in terms of implication and falsehood as

$$\neg A \equiv A \Rightarrow \perp \tag{I.243}$$

Topological

... is a Heyting algebra, where $\mathcal{O}(X)$ is the family of of all open sets in X, and partial ordering $\subseteq$ is defined by set inclusion. In $\mathcal{O}(X)$ meet and join are just set-theoretical intersection and union, while the implication operation is given by

$$U \Rightarrow V = \text{ interior of } (X - U) \cup V.$$

taken from B . Residuated Lattices, B1. Algebras for Logics.

Any Algebra Determines a Logic

For each Heyting algebra $\mathbf{H}$, define $L(\mathbf{H}) = \{\psi(t) : P \models t = 1\}$. Then, $L(\mathbf{H})$ is a logic over intuitionistic logic **Int**.

Any Logic is Determined by an Algebra

For each logic $\mathbf{L}$ over **Int**, there is an algebra $\mathbf{Q}$ such that $\mathbf{L} = L(\mathbf{Q})$. For instance, take Lindenbaum algebra of $\mathbf{L}$

Lattice $\longrightarrow$ Algebra $\longrightarrow$ Underlying Logic

I.10 Biblioliographical notes

In this chapter I have relied on the following references:

Arun Ram Quantum groups: *A survey of definitions, motivations, and results.*

I.11 Worked Exercies and Details

Proofs

(a) Prove the centre of $\mathbf{g}$ is an ideal.

(b) Show this is precisely the kernal of the adjoint representation of $\mathbf{g}$.

Answers

(a) Must show that

$$[[g_1, h_1], [g_2, h_2]] = [[g_2, h_2], [g_1, h_1]] \tag{I.244}$$

for all $g_1, g_2 \in \mathbf{g}$ and for all h_1, h_2 in the set of commuting elements. Use $[ab, c] = abc - cab = a[b, c] + [a, c]b$ to write

Proofs

(a) Prove

(b)

Answers

(a)

Proofs

ideal generated by $x \otimes x$

$$k \oplus \mathcal{L} \oplus \mathcal{L} \otimes \mathcal{L} \oplus \dots \tag{I.245}$$

Proofs

$$[\Delta]A = \Delta A + \mathbf{1} \otimes x + x \otimes \mathbf{1} = [\Delta A] \quad \text{where } x \in I, \tag{I.246}$$

$$[\mathrm{id}]A = \mathrm{id}A + \mathbf{1} \otimes x + x \otimes \mathbf{1} = [A]. \tag{I.247}$$

$$\begin{aligned} ([\Delta] \otimes [\mathrm{id}]) \circ [\Delta] &= ([\Delta] \otimes [\mathrm{id}]) \circ (\Delta A + \mathbf{1} \otimes x + x \otimes \mathbf{1}) \\ &= \end{aligned} \tag{I.248}$$

$$([\Delta] \otimes [\mathrm{id}]) \circ [\Delta] = ([\mathrm{id}] \otimes [\Delta]) \circ [\Delta] \tag{I.249}$$

Explicate example Hopf algebra.

The multiplication operation

$$m \circ (\sigma_a \otimes \sigma_b) = m_{ab}^c \sigma_c \tag{I.250}$$

$$
\begin{aligned}
m &\circ (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
&= m \circ (\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}) \\
&= m^0_{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m^1_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + m^2_{12} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + m^3_{12} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{I.251}
$$

$$
\begin{aligned}
m &\circ (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= m^0_{02} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m^1_{02} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + m^2_{02} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + m^3_{02} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{I.252}
$$

$$
\begin{aligned}
m^k_{ij} &= i\epsilon_{ijk}, \;\text{ for } i,j,k = 1,2,3, \;\text{ for } i \neq j. \\
m^0_{aa} &= 1, \qquad \text{for } a = 0,1,2,3 \;, \text{(no summation)} \\
m^b_{0a} &= m^b_{a0} = \delta^b_a, \;\text{ for } a,b = 0,1,2,3.
\end{aligned} \tag{I.253}
$$

$$
m^0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; m^1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; m^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; m^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{I.254}
$$

The coproduct

$$
\Delta(\sigma_a) = \sigma_a \otimes e + e \otimes \sigma_a = \mu^{bc}_a \sigma_b \otimes \sigma_c \tag{I.255}
$$

$$
\Delta(\sigma_2) = \sigma_2 \otimes e + e \otimes \sigma_2 = \begin{pmatrix} 0 & -i & -i & -i \\ i & 0 & -i & -i \\ i & i & 0 & -i \\ i & i & i & 0 \end{pmatrix} \tag{I.256}
$$

Proofs

Express as matrix equations

$$
\begin{aligned}
m \circ (e_i \otimes e_j) &= m_{ij}^k e_k, \quad \Delta(e_i) = \mu_i^{jk} e_j \otimes e_k \\
\imath(k) = i^j e_j, \quad \epsilon(e_i) &= \epsilon_i, \quad S(e_i) = S_i^j e_j
\end{aligned} \tag{I.257}
$$

Answers

(1) (I.62)

$$
\begin{aligned}
m \circ (\mathrm{id}_A \otimes m)(e_i \otimes e_j \otimes e_k) = m \circ (e_i \otimes m_{jk}^l e_l) = m_{jk}^l m_{il}^m e_m \\
m \circ (m \otimes \mathrm{id}_A)(e_i \otimes e_j \otimes e_k) = m \circ (m_{ij}^l e_l \otimes e_k) = m_{ij}^l m_{lk}^m e_m
\end{aligned} \tag{I.258}
$$

$$
\Longrightarrow \qquad m_{jk}^l m_{il}^m = m_{ij}^l m_{lk}^m \tag{I.259}
$$

(2) (I.63)

$$
(\mathrm{id}_A \otimes \Delta) \circ \Delta(e_i) = \mu_i^{jk} (\mathrm{id}_A \otimes \Delta) \circ (e_j \otimes e_k) = \mu_i^{jk} \mu_k^{lm} (e_j \otimes e_l \otimes e_m) \tag{I.260}
$$

$$
\begin{aligned}
(\Delta \otimes \mathrm{id}_A) \circ \Delta \circ e_i &= \mu_i^{jk} (\Delta \otimes \mathrm{id}_A) \circ (e_j \otimes e_k) = \mu_i^{jk} \mu_j^{lm} (e_l \otimes e_m \otimes e_k) \\
&= \mu_i^{km} \mu_k^{jl} (e_j \otimes e_l \otimes e_m)
\end{aligned} \tag{I.261}
$$

$$
\Longrightarrow \qquad \mu_i^{jk} \mu_k^{lm} = \mu_i^{km} \mu_k^{jl} \tag{I.262}
$$

(3) (I.64) **unit condition,**

(4) (I.65) **counit condition,** $\epsilon : A \to k$

$$
m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta(a) = \mu_i^{jk} m \circ (\mathrm{id}_A \otimes \epsilon) \circ e_j \otimes e_k = \mu_i^{jk} m_{lm}^k e_j \epsilon(e_k) \tag{I.263}
$$

(5) (I.66) **Δ is an algebra homomorphism,** $\epsilon : A \to k$

$$
\Delta \circ m = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta), \tag{I.264}
$$

(6) (I.67) ϵ **is an algebra homomorphism,** $\epsilon : A \to k$, $\epsilon(e_i) \mapsto k_i$

$$
\epsilon \circ m(e_i \otimes e_j) = \epsilon \otimes \epsilon(e_i \otimes e_j), \tag{I.265}
$$

$$
m_{ij}^k k_k = k_i k_j \tag{I.266}
$$

(7) (I.68) $S(e_i) = S_i^j e_j$

$$m\circ(\mathrm{id}_A\otimes S)\circ\Delta(e_i)=\mu_i^{jk}m\circ(\mathrm{id}_A\otimes S)\circ(e_j\otimes e_k)=\mu_i^{jk}S_k^l m\circ(e_j\otimes e_l)=(\mu_i^{jk}S_k^l m_{jl}^m)\ e_m \quad (I.267)$$

$$m(S\otimes \mathrm{id}_A)\circ\Delta(e_i)=\mu_i^{jk}S_j^l m\circ(e_l\otimes e_k)=(\mu_i^{jk}S_j^l m_{lk}^m)\ e_m \quad (I.268)$$

$$\imath\circ\epsilon(e_i)=\imath(k)=k\mathrm{id}_A \quad (I.269)$$

$$\mu_i^{jk}S_k^l m_{jl}^m=\mu_i^{jk}S_j^l m_{lk}^m= \quad (I.270)$$

Proofs

Verify that $(A,m,\Delta^{op},\epsilon,\imath,S^{-1})$ forms a Hopf algebra,

Answer

$(A,m,\Delta^{op},\epsilon,\imath,S^{-1})$ forms a Hopf algebra if:

$$\begin{array}{ll}
(1) & m\circ(\mathrm{id}_A\otimes\epsilon)\circ\Delta^{op}=m\circ(\epsilon\otimes\mathrm{id}_A)\circ\Delta^{op}=\mathrm{id}_A\\
(2) & (\mathrm{id}_A\otimes\Delta^{op})\circ\Delta^{op}=(\Delta^{op}\otimes\mathrm{id}_A)\circ\Delta^{op},\\
(3) & \Delta^{op}\circ m=(m\otimes m)\circ(\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_A)\circ(\Delta^{op}\otimes\Delta^{op}),\\
(4) & m\circ(\mathrm{id}_A\otimes S^{-1})\circ\Delta^{op}=m(S^{-1}\otimes\mathrm{id}_A)\circ\Delta^{op}=\imath\circ\epsilon.
\end{array} \quad (I.271)$$

Assume (2) is true,

$$\Delta^{op}(e_i)=\mu_i^{jk}e_k\otimes e_j=\mu_i^{kj}e_j\otimes e_k \quad (I.272)$$

$$(\mathrm{id}_A\otimes\Delta^{op})\circ\Delta^{op}(e_i)=\mu_i^{kj}(\mathrm{id}_A\otimes\Delta^{op})\circ(e_j\otimes e_k)=\mu_i^{kj}\mu_k^{ml}(e_j\otimes e_l\otimes e_m) \quad (I.273)$$

$$\begin{aligned}(\Delta^{op}\otimes\mathrm{id}_A)\circ\Delta^{op}\circ e_i &= \mu_i^{kj}(\Delta^{op}\otimes\mathrm{id}_A)\circ(e_j\otimes e_k)=\mu_i^{kj}\mu_j^{ml}(e_l\otimes e_m\otimes e_k)\\ &= \mu_i^{km}\mu_k^{jl}(e_j\otimes e_l\otimes e_m)\end{aligned} \quad (I.274)$$

$$\Longrightarrow \qquad \mu_i^{kj}\mu_k^{ml}=\mu_i^{mk}\mu_k^{lj} \quad (I.275)$$

If we swap around m and j, we have agreement with (I.262) and so we have verified (2).

Assume (4) is true,

The same as in but we swap the upper two indices of μ replace S by its inverse, so that we arrive at the condition:

$$\mu_i^{kj}(S^{-1})_k^l m_{jl}^m = \mu_i^{kj}(S^{-1})_j^l m_{lk}^m = \tag{I.276}$$

$$\mu_i^{kj}(S^{-1})_k^l m_{jl}^m = \mu_i^{kj}(S^{-1})_j^l m_{lk}^m = \tag{I.277}$$

bf Prove:

Verify that $(A, m^{op}, \Delta, \epsilon, \imath, S^{-1})$ forms a Hopf algebra,

Proofs

Proof:

1

$$\begin{aligned} 0 &= [(z_1^2 - z_2^2)\partial_1 + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) \\ &= [(z_1 + z_2)(z_1 - z_2)\partial_1 + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) \\ &= [-(z_1 + z_2)(h_1 + h_2) + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) \end{aligned}$$

2

Proofs

Prove the equivalence of (I.184) and (I.185).

proof (1)

$x \wedge y = x$ and $x \vee y = y$ are equivalent. $x \wedge y = x$, then

$$\begin{aligned} x \vee y &= (x \wedge y) \vee y \\ &= (y \wedge x) \vee y \\ &= y, \end{aligned} \tag{I.278}$$

and similarly $x \vee y = y$ implies $x \wedge y$. Since

$x \wedge x = x$, we have $x \leq x$

We have proved that $\leq$ is a partially order relation. We now show that $x \wedge y$ is the greatest lower bound of x and y. Since

$(x \wedge y) = x$ and $(x \wedge y) \vee y = (x \vee y) \wedge y = y$

Now we show that if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$ as it should to be.

$z \wedge x = z$ and $z \wedge y = z$, then

$$\begin{aligned} z \wedge (x \wedge y) &= (z \wedge x) \wedge y \\ &= z \wedge y \\ &= y, \end{aligned} \tag{I.279}$$

so $z \leq x \wedge y$

proof ()

First note that by (I.184) $w \wedge (x \vee z) = (w \wedge x) \vee (w \wedge z)$. We write $w = (x \vee z)$

$$\begin{aligned} (x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\ &= x \vee [(x \vee y) \wedge z] \\ &= x \vee [(x \wedge z) \vee (y \wedge z)] \qquad \text{by (I.184) again} \\ &= [x \vee (x \wedge z)] \vee (y \wedge z) \\ &= x \vee (y \wedge z), \end{aligned} \tag{I.280}$$

Proofs

an analogous definition can be made in $\mathbf{K} - \mathbf{v.s.}^{op}$, but then defines in $\mathbf{K} - \mathbf{v.s.}$ a dual to the definition of an algebra.

Proofs

$$\bigvee A := a_1 \vee a_2 \vee a_3 \vee \cdots \vee a_N \tag{I.281}$$

say $a_2 \leq a_1$, $a_1 \leq a_3$

$$\begin{aligned} \bigvee A &:= (a_1 \vee a_2) \vee a_3 \vee a_4 \\ &= (a_1 \vee a_3) \vee a_4 \\ &= a_3 \vee a_4 \end{aligned} \tag{I.282}$$

complete homomorphism if for every $A \subseteq B$

$f(\bigvee A) = \bigvee\{f(a)|a \in A\}$ and $f(\bigwedge A) = \bigwedge\{f(a)|a \in A\}$

Exercise. Show $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$. Conclude that a complete homomorphism preserves 1,0 and complements.

$\bigwedge \emptyset = \emptyset = \emptyset \wedge 1 = 1$

$\bigvee \emptyset = 0 \bigvee \emptyset = 0$

Exercise. Show that $\bigwedge A = \neg \bigvee \{\neg a | a \in A\}$ and conclude that if a function preserves all $\bigvee$'s, 1 and complements, it is a complete homomorphism.

Wigner's theorem

Wigner's unitary-antiunitary theorem. Let H be a complex Hilbert space and $\alpha : H \to H$ be a surjective (onto) map with the property that

$$| < \alpha x, \alpha y > | = | < x, y > | \quad (x, y \in H).$$

Then α is of the form

$$\alpha x = \varphi(x) U x \quad (x, y \in H)$$

where $U : H \to H$ is either unitary or antiunitary operator (that is, U is either an inner product preserving bijection (onto and one-to-one) or a bijection conjugate linear map with the property that $< Ux, Uy >=< x, y >$ for all $x, y \in H$) and $\varphi : H \to \mathbb{C}$ is a so-called phase-function which means that its values are modulus one.

Details: Matched pairs and Bicrossproducts.

S. Majid, *Algebraic Approach to Quantum Gravity.*

hepth0604130 S. Majid, *Algebraic Approach to Quantum Gravity II: Noncommutative Spacetime*, [hep-th/0604130].

hepth0604132 S. Majid, *Algebraic Approach to Quantum Gravity III: Noncommutative Riemannian Geometry*, [hep-th/0604132].

Double cross product $X \simeq G \bowtie M$

$$h \triangleleft e = h, \qquad (h \triangleleft g_1) \triangleleft g_2 = h \triangleleft (g_1 g_2) \tag{I.283}$$
$$e \triangleright g = g, \qquad h_1 \triangleright (h_2 \triangleright g) = (h_1 h_2) \triangleright g \tag{I.284}$$
$$e \triangleleft g = e, \qquad (h_1 h_2) \triangleleft g = (h_1 \triangleleft (h_2 \triangleright g))(h_2 \triangleleft g) \tag{I.285}$$
$$h \triangleright e = e, \qquad h \triangleright (g_1 g_2) = (h \triangleright g_1)((h \triangleleft g_1) \triangleright g_2) \tag{I.286}$$

where the first two equations say that we have an action, .

Having such a pair, one can introduce the following group structure on $G \times H$:

$$\begin{aligned} \{g_1, h_1\} \circ \{g_2, h_2\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\}, \\ \{g, h\}^{-1} &= \{h^{-1} \triangleright g^{-1}, h^{-1} \triangleleft g^{-1}\} \end{aligned} \tag{I.287}$$

Subgroups

For $g_1 = g_2 = e$

$$\begin{aligned} \{e, h_1\} \circ \{e, h_2\} &= \{e(h_1 \triangleright e), (h_1 \triangleleft e)h_2\} \\ &= \{e, h_1 h_2\} \end{aligned} \tag{I.288}$$

For $h_1 = h_2 = e$

$$\begin{aligned} \{g_1, e\} \circ \{g_2, e\} &= \{g_1(e \triangleright g_2), (e \triangleleft g_2)e\} \\ &= \{g_1 g_2, e\} \end{aligned} \tag{I.289}$$

special cases

Semigroups are a special case of a matched pair: Say $h \triangleleft g = h$ for all $h \in H$ and $g \in G$

$$\begin{aligned} \{g_1, h_1\} \circ \{g_2, h_2\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\} \\ &= \{g_1(h_1 \triangleright g_2), h_1 h_2\} \end{aligned} \tag{I.290}$$

if $h_1 \triangleright g_2 = h_1 g_2 h_1^{-1}$ then it is a semigroup. Now, say $h \triangleright g = g$ for all $h \in H$ and $g \in G$

$$\begin{aligned} \{g_1, h_1\} \circ \{g_2, h_2\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\} \\ &= \{g_1 g_2, (h_1 \triangleleft g_2)h_2\} \end{aligned} \tag{I.291}$$

if $h_1 \triangleleft g_2 = g_2 h_1 g_2^{-1}$ then it is a semigroup.

Check Verify inverse:

$$
\begin{aligned}
\{g,h\} \circ \{g,h\}^{-1} &= \{g,h\} \circ \{h^{-1} \triangleright g^{-1}, h^{-1} \triangleleft g^{-1}\} \\
&= \{g(h \triangleright (h^{-1} \triangleright g^{-1})), (h \triangleleft (h^{-1} \triangleright g^{-1})(h^{-1} \triangleleft g^{-1})\} \\
&= \{g((hh^{-1}) \triangleright g^{-1}), (hh^{-1}) \triangleleft g^{-1}\} \\
&= \{g(e \triangleright g^{-1}), e \triangleleft g^{-1}\} \\
&= \{gg^{-1}, e\} \\
&= \{e, e\}
\end{aligned} \tag{I.292}
$$

$$
\begin{aligned}
\{g,h\}^{-1} \circ \{g,h\} &= \{h^{-1} \triangleright g^{-1}, h^{-1} \triangleleft g^{-1}\} \circ \{g,h\} \\
&= \{(h^{-1} \triangleright g^{-1})((h^{-1} \triangleleft g^{-1}) \triangleright g), ((h^{-1} \triangleleft g^{-1}) \triangleleft g)h\} \\
&= \{h^{-1} \triangleright (g^{-1}g), h^{-1} \triangleleft (g^{-1}g)h\} \\
&= \{h^{-1} \triangleright e, (h^{-1} \triangleleft e)h\} \\
&= \{e, hh^{-1}\} \\
&= \{e, e\}
\end{aligned} \tag{I.293}
$$

Verify associativity:

$$
\begin{aligned}
(\{g_1,h_1\} \circ \{g_2,h_2\}) \circ \{g_3,h_3\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\} \circ \{g_3,h_3\} \\
&= \{(g_1(h_1 \triangleright g_2))\,(\,((h_1 \triangleleft g_2)h_2) \triangleright g_3), (((h_1 \triangleleft g_2)h_2) \triangleleft g_3)h_3\}
\end{aligned} \tag{I.294}
$$

$$
\begin{aligned}
\{g_1,h_1\} \circ (\{g_2,h_2\} \circ \{g_3,h_3\}) &= \{g_1,h_1\} \circ \{g_2(h_2 \triangleright g_3), (h_2 \triangleleft g_3)h_3\} \\
&= \{g_1(h_1 \triangleright (g_2(h_2 \triangleright g_3))\,), (h_1 \triangleleft (g_2(h_2 \triangleright g_3))\,)((h_2 \triangleleft g_3)h_3)\}
\end{aligned} \tag{I.295}
$$

Consider the "h" term in the last line of (I.294)

$$
\begin{aligned}
((h_1 \triangleleft g_2)h_2) \triangleleft g_3 &= ((h_1 \triangleleft g_2) \triangleleft (h_2 \triangleright g_3))(h_2 \triangleleft g_3) \\
&= (h_1 \triangleleft (g_2(h_2 \triangleright g_3)))(h_2 \triangleleft g_3)
\end{aligned} \tag{I.296}
$$

where we used (I.285) in the first line and (I.283) in the second. Consider the "g" term in the last line of (I.294)

$$
\begin{aligned}
h_1 \triangleright (g_2(h_2 \triangleright g_3)) &= (h_1 \triangleright g_2)((h_1 \triangleleft g_2) \triangleright (h_2 \triangleright g_3)) \\
&= (h_1 \triangleright g_2)(\,((h_1 \triangleleft g_2)h_2) \triangleright g_3))
\end{aligned} \tag{I.297}
$$

where we used (I.286) in the first line and (I.284) in the second. Substituting both these results back into the equations they were taken from proves associativity.

□

Conversely For a group X, which factorizes into two subgroups G and H, we recover a matched pair G and H, with left and right actions derived from (I.287).

$$\begin{aligned}\{g,h\} &= \{g,h\}\circ\{e,e\} \\ &= \{g(h\triangleright e),(h\triangleleft e)e\},\end{aligned}\tag{I.298}$$

implying $h\triangleright e = e$ and $h\triangleleft e = h$. Similarly,

$$\begin{aligned}\{g,h\} &= \{e,e\}\circ\{g,h\} \\ &= \{e(e\triangleright g),(e\triangleleft g)h\},\end{aligned}\tag{I.299}$$

implies $e\triangleright g = g$ and $e\triangleleft g = e$.

from associativity of the group we can derive the action and matched pair conditions. Put $g_1 = g_2 = e$

$$\begin{aligned}(\{e,h_1\}\circ\{e,h_2\})\circ\{g_3,h_3\} &= \{e(h_1\triangleright e),(h_1\triangleleft e)h_2\}\circ\{g_3,h_3\} \\ &= \{(e(h_1\triangleright e))\ (\ ((h_1\triangleleft e)h_2)\triangleright g_3),(((h_1\triangleleft e)h_2)\triangleleft g_3)h_3\} \\ &= \{((h_1h_2)\triangleright g_3),((h_1h_2)\triangleleft g_3)h_3\}\end{aligned}\tag{I.300}$$

$$\begin{aligned}\{e,h_1\}\circ(\{e,h_2\}\circ\{g_3,h_3\}) &= \{e,h_1\}\circ\{e(h_2\triangleright g_3),(h_2\triangleleft g_3)h_3\} \\ &= \{e(h_1\triangleright(e(h_2\triangleright g_3))\ ),(h_1\triangleleft(e(h_2\triangleright g_3))\ )((h_2\triangleleft g_3)h_3)\} \\ &= \{h_1\triangleright(h_2\triangleright g_3))\ ),(h_1\triangleleft(h_2\triangleright g_3)\ )((h_2\triangleleft g_3)h_3)\}\end{aligned}\tag{I.301}$$

implying

$$(h_1h_2)\triangleright g_3 = h_1\triangleright(h_2\triangleright g_3)$$

and

$$(h_1h_2)\triangleleft g_3 = (h_1\triangleleft(h_2\triangleright g_3))(h_2\triangleleft g_3)$$

$h_2 = h_3 = e$

$$\begin{aligned}(\{g_1,h_1\}\circ\{g_2,e\})\circ\{g_3,e\} &= \{g_1(h_1\triangleright g_2),(h_1\triangleleft g_2)e\}\circ\{g_3,e\} \\ &= \{(g_1(h_1\triangleright g_2))\ (\ ((h_1\triangleleft g_2)e)\triangleright g_3),(((h_1\triangleleft g_2)e)\triangleleft g_3)e\} \\ &= \{g_1(h_1\triangleright g_2)((h_1\triangleleft g_2)\triangleright g_3),(h_1\triangleleft g_2)\triangleleft g_3\}\end{aligned} \tag{I.302}$$

$$\begin{aligned}\{g_1,h_1\}\circ(\{g_2,e\}\circ\{g_3,e\}) &= \{g_1,h_1\}\circ\{g_2(e\triangleright g_3),(e\triangleleft g_3)e\} \\ &= \{g_1(h_1\triangleright(g_2(e\triangleright g_3))\ ),(h_1\triangleleft(g_2(e\triangleright g_3))\ )((e\triangleleft g_3)e)\} \\ &= \{g_1(h_1\triangleright(g_2g_3)\ ),h_1\triangleleft(g_2g_3)\}\end{aligned} \tag{I.303}$$

implying

$$h_1\triangleright(g_2g_3) = (h_1\triangleright g_2)((h_1\triangleleft g_2)\triangleright g_3)$$

$$h_1\triangleleft(g_2g_3) = (h_1\triangleleft g_2)\triangleleft g_3$$

$\square$

Consider $hg \in X$ where $h \in H$, $g \in G$. By unique factorization assumption, this element is equal to $(h\triangleright g)(h\triangleleft g)$ for some elements $h\triangleright g \in G$ and $h\triangleleft g \in H$. This defines the maps $\triangleright : H \times G \to G$ and $\triangleleft : H \times G \to H$.

$\square$

Bicrosproduct

The backreaction of H on G can be used to make a semidirect product structure and render a Hopf algebra

$\mathbb{C}[H]\blacktriangleright\!\!\triangleleft U(\mathfrak{g})$

Appendix J

Category Theory and Topos

J.1 A Very Gentle Introduction to Category Theory

John Baez website http://math.ucr.edu/home/baez/README.html

In especially simple and clear way.

sets
functions
diagrams of sets and functions
categories
functors
natural transformations
diagrams of categories, functors and natural transformations 2-categories
2-functors
2-natural transformations
modifications

understand an object by placing it in a category and studying its relation with other objects of the same category by their morphisms, or related categories through functors.

For brevity we will simplify the situation considerably, leaving out significant details.

pass from a an element-based approach to an arrow-based one approach to an arrow based one in the category of sets.

Examples of categories

Category	obects	morphism
Set	sets	function
Top	topology	homeomorphism
Grp	group	homomorphism
$\mathcal{P}$	elements	$p \leq q$
Dgrph	vertices	paths

Example Every directed graph can be regarded as a category if we take the vertices to be the objects and the arrows to be the paths in the graph.

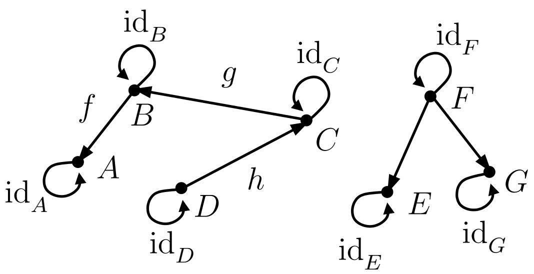


Figure J.1: Every directed graph can be regarded as a category if we take the vertices to be the objects and the arrows to be the paths in the graph.

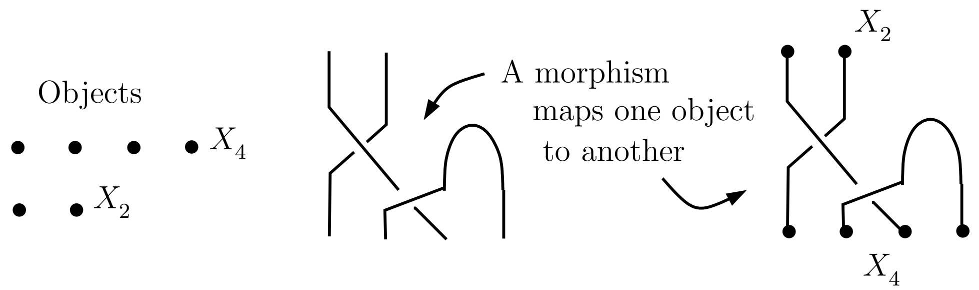


Figure J.2: is in Hom(4, 2)

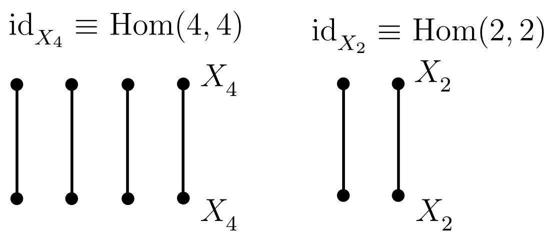


Figure J.3: Different identities X_2 id_{X_2}.

Hom(4, 2) =

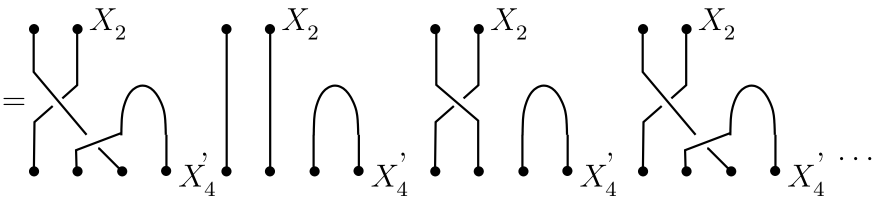


Figure J.4: Different identities X_2 id_{X_2}. is in $\mathrm{Hom}(4, 2)$

Things to note about these:

1. There are objects and morphisms between them.

2. Composition may not be defined for all pairs of objects.

3. If there is an arrow from X to Y, there may not be an arrow from Y to X.

4. When composition is defined between object X and object Y, and composition between object Y and object Z, composition between X and Y is defined.

5. Objects have different identities. We will denote the identity of the object X as id_X.

Definition A category $\mathcal{C}$ consists of "objects", $\mathfrak{ob}C$, and "morhisms" (or "arrows"), f, between them, such that

(i) If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morhisms , then there exists a morphism $g \circ f : C_1 \to C_3$.

The morphisms are often denoted $\mathrm{Hom}(C_1, C_2)$ for f. The composition of morphisms is then expressed as

$$\mathrm{Hom}(C_1, C_2) \times \mathrm{Hom}(C_2, C_3) \to \mathrm{Hom}(C_1, C_3) \tag{J.1}$$

(ii) It is assumed that the identity map for C, $\mathrm{id} : C \to C$, is a morphism for every object C of $\mathcal{C}$.

The axioms on the morphisms and composition are very weak, with no conditions imposed on the objects other than they belong to a certain class. This so means that many diverse mathematical structures can be seen as a category.

Example A **monoid** is a set X with an operation called multiplication xy for $x, y \in X$, which is associative and has unit $e \in X$, satisfying $ex = xe = x$ for all. A monoid can be regarded as a category with one object (i.e. X), and an arrow for every $x \in X$.

As a group G is a special case of a monoid, any group can be thought of as a category with the single object G and the elements of the group $g \in G$ as the arrows.

Example **Set** is the category which has the class of all sets as objects, and functions between sets as arrows.

Example **Grp** is the category which has the class of all groups as objects, and functions between between groups as arrows.

Example A partially ordered set is . This can be regarded as a category in which the objects are the elements of $\mathcal{P}$; and if $p, q \in \mathcal{P}$, an arrow from p to q is defined to exist if $p \leq q$.

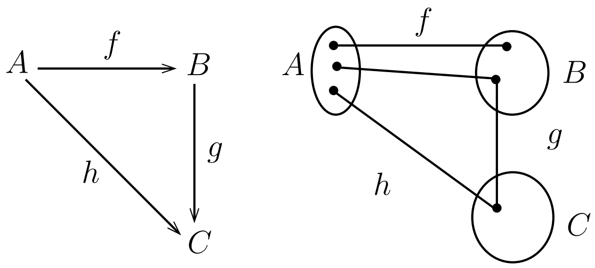


Figure J.5: x

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

commutes if and only if $g(f(x)) = h(x)$ for all $x \in X$. We write

$$g \circ f = h. \tag{J.2}$$

Definition The 'opposite' of a category $\mathcal{C}$, denoted $\mathcal{C}^{op}$, is obtained by reversing the arrows of $\mathcal{C}$, while keeping the same objects. The arrow $f : A \longrightarrow B$ in $\mathcal{C}$ becomes the arrow $f^{op} : B \longrightarrow A$ in $\mathcal{C}^{op}$. Composition is ordered the other way around. Insted of

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{J.3}$$

we have

$$C \xrightarrow{g^{op}} B \xrightarrow{f^{op}} A \tag{J.4}$$

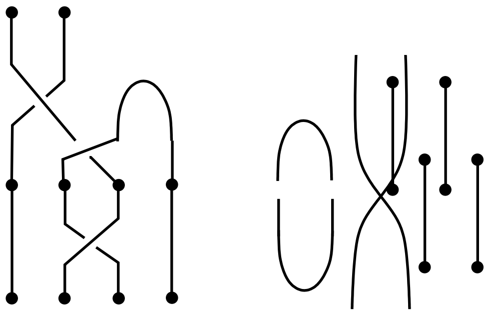


Figure J.6: tangleCat4 is in $\mathrm{Hom}(4, 2)$

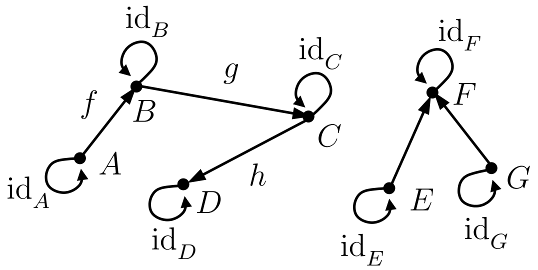


Figure J.7: The ‘opposite’ category of the graph in Fig(J.1).

equivalences a group with a different way of renaming the elements does not really give you back a different group.

Example A category where the objects are discrete events and the arrows are causal relations between events. $p \to q$ means p preceeds q or $p \leq q$. Arrows are closed under composition because $p \to q$ and $q \to r$ implies $p \to r$. Arrows are associative $(p \to q) \to r = p \to (q \to r)$.

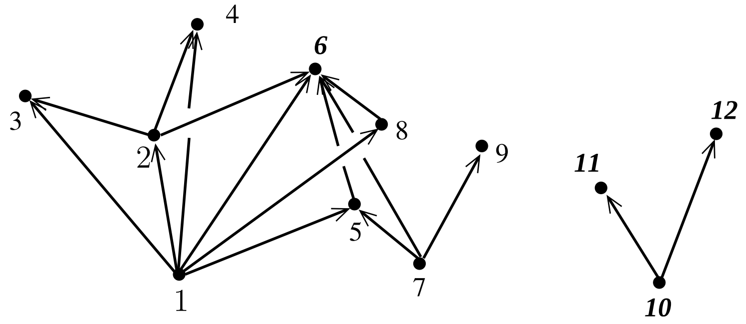


Figure J.8: causalsetex1. Taken from [366].

Definition Functors F are the structure preserving assignments between categories. In other words, a functor is a *homomorphism* between categories.

1. to each object A in $\mathcal{C}$, there is an object in $\mathcal{D}$ $F(A)$;
2. to each object arrow $f : B \to A$ in $\mathcal{C}$
3. $F(1_A) = id_{F(A)}$

$$F(f \circ g) = F(f) \circ F(g). \tag{J.5}$$

We have just encountered in the last section an important example of a functor, that of the fundamental group.

Example

$$Past : \longrightarrow \mathbf{Set} \tag{J.6}$$

Past as a functor it outputs the set $Past(p)$ for the object p for each p, (see fig.(J.1)). For each causal relation $p \to q$ and outputs the function that takes you from the set $Past(p)$ to the set $Past(q)$.

that assigns to each $p \in \mathcal{C}$ its past

$$Past(p) = \{r \in \mathcal{C} : r < p\}. \tag{J.7}$$

and to each causal relation, when it appears in the causal set, a function

$$Past_{pq} : Past(p) \longrightarrow Past(q), \tag{J.8}$$

which includes the identity map $Past_{pp} : Past(p) \longrightarrow Past(p)$.

Theorem J.1.1 *If $f : X \to Y$ is a homeomorphism of X onto Y, the induced homomorphism $f_* : \pi(X, p) \to \pi(Y, fp)$ is an isomorphism onto for any basepoint p in X.*

If pathwise connected topological spaces X and Y are homeomorphic, their fundamental groups are isomorphic. This can be expressed as saying their is a functor from the category of topological spaces to the category of groups. The functor F map a topological space X to the fundamental group $\pi(X, p)$, i.e., $F(X) = \pi(X, p)$ and maps homeomorphisms into isomorphisms, i.e., $F(f) = f_*$.

One can even define maps between functors, and so on and on. We then come into the wonderful world of *n-categories*, which is, John Baez mathematics underlying GR and quantum mechanics. Finds application in many things including also homotopy theory...

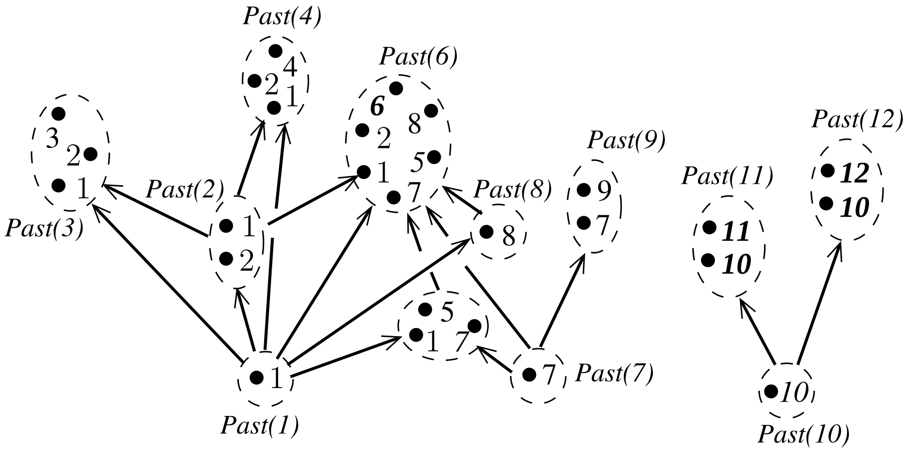


Figure J.9: causalsetex2. Taken from [366].

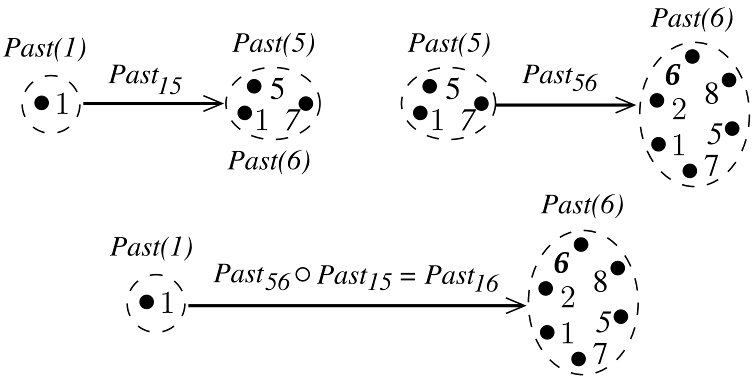


Figure J.10: causalsetex2A. Composition. Associativity.

Definition Natural Transformations Suppose we have two functors F and G from category $\mathcal{C}$ to the category $\mathcal{D}$. Then a *natural transformation* N from F to G assigns to each object X in $\mathcal{C}$ a morphism $N(X) :\to G(X)$ such that this diagram commutes:

Example "abelianzation", which maps a group to the abelian group. $ab = ba + [a, b]$ so that $ab \sim ba$ so that under induced group multiplication $[a][b] = [ab] = [ba] = [b][a]$. The category $\mathcal{C}$ has only one object, which is H, and the morphisms f are the isomorphisms of the group H. The functors F and G map to the category $\mathcal{D}$, which are themselves groups. The *natural transformation, N, is the "abelianzation"* of $F(H)$ and $G(H)$.

We say a category is small if the collection of objects is a set. The category of all small categories is denoted **Cat**.

A notion of isomorphism only makes sense if the object has some internal structure, for example an isomorphism between groups G and H is when every element of the group G

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "F(\mathcal{C})"] \arrow[rr, bend right, "G(\mathcal{C})"'] \arrow[rr, phantom, "\Downarrow \eta"] & & \mathcal{D}
\end{tikzcd}
$$

Figure J.11: NaturalTr0. An arrow η that maps the functor F to the functor G.

$$
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "n(X)"'] & F(Y) \arrow[d, "n(Y)"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}
$$

Figure J.12: Natural transformation.

is mapped to one element of H. When the objects of a category have no internal structure there is no such thing as an isomorphism. In particular, there no notion of isomorphism between morhpisms!

Example The set of functors $\mathrm{Hom}[\mathcal{P}, G]$. G is a finite connected and compact Lie group.

$\mathrm{Hom}[\mathcal{P}, G]$ is the set of all functors from the groupoid $\mathcal{P}$ to the group G.

Universal Arrows

$u : T \to T'$ satisfying $u \circ h = h'$. Similarly, by the universal property of h' there is a map $u' : T' \to T$ satisfying $u' \circ h' = h$.

We have $u' \circ u : T \to T$ with $(u' \circ u) \circ h = \mathrm{id}_T \circ h$. By uniqueness if arrow, we have that $u' \circ u = \mathrm{id}_T$. Similarly, $u \circ u' = \mathrm{id}_{T'}$. Hence, $T \cong T'$.

Definition Isomorphism (iso) Two objects A and B in a category are said to be *isomorphic* if there exists arrows $f : A \to B$ and $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

terminal object is when for every object there is one map to

if there is more than one object and a iso between them then both are terminal isomorphic - so the same as far as the categorical perspective.

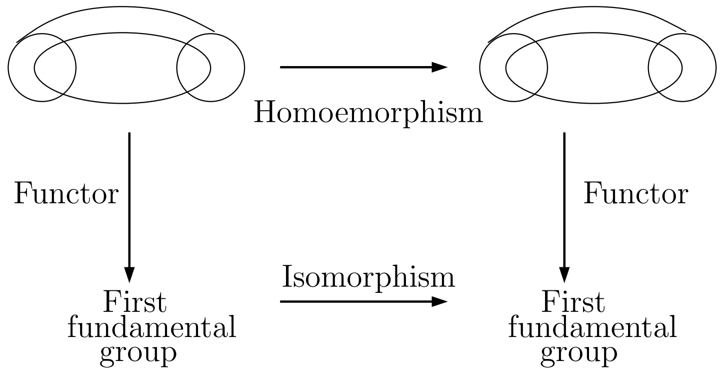


Figure J.13: Natural transformation.

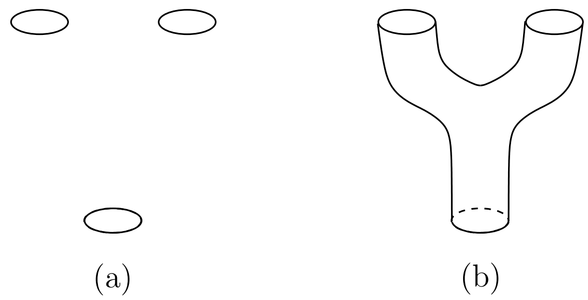


Figure J.14: Circles are the objects, cobordism is morphism from botoom circle to top circles. 1-d manifolds (circles) represent "space" and cobordism "spacetime".

T and Q with f between T and Q $f : T \to Q$. $u : A \to T$ and $f^{-1} : Q \to T$. Then $f \circ u : A \to Q$ for all A. $f \circ u = v$, $v : A \to Q$ for all A and so Q is also a terminal object.

There is only one morphism from Q to Q and that is the identity id_Q. So T is iso to Q.

But pick one of them and stick with it.

Products

Given two objects A and B, their **product** (if it exists) consists of an object

Definition Product diagram consists of an object P together with to arrows $p_1 : P \to A$ and $p_2 : P \to B$ which has the following properties. Given any object T with arrows $h : T \to A$ and $k : T \to B$, there exists a unique arrow $u : T \to P$ which makes the diagram fig J.1 (b) commute.

Theorem J.1.2 *If P, p_1, p_2 is a product diagram for A and B, and there is an iso $f : Q \xrightarrow{\sim} P$, then Q, q_1, q_2 is alos a product diagram for A and B.*

$$
\begin{tikzcd}
X \times Y \arrow[r] \arrow[dr, "h"'] & T \arrow[d, "u"] \\
& T'
\end{tikzcd}
$$

Figure J.15: universal1.

$$
\begin{tikzcd}
& T \arrow[d, "u"] \\
X \times Y \arrow[ur, "h"] \arrow[r, "h'"] \arrow[dr, "h"'] & T' \arrow[d, "u'"] \\
& T
\end{tikzcd}
$$

Figure J.16: universal2.

from fig. J.1(c) composes $f \circ v : T \xrightarrow{\sim} P$ so that by uniqueness $f \circ v = u$. As u is unique and f was given, $v = f^{-1} \circ u$ is the unique arrow from T to Q composing with $p_1 \circ f$ to give h, and with $p_2 \circ f$ to give k.

J.1.1 Monoidal Categories

Topological qunatum field theory TQFT.

A monoidal category has tensor products of objects and morphisms satisfying the usual axioms, and an object **1** playing the role of identity for the tensor product.

J.2 Topos

If we wish to make the context dependence the central feature of the mathematical formulation the maths that does this is topos!

speculated application to social, political, economic worlds [13].

pass from a an element-based approach to an arrow-based one approach to an arrow based one in the category of sets.

Non-Boolean logic does not assume every statement can be judged true or false, there are some statements upon which one cannot decide. An example of a physical situation where the underlying logic would be non-Boolean is in cosmology where observers can only make judgements upon statements that have to do with their backwards light cone.

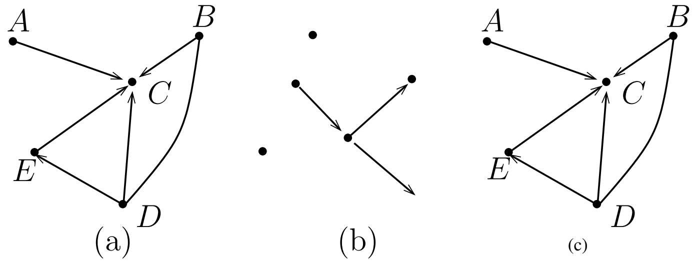


Figure J.17: In especially simple and clear way. (a) Terminal object **Dgrph**. (b) Initial.

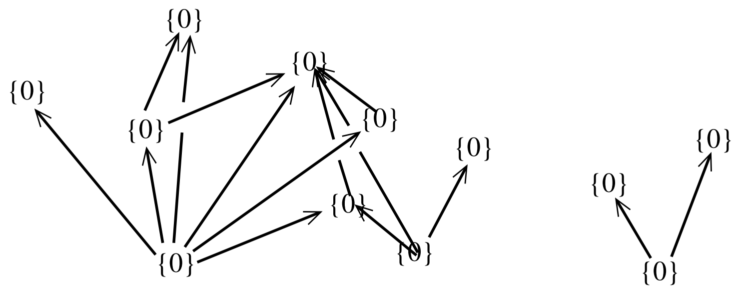


Figure J.18: causalsetex3. Taken from [366].

technique from categorial maths (presheaves) to which we give the physical interpretation of context-dependence.

$$
\begin{array}{rcl}
\text{fixed assumptions} & \rightarrow & \text{physical statements} \\
 & \text{to} & \\
\text{Context category} & \rightarrow & \text{category of physical statements}
\end{array}
$$

we need physical statements/quantum theory that are observer-dependent. In relational interpretations of quantum theory, context category = all possible systems/enviroments.

Functors between two categories preserve the structure of the first category. In other words, a functor is a homomorphism between categories.

J.2.1 Basics of Topos

To a given proposition $\mathcal{P}$, we assign a subset A with the interpretation that an element belongs to A if and only if it satisfies the proposition $\mathcal{P}$. In classical logic two-valued logic admit two distinct truth values, namely truth and untruth.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{J.9}$$

$$
\begin{tikzcd}
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T \arrow[dl, "h"'] \arrow[d, "u"] \arrow[dr, "k"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

(a) (b)

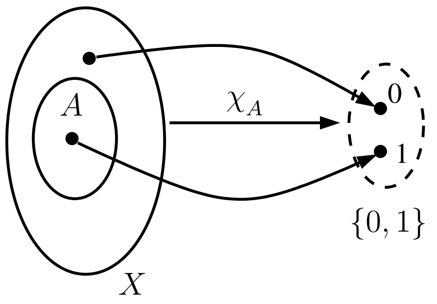
Figure J.19: A product diagram (a) is defined, if u for all T, there an exists a unique arrow u that makes diagram (b) commute.

$$
\begin{tikzcd}
 & T \arrow[dl, "h"'] \arrow[d, "u"] \arrow[dr, "k"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
\text{ and }
\begin{tikzcd}
Q \arrow[r, shift left, "f"] & P \arrow[l, shift left, "f^{-1}"]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & T \arrow[dl, "h"'] \arrow[d, "f^{-1} \circ u"] \arrow[dr, "k"] & \\
A & Q \arrow[l, "p_1 \circ f"] \arrow[r, "p_2 \circ f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T \arrow[ddl, "h"'] \arrow[d, "v"] \arrow[ddr, "k"] & \\
 & Q \arrow[d, "f"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

(a) (b) (c)

Figure J.20: Proof of Theorem J.1.2. from (c) we see that $f \circ v = u$ is the unique arrow that makes the diagram commute.

$$A = \chi_A^{-1}(1) \tag{J.10}$$



Note that in nthis setting we will refer to A as the subset of X meaning that there is an inclusion function from A to X.

From a purely technical point of view, it does not matter what truth values and are, so long as they are distinct.

topos: To find out more about the subject to John Baez homepage (http://math.ucr.edu/home/baez/README.html) "fun stuff" and then to the link topos.

1. an initial object (an object like the empty set)
2. a terminal object (an object like a set with one element)

3. binary coproducts (something like the disjoint union of two sets)
4. binary products (something like the Cartesian product of two sets)
5. equalizers (something like the subset of X consisting of all elements x such that $f(x) = g(x)$, where $f, g : X \to Y$)
6. coequalizers (something like the quotient set of X where two elements $f(y)$ and $g(y)$ are identified, where $f, g : Y \to X$)

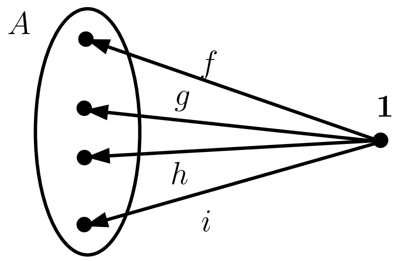
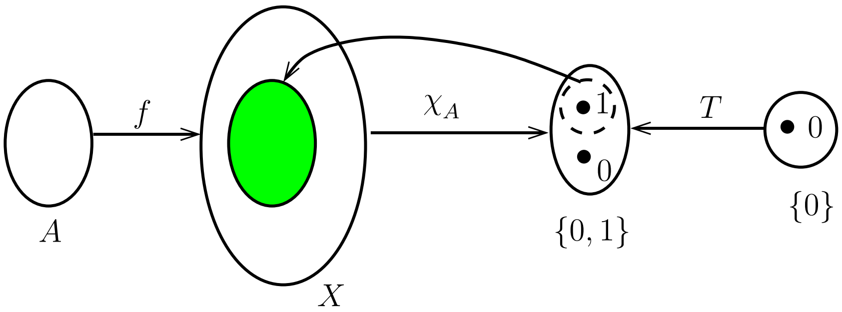


Figure J.21: one-to-one correspondence of elements of A to functions from the terminal set to A.

(3) analogy of $\{0, 1\}$ denoted Ω, intuitively, the elements of Ω are the answers to a natural 'multiple-choice questions' about objects in the objects in the topos, just as "$x \in X$" is a natural for sets.



Category theory notes for ESSLLI

binary products

binary coproducts (something like the disjoint union of two sets) or sums that allow the specification of alternatives

The category of presheaves on $\mathcal{C}$, $\mathbf{Set}^{\mathcal{C}}$, forms a topos.

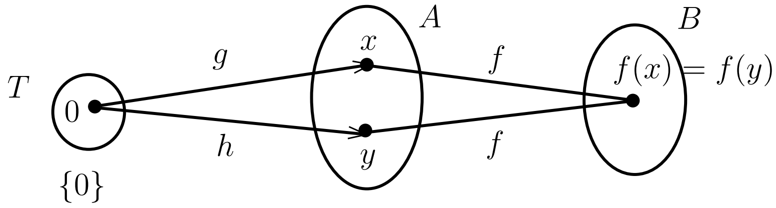
coming from [361]

We will include more details.

Definition An isomorphism (or iso) is a pair of arrows

$$
\begin{tikzcd}
A \arrow[r, shift right, "g"'] & B \arrow[l, shift right, "f"']
\end{tikzcd}
$$

such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. If $f : A \to B$ is an arrow of an ico pair,we say that f is an isomorphism (or f is an iso), and write $f : A \cong B$.



$$g(0) = x \qquad h(0) = y \tag{J.11}$$

f is one-to-one or injective.

Definition The arrow f in the category $\mathcal{C}$ is mono (monic) if and only if for any pair of arrows g, h

$$
\begin{tikzcd}
A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "f"] & C
\end{tikzcd}
$$

we have that $f \circ g = g \circ f$ implies $g = h$.

Definition The arrow f in the category $\mathcal{C}$ is ***epi*** if and only if for any pair of arrows g, h Z

we have that $g \circ f = f \circ h$ implies $g = h$. We say that it is left cancellable??.

Definition Generalized element: We can think of an arrow to any object B, say $x : A \to B$, as a kind of element of B. x ***the generalized element*** of B

Summary:

$$
\begin{aligned}
x \in_T A,\ y \in_T A \qquad & f(x) = f(y) \quad \Rightarrow x = y \\
x : T \to A,\ y : T \to A \qquad & f \circ x = f \circ y \Rightarrow x = y
\end{aligned} \tag{J.12}
$$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

Definition (Locally Small). A *category* $\mathcal{C}$ is called locally small if for all objects $X, Y \in \mathcal{C}$, the collection $\mathcal{C}(X, Y)$ is a set.

Definition (Small). A *category* $\mathcal{C}$ is called ***small*** if in addition as locally small, the collection of all objects is a *set*.

??In a mere set elements are either the same or different; there is nothing else to say. The category **Sets** of all sets is not small because there is no set of all sets.??

If the function f's domain is restricted to the shaded region then $f \circ g = f \circ h$ then the fact that would not imply $g = h$, hence f must at least be onto (surjective). In set theory the monic is an inclusive and one-to-one function

Pullbacks

Given any two arrows f and g with the same codomain

$$\begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

Figure J.22: pullback1.

their pullback consists of a pair of arrows p_1, p_2 and an object T such that $f \circ p_1 = p_2 \circ p_2$

$$\begin{tikzcd} D \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Figure J.23: pullback2.

such that for every pair of arrows $h : T \to A$ and $k : T \to B$ that makes the outer square commutes, i.e. $f \circ h = g \circ k$, there exists a unique arrow $uT :\to D$ $p_1 \circ u = h$ and $p_2 \circ u = k$

$$\begin{tikzcd}
T \arrow[dr, "u"] \arrow[drr, "k"] \arrow[ddr, "h"'] & & \\
& D \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Figure J.24: pullback3.

$$\begin{tikzcd}
D \arrow[r, "k"] \arrow[d, "h"'] & E \arrow[r, "m"] \arrow[d, "i"] & F \arrow[d, "j"] \\
A \arrow[r, "g"'] & B \arrow[r, "f"'] & C
\end{tikzcd}$$

Figure J.25: pullback10.

Equalizers

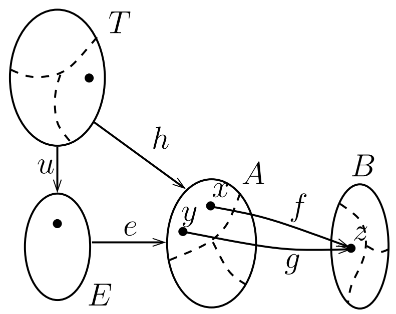


Figure J.26: The equalizer in the category **Set**.

In **Set**, the equalizer of f and g is the set of points x such that $f(x) = g(x)$. In **Grp** it is the kernal of a homomorphism $f : G \to H$ is the equalizer of f and the constant map at the group identity.

Subobject Classifier

a subobject A of X that represents the part of X on which b is true. That b is true is to say that the following diagram commutes

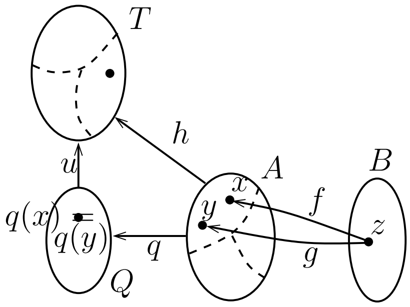


Figure J.27: The coequalizer in the category **Set**.

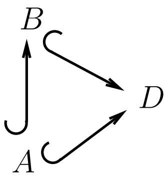


Figure J.28: inclusion. $A \hookrightarrow B$ and $B \hookrightarrow D$ implies $A \hookrightarrow D$. subobject of D

J.2.2 Cartesian Closed Categories

We used the notion of an element using maps from the final object. A cartesian closed category is one in which the arrows can also be seen as being functions. function space, evaluation, composition, etc.

Given sets A and B we have in **Set** the colection of all functions from A to B we denote

$$B^A \tag{J.13}$$

the evaluation map is

$$ev : B^A \times A \to B \tag{J.14}$$

written out in the notation this takes the more familiar form of evaluation:

$$ev(< f, x >) = f(x) \tag{J.15}$$

ev the *evaluation* function.

, a mono b :? →? corresponds to a subset of A and χ_A is the charaeristic morphism ?.

$$
\begin{tikzcd}
A \arrow[r, "\chi_A"] \arrow[d, "!_A"'] & X \arrow[d, "b"] \\
\mathbf{1} \arrow[r, "\mathbf{true}"'] & \{0,1\}
\end{tikzcd}
$$

Figure J.29: subobject of .

$$
\begin{tikzcd}
X \arrow[dr, "x_b"] \arrow[drr, "x"] \arrow[ddr, "!_X"'] & & \\
& A \arrow[r, "\chi_A"] \arrow[d, "!_A"'] & X \arrow[d, "b"] \\
& \mathbf{1} \arrow[r, "\text{true}"'] & \{0,1\}
\end{tikzcd}
$$

Figure J.30: A pullback of b along **true**.

Definition (**Topos**). An elementary topos is a category such that

1. is finitely complete
2. is finitely co-complete,
3. has exponentiation,
4. with a **sub-object classifier**.

Heyting Algebra of $Sub(X)$

Definition (**Topos**). A topos is a **cartesian closed caregory** with a **sub-object classifier**.

The arrow χ_A is called the charateristic morphism of b, this notation comes from the situation in **Set**, where $\Omega = \{0, 1\}$.

$m \leq m'$ if there is $f : Y \to Y'$ satisfying $m' \circ f = m$, i.e.,

$$m' \circ f = m \quad \Longleftrightarrow \quad m \leq m'. \tag{J.16}$$

We have a partial order $Sub(X)$ ordered by the smaller-than relation.

$$
\begin{tikzcd}
A \arrow[r, "\chi_A"] \arrow[d, "!_A"'] & X \arrow[d, "b"] \\
\mathbf{1} \arrow[r, "t"'] & \Omega
\end{tikzcd}
$$

Figure J.31: A pullback of b along t.

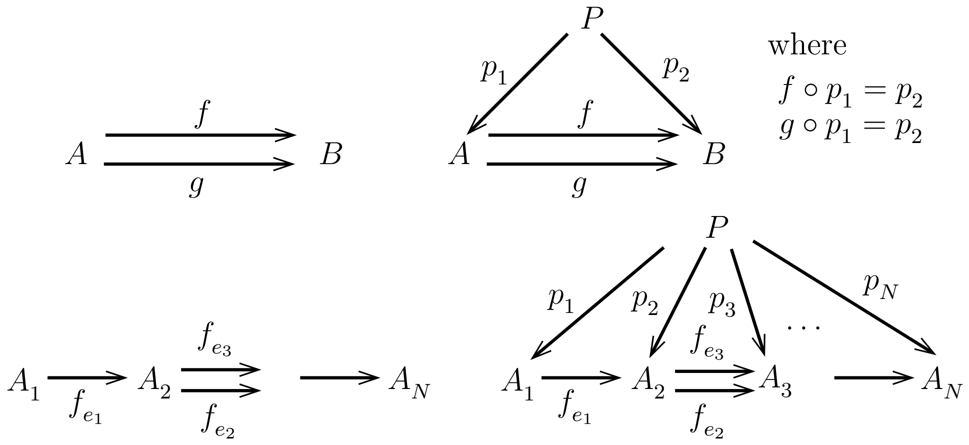


Figure J.32: cones3. (a) $f \circ p_1 = p_2$ and $g \circ p_1 = p_2$. (b) $f_{e_1} \circ p_1 = p_2$, $f_{e_2} \circ p_2 = p_3$, $f_{e_3} \circ p_2 = p_3$, etc

As there always exists f for any subobject m such that

$$\mathrm{id}_X \circ f = m, \tag{J.17}$$

, (i.e. $f = m$), id_X is the greatest element of $Sub(X)$.

$Y \cap Y'$ given by the pullback fig(J.2.2). This seen by noting that for any n, n' such that $n \circ m \leq m''$ and $n' \circ m' \leq m''$, there exists, by defintion of a pullback, a (unique) arrow $u : Y \cap Y' \to X$ such that $m'' \circ u = (m \circ n)$ and $m'' \circ u = (m' \circ n')$, see fig(J.2.2). Hence, $m \circ n \leq m''$ and $m' \circ n' \leq m''$.

A Heyting algebra $\mathbf{H}$ is a latice with greatest and least elements in the which the meet $a \wedge b$ has an implication operator, $\to$, satisfying

$$a \wedge b \leq c \quad a \leq b \to c \tag{J.18}$$

$$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b) \tag{J.19}$$

$$\begin{tikzcd}
B^A \times A \arrow[r, "ev"] & B \\
C \times A \arrow[u, "\tilde{g} \times \mathrm{id}_A"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Figure J.33: exponat. in the category **Set**.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "s"'] & 1 \arrow[d, "t"] \\
A \arrow[r] & \Omega
\end{tikzcd}$$

Figure J.34: .

Categorically this corresponds to a ccc with finite coproducts.

Clearly $m \leq m \cup m'$ and $m' \leq m \cup m'$;

$$\begin{aligned} m \cup m' \circ f &= m \\ m \cup m' \circ f' &= m' \end{aligned} \tag{J.20}$$

where f is formed by composition of the the top arrow in fig.(J.2.2) with e and f' is formed by composition of the the left arrow with e.

Lemma J.2.1 *In a topos $\mathcal{E}$ each pair of elements of Sub(X) has a least upper bound.*

Suppose there is a subobject $n : Z \rightarrowtail X$ such that $m, m' \leq n$, we wish to show that for any such arrow $m \vee m' \leq n$. By definition of the coproduct there is a unique arrow u from $Y + Y'$ to Z such that the diagram fig.(J.2.2) commutes. This means that $n \circ u$ is a factorization of $[m, m']$.

Since $(m \vee m') \circ e$ is the least such factorization(fig.(J.2.2)), there is an arrow $s : Y \vee Y' \to Z$ such that $n \circ s = m \vee m'$ which shows that $m \vee m' \leq n$.

saying $\wedge$ is the same as saying that it has a right adjoint.

A poset **H** is a Heyting algebra if we have elements

$$
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, tail, "m"'] & Q \arrow[dl, tail, "n"] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "f"] \arrow[d, tail, "m"'] & Q \arrow[dl, tail, "n"] \\
B \arrow[r, tail, "m'"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f \circ h"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, tail, "n"] \\
B \arrow[r, tail, "m'"'] & X
\end{tikzcd}
$$

Figure J.35: Submpinm. smaller than.

$$
\begin{tikzcd}
Z \arrow[dr, "u"] \arrow[drr, "n"] \arrow[ddr, "n'"'] & & \\
& Y \cap Y' \arrow[r, tail] \arrow[d, tail] \arrow[dr, tail, "m''"] & Y \arrow[d, tail, "m"] \\
& Y' \arrow[r, tail, "m'"'] & X
\end{tikzcd}
$$

Figure J.36: Heytlattice. Proof that $Y \cap Y'$ is the greatest lower bound of Y and Y'.

$$T \in \mathbf{H}, \qquad \perp \in \mathbf{H} \tag{J.21}$$

and operations

$$\wedge : \mathbf{H} \times \mathbf{H} \to \mathbf{H}, \qquad \vee : \mathbf{H} \times \mathbf{H} \to \mathbf{H}, \qquad \to : \mathbf{H} \times \mathbf{H} \to \mathbf{H} \tag{J.22}$$

such that $(\mathbf{H}, \wedge, \vee, T, \perp)$ is a lattice and

$$c \wedge a \leq b \quad c \leq a \to b \tag{J.23}$$

for all $a, b, c \in \mathbf{H}$.

$$
\begin{aligned}
\top &= \mathrm{id}_1 \\
\perp &= 0 \rightarrowtail 1 && \text{the uniqe arrow from the intial object.} \\
A \wedge B &= A \times_{\mathcal{E}} B && \text{the pillback of the A and B.} \\
A \vee B &= \mathrm{Im}(A + B) \\
A \to B &= B^A \rightarrowtail 1
\end{aligned} \tag{J.24}
$$

$$\begin{tikzcd}
Y+Y' \arrow[dr, "e"] & & Y \arrow[ll, tail] \arrow[dd, tail, "m"] \\
& Y \cup Y' \arrow[dr, tail, "m \cup m'"'] & \\
Y' \arrow[uu, tail] \arrow[rr, tail, "m'"'] & & X
\end{tikzcd}$$

Figure J.37: Heytlattice2. Proof that $Y \cup Y'$ is the least upper bound of Y and Y'.

$$\begin{tikzcd}
Y+Y' \arrow[dr, "u"] & & Y \arrow[ll, tail] \arrow[dd, tail, "m"] \\
& Z \arrow[dr, tail, "n"] & \\
Y' \arrow[uu, tail] \arrow[rr, tail, "m'"'] & & X
\end{tikzcd}$$

Figure J.38: Heytlattice3. Proof that $Y \cup Y'$ is the least upper bound of Y and Y'.

Theorem J.2.2 *For any object X in a topos $\mathcal{E}$, the partially ordered set $Sub_{\mathcal{E}}(X)$ is a Heyting algebra.*

$\mathrm{Sub}_{\mathcal{E}}(X) \cong \mathrm{Sub}_{\mathcal{E}/X}(1)$ together with the fact that $\mathcal{E}/X$ is a topos to conclude that $\mathrm{Sub}_{\mathcal{E}}(X)$ is a Heyting algebra.

Internal Heyting Algebra

subobject lattices $\mathrm{Sub}(X)$ in topos $\mathcal{E}$ induces an internal Heyting algebra on Ω via the isomorphism

$$\mathrm{Sub}_X \cong \mathcal{E}(X, \Omega) \tag{J.25}$$

Presheaves

[361],

The functor category $\mathrm{Set}^{\mathcal{C}^{op}}$ which is also denoted $\hat{\mathcal{C}}$, has presheaves as objects and natural transformations as arrows.

$$
\begin{tikzcd}
Y + Y' & Z \arrow[l, tail] \arrow[d, tail, "m"] \\
Y' \arrow[u, tail] \arrow[ur, "s"] \arrow[r, tail, "m'"'] & X
\end{tikzcd}
$$

Figure J.39: Heytlattice3A. Proof that $Y \cup Y'$ is the least upper bound of Y and Y'.

Yoneda Lemma

A remind of the concepts that will be involved in the statement of the Yoneda lemma:

Let $\mathcal{C}$ be a locally small category (i.e. for every pair of objects A and B the class of arrows $\mathcal{C}(A, B)$ ia a set). For any object A in $\mathcal{C}$ there is a functor $H_A : \mathcal{C}^{op} \to \text{Set}$ which assigns to any object B the set $\mathcal{C}(A, B)$. Any arrow $f : A' \to A$ gives by composition a function $\mathcal{C}(A', A) \to \mathcal{C}(A'', A')$. That is, if $g : A'' \to A'$ then $f \circ g : A'' \to A$ or if $f \in \mathcal{C}(A', A)$ and $g \in \mathcal{C}(A'', A')$ then $f \circ g \in \mathcal{C}(A'', A)$.

$$
Y_{\mathcal{C}}(B) = \text{Hom}(-, B) : \mathcal{C}^{op} \to \text{Set} \tag{J.26}
$$

For any two categoeires $\mathcal{A}$ and $\mathcal{B}$ there is a category $\mathcal{B}^{\mathcal{A}}$ (also denoted as $[\mathcal{A}, \mathcal{B}]$), whose objects are functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are natural transformations.

X is a functor $\mathcal{C}^{op} \to$**Set**.

Definition Let $\mathcal{C}$ be a locally small category. Then

$$
[\mathcal{C}^{op}, \text{Set}](H_A, X) \equiv \text{Set}^{\mathcal{C}^{op}}(H_A, X) \cong X(A) \tag{J.27}
$$

$$
\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, bend left=50, "H_A"{name=U}] \arrow[r, bend right=50, "X"'{name=D}] & \mathbf{Set}
\arrow[from=U, to=D, Rightarrow, shorten=2pt]
\end{tikzcd}
$$

Figure J.40: Yoneda1. Left hand sidee of J.27.

Definition A representation for a functor is $F : \mathcal{C}^{op} \to$**Set** consists of an object $C \in \mathcal{C}$ and a natural isomorphism

$$\eta_C = (-, C) \cong F. \tag{J.28}$$

If there exists a representation for F then we say the functor F is representable.

Sieves

$$\Omega(p) = \{\text{sieves on } p\} \tag{J.29}$$

Definition (Subfunctor). A functor $A : \mathcal{C}^{op} \to \mathbf{Set}$ is a subfunctor

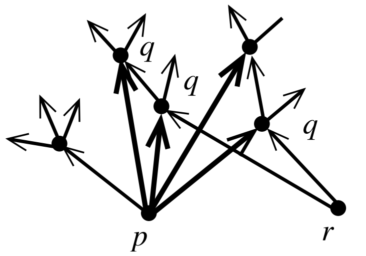


Figure J.41: caustruth1. Taken from [366].

J.2.3 Quantum Causal Histories

The *causal set* $\mathcal{C}$ is a category in which the objects are events $p, q, r \ldots$ and arrows are causal relations between events: $p \leq q$.

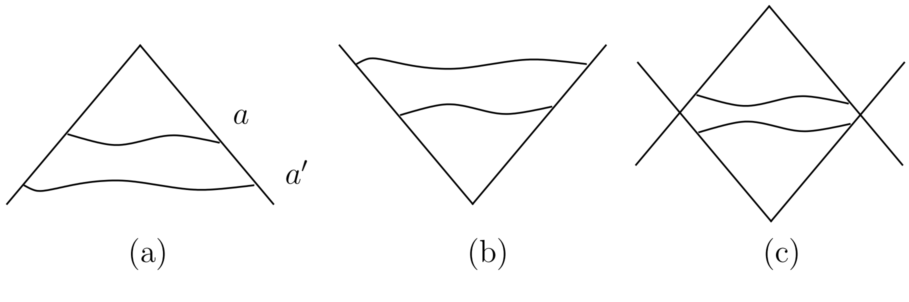


Figure J.42: .

1. The causal past of some event p is the set of all events $r \in \mathcal{C}$ with $r \leq p$. We denote the causal past of p by $P(p)$.

2. The causal future of p is the set of all events $q \in \mathcal{C}$ with $p \leq q$. We denote it by $F(p)$.

3. An *acausal set*, denoted $a, b, c, ...$, is a set of events in $\mathcal{C}$ that are all causally unrelated to each other. Acausal sets plays the role of spacelike hypersurfaces in causal stes.

4. The acausal set a is a complete past of the event p when every event in the causal past $P(p)$ of p is related to some event in a. It is not possible to add an event from $(P(p) - a)$ to a and produce a new acausal set.

5. Similarly, an acausal set b is a complete future of p when every event in the causal future $F(p)$ of p is related to some event in b.

6. A chain is a subset of a poset such that they are connected, this motivates A maximal antichain in the causal set C is an acausal set A such that every event in $(\mathcal{C} - A)$ is causally related to some event in A.

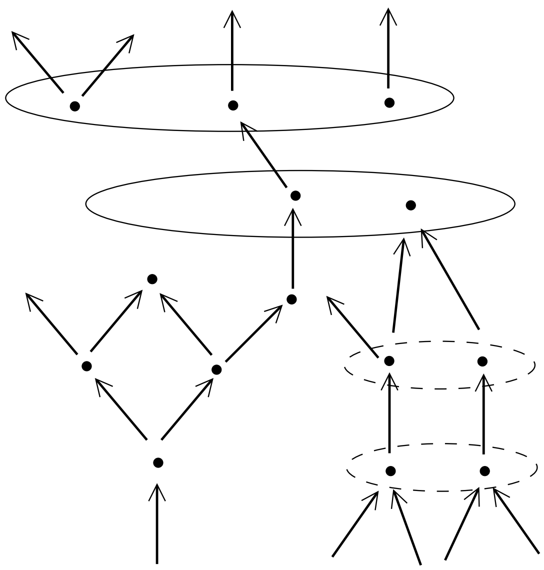


Figure J.43: .

$p = a\mathbf{Set}$:

objects: sets of events $\{r \leq p\}$
arrows: inclusions $\{r \leq p\} \subseteq \{r \leq q\}$ when $p \leq q$.

see definition of a presheaf in the maths glossary.

J.2.4 Consistent Histories Interpretation

Kochen-Specker

J.2.5 Application to Rovelli's Relational Quantum Mechanics

may require the use of more than one Hilbert space, in a way that cannot be 'reduced' to a single Hilbert space.

Initial state of A is $|ON>$. There is an operator $\hat{U}$ in $\mathcal{H}_e \otimes \mathcal{H}_A$.

$$\hat{U} : |\text{spin} > \otimes|\text{ON} > \tag{J.30}$$

Correlated

$$\begin{array}{rcl} (a|\uparrow> +b & \to & a|\uparrow> \otimes|\text{``}\uparrow\text{''} > +b|\downarrow> \otimes|\text{``}\downarrow\text{''} > \\ |\uparrow> \otimes|\text{ON} > & \to & |\uparrow> \otimes|\text{``}\uparrow\text{''} > \\ |\downarrow> \otimes|\text{ON} > & \to & |\downarrow> \otimes|\text{``}\downarrow\text{''} > \end{array} \tag{J.31}$$

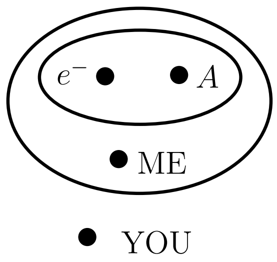


Figure J.44: ME YOU

If I'm going to look consistently say that A the apparatus a new interaction operator of in the Hilbert space of all three of us.

$$\mathcal{H}_e \otimes \mathcal{H}_A \otimes \mathcal{H}_{ME}$$

I should not be passed on a collapsed state by YOU.

When are you going to go from generally a state in the tensor product to a particular one and when does it make sense.

all this makes sense if you tell me which system is supposed to say what about which system. Have all possible

My Hilbert space of all possible systems and subsystems.

a partially ordered set.

Sieves will come from: If you see ME saying e^- is $|\uparrow>$, then you know that A showed $|\text{``}\uparrow\text{''} >$ and e^- is $|\uparrow>$. i.e., you projected on a particular state implies every **subsystem** of YOU is projected to a particular state.

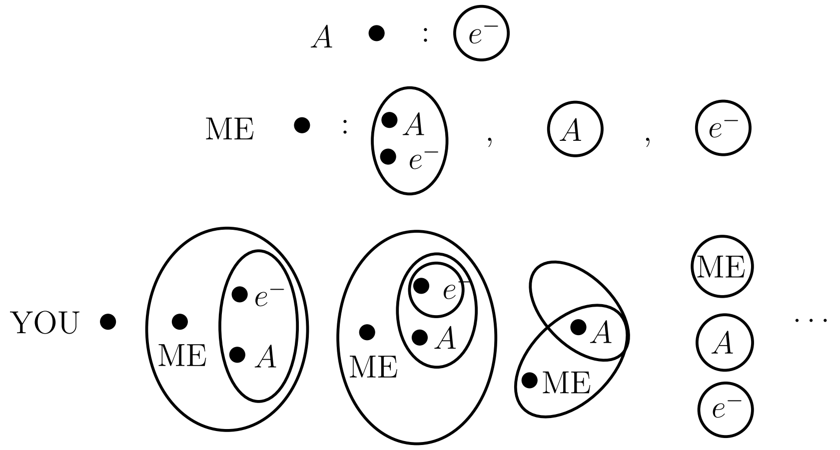


Figure J.45: Partially ordered set. Context for all possible systems and subsystems.

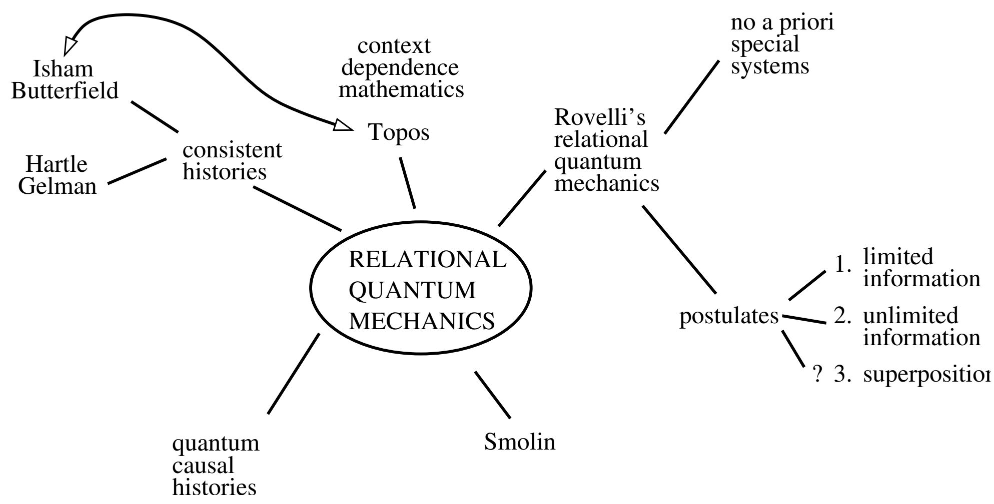


Figure J.46: Relational quantum mechanics.

J.3 Biblioliographical notes

In this chapter I have relied on the following references:

Arun Ram Quantum groups: *A survey of definitions, motivations, and results.*

J.4 Worked Exercises

Proofs

the arrow $f : X \to Y$ in the category $\mathcal{C}$

Given that for any g and h

$$Z \xrightarrow{g,h} X \xrightarrow{f} Y \tag{J.32}$$

we have that $f \circ g = f \circ h$ implies $g = h$.

1. (i) Show that the composition of monos are mono.

(ii) Show that if the composition $f \circ e$ is mono then e is mono.

Answers

1. (i) We have the arrows $e : W \to X$ and $f : X \to Y$ are mono.

we want to show that for any pair g and h

$$Z \xrightarrow{g,h} X \xrightarrow{f \circ e} Y \tag{J.33}$$

we have that $(f \circ e) \circ g = (f \circ e) \circ h$ implies $h = g$.

$$Z \xrightarrow{g,h} W \xrightarrow{e} X \xrightarrow{f} Y \tag{J.34}$$

$$Z \xrightarrow{e \circ g, e \circ h} W \xrightarrow{f} Y \tag{J.35}$$

Since composition is associative the condition $(f \circ e) \circ g = (f \circ e) \circ h$ is the same as $f \circ (e \circ g) = f \circ (e \circ h)$. As f is mono we have: $e \circ g = e \circ h$. Now, as e is mono for any g and h

$$Z \xrightarrow{g,h} W \xrightarrow{e} X \tag{J.36}$$

we have that $e \circ g = e \circ h$ implies $g = h$.

(ii) We want to show that $e \circ g = e \circ h$ implies $g = h$. Suppose $e \circ g = e \circ h$, applying f to both sides gives $f \circ (e \circ g) = f \circ (e \circ h)$ or $(f \circ e) \circ g = (f \circ e) \circ h$ and since the composition $f \circ e$ is mono $g = h$, hence e is mono too.

2. (i)

$$W \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g,h} Z \tag{J.37}$$

The condition $g \circ (f \circ e) = h \circ (f \circ e)$ is the same as $(g \circ f) \circ e = (h \circ f) \circ e$. Now, since e is epi we have $g \circ f = h \circ f$. As f is epi $g = h$, and hence we have shown the composition $f \circ e$ is epi too.

implies $g = h$.

$\boxed{}$

Products.

Consider any arrows $a : T \to E$ and $b : T \to E$ such that $e \circ a = e \circ b$. We wish to show this implies $a = b$.

$$\begin{tikzcd} T \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

(a)

$$\begin{tikzcd} T \arrow[d, "a"'] \arrow[dr, "h = a \circ e"] & & \\ E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

(b)

Figure J.47: Steps in proof of theorem .

we could have done the same for b in plase of a, and as $b \circ e = a \circ e$, the unique arrow that makes this diagram the same equalizer is b. Hence $a = b$.

Products.

1 Show there is a

2 Show there is a

Answer

$$\begin{tikzcd} & Q \arrow[dl, "p_1"'] \arrow[d, "u"] \arrow[dr, "p_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

(a)

$$\begin{tikzcd} A & Q \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\ & P \arrow[ul, "p_1"] \arrow[u, "v"'] \arrow[ur, "p_2"'] & \end{tikzcd}$$

(b)

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[d, "u \circ v"] \arrow[dr, "p_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & A \end{tikzcd} \Rightarrow u \circ v = \mathrm{id}_A$$

(c)

Figure J.48: Steps in proof of theorem **??**.

$$\begin{tikzcd} & A \times B \arrow[dl, "h \circ p_1"'] \arrow[d, "h \times k"] \arrow[dr, "k \circ p_2"] & \\ A & A' \times B' \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Figure J.49: Implies $h \times k$ is unique E.7.

Twist arrows are natural

Prove:

$$tw \circ f \times g = g \times f \circ tw \tag{J.38}$$

$$\begin{tikzcd}
A \times B \arrow[r, "\sim"] \arrow[d, "f \times g"'] & B \times A \arrow[d, "f \times g"] \\
A' \times B' \arrow[r, "\sim"] & B' \times A'
\end{tikzcd}$$

Figure J.50: .

Answer

$$\begin{tikzcd}
A \times B \arrow[r, "f \times g"] \arrow[dr, "p_2"'] & A' \times B' \arrow[r, "\sim"] \arrow[dr, "p'_2"'] & B' \times A' \arrow[d, "q'_1"] \\
& B \arrow[r, "g"'] & B'
\end{tikzcd}$$

Figure J.51: prod9.

(a) (b)

$$\begin{tikzcd}
A \times B \arrow[r, "\sim"] \arrow[dr, "p_2"'] & B \times A \arrow[r, "g \times f"] \arrow[d, "q_1"] & B' \times A' \arrow[d, "q'_1"] \\
& B \arrow[r, "g"'] & B'
\end{tikzcd}$$

Figure J.52: prod10.

$$q'_1 \circ tw \circ f \times g = q'_1 \circ g \times f \circ tw \tag{J.39}$$

Quotient categories.

$p_1 \circ e$ $p_2 \circ e$

$$\begin{tikzcd} A \times B \arrow[r, "g \circ p_2"] \arrow[d, "f \times g"'] & B' \\ A' \times B' \arrow[r, "\sim"'] & B' \times A' \arrow[u, "q_1'"'] \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[r, "\sim"] \arrow[d, "g \circ p_2"'] & B \times A \arrow[d, "g \times f"] \\ B' & B' \times A' \arrow[l, "q_1'"] \end{tikzcd}$$

Figure J.53: prod11.

$$\begin{tikzcd} E \arrow[r, "p_1 \circ e"] \arrow[d, "p_2 \circ e"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

Figure J.54: .

$$p_1 \circ e \tag{J.40}$$

$p_1 \circ e$ $p_2 \circ e$

$$p_1 \circ e \tag{J.41}$$

The morphism e equalizes f and g because we have

$$f \circ e = p_1 \circ < f, g > \circ e = (p_1 \circ \Delta_B) \circ f' = \mathrm{id}_B \circ f' = f' \tag{J.42}$$
$$g \circ e = p_2 \circ < f, g > \circ e = (p_2 \circ \Delta_B) \circ f' = \mathrm{id}_B \circ f' = f' \tag{J.43}$$

Quotient categories.

For

i) $f, g : X \to Y$ and $h : Y \to Z$, if $f \sim_{X,Y} g$ then $h \circ f \sim_{X,Y} h \circ g$

ii) $f : X \to Y$ and $g, h : Y \to Z$, if $g \sim_{Y,Z} g$ then $g \circ f \sim_{X,Z} h \circ f$.

2 Show there is a functor $\mathcal{C} \to \mathcal{C}/\sim$, which takes each arrow of $\mathcal{C}$ to its equivalence class.

$$
\begin{tikzcd}
 & T \arrow[dl, "h"'] \arrow[d, "< h, k >"] \arrow[dr, "k"] & \\
A & A\times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
T \arrow[d, "u"'] \arrow[dr, "< h, k >"] & & & \\
E \arrow[r, tail, "e"] & A\times B \arrow[r, "p_1"] \arrow[dr, "p_2"'] & A \arrow[r, "f"] & C \\
 & & B \arrow[ur, "g"'] &
\end{tikzcd}
$$

(b)

Figure J.55: this one.

$$
\begin{tikzcd}
T \arrow[d, "u"] \arrow[dr, "< h, k >"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
E \arrow[r, "e"] & A\times B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "f"] \\
 & A \arrow[r, "g"'] & C
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
T \arrow[dr, "u"'] \arrow[drr, "k"] \arrow[ddr, "h"'] & & \\
 & E \arrow[r, "p_2 \circ e"'] \arrow[d, "p_1 \circ e"] & B \arrow[d, "f"] \\
 & A \arrow[r, "g"'] & C
\end{tikzcd}
$$

(b)

Figure J.56: this one.

Answer

We need to show that composition is representation independent, i.e., that

for $[f]_{X,Y} : X \to Y$, $[g]_{Y,Z} : Y \to Z$ and $[h]_{Y,Z} : Y \to Z$ then: $[h]_{Y,Z} \circ [f]_{X,Y} = [h]_{Y,Z} \circ [g]_{X,Y}$

and for $[g]_{X,Y} : X \to Y$, $[h]_{Y,Z} : Y \to Z$ and $[f]_{Y,Z} : Y \to Z$ then: $[g]_{Y,Z} \circ [f]_{X,Y} = [h]_{Y,Z} \circ [f]_{X,Y}$

condition 1. above is the same as $[f]_{X,Y} = [g]_{X,Y}$, if $[h \circ f]_{X,Y} = [h \circ g]_{X,Y}$

$$[h]_{Y,Z} \circ [f]_{X,Y} = [h \circ f]_{X,Z} = [h \circ g]_{X,Z} = [h]_{Y,Z} \circ [g]_{X,Y} \tag{J.44}$$

$$[g]_{Y,Z} \circ [f]_{X,Y} = [h \circ f]_{X,Z} = [g \circ f]_{X,Z} = [g]_{Y,Z} \circ [f]_{X,Y} \tag{J.45}$$

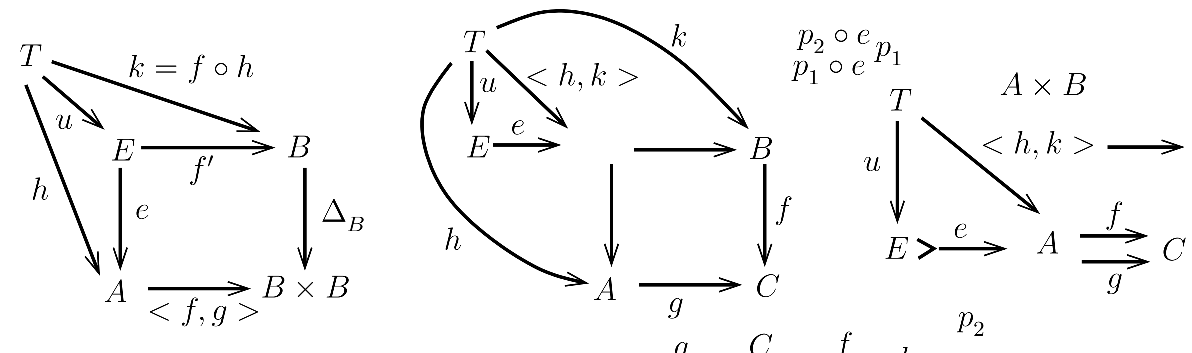
2 $F : \mathcal{C} \to \mathcal{C}/\sim$. Does $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

Proofs

From the diagram fig(E.7), it is easy to see that

$$
\begin{tikzcd}
T \arrow[dr, "u"] \arrow[drr, "k = f \circ h"] \arrow[ddr, "h"'] & & \\
& E \arrow[r, "f'"'] \arrow[d, "e"] & B \arrow[d, "\Delta_B"] \\
& A \arrow[r, "<f, g>"'] & B \times B
\end{tikzcd}
$$

Figure J.57: this one.



(a) (b) (c)

Figure J.58: this one.

$$
g \circ p_2 \circ h = g \circ p_2 \circ k, \tag{J.46}
$$

p_1 can be composed with k or h p_2 can be composed with k or h to form arrows from T to A and from T to B respectively.

$$
p_1 \circ k = p_1 \circ h, \qquad p_2 \circ k = p_2 \circ h \tag{J.47}
$$

Therefore the two diagrams in figs J.4(a) and (b) coincide except possibly for the arrows h and k. However, by definition of a pullback there exists a unique arrow $u : T \to A \times_C B$ such that diagrams shown in figs(a) and (b) should commute. Hence $h = k$.

co-products

Proofs

Obviously the conditions $f \circ p_1 = p_3$ and $g \circ p_2 = p_3$ are equivalent to the condition that the square in diagram fig(J.4) commutes. Similar observation for the cone T q_1, q_2. Whence, the condition that C, p_1, p_2 be a limit is equalent to the condition that C, p_1, p_2 be a pullback.

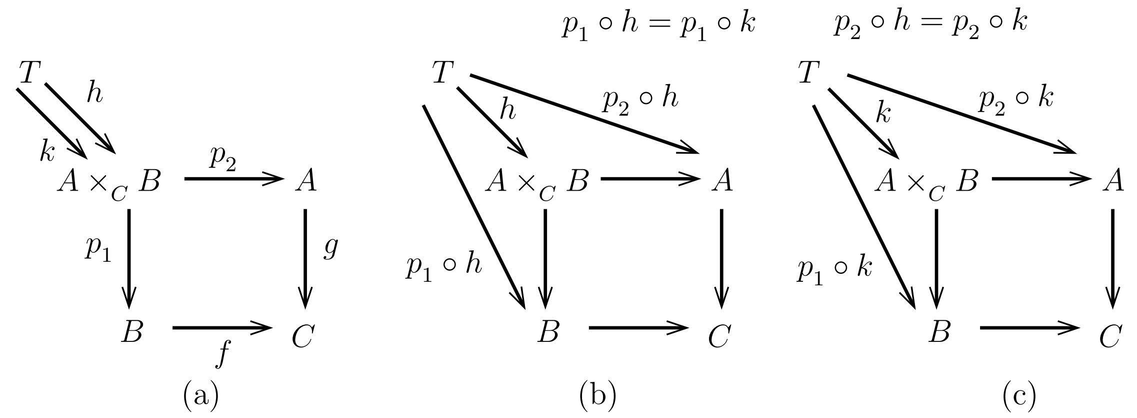


Figure J.59: Since $p_1 \circ k = p_1 \circ h$ and $p_2 \circ k = p_2 \circ h$, $h = k$.

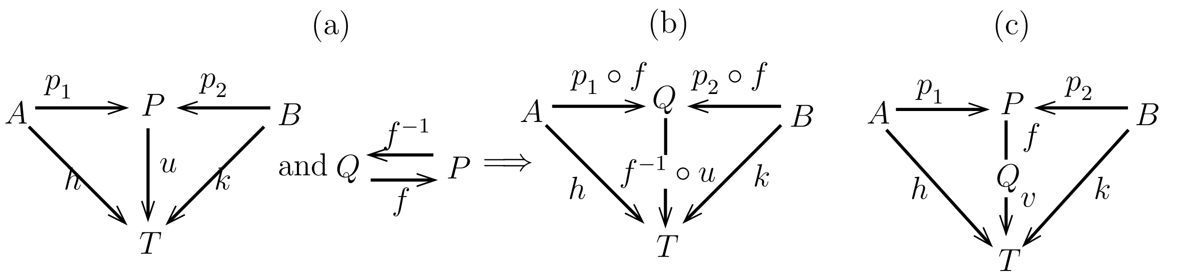


Figure J.60: Proof of Theorem J.1.2. from (c) we see that $f \circ v = u$ is the unique arrow that makes the diagram commute.

Limits as Equalizers of Products

A T-element $x_i \in_T A_i$ for each vertex i for every $f_e : A_j \to A_k$ we have $f_e(x_j) = x_k$.

Then D has a limit if and only if there is an equalizer for r and s.

Consider any cone T, $h_i : T \to A_i$ over D.

Questions on intersection pullbacks

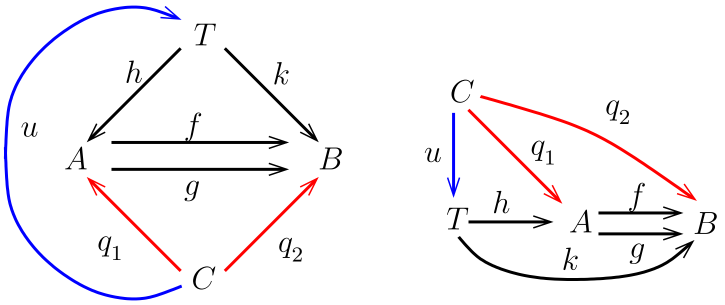
1.

2.(a) Show that for any X, and any $x \in_X A$ that $x \in i \cap j$ if and only if $x \in i$ and $x \in j$.

(b) Explain why this is just the defition of a pullback in this notation.

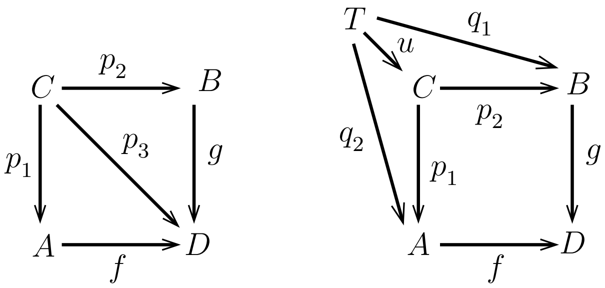
Proof

1.



(a) (b)

Figure J.61: limits3.



(a) (b)

Figure J.62: limits4.

Conversely, suppose the outer square is a pullback $q : T \to E$ and $t : T \to A$ with $i \circ q = f \circ t$.

So that there is a unique $u : T \to D$ with $t = h \circ u$ and $m \circ q = (m \circ k) \circ u$, see fig(J.4(a)).

Thus $m \circ (k \circ u) = m \circ q$ and $i \circ (k \circ u) = i \circ q$. As m and i are the projection arrows of a pullback, $q = k \circ u$. Then $t = h \circ u$ and $q = k \circ u$. Must prove u is unique.

2.

for any object X, and any $x \in_X A$ has $x \in i \cap j$ if and only if $x \in i$ and $x \in j$.

Say there is h such that $x = i \circ h$ and $x = j \circ k$, (i.e. $x \in i$ $x \in j$). Then the outer square commutes, i.e., $i \circ h = j \circ k$. From the definition of a pullback we know that there is a (unique) arrow $u : X \to S \cap T$ such that $x = i \cap j \circ u$.

Conversely, suppose there is an arrow $u : X \to S \cap T$ such that $x = i \cap j \circ u$.

3.

Note $m \circ q = f \circ p$. Since $n \circ (h \circ q) = f \circ p$, see fig (J.4(a)). Now as F is a pullback, there is a unique $u : E \to F$ with $p = r \circ u$ and $h \circ q = s \circ u$, (J.4(b)). Check that the whole diagram commutes.

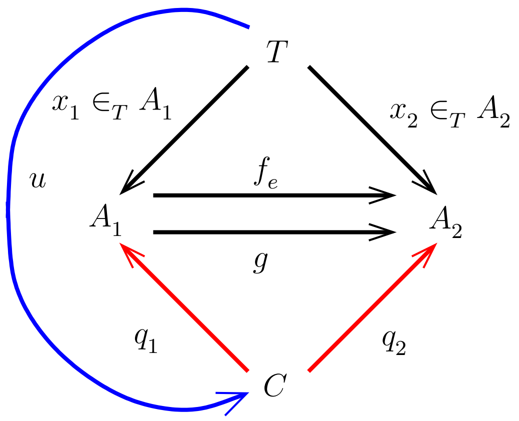


Figure J.63: limits5. The T-element $x_1 \in_T A_1$ for the vertex 1 for $f_e : A_1 \to A_2$ we have $f_e(x_1) = x_2$

Heyting Algebras

Proofs

Show that any Heyting algebra is a *distributive* lattice:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{J.48}$$

for all $a, b, c \in \mathbf{H}$.

Sub(X) forms a Heyting Algebra

Questions

1. Prove that the pullback $Y \cap Y'$ is the greatest lower bound for Y and Y'.

Proof

1.

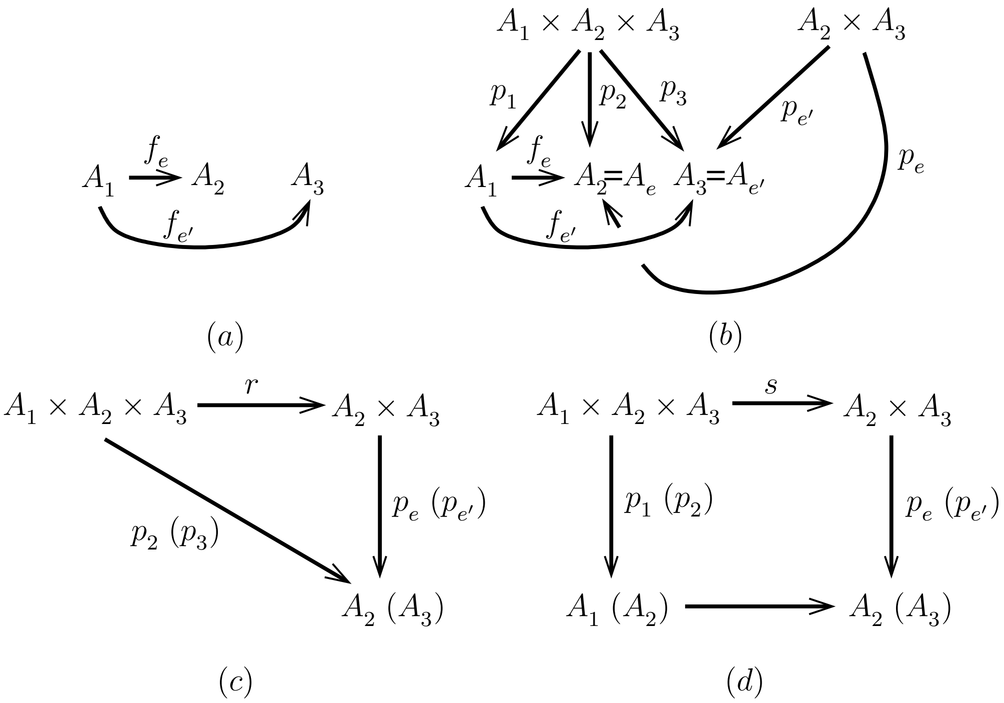


Figure J.64: limEqProd. (a) For diagram D. (b)

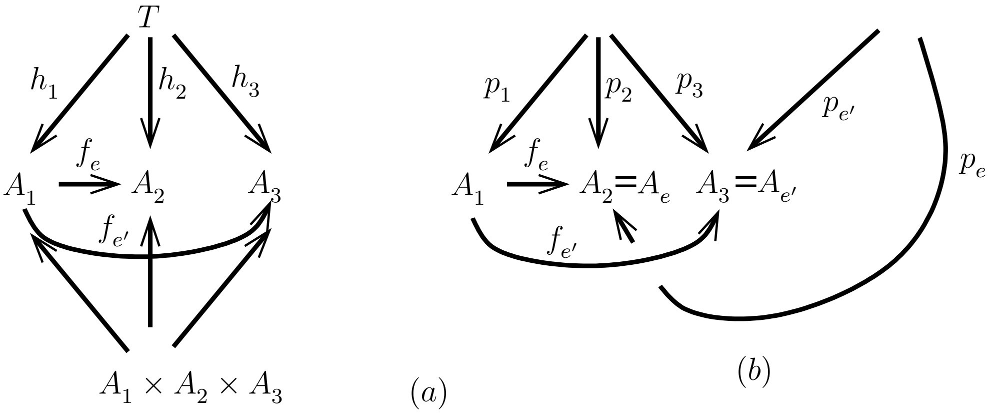
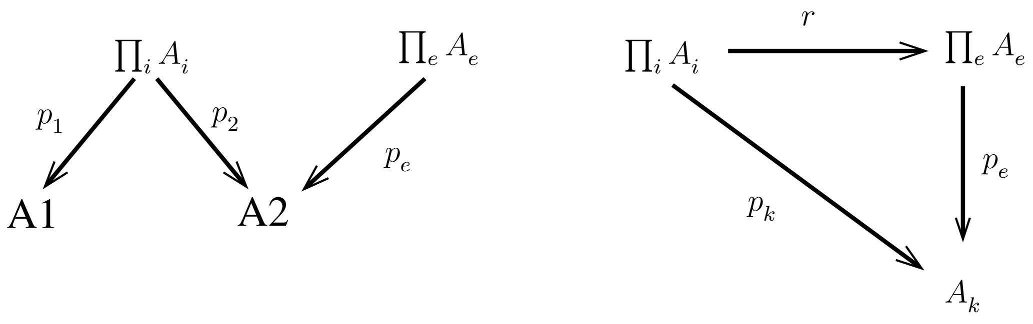
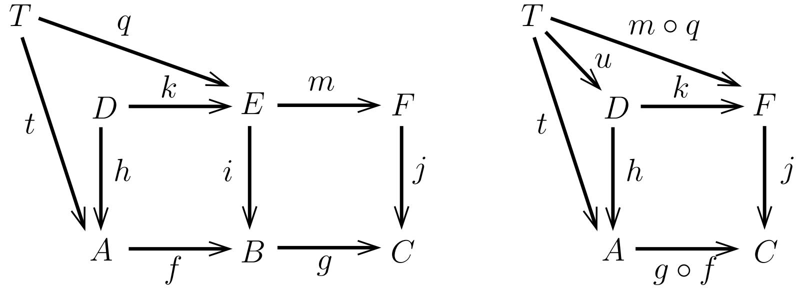


Figure J.65: limEqProd1. (a) For diagram D. (b)



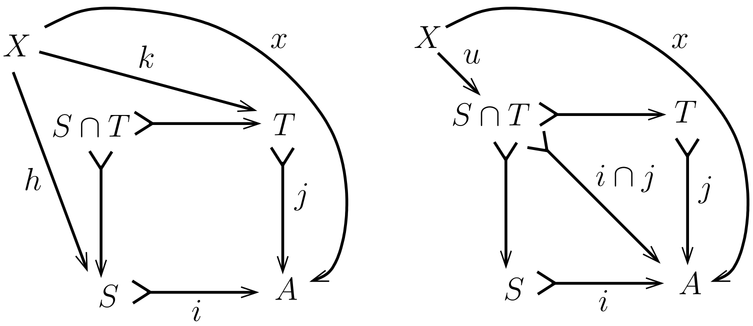
(a) (b)

Figure J.66: limEqProd2.



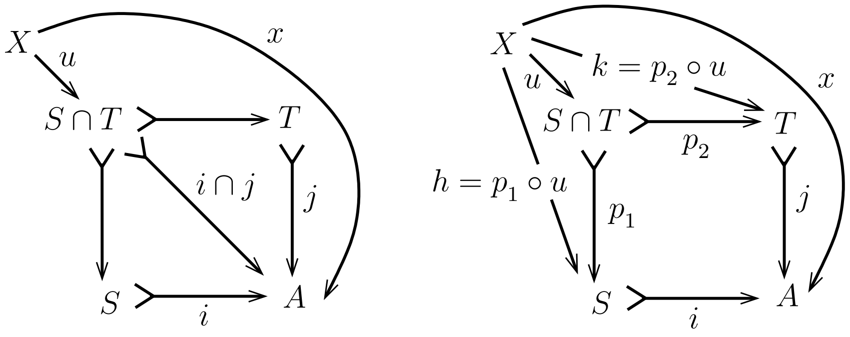
(a) (b)

Figure J.67: pullback11.



(a) (b)

Figure J.68: intersectcomp1.



(a) (b)

Figure J.69: intersectcomp2.

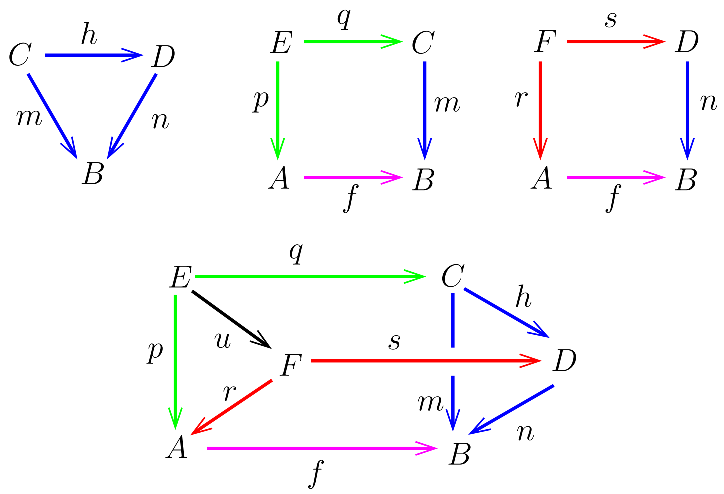
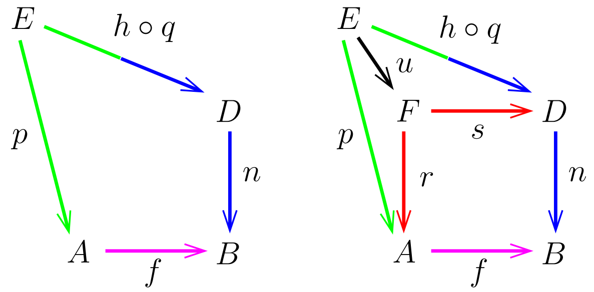


Figure J.70: pullback4.



(a) (b)

Figure J.71: pullback5.

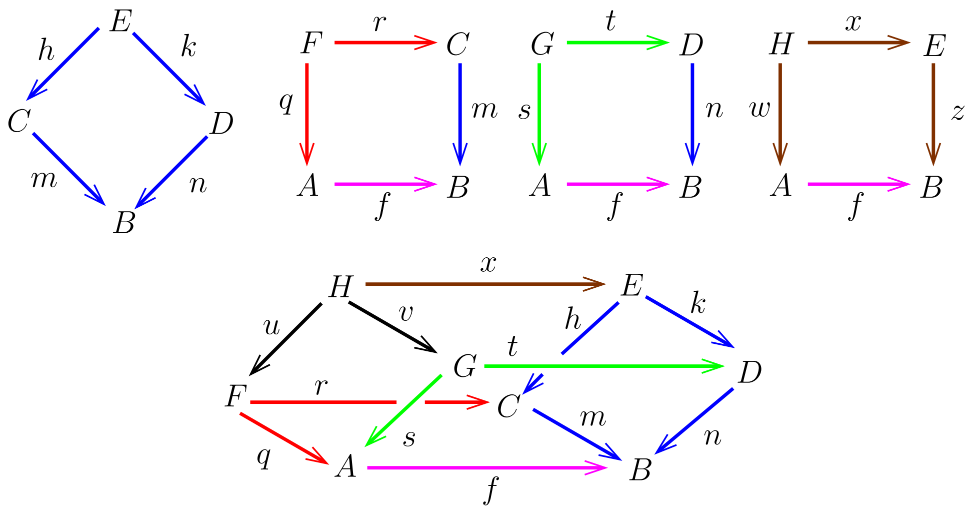


Figure J.72: pullback6.

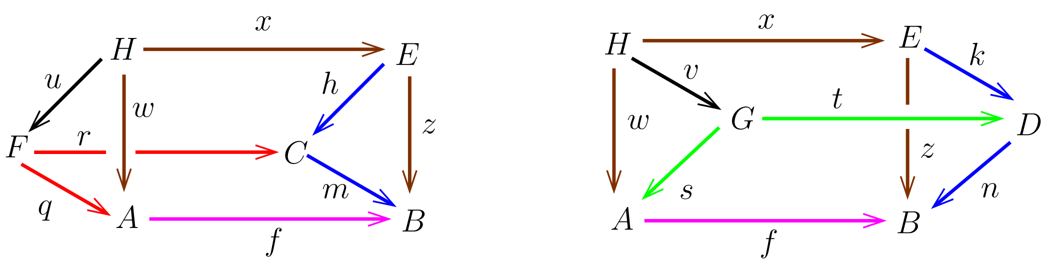


Figure J.73: pullback7.

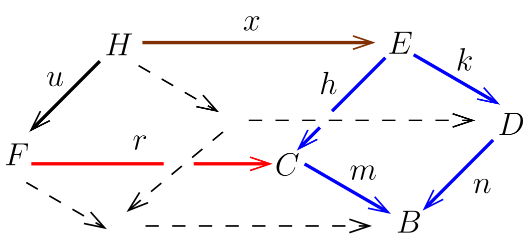


Figure J.74: pullback8.

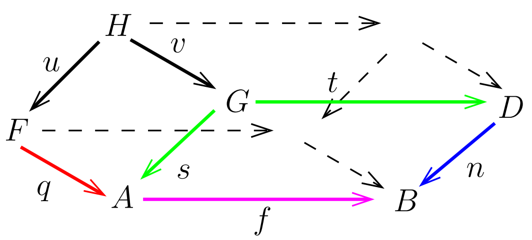


Figure J.75: pullback9.

Appendix K

Loop Quantum Gravity Quantization of String Theory

K.1 Introduction

String theory starts with the idea that fundamental particles are not point-like but are excitations of a one dimensional string. To remove the tachyon, one adds fermions and requires that the string be supersymmetric.

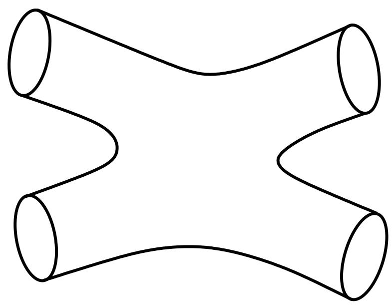


Figure K.1: stringfig1.

In defining a perturbative expansion, they must make a choice for the background metric of the spacetime will be. They don't really want the spacetime metric to appear at all in the formulation of the theory. They want it to be something that emerges as a prediction of the theory.

Argue that it is natural to first address these difficulties in the simplest possible scenario of general relativity. What is learnt will be useful in the future to tackle more elaborate theories of string theory.

surfaces swept out by a string in spacetime (called world sheets). In analogy with minimization of the proper time of a particle solutions to the equations of motion are surface swept out that are extremal with respect to the Minkowski metric.

Since the process of first quantization a string is equivalent to second quantization of a point particle theory on teh worldsheet that possesses gauge symmetries, it is a quantum field theory.

K.1.1 Unification via Strings

One of the wonderful things about string theory is the natural way it gives rise to a unification of all the particles and forces.

there should be a massless particle of spin 2 arising from the vibration of the strings. It was appropriate to identify this massless field with gravity.

"The vibrations of a string include states that correpsond to all the known kinds of matter and forces. The graviton, the particle that carries the gravitational force, comes out of the vibrations of loops (i.e., closed strings). The phton, carrier of the electromagnetic force, also emerges from the vibrations of a string. The more complicated gauge fields, in terms of our understanding of the strong and weak nuclear forces, also comes out automatically; that is, string theory predicts generally that there are gauge fields similar to these, although it does not predict the particular mix of forces we see in nature."

The equations of motion do not determine the

parametrization is a symmetry transformation that does not change the physical state of the system. We will call this parameter space the *abstract world-sheet*. Puting it into the same class of theories as General Relativity - the *background independent* techniques of loop quantum gravity.

of the closed bosonic string in flat Minkowskian spacetime.

K.2 Unresolved Problems in Conventional String Theory

Among thye fields that live on the worldsheet are the imbedding coordinates, X^a, where $X^a(\sigma)$ are coordinates in the background spacetime and σ are the two dimensional coordinates on the worldsheet. The original form of the string action seemd difficult to quantize. Progress was made when Polyakov werote the worldsheet theory in terms of the metric on the worldsheet $h_{\alpha\beta}$, so as to form the action,

$$\int d^2\sigma\sqrt{h}h^{\alpha\beta}(\partial_\alpha X^a)(\partial_\beta X^b)g_{ab}(X(\sigma))$$

where g_{ab} is the metric of the background metric. The two dimensional coordinates of the worldsheet an be fixed so that

$$h_{\alpha\beta} = \eta_{\alpha\beta} e^{\phi},$$

where η is the metric of flat, two dimensional spacetime. This leaves unfixed the third component of the metric, represented by ϕ, which we see here is the conformal factor.

Polyakov noticed that the quantization of the X^a fields on the worldsheet will in general give rise to dependence on ϕ in spite of the fact that the calssical actionis conformally invariant. However, it was noticed that this anomaly is absent if the dimension of the (Minkowski) background spacetime is 26 diemnsions!

Supersymmetry gives rise to additional necesary conditions on the background spacetime that it must satisfy.

K.2.1 String Theory in Non-Statinary Spacetimes

For there to be a consistent quantum mechanical description of strings moving and interacting in that geometry it turns out that a necessary condition for the string theory to be consistent is that, to a certain approximation, the geometry is a solution to the equations of a higher dimensional version of general relativity. So in a sense the the equations of general relativity come about from conditions for a string to move consistently.

More precisely, a necessary condition for a perturbative theory to be consisitent is that the two dimensional world sheet quantum field theory that defines the theory be conformally invariant. This maens that the conformal anomaly on the two dimensional worldsheet vanishes. To leading order in l_{string} this condition is equivalent to the Einstein equations of the background manifold.

However, this is the situation only in the original twenty-six dimensional bosonic string theory. Worldsheet supersymmetry $\rightarrow$ space-time supersymmetry. Space-time supersymmetry requires background spacetime geometry admits a timelike Killing vector field, that is, that the background spacetime geometry is not evolving with time!

In the absence of a construction of string theory on general time-dependent spacetime or compelling arguent for its existence, it can not be asserted that all of general relativity can be derived from string theory.

However, the LQG string does not require supersymmetry, and hence could be viable in time dependent background spacetime geometries.

Supersymmetry requires fermions and bosons to come in pairs consisting of one of each, with the same mass. This is not observed in nature.

K.2.2 Problems Arising From Extra Dimensions

Problems?!! Yes.

Roger Penrose argues that most of the compactified spaces that extra dimensions curl up into will quickly colapse to singularities. To show this, he apllies to the spacetime backgrounds of these string theories the Hawking-Penrose theorems in cosmological solutions.

K.2.3 Positive Cosmological Constant and the Landscape (More is Less)

If we want the theory to give a positive cosmological constant, so as to agree with observation, there is at present evidence for around 10^{500} such theories.

An attempt to construct a unique theory of nature leads instead to 10^{500} theories.

Gary Horowitz and colleagues have found a possible instability that afflicts not just the landscape of new theories but all solutions that involve the six-dimensional Calabi-Yau spaces.

K.3 The Approach to Quantization

Quantisation is the problem of deriving the mathematical framework of a quantum mechanical system from the mathematical framework of the corresponding classical mechanical system. A method of quantisation must contain a map $\mathcal{Q}$ from the set of classical observables to the set of quantum observables with the following properties:

The steps in quantization.

$$\mathcal{Q} : f \to \mathcal{Q}(f) \tag{K.1}$$

of operators $\mathcal{Q}(f)$ on some Hilbert space to classical observerables f.

1) Lineararity,

$$\mathcal{Q}(\alpha f + g) = \alpha \mathcal{Q}(f) + \mathcal{Q}(g), \quad \text{for all } \alpha \in \mathbb{C}, f, g \in C^{\infty}(\mathcal{M}), \tag{K.2}$$

2) The constant function 1 is mapped to the identity operator or matrix $\mathbb{I}$,

$$\mathcal{Q}(1) = \mathbb{I}. \tag{K.3}$$

3)

$$\mathcal{Q}(f)^* = \mathcal{Q}(f). \tag{K.4}$$

In the early days of quantum theory, Dirac proposed that the quantum analogue of the Poisson bracket of two classical observables should be the commutator of the corresponding quantum observables.

4)

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar \mathcal{Q}(\{f, g\}). \tag{K.5}$$

5) If $\{f_1, f_2, \dots, f_k\}$ is a complete set of observerables, $\{\mathcal{Q}(f_1 0, \mathcal{Q}(f_2), \dots, \mathcal{Q}(f_k)\}$ is a complete set of operators.

However, there is a problem. If we demand some particular complete set of observerables to satisfy (K.5), then some alternative complete set of observerables don't neccessarly satisfy (K.5), there may be extra higher order terms in $\hbar^2$ resulting in a kind of deformation quantization for this other complete set of observerables. So it is not ruled out that different choices of complete sets will lead to inequivalent quantum theories (i.e., to inequivalent predictions for the results of experiments). There isn't an axiomatic scheme for quantization that will tell us how to find a complete set of observerables, or which ones to choose. Extraneous considerations are required to construct the quantum theory, such as symmetries of the classical theory which may make one complete set more 'natural' than another.

We will use an algebraic approach to quantizing the bosonic string, the approach known as algebraic quantum field theory to be precise. This is a formulation of QFT on an axiomatic basis: that is, starting from what seem to be physically necessary and mathematically precise principles which any QFT would have to satisfy, and then finding QFTs which actually satisfy them.

An algebraic quantum theory is specified by giving the algebraic structure of the observables but not the Hilbert space on which they act. GNS construction is used to obtain a Hilbert space with set of operators which realizes a representation of the observable operator algebra.

For the LQG string again the Poincare group is represented unitarily but weakly discontinuously. However, we can approximate the generators in terms of teh corresponding Weyl operators using some tiny but finite parameter ϵ. Since there are a finite number of operators in the corresponding C^*-algebra, an appeal to Fell's theorem and using the continuity of the Weyl operators in the Fock representation guarentees that we find a state in the folium of the LQG string with respect to which the expectation values of the approximate Poincare generators coincide with their vacuum (or higher excited state) expectation values in the Fock representation to arbitrary precision δ.

K.4 String Theory

K.5 Nambu-Goto String

The functions $X^\mu(\sigma,\tau)$ specify points in Minkowskian spacetime of the string worldsheet. They map from a two dimensional parameter space (the abstract worldsheet), to coordinates in Minkowski spacetime.

$$ds^2 = h_{ab} d\xi^a d\xi^b \tag{K.6}$$

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} d\xi^a d\xi^b \tag{K.7}$$

hence

$$h_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \tag{K.8}$$

The action is

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi (-h)^{1/2} \tag{K.9}$$

where $h = \det h_{ab}$ and α is related to the string tension as $T = \frac{1}{2\pi\alpha'}$.

$$\begin{aligned} ||\, d\xi^1 \times\, d\xi^2|| &= ||\, d\xi^1||\; ||\, d\xi^2||\sin\theta \\ &= \sqrt{||\, d\xi^1||^2||\, d\xi^2||^2(1-\cos^2\theta)} \\ &= \sqrt{||\, d\xi^1||^2||\, d\xi^2||^2 - (||\, d\xi^1 \cdot\, d\xi^2||)^2} \\ &= \sqrt{|q_{11}q_{22} - (q_{12})^2|}\, d\sigma_1\, d\sigma_2 \\ &= \sqrt{\det q}\, d^2\sigma \end{aligned} \tag{K.10}$$

$$A(S) = \int d^2\sigma \sqrt{-\det h} \tag{K.11}$$

This is the so-called Nambu-Goto action. by using the matrix h_{ab}:

$$(h_{ab}) := \begin{pmatrix} (\dot{X})^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & (X')^2 \end{pmatrix}. \tag{K.12}$$

$$\begin{aligned} S_{NG} &= -T\int_{\Sigma}(\partial_\tau X^\mu \partial_\sigma X_\mu - \partial_\sigma X^\mu \partial_\tau X_\mu)^{1/2}\ d\sigma\, d\tau, \\ &= -T\int_{\Sigma}(\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu})^{1/2}\ d\sigma\, d\tau. \end{aligned} \tag{K.13}$$

Symmetries of the Nambo-Goto action

D-dimensional Poincare invariance

$$X'^\mu(\xi) = \Lambda^\mu_\nu X^\nu(\xi) + a^\mu \tag{K.14}$$

Two-dimensional diffeomorphism invariance - reparametrization of the worldsheet

$$\begin{aligned} \xi'^\mu &= \xi'^\mu(\xi) \\ X'^\mu(\xi') &= X^\mu(\xi). \end{aligned} \tag{K.15}$$

$$\begin{aligned} \pi_\mu &:= \alpha' \frac{\delta S}{\delta \dot{X}^\mu(y)} \\ &:= \alpha' \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \ddot{X}^\mu}\right)\right) \\ &= \alpha' \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \quad \text{as there areno second derivatives of } X \text{ in the action.} \end{aligned} \tag{K.16}$$

$$\begin{aligned} \pi^\tau_\mu &:= \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X') - (\dot{X})^2(X')^2}} \\ \pi^\sigma_\mu &:= \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X') - (\dot{X})^2(X')^2}} \end{aligned} \tag{K.17}$$

$$\{X^\mu(t,x), X^\nu(t,x')\} = \{\pi_\mu(t,x), \pi_\nu(t,x')\} = 0, \{\pi_\mu(t,x), X^\nu(t,x')\} = \alpha'\delta^\nu_\mu \delta(x,x'). \tag{K.18}$$

$$
\begin{aligned}
D_a &:= \pi_\mu X^\mu_{,a} = 0, \\
C &:= \frac{1}{2}(\eta^{\mu\nu}\pi_\mu\pi_\nu + \det(q)) = 0
\end{aligned}
\tag{K.19}
$$

where

$$
q_{ab}(y) = [(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})^{1/2}](y) \tag{K.20}
$$

K.6 The Polyakov String

The Nambo-Goto action is not very convenient for quantizing the world sheet theory. It is replaced by another action, which is classically equivalent, the Polyakov action. To do this one needs to introduce another metric $g_{ab}(\sigma,\tau)$ which is taken to be independent of the induced metric h_{ab}. The action for the world sheet

$$
S = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi(-g)^{1/2}g^{ab}(\sigma,\tau)\partial_a X^\mu \partial_b X^\mu \eta_{\mu\nu} \tag{K.21}
$$

with $g = \det(g_{ab})$.

The new metric $g_{ab}(\sigma,\tau)$ is to be thought of as a metric introduced onto the abstract worldsheet. It is easy to see that if the Polyakov action is to invariant under reparmetization, the metric $g_{cd}(\xi)$ must transform as

$$
g'_{ab}(\xi') = \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi). \tag{K.22}
$$

Polyakov action

$\delta S/\delta g_{ab}$. We will need to use the identity

$$
\delta g = -g g_{ab}\delta g^{ab} \quad \Rightarrow \quad \delta(-g)^{1/2} = \frac{1}{2}(-g)^{1/2} g_{ab}\delta g^{ab}. \tag{K.23}
$$

$$
\begin{aligned}
\delta S &= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \left\{\delta(-g)^{1/2}g^{ab} + (-g)^{1/2}\delta g^{ab}\right\}\partial_a X^\mu \partial_b X^\mu \\
&= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi (-g)^{1/2}\delta g^{ab}\left\{-\frac{1}{2}g_{ab}g^{cd}\partial_c X^\mu \partial_d X^\mu + \partial_a X^\mu \partial_b X^\mu\right\} \\
&= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi (-g)^{1/2}\delta g^{ab}\left(-\frac{1}{2}g_{ab}g^{cd}h_{cd} + h_{ab}\right)
\end{aligned}
\tag{K.24}
$$

equations of motion which fixes a relation between the two metrics h_{ab} and g_{ab}

$$h_{ab} = \frac{1}{2} g_{ab} g^{cd} h_{cd}. \tag{K.25}$$

Taking the determinant of both sides

$$(-h)^{1/2} = \frac{1}{2}(-g)^{1/2} g^{cd} h_{cd} \tag{K.26}$$

$$-\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi(-h)1/2 \equiv S_{NG}[X(\xi)]. \tag{K.27}$$

So on-shell the Nambu-Goto and Polyakov theories are equaivalent. the more directly geometric Nambu-Goto string.

In the Polyakov string Weyl invariance introduced artificially by hand then factored out later.

Symmetries of the Polyakov action

$\rightarrow$

introduce τ and σ to describe the motion of the string

The parameterization is arbitrary - a different parameterization of the same path is physically equivalent, and all physical quantities must be independent of this choice. That is, for any two monotonic function $(\tau'(\tau), \sigma'(\sigma))$ and (τ, σ) are the same, where

$$X^a(\tau'(\tau), \sigma'(\sigma)) = X^a(\tau, \sigma) \tag{K.28}$$

$\leftarrow$

D-dimensional Poincare invariance

$$\begin{aligned} X'^\mu(\xi) &= \Lambda^\mu_\nu X^\nu(\xi) + a^\mu \\ g'_{ab}(\xi) &= g_{ab}(\xi) \end{aligned} \tag{K.29}$$

Two-dimensional reparametrization of the worldsheet

$$\begin{aligned}
\xi'^{\mu} &= \xi'^{\mu}(\xi) \\
X'^{\mu}(\xi') &= X^{\mu}(\xi) \\
g'_{ab}(\xi') &= \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi).
\end{aligned} \tag{K.30}$$

Two-dimensional Weyl invariance

$$\begin{aligned}
X'^{\mu}(\xi) &= X^{\mu}(\xi) \\
g'_{ab}(\xi) &= \Omega(\xi) g_{ab}(\xi)
\end{aligned} \tag{K.31}$$

quantize a bunch of decoupled simple harmonic oscillators - string oscillation modes.

Conformal invariance is at he heart of the finiteness of string theory- apparently.

K.6.1 Conformal Invariance

Their formulas look so alike that the two are easily mixed up. This often leads to the statements in the literature like:

"conformal transformations are coordinate transformations which affect a Weyl transformation"

even though, as one of the first things one learns in a course on GR, coordinate transformations leave lengths invariant! - we are taught that the metric $g_{\mu\nu}$ transforms in such a way as to leave $d\tau^2$ unchanged, where $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$. The transformation which affects a Weyl transformation is an *active diffeomorphism* not a coordinate transformation. Consider an active diffeomorphism which sends the point x to the point x' s.t. $x^\mu = f^\mu(x'^\nu)$, this has the effect on the metric

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial f^\rho}{\partial x'^\mu}\frac{\partial f^\sigma}{\partial x'^\nu} g_{\rho\sigma}(f(x')). \tag{K.32}$$

We require the LHS be proportional to $g_{\mu\nu}$.

subsectionReparametrization symmetry to Conformal symmetry

Stretching represents continouis world-sheet coordinate transformations.

The group of general coordinate transformations has as a subgroup conformal transformations (which can be obtained as a residual gauge invariance upon covariant gauge fixing))

K.6.2 Polymer Reresentation

A wide sread expectation belief (not just with LQG people) that geometry will become discrete. as the quantum atom resolved the problem of .. space takes on a natural discretness - and so the momentum oerator is *not* going to be a well-defined operator anymore. It has been reffered to a “pathelogical” but this is physically desirable atribute.

K.7 Worldsheet Background Independence of the String

Background Independence of the Nambu-Goto String

Recall, that the active transformation changes the metric. There is no world-sheet metric so under the active transformation we do generate a new solution.

Same argument apllies to the Polyakov string - world-sheet metric $g_{ab}(\tau, \sigma)$ - if the theory is to be deterministic it should be background independent.

Symmetries of the Nambo-Goto string

not just invariance under coordinate transformations (passive diffeomorphisms)

$$\begin{aligned} \xi'^{\mu} &= \xi'^{\mu}(\xi) \\ X'^{\mu}(\xi') &= X^{\mu}(\xi) \end{aligned} \tag{K.33}$$

but also invariance under active diffeomorphisms

$$\begin{aligned} \xi'^{\mu} &= \xi'^{\mu}(\xi) \\ X^{\mu}(\xi) &\to X^{\mu}(\xi') \end{aligned} \tag{K.34}$$

Background Independence of the Polyakov String

The metric $h_{ab}(x)$ the string worldsheet picks up from the Minkowski metric η_{ab} of the target space.

shift of emphesis to distances on the abstract world-sheet.

Symmetris of the Polyakov string

not just invariance under coordinate transformations (passive diffeomorphisms). Two-dimensional reparametrization of the worldsheet

$$\begin{aligned}
\xi'^{\mu} &= \xi'^{\mu}(\xi) \\
X'^{\mu}(\xi') &= X^{\mu}(\xi) \\
g'_{ab}(\xi') &= \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi).
\end{aligned} \tag{K.35}$$

but also invariance under active diffeomorphisms

$$\begin{aligned}
\xi'^{\mu} &= \xi'^{\mu}(\xi) \\
g_{ab}(\xi) &\to g_{ab}(\xi') \\
X^{\mu}(\xi) &\to X^{\mu}(\xi')
\end{aligned} \tag{K.36}$$

Transformation under a *passive* diffeomorphism:

$$\xi'^a = \xi^a + \epsilon v^a \partial(\xi) \tag{K.37}$$

$$\frac{\partial \xi^c}{\partial \xi'^a} = \delta^c_a + \epsilon \partial_a v^c(\xi) \tag{K.38}$$

$$g'_{ab}(\xi') = (\delta^c_a + \epsilon \partial_a v^c)(\delta^d_a + \epsilon \partial_a v^d) g_{ab}(\xi) = g_{ab}(\xi) + \epsilon(\partial_a v^c + \partial_a v^d) g_{ab}(\xi) \tag{K.39}$$

$$(\delta g)^P_{ab} = g'_{ab}(\xi') - g_{ab}(\xi) = \epsilon(\partial_a v^c + \partial_a v^d) g_{ab}(\xi) \tag{K.40}$$

Transformation under an *active* diffeomorphism:

$$(\delta g)^A_{ab} = \epsilon(v^c \partial_b g_{ac} + v^c \partial_a g_{bc}) \tag{K.41}$$

The Lie derivative generates active diffeomorphisms.

$$\begin{aligned}
\{D(u), D(v)\} &= \alpha' D(\mathcal{L}_u v) = \alpha' D(uv') \\
\{D(u), C(v)\} &= \alpha' C(\mathcal{L}_u v) = \alpha' C(uv') \\
\{C(u), C(v)\} &= \alpha' \int dx (u'v - uv') D
\end{aligned} \tag{K.42}$$

$C(u) = \int dx u(x) C(x)$ and $D(v) = \int dx v(x) D(x)$

Implications

The theory is finite because of the background independence of the Nambo-Goto string for the same reason as LQG is finite - “short and large distances are gauge equivalent”.

Physical observables Poisson commute with all the constraints.

$\mathcal{P}_-^R$ are not physical because they evaluate the strings momentum and tension energy at a given value of σ; as in GR the value that a field takes at a point can have no physical meaning.

differential geometry is coordinate independent and so active diffeomorphisms are the only kind of diffeomorphisms.

K.8 Standard Quantization of the Polyakov String

Light-cone gauge fixing. Light-cone coordinates

$$\begin{aligned} X^{\pm} &= \frac{1}{\sqrt{2}}(X^0 \pm X^1) \\ X^i & \quad I = 1, 2, \dots, D-1. \end{aligned} \tag{K.43}$$

$$A^\mu B_\mu = -A^+ B^- - A^- B^+ + A^i B^i \tag{K.44}$$

so

$$A_- = -A^+, \quad A_+ = -A^-, \quad A_i = A^i \tag{K.45}$$

K.8.1 Oscillator Expansion

$$X^i(\sigma, t) = X^i_L(\sigma + t) + X^i_R(\sigma - t) \tag{K.46}$$

$$\begin{aligned} X^i_L(\sigma + t) &= \frac{x^i}{2} + \frac{p_i}{2p^+}(t + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbf{Z} - \{0\}} \frac{\alpha^i_n}{n} e^{-2\pi i n(\sigma + t)/\ell} \\ X^i_R(\sigma - t) &= \frac{x^i}{2} + \frac{p_i}{2p^+}(t - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbf{Z} - \{0\}} \frac{\tilde{\alpha}^i_n}{n} e^{2\pi i n(\sigma - t)/\ell} \end{aligned} \tag{K.47}$$

$$[x^-, p^+] = -i, \quad [x^i, p_j] = i\delta_{ij}, \quad [\alpha^i_m, \alpha^j_n] = [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m\delta_{ij}\delta_{m,-n} \quad [\alpha^i_m, \tilde{\alpha}^j_n] = 0 \tag{K.48}$$

K.8.2 Gupta-Bleuer Covariant Quantization

The price of maintaining manifest Lorentz covariance is the quantization of non-dynamical degrees of freedom and the introduction of ghosts.

Since the metric is not positive or negative definite, ghost states are introduced which can destroy unitarity.

Do we require unitary evolution with respect to the unphysical parameter τ - probabilities of observable quantities dont depend on it. What physical reason do we have for evolution with respect to an unphysical parameter??

K.9 Dirac Observables of the Nambu-Goto String

The constraints of the theory are represented as quantum operators in the kinematic representation and physical states are those states that are anihilated by the set of constraints. Evolution is generated by constraints, thus implying that Dirac observables are constants of motion. Hence finding enough Dirac observables is equivalent to finding a complete set of constants of motion for the theory. In the case of the Einstein's equations this close to immpossible, however, in the case of the Nambu-Goto string a complete set of constants of motion has been found by Pohlmeyer.

K.10 Invariant charges of the Nambu-Goto String

Lax Pairs Method for Constructing Invariants of Integrable Systems

$$\frac{dL}{dt} \equiv \dot{L} = [M, L]. \tag{K.49}$$

$$L(t) = g(t)L(0)g(t)^{-1} \tag{K.50}$$

where the invertible matrix is determined by the equation

$$M = \frac{dg}{dt}g^{-1}. \tag{K.51}$$

They allow the construction ofconserved quantities.

$$I(L(t)) = I(g(t)L(0)g(t)^{-1}) = I(L(0)) \tag{K.52}$$

such functions are functions of the eigenvalues of L.

$$\mathrm{Tr}(L^n) \tag{K.53}$$

integrability of the system in the sence of Lioville:

(1) the number of independent conserved quantities equals the number of degrees of freedom, and that

(2) these conserved quantities are in involution - are in involution means F_i

$$\{F_i, F_j\} = 0 \tag{K.54}$$

K.10.1 Invariant charges of the Nambu-Goto String

$$\begin{aligned} D &:= 2\pi_\mu X^{\mu\prime}, \\ C &:= \eta^{\mu\nu}\pi_\mu\pi_\nu + \eta_{\mu\nu}X^{\mu\prime}X^{\nu\prime} \equiv \eta^{\mu\nu}\pi_\mu\pi_\nu + \det(q) \end{aligned} \tag{K.55}$$

first step is to introduce a new set of constraints called the Virasoro constraints

$$V_\pm(u) := \pm\frac{1}{2\alpha'}\int_{S^1} dx\ u(C \pm D) \equiv \pm\frac{1}{2\alpha'}\int_{S^1} dx\ \xi\eta_{\mu\nu}Y^\mu_\pm Y^\nu_\pm \tag{K.56}$$

where u is a smearing function and

$$Y^\mu_\pm := \eta^{\mu\nu} \pm X^{\mu\prime}. \tag{K.57}$$

The constraint algebra simplifies to

$$\{V_\pm(u), V_\mp(v)\} = 0, \quad \{V_\pm(u), V_\pm(v)\} = V_\pm(s(u,v)) \tag{K.58}$$

where $(s(u,v)) = u'v - uv'$.

$$\{V_\pm(u), Y_\mp(f)\} = 0, \quad \{V_\pm(u), Y_\pm(f)\} = Y_\pm(uf'). \tag{K.59}$$

$$\begin{aligned} p_\mu &= \frac{1}{\alpha'}\int_{S^1} dx\ \pi_\mu \\ J^{\mu\nu} &= \frac{1}{\alpha'}\int_{S^1} dx[X^\nu \eta^{\mu\rho}\pi_\rho - X^\mu \eta^{\nu\rho}]. \end{aligned} \tag{K.60}$$

It is straightforward to check that p_μ, $J^{\mu\nu}$ have vanishing Poisson brackets with the $V_\pm(u)$ and are thus strong Dirac observables.

$$\{p_\mu, Y^\nu_\pm(x)\} = 0, \quad \{J^{\mu\nu}, Y^\rho_\pm(x)\} = (\eta^{\mu\rho}Y^\nu_\pm - \eta^{\nu\rho}Y^\mu_\pm)(x) \tag{K.61}$$

K.10.2 Algebra of Invariants

The integrability condition for such a Lax pair M, A is the zero curvature condition

$$F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]} + [A_\alpha, A_\beta] = 0 \tag{K.62}$$

$$\mathcal{H}(u,v) = C(u) + D(v), \tag{K.63}$$

introduce new variables to use instead of the canonical ones - momenta of the left or right movers

$$Y^\mu_\pm = \pi^\mu \pm X^{\mu\prime}, \tag{K.64}$$

$$\dot{Y}_\pm = \{\mathcal{H}(u,v), Y^\mu_\pm\} = [(v \pm u)Y^\mu_\pm]'. \tag{K.65}$$

More explicitly this equation reads

$$\frac{d}{dt}\int_{S^1} dx f_\mu(x) Y^\mu_\pm(t,x) = \int_{S^1} dx f_\mu(x)\frac{\partial}{\partial x}\big(Y^\mu_\pm(t,x)(u(x) \pm v(x))\big) \tag{K.66}$$

"time" evolution.

The invariants that we construct will be functions of $Y^\mu_\pm$, and so we will have to construct

Let τ_I be a basis of $GL(n, \mathbf{R})$ and T^I_μ some complex valued matries.

W define the following connection

$$A_x^{\pm} = Y_{\pm}^{\mu} T_{\mu}^{I} \tau_I \equiv Y_{\pm}^{\mu} T_{\mu}, \quad A_t^{\pm} = (v \pm u) A_x^{\pm}. \tag{K.67}$$

The zero curvature condition then mimics the equations of motion

$$F_{tx}^{\pm} = \partial_t A_x^{\pm} - \partial_x A_t^{\pm} + [A_t^{\pm}, A_x^{\pm}] = \partial_t A_x^{\pm} - \partial_x A_t^{\pm} = \partial_t A_x^{\pm} - \{\mathcal{H}(u,v), A_x^{\pm}\} = 0. \tag{K.68}$$

Need suitable matrix to form a Lax pair.

$$h_c(A^{\pm}) = \mathcal{P} \exp\left(\int_c ds \frac{dc^{\alpha}}{ds} A_{\alpha}^{\pm}\right). \tag{K.69}$$

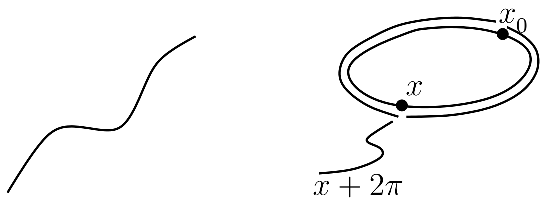


Figure K.2: The.

$$\partial_{\alpha} h_{t,x}(A^{\pm}) := (h_{e_{t,x}}(A^{\pm}))^{-1} h_{e'_{t,x}}(A^{\pm}) \tag{K.70}$$

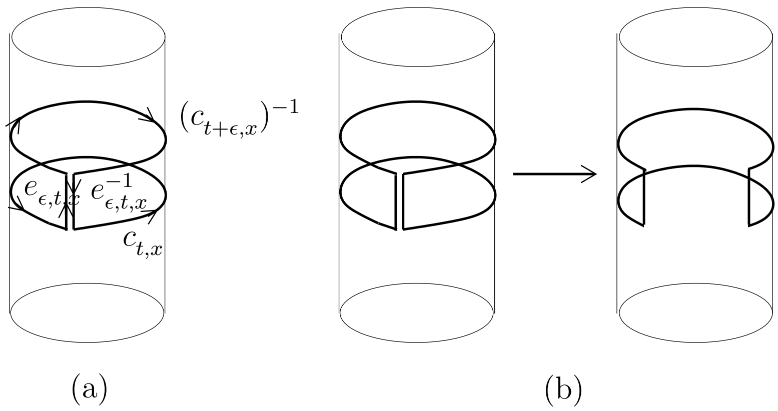


Figure K.3: (a) $c_{t,x} \circ e_{\epsilon,t,x} \circ c^{-1}_{t+\epsilon,x} \circ e^{-1}_{\epsilon,t,x}$. (b) The curve is easily seen to be contractable and hence, because the connection $A^{\pm}$ is flat, its holonomy is equal to one.

On the other hand

$$\partial_{\alpha} h_{t,x}(A^{\pm}) = [h_{t,x}(A^{\pm}), A_{\alpha}^{\pm}(t,x)] \tag{K.71}$$

$$2\partial_{[\alpha}\partial_{\beta]}h_{t,x}(A^{\pm}) = [h_{t,x}(A^{\pm}), F^{\pm}_{\alpha\beta}] \tag{K.72}$$

The zero curvature condition is the integrability condition for the equations

$$T^{n}_{t,x} := \mathrm{Tr}\left([h_{t,x}(A^{\pm})]^{n}\right) \tag{K.73}$$

K.10.3 Pohlmeyer Charges

$$Z_{\pm}^{\mu_1\mu_2\ldots\mu_N} = [R^{\mu_1\mu_2\ldots\mu_N}(x) + R^{\mu_2\mu_N\ldots\mu_1}(x) + \cdots + \ldots R^{\mu_N\mu_1\ldots\mu_{N-1}}(x)] \tag{K.74}$$

where

$$R^{\mu_1\mu_2\ldots\mu_N}(x) = \int_{x}^{x+2\pi} dx_1 \int_{x_1}^{x+2\pi} dx_2 \ldots \int_{x_N-1}^{x+2\pi} dx_N Y_{\pm}^{\mu_1}(x_1)\ldots Y_{\pm}^{\mu_1}(x_N). \tag{K.75}$$

It can be shown that, together with $J^{\mu\nu}$, the invariants $Z_{\pm}$ provide a *complete* system of invariants for the string in the sense that one can reconstruct $X^{\mu}(t,x)$ up to gauge transformations (parametrzations) and up to translations in the direction of p_{μ}.

K.11 Worldsheet Background Independent Quantization of the Closed Bosonic Nambu-Goto String

K.11.1 Introduction

in order to highlight the differences and similarities in the two different quantization procedures.

K.11.2 Background Independent Quantization of the Nambu-Goto String

Canonical Quantization

For finite dimensional Hilbert spaces the trace of a commutator should be zero

$$< \psi_n|\hat{P}\hat{Q} - QP|\psi_n >= 0 \neq -i\hbar < \psi_n|\mathbf{I}|\psi_n > \tag{K.76}$$

$$
\begin{aligned}
[Q^n, P] &= Q^n P - PQ^n \\
&= Q^{n-1}(QP - PQ) + Q^{n-1}PQ - PQ^n \\
&= iQ^{n-1} + Q^{n-2}(QP - PQ)Q + Q^{n-2}PQ^2 - PQ^n \\
&= \quad \dots \\
&= i(n-1)Q^{n-1} + PQ^n - PQ^n \\
&= inQ^{n-1}
\end{aligned}
\tag{K.77}
$$

from this we find that

$$
\begin{aligned}
\|inQ^{n-1}\| = n\|Q^{n-1}\| &= \|[Q^n, P]\| \\
&= \|Q^n P - PQ^n\| \\
&\leq 2\|P\|\|Q^n\| \\
&\leq 2\|P\|\|Q\|\|Q^{n-1}\|
\end{aligned}
\tag{K.78}
$$

which implies that $n \leq 2\|P\|\|Q\|$ for any positive integer $n \in \mathbb{N}$.

so as not to overburden the student with functional analytic issues.

Weyl-Heisenburg algebra

$$
e^{i\alpha P} e^{i\beta Q} = e^{i\beta Q} e^{i\alpha P} e^{i\alpha\beta} \tag{K.79}
$$

this is called the Weyl relation.by differentiating with respect to α andβ and then setting $\alpha = \beta = 0$ we recover the original Heisenberg relation.

$$
U(t)V(s) = V(s)U(t)e^{its} \tag{K.80}
$$

Any representation of the Weyl rerlation is unitarily equivalent to a direct sum of copies of the Schrödinger representation.

von Neumann "Mathematical Foundations of Quantum Mechanics"

$3 + 1$ dimensions

determinisic then, it must be background independent (by background of course we mean the abstract world-sheet) .

Quantum superpositions will result in quantum fluctuations in the world-sheet area. world-sheet area operators - do they have a discrete spectrum? What are the properties of the area operators in the standard quantization of string theory?

A dynamical geometric theory - dont we expect distances and areas to be quantized? If so, should not the absence of a well defined momentum operator to be physically desirable feature? Why is the momentum operator well defined in the standard string quantization with its Fock space Hilbert space?

differential geometry is coordinate independent and so active diffeomorphisms are the only kind of diffeomorphisms. Thiemann uses the coordinate independent formulism so when he talks of diffeomorphisms he is always reffering to active diffeomorphisms.

K.12 The Closed, Bosonic LQG-String

and one can discuss different representations of this algebra afterwards taking care of different physical circumstances.

As we have sketched in section ??, a powerful tool to arrive at such representations is the GNS construction. Hence we will be looking at representations that arise from a linear functional ω on $\mathcal{A}$ whihc is invariant under all the automorphisms α.

here we collect the notation:

Given any element a of the abstract algerba $\mathfrak{U}$, $$\pi_\omega(a)$$ is the corresponding operator on a Hilbert space of some specific representation. $\omega \sim$

$$f_\mu^{I,k}(x) = k_\mu \chi_I(x) \tag{K.81}$$

$$\{Y_\epsilon^k(I), Y_{\epsilon'}^{k'}(I')\} = \epsilon\alpha'\delta_{\epsilon,\epsilon'}\eta^{\mu\nu}k_\mu k'_\nu \int_{S^1} dx((\partial_x\chi_{I'})\chi_I - \chi_{I'}\partial_x\chi_I) \tag{K.82}$$

$$(\partial_x\chi_{I'}(x))\chi_I(x) = \{-\delta(x, b(I')) + \delta(x, f(I')\}\chi_I(x) \tag{K.83}$$

where $b(I')$ and $f(I')$ denote begining and final point of I' respectively, and similarly for the second term of (K.82). It is easily found that

$$\{Y_\epsilon^k(I), Y_{\epsilon'}^{k'}(I')\} = \epsilon\alpha'\delta_{\epsilon,\epsilon'}\eta^{\mu\nu}k_\mu k'_\nu\{[\chi_I]_{\partial I'} - [\chi_{I'}]_{\partial I}\} =: \alpha \tag{K.84}$$

The kinematic quantum algebra will be generated from the Weyl elements

$$\hat{W}_\epsilon^k(I) := e^{i\hat{Y}_\epsilon^k} \tag{K.85}$$

for which we require canonical commutation relations induced from

$$[\hat{Y}^k_\epsilon(I), \hat{Y}^{k'}_{\epsilon'}(I')] = i\hbar\{Y^k_\epsilon, \widehat{Y^{k'}_{\epsilon'}}\} \tag{K.86}$$

It follows from the Baker-Cambell-Hausdorff formula that

$$W^k_\epsilon(I)W^{k'}_{\epsilon'}(I') = \exp(-\ell_s^2\alpha(\epsilon, k, I; \epsilon, k'I')/2)\exp(i[Y^k_\epsilon(I) + Y^{k'}_{\epsilon'}(I')]) \tag{K.87}$$

since

$$Y^k_\pm(I) + Y^{k'}_\pm(I') = Y^{k+k'}_\pm(I \cap I') + Y^k_\pm(I - I') + Y^{k'}_\pm(I' - I) \tag{K.88}$$

we see that a general element of U can be written as a finite, complex linear combination of elemnets of the form

$$W^{k_1 \ldots k_M}_+(I_1, \ldots, I_M)W^{l_1 \ldots l_M}_-(J_1, \ldots, J_M) \tag{K.89}$$

where

$$W^{k_1 \ldots k_M}_+(I_1, \ldots, I_M) = \exp(i[\sum_{m=1} MY^{k_m}_\pm(I_m)]) \tag{K.90}$$

where all the I_m are non-empty and mutually non-overlapping

The general **Weyl Relations** are then given by

$$\begin{aligned} W_+(s_1)W_-(s_1')W_+(s_2)W_-(s_2') &= e^{-\frac{i}{2}\ell_s^2[\alpha(s_1,s_2)-\alpha(s_1',s_2')]}W_+(s_1+s_2)W_-(s_1'+s_2') \\ \left[W_+(s_1)W_-(s_2)\right]^* &= W_+(-s_1)W_-(-s_2) \end{aligned} \tag{K.91}$$

$$h^k_I(X) := \exp(2ik_\mu \int_I dx X^{\mu\prime}) = W^k_+(I)W^{-k}_-(I) \tag{K.92}$$

$$F^k_I(\pi) := \exp(2ik_\mu \int_I dx 1_\mu) = W^k_+(I)W^k_-(I) \tag{K.93}$$

We can combine the local gauge group generated by the Virasoro conatraints and the global Poincare group which itself is the semi-direct product of the Lorentz group with the translation group.

$$G := \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \times \mathcal{P} \tag{K.94}$$

K.13 Implementation of Pohlmeyer Charges

Algebraic properties:

$$\begin{aligned}
\pi_\omega(Z^{k_1\ldots k_n}_{\pm,M}) &= \pi_\omega(R^{k_2\ldots k_n k_1}_{\pm,M}) + \pi_\omega(R^{k_1\ldots k_n}_{\pm,M}) + \cdots + \pi_\omega(R^{k_n\ldots k_1 k_{n-1}}_{\pm,M}) \\
(R^{k_2\ldots k_n k_1}_{\pm,M}) &= \sum_{1\le m_1\le\cdots\le m_n\le M} \frac{1}{n!}\sum_{\pi\in S_n} \pi_\omega(Y^{k_{\pi(1)}}_\pm(I_{m_{\pi(1)}}))\ldots(Y^{k_{\pi(n)}}_\pm(I_{m_{\pi(n)}}))
\end{aligned} \tag{K.95}$$

$$\pi_\omega(Z^{k_1}_{\pm,M})\pi_\omega(Z^{k_2k_3}_{\pm,M}) \tag{K.96}$$

up to commutators, to the expected expression $\pi_\omega(Z^{k_1k_2k_3}_\pm) + \pi_\omega(Z^{k_2k_3k_3}_\pm)$, while the there are two remaining terms that converge in the semiclassical limit to path ordered integrals of the form

$$\frac{1}{M}\int_{x_1\ge x_2} d^2x Y^{k_1}_\pm(x_2) Y^{k_2}_\pm(x_2) Y^{k_3}_\pm(x_3) \tag{K.97}$$

and thus vanish in the large M linit, see below. For the general reations we get similar correction terms whose number depends only on n and which are therefore suppressed compared to the correct leading term $M\to\infty$ and $\ell_s\to 0$.

$$Z^{k_1\ldots k_n}_\pm := k_{\mu_1}\ldots k_{\mu_n} Z^{\mu_1\ldots\mu_n}_\pm \tag{K.98}$$

$$\frac{W^k_\pm([a,b]) - W^{-k}_\pm([a,b])}{2i[b-a]} = k_\mu Y^\mu_\pm(\frac{a+b}{2}) + \mathcal{O}((b-a)^2) \tag{K.99}$$

$$\pi_\omega(Z^{k_1\ldots k_n}_\pm) = \lim_{\mathcal{P}\to S^1} \pi_\omega(Z^{k_1\ldots k_n}_{\pm,\mathcal{P}}) \tag{K.100}$$

where $\mathcal{P} = \{I_m; m = 1,\ldots,M := |\mathcal{P}|\}$ is any partition of $[2\pi]$ into consecutive intervals

$$\begin{aligned}
\pi_\omega(Z^{k_1\ldots k_n}_{\pm,\mathcal{P}}) &= \pi_\omega(R^{k_1\ldots k_n}_{\pm,\mathcal{P}}(0)) + \pi_\omega(R^{k_2\ldots k_n k_1}_{\pm,\mathcal{P}}(0)) + \cdots + \pi_\omega(R^{k_n k_1\ldots k_{n-1}}_{\pm,\mathcal{P}}(0)) \\
\pi_\omega(R^{k_2\ldots k_n k_1}_{\pm,\mathcal{P}}(0)) &= \sum_{m_1=1}^{M}\sum_{m_2=m_1}^{M}\cdots\sum_{m_n=m_{n-1}}^{M} \frac{1}{(2i)^n n!}\sum_{\pi\in S_n}\times \\
&\quad [\pi_\omega(W^{k_{\pi(1)}}_\pm(I_{m_{\pi(1)}})) - \pi_\omega(W^{-k_{\pi(1)}}_\pm(I_{m_{\pi(1)}}))]\cdots\times \\
&\quad \cdots\times[\pi_\omega(W^{k_{\pi(1)}}_\pm(I_{m_{\pi(n)}})) - \pi_\omega(W^{-k_{\pi(1)}}_\pm(I_{m_{\pi(n)}}))]
\end{aligned} \tag{K.101}$$

$$f^s_\mu(x) = \sum_{I \in \gamma(s)} \chi_I(x) k^I_\mu(s) \tag{K.102}$$

such that $\exp(iY_\pm(f^s)) = W_\pm(s)$.

$$W^{k+k'}_\pm((I_m),(I_{m'})) := \exp(Y^k_\pm(I_m) + Y^{k'}_\pm(I_{m'})) \tag{K.103}$$

K.14 Classical Limit of the Quantum Pohlmeyer Charges

$$W_\pm(s, m_0) := \exp(i \sum_{I \in \gamma} k^I_\mu(s) \int_I dx[\pi^\mu_0(x) \pm X^{\mu\prime}_0]) \tag{K.104}$$

Adjointneess Relatons

The hilbert space can be written as an uncoountable direct sum

$$\mathcal{H}^\pm_\omega = \overline{\oplus_\gamma \mathcal{H}^\pm_{\omega,\gamma}} \tag{K.105}$$

Algebraic Properties

$$\pi_\omega(Y^k_\pm(I)) := \frac{1}{2i}[\pi_\omega(W^k_\pm(I)) - \pi_\omega(W^{-k}_\pm(I))] \tag{K.106}$$

K.15 Worked Examples and Details

Details: Diff .

$$\{Y^\mu_\pm, Y^\nu_\pm\} = \{\eta^{\mu\rho}\pi_\rho \pm X^{\mu\prime}, \eta^{\mu\rho}\pi_\rho \pm X^{\mu\prime}\} \tag{K.107}$$

we get four terms however two of which are $\{\pi_\rho, \pi_\sigma\}$ and $\{X^{\mu\prime}, X^{\nu\prime}\}$ which are both zero, so

$$\begin{aligned} \{Y^\mu_\pm, Y^\nu_\pm\} &= \pm\,[\eta^{\mu\rho}\partial_y\{\pi_\rho(x), X^\nu(y)\} + \eta^{\nu\sigma}\partial_x\{X^\nu(x), \pi_\sigma(y)\}] \\ &= \pm\alpha'\eta^{\mu\nu}\,[\partial_y\delta(x,y) - \partial_x\delta(x,y)] \end{aligned} \tag{K.108}$$

Similarly, we find

$$
\begin{aligned}
\{Y^\mu_\pm, Y^\nu_\mp\} &= \{\eta^{\mu\rho}\pi_\rho \pm X^{\mu\prime}, \eta^{\mu\rho}\pi_\rho \mp X^{\mu\prime}\} \\
&= \pm\alpha'\eta^{\mu\nu}\left[\partial_y\delta(x,y) + \partial_x\delta(x,y)\right] \qquad \text{(K.109)}
\end{aligned}
$$

First note:

$$
\begin{aligned}
\eta_{\mu\nu}\eta_{\sigma\rho}\{Y^\mu_\pm(x)Y^\nu_\pm(x), Y^\sigma_\pm(y)Y^\rho_\pm(y)\} &= 2\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)\{Y^\nu_\pm(x), Y^\sigma_\pm(y)Y^\rho_\pm(y)\} \\
&= 4\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\{Y^\nu_\pm(x), Y^\sigma_\pm(y)\} \qquad \text{(K.110)}
\end{aligned}
$$

$$
\begin{aligned}
\{V_\pm(u), V_\pm(v)\} &= \frac{1}{4\alpha'^2}\int_{S^1} dx \int_{S^1} dy u(x)\eta_{\mu\nu}v(y)\eta_{\sigma\rho}\{Y^\mu_\pm(x)Y^\nu_\pm(x), Y^\sigma_\pm(y)Y^\rho_\pm(y)\} \\
&= \frac{1}{\alpha'^2}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\{Y^\nu_\pm(x), Y^\sigma_\pm(y)\} \\
&= \pm\frac{1}{\alpha'}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\eta^{\nu\rho}[\partial_y\delta(x,y) - \partial_x\delta(x,y)] \\
&= \pm\frac{1}{\alpha'}\int_{S^1} dx(u'v - uv')\eta_{\mu\nu}Y^\mu_\pm(x)Y^\nu_\pm(y) \\
&= V_\pm(u'v - uv'). \qquad \text{(K.111)}
\end{aligned}
$$

$$
\begin{aligned}
\{V_\pm(u), V_\mp(v)\} &= \frac{1}{\alpha'^2}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\mp(y)\{Y^\nu_\pm(x), Y^\sigma_\mp(y)\} \\
&= \pm\frac{1}{\alpha'}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\eta^{\nu\rho}[\partial_y\delta(x,y) + \partial_x\delta(x,y)] \\
&= \pm\frac{1}{\alpha'}\int_{S^1} dx\partial_x(uvY^\mu_\pm(x)Y^\nu_\pm(x))\eta_{\mu\nu} \\
&= = 0. \qquad \text{(K.112)}
\end{aligned}
$$

(K.108)

$$
\begin{aligned}
\{Y_\pm(f), Y_\pm(g)\} &= \int_{S^1} dx \int_{S^1} dy f_\mu(x)g_\nu(y)\{Y^\mu_\pm(x), Y^\nu_\pm(y)\} \\
&= \pm\alpha'\int_{S^1} dx \int_{S^1} dy f_\mu(x)g_\nu(y)\eta^{\mu\nu}[\partial_y\delta(x,y) - \partial_x\delta(x,y)] \\
&= \pm\alpha'\eta^{\mu\nu}\int_{S^1} dx\left(\partial_x f_\mu(x)g_\nu(x) - f_\mu\partial_x g_v(x)\right) \qquad \text{(K.113)}
\end{aligned}
$$

$$
\begin{aligned}
\{Y_\pm(f), Y_\mp(g)\} &= \pm\alpha'\eta^{\mu\nu}\int_{S^1} dx\,(\partial_x f_\mu(x) g_\nu(x) + f_\mu \partial_x g_v(x)) \\
&= \pm\alpha'\eta^{\mu\nu}\int_{S^1} dx \partial_x (f_\mu(x) g_\nu(x)) \\
&= \pm\alpha'\eta^{\mu\nu}[f_\mu(x) g_\nu(x)]_0^{2\pi} = 0
\end{aligned} \tag{K.114}
$$

Details: Constraint algebra im terms of Virasoro constraints.

Check:

(a) $\{V_\pm(u), Y_\mp(f)\} = 0$ and $\{V_\pm(u), Y_\pm(f)\} = Y_\pm(uf')$

(b) $[\alpha_u^\pm(t)](Y_\mp(x)) = Y_\mp(x), \quad [\alpha_u^\pm(t)](Y_\pm(x)) = ([]Y_\pm)(x)$

Proof:

(a)

$$
\begin{aligned}
\{V_\pm(u), Y_\mp(f)\} &= \pm\frac{1}{\alpha'}\int_{S^1} dx \int_{S^1} dy\; u(x) f_\sigma(y)\; \eta_{\mu\nu}\left(Y_\pm^\nu(x)\{Y_\pm^\mu(x), Y_\mp^\sigma(y)\}\right) \\
&= \int_{S^1} dx \int_{S^1} dy\; u(x) f_\sigma(y)\; \eta_{\mu\nu}\eta^{\mu\sigma} Y_\pm^\nu(x)(\partial_y \delta(x,y) + \partial_x \delta(x,y)) \\
&= -\int_{S^1} dx[u(x) Y_\pm^\nu(x)\partial_x f_\mu(x) + f_\mu(x)\partial_x(u(x) Y_\pm^\nu(x))] \\
&= -\int_{S^1} dx[u(x) Y_\pm^\mu(x)\partial_x f_\mu(x) - u(x) Y_\pm^\nu(x)\partial_x f_\nu(x)] \\
&= 0.
\end{aligned} \tag{K.115}
$$

Noting first that $\{Y_\pm^\mu Y_\pm^\nu, Y_\pm^\sigma\} = Y_\pm^\nu\{Y_\pm^\mu, Y_\pm^\sigma\} + Y_\pm^\mu\{Y_\pm^\nu, Y_\pm^\sigma\}$

$$
\begin{aligned}
\{V_\pm(u), Y_\pm(f)\} &= \pm\frac{1}{2\alpha'}\int_{S^1} dx \int_{S^1} dy\; u(x) f_\sigma(y)\; \eta_{\mu\nu} 2\left(Y_\pm^\nu(x)\{Y_\pm^\mu(x), Y_\pm^\sigma(y)\}\right) \\
&= \int_{S^1} dx \int_{S^1} dy\; u(x) f_\sigma(y)\; \eta_{\mu\nu}(\eta^{\mu\sigma} Y_\pm^\nu(x)\partial_y \delta(x,y) - \partial_x \delta(x,y)) \\
&= \int_{S^1} dx f_\mu(x)\; \partial_x(u(x) Y_\pm^\nu(x)) - u(x) Y_\pm^\nu(x)\; \partial_x f_\mu(x) \\
&= -2\int_{S^1} dx Y_\pm(x) u(x)\partial_x f_\mu(x) \\
&\equiv -2 Y_\pm(uf')
\end{aligned} \tag{K.116}
$$

in the first line we used that $\eta_{\mu\nu}Y^\nu_\pm\{Y^\mu_\pm, Y^\sigma_\pm\} = \eta_{\mu\nu}Y^\mu_\pm\{Y^\nu_\pm, Y^\sigma_\pm\}$, and in the second line we used (K.108).

(b)

From $\{V_\pm(u), Y_\mp(x)\} = 0$ it is obvious that

$$\{V_\pm(u), Y_\mp(x)\}_{(n)} = 0 \quad \text{for all } n \geq 1 \tag{K.117}$$

hence the only non-zero term in the sum (E.7) is the first $n = 0$.

$$\{V_\pm(u), Y^\mu_\pm(x)\}_{(2)} = \{V_\pm(u), \partial_x u(x) Y^\mu_\pm(x)\} = \partial_x(\partial_x u(x))Y^\rho_\pm(x) \tag{K.118}$$

$$\{V_\pm(u), Y_\pm(f)\}_{(n)} = \frac{\partial^n u(x)}{\partial x^n} Y_\pm(x) \tag{K.119}$$

where $u_{(n)}$ is defined inductively as $u^{(n+1)} = \partial_x(u_{(n)})$

$$\{V_\pm(u), Y_\pm(f)\}_{(n)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{\partial^n u(x)}{\partial x^n} Y^\mu_\pm(x) \tag{K.120}$$

$$[\alpha^+_u(t)](F) = F[(\varphi^u_t)^* Y_+, Y_-], \quad [\alpha^-_u](F) = F[Y_+, \varphi^* Y_-] \tag{K.121}$$

Details: Diff .

Prove:

$$\begin{aligned} \alpha_a(Y_\pm) &:= \sum_{n=0}^{\infty} \frac{1}{n!}\{a^\mu p_\mu, Y_\pm\}_{(n)} = Y_\pm & \text{(K.122)} \\ \alpha_\Lambda(Y_\pm) &:= \sum_{n=0}^{\infty} \frac{1}{n!}\{\Lambda_{\mu\nu}J^{\mu\nu}, Y_\pm\}_{(n)} = \exp(\Lambda_{\mu\nu}\tau^{\mu\nu}) \cdot Y_\pm & \text{(K.123)} \end{aligned}$$

Proof:

part I

It is obvious from the Poisson brackets that

$$\{\pi_\mu(y), Y^\nu_\pm(x)\} = \pm\alpha'\delta^\nu_\mu \partial_x \delta(x, y), \quad \{X^\mu(y), Y^\nu_\pm(x)\} = \alpha' \eta^{\mu\nu}\delta(x, y). \tag{K.124}$$

$$
\begin{aligned}
\{p_\mu, Y^\nu_\pm(x)\} &= \frac{1}{\alpha'}\int_{S^1} dy\{\pi_\mu(y), Y^\nu_\pm(x)\} \\
&= \pm\delta^\nu_\mu\,\partial_x \int_{S^1} dy\delta(x,y) \\
&= \pm\delta^\nu_\mu\,\partial_x 1 = 0.
\end{aligned} \tag{K.125}
$$

$$
\begin{aligned}
\{J^{\mu\nu}, Y^\rho_\pm(x)\} &= \frac{1}{\alpha'}\int_{S^1} dy\{X^\nu(y)\eta^{\mu\sigma} - X^\mu(y)\eta^{\nu\sigma})\pi_\sigma(y), Y^\rho_\pm(x)\} \\
&= \frac{1}{\alpha'}\int_{S^1} dy(\eta^{\mu\sigma}\{\pi_\sigma(y)X^\nu(y), Y^\rho_\pm(x)\} - \mu \leftrightarrow \nu) \\
&= \frac{1}{\alpha'}\int_{S^1} dy(\eta^{\mu\sigma}(\pi_\sigma(y)\{X^\nu(y), Y^\rho_\pm(x)\} + \{\pi_\sigma(y), Y^\rho_\pm(x)\}X^\nu(y)) - \mu \leftrightarrow \nu) \\
&= \int_{S^1} dy(\eta^{\mu\sigma}(\pi_\sigma(y)\eta^{\nu\rho}\delta(x,y) \pm \delta^\rho_\sigma\partial_x\delta(x,y)X^\nu(y)) - \mu \leftrightarrow \nu) \\
&= \eta^{\mu\rho}(\eta^{\nu\sigma}\pi_\sigma \pm \partial_x X^\nu) - \mu \leftrightarrow \nu \\
&= (\eta^{\mu\rho}Y^\nu_\pm - \eta^{\nu\rho}Y^\mu_\pm)(x).
\end{aligned} \tag{K.126}
$$

part II

$$
\begin{aligned}
\{p_\nu, V_\pm(u)\} &= \frac{1}{2\alpha'^2}\int_{S^1} dxu(x)\eta_{\mu\nu}\{p_\nu, Y^\mu_\pm(x)Y^\nu_\pm(x)\} \\
&= \frac{1}{\alpha'}\int_{S^1} dxu(x)\eta_{\mu\nu}\{p_\nu, Y^\mu_\pm(x)\}Y^\nu_\pm(x) = 0.
\end{aligned} \tag{K.127}
$$

$$
\begin{aligned}
\{J^{\mu\nu}, V_\pm(u)\} &= \frac{1}{\alpha'}\int_{S^1} dxu(x)Y^\sigma_\pm(x)\{J^{\mu\nu}, Y^\sigma_\pm(x)\} \\
&= \frac{1}{\alpha'}\int_{S^1} dxu(x)Y^\sigma_\pm(x)(\eta^{\mu\rho}Y^\nu_\pm(x) - \eta^{\nu\rho}Y^\mu_\pm(x) \\
&= \frac{1}{\alpha'}\int_{S^1} dxu(x)(Y^\mu_\pm(x)Y^\nu_\pm(x) - Y^\nu_\pm(x)Y^\mu_\pm(x))
\end{aligned} \tag{K.128}
$$

$$
\begin{aligned}
\{\Lambda_{\mu\nu}J^{\mu\nu}, Y_\pm\}_{(2)} &\equiv \{\Lambda_{\mu_2\nu_2}J^{\mu_2\nu_2}, \{\Lambda_{\mu_1\nu_1}J^{\mu_1\nu_1}, Y_\pm\}\} \\
&= \{\Lambda_{\mu_2\nu_2}J^{\mu_2\nu_2}, \eta^{\mu_1\rho_1}Y^{\nu_1}_\pm - \eta^{\nu_1\rho_1}Y^{\mu_1}_\pm\}
\end{aligned} \tag{K.129}
$$

Details: ‘time” evolution .

Prove:

$$\dot{Y}_\pm = \{\mathcal{H}(u,v), Y_\pm\} = [(v \pm u)Y^\mu_\pm]' \tag{K.130}$$

explicitly this equation reads

$$\frac{d}{dt}\int_{S^1} dx f_\mu(x) Y^\mu_\pm(t,x) = \int_{S^1} dx f_\mu(x) \frac{\partial}{\partial x}\big(Y^\mu_\pm(t,x)(u(x) \pm v(x))\big) \tag{K.131}$$

Proof:

We consider $Y_+(f)$ and $Y_-(f)$ separately and using (K.59)

$$\begin{aligned}
\{\mathcal{H}(u,v), Y_+(f)\} &\equiv \{V_+(v+u) + V_-(v-u), Y_+(f)\} \\
&= Y_+((u+v)f') + 0 \\
&= \int_{S^1} dx Y^\mu_+(x)(u+v)(x)\partial_x f_\mu(x) \\
&= -\int_{S^1} dx f_\mu(x)\ \partial_x(Y^\mu_+(x)(u+v)(x)) \\
&\equiv -[(v+u)Y^\mu_+]'(f),
\end{aligned} \tag{K.132}$$

$$\begin{aligned}
\{\mathcal{H}(u,v), Y_-(f)\} &\equiv \{V_+(v+u) + V_-(v-u), Y_-(f)\} \\
&= 0 + Y_-((u-v)f') \\
&= \int_{S^1} dx Y^\mu_-(x)(u-v)(x)\partial_x f_\mu(x) \\
&= -\int_{S^1} dx f_\mu(x)\ \partial_x(Y^\mu_-(x)(u-v)(x)) \\
&\equiv [(v-u)Y^\mu_-]'(f)
\end{aligned} \tag{K.133}$$

Details: Diff .

Prove:

(a) $\partial_x h_{t,x}(A^\pm) = [h_{t,x}, A^\pm_x(t,x)]$

(b) $\partial_t h_{t,x}(A^\pm) = [h_{t,x}, A^\pm_t(t,x)]$

(c) Using the functional chain rule:

$$\frac{\delta F(x)}{\delta G(z)} \equiv \int dy\, \frac{\delta F(x)}{\delta J(y)}\, \frac{\delta J(y)}{\delta G(z)} \tag{K.134}$$

to prove $\{\mathcal{H}(u,v), h_{t,x}(A^{\pm})\} = \partial_t h_{t,x}(A^{\pm})$

Proof:

(a)

$$\begin{aligned}
\partial_x h_{t,x} &= \partial_x[(h_{e_{t,x}})^{-1} h_{e'_{t,x}}] \\
&= (h_{e_{t,x}})^{-1}\partial_x h'_{e_{t,x}} + (\partial_x (h_{e_{t,x}})^{-1}) h'_{e_{t,x}} \\
&= (h_{e_{t,x}})^{-1}\partial_x h'_{e_{t,x}} - (h_{e_{t,x}})^{-1}\partial_x h_{e'_{t,x}} (h_{e_{t,x}})^{-1} h_{e'_{t,x}} \\
&= (h_{e_{t,x}})^{-1}\partial_x h_{e'_{t,x}} A^{\pm} - A^{\pm}(h_{e_{t,x}})^{-1}\partial_x h_{e'_{t,x}} \\
&= [h_{t,x}, A^{\pm}_x(t,x)]
\end{aligned} \tag{K.135}$$

(b)

$c_{t,x} \circ e_{\epsilon,t,x} \circ c^{-1}_{t+\epsilon,x} \circ e^{-1}_{\epsilon,t,x}$ is contractible. Now, zero curvature we make the holonomy of the this contractible curve equal to one:

$$h_{t,x}(A^{\pm}) h_{\epsilon,t,x}(A^{\pm}) h^{-1}_{t+\epsilon,x}(A^{\pm}) h^{-1}_{\epsilon,t,x}(A^{\pm}) = 1 \tag{K.136}$$

Recall that $h_{t,x}(A^{\pm}) := h_{t+\epsilon,x}$ so that we have from (K.136)

$$h_{t,x}(A^{\pm}) h_{\epsilon,t,x}(A^{\pm}) = h_{\epsilon,t,x}(A^{\pm}) h_{t+\epsilon,x}(A^{\pm}) \tag{K.137}$$

$$h_{\epsilon,t,x}(A^{\pm}) := \mathcal{P}\exp(\int_{t,x}^{t+\epsilon,x} ds A^{\pm}) = \mathbf{1} + \epsilon A^{\pm} + \mathcal{O}(\epsilon^2) \tag{K.138}$$

Putting all this together we have

$$\begin{aligned}
\partial_t h_{t,x}(A^{\pm}) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\, h_{t+\epsilon,x}(A^{\pm}) - \partial_t h_{t,x}(A^{\pm})\,\right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(h_{e_\epsilon,t,x}(A^{\pm})^{-1}[h_{e_\epsilon,t,x}(A^{\pm})^{-1} h_{t+\epsilon,x}(A^{\pm})] - h_{t,x}(A^{\pm})\right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(h_{e_\epsilon,t,x}(A^{\pm})^{-1}[h_{t,x}(A^{\pm})^{-1} h_{e_\epsilon,t,x}(A^{\pm})] - h_{t,x}(A^{\pm})\right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(1 - \epsilon A^{\pm}\right) h_{t,x}(A^{\pm}) \left(1 + \epsilon A^{\pm}\right) \\
&= [h_{t,x}, A^{\pm}_t(t,x)]
\end{aligned} \tag{K.139}$$

(c)

$$
\begin{aligned}
\{\mathcal{H}(u,v), h_{t,x}(A^{\pm})\} &= \int_{S^1} dy\, \{\mathcal{H}(u,v), Y^{\mu}_{\pm}(t,y)\}\, \frac{\delta h_{t,x}(A^{\pm})}{\delta Y^{\mu}_{\pm}(t,y)} \\
&=: \int_{S^1} dy\, [\partial_t Y^{\mu}_{\pm}(t,y)]\, \frac{\delta h_{t,x}(A^{\pm})}{\delta Y^{\mu}_{\pm}(t,y)} \\
&= \partial_t h_{t,x}(A^{\pm})
\end{aligned}
\quad (K.140)
$$

Details: Diff .

$$\partial_x h_{t,x} = [h_{t,x}, A^{\pm}_x] \quad (K.141)$$

$$\partial_t h_{t,x} = [h_{t,x}, A^{\pm}_t] \quad (K.142)$$

differentiating (K.141) with respect to t gives

$$
\begin{aligned}
\partial_t \partial_x h_{t,x} &= [\partial_t h_{t,x}, A^{\pm}_x] + [h_{t,x}, \partial_t A^{\pm}_x] \\
&= [[h_{t,x}, A^{\pm}_t], A^{\pm}_x] + [h_{t,x}, \partial_t A^{\pm}_x]
\end{aligned}
\quad (K.143)
$$

where we have substituted (K.142) in the second line of (K.143). Differentiating (K.142) with respect to x gives, doing

$$\partial_x \partial_t h_{t,x} = [[h_{t,x}, A^{\pm}_x], A^{\pm}_t] + [h_{t,x}, \partial_t A^{\pm}_t] \quad (K.144)$$

subtracting (K.144) from (K.143) gives

$$(\partial_t \partial_x - \partial_x \partial_t) h_{t,x} = [h_{t,x}, \partial_t A^{\pm}_x - \partial_x A^{\pm}_t] + [[h_{t,x}, A^{\pm}_t], A^{\pm}_x] - [[h_{t,x}, A^{\pm}_x], A^{\pm}_x] \quad (K.145)$$

now using the Jacobi identity

$$[[h_{t,x}, A^{\pm}_t], A^{\pm}_x] + [[A^{\pm}_x, h_{t,x}], A^{\pm}_t] + [[A^{\pm}_t, A^{\pm}_x], h_{t,x}] \equiv 0 \quad (K.146)$$

in (K.145) we get our final result

$$
\begin{aligned}
(\partial_t\partial_x - \partial_x\partial_t)h_{t,x} &= [h_{t,x}, \partial_t A_x^\pm - \partial_x A_t^\pm + [A_t^\pm, A_x^\pm]] \\
&\equiv [h_{t,x}, F_{\alpha\beta}^\pm]. \qquad \text{(K.147)}
\end{aligned}
$$

Details: Pohlmeyer Charges.

Pohlmeyer Charges

$$
\{\mathcal{P}^\mu(\sigma), \mathcal{P}^\nu(\sigma')\} = -\eta_{\mu\nu}\delta'(\sigma - \sigma') \qquad \text{(K.148)}
$$

The modes of the Virasoro constraints are

$$
L_m := \frac{1}{2}\int d\sigma e^{-im\sigma}\eta_{\mu\nu}\mathcal{P}^\mu(\sigma)\mathcal{P}^\nu(\sigma) \qquad \text{(K.149)}
$$

$$
\{L_m, \mathcal{P}^\mu(\sigma)\} = \left(e^{-im\sigma}\mathcal{P}^\mu(\sigma)\right)' \qquad \text{(K.150)}
$$

The **Pohlmeyer invariants** $Z^{\mu_1\cdots\mu_N}$ are defined as

$$
Z^{\mu_1\cdots\mu_N}(\mathcal{P}) := \frac{1}{N}\int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \ldots \int_{\sigma^N}^{\sigma^1+2\pi} d\sigma^N \mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\ldots\mathcal{P}^{\mu_N}(\sigma^N) \qquad \text{(K.151)}
$$

Poisson commute with all L_m.

First note that if $F(\sigma^1, \sigma^2, \cdots, \sigma^N$ is any function which is p-eriodic with period 2π in each of its N arguments, the cyclically permutated path-ordered integral over F is equal to the integral used in (K.151)

$$
\begin{aligned}
\Bigg[\int_{0<\sigma^1<\sigma^2<\cdots<\sigma^N<2\pi} d^N\sigma \quad &+ \quad \int_{0<\sigma^N<\sigma^1<\cdots<\sigma^{N-1}<2\pi} d^N\sigma \\
&+ \quad \cdots + \int_{0<\sigma^{N-1}<\sigma^N<\cdots<\sigma^{N-2}<2\pi} d^N\sigma\Bigg] F(\sigma^1, \sigma^2, \ldots, \sigma^N) \\
&= \quad \int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^1 \cdots \int_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N F(\sigma^1, \sigma^2, \ldots, \sigma^N)
\end{aligned} \qquad \text{(K.152)}
$$

This follows...

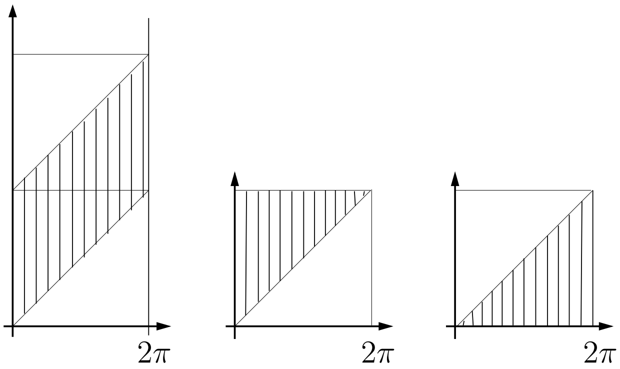


Figure K.4: .

implies cyclic permutation

$$Z^{\mu_1\mu_2\mu_3}(\mathcal{P}) = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3 \left(\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\mathcal{P}^{\mu_3}(\sigma^3)+ \right.$$
$$\left. \mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2)\mathcal{P}^{\mu_2}(\sigma^3) + \mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2)\mathcal{P}^{\mu_1}(\sigma^3)\right). \tag{K.153}$$

It can be used to express the variation as

$$\delta Z^{\mu_1\mu_2\mu_3} = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3 \left(\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\delta\mathcal{P}^{\mu_3}(\sigma^3)+ \right.$$
$$\left. \mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2)\delta\mathcal{P}^{\mu_2}(\sigma^3) + \mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2)\delta\mathcal{P}^{\mu_1}(\sigma^3)\right). \tag{K.154}$$

because we can cyclically permute the integration variables. But if one now sets $\delta\mathcal{P}(\sigma) = \{L_m, \mathcal{P}^\mu(\sigma)\}$ one gets, using (K.150)

$$\delta Z^{\mu_1\mu_2\mu_3} = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3 \left(\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)(\xi\mathcal{P}^{\mu_3}(\sigma^3))'+ \right.$$
$$\left. \mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2)(\xi\mathcal{P}^{\mu_2}(\sigma^3))' + \mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2)(\xi\mathcal{P}^{\mu_1}(\sigma^3))'\right). \tag{K.155}$$

integrating over σ^3, which is straightforward as

$$\int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3 (\xi\mathcal{P}(\sigma^3))' = \left[\xi\mathcal{P}(\sigma^3)\right]_{\sigma^2}^{\sigma^1+2\pi} \tag{K.156}$$

where for brevity we have set $\xi = e^{-im\sigma}$

$$
\begin{aligned}
\delta Z^{\mu_1\mu_2\mu_3} = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \qquad & \big(\xi\mathcal{P}^{\mu_3}\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2) - \mathcal{P}^{\mu_1}(\sigma^1)\xi\mathcal{P}^{\mu_2}\mathcal{P}^{\mu_3}(\sigma^2) \\
+ \quad & \xi\mathcal{P}^{\mu_2}\mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2) - \mathcal{P}^{\mu_3}(\sigma^1)\xi\mathcal{P}^{\mu_1}\mathcal{P}^{\mu_2}(\sigma^2) \\
+ \quad & \xi\mathcal{P}^{\mu_1}\mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2) - \mathcal{P}^{\mu_2}(\sigma^1)\xi\mathcal{P}^{\mu_1}\mathcal{P}^{\mu_3}(\sigma^2)\big) \,.
\end{aligned} \tag{K.157}
$$

and used the notation $\xi\mathcal{P}^{\mu_3}\mathcal{P}^{\mu_1}(\sigma^1) \equiv \xi(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^1)$, etc.

$$
\{L_m, Z^{\mu_1\cdots\mu_N}(\mathcal{P})\} = 0 \tag{K.158}
$$

$$
W^{\mathcal{P}}[A] = \mathrm{Tr}P\exp\left(\int_0^{2\pi} A\cdot\mathcal{P}(\sigma)\; d\sigma\right) \tag{K.159}
$$

$$
W^{\mathcal{P}}[A] = \sum_{n=0}^{\infty} Z^{\mu_1\cdots\mu_n}\mathrm{Tr}(A_{\mu_1}\ldots A_{\mu_n}) \tag{K.160}
$$

Details: Pohlmeyer Charges.

one would like to define

$$
\pi_\omega(Z_{\pm}^{k_1\ldots k_n}) := \lim_{|\mathcal{P}|\to S^1} \pi_\omega(Z_{\pm,\mathcal{P}}^{k_1\ldots k_n}) \tag{K.161}
$$

show that we have

(1) $\pi_\omega(Z_{\pm}^{k_1\ldots k_n}) = 0$ in the weak operator topology on $\mathcal{H}_\omega$ while

(2) $\pi_\omega(Z_{\pm}^{k_1\ldots k_n}) = \infty$ in the strong operator topology on $\mathcal{H}_\omega$, the divergence being of order of $M^{n/2}$.

Details: .

$$
\alpha(\epsilon, k, I; \epsilon', k', I') := \epsilon\alpha'\delta_{\epsilon,\epsilon'}\eta^{\mu\nu}k_\mu k_{\mu'}\alpha(I_m, I_{m'}) \tag{K.162}
$$

where

$$\alpha(I_m, I_{m'}) = (\chi_{I_{m'}})|_{\partial I_m} - (\chi_{I_m})|_{\partial I_{m'}} = \begin{cases} 0 & m = m' \text{ or } |m-m'| > 1 \\ -1 & m = m'+1 \\ 1 & m' = m+1 \end{cases} \tag{K.163}$$

$$\pi_\omega(W_\pm^{k+k'}((I_m),(I_{m'}))) := \pi_\omega\left(\exp(Y_\pm^k(I_m) + Y_\pm^{k'}(I_{m'}))\right) \tag{K.164}$$

$$\begin{aligned} [\pi_\omega(W_\pm^k(I_m)), \pi_\omega(W_\pm^{k'}(I'))] &= \pi_\omega(W_\pm^k(I))\pi_\omega(W_\pm^{k'}(I_{m'})) - \pi_\omega(W_\pm^{k'}(I_{m'}))\pi_\omega(W_\pm^k(I)) \\ &= (e^{-\ell_s^2\alpha(k,I;k',I')/2} - e^{-\ell_s^2\alpha(k',I';k,I)/2})\ \pi_\omega\left(\exp(Y_\pm^k(I_m) + Y_\pm^{k'}(I_{m'}))\right) \\ &= -2i\sin(-\ell_s^2\alpha(k,I;k',I')/2)\ \pi_\omega(W_\pm^{k+k'}((I_m),(I_{m'}))) \end{aligned} \tag{K.165}$$

$$\begin{aligned} [\pi_\omega(Y_\pi^k(I_m)), \pi_\omega(Y_\pi^l(I_{m'}))] &= -\frac{1}{4}[\pi_\omega(W_\pm^k(I_m)) - \pi_\omega(W_\pm^{-k}(I_{m'})), \pi_\omega(W_\pm^l(I_{m'})) - \pi_\omega(W_\pm^{-l}(I_{m'}))] \\ &= -\frac{1}{4}\Big\{[\pi_\omega(W_\pm^k(I_m)), \pi_\omega(W_\pm^l(I_{m'}))] - [\pi_\omega(W_\pm^k(I_m))\pi_\omega(W_\pm^{-l}(I_{m'}))] \\ &- [\pi_\omega(W_\pm^{-k}(I_{m'})), \pi_\omega(W_\pm^l(I_{m'}))] + [\pi_\omega(W_\pm^{-k}(I_{m'})), \pi_\omega(W_\pm^{-l}(I_{m'}))]\Big\} \end{aligned} \tag{K.166}$$

$$[\pi_\omega(Y_\pi^k(I_m)), \pi_\omega(Y_\pi^l(I'_m))] = -\frac{i}{2}\sin(\mp\ell_s^2[k\cdot l]\alpha(I_m, I_{m'})/2)\times \tag{K.167}$$

Details: Pohlmeyer Charges.

Abbreviating

$$\begin{aligned} F_\pm^{k_1k_2k_3} &:= \pi_\omega(Y_\pm^{k_1}(I_{m_1}))\times \\ &\times [\pi_\omega(Y_\pm^{k_2}(I_{m_2}))\pi_\omega(Y_\pm^{k_3}(I_{m_3})) + \pi_\omega(Y_\pm^{k_3}(I_{m_3}))\pi_\omega(Y_\pm^{k_2}(I_{m_2})) \\ &+ \pi_\omega(Y_\pm^{k_3}(I_{m_2}))\pi_\omega(Y_\pm^{k_2}(I_{m_3})) + \pi_\omega(Y_\pm^{k_2}(I_{m_3}))\pi_\omega(Y_\pm^{k_3}(I_{m_2}))]. \end{aligned} \tag{K.168}$$

$$\begin{aligned}\pi_\omega(Z^{k_1}_{\pm,M})\pi_\omega(Z^{k_2k_3}_{\pm,M}) &= \frac{1}{2}\sum_{m_1=1}^{M}\sum_{1\le m_2\le m_3\le M} F^{k_1k_2k_3}_{\pm} \\ &= \frac{1}{2}\left(\sum_{1\le m_1\le m_2\le m_3\le M} + \sum_{1\le m_2<m_1\le m_3\le M} + \sum_{1\le m_2\le m_3<m_1\le M}\right) F^{k_1k_2k_3}_{\pm}\end{aligned} \tag{K.169}$$

the case $m_1 = m_2 = m_3$ is taken care of in the first summation. We can change the condition $< m_1$ in the summations of the last two terms to include equality, i.e. $\le m_1$, by subtracting off two terms

$$\begin{aligned}\pi_\omega(Z^{k_1}_{\pm,M})\pi_\omega(Z^{k_2k_3}_{\pm,M}) &= \frac{1}{3!}\left(\sum_{1\le m_1\le m_2\le m_3\le M} + \sum_{1\le m_2\le m_1\le m_3\le M} + \sum_{1\le m_2\le m_3\le m_1\le M}\right) 3F^{k_1k_2k_3}_{\pm} \\ &- \frac{1}{2}\left(\sum_{1\le m_2\le m_3\le M} + \sum_{1\le m_2\le m_3\le M}\right) F^{k_1k_2k_3}_{\pm}.\end{aligned} \tag{K.170}$$

Details: Pohlmeyer Charges.

$$\sum_{0\le m_1\le m_2\le M} \ldots = \sum_{m_1,m_2=0}^{M} \Theta(m_2-m_1)\ \ldots \tag{K.171}$$

$$[\pi_\omega \tag{K.172}$$

Details: Classical Limit of the Pohlmeyer Charges.

$$\psi^{\pm}_{\gamma,L,m_0} := \sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s')}\overline{W_\pm(s')}\pi_\omega(W_\pm(s'+s_0))\Omega_\omega \tag{K.173}$$

$$\pi_\omega(W_\pm(s))\pi_\omega(W_\pm(s'+s_0)) = e^{\mp i\ell_s^2\alpha(s,s_0+s')/2}\pi_\omega(W_\pm(s+s'+s_0)) \tag{K.174}$$

$$\begin{aligned}
\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0} &= \sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s')}\overline{W_\pm(s',m_0)}e^{\mp i\ell_s^2\alpha(s,s_0+s')/2}\pi_\omega(W_\pm(s+s'+s_0))\Omega_\omega \\
&= \sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'-s)}\overline{W_\pm(s'-s,m_0)}e^{\mp i\ell_s^2\alpha(s,s_0+s'-s)/2}\pi_\omega(W_\pm(s'+s_0))\Omega_\omega
\end{aligned} \tag{K.175}$$

where we have used the translation invariance of the lattice to change s' to $s'-s$. From the bilinearity and antisymmetry of the function

$$\alpha(s,s') := [k^I(s)\cdot k^{I'}(s')]\alpha(I,I') \tag{K.176}$$

we have that

$$\begin{aligned}
\alpha(s,s+s'+s_0) &= \alpha(s,s_0)+\alpha(s,s')+\alpha(s,-s) \\
&= \alpha(s,s_0)+\alpha(s,s')+0,
\end{aligned} \tag{K.177}$$

and hence

$$\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0} = W_\pm(s,m_0)e^{\mp i\ell^2\alpha(s,s_0)/2}\sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'-s)}\overline{W_\pm(s',m_0)}e^{\mp i\ell^2\alpha(s,s')/2}\pi_\omega(W_\pm(s'+s_0))\Omega_\omega \tag{K.178}$$

$$\frac{<\psi^\pm_{\gamma,L,m_0}|\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0}>}{\|\psi^\pm_{\gamma,L,m_0}\|^2} \tag{K.179}$$

$$\begin{aligned}
\|\psi^\pm_{\gamma,L,m_0}\|^2 &= \sum_{s'\in S_{\gamma,L}}\sum_{s''\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'+s'')}W_\pm(s'')\overline{W_\pm(s')} <\pi_\omega(W_\pm(s''+s_0))\Omega_\omega|\pi_\omega(W_\pm(s'+s_0))\Omega_\omega> \\
&= \sum_{s'\in S_{\gamma,L}}\sum_{s''\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'+s'')}\delta_{s',s''} \\
&= \sum_{s'\in S_{\gamma,L}} e^{-t\lambda(s')}
\end{aligned} \tag{K.180}$$

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[401] M. Dutdevich, *Quantum Geometry and new concept of space.*

[402] John R. Klauder, *Overview of Affine Quantum Gravity*, invited contribution to IJGMMP, [gr-qc/0507113].

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[403] J.S. Milne , *Group Theory.*

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[410] **Stuff**

Penn State Gravitation Seminars (www.phys.psu.edu/events/index.html)

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[411] State Sum Models for Quantum Gravity IGPG Seminar by Prof. John Barrett from University of Nottingham Thursday at 1:00 PM in (8/27/1998)

[412] Internal Logic of Casual Sets: What the Universe Looks Like from the Inside Relativity Seminar by Dr. Fotini Markopoulou from Penn State Thursday at 1:00 PM in (10/29/1998)

couldn't listen to this one - software problem

[413] Group Averaging: A Uniqueness Theorem IGPG Seminar by Dr. Donald Marolf from Syracuse University Monday at 1:00 PM in (11/2/1998)

may have listened to this one - can't remember

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[425] Quantum Cosmology Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (11/8/2002)

haven't listened to this one yet

[426] Towards QFT on Quantum Geometry I: Ideas, Formalism, Conceptual Issues, Open Questions Gravity Seminar by Dr. Hanno Sahlmann from Penn State University Friday at 11:00 AM in 318 Osmond Laboratory (12/6/2002)

haven't listened to this one yet

[427] Towards QFT on Quantum Geometry II: Some Concrete Calculations Gravity Seminar by Dr. Hanno Sahlmann from Penn State University Friday at 11:00 AM in 318 Osmond Laboratory (12/13/2002)

[428] Black Hole Entropy: Inclusion of Distortion and Angular Momentum Gravity Seminar by Dr. Abhay Ashtekar from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/17/2003)

[429] Modeling the Low energy Limit of Loop Quantum Gravity: Shadow States and Quantum Mechanics Gravity Seminar by Josh Willis from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/24/2003)

[430] Quantum Cosmology: An Overview IGPG Seminar by Dr. Martin Bojowald from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (1/27/2003)

[431] Quantum Cosmology: Formalism Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/31/2003)

haven't listened to this one yet

[432] Quantum Gravity: Overview and Status IGPG Seminar by Dr. Abhay Ashtekar from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (2/3/2003)

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[433] Quantum Cosmology: Applications Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (2/7/2003)

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[434] Representations of the Sahlmann Star Algebra Gravity Seminar by Dr. Jerzy Lewandowski from University of Warsaw (Poland) Friday at 11:00 AM in 318 Osmond Laboratory (2/28/2003)

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[435] How Black Holes Grow Gravity Seminar by Dr. Badri Krishnan from Albert Einstein Institute Friday at 11:00 AM in 318 Osmond Laboratory (4/11/2003)

haven't listened to this one yet

[436] Diffeomorphism Invariance in the Smooth Category Gravity Seminar by Christian Fleischhack from Leipzig University and Penn State Monday at 3:00 PM in 318 Osmond Lab (4/28/2003)

haven't listened to this one yet

[437] An Update on Dynamical Horizons: Applications and Open Issues Gravity Seminar by Abhay Ashtekar from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (9/19/2003)

haven't listened to this one yet

[438] 2+1 Gravity and the Physical Scalar Product from Spin Foams Gravity Seminar by Alejandro Perez from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (10/24/2003)

haven't listened to this one yet

[439] Quadrupole Moments in Isolated Horizons Gravity Seminar by Jonathan Engle from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (10/31/2003)

haven't listened to this one yet

[440] Dynamics in Loop Quantum gravity as a Sum Over Histories of Quantum Geometry IGPG Seminar by Alejandro Perez from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (1/26/2004)

haven't listened to this one yet

[441] Discrete Quantum Gravity: Recent Developments and Applications Gravity Seminar by Rodolfo Gambini from University of Uruguay Friday at 11:00 AM in 318 Osmond Laboratory (2/6/2004)

[442] Master Constraint Programme for Loop Quantum Gravity Gravity Seminar by Thomas Thiemann from Perimeter Institute Monday at 12:00 PM in 318 Osmond Laboratory (2/9/2004) Audio, Presentation

[443] Testing the Master Constraint Programme for Loop Quantum Gravity Gravity Seminar by Bianca Dittrich from Perimeter Institute Friday at 11:00 AM in 318 Osmond Laboratory (2/13/2004)

[444] Goals and Achievements of Twistor Theory IGPG Seminar by Sir Roger Penrose from Oxford University and Penn State Monday at 3:00 PM in 318 Osmond (4/12/2004)

[445] Black Hole Entropy Gravity Seminar by Jerzy Lewandowski from University of Warsaw (Poland) Friday at 11:00 AM in 318 Osmond (9/3/2004)

[446] Black Hole Evaporation and Information Loss: Recent Advances IGPG Seminar by Abhay Ashtekar from Penn State Monday at 3:00 PM in 318 Osmond (9/20/2004)

[447] Free Vacuum for Loop Quantum Gravity Gravity Seminar by Florian Conrady from University of Marseille Friday at 9:00 AM in 318 Osmond (10/22/2004)

[448] Jorge Pullin, *Consistent discretization for Classical and Quantum General Relativity*, (17/01/05).

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