Baker-Campbell-Hausdorff Formula

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Chapter 1

Baker-Campbell-Hausdorff Formula

1.1 Introduction

Recall the power series;

$$\exp X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots, \tag{1.1}$$

we wish to study

$$log(\exp X \exp Y) \tag{1.2}$$

which will yields an expression for C such that $\exp C = (\exp X)(\exp Y)$. The problem is that X and Y need not commute. For example, if we retain only the linear and constant terms in the series we find

$$log[(1 + X + \cdots)(1 + Y + \cdots)] = log(1 + X + Y + \cdots) = X + Y + \cdots \tag{1.3}$$

On the other hand, if we go to to terms of second order, the non-commutativity begins enter:

$$\begin{aligned}
&log[(I + X + \frac{1}{2}X^2 + \cdots)(I + Y + \frac{1}{2}Y^2 + \cdots)] \\
&\quad = log[I + X + Y + \frac{1}{2}X^2 + XY + \frac{1}{2}Y^2 + \cdots] \\
&\quad = X + Y + \frac{1}{2}X^2 + XY + \frac{1}{2}Y^2 - \frac{1}{2}(X + Y + \cdots)^2 + \cdots \\
&\quad = X + Y + \frac{1}{2}[X, Y] + \cdots
\end{aligned} \tag{1.4}$$

Collecting the terms of degree three

$$
\begin{aligned}
&log[(I+X+\frac{1}{2}X^2+\frac{1}{3!}X^3+\cdots)(I+Y+\frac{1}{2}Y^2+\frac{1}{3!}Y^3+\cdots)]\\
&\quad = log[I+X+Y+\frac{1}{2}X^2+XY+\frac{1}{2}Y^2+\frac{1}{6}X^3+\frac{1}{2}X^2Y+\frac{1}{2}XY^2+\frac{1}{6}Y^3+\cdots]\\
&\quad = X+Y+\frac{1}{2}X^2+XY+\frac{1}{2}Y^2+\frac{1}{6}X^3+\frac{1}{2}X^2Y+\frac{1}{2}XY^2+\frac{1}{6}Y^3\\
&\quad -\frac{1}{2}(X+Y+\frac{1}{2}X^2+XY+\frac{1}{2}Y^2+\cdots)^2+\frac{1}{3}(X+Y+\cdots)^3\cdots\\
&\quad = X+Y+(\frac{1}{2}X^2+XY+\frac{1}{2}Y^2)-\frac{1}{2}(X+Y)^2\\
&\quad +(\frac{1}{6}X^3+\frac{1}{2}X^2Y+\frac{1}{2}XY^2+\frac{1}{6}Y^3)\\
&\quad -\frac{1}{2}(X^3+\frac{3}{2}X^2Y+\frac{3}{2}XY^2+\frac{1}{2}Y^2X+\frac{1}{2}YX^2+XYX+YXY+Y^3)\\
&\quad +\frac{1}{3}(X^3+X^2Y+XY^2+Y^2X+YX^2+XYX+YXY+Y^3)+\cdots\\
&\quad = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}(X^2Y+XY^2+Y^2X+YX^2-2XYX-2YXY)+\cdots
\end{aligned}
\tag{1.5}
$$

We can rearange

$$
\begin{aligned}
&\frac{1}{12}(X^2Y+XY^2+Y^2X+YX^2-2XYX-2YXY)\\
&\quad = \frac{1}{12}\big((X^2Y-XYX)-(XYX-YX^2)+(Y^2X-YXY)-(YXY-XY^2)\big)\\
&\quad = \frac{1}{12}(X[X,Y]-[X,Y]X+Y[Y,X]-[Y,X]Y)\\
&\quad = \frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]].
\end{aligned}
\tag{1.6}
$$

The content of the Campbell-Baker-Hausdorff formula is that the series $log(\exp X \exp Y)$ can be expressed entirely in terms of successive Lie brackets of X and Y.

1.2 Dynkin Formula

$$
log(\exp X \exp Y) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum\frac{1}{i_1+j_1+\cdots+i_k+j_k}\frac{[X^{(i_1)}Y^{(j_1)}\cdots X^{(i_k)}Y^{(j_k)}]}{i_1!j_1!\cdots i_k!j_k!}
\tag{1.7}
$$

where, and the following notation has been used

$$[X^{(i_1)}Y^{(j_1)}\cdots X^{(i_k)}Y^{(j_k)}] = [\underbrace{X,[X,\ldots[X}_{i_1},\underbrace{[Y,[Y,\cdots,[Y}_{j_1},\cdots[\underbrace{X,[X,\cdots[X}_{i_k},\underbrace{[Y,[Y,\cdots,[Y}_{j_k}]]\cdots]] \tag{1.8}$$

Since $[A,A]=0$, the term is zero if $j_k>1$ or if $j_k=0$ and $i_k>1$.

The first few terms are

$$\begin{aligned} Z(X,Y) &= log(\exp X \exp Y) \\ &= X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]-[Y,[Y,X]]) \\ &\quad -\frac{1}{24}[Y,[X,[X,Y]]] \\ &\quad -\frac{1}{720}([Y,[Y,[Y,[Y,X]]]]+[X,[X,[X,[X,Y]]]]) \\ &\quad +\frac{1}{360}([X,[Y,[Y,[Y,X]]]]+[Y,[X,[X,[X,Y]]]]) \\ &\quad -\frac{1}{120}([Y,[X,[Y,[X,Y]]]]+[X,[Y,[X,[Y,X]]]])+\cdots \end{aligned} \tag{1.9}$$

1.3 Derivation

1.3.1 Derivative of the exponential map

$$\frac{d}{dt}e^{X(t)} = e^X\frac{1-e^{ad_X}}{ad_X}\frac{dX(t)}{dt}. \tag{1.10}$$

where ad_X is the linear transformation of the Lie algebra given by $ad_X(Y)=[X,Y]$. It is the adjoint action of a Lie algebra on itself. The fraction $\frac{1-\exp(-ad_X)}{ad_X}$ is given by the power series

$$\frac{1-\exp(-ad_X)}{ad_X} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(ad_X)^k \tag{1.11}$$

Proof

Define

$$\Gamma(s,t) = e^{-sX(t)}\frac{\partial}{\partial t}e^{sX(t)}. \tag{1.12}$$

Let Ad denote the adjoint action of the group on its Lie algebra. The action $Ad_A = AXA^{-1}$ where $A \in G$, $X \in \mathbf{g}$. A relationship between Ad and ad is given by

$$Ad_{e^X} = e^{ad_X}, \quad X \in \mathbf{g}. \tag{1.13}$$

Proof of (1.13):

Define

$$f(\lambda) = e^{\lambda X}Ae^{-\lambda X} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}C_n. \tag{1.14}$$

We have $f(\lambda = 0) = C_0 = A$ and

$$\begin{aligned}
\frac{d}{d\lambda}e^{\lambda X}Ae^{-\lambda X} &= e^{\lambda X}[X,A]e^{-\lambda X} \\
\frac{d^2}{d\lambda^2}e^{\lambda X}Ae^{-\lambda X} &= e^{\lambda X}[X,[X,A]]e^{-\lambda X} \\
\cdots &= \cdots \\
\frac{d^m}{d\lambda^m}e^{\lambda X}Ae^{-\lambda X} &= e^{\lambda X}((ad_X)^m(A))e^{-\lambda X}
\end{aligned} \tag{1.15}$$

implies

$$e^{\lambda X}Ae^{-\lambda X} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(ad_X)^m(A). \tag{1.16}$$

and so

$$Ad_{e^X}(A) = e^XAe^{-X} = \sum_{n=0}^{\infty}\frac{1}{n!}(ad_X)^m(A) = e^{ad_X}(A), \tag{1.17}$$

establishing the result.

Algebraic proof of (1.13):

We will first show that by induction that

$$(ad_X)^n(A) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} X^k Y (-X)^{n-k}. \tag{1.18}$$

Assume the result for m, then

$$\begin{aligned}
(ad_X)^{m+1}(A) &= ad_X \left(\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k A (-X)^{m-k} \right) \\
&= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} ad_X (X^k A (-X)^{m-k}) \\
&= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} [X, X^k A (-X)^{m-k}] \\
&= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (X^k A (-X)^{m+1-k} + X^{k+1} A (-X)^{m-k}) \\
&= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k A (-X)^{m+1-k} + \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^{k+1} A (-X)^{m-k} \\
&= A(-X)^{m+1} + \sum_{k=1}^{m} \frac{m!}{k!(m-k)!} X^k A (-X)^{m+1-k} \\
&\quad + \sum_{k=1}^{m+1} \frac{m!}{(k-1)!(m+1-k)!} X^k A (-X)^{m+1-k} \\
&= A(-X)^{m+1} + \sum_{k=1}^{m} \frac{(m+1)!}{k!(m+1-k)!} X^k A (-X)^{m+1-k} + X^{m+1} A \\
&= \sum_{k=0}^{m+1} \frac{(m+1)!}{k!(m+1-k)!} X^k A (-X)^{m+1-k}.
\end{aligned} \tag{1.19}$$

We check the result for $m = 1$:

$$\begin{aligned}
\sum_{k=0}^{1} \frac{1!}{k!(1-k)!} X^k A (-X)^{1-k} &= XA + A(-X) \\
&= ad_X(A).
\end{aligned} \tag{1.20}$$

Now by direct computation

$$
\begin{aligned}
Ad_{e^X}(A) &= e^X A e^{-X} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{X^m}{m!} A \frac{(-X)^n}{n!} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{1}{k!(m-k)!} X^k A (-X)^{m-k} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k A (-X)^{m-k} \right) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} (ad_X)^m (A) \\
&= e^{ad_X}(A) \qquad (1.21)
\end{aligned}
$$

where we introduced the index $k = m + n$.

Using the product rule twice with (1.12)

$$
\begin{aligned}
\frac{\partial \Gamma}{\partial s} &= \frac{\partial}{\partial s}\left(e^{-sX(t)} \frac{\partial}{\partial t} e^{sX(t)} \right) \\
&= e^{-sX(t)}(-X)\frac{\partial}{\partial t} e^{sX(t)} + e^{-sX(t)} \frac{\partial}{\partial t}[X(t) e^{sX(t)}] \\
&= e^{-sX} \frac{dX}{dt} e^{sX}. \qquad (1.22)
\end{aligned}
$$

By (1.13)

$$
\frac{\partial \Gamma}{\partial s} = Ad_{e^{-sX}} X' = e^{ad_X} X'. \qquad (1.23)
$$

Integration yields

$$
\Gamma(1,t) = e^{-X(t)} \frac{\partial}{\partial t} e^{X(t)} = \int_0^1 \frac{\partial \Gamma}{\partial s} ds = \int_0^1 e^{ad_X} X' ds. \qquad (1.24)
$$

Using the formal power series to expand the exponential, integrating term by term, and finally

$$
\begin{aligned}
\Gamma(1,t) &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{k!} (ad_X)^k \frac{dX}{dt} ds \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (ad_X)^k \frac{dX}{dt} \\
&= \frac{1-\exp(-ad_X)}{ad_X} \frac{dX}{dt}, \qquad (1.25)
\end{aligned}
$$

and the result follows.

1.3.2 General Case

The formula in the general case is given by

$$\frac{d}{dt}\exp(C(t)) = \exp(C)\phi(\exp(C))C', \tag{1.26}$$

where

$$\phi(z) = \frac{e^z - 1}{z} = 1 + \frac{1}{2!}z + \frac{1}{3!}z^3 + \ldots, \tag{1.27}$$

which formally reduces to

$$\frac{d}{dt}\exp(C(t)) = \exp(C)\frac{1-e^{-ad_C}}{ad_C}\frac{dC(t)}{dt} \tag{1.28}$$

1.4 Derivation of Dynkin's series formula

If $Z(t)$ is defined such that

$$Z(t) = e^{tX}e^{tY}, \tag{1.29}$$

an expression for $Z(1) = log(\exp X \exp Y)$, the BCH formula, can be derived.

$$
\begin{aligned}
e^{-Z(t)} \frac{de^{Z(t)}}{dt} &= e^{-tY} e^{-tX} (e^{tX} X e^{tY} + e^{tX} e^{tY} Y) \\
&= e^{-tY} X e^{tY} + Y \\
&= e^{-tad_Y} X + Y. \qquad (1.30)
\end{aligned}
$$

By the general formula (1.26),

$$
e^{-Z(t)} \frac{de^{Z(t)}}{dt} = \frac{1 - e^{-ad_Z}}{ad_Z} \frac{dZ(t)}{dt} \qquad (1.31)
$$

Using this in (1.30) gives

$$
\frac{1 - e^{-ad_Z}}{ad_Z} \frac{dZ(t)}{dt} = e^{-tad_Y} X + Y. \qquad (1.32)
$$

Hence, formally,

$$
\frac{dZ(t)}{dt} = \frac{ad_Z}{1 - e^{-ad_Z}} (e^{-tad_Y} X + Y). \qquad (1.33)
$$

However, using the relationship between Ad and ad given by (1.13) we have,

$$
\begin{aligned}
e^{ad_Z}(A) &= Ad_Z(A) \\
&= e^Z A e^{-Z} \\
&= e^{tX} e^{tY} A e^{-tY} e^{-tX} \\
&= e^{tX} Ad_{tY}(A) e^{-tX} \\
&= Ad_{tX} Ad_{tY}(A) \\
&= e^{tad_X} e^{tad_Y}(A). \qquad (1.34)
\end{aligned}
$$

Using this in (1.33) gives

$$
\begin{aligned}
\frac{dZ(t)}{dt} &= \frac{ad_Z}{e^{ad_Z} - 1} e^{ad_Z} (e^{-tad_Y} X + Y) \\
&= \frac{ad_Z}{e^{ad_Z} - 1} e^{tad_X} e^{tad_Y} (e^{-tad_Y} X + Y) \\
&= \frac{ad_Z}{e^{ad_Z} - 1} (X + e^{tad_X} Y). \qquad (1.35)
\end{aligned}
$$

Now since

$$\begin{aligned} ad_Z &= log(\exp(ad_Z)) \\ &= log(1+\exp(ad_Z)-1) \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(\exp(ad_Z)-1)^n, \quad \|ad_Z\| < log2 \end{aligned} \tag{1.36}$$

it follows

$$\frac{dZ(t)}{dt} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(\exp(ad_Z)-1)^n(X+e^{tad_X}Y) \tag{1.37}$$

Changing the summation index in to $k = n-1$ and expanding

$$\frac{dZ}{dt} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\left\{(e^{ad_{tX}}e^{ad_{tY}}-1)^kX + (e^{ad_{tX}}e^{ad_{tY}}-1)e^{ad_{tX}}Y\right\}. \tag{1.38}$$

The log-series and the exp-series are given by

$$log(A) = \sum_{k=1}^{\infty}(A-I)^k, \quad \text{and} \quad e^X = \sum_{k=0}^{\infty}\frac{X^k}{k!} \tag{1.39}$$

respectively. These series expansions imply

$$\begin{aligned} (e^{ad_{tX}}e^{ad_{tY}}-1)^kA &= \left(\sum_{i=0}^{\infty}\frac{t^i(ad_X)^i}{i!}\sum_{j=0}^{\infty}\frac{t^j(ad_Y)^j}{j!} - I\right)^k A \\ &= \left(\sum_{i,j\geq 0, i+j>1}^{\infty} t^{i+j}\frac{(ad_X)^i(ad_Y)^j}{i!j!}\right)^k A \\ &= \sum_{s\in S_k} t^{i_1+j_1+\cdots+i_k+j_k}\frac{ad_X{}^{i_1}ad_Y{}^{j_1}\cdots ad_X{}^{i_k}ad_Y{}^{j_k}}{i_1!j_1!\cdots i_k!j_k!}A \\ & \qquad i_r, j_r \geq 0, i_r+j_r > 0, 1 \leq r \leq k. \end{aligned} \tag{1.40}$$

Using this result in (1.38)

$$
\begin{aligned}
\frac{dZ}{dt} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{s \in S_k, i_{k+1} \geq 0} \left\{ t^{i_1+j_1+\cdots+i_k+j_k} \frac{{ad_X}^{i_1} {ad_Y}^{j_1} \cdots {ad_X}^{i_k} {ad_Y}^{j_k}}{i_1! j_1! \cdots i_k! j_k!} X \right. \\
& \qquad \left. + \, t^{i_1+j_1+\cdots+i_k+j_k+i_{k+1}} \frac{{ad_X}^{i_1} {ad_Y}^{j_1} \cdots {ad_X}^{i_k} {ad_Y}^{j_k} {ad_X}^{i_{k+1}}}{i_1! j_1! \cdots i_k! j_k! i_{k+1}!} Y \right\}, \\
& \qquad i_r, j_r \geq 0, i_r + j_r > 0, 1 \leq r \leq k.
\end{aligned} \tag{1.41}
$$

or by switching notation

$$
\begin{aligned}
\frac{dZ}{dt} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{s \in S_k, i_{k+1} \geq 0} \left\{ t^{i_1+j_1+\cdots+i_k+j_k} \frac{[X^{(i_1)} Y^{(j_1)} \ldots X^{(i_k)} Y^{(j_k)} X]}{i_1! j_1! \cdots i_k! j_k!} \right. \\
& \qquad \left. + \, t^{i_1+j_1+\cdots+i_k+j_k+i_{k+1}} \frac{[X^{(i_1)} Y^{(j_1)} \ldots X^{(i_k)} Y^{(j_k)} X^{(i_{k+1})} Y]}{i_1! j_1! \cdots i_k! j_k! i_{k+1}!} \right\}, \\
& \qquad i_r, j_r \geq 0, i_r + j_r > 0, 1 \leq r \leq k.
\end{aligned} \tag{1.42}
$$

Now integrate $Z = Z(1) = \int_0^1 \frac{dZ}{dt} dt$, using $Z(0) = 0$,

$$
\begin{aligned}
Z &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{s \in S_k, i_{k+1} \geq 0} \left\{ \frac{1}{i_1 + j_1 + \cdots + i_k + j_k} \frac{[X^{(i_1)} Y^{(j_1)} \ldots X^{(i_k)} Y^{(j_k)} X]}{i_1! j_1! \cdots i_k! j_k!} \right. \\
& \qquad \left. + \frac{1}{i_1 + j_1 + \cdots + i_k + j_k + i_{k+1}} \frac{[X^{(i_1)} Y^{(j_1)} \ldots X^{(i_k)} Y^{(j_k)} X^{(i_{k+1})} Y]}{i_1! j_1! \cdots i_k! j_k! i_{k+1}!} \right\}, \\
& \qquad i_r, j_r \geq 0, i_r + j_r > 0, 1 \leq r \leq k.
\end{aligned} \tag{1.43}
$$

We can write this as

$$
\begin{aligned}
Z &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{s \in S_k, i_{k+1} \geq 0} \\
& \left\{ \frac{1}{i_1 + j_1 + \cdots + i_k + j_k + (i_{k+1} = 1) + (j_{k+1} = 0)} \frac{[X^{(i_1)} Y^{(j_1)} \ldots X^{(i_k)} Y^{(j_k)} X^{(i_{k+1}=1)} Y^{(j_{k+1}=0)}]}{i_1! j_1! \cdots i_k! j_k! (i_{k+1} = 1)! (j_{k+1} = 0)!} \right. \\
& \qquad \left. + \frac{1}{i_1 + j_1 + \cdots + i_k + j_k + i_{k+1} + (j_{k+1} = 1)} \frac{[X^{(i_1)} Y^{(j_1)} \ldots X^{(i_k)} Y^{(j_k)} X^{(i_{k+1})} Y^{(j_{k+1}=1)}]}{i_1! j_1! \cdots i_k! j_k! i_{k+1}! (j_{k+1} = 1)} \right\}, \\
& \qquad i_r, j_r \geq 0, i_r + j_r > 0, 1 \leq r \leq k.
\end{aligned} \tag{1.44}
$$

But this is equal to

$$
\begin{aligned}
Z &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{s \in S_k, i_{k+1} \geq 0} \\
&\quad \left\{ \frac{1}{i_1 + j_1 + \cdots + i_k + j_k + i_{k+1} + j_{k+1}} \frac{[X^{(i_1)} Y^{(j_1)} \cdots X^{(i_k)} Y^{(j_k)} X^{(i_{k+1})} Y^{(j_{k+1})}]}{i_1! j_1! \cdots i_k! j_k! i_{k+1}! j_{k+1}!} \right\} \\
&\qquad i_r, j_r \geq 0, i_r + j_r > 0, 1 \leq r \leq k+1. \qquad (1.45)
\end{aligned}
$$

using the fact that $[A, A] = 0$.

Finally, shifting the index, $k \to k-1$,

$$
\begin{aligned}
Z &= log(\exp X \exp Y) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{s \in S_k} \frac{1}{i_1 + j_1 + \cdots + i_k + j_k} \frac{[X^{(i_1)} Y^{(j_1)} \cdots X^{(i_k)} Y^{(j_k)}]}{i_1! j_1! \cdots i_k! j_k! i_{k+1}! j_{k+1}!}, \\
&\qquad i_r, j_r \geq 0, i_r + j_r > 0, 1 \leq r \leq k. \qquad (1.46)
\end{aligned}
$$

establishing Dynkin's formula.

1.5 Expansion

$$
\begin{aligned}
Z(X, Y) &= log(\exp X \exp Y) \\
&= \sum \frac{1}{i_1 + j_1} \frac{[X^{(i_1)} Y^{(j_1)}]}{i_1! j_1!} \\
&\quad - \sum \frac{1}{2(i_1 + j_1 + i_2 + j_2)} \frac{[X^{(i_1)} Y^{(j_1)} X^{(i_2)} Y^{(j_2)}]}{i_1! j_1! i_2! j_2!} \\
&\quad + \sum \frac{1}{3(i_1 + j_1 + i_2 + j_2 + i_3 + j_3)} \frac{[X^{(i_1)} Y^{(j_1)} X^{(i_2)} Y^{(j_2)} X^{(i_3)} Y^{(j_3)}]}{i_1! j_1! i_2! j_2! i_3! j_3!} \\
&\quad + \sum \frac{1}{3(i_1 + j_1 + i_2 + j_2 + i_3 + j_3 + i_4 + j_4)} \frac{[X^{(i_1)} Y^{(j_1)} X^{(i_2)} Y^{(j_2)} X^{(i_3)} Y^{(j_3)} X^{(i_4)} Y^{(j_4)}]}{i_1! j_1! i_2! j_2! i_3! j_3! i_4! j_4!} \\
&\quad + \cdots \qquad (1.47)
\end{aligned}
$$

$$
\begin{aligned}
Z(X,Y) &= log(\exp X \exp Y) \\
&= X + Y + \frac{1}{2}[X,Y] + \cdots \\
&\quad - \frac{1}{2(1+0+0+1)}[X,Y] - \frac{1}{2(1+0+0+1)}[Y,X] \\
&\quad - \frac{1}{2(1+1+1+0)}[X,[X,Y]] - \frac{1}{2(1+0+1+1)}[X,[Y,X]] \\
&\quad - \frac{1}{2(0+1+1+1)}[Y,[Y,X]] - \ldots \\
&\quad - \frac{1}{2(2+0+0+1)}\frac{[X,[X,Y]]}{2!} - \frac{1}{2(0+2+0+1)}\frac{[Y,[Y,X]]}{2!} \\
&\quad + \frac{1}{3(1+0+0+1+1+0)}[X,[Y,X]] + \frac{1}{3(1+0+0+1+0+1)}[X,[X,Y]] \\
&\quad + \frac{1}{3(0+1+0+1+0+1)}[Y,[X,Y]] + \cdots \\
&\quad + \cdots \\
&= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots
\end{aligned} \tag{1.48}
$$

1.6 Special Case: $[X,[X,Y]] = [Y,[X,Y]] = 0$

We wish to prove that

$$
\exp(X+Y) = \exp(X)\exp(Y)\exp(C/2) \tag{1.49}
$$

where $C = [X,Y]$.

1.6.1 Simple Proof

From () we have

$$
\begin{aligned}
\exp(\lambda X) Y \exp(-\lambda X) &= Y + \lambda[X,Y] + \frac{\lambda^2}{2!}[X,[X,Y]] + \ldots \\
&= Y + \lambda[X,Y]
\end{aligned} \tag{1.50}
$$

where we have used $[X,[X,Y]] = 0$.

Now differentiating

$$g(\lambda) = \exp(\lambda X)\exp(\lambda Y) \tag{1.51}$$

with respect to λ yields

$$\begin{aligned} \frac{d}{d\lambda} g(\lambda) &= X\exp(\lambda X)\exp(\lambda Y) + \exp(\lambda X) Y \exp(-\lambda X)\exp(\lambda X)\exp(\lambda Y) \\ &= (X + Y + \lambda[X,Y])g(\lambda) \end{aligned} \tag{1.52}$$

where we have used (1.50). A natural "guess" for a solution

$$g(\lambda) = \exp\left(\lambda X + \lambda Y + \frac{\lambda^2}{2}[X,Y]\right) \tag{1.53}$$

does indeed work when we assume $[Y,[X,Y]] = 0$. Combining (1.51) and (1.53) gives;

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y]\right) \tag{1.54}$$

for $\lambda = 1$, which is the desired result.

1.6.2 Algebraic Proof Involving Mathematical Induction

We split the proof into a number of steps.

Step 1:

Substituting

$$YX = XY - C \tag{1.55}$$

into, for example,

$$\begin{aligned} XYYYX &= XYYXY - XYYC \\ &= X(YYXY) - CXYY \\ &= X(YXYY - YCY) - CXYY \\ &= X(YXYY) - 2CXYY \\ &= X(XYYY - CYY) - 2CXYY \\ &= XXYYY - 3CXYY. \end{aligned} \tag{1.56}$$

It should be clear that ant sequence of X's and Y's can be explained in series whose terms are of the form

$$C^l X^m Y^n. \tag{1.57}$$

We define a normal ordering

$$:(X+Y)^n:\equiv \sum_{m=0}^{n} \binom{n}{m} X^m Y^{n-m}. \tag{1.58}$$

This is to make the calculation a little more tractable, it allows us to write

$$\begin{aligned}
\exp(X)\exp(Y) &= \left(\mathbb{1} + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots\right)\left(\mathbb{1} + Y + \frac{1}{2!}Y^2 + \frac{1}{3!}Y^3 + \cdots\right) \\
&= \mathbb{1} + (X+Y) + \left(\frac{1}{2!}X^2 + XY + \frac{1}{2!}Y^2\right) \\
&\quad + \left(\frac{1}{3!}X^3 + \frac{1}{2!}\frac{1}{1!}X^2Y + \frac{1}{1!}\frac{1}{2!}XY^2\frac{1}{3!}Y^3\right) + \ldots \\
&\quad \cdots + \frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m} X^m Y^{n-m} + \ldots \\
&= \mathbb{1} + :X+Y: + \frac{1}{2!}:(X+Y)^2: + \frac{1}{3!}:(X+Y)^3: + \cdots
\end{aligned} \tag{1.59}$$

To prove the BCH formula is to show

$$\exp(X+Y) =: \exp(X+Y) : \exp(C/2) \tag{1.60}$$

Step 2:

We show by induction that

$$(X+Y)^{n+1} = X(X+Y)^n + (X+Y)^n Y - nC(X+Y)^{n-1}. \tag{1.61}$$

Proof: First we assume for $(X+Y)^m$ that

$$(X+Y)^m = X(X+Y)^{m-1} + (X+Y)^{m-1}Y - (m-1)C(X+Y)^{m-2} \tag{1.62}$$

and we multiply both sides from the left by $X+Y$,

$$\begin{aligned}(X+Y)(X+Y)^m &= (X+Y)X(X+Y)^{m-1}+(X+Y)(X+Y)^{m-1}Y \\ &\quad -(m-1)(X+Y)C(X+Y)^{m-2} \qquad (1.63)\end{aligned}$$

This immediately simplfies

$$(X+Y)^{m+1}=(X+Y)X(X+Y)^{m-1}+(X+Y)^m Y-(m-1)C(X+Y)^{m-1} \qquad (1.64)$$

After a bit of rearrangement we have

$$\begin{aligned}(X+Y)^{m+1} &= \left[X(X+Y)-C\right](X+Y)^{m-1}+(X+Y)^m Y-(m-1)C(X+Y)^{m-1} \\ &= X(X+Y)^m+(X+Y)^m Y-mC(X+Y)^{m-1}. \qquad (1.65)\end{aligned}$$

It is easy to check that for $m=2$ that

$$(X+Y)^2=X(X+Y)+(X+Y)Y-C. \qquad (1.66)$$

Step 3:

By the argument any combination of X's and Y's can be written in (1.56)

$$(X+Y)^n=\sum_{m=0}^{[n/2]}\alpha_{n,m}C^m:(X+Y)^{n-2m}: \qquad (1.67)$$

where

$$[n/2]=\begin{cases} n/2, & n \text{ even} \\ (n-1)/2 & n \text{ odd.}\end{cases} \qquad (1.68)$$

Thi is called a floor function and it gauarntees that $[n/2]$ is always an integer and that $n-2m$ does not become negative.

We now derive a recursion relation using (1.61) to find the α_{nm}'s. We have

$$\sum_{m=0}^{[(n+1)/2]} \alpha_{n+1,m} C^m :(X+Y)^{n+1-2m}: = X \sum_{m=0}^{[n/2]} \alpha_{n,m} C^m :(X+Y)^{n-2m}:$$
$$\sum_{m=0}^{[n/2]} \alpha_{n,m} C^m :(X+Y)^{n-2m}: Y - nC \sum_{m=0}^{[(n-1)/2]} \alpha_{n-1,m} C^m :(X+Y)^{n-1-2m}: \tag{1.69}$$

By noting

$$\begin{aligned}
X:(X+Y)^p:+:(X+Y)^p:Y &= \sum_{q=0}^{p} \binom{p}{q} \left[X^{p+1}Y^{q-p} + X^p Y^{p+1-q}\right] \\
&= X^{n+1} + \sum_{q=1}^{p} \left[\binom{p}{q-1} + \binom{p}{q}\right] X^{p+1} Y^{p+1-q} + Y^{n+1} \\
&= \sum_{q=0}^{p+1} \binom{p+1}{q} X^{p+1} Y^{p+1-q} \\
&= :(X+Y)^{p+1}:
\end{aligned} \tag{1.70}$$

(1.69) simplifies to

$$\sum_{m=0}^{[(n+1)/2]} \alpha_{n+1,m} C^m :(X+Y)^{n+1-2m}: = \sum_{m=0}^{[n/2]} \alpha_{n,m} C^m :(X+Y)^{n+1-2m}:$$
$$-nC \sum_{m=0}^{[(n-1)/2]} \alpha_{n-1,m} C^m :(X+Y)^{n-1-2m}: \tag{1.71}$$

which then becomes

$$\alpha_{n+1,0} C^0 :(X+Y)^{n+1}: + \sum_{m=1}^{[(n+1)/2]} \alpha_{n+1,m} C^m :(X+Y)^{n+1-2m}:$$
$$= \alpha_{n,0} C^0 :(X+Y)^{n+1}: + \sum_{m=1}^{[n/2]} \alpha_{n,m} C^m :(X+Y)^{n+1-2m}:$$
$$-nC \sum_{m=0}^{[(n-1)/2]} \alpha_{n-1,m} C^m :(X+Y)^{n-1-2m}: \tag{1.72}$$

or

$$
\begin{aligned}
&\alpha_{n+1,0} : (X+Y)^{n+1} : + \sum_{m=0}^{[(n-1)/2]} \alpha_{n+1,m+1} C^{m+1} : (X+Y)^{n-1-2m} : \\
&= \alpha_{n,0} : (X+Y)^{n+1} : + \sum_{m=0}^{[(n-2)/2]} \alpha_{n,m+1} C^{m+1} : (X+Y)^{n-1-2m} : \\
&-n \sum_{m=0}^{[(n-1)/2]} \alpha_{n-1,m} C^{m+1} : (X+Y)^{n-1-2m} :
\end{aligned}
\tag{1.73}
$$

We consider n even and n odd separatly. Firtst n even. The upper value in two of the sums is $[(n-1)/2] = (n-2)/2$ and the upper value in the other sum is $[(n-2)/2] = (n-2)/2$, meaning the upper value in all summations is the same. We arrive at

$$
\alpha_{n+1,m+1} = \alpha_{n,m+1} - n\alpha_{n-1,m} \tag{1.74}
$$

(provided $\alpha_{n+1,0} = \alpha_{n,0}$). This recursion relation is solved by

$$
\alpha_{n,m} = \left(-\frac{1}{2}\right)^m \frac{n!}{(n-2m)!m!}. \tag{1.75}
$$

Proof:

$$
\begin{aligned}
\alpha_{n,m+1} - n\alpha_{n-1,m} &= \left(-\frac{1}{2}\right)^{m+1} \frac{n!}{(n-2-2m)!(m+1)!} - \left(-\frac{1}{2}\right)^m n \frac{(n-1)!}{(n-1-2m)!m!} \\
&= \left(-\frac{1}{2}\right)^{m+1} \frac{n!}{(n-1-2m)!(m+1)!} [(n-1-2m) - (-2)(m+1)] \\
&= \left(-\frac{1}{2}\right)^{m+1} \frac{(n+1)!}{(n-2m-1)!(m+1)!} \\
&= \alpha_{n+1,m+1}.
\end{aligned}
\tag{1.76}
$$

(and the condition $\alpha_{n+1,0} = \alpha_{n,0}$ is obviously satisfied).

We now consider n odd. The upper value in two of the sums in (1.73) is $[(n-1)/2] = (n-1)/2$ and the upper value in the other sum is $[(n-2)/2] = (n-3)/2$, meaning the upper value in the summations is not the same. We must consider the condition

$$\alpha_{n+1,(n-1)/2+1} = -n\alpha_{n-1,n-1/2} \tag{1.77}$$

separaletly. This can easily be shown to be satisfied by (1.75):

$$\begin{aligned}
-n\alpha_{n-1,n-1/2} &= -n\left(-\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{(n-1)!}{((n-1)-2(n-1)/2)!\,((n-1)/2)!} \\
&= 2\left(-\frac{1}{2}\right)^{\frac{n-1}{2}+1} \frac{n!}{0!\,((n-1)/2)!} \\
&= \left(-\frac{1}{2}\right)^{\frac{n-1}{2}+1} \frac{(n+1)!}{n+1} 2\frac{(n+1)/2}{((n+1)/2)!} \\
&= \left(-\frac{1}{2}\right)^{\frac{n-1}{2}+1} \frac{(n+1)!}{0!\,((n-1)/2+1)!} \\
&= \alpha_{n+1,(n-1)/2+1}.
\end{aligned} \tag{1.78}$$

(and again the condition $\alpha_{n+1,0} = \alpha_{n,0}$ is obviously satisfied). Thus (1.75) has been established in general.

Now substituting (1.75) into (1.67) gives

$$(X+Y)^n = \sum_{m=0}^{[n/2]} \left(-\frac{1}{2}\right)^m \frac{n!}{(n-2m)!m!} C^m : (X+Y)^{n-2m} : \tag{1.79}$$

and using this we have

$$\begin{aligned}
\exp(X+Y) &= \sum_{n=0}^{\infty} \frac{1}{n!}(X+Y)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{[n/2]} \left(-\frac{1}{2}\right)^m \frac{n!}{(n-2m)!m!} C^m : (X+Y)^{n-2m} : \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} \frac{(-C/2)^m}{m!} \frac{: (X+Y)^{n-2m} :}{(n-2m)!}
\end{aligned} \tag{1.80}$$

We reorder the summations:

$$
\begin{aligned}
\exp(X+Y) &= \underbrace{\sum_{n=0}^{\infty}\frac{:(X+Y)^n:}{n!}}_{\textit{all } m=0 \textit{ terms}}+\underbrace{\sum_{n=2}^{\infty}(-C/2)\frac{:(X+Y)^{n-2}:}{(n-2)!}}_{\textit{all } m=1 \textit{ terms}}+ \\
&\quad +\underbrace{\sum_{n=4}^{\infty}\frac{(-C/2)^2}{2!}\frac{:(X+Y)^{n-4}:}{(n-4)!}}_{\textit{all } m=2 \textit{ terms}}+\cdots \\
&= \left(\mathbb{1}+(-C/2)+\frac{1}{2!}(-C/2)^2+\ldots\right)\sum_{n=0}^{\infty}\frac{:(X+Y)^n:}{n!} \\
&= \exp(X)\exp(Y)\exp(-C/2). \qquad (1.81)
\end{aligned}
$$

Hence we arrive at

$$
\exp(X+Y)=\exp(X)\exp(Y)\exp(-C/2). \qquad (1.82)
$$