Derivation of Black Hole Solutions

Contents

1 The Schwarzschild Metric **6**

2 Schwarzschild Black Hole **17**

2.1 Eddington-Finkelstein Coordinates 17

2.1.1 Advanced time parameter 18

3 Internal Schwarzschild Solution **20**

3.1 Introduction 20

3.2 Perfect Fluid Schwarzschild Solution 22

4 Penrose-Carter diagrams **26**

4.1 Geodesics under conformal transformations 26

4.2 Penrose-Carter Diagram for Minkowski Spacetime. 27

4.3 Maximal Extensions 30

4.4 The Kruskal Solution 30

5 Charged Balck Holes **31**

5.0.1 Solving the Maxwell equations and the electric charge 33

5.0.2 Calculation of the energy-momentum tensor 38

5.0.3 Solving the field equations 40

5.1 Event Horizons 41

5.2 Analogue of Eddington-Finkelstein Coordinates 42

5.3 Penrose Diagram . . . 44
5.4 Double null coordinates . . . 44
5.5 Maximal extension . . . 45

6 Rotating Black Holes **48**

6.1 Field Equations . . . 48
6.1.1 Degenerate metric . . . 48
6.1.2 Order m^4 equations: . . . 51
6.1.3 Order m^3 equations: . . . 51
6.1.4 Equation from linear in m equation . . . 54
6.1.5 Order m^2 equations . . . 55
6.1.6 Stationary field equations from the m order equation . . . 59
6.1.7 Laplace and eikonal equations . . . 69
6.1.8 Introduce the complex function γ . . . 72
6.1.9 Expression for λ . . . 74
6.1.10 Expression for l_0 . . . 76
6.2 The Kerr Solution . . . 77
6.2.1 A solution to the field equations . . . 77
6.2.2 The Schwarzschild solution . . . 78
6.2.3 The Kerr solution . . . 79
6.3 Independence of Metric on Angular Variable φ . . . 83
6.4 Boyer-Lindquist Coordinates . . . 86
6.5 Interptetation as Rotating Body . . . 88
6.6 Basic Propertire of the Kerr Solution . . . 89
6.6.1 Boyer-Lindquist cordinates . . . 89
6.6.2 Cartesian Coordinates . . . 90
6.7 Singularities and Event Horizons . . . 90

A Derivation of the Full Field Equations of General Relativity **92**

A.1 Mathematical Preliminaries 93

A.1.1 The Metric Connection 93

A.1.2 Tensor Densities 94

A.1.3 Matrix Identities 94

A.1.4 Metric Identities 95

A.1.5 Palatini Equation 97

A.2 The Einstein Lagrangian 98

A.3 The Full Field Equations - Inclusion of Matter and Radiation 100

A.4 The Energy-Momentum Tensor for Dust, a Perfect Fluid 101

A.4.1 Source term: From Newton's Theory to Special Relativity 101

A.4.2 Dust 102

A.4.3 Dust Lagrangian 105

A.4.4 Perfect Fluid 106

A.5 The Energy-Momentum Tensor for the Electro-Magnetic Field 108

A.5.1 Reminder of Maxwell's Equations 108

A.5.2 Potential Formulation of Maxwell's Equations 110

A.5.3 Action for Maxwell Equations and Minimal Coupling to Charged Matter 112

A.5.4 The Maxwell Energy-momentum Tensor 114

A.6 Einstein-Maxwell Equations 116

List of Figures

4.1 Penrose diagram of Minkowski spacetime 29

4.2 Penrose diagram of the Kruskal solution 30

4.3 Penrose diagram of a black hole. 30

5.1 Penrose diagram of a black hole. 47

6.1 Rotating blackhole 90

Chapter 1

The Schwarzschild Metric

The most general spherically symmetric metric is

$$d\tau^2 = A(t,r)dt^2 - B(t,r)dr^2 - 2C(t,r)dtdr - D(t,r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.1}$$

We introduce a new radial coordinate

$$r' = D^{1/2}(t,r)$$

The line element then becomes

$$d\tau^2 = A'(t,r')dt^2 - B'(t,r')dr'^2 - 2C'(t,r')dtdr - r'^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.2}$$

Consider the differential

$$A'(t,r')dt - B'(t,r')dr'$$

The theory of ordinary differential equations tells us that there exists a function (so called integration factor) $f(t,r')$ such that

$$\frac{\partial t'}{\partial t} = A'(t,r')f(t,r'), \qquad \frac{\partial t'}{\partial r'} = -B'(t,r')f(t,r')$$

for some $t' = t'(t,r')$, making t' a perfect differential:

$$\begin{aligned} dt' &= \frac{\partial t'}{\partial t}dt + \frac{\partial t'}{\partial r'}dr' \\ &= (A'(t,r')dt - B'(t,r')dr')f(t,r') \end{aligned} \tag{1.3}$$

We need to know something of integrating factors and perfect differentials.

Proof:

Consider the ordinary differential equation of first order in two variables:

$$(\frac{dy}{dx} - \frac{Y(x,y)}{X(x,y)}) = g(x,y). \tag{1.4}$$

Observe that

$$\frac{d}{dx}\left(ye^{-\int^x Y(x',y)/X(x',y)dx'}\right) = \left(\frac{dy}{dx}e^{-\int^x Y(x',y)/X(x',y)dx'} - \frac{Y(x,y)}{X(x,y)}e^{-\int^x Y(x')/X(x')dx'}\right)$$

Thus we multiply a given differential equation by the factor

$$e^{-\int^x Y(x',y)/X(x',y)dx'}$$

which turns the LHS of the equation into a perfect differential

$$\frac{d}{dx}\left(ye^{-\int^x Y(x',y)/X(x',y)dx'}\right) = g(x,y)e^{-\int^x Y(x',y)/X(x',y)dx'}$$

or

$$\begin{aligned} d\left(y(x,y)e^{-\int^x Y(x',y)/X(x',y)dx'}\right) &= (dy - \frac{Y(x,y)}{X(x,y)}dx)e^{-\int^x Y(x')/X(x')dx'} \\ &= (X(x,y)dy - Y(x,y)dx)(e^{-\int^x Y(x',y)/X(x',y)dx'}/X(x,y)) \end{aligned} \tag{1.5}$$

Put

$$y'(x,y) = y(x,y)e^{-\int^x Y(x',y)/X(x',y)dx'}$$

and

$$f(x,y) = e^{-\int^x Y(x',y)/X(x',y)dx'}/X(x,y)$$

then we have

$$dy' = (X(x,y)dy - Y(x,y)dx)f(x,y)$$

□

We use this result to define a new time coordinate t' for which

$$dt' = (A'(t,r')dt - B'(t,r')dr')f(t,r') \tag{1.6}$$

Squaring, we obtain

$$dt'^2 = (A'^2dt^2 - 2A'B'dtdr' + B'^2dr'^2)f^2(t,r')$$

or

$$A'dt^2 - 2B'dtdr' = A'^{-1}f^{-2}dt'^2 - A'^{-1}B'^2dr'^2$$

the line element then becomes

Dropping the primes, the line element then becomes

$$d\tau^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.7}$$

Dropping the primes, the line element then becomes

$$d\tau^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.8}$$

where

$$\nu = \nu(t,r), \qquad \lambda = \lambda(t,r) \tag{1.9}$$

The metric components are

$$g_{00} = A, \quad g_{11} = -B, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \tag{1.10}$$

$$g_{00} = e^{\nu}, \quad g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \tag{1.11}$$

The metric is diagonal so that $g^{00} = 1/g_{00}$. The coordinates are labelled

$$x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi.$$

By definition

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}).$$

There are $4^3/2 = 32$ independent Γ^a_{bc}s. However many will be zero. For example, it is obvious that if all components are distinct the connection vanishes.

Let us do some examples.

$$\begin{aligned} \Gamma^0_{01} &= \frac{1}{2} g^{0d}(g_{0d,1} + g_{1d,0} - g_{01,d}) \\ &= \frac{1}{2} g^{00}(g_{00,1} + g_{10,0} - g_{01,0}) \\ &= \frac{\nu'}{2} \end{aligned} \tag{1.12}$$

where we have used that the metric is diagonal and the prime denotes differentiation with respect to r.

Consider Γ^1_{00}

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2} g^{11}(g_{01,0} + g_{01,0} - g_{00,1}) \\ &= -\frac{1}{2} g^{11} g_{00,1} \\ &= \frac{e^{-\lambda}}{2} \nu' e^{\nu} \\ &= \frac{e^{\nu-\lambda}}{2} \nu' \end{aligned} \tag{1.13}$$

Consider Γ^1_{11}

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2} g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) \\ &= \frac{1}{2} g^{11} g_{11,1} \\ &= \frac{e^{-\lambda}}{2} \lambda' e^{\lambda} \\ &= \frac{\lambda'}{2} \end{aligned} \tag{1.14}$$

Consider Γ^1_{22}

$$\begin{aligned}\Gamma^1_{22} &= \frac{1}{2}g^{11}(g_{21,2} + g_{21,b} - g_{22,1}) \\ &= -\frac{1}{2}g^{11}g_{22,1} \\ &= \frac{e^{-\lambda}}{2}(-2r) \\ &= -re^{-\lambda} \end{aligned} \tag{1.15}$$

Consider Γ^2_{12}

$$\begin{aligned}\Gamma^2_{12} &= \frac{1}{2}g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) \\ &= \frac{1}{2}g^{22}g_{22,1} \\ &= \frac{1}{2r^2}2r \\ &= 1/r \end{aligned} \tag{1.16}$$

Consider Γ^3_{13}

$$\begin{aligned}\Gamma^3_{13} &= \frac{1}{2}g^{33}(g_{13,3} + g_{33,1} - g_{13,3}) \\ &= \frac{1}{2}g^{33}g_{33,1} \\ &= -\frac{1}{2r^2\sin^2\theta}(-2r\sin^2\theta) \\ &= 1/r. \end{aligned} \tag{1.17}$$

Consider Γ^1_{33}

$$\begin{aligned}\Gamma^1_{33} &= \frac{1}{2}g^{1d}(g_{3d,3} + g_{3d,3} - g_{33,d}) \\ &= \frac{1}{2}g^{11}(g_{31,3} + g_{31,3} - g_{33,1}) \\ &= -g^{11}g_{33,1} \\ &= -r\sin^2\theta/B. \end{aligned} \tag{1.18}$$

Consider Γ^3_{32}

$$\begin{aligned}\Gamma^3_{32} &= \frac{1}{2}g^{33}(g_{33,2} + g_{23,3} - g_{32,3}) \\ &= \frac{1}{2}g^{33}g_{33,2} \\ &= -\frac{1}{2r^2\sin^2\theta}\frac{\partial}{\partial\theta}(-r^2\sin^2\theta) \\ &= \frac{\cos\theta}{\sin\theta} = \cot\theta. \end{aligned} \tag{1.19}$$

Consider Γ^3_{33}

$$\begin{aligned}\Gamma^3_{33} &= \frac{1}{2}g^{33}(g_{33,3} + g_{33,3} - g_{33,3}) \\ &= \frac{1}{2}g^{33}g_{33,3} = 0 \end{aligned} \tag{1.20}$$

As none of the metric components depend on ϕ. There are many other examples that turn out to be zero.

Consider Γ^0_{00}

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{2}g^{00}(g_{00,0} + g_{00,0} - g_{00,0}) \\ &= \frac{1}{2}g^{00}g_{00,0} \\ &= \frac{e^{-\nu}}{2}\dot{\nu}e^{\nu} \\ &= \frac{\dot{\nu}}{2} \end{aligned} \tag{1.21}$$

Consider Γ^0_{11}

$$\begin{aligned}\Gamma^0_{01} &= \frac{1}{2}g^{00}(g_{00,1} + g_{10,0} - g_{01,0}) \\ &= \frac{1}{2}g^{00}g_{00,1} \\ &= \frac{e^{-\nu}}{2}\nu' e^{\nu} \\ &= \frac{\nu'}{2} \end{aligned} \tag{1.22}$$

Consider Γ^0_{11}

$$\begin{aligned}\Gamma^0_{11} &= \frac{1}{2}g^{00}(g_{10,1}+g_{10,1}-g_{11,0}) \\ &= -\frac{1}{2}g^{00}g_{11,0} \\ &= \frac{e^{\lambda-\nu}}{2}\dot{\lambda} \end{aligned} \tag{1.23}$$

Consider Γ^1_{00}

$$\begin{aligned}\Gamma^1_{00} &= \frac{1}{2}g^{11}(g_{01,0}+g_{01,0}-g_{00,1}) \\ &= -\frac{1}{2}g^{11}g_{00,1} \\ &= -\frac{e^{\nu-\lambda}}{2}\nu' \end{aligned} \tag{1.24}$$

Consider Γ^1_{01}

$$\begin{aligned}\Gamma^1_{01} &= \frac{1}{2}g^{11}(g_{01,1}+g_{11,0}-g_{01,1}) \\ &= \frac{1}{2}g^{11}g_{11,0} \\ &= \frac{e^{-\lambda}}{2}\dot{\lambda}e^{\lambda} \\ &= \frac{\dot{\lambda}}{2} \end{aligned} \tag{1.25}$$

We find altogether:

$$
\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}e^{\nu-\lambda}\nu' \\
\Gamma^1_{11} &= \frac{1}{2}\lambda' \\
\Gamma^1_{22} &= -re^{-\lambda}, \\
\Gamma^1_{33} &= -re^{-\lambda}\sin^2\theta, \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = 1/r \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \\
\Gamma^3_{32} &= \Gamma^3_{23} = \cot\theta \\
\Gamma^0_{00} &= \frac{\dot{\nu}}{2}, \\
\Gamma^0_{01} &= \frac{\nu'}{2} \\
\Gamma^0_{11} &= \frac{e^{\lambda-\nu}}{2}\dot{\lambda} \\
\Gamma^1_{00} &= -\frac{e^{\nu-\lambda}}{2}\nu' \\
\Gamma^1_{01} &= \frac{1}{2}\dot{\lambda}
\end{aligned}
\tag{1.26}
$$

All others being zero.

We can calculate the Riemann tensor using

$$
R^a{}_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \tag{1.27}
$$

$$
R^a{}_{bcd} = -R^a{}_{bdc} \tag{1.28}
$$

The non-vanishing components of the Riemann tensor are

$$
\begin{aligned}
R^0{}_{101} &= e^{\lambda-\nu}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\nu'\dot{\lambda}}{4}\right) + \left(\frac{\nu'\lambda'}{4} - \frac{\nu''}{2} - \frac{\nu'^2}{4}\right) \\
R^0{}_{202} &= -re^{-\lambda}\nu'/2 \\
R^0{}_{303} &= -re^{-\lambda}\sin^2\theta\nu'/2 \\
R^0{}_{212} &= -re^{-\nu}\dot{\lambda}/2 \\
R^0{}_{313} &= -re^{-\nu}\sin^2\theta\dot{\lambda}/2 \\
R^1{}_{212} &= re^{-\lambda}\lambda'/2 \\
R^1{}_{313} &= re^{-\lambda}\sin^2\theta\lambda'/2 \\
R^2{}_{323} &= (1-e^{-\lambda})\sin^2\theta
\end{aligned}
\tag{1.29}
$$

We use these to calculate the components of the Ricci tensor.

$$R_{ab} = \partial_b \Gamma^c_{ac} - \partial_c \Gamma^c_{ab} + \Gamma^d_{ac}\Gamma^c_{bd} - \Gamma^d_{ab}\Gamma^c_{dc} \tag{1.30}$$

The non-vanishing components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= \left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\nu}\dot{\lambda}}{4}\right) + e^{\lambda-\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{1}{r}\nu'\right) && (1.31)\\
R_{11} &= -\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} - \frac{1}{r}\lambda'\right) + e^{\lambda-\nu}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\nu}\dot{\lambda}}{4}\right) && (1.32)\\
R_{01} &= \frac{1}{r}\dot{\lambda} && (1.33)\\
R_{22} &= e^{-\lambda}\left(\frac{r}{2}(\lambda' - \nu') - 1\right) + 1 && (1.34)\\
R_{33} &= R_{22}\sin^2\theta && (1.35)
\end{aligned}$$

(1.33) vanishing implies

$$\lambda = \lambda(r).$$

From R_{00} and R_{11} vanishing independently

$$0 = e^{\lambda-\nu}R_{00} + R_{11} = \frac{1}{r}(\lambda' + \nu') \tag{1.36}$$

implying

$$\lambda + \nu = h(t) \tag{1.37}$$

where $h(t)$ is an arbitrary function of integration. As λ is purely a function of r, (1.34) is an ordinary differential equation, which we write

$$e^{-\lambda} - re^{-\lambda}\lambda' = 1,$$

or

$$(re^{-\lambda})' = 1.$$

Integrating, we get

$$re^{-\lambda} = r + Const.$$

so

$$e^{\lambda} = (1 + Const./r)^{-1}. \tag{1.38}$$

Note

$$e^{\nu} = e^{h(t)-\lambda}$$

The line element becomes

$$ds^2 = e^{h(t)}\left(1 + \frac{Const.}{r}\right)dt^2 - \frac{dr^2}{1 + Const./r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.39}$$

Write

$$t' = \int^t e^{h(u)/2}du \tag{1.40}$$

then

$$dt'^2 = e^{h(t)}dt^2$$

Dropping primes the most general spherically symmetric solution of the vacuum field equations is

$$ds^2 = \left(1 + \frac{Const.}{r}\right)dt^2 - \frac{dr^2}{1 + Const./r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.41}$$

All of the metric components are independent of a time coordinate. We have therefore proven that any spherically symmetric vacuum metric is stationary.

Now consider the Newtonian limit. A point mass M situated at the origin in Newtonian theory gives rise to a potential

$$\phi = -GM/r$$

Using this in the weak field limit gives

$$g_{00} \simeq 1 + 2\phi/c^2 = 1 - 2GM/c^2 r.$$

We see that the constant turns out to be $-2GM/c^2$

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{dr^2}{1 - 2GM/c^2 r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.42}$$

Chapter 2

Schwarzschild Black Hole

2.1 Eddington-Finkelstein Coordinates

In the Eddington-Finkelstein coordinates radial null geodesics become straight lines

$$\bar{t} = t + 2m\ln(r-2m) \tag{2.1}$$

$$dt = d\bar{t} - \frac{2m}{r-2m}dr \tag{2.2}$$

Squaring gives

$$\begin{aligned}\left(1-\frac{2M}{r}\right)dt^2 &= \left(1-\frac{2M}{r}\right)\left[dt - \frac{2m}{r}\left(1-\frac{2m}{r}\right)^{-1}dr\right]^2 \\ &= \left(1-\frac{2M}{r}\right)d\bar{t}^2 - 2\frac{2m}{r}d\bar{t}dr + \frac{4m^2}{r^2}\left(1-\frac{2m}{r}\right)^{-1}dr^2\end{aligned} \tag{2.3}$$

Now we substitute it into the Schwarzschild line element.

$$\begin{aligned}ds^2 &= \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2) \\ &= \left(1-\frac{2M}{r}\right)d\bar{t}^2 - \frac{4m}{r}d\bar{t}dr - \left(1-\frac{4m^2}{r^2}\right)\left(1-\frac{2m}{r}\right)^{-1}dr^2 \\ &\quad -r^2(d\theta^2+\sin^2\theta d\phi^2) \\ &= \left(1-\frac{2M}{r}\right)d\bar{t}^2 - \frac{4m}{r}d\bar{t}dr - \left(1-\frac{2M}{r}\right)dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2)\end{aligned} \tag{2.4}$$

and obtain the line element

$$ds^2 = \left(1 - \frac{2M}{r}\right)d\bar{t}^2 - \frac{4m}{r}d\bar{t}dr - \left(1 - \frac{2M}{r}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5}$$

Notice that in this new coordinate system we no longer have a coordinate signularity at $R = 2M$. In fact it is regular for the whole range $0 < r < 2m$.

2.1.1 Advanced time parameter

As an external observer we cannot chart the course of a particle upon entering the event horizon. However, a particle entering would pass through the event horizon unaware of anything strange. The singularity at $R = 2M$ of the Schwarzschild metric is unphysical, it is simply a coordinate singularity.

To show it isn't a physical singularity, let us make a change of coordinates and derive a new metric. We will keep r, θ but replace t with

$$t = v - r - 2M \ \ln\left|\frac{r}{2M} - 1\right| \tag{2.6}$$

Let us find dt and substitute it back into the Schwarzschild line element. We begin by taking the derivative of t,

$$\begin{aligned} dt &= dv - dr - \frac{dr}{r/2M - 1} \\ &= dv - \left(\frac{r/2M - 1 + 1}{r/2M - 1}\right)dr \\ &= dv - \frac{rdr}{r - 2M} \\ &= dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \end{aligned} \tag{2.7}$$

Squaring this we get

$$dt^2 = dv^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dvdr + \left(1 - \frac{2M}{r}\right)^{-2} dr^2. \tag{2.8}$$

Now we substitute it into the Schwarzschild line element.

$$
\begin{aligned}
ds^2 &= \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{1-2M/r} - r^2(d\theta^2+\sin^2\theta d\phi^2) \\
&= \left(1-\frac{2M}{r}\right)\left[dv^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dvdr + \left(1-\frac{2M}{r}\right)^{-2} dr^2\right] \\
&\quad -\left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2) \\
&= \left(1-\frac{2M}{r}\right)dv^2 - 2dvdr - r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (2.9)
\end{aligned}
$$

We have the line element in the Eddington-Finkelstein coordinates

$$
ds^2 = \left(1-\frac{2M}{r}\right)dv^2 - 2dvdr - r^2(d\theta^2+\sin^2\theta d\phi^2). \qquad (2.10)
$$

Notice that in this new coordinate system we no longer have a coordinate signularity at $R = 2M$. We are now able to explore what happens inside the Schwarzschild black hole. Notice however we still have a singularity at $r = 0$. This is a real physical singularity, and not due to the coordinates used.

Chapter 3

Internal Schwarzschild Solution

3.1 Introduction

Before solving Einstein's equations, we look at Newtonian theory, and specifically the case of self-gravitating fluid in hydrostatic equillibrium. Much carries over to the relativistic theory.

In Newton's theory the equations equations for a self-gravitating fluid are

$$\rho[\dot{V}_i + V_j \nabla_j V_i] = -\nabla_i p - \rho \nabla_i \Phi, \tag{3.1}$$

together with the Newton-Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho. \tag{3.2}$$

For a fluid in static equilibrium the velocity three- vector V_i vanishes, and we are left with

$$\nabla_i p = -\rho \nabla_i \Phi, \tag{3.3}$$

represeting the balance of pressure and gravity. Newton's theeory of a self-gravitating fluid in static equilibrium satisfies (3.3) and (3.2).

We are interesed in the case where we have spherical symmetry as we wish to model a non-relativistic star.

With spherical symmetry for the arbitrary function $f(r)$ we have

$$\begin{aligned}\nabla_i f(r) &= \frac{\partial r}{\partial x_i}\frac{d}{dr}f(r) \\ &= \frac{x_i}{r}f'(r) \end{aligned} \tag{3.4}$$

where $r^2 = x_i x_i$ and the prime denotes d/dr. Furthermore,

$$\nabla_i \nabla_i \Phi f(r) = f'' + \frac{2}{r}f'. \tag{3.5}$$

Thus if p, ρ, and Φ are functions of r alone then (3.2) and (3.3) read

$$\Phi'' + \frac{2}{r}\Phi' = 4\pi G\rho \tag{3.6}$$

and

$$p' = -\rho\Phi'. \tag{3.7}$$

We wish to eliminate Φ so as to obtain an equation relating ρ ad p. We will impose the condition that the fields p, ρ, and Φ and their derivatives are non-singular throughout the star and its environment. Multiplying (3.6) by r^2 we have

$$r^2\Phi'' + 2r\Phi' = (r^2\Phi)'' = 4\pi G\rho r^2$$

Integrating we have

$$\Phi' = \frac{4\pi G}{r^2}\int_0^r \rho(\xi)\xi^2 d\xi + \frac{k}{r}\rho \tag{3.8}$$

where k is a constant. However, by our assumption that $\Phi'(0)$ is non-singular, we must have $k = 0$.

If we write

$$m(r) = 4\pi \int_0^r \rho(\xi)\xi^2 d\xi \tag{3.9}$$

for the total mass up to the radius r, we have

$$\Phi' = \frac{Gm(r)}{r^2}. \tag{3.10}$$

Finally, by substituting (3.11) into (3.7) we get

$$p' = -\frac{G\rho m(r)}{r^2}. \tag{3.11}$$

This is the fundamental equation of hydrostatic equillibrium for spherically symmetric self-gravitating system of fluid in Newton's theory.

3.2 Perfect Fluid Schwarzschild Solution

Perfect Fluid Schwarzschild Solution

The metric has the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.12}$$

We need to solve the full Einstein equation,

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}. \tag{3.13}$$

The Ricci scalar is giveb by

$$R = e^{-\lambda}\left(\nu'' + \frac{1}{2}\nu' 2 - \frac{1}{2}\nu'\lambda' + \frac{2}{r}(\nu' - \lambda') + \frac{2}{r^2}\right) - \frac{2}{r^2}. \tag{3.14}$$

The nonvanishing components of the Einstein tensor G_{ab} are given by

$$\begin{aligned} G_{00} &= \frac{1}{r^2}e^{\nu-\lambda}\left(r\lambda' - 1 + e^{\lambda}\right) \\ G_{11} &= \frac{1}{r^2}\left(r\nu' + 1 - e^{\lambda}\right) \\ G_{22} &= r^2 e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{1}{2r}(\nu' - \lambda')\right) \\ G_{33} &= \sin^2\theta G_{22} \end{aligned} \tag{3.15}$$

The star is modelled by a perfect fluid

$$T_{ab} = (\rho + p)U_a U_b - p g_{ab}. \tag{3.16}$$

where ρ and p are the energy density and uniform pressure as measured in the rest frame of the fluid, and U is the fluid four velocity - assumed to be

$$U_a = (e^{\nu/2}, 0, 0, 0) \tag{3.17}$$

$U^a U_a = 1$

$$T_{ab} = \begin{pmatrix} e^{\nu}\rho & 0 & 0 & 0 \\ 0 & e^{\lambda}p & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta\, p \end{pmatrix} \tag{3.18}$$

Einstein's equations read

$$\frac{1}{r^2} e^{-\lambda}\left(r\lambda' - 1 + e^{\lambda}\right) = 8\pi G\rho, \tag{3.19}$$

$$\frac{1}{r^2} e^{-\lambda}\left(r\nu' + 1 - e^{\lambda}\right) = 8\pi G p, \tag{3.20}$$

$$e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{1}{2r}(\nu' - \lambda')\right) = 8\pi G p \tag{3.21}$$

Note that the first of these equations involves only $\lambda(r)$ and the density $\rho(r)$. Write

$$m(r) = \frac{1}{2G}(r - re^{-\lambda}) \tag{3.22}$$

then the first equation becomes

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{3.23}$$

which that can be integrated

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'. \tag{3.24}$$

Here, ρ is the density and so we interpret $m(r)$ as the total mass of the star enclosed by a sphere of radius r. Say or star extends to a radius R, after which spacetime is described by the Schwarzschild solution. We must have

$$M = m(R) = 4\pi \int_0^R \rho(r')r^2 dr. \tag{3.25}$$

where M is the Schwarzschild mass.

$$e^{\lambda} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1} \tag{3.26}$$

so that the line element becomes

$$ds^2 = e^{\nu(r)}dt^2 - \left(1 - \frac{2Gm(r)}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.27}$$

Equation (3.20) becomes

$$\frac{1}{r^2}\left(1 - \frac{2Gm(r)}{r}\right)(r\nu' + 1) - \frac{1}{r^2} = 8\pi G p,$$

or

$$(r - 2Gm(r))\,(r\nu' + 1) = r + 8\pi G p r^3,$$

or

$$(r\nu' + 1) = \frac{r + 8\pi G p r^3}{(r - 2Gm(r))},$$

and so

$$\frac{d\nu}{dr} = \frac{2Gm(r) + 8\pi G p r^3}{r\,(r - 2Gm(r))} \tag{3.28}$$

From the energy-momentun $\nabla_a T^{ab} = 0$, the component $b = 1$ gives

$$(\rho + p)\frac{d\nu}{dr} = -\frac{dp}{dr} \tag{3.29}$$

we use this to eliminate ν from (3.28):

$$\frac{dp}{dr} = -\frac{(\rho + p)[2Gm(r) + 8\pi G p r^3]}{r\,(r - 2Gm(r))} \tag{3.30}$$

This is the equation of hydrostatic equilibrium, describing the balance between compression due to gravity and the pressure gradiant force in the opposite direction.

Chapter 4

Penrose-Carter diagrams

Let $\mathcal{M}$ denote physical space-time with metric g_{ab}. The idea is to construct another "unphysical" manifold $\tilde{\mathcal{M}}$ with boundary $\mathcal{I}$ and metric $\tilde{g}_{ab}$, such that $\mathcal{M}$ is conformal to the interior of $\tilde{\mathcal{M}}$ with

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{4.1}$$

(where Ω is the conformal factor) and so that the "infinity" of $\mathcal{M}$ is represented by the finite hypersurface $\mathcal{I}$. We realise the whole physical spacetime $\mathcal{M}$ as a subset of the unphysical spacetime $\tilde{\mathcal{M}}$.

Asymptotic properties of $\mathcal{M}$ and of fields in $\mathcal{M}$ can be investigated by studying $\mathcal{I}$, and the local behaviour of the fields at $\mathcal{I}$ provided the relavent information is conformally invariant.

We will show that the null geodesics of conformally related metrics are the same, and hence have the same causal structure.

In chapter 7 we use such diagrams when investigating Hawking radiation.

4.1 Geodesics under conformal transformations

We denote the inverse metric to $\tilde{g}_{ab}$ by $\tilde{g}^{ab}$. Obviously,

$$\tilde{g}^{ab} = \Omega^{-2} g^{ab}.$$

Let $\tilde{\nabla}_a$ denote the derivative operator associated with $\tilde{g}_{ab}$.

$$\tilde{\nabla}_a \tilde{g}_{ab} = 0 \tag{4.2}$$

implies

$$\tilde{\Gamma}^a_{bc} = \frac{1}{2}\tilde{g}^{ad}(\nabla_b \tilde{g}_{cd} + \nabla_c \tilde{g}_{bd} - \nabla_d \tilde{g}_{bc}) \tag{4.3}$$

But since $\nabla_a g_{bc} = 0$, we have

$$\nabla_b \tilde{g}_{cd} = \nabla_b(\Omega^2 g_{cd}) = 2\Omega g_{cd}\nabla_b \Omega \tag{4.4}$$

$$\begin{aligned} \tilde{\Gamma}^a_{bc} &= \Omega^{-1} g^{ad}(g_{cd}\nabla_b \Omega + g_{bd}\nabla_c \Omega - g_{bc}\nabla_d \Omega) \\ &= 2\delta^a_{(b}\nabla_{c)} \ln \Omega - g_{bc}\nabla_d \ln \Omega \end{aligned} \tag{4.5}$$

The tangent, v^a, to an affinely parameterised geodesic γ with respect to ∇_b satifies

$$v^b \nabla_b v^a = 0 \tag{4.6}$$

Hence

$$\begin{aligned} v^b \tilde{\nabla}_b v^a &= v^b \nabla_b v^a + v^b \tilde{\Gamma}^a_{bc} v^c \\ &= v^a(2v^c \nabla_c \ln \Omega) - (g_{bc} v^b v^c) g^{ad} \nabla_d \ln \Omega \end{aligned} \tag{4.7}$$

Thus, γ fails to be a geodesic with respect to $\tilde{\nabla}_b$ unless $g_{bc} v^b v^c = 0$, in which case it is the non-affinely parameterised geodesic equation - the RHS is proportional to v^a. Hence we have proved the result.

4.2 Penrose-Carter Diagram for Minkowski Spacetime.

As a first example to illustrate the idea of a Penrose diagram we consider the procedure for Minkowski spacetime.

We introduce coordinates

$$v = t + r, \tag{4.8}$$

$$w = t - r, \tag{4.9}$$

It is obvious that

$$-\infty < v < \infty, \quad -\infty < w < \infty \tag{4.10}$$
$$v \geq w, \tag{4.11}$$

the Minkowski spacetime becomes

$$ds^2 = dvdw - \frac{1}{4}(v-w)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.12}$$

We define new coordinates p and q by

$$p = \tan^{-1} v, \tag{4.13}$$
$$q = \tan^{-1} w, \tag{4.14}$$

with coordinate ranges

$$-\frac{1}{2}\pi < p < \frac{1}{2}\pi, \quad -\frac{1}{2}\pi < q < \frac{1}{2}\pi \tag{4.15}$$
$$p \geq q. \tag{4.16}$$

$$ds^2 = \frac{1}{4}\sec^2 p \ \sec^2 q[4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.17}$$

and the line element of the unphysical metric is

$$d\tilde{s}^2 = 4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.18}$$

$$\Omega = \frac{1}{4}\sec^2 p \ \sec^2 q \tag{4.19}$$

We introduce coordinates

$$t' = p + q, \tag{4.20}$$
$$r' = p - q, \tag{4.21}$$

The unphysical metric is now

$$d\tilde{s}^2 = dt'^2 - dr'^2 - \sin^2 r'(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.22}$$

subject to the coordinate range

$$-\pi < t' + r' < \pi, \tag{4.23}$$
$$-\pi < t' - r' < \pi, \tag{4.24}$$
$$r' \geq 0. \tag{4.25}$$

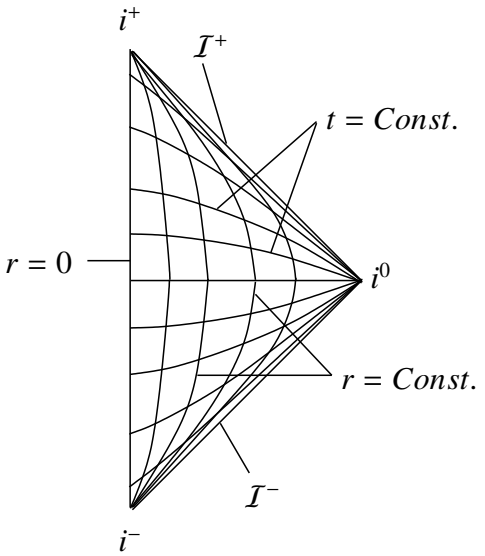

Figure 4.1: Penrose diagram of Minkowski spacetime

The whole of Minkowski spacetime has been shrunk into a finite or compact region. The process is called conformal compactification.

$\mathcal{I}^+$ is future null infinity,

$\mathcal{I}^-$ is past null infinity,

i^+ is future timelike infinity,

i^- is past timelike infinity,

i^0 is past spacelike infinity.

4.3 Maximal Extensions

4.4 The Kruskal Solution

The coordinate choice most useful for path integral investigation into black hole radiation.

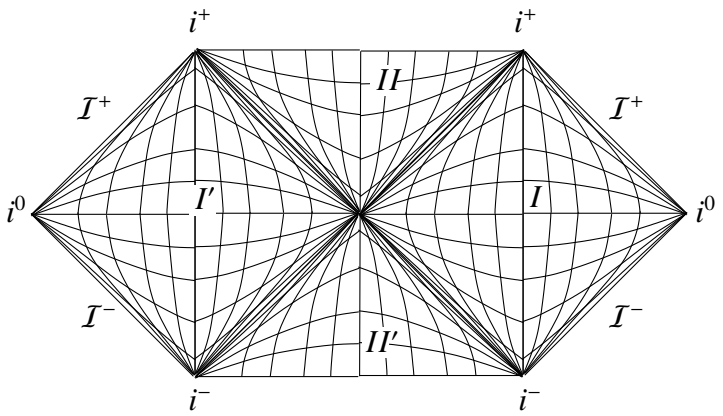

Figure 4.2: Penrose diagram of the Kruskal solution

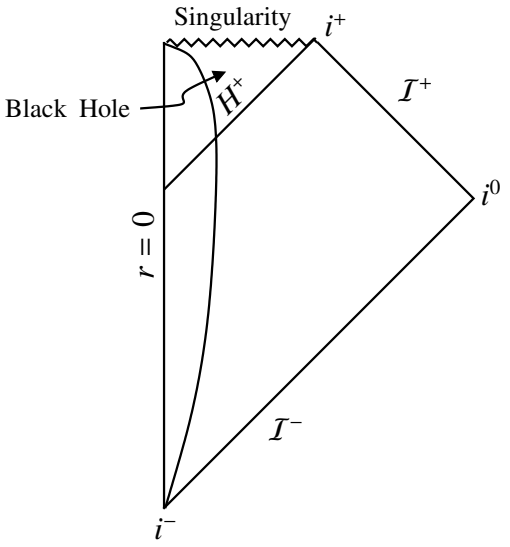

Figure 4.3: Penrose diagram of a black hole.

Chapter 5

Charged Balck Holes

We now obtain the Reissener-Nordstrom solution that describes a charged non-rotating black hole. We look for a static, asymptotically flat, shrerically symmetric solution of the Einstein-Maxwell field equations. The Einstein-Maxwell equations are

$$G_{ab} = 8\pi T_{ab} \tag{5.1}$$

where T_{ab} is the energy-momentum tensor of electromagnetism

$$T_{ab} = \frac{1}{4\pi}(-g^{cd}F_{ac}F_{bd} + \frac{1}{4}g_{ab}F_{cd}F^{cd}) \tag{5.2}$$

and where F_{ab} is the field strength tensor. Note that T_{ab} has zero trace,

$$T = g^{ab}T_{ab} = \frac{1}{4\pi}(-g^{ab}g^{cd}F_{ac}F_{bd} + \frac{1}{4}g^{ab}g_{ab}F_{cd}F^{cd}) = 0. \tag{5.3}$$

This implies the vanishing of the Ricci scalar as the trace of the Einstein tensor must vanish

$$R - \frac{1}{2}4R = T^{a}{}_{a} = 0 \quad \Rightarrow R = 0$$

We can then instead use

$$R_{ab} = 8\pi T_{ab} \tag{5.4}$$

In source-free regions the tensor F_{ab} must satisfy Maxwell's equations

$$\nabla_b F^{ab} = 0, \tag{5.5}$$
$$\partial_{[a} F_{bc]} = 0. \tag{5.6}$$

We can assume there are coordinates (t, r, θ, ϕ) so thay the metric reduces to the form

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.7}$$

If we impose the condition that the solution is static, then this requires ν and λ are functions of r only,

$$\nu = \nu(r), \qquad \lambda = \lambda(r). \tag{5.8}$$

We also assume the solution to be asymptotically flat.

We read off from (5.7) that

$$g_{ab} = \begin{pmatrix} e^{\nu} & 0 & 0 & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \tag{5.9}$$

and so

$$g^{ab} = \begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2\theta \end{pmatrix} \tag{5.10}$$

and

$$\sqrt{|g|} = \sqrt{-\det(g_{ab})} = e^{\frac{1}{2}(\nu+\lambda)} r^2 \sin\theta. \tag{5.11}$$

The non-zero components of the Ricci tensor R_{ab} are then

$$
\begin{aligned}
R_{00} &= e^{\lambda-\nu}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{1}{r}\nu'\right)\\
R_{11} &= -\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}-\frac{1}{r}\lambda'\right)\\
R_{22} &= e^{-\lambda}\left(\frac{r}{2}(\lambda'-\nu')-1\right)+1\\
R_{33} &= R_{22}\sin^2\theta
\end{aligned}
\tag{5.12}
$$

which follow from (1.31)-(1.35) and the assumption that $\nu = \nu(r)$ and $\lambda = \lambda(r)$.

5.0.1 Solving the Maxwell equations and the electric charge

In spherical polar coordinates the Maxwell tensor takes the form

$$
F_{ab} = E(r)\begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.13}
$$

To confirm this ansatz we will take the definition of the Maxwell tensor in Minkoski coordinates (t, x, y, z):

$$
F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z\\ E_x & 0 & B_z & -B_y\\ E_y & -B_z & 0 & B_x\\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{5.14}
$$

and find its components in spherical polar coordinates (t, r, θ, ϕ)

$$
\begin{aligned}
t &= t\\
x &= r\sin\theta\cos\phi\\
y &= r\sin\theta\sin\phi\\
z &= r\cos\theta
\end{aligned}
\tag{5.15}
$$

via the tansformation formula

$$
F'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}F_{\sigma\rho}, \tag{5.16}
$$

where we easily find that,

$$\frac{\partial x}{\partial x'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{5.17}$$

while inserting the assumption that the elecric field is radial:

$$\begin{aligned} E_x &= E(r)\sin\theta\cos\phi \\ E_y &= E(r)\sin\theta\sin\phi \\ E_z &= E(r)\cos\theta. \end{aligned} \tag{5.18}$$

We obtain from (5.16)

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ 0 & -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \\ &\begin{pmatrix} 0 & -E(r)\sin\theta\cos\phi & -E(r)\sin\theta\sin\phi & -E(r)\cos\theta \\ E(r)\sin\theta\cos\phi & 0 & 0 & 0 \\ E(r)\sin\theta\sin\phi & 0 & 0 & 0 \\ E(r)\cos\theta & 0 & 0 & 0 \end{pmatrix} \\ &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix} \\ = &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ 0 & -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \\ &\begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r)\sin\theta\cos\phi & 0 & 0 & 0 \\ E(r)\sin\theta\sin\phi & 0 & 0 & 0 \\ E(r)\cos\theta & 0 & 0 & 0 \end{pmatrix} \\ = &\begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{5.19}$$

confirming the ansatz.

Raising the indices using the inverse metric, we obtain

$$\begin{aligned} F^{ab} &= E(r)\begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\qquad \begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \\ &= E(r)\begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} 0 & e^{-\lambda} & 0 & 0 \\ e^{-\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= e^{-\nu-\lambda}E(r)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{5.20}$$

We verify that (5.6) is automatically satisfied using (5.13). We can use $F_{ab} = -F_{ba}$ to write

$$\partial_{[a}F_{bc]} = \frac{2}{3!}(\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca}) \tag{5.21}$$

We need only check for

$$\begin{aligned} &a = 0, b = 1, c = 2 \\ &a = 0, b = 1, c = 3 \\ &a = 0, b = 2, c = 3 \\ &a = 1, b = 2, c = 3. \end{aligned} \tag{5.22}$$

First

$$\begin{aligned} \partial_{[0}F_{12]} &= \frac{2}{3!}(\partial_0 F_{12} + \partial_2 F_{01} + \partial_1 F_{20}) \\ &= 0 \end{aligned} \tag{5.23}$$

then

$$\begin{aligned} \partial_{[0}F_{13]} &= \frac{2}{3!}(\partial_0 F_{13} + \partial_3 F_{01} + \partial_1 F_{30}) \\ &= 0 \end{aligned} \tag{5.24}$$

then

$$\begin{aligned} \partial_{[0}F_{23]} &= \frac{2}{3!}(\partial_0 F_{23} + \partial_3 F_{02} + \partial_2 F_{30}) \\ &= 0 \end{aligned} \tag{5.25}$$

and lastly

$$\begin{aligned} \partial_{[0}F_{23]} &= \frac{2}{3!}(\partial_0 F_{23} + \partial_3 F_{02} + \partial_2 F_{30}) \\ &= 0. \end{aligned} \tag{5.26}$$

To write out (5.5) we will use the following expression that holds for any antisymmetric rank two tensor $X^{ab} = -X^{ba}$

$$\nabla_a X^{ab} = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}X^{ab}). \tag{5.27}$$

To prove this result we need

$$\Gamma^a_{ab} = \frac{1}{\sqrt{|g|}}\partial_b \sqrt{|g|}. \tag{5.28}$$

Observe for a rank two tensor X^{ab}

$$\begin{aligned} \nabla_a X^{ab} &= \partial_a X^{ab} + \Gamma^a_{ac}X^{cb} + \Gamma^b_{ac}X^{ac} \\ &= \partial_a X^{ab} + \frac{1}{\sqrt{|g|}}(\partial_c \sqrt{|g|})X^{cb} + \Gamma^b_{ac}X^{ac} \\ &= \frac{1}{\sqrt{|g|}}\partial_c(\sqrt{|g|}X^{cb}) + \Gamma^b_{ac}X^{ac} \end{aligned} \tag{5.29}$$

Since we require the tensor X^{ab} to be antisymmetric the last term vanishes and we have established the result.

We now apply it to the tensor F^{ab}:

$$\begin{aligned} \nabla_a F^{ab} &= \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}F^{ab}) \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_a(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{ab}) \end{aligned} \tag{5.30}$$

From the t component of Maxwell's equation (5.5) and using (5.20) we must have

$$\begin{aligned} 0 &= \nabla_a F^{a0} \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_a(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{a0}) \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2}\partial_1(e^{\frac{1}{2}(\nu+\mu)}r^2F^{10}) \\ &= -\frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2}\partial_1(e^{-\frac{1}{2}(\nu+\mu)}r^2E). \end{aligned} \tag{5.31}$$

Integrating this gives

$$E = e^{\frac{1}{2}(\nu+\lambda)}\epsilon/r^2 \tag{5.32}$$

where ϵ is the constant of integration. Assuming the solution is asymptotically flat requires

$$\nu, \lambda \to 0 \quad \text{as} \quad r \to \infty \tag{5.33}$$

and so asymptotically

$$E \sim \epsilon/r^2$$

We therefore interpret ϵ as the charge of the blackhole.

We confirm the other Maxwell's eqations. From the r component

$$\begin{aligned}\nabla_a F^{a1} &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_a(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{a1}) \\ &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_0(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{01}) \\ &= 0 \end{aligned} \tag{5.34}$$

as there is no time-dependence. The other two are zero by spherical symmetry assumption. In particular as $F^{a2} = 0$

$$\begin{aligned}\nabla_a F^{a2} &= \frac{e^{-\frac{1}{2}(\nu+\mu)}}{r^2\sin\theta}\partial_a(e^{\frac{1}{2}(\nu+\mu)}r^2\sin\theta F^{a2}) \\ &= 0. \end{aligned} \tag{5.35}$$

Similarly for $\nabla_a F^{a3} = 0$.

5.0.2 Calculation of the energy-momentum tensor

We employ the ansatz (5.13) together with (5.9) and (5.10) to compute the Maxwell energy momentum tensor:

$$T_{ab} = \frac{1}{4\pi}(-g^{cd}F_{ac}F_{bd} + \frac{1}{4}g_{ab}F_{cd}F^{cd}). \tag{5.36}$$

First we compute

$$\begin{aligned}F_{cd}F^{cd} &= g^{ce}g^{df}F_{cd}F_{ef} \\ &= g^{0e}g^{1f}F_{01}F_{ef} + g^{1e}g^{0f}F_{10}F_{ef} \\ &= 2g^{0e}g^{1f}F_{01}F_{ef} \\ &= 2g^{00}g^{11}F_{01}F_{01} \\ &= -2e^{-\nu-\lambda}E^2. \end{aligned} \tag{5.37}$$

We find the componets of T_{ab}: First T_{00}

$$
\begin{aligned}
T_{00} &= \frac{1}{4\pi}(-g^{cd}F_{0c}F_{0d} + \frac{1}{4}g_{00}F_{cd}F^{cd}) \\
&= \frac{1}{4\pi}(-g^{11}F_{01}F_{01} + \frac{1}{4}g_{00}F_{cd}F^{cd}) \\
&= \frac{1}{4\pi}(e^{-\lambda}E^2 - \frac{1}{2}e^{\nu}e^{-\nu-\lambda}E^2) \\
&= \frac{1}{8\pi}e^{-\lambda}E^2 \qquad (5.38)
\end{aligned}
$$

Next T_{11}

$$
\begin{aligned}
T_{11} &= \frac{1}{4\pi}(-g^{cd}F_{1c}F_{1d} + \frac{1}{4}g_{11}F_{cd}F^{cd}) \\
&= \frac{1}{4\pi}(-g^{00}F_{10}F_{10} + \frac{1}{4}g_{11}F_{cd}F^{cd}) \\
&= -\frac{1}{4\pi}(e^{-\nu}E^2 - \frac{1}{2}e^{\lambda}e^{-\nu-\lambda}E^2) \\
&= -\frac{1}{8\pi}e^{-\nu}E^2 \qquad (5.39)
\end{aligned}
$$

Then T_{22}

$$
\begin{aligned}
T_{22} &= \frac{1}{4\pi}(-g^{cd}F_{2c}F_{2d} + \frac{1}{4}g_{22}F_{cd}F^{cd}) \\
&= \frac{1}{16\pi}g_{22}F_{cd}F^{cd} \\
&= \frac{1}{8\pi}r^2e^{-\nu-\lambda}E^2. \qquad (5.40)
\end{aligned}
$$

It is easy to see that $T_{33} = T_{22}\sin^2\theta$. We compute T_{01}:

$$
\begin{aligned}
T_{01} &= \frac{1}{4\pi}(-g^{cd}F_{0c}F_{1d} + \frac{1}{4}g_{01}F_{cd}F^{cd}) \\
&= -\frac{1}{4\pi}g^{10}F_{01}F_{10} \\
&= 0. \qquad (5.41)
\end{aligned}
$$

In acordance with (5.12), (note in the derivation of the Schwarzschild solution that $R_{01} = 8\pi T_{01} = 0$ implied $\lambda(t, r) = \lambda(r)$ by (1.33)).

The other components of T_{ab} are easily seen to be zero, which are in accordance with (5.12).

5.0.3 Solving the field equations

The equations $R_{00} = 8\pi T_{00}$ and $R_{11} = 8\pi T_{11}$ can be combined

$$e^{\lambda-\nu}R_{00} + R_{11} = 8\pi[e^{\lambda-\nu}T_{00} + T_{11}] \tag{5.42}$$

By (5.12), (5.38) and (5.39), from the 00 and 11 equations we get

$$\lambda' + \nu' = 0 \tag{5.43}$$

which by (5.33) means that

$$\lambda = -\nu. \tag{5.44}$$

The 22 equation

$$R_{22} = 8\pi T_{22} = r^2 e^{-\nu-\lambda}E^2$$

becomes on using (5.12), (5.32) and (5.44)

$$e^{\nu}(-r\nu' - 1) + 1 = r^2E^2 = \epsilon^2/r^2 \tag{5.45}$$

or

$$(re^{\nu})' = 1 - \epsilon^2/r^2 \tag{5.46}$$

integrating

$$e^{\nu} = 1 + Const./r + \epsilon^2/r^2 \tag{5.47}$$

(note that because $R_{33} = R_{22}\sin^2\theta$ and $T_{33} = T_{22}\sin^2\theta$ the field equation $R_{33} = 8\pi T_{33}$ gives the same equation as $R_{22} = 8\pi T_{22}$). We obtain the line element

$$ds^2 = \left(1 + \frac{Const.}{r} + \frac{\epsilon^2}{r^2}\right)dt^2 - \left(1 + \frac{Const.}{r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.48}$$

When $\epsilon = 0$, this reduces to the Schwarzschild line element and so

$$Const. = -2MG/c^2.$$

We obtain the Reissner-Nordstrom solution

$$ds^2 = \left(1 - \frac{2MG}{c^2 r} + \frac{\epsilon^2}{r^2}\right) dt^2 - \left(1 - \frac{2MG}{c^2 r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.49}$$

5.1 Event Horizons

Recall that at the event horizon was the point after which the coefficients of dr and dt change signs.

The horizon function $H(r)$ is given by

$$H(r) = \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) \tag{5.50}$$

is quadratic with two distinct roots. Write

$$r^2 H(r) = r^2 - 2mr + \epsilon^2 = (r - m)^2 + \epsilon^2 - M^2. \tag{5.51}$$

Notice that we must have $\epsilon^2 \leq m^2$ otherwise the roots would be complex.

The two roots are given by

$$\begin{aligned} r_+ &= m + (m^2 - \epsilon^2)^{1/2} \\ r_- &= m - (m^2 - \epsilon^2)^{1/2} \end{aligned} \tag{5.52}$$

These two roots correspond to two different event horizons, one at r_+ and the other at r_-.

The line element has three regular regions

$$\begin{aligned} &I. && r_+ < r < \infty, \\ &II. && r_- < r < r_+, \\ &III. && 0 < r < r_-. \end{aligned} \tag{5.53}$$

The outer horizon at r_+ is much like the event horizon of $2m$ for a Schwarzschild black hole.

Space and time change roles.

At the inner horizon, the Cauchy horizon, space and time change roles again.

Inside the Cauchy horizon the sigularity is space-like.

As the more charge is entered into the black hole, the inner event horizon gets larger, while the outer event horizon starts to shrink. When we reach the maximum possible charge, i.e. $\epsilon^2 = m^2$, the two horizons merge and only the regions *I* and *III* exist.

5.2 Analogue of Eddington-Finkelstein Coordinates

$\epsilon^2 < m^2$

$$t = \bar{t} - \frac{r_+^2}{r_+ - r_-}\ln(r - r_+) + \frac{r_-^2}{r_+ - r_-}\ln(r - r_-) \tag{5.54}$$

$$r_+ - r_- = 2(m^2 - \epsilon^2)^{1/2}$$

$$\begin{aligned} r_+ - r_- &= 2(m^2 - \epsilon^2)^{1/2} \\ r_+ + r_- &= 2m \qquad (5.55) \\ r_+ r_- &= \epsilon^2 \qquad (5.56) \end{aligned}$$

$$\begin{aligned} dt &= d\bar{t} - \frac{r_+^2}{r_+ - r_-}\frac{dr}{r - r_+} + \frac{r_-^2}{r_+ - r_-}\frac{dr}{r - r_-} \\ &= d\bar{t} - \frac{dr}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} + \frac{r_-^2}{r - r_-}\right) \end{aligned} \tag{5.57}$$

$$dt^2 = d\bar{t}^2 - 2\frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)d\bar{t}dr + \frac{dr^2}{(r_+ - r_-)^2}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)^2 \tag{5.58}$$

Notice that the horizon function is

$$r^2 H(r) = (r - r_+)(r - r_-)$$

$$
\begin{aligned}
ds^2 &= \frac{(r-r_+)(r-r_-)}{r^2}dt^2 - \frac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2) \\
&= \frac{(r-r_+)(r-r_-)}{r^2}d\bar{t}^2 - 2\frac{(r-r_+)(r-r_-)}{r^2}\frac{1}{r_+-r_-}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right)d\bar{t}dr \\
&\quad +\frac{(r-r_+)(r-r_-)}{r^2}\frac{dr^2}{(r_+-r_-)^2}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right)^2 \\
&\quad -\frac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2)
\end{aligned} \tag{5.59}
$$

Consider the coefficient of $-2d\bar{t}dr$:

$$
\begin{aligned}
&\frac{(r-r_+)(r-r_-)}{r^2}\frac{1}{r_+-r_-}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right) \\
&= \frac{1}{r^2(r_+-r_-)}\left(r_+^2(r-r_-)-r_-^2(r-r_+)\right) \\
&= \frac{1}{r^2(r_+-r_-)}\left(r(r_+^2-r_-^2)-(r_+^2r_- - r_-^2r_+)\right) \\
&= \frac{r_++r_-}{r}-\frac{r_+r_-}{r^2} \\
&= \frac{2m}{r}-\frac{\epsilon^2}{r^2}
\end{aligned} \tag{5.60}
$$

Consider the coefficient of $-dr^2$:

$$
\begin{aligned}
&-\frac{(r-r_+)(r-r_-)}{r^2}\frac{1}{(r_+-r_-)^2}\left(\frac{r_+^2}{r-r_+}-\frac{r_-^2}{r-r_-}\right)^2+\frac{r^2}{(r-r_+)(r-r_-)} \\
&= -\frac{r^2}{(r-r_+)(r-r_-)}\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2+\frac{r^2}{(r-r_+)(r-r_-)} \\
&= \frac{r^2}{(r-r_+)(r-r_-)}\left[1-\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2\right] \\
&= \frac{1}{1-2m/r-\epsilon^2/r^2}\left[1-\left(\frac{2m}{r}-\frac{\epsilon^2}{r^2}\right)^2\right] \\
&= 1+\frac{2m}{r}-\frac{\epsilon^2}{r^2}
\end{aligned} \tag{5.61}
$$

Introduce the function f by

$$f = 1 - g_{00} = \frac{2m}{r} - \frac{\epsilon^2}{r^2} \tag{5.62}$$

Then the line element can becomes

$$ds^2 = (1-f)d\bar{t}^2 - 2fd\bar{t}dr - (1+f)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.63}$$

Notice we no longer have any coordinate singularities.

5.3 Penrose Diagram

5.4 Double null coordinates

Introduce double null coordinates

$$v = \bar{t} + r, \qquad w = 2t - v, \tag{5.64}$$

this implies

$$v = t + r^*, \qquad w = t - r^*, \tag{5.65}$$

where

$$r^* = r + \frac{r_+^2}{r_+ - r_-}\ln(r - r_+) - \frac{r_-^2}{r_+ - r_-}\ln(r - r_-) \tag{5.66}$$

Obviously

$$\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)dvdw = \frac{(r-r_+)(r-r_-)}{r^2}(dt^2 - dr^{*2}) \tag{5.67}$$

The coefficient of dt^2 can seen immediately to be correct. Now

$$dr^* = \left[1 + \frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)\right]dr \tag{5.68}$$

Squaring

$$dr^{*2} = \left[1 + 2\frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right) + \frac{1}{(r_+ - r_-)^2}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)^2\right] dr^2 \tag{5.69}$$

$$\begin{aligned}
& \frac{(r - r_+)(r - r_-)}{r^2} dr^{*2} \\
= & \Big[\frac{(r - r_+)(r - r_-)}{r^2} dr^{*2} \\
& + 2\frac{(r - r_+)(r - r_-)}{r^2}\frac{1}{r_+ - r_-}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right) \\
& + \frac{(r - r_+)(r - r_-)}{r^2}\frac{1}{(r_+ - r_-)^2}\left(\frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right)^2\Big] dr^2 \\
= & \frac{r^2}{(r - r_+)(r - r_-)}\Big[\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)^2 + \\
& + 2\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)\left(\frac{2m}{r} - \frac{\epsilon^2}{r^2}\right) + \left(\frac{2m}{r} - \frac{\epsilon^2}{r^2}\right)^2\Big] dr^2 \\
= & \frac{r^2}{(r - r_+)(r - r_-)}\Big[1 - 2\left(\frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) + \left(\frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)^2 \\
& + 2\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)\left(\frac{2m}{r} - \frac{\epsilon^2}{r^2}\right) + \left(\frac{2m}{r} - \frac{\epsilon^2}{r^2}\right)^2\Big] dr^2 \\
= & \frac{r^2}{(r - r_+)(r - r_-)} dr^2.
\end{aligned} \tag{5.70}$$

Therefore

$$\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) dvdw = \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2$$

Thus the line element can be written in double null coordinates as

$$\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) dvdw - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.71}$$

5.5 Maximal extension

$\epsilon^2 < m^2$, we define new coordinates

$$v'' = \tan^{-1}\left(\exp\frac{r_+ - r_-}{4r_+^2}v\right), \qquad w'' = \tan^{-1}\left(-\exp\frac{r_+ - r_-}{4r_+^2}w\right) \tag{5.72}$$

Inverting we see

$$v = \frac{4r_+^2}{r_+ - r_-}\ln\ \tan v'', \qquad w = -\frac{4r_+^2}{r_+ - r_-}\ln\ \tan w'' \tag{5.73}$$

As

$$d\ln\tan x = \frac{d\tan x}{\tan x} = \frac{1}{\sin x\cos x}dx = \frac{2}{\sin 2x}dx = 2\mathrm{cosec}2x\ dx$$

we get

$$dvdw = -4^2 \times 2^2\left(\frac{4r_+^2}{r_+ - r_-}\right)^2 \mathrm{cosec}2v''\mathrm{cosec}2w''\ dv''dw''$$

Consider the product $\tan v''\tan w''$

$$\begin{aligned}
& \tan v''\tan w'' \\
= & -\exp\left(\frac{r_+ - r_-}{4r_+^2}v\right)\exp\left(\frac{r_- - r_+}{4r_+^2}w\right) \\
= & -\exp\left(\frac{r_+ - r_-}{4r_+^2}(v-w)\right) \\
= & -\exp\left(\frac{r_+ - r_-}{2r_+^2}r^*\right) \\
= & -\exp\left(\frac{r_+ - r_-}{2r_+^2}r\right) \\
\times & \exp\left(\frac{r_+ - r_-}{2r_+^2}\left[\frac{r_+^2}{r_+ - r_-}\ln(r - r_+)\right]\right)\exp\left(\frac{r_+ - r_-}{2r_+^2}\left[\frac{r_-^2}{r_+ - r_-}\ln(r - r_-)\right]\right) \\
= & -\exp\left(\frac{r_+ - r_-}{2r_+^2}r\right)(r - r_+)^{1/2}(r - r_-)^{r_-^2 1/2r_+^2}.
\end{aligned} \tag{5.74}$$

Therefore the line element has the form

$$\begin{aligned}
ds^2 &= -64\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)\frac{4r_+^4}{(r_+ - r_-)^2}\mathrm{cosec}2v''\mathrm{cosec}2w''\ dv''dw'' \\
&= -r^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned} \tag{5.75}$$

where r is defined implicitly by

$$\tan v'' \tan w'' = -\exp\left(\frac{r_+ - r_-}{2r_+^2} r\right)(r - r_+)^{1/2}(r - r_-)^{r_-^2/2r_+^2}. \tag{5.76}$$

This line element is the analogue of the Kruskel solution and represents the maximal analytic extension of the Reissner-Nordstrm solution for $\epsilon^2 < m^2$.

The Penrose diagram for this maximal extension is shown in fig (5.5)

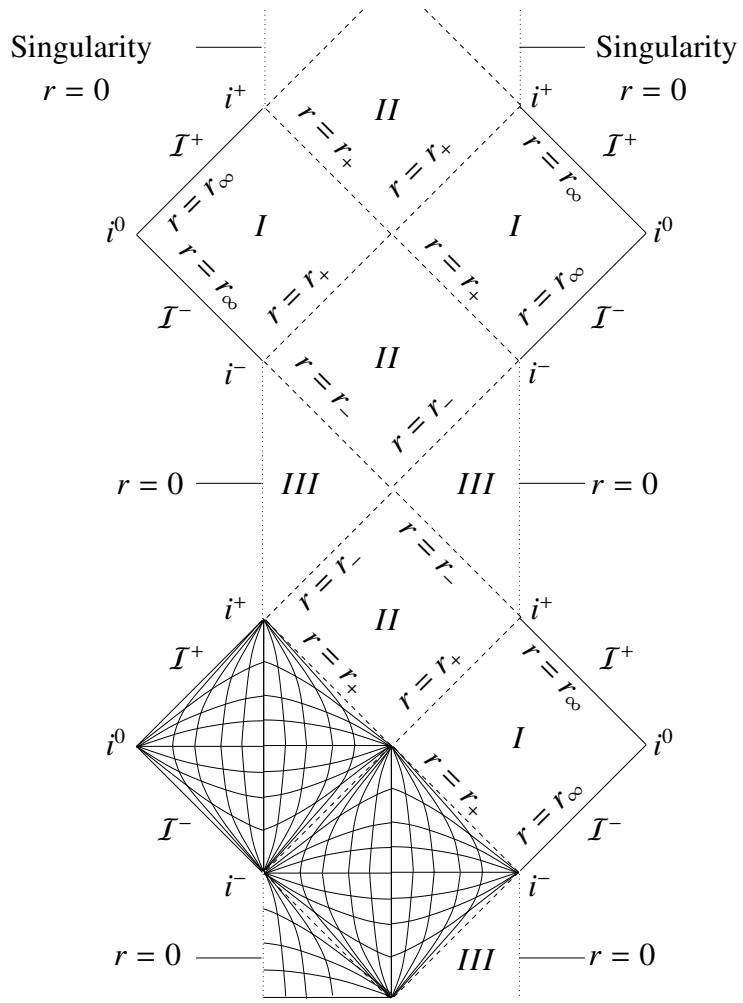

Figure 5.1: Penrose diagram of a black hole.

Chapter 6

Rotating Black Holes

6.1 Field Equations

6.1.1 Degenerate metric

A metric of the form

$$g_{ab} = \eta_{ab} - 2ml_a l_b, \quad l_a l_b \eta^{ab} = 0, \quad m \text{ is arbitrary.} \tag{6.1}$$

is called a degenerate metric. The Schwarzschild metric is an example as can be seen from the Eddington-Finkelstein form (2.5)

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) d\bar{t}^2 - \frac{4m}{r} d\bar{t} dr - \left(1 - \frac{2M}{r}\right) dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= d\bar{t}^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{2m}{r}(d\bar{t} + dr)^2 \\ &= d\bar{t}^2 - (dx^2 + dy^2 + dz^2) - \frac{2m}{r}\left(d\bar{t} + \frac{xdx + ydy + zdz}{r}\right)^2 \end{aligned} \tag{6.2}$$

Therefore the Schwarzschild metric can be written in degenerate form with l_a given by

$$l_a = \frac{1}{\sqrt{r}}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{6.3}$$

It turns out that the solution for a rotating black-hole is also of the form of a degenerate metric. Let us proceed. Define

$$l^a := \eta^{ab} l_b. \tag{6.4}$$

The inverse matrix of g_{ab} is

$$g^{ab} = \eta^{ab} + 2ml^a l^b \tag{6.5}$$

as is easily seen:

$$\begin{aligned} g_{ac}g^{cb} &= (\eta_{ac} - 2ml_a l_c)(\eta^{cb} + 2ml^c l^b) \\ &= \delta_a{}^b + 2ml_a l^b - 2ml^b l_a + 4m^2 l_a l^b l_c l^c \\ &= \delta_a{}^b \end{aligned}$$

It follows from (6.5) that the contravariant four vector corresponding to l_a is the same as $\eta^{ab} l_b$:

$$l^a := g^{ac} l_c = \eta^{ac} l_c. \tag{6.6}$$

Therefore its indices han be raised and lowered by either the true metric or the Lorentz metric. Since l_a is null it has the property

$$0 = \frac{1}{2}\partial_d(\eta^{ab} l_a l_b) = l^c \partial_d l_c = l_c \partial_d l^c. \tag{6.7}$$

We can consider, g, the determinat of the metric. At any point l_a is a flat-space null vector: $l_a l_b \eta^{ab} = 0$. We can perform a proper rotation of coordinates in three-space that leaves η^{ab} invariant and brings l_a into the form

$$\begin{pmatrix} a \\ a \\ 0 \\ 0 \end{pmatrix}, \tag{6.8}$$

see

$$(a, a, 0, 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ a \\ 0 \\ 0 \end{pmatrix} = 0. \tag{6.9}$$

In this system we have

$$
\begin{aligned}
g &= \begin{vmatrix} 1-2ma^2 & -2ma^2 & 0 & 0 \\ -2ma^2 & -1-2ma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 1-2ma^2 & -2ma^2 \\ -2ma^2 & -1-2ma^2 \end{vmatrix} \\
&= -(1-2ma^2)(1+2ma^2) - 4m^2a^4 \\
&= -1
\end{aligned} \tag{6.10}
$$

Since a three-dimensional rotation has a unit Jacobian, the metric transforms as a scalar under this transformation. Thus, $g = -1$ for any degenerate metric. It follows that

$$
\Gamma^a_{ca} = \frac{\partial}{\partial x^c} \ln \sqrt{-g} = 0 \tag{6.11}
$$

This simplifies the field equations $R_{ab} = 0$,

$$
R_{ab} = -\partial_c \Gamma^c_{ab} + \Gamma^d_{ac}\Gamma^c_{db} = 0 \tag{6.12}
$$

Now R_{ab} involves different powers of m and since m is arbitrary each order must vanish separately. If we note that the Christoffel symbol of the first kind, $\Gamma_{ab,d}$, is linear in m, power counting is easy and we isolate terms corresponding to different powers of m,

$$
\begin{aligned}
R_{ab} &= -\partial_c \Gamma^c_{ab} + \Gamma^d_{ac}\Gamma^c_{db} \\
&= -\partial_c (g^{cd}\Gamma_{ab,d}) + g^{cd}\Gamma_{ea,d}\, g^{ef}\Gamma_{cb,f} \\
&= -\partial_c ((\eta^{cd} + 2ml^c l^d)\Gamma_{ab,d}) + (\eta^{cd} + 2ml^c l^d)(\eta^{ef} + 2ml^e l^f)\, \Gamma_{ea,d}\Gamma_{cb,f} \\
&= -\eta^{cd}\partial_c \Gamma_{ab,d} \\
&\quad -2m\partial_c (l^c l^d \Gamma_{ab,d}) + \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} \\
&\quad +2m(l^c l^d \eta^{ef} + l^e l^f \eta^{cd})\Gamma_{ea,d}\Gamma_{cb,f} \\
&\quad +4m^2 l^c l^d l^e l^f \Gamma_{ea,d}\Gamma_{cb,f}
\end{aligned} \tag{6.13}
$$

We just rewrite one term by swapping the dummy variables d and f:

$$
l^c l^d \eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} = l^c l^f \eta^{ed}\Gamma_{ea,f}\Gamma_{cb,d}
$$

This gives four sets of ten equations:

$$
\begin{aligned}
&-\eta^{cd}\partial_c\Gamma_{ab,d} = 0 && \mathcal{O}(m)\\
&2m\partial_c(l^c l^d\Gamma_{ab,d}) - \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} = 0 && \mathcal{O}(m^2)\\
&l^e l^f\eta^{cd}\Gamma_{ea,d}\Gamma_{cb,f} + l^c l^f\eta^{ed}\Gamma_{ea,f}\Gamma_{cb,d} = 0 && \mathcal{O}(m^3)\\
&4m^2 l^c l^d l^e l^f\Gamma_{ea,d}\Gamma_{cb,f} = 0 && \mathcal{O}(m^4)
\end{aligned}
\tag{6.14}
$$

6.1.2 Order m^4 equations:

We need to satisfy

$$l^c l^d l^e l^f\Gamma_{ea,d}\Gamma_{cb,f} = 0$$

From the form of the degenerate metric (6.1) we have

$$\partial_a g_{bc} = -2m\partial_a(l_b l_c). \tag{6.15}$$

We will use this a number of times in what follows. Consider the part $l^d l^e\Gamma_{ea,d}$

$$
\begin{aligned}
l^d l^e\Gamma_{ea,d} &= l^d l^e\frac{1}{2}l^e(\partial_e g_{ad} + \partial_a g_{ed} - \partial_d g_{ea})\\
&= -ml^d l^e(\partial_e(l_d l_a) + \partial_a(l_d l_e) - \partial_d(l_e l_a))\\
&= -m[l^e l_a(l^d\partial_e l_d) + (l^d l_d)l^e\partial_e l_a +\\
&\quad + (l^e l_e)l^d\partial_a l_d + (l^d l_d)l^e\partial_a l_e\\
&\quad -l^d l_a(l^e\partial_d l_e) - (l^e l_e)l^d\partial_d l_a\ ]\\
&= 0
\end{aligned}
\tag{6.16}
$$

Therefore the m^4 equations are automatically stisfied by degenerate metrics.

6.1.3 Order m^3 equations:

$$l^e l^f\eta^{cd}\Gamma_{ea,d}\Gamma_{cb,f} + l^c l^f\eta^{ed}\Gamma_{ea,f}\Gamma_{cb,d} = 0 \tag{6.17}$$

First note that

$$l^c l^f\eta^{ed}\Gamma_{ea,f}\Gamma_{cb,d} = l^e l^f\eta^{cd}\Gamma_{eb,d}\Gamma_{ca,f} \tag{6.18}$$

shows the second term in on the LHS (6.17) is the same as the first term with a and b exchanged. Thus the field equations are

$$l^e l^f \eta^{cd} \Gamma_{ea,d} \Gamma_{cb,f} + a \leftrightarrow b = 0 \tag{6.19}$$

Consider

$$\begin{aligned} l^e \Gamma_{ea,d} &= \frac{1}{2} l^e (\partial_e g_{ad} + \partial_a g_{ed} - \partial_d g_{ea}) \\ &= -m l^e [\partial_e (l_a l_d) + \partial_a (l_e l_d) - \partial_d (l_e l_a)] \\ &= -m l^e \partial_e (l_a l_d), \end{aligned} \tag{6.20}$$

where again we have used (6.15), and also consider

$$\begin{aligned} l^f \Gamma_{cb,f} &= \frac{1}{2} l^f (\partial_c g_{bf} + \partial_b g_{cf} - \partial_f g_{cb}) \\ &= m l^f \partial_f (l_c l_b). \end{aligned} \tag{6.21}$$

Combining them we have

$$\begin{aligned} l^e l^f \eta^{cd} \Gamma_{ea,d} \Gamma_{cb,f} &= -m^2 l^e l^f \eta^{cd} \partial_e (l_a l_d) \partial_f (l_c l_b) \\ &= -m^2 l^e l^f \eta^{cd} [l_d \partial_e l_a + l_a \partial_e l_d][l_b \partial_f l_c + l_c \partial_f l_b] \\ &= -m^2 l^e l^f \eta^{cd} [l_d l_b (\partial_e l_a)(\partial_f l_c) + l_a l_b (\partial_e l_d)(\partial_f l_c) + l_a l_d \partial_e l_c \partial_f l_b]. \end{aligned} \tag{6.22}$$

Let us consider the terms separately, first,

$$\begin{aligned} -m^2 l^e l^f \eta^{cd} l_d l_b (\partial_e l_a)(\partial_f l_c) &= ... \eta^{cd} l_d \partial_f l_c \\ &= ... l^c \partial_f l_c \\ &= 0. \end{aligned} \tag{6.23}$$

where we have used (6.7). Next

$$-m^2 l^e l^f \eta^{cd} l_a l_b (\partial_e l_d)(\partial_f l_c) = -m^2 l_a l_b (l^e \partial_e l_d)(l^f \partial_f l^d) \tag{6.24}$$

by a simple rearangement of terms. Next

$$\begin{aligned} -m^2 l^e l^f \eta^{cd} l_a l_d \partial_e l_c \partial_f l_b &= ... \eta^{cd} l_d \partial_e l_c \\ &= ... l^c \partial_e l_c \\ &= 0. \end{aligned} \tag{6.25}$$

where we have used (6.7) again.

Therefore

$$l^e l^f \eta^{cd} \Gamma_{ea,d} \Gamma_{cb,f} = -m^2 l_a l_b (l^e \partial_e l_d)(l^f \partial_f l^d) \tag{6.26}$$

Using this in (6.19), the m^3 equations lead to

$$-m^2 l_a l_b (v^c v_c) = 0 \tag{6.27}$$

where

$$v^c := l^d \partial_d l^c. \tag{6.28}$$

Therefore v^c is a null vector. It is also orthogonal to the null vector l^c, as easily seen

$$v^b l_b = (l^c \partial_c l^b) l_b = l^c (l_b \partial_c l^b) = 0 \tag{6.29}$$

where we have used (6.7). Notice that

$$\begin{aligned} v_a &= g_{ab} v^b \\ &= (\eta_{ab} - 2m l_a l_b) v^b \\ &= \eta_{ab} v^b \end{aligned} \tag{6.30}$$

so that the indices of v^c can be lowered and raised with the Lorentz metric, as with l^c. At any chosen point l^a and v^a may be written

$$l^b = (|\vec{l}|, \vec{l}), \qquad v^b = (|\vec{v}|, \vec{v}) \tag{6.31}$$

where $\vec{l}$ an $\vec{v}$ are ordinary three-vectors in Eucldean space. If θ is the angle between $\vec{l}$ an $\vec{v}$ then

$$\cos\theta = \frac{\vec{l}\cdot\vec{v}}{|\vec{l}|\,|\vec{v}|} \tag{6.32}$$

Now because l^b and v^b are orthogonal (from (6.29)),

$$\begin{aligned} l^b v_b &= l^a v^b \eta_{ab} = l^0 v^0 - \vec{l}\cdot\vec{v} \\ &= |\vec{l}||\vec{v}|(1-\cos\theta) = 0 \end{aligned} \tag{6.33}$$

Thus $\cos\theta = 1$, and so $\vec{v}$ is parallel to $\vec{l}$ at any given point. We may therefore write

$$v^b = l^c \partial_c l^b = -A(x^a) l^b \tag{6.34}$$

where A is a scalar field.

We will next consider the linear order in m equations and return to the order m^2 equations later, where we will show that they are identically satisified.

6.1.4 Equation from linear in m equation

Using (6.15) in the linear in m equation

$$-\eta^{cd}\partial_c \Gamma_{ab,d} = 0 \tag{6.35}$$

gives

$$\begin{aligned} -\eta^{cd}\partial_c\Gamma_{ab,d} &= -\eta^{cd}\partial_c[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}] \\ &= 2m\eta^{cd}[-\partial_c\partial_d(l_a l_b) + \partial_c\partial_a(l_b l_d) + \partial_c\partial_b(l_a l_d)] \\ &= 2m[-\Box^2(l_a l_b) + \eta^{cd}\partial_c\partial_a(l_b l_d) + \eta^{cd}\partial_c\partial_b(l_a l_d)] \\ &= 0 \end{aligned} \tag{6.36}$$

where we have introduced the D'Alembertian opertator

$$\eta^{cd}\partial_c\partial_d = \Box^2 = \frac{\partial^2}{\partial x^{02}} - \nabla^2.$$

Define

$$L := -\partial_a l^a. \tag{6.37}$$

Expanding $\eta^{cd}\partial_c(l_b l_d)$ gives

$$\begin{aligned} \eta^{cd}\partial_c(l_b l_d) &= \eta^{cd} l_d \partial_c l_b + \eta^{cd} l_b \partial_c l_d \\ &= -((-l^c \partial_c l_b) - (l_b \partial_c l^c)) \\ &= -(L + A) l_b \end{aligned} \tag{6.38}$$

implying

$$\eta^{cd}\partial_c\partial_a(l_b l_d) = \partial_a(\eta^{cd}\partial_c(l_b l_d)) = -\partial_a[(L + A) l_b]$$

Substituting this into equation (6.36) we obtain the equation for l_a

$$-\Box^2(l_a l_b) = \partial_b[(L + A) l_a] + \partial_a[(L + A) l_b] \tag{6.39}$$

We use this equation in the following to prove the order m^2 equations are satisfied, proving the order m^2 equations are automatically satisfied by any solution to (6.39). The entire content of the field equations are thus embodied in (6.39). After dealing with the m^2 equations we will specialise (6.39) to the stationary case where all x^0 derivatives vanish.

6.1.5 Order m^2 equations

Recall the order m^2 equations

$$2m\partial_c(l^c l^d \Gamma_{ab,d}) - \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} = 0. \tag{6.40}$$

Consider the first term, again we use (6.15), we have

$$\begin{aligned} 2m\partial_c(l^c l^d \Gamma_{ab,d}) &= \frac{2m}{2}\partial_c[l^c l^d(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})] \\ &= -2m^2\partial_c[l^c l^d(\partial_a(l_b l_d) + \partial_b(l_a l_d) - \partial_d(l_a l_b))] \end{aligned} \tag{6.41}$$

Looking at the first term in the brackets

$$l^d\partial_a(l_b l_d) = l_b(l^d\partial_a l_d) + (l^d l_d)\partial_a l_b = 0$$

and similarly the second

$$l^d \partial_b (l_a l_d) = 0.$$

So

$$\begin{aligned}
2m\partial_c(l^c l^d \Gamma_{ab,d}) &= 2m^2 \partial_c [l^c l^d (l_b \partial_d l_a + l_a \partial_d l_b)] \\
&= 2m^2 \partial_c [l^c l^d l_b \partial_d l_a + l^c l^d l_a \partial_d l_b)] \\
&= 2m^2 \partial_c [(l^d \partial_d l_a) l^c l_b + (l^d \partial_d l_b) l^c l_a] \\
&= 2m^2 \partial_c [(-A l_a) l^c l_b + (-A l_b) l^c l_a] \\
&= -4m^2 \partial_c [(A l^c)(l_a l_b)] \\
&= -4m^2 [l_a l_b \partial_c (A l^c) + A l^c \partial_c (l_a l_b)] \\
&= -4m^2 [l_a l_b \partial_c (A l^c) + A l^c l_b \partial_c l_a + A l^c l_a \partial_c l_b] \\
&= -4m^2 [l_a l_b \partial_c (A l^c) + A l_b (l^c \partial_c l_a) + A l_a (l^c \partial_c l_b)] \\
&= -4m^2 [l_a l_b \partial_c (A l^c) - 2A^2 l_a l_b] \\
&= -4m^2 l_a l_b [\partial_c (A l^c) - 2A^2] \qquad (6.42)
\end{aligned}$$

So that

$$2m\partial_c(l^c l^d \Gamma_{ab,d}) = -2m^2 l_a l_b [2\partial_c (A l^c) - 4A^2] \qquad (6.43)$$

Now lets us look at the second term in (6.40)

$$-\eta^{cd} \eta^{ef} \Gamma_{ea,d} \Gamma_{cb,f}.$$

We have

$$\begin{aligned}
-\eta^{cd} \eta^{ef} \Gamma_{ea,d} \Gamma_{cb,f} &= -\frac{\eta^{cd} \eta^{ef} 4m^2}{4} [\partial_e (l_a l_d) + \partial_a (l_e l_d) - \partial_d (l_e l_a)] \times \\
&\quad \times [\partial_c (l_b l_f) + \partial_b (l_c l_f) - \partial_f (l_c l_b)] \\
&= -m^2 \eta^{cd} \eta^{ef} [\partial_e (l_a l_d) \partial_c (l_b l_f) + \partial_e (l_a l_d) \partial_b (l_c l_f) - \partial_e (l_a l_d) \partial_f (l_c l_b) \\
&\qquad \partial_a (l_e l_d) \partial_c (l_b l_f) + \partial_a (l_e l_d) \partial_b (l_c l_f) - \partial_a (l_e l_d) \partial_f (l_c l_b) \\
&\qquad -\partial_d (l_e l_a) \partial_c (l_b l_f) - \partial_d (l_e l_a) \partial_b (l_c l_f) + \partial_d (l_e l_a) \partial_f (l_c l_b)].
\end{aligned} \qquad (6.44)$$

Tedious but straightforward calculations give

$$
\begin{aligned}
&1. && \eta^{cd}\eta^{ef}\partial_e(l_a l_d)\partial_c(l_b l_f) = l_a l_b[3A^2 + \partial_d l^c \partial_c l^d] \\
&2. && \eta^{cd}\eta^{ef}\partial_e(l_a l_d)\partial_b(l_c l_f) = 0 \\
&3. && -\eta^{cd}\eta^{ef}\partial_e(l_a l_d)\partial_f(l_c l_b) = -l_a l_b(\partial_d l^c \partial^d l_c) \\
&4. && \eta^{cd}\eta^{ef}\partial_a(l_e l_d)\partial_c(l_b l_f) = 0 \\
&5. && \eta^{cd}\eta^{ef}\partial_a(l_e l_d)\partial_b(l_c l_f) = 0 \\
&6. && -\eta^{cd}\eta^{ef}\partial_a(l_e l_d)\partial_f(l_c l_b) = 0 \\
&7. && -\eta^{cd}\eta^{ef}\partial_d(l_e l_a)\partial_c(l_b l_f) = -l_a l_b(\partial_d l^c \partial^d l_c) \\
&8. && -\eta^{cd}\eta^{ef}\partial_d(l_e l_a)\partial_b(l_c l_f) = 0 \\
&9. && \eta^{cd}\eta^{ef}\partial_d(l_e l_a)\partial_f(l_c l_b) = l_a l_b[3A^2 + \partial_d l^c \partial_c l^d]
\end{aligned} \tag{6.45}
$$

so that

$$
\begin{aligned}
-\eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} &= -m^2[1. + \cdots + 9.] \\
&= -2m^2 l_a l_b[3A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c]
\end{aligned} \tag{6.46}
$$

Putting together (6.43) and (6.46)

$$
\begin{aligned}
& 2m\partial_c(l^c l^d \Gamma_{ab,d}) - \eta^{cd}\eta^{ef}\Gamma_{ea,d}\Gamma_{cb,f} \\
= \; & -2m^2 l_a l_b[2\partial_c(l^c A) - 4A^2] \\
& - 2m^2 l_a l_b[3A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c] \\
= \; & -2m^2 l_a l_b[2\partial_c(l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c]
\end{aligned} \tag{6.47}
$$

we see the m^2 order equations vanish when

$$
l_a l_b[2\partial_c(l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c] = 0. \tag{6.48}
$$

which implies the scalar equation

$$
2\partial_c(l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c = 0. \tag{6.49}
$$

Let us consider the third term, using $l^d \partial_d l^c = -A l^c$ and $L = -\partial_c l^c$

$$
\begin{aligned}
\partial_d l^c \partial_c l^d &= \partial_c(l^d \partial_d l^c) - l^d \partial_c \partial_d l^c \\
&= \partial_c(-A l^c) + l^d \partial_d L \\
&= \partial_c[(L - A) l^c] + L^2
\end{aligned} \tag{6.50}
$$

Similarly we find for the fourth term using $l^c \partial^d l_c = 0$

$$\begin{aligned} \partial_d l^c \partial^d l_c &= \partial_d (l^c \partial^d l_c) - l^c \partial_d \partial^d l_c \\ &= -l^c \partial_d \partial^d l_c \end{aligned} \tag{6.51}$$

Expanding (6.39) gives

$$\begin{aligned} &-l_a \partial_d \partial^d l_b - l_b \partial_d \partial^d l_a - 2(\partial^d l_a)(\partial_d l_b) \\ &\quad = l_b \partial_a (L + A) + l_a \partial_b (L + A) + (L + A)(\partial_b l_a + \partial_a l_b). \end{aligned} \tag{6.52}$$

Contracting with l^a gives

$$\begin{aligned} -l_b l^a \partial_d \partial^d l_a &= l^a l_b \partial_a (L + A) + (L + A) l^a \partial_a l_b \\ &= l_b l^a \partial_a (L + A) - (L + A) A l_b \end{aligned} \tag{6.53}$$

(where we have used $l^a l_a = 0$ as well as $l^a \partial_b l_a = 0$ and $l^a \partial_a l_b = -A l_b$). As l_b is a common factor we get

$$\begin{aligned} -l^a \partial_d \partial^d l_a &= l^a \partial_a (L + A) - (L + A) A \\ &= \partial_a [(L + A) l^a] - \partial_a l^a (L + A) - (L + A) A \\ &= \partial_a [(L + A) l^a] + L^2 - A^2 \end{aligned} \tag{6.54}$$

(where we have used the definition $L := -\partial_a l^a$). By using (6.51) the left hand side of this equation can be replaced by $\partial_d l^c \partial^d l_c$. Thus the fourth term of (6.49) can be written

$$\partial_d l^c \partial^d l_c = L^2 - A^2 + \partial_a [(L + A) l^a]. \tag{6.55}$$

We now sustitute (6.50) and (6.55) into the LHS of the scalar equation (6.49), to obtain

$$\begin{aligned} &2\partial_c (l^c A) - A^2 + \partial_d l^c \partial_c l^d - \partial_d l^c \partial^d l_c = \\ &\quad 2\partial_c (l^c A) - A^2 + (\partial_a [(L - A) l^a] + L^2) - (\partial_a [(L + A) l^a] + L^2 - A^2) \\ &\quad 2\partial_c (l^c A) - 2\partial_c (l^c A) \\ &\quad 0 \end{aligned} \tag{6.56}$$

Thus any solution of (6.39) makes the m^2 equations vanish identically.

6.1.6 Stationary field equations from the m order equation

We have found that the full content of the field equations is embodied in

$$-\Box^2(l_a l_b) = \partial_b[(L+A)l_a] + \partial_a[(L+A)l_b]. \tag{6.57}$$

We now consider the stationary, or time-independent, case. We will find a simlpification to the algebraic manipulations will occur with the introduction of the three-vector λ_j via

$$l_a = l_0(1, \lambda_1, \lambda_2, \lambda_3) \tag{6.58}$$

Since l_a is a flat-space null vector ($l_a l_b \eta^{ab} = 0$), λ_j is a flat-space unit vector,

$$\vec{\lambda}^2 = 1.$$

For $a = b = 0$, in the time-independent case we are now considering, (6.57) reduces to

$$\nabla^2(l_0^2) = 0 \tag{6.59}$$

For $a = 0$, $b = j \neq 0$ (6.57) reduces to

$$\nabla^2(l_0^2\lambda_j) = \partial_j[(L+A)l_0] \tag{6.60}$$

For $a = i \neq 0$, $b = j \neq 0$ (6.57) reduces to

$$\nabla^2(l_0^2\lambda_i\lambda_j) = \partial_j[(L+A)l_0\lambda_i] + \partial_i[(L+A)l_0\lambda_j] \tag{6.61}$$

We can take (6.61) and simplify it to a first-order diferential equation by using (6.60) and (6.59). First expand the RHS of (6.61) and then use (6.60),

$$\begin{aligned}
\nabla^2(l_0^2\lambda_i\lambda_j) &= \partial_j[(L+A)l_0\lambda_i] + \partial_i[(L+A)l_0\lambda_j] \\
&= \lambda_i\partial_j[(L+A)l_0] + \lambda_j\partial_i[(L+A)l_0] + (L+A)l_0(\partial_j\lambda_i + \partial_i\lambda_j) \\
&= \lambda_i\nabla^2(l_0^2\lambda_j) + \lambda_j\nabla^2(l_0^2\lambda_i) + (L+A)l_0(\partial_j\lambda_i + \partial_i\lambda_j)
\end{aligned} \tag{6.62}$$

Rearanging gives

$$\partial_j\lambda_i + \partial_i\lambda_j = \frac{1}{(L+A)l_0}[\nabla^2(l_0^2\lambda_i\lambda_j) - \lambda_i\nabla^2(l_0^2\lambda_j) - \lambda_j\nabla^2(l_0^2\lambda_i)] \tag{6.63}$$

Let us expand the derivatives in $\nabla^2(l_0^2\lambda_i\lambda_j)$:

$$\begin{aligned}
\nabla^2(l_0^2\lambda_i\lambda_j) &= \partial_k\partial_k(l_0^2\lambda_i\lambda_j) \\
&= \partial_k[\lambda_i\lambda_j\partial_k(l_0^2) + l_0^2\lambda_j\partial_k\lambda_i + l_0^2\lambda_i\partial_k\lambda_j] \\
&= \lambda_i\lambda_j\nabla^2(l_0^2) + \lambda_j\partial_k(l_0^2)(\partial_k\lambda_i) + \partial_k(l_0^2)\lambda_i\partial_k\lambda_j \\
&+ \partial_k(l_0^2)(\partial_k\lambda_i)\lambda_j + l_0^2(\partial_k\partial_k\lambda_i)\lambda_j + l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \\
&+ \partial_k(l_0^2)\lambda_i\partial_k\lambda_j + l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) + l_0^2\lambda_i\partial_k\partial_k\lambda_j \\
&= \lambda_i\lambda_j\nabla^2(l_0^2) + 2(\partial_k(l_0^2))(\lambda_j\partial_k\lambda_i + \lambda_i\partial_k\lambda_j) \\
&+ l_0^2(\lambda_j\nabla^2\lambda_i + \lambda_i\nabla^2\lambda_j) + 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j). \qquad (6.64)
\end{aligned}$$

Let us expand the derivatives in $\lambda_i\nabla^2(l_0^2\lambda_j)$:

$$\begin{aligned}
\lambda_i\nabla^2(l_0^2\lambda_j) &= \lambda_i\partial_k\partial_k(l_0^2\lambda_j) \\
&= \lambda_i\partial_k((\partial_k l_0^2)\lambda_j + l_0^2\partial_k\lambda_j) \\
&= \lambda_i\lambda_j\nabla^2(l_0^2) + 2(\partial_k(l_0^2))(\lambda_i\partial_k\lambda_j) + l_0^2\ \lambda_i\nabla^2\lambda_j. \qquad (6.65)
\end{aligned}$$

With these results, let us expand the content of the square brackets on the RHS of (6.63)

$$\begin{aligned}
&\nabla^2(l_0^2\lambda_i\lambda_j) - \lambda_i\nabla^2(l_0^2\lambda_j) - \lambda_j\nabla^2(l_0^2\lambda_i) \\
= &(\nabla^2(l_0^2))\lambda_i\lambda_j + 2\partial_k(l_0^2)(\lambda_j\partial_k\lambda_i + \lambda_i\partial_k\lambda_j) \\
+ &\ l_0^2(\lambda_j\nabla^2\lambda_i + \lambda_i\nabla^2\lambda_j) + 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \\
- &\ \lambda_i\lambda_j\nabla^2(l_0^2) - 2(\partial_k l_0^2)\lambda_i\partial_k\lambda_j - l_0^2\lambda_i\nabla^2\lambda_j \\
- &\ \lambda_j\lambda_i\nabla^2(l_0^2) - 2(\partial_k l_0^2)\lambda_j\partial_k\lambda_i - l_0^2\lambda_j\nabla^2\lambda_i \\
= &-(\nabla^2(l_0^2))\lambda_i\lambda_j + 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \\
= &\ 2l_0^2(\partial_k\lambda_i)(\partial_k\lambda_j) \qquad (6.66)
\end{aligned}$$

where we used (6.59) in the last line. We get

$$\partial_j\lambda_i + \partial_i\lambda_j = \frac{2l_0}{L+A}(\partial_k\lambda_i)(\partial_k\lambda_j) \qquad (6.67)$$

Define

$$p := \frac{L+A}{2l_0} \qquad (6.68)$$

then

$$\partial_j \lambda_i + \partial_i \lambda_j = \frac{1}{p}(\partial_k \lambda_i)(\partial_k \lambda_j) \tag{6.69}$$

The gravitational field is now described by (6.59), (6.60), and (6.69).

Let

$$M_{ik} := \partial_k \lambda_i \tag{6.70}$$

Then (6.69) becomes

$$M_{ij} + M_{ji} = \frac{1}{p} M_{ik} (M^T)_{kj} \tag{6.71}$$

or

$$M + M^T = \frac{1}{p} M M^T \tag{6.72}$$

The constant length of λ_j, implies

$$\frac{1}{2}\partial_k(\lambda_i \lambda_i) = \lambda_i \partial_k \lambda_i = 0 \tag{6.73}$$

or

$$M^T \lambda = 0. \tag{6.74}$$

That is $\vec{\lambda}$ is in the null space of M^T. Moreover, $l^c \partial_c l_b = -A l_b$ with $b = 0$ gives

$$\lambda_j \partial_j l_0 = A \tag{6.75}$$

and with $b = k \neq 0$

$$\lambda_j \partial_j (l_0 l_k) = A \lambda_k \tag{6.76}$$

where we used the definition of λ given in (6.58), but

$$\begin{aligned}\lambda_j\partial_j(l_0\lambda_k) &= l_0(\lambda_j\partial_j\lambda_k) + \lambda_k\lambda_j\partial_j l_0 \\ &= l_0(\lambda_j\partial_j\lambda_k) + A\lambda_k \end{aligned} \tag{6.77}$$

where we used (6.75). By comparing (6.76) and (6.77) we see that

$$\lambda_j\partial_j\lambda_k = 0 \tag{6.78}$$

or

$$M\lambda = 0. \tag{6.79}$$

We will now be able to solve (6.72), (6.74), and (6.79) for M as a function of $\vec{\lambda}$.

Consider a rotation such that

$$R\lambda = \lambda' \tag{6.80}$$

where

$$\lambda' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{6.81}$$

If $\vec{\lambda}$ is in the null space of M and M^T, then $\vec{\lambda}'$ is in the null space of M' and M'^T, where

$$M' = RMR^T, \qquad M'^T = RM^TR^T. \tag{6.82}$$

Now a rotation matrix satisfies $RR^T = I$. Let us write this out in component form,

$$RR^T = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}\begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Writing out some of the terms

$$\begin{aligned} R_{11}R_{11} + R_{12}R_{12} + R_{13}R_{13} &= 1 \\ R_{21}R_{11} + R_{22}R_{12} + R_{23}R_{13} &= 0 \\ \vdots \quad + \quad \vdots \quad + \quad \vdots &= \vdots \end{aligned}$$

Generally:

$$R_{i1}R_{k1} + R_{i2}R_{k2} + R_{i3}R_{k3} = \delta_{ik} \tag{6.83}$$

In other words the three rows constitue three orthonormal vectors in 3 spacial dimensions. Define

$$\begin{aligned} \vec{R}_1 &= (R_{11}, R_{12}, R_{13}) \\ \vec{R}_2 &= (R_{21}, R_{22}, R_{23}) \\ \vec{R}_3 &= (R_{31}, R_{32}, R_{33}) \end{aligned}$$

We are considering rotation such that

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.84}$$

Notice that you could exchange the second and third rows of R without changing (6.84). Let us choose the ordering such that:

$$\vec{R}_2 \times \vec{R}_3 = \vec{R}_1. \tag{6.85}$$

We will use this later. From that

$$\lambda' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and that $M'\lambda' = 0$ and $M'^T\lambda' = 0$ we see that M' must have the form

$$M' = \begin{pmatrix} 0 & a & b \\ c & N'_{11} & N'_{12} \\ d & N'_{21} & N'_{22} \end{pmatrix} \tag{6.86}$$

The values a, b, c, d can be determined from the property of M',

$$M' + M'^T = \frac{1}{p} M' M'^T \tag{6.87}$$

We have first

$$M' + M'^T = \begin{pmatrix} 0 & a+c & b+d \\ a+c & \cdot & \cdot \\ b+d & \cdot & \cdot \end{pmatrix} \tag{6.88}$$

but then

$$\begin{aligned} \frac{1}{p} M' M'^T &= \frac{1}{p} \begin{pmatrix} 0 & a & b \\ c & \cdot & \cdot \\ d & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 & c & d \\ a & \cdot & \cdot \\ b & \cdot & \cdot \end{pmatrix} \\ &= \frac{1}{p} \begin{pmatrix} a^2+b^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{aligned} \tag{6.89}$$

which implies $a^2 + b^2 = 0$ or $a = b = 0$. Now consider $\frac{1}{p} M' M'^T$ again:

$$\begin{aligned} \frac{1}{p} M' M'^T &= \frac{1}{p} \begin{pmatrix} 0 & 0 & 0 \\ c & \cdot & \cdot \\ d & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 & c & d \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix} \\ &= \frac{1}{p} \begin{pmatrix} 0 & \cdot & \cdot \\ 0 & \cdot & \\ 0 & & \end{pmatrix} \end{aligned} \tag{6.90}$$

implying $c = d = 0$. So M' must be of the form

$$M' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & N'_{11} & N'_{21} \\ 0 & N'_{12} & N'_{12} \end{pmatrix} \tag{6.91}$$

The matrix N' satisfies the same equation, (6.87), as M':

$$N' + N'^T = \frac{1}{p} N' N'^T \tag{6.92}$$

Write

$$U = I - \frac{1}{p} N'$$

then the relation (6.92) becomes the following relation for U,

$$\begin{aligned} UU^T &= \left(I - \frac{1}{p}N'\right)\left(I - \frac{1}{p}N'^T\right) \\ &= I - \frac{1}{p}\left(N' + N'^T - \frac{1}{p}N'N'^T\right) \\ &= I \end{aligned} \tag{6.93}$$

Put

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.94}$$

Then $UU^T = I$ reads

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} &= \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{6.95}$$

$$\begin{aligned} a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1 \\ ac + bd &= 0 \end{aligned} \tag{6.96}$$

Now we consider what the result is of taking product $U^T U$:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \tag{6.97}$$

From

$$det\ U det\ U^T = 1$$

we have

$$det\ U = \mp 1$$

or

$$ad - bc = \mp 1$$

multiplying this by c

$$acd - bc^2 = \mp c$$

and using $ac = -bd$ gives

$$-bd^2 - bc^2 = \mp c$$

or

$$-b(c^2 + d^2) = \mp c$$

and as $c^2 + d^2 = 1$ we therefore have

$$b = \pm c. \tag{6.98}$$

Using this in (6.97) we have

$$\begin{aligned} a^2 + c^2 &= a^2 + b^2 = 1 \\ b^2 + d^2 &= c^2 + d^2 = 1 \\ ab + cd &= a(\pm c) + (\pm b)d \\ &= \pm(ac + bd) = 0 \end{aligned} \tag{6.99}$$

therefore

$$U^T U = I.$$

Now $UU^T = U^T U = I$ is the well known condition for unitary matrices. Summarising

$$U = I - \frac{1}{p}N', \quad UU^T = U^T U = I \tag{6.100}$$

Since N' is a 2×2 real matrix, it is therefore either a proper rotation or a an improper rotation, that is, a rotation plus inversion. Thus it may be written

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{6.101}$$

The first case corresponds to $detU = +1$ (a proper rotation) and the second $detU = -1$ (an improper rotation). The first case leads to interesting results. For N' and M' we have

$$N' = p\begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix} \tag{6.102}$$

$$M' = p\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\cos\theta & \sin\theta \\ 0 & -\sin\theta & 1-\cos\theta \end{pmatrix} \tag{6.103}$$

We now need to rotate back to the originl coordinates to get

$$M = R^T M' R.$$

The simple form of M' allows us to write

$$\begin{aligned} M_{ik} &= R_{li}R_{jk}M'_{lj} \\ &= R_{2i}R_{2k}M'_{22} + R_{3i}R_{3k}M'_{33} \\ &+ R_{2i}R_{3k}M'_{23} + R_{3i}R_{2k}M'_{32} \\ &= p(1-\cos\theta)(R_{2i}R_{2k} + R_{3i}R_{3k}) + p\sin\theta(R_{2i}R_{3k} - R_{3i}R_{2k}) \end{aligned} \tag{6.104}$$

Let us look at properties of the rotation matrix

$$R^T R = \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix}\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Writing out some terms

$$\begin{aligned} R_{11}R_{11} + R_{21}R_{21} + R_{31}R_{31} &= 1 \\ R_{12}R_{11} + R_{22}R_{21} + R_{32}R_{31} &= 0 \\ \vdots \quad + \quad \vdots \quad + \quad \vdots &= \vdots \end{aligned}$$

Or generaly:

$$R_{1i}R_{1k} + R_{2i}R_{2k} + R_{3i}R_{3k} = \delta_{ik}. \tag{6.105}$$

Now take (6.85), which can be written in component form

$$\epsilon_{lmn}R_{2m}R_{3n} = R_{1l},$$

contracting with ϵ_{ijl} gives

$$\begin{aligned} \epsilon_{ijl}\epsilon_{lmn}R_{2m}R_{3n} &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})R_{2m}R_{3n} \\ &= R_{2i}R_{3j} - R_{3i}R_{2j} \end{aligned} \tag{6.106}$$

meaning

$$R_{2i}R_{3j} - R_{3i}R_{2j} = \epsilon_{ijl}R_{1l} \tag{6.107}$$

Let us write $R_{1i} = R_i$. Using the above results in (6.104) gives

$$M_{ik} = p(1 - \cos\theta)(\delta_{ik} - R_iR_k) + p\sin\theta\epsilon_{ikl}R_l. \tag{6.108}$$

Now we had

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.109}$$

implying

$$R_{11}\lambda_1 + R_{12}\lambda_2 + R_{13}\lambda_3 = 1$$

or

$$R_{1i}\lambda_i = 1$$

Using the notation $R_{1i} = R_i$, this becomes

$$\vec{R} \cdot \vec{\lambda} = 1, \tag{6.110}$$

where $\vec{R}$ denotes the vector with components R_i. As both $\vec{R}$ and $\vec{\lambda}$ are unit vectors,

$$\vec{R} \cdot \vec{\lambda} = \cos\varphi = 1. \tag{6.111}$$

implies $\varphi = 0$ and hence

$$\vec{R} = \vec{\lambda} \tag{6.112}$$

or $R_i = \lambda_i$. Then we have our end result

$$M_{ik} = p(1 - \cos\theta)(\delta_{ik} - \lambda_i\lambda_k) + p\sin\theta\epsilon_{ikl}\lambda_l. \tag{6.113}$$

We have now replaced the non-linear implicit relation (6.69) by the above simple explicit expression for $\partial_k\lambda_i$.

6.1.7 Laplace and eikonal equations

We rewrite (6.113) in terms of new parameters α and β

$$\partial_k\lambda_i = \alpha(\delta_{ik} - \lambda_i\lambda_k) + \beta\epsilon_{ikl}\lambda_l. \tag{6.114}$$

It will turn out that α and β determine the metric. A number of important three-vector realtion follow directly from (6.114). Set $i = k$ and sum

$$\begin{aligned} \partial_i\lambda_i &= \alpha(\delta_{ii} - \lambda_i\lambda_i) + \beta\epsilon_{iil}\lambda_l \\ &= \alpha(3 - 1) + 0 \end{aligned}$$

giving

$$\nabla \cdot \lambda = 2\alpha. \tag{6.115}$$

Multiplying (6.114) by ϵ_{jki} and summing over i and k gives

$$\begin{aligned}\epsilon_{jki}\partial_k\lambda_i &= \alpha(\delta_{ik}-\lambda_i\lambda_k)\epsilon_{jki}+\beta\epsilon_{ikl}\epsilon_{jki}\lambda_l \\ &= \beta\epsilon_{ikl}\epsilon_{jki}\lambda_l \\ &= -2\beta\lambda_j\end{aligned}$$

or

$$\nabla\times\lambda = -2\beta\lambda. \tag{6.116}$$

The Laplacian of λ can be obtaind in two ways. First differentiating by (6.114) with respect to x^k gives

$$\begin{aligned}\nabla^2\lambda_i &= \partial_k[\alpha(\delta_{ik}-\lambda_i\lambda_k)+\beta\epsilon_{ikl}\lambda_l] \\ &= \partial_i\alpha-\lambda_i(\lambda_k\partial_k\alpha)-\alpha\lambda_k\partial_k\lambda_i-\alpha\lambda_i\partial_k\lambda_k+\epsilon_{ikl}\lambda_l\partial_k\beta+\beta\epsilon_{ikl}\partial_k\lambda_l \\ &= \partial_i\alpha-\lambda_i(\nabla\alpha\cdot\lambda)-\alpha\lambda_i(\nabla\cdot\lambda)+\beta(\nabla\times\lambda)_i+(\nabla\beta\times\lambda)_i \\ &= \partial_i\alpha-\lambda_i(\nabla\alpha\cdot\lambda)-2(\alpha^2+\beta^2)\lambda_i+(\nabla\beta\times\lambda)_i\end{aligned}$$

where we used (6.115), (6.116), and

$$\begin{aligned}\lambda_k\partial_k\lambda_i &= \alpha(\lambda_i-\lambda_i\lambda\cdot\lambda)+\beta\epsilon_{ikl}\lambda_k\lambda_l \\ &= 0\end{aligned} \tag{6.117}$$

(which is just $M\lambda = 0$). We have obtained our first equation for $\nabla^2\lambda$:

$$\nabla^2\lambda = \nabla\alpha-\lambda(\nabla\alpha\cdot\lambda)-2(\alpha^2+\beta^2)\lambda+\nabla\beta\times\lambda \tag{6.118}$$

To obtain the alternative equation we first derive the vector identity

$$\nabla\times(\nabla\times\lambda) = \nabla(\nabla\cdot\lambda)-\nabla^2\lambda$$

from

$$\begin{aligned}[\nabla\times(\nabla\times\lambda)]_m &= \epsilon_{mni}\partial_n(\epsilon_{ikl}\partial_k\lambda_l) \\ &= \epsilon_{mni}\epsilon_{ikl}\partial_n\partial_k\lambda_l \\ &= (\delta_{mk}\delta_{nl}-\delta_{ml}\delta_{nk})\partial_n\partial_k\lambda_l \\ &= \partial_m(\partial_n\lambda_n)-\partial_n\partial_n\lambda_m.\end{aligned}$$

We then have

$$\begin{aligned}\nabla^2\lambda &= \nabla(\nabla\cdot\lambda) - \nabla\times(\nabla\times\lambda)\\ &= \nabla(2\alpha) - \nabla\times(-2\beta\lambda)\\ &= 2\nabla\alpha + 2(\nabla\beta\times\lambda) + 2\beta\nabla\times\lambda\\ &= 2\nabla\alpha + 2(\nabla\beta\times\lambda) - 4\beta^2\lambda \qquad (6.119)\end{aligned}$$

where again we used (6.115) and (6.116). We equate the expressions (6.118) and (6.119)

$$\begin{aligned}&\nabla\alpha - \lambda(\nabla\alpha\cdot\lambda) - 2(\alpha^2+\beta^2)\lambda + \nabla\beta\times\lambda\\ &= 2\nabla\alpha + 2(\nabla\beta\times\lambda) - 4\beta^2\lambda\end{aligned}$$

from which we obtain

$$\nabla\alpha = -\nabla\beta\times\lambda - \lambda(\nabla\alpha\cdot\lambda) - 2(\alpha^2-\beta^2)\lambda. \qquad (6.120)$$

Dotting this with λ gives

$$\nabla\alpha\cdot\lambda = -(\nabla\alpha\cdot\lambda) - 2(\alpha^2-\beta^2)$$

and rearanging we have

$$\nabla\alpha\cdot\lambda = \beta^2 - \alpha^2 \qquad (6.121)$$

and substituting this into (6.120) gives

$$\nabla\alpha = (\beta^2-\alpha^2)\lambda - \nabla\beta\times\lambda \qquad (6.122)$$

Equations (6.121) and (6.122) are very important.

Equations analogous to (6.121) and (6.122) with β replacing α on the left hand side can be obtained. The divergence of (6.116) is zero, implying the divergence of $\beta\lambda$ is zero,

$$\nabla\cdot(\beta\lambda) = \beta\nabla\cdot\lambda + \nabla\beta\cdot\lambda = 0 \qquad (6.123)$$

using (6.115) in this gives

$$\nabla\beta \cdot \lambda = -2\alpha\beta \tag{6.124}$$

Now cross (6.122) with λ and use $(\nabla\beta \times \lambda) \times \lambda = -\lambda \cdot \lambda\nabla\beta + \lambda(\lambda \cdot \nabla\beta)$. We obtain

$$\nabla\beta = \lambda(\lambda \cdot \nabla\beta) + (\nabla\alpha \times \lambda) \tag{6.125}$$

which usong (6.124) becomes

$$\nabla\beta = -2\alpha\beta\lambda + (\nabla\alpha \times \lambda). \tag{6.126}$$

6.1.8 Introduce the complex function γ

Eqations (6.121), (6.122), (6.124), and (6.126) are very important and can be expressed in a more concise way by the introduction of the complex function $\gamma = \alpha + i\beta$:

$$\begin{aligned} \nabla\gamma \cdot \lambda &= \nabla\alpha \cdot \lambda + i\nabla\beta \cdot \lambda \\ &= [\beta^2 - \alpha^2] + i[-2\alpha\beta] \\ &= -(\alpha + i\beta)^2 \\ &= -\gamma^2 \end{aligned} \tag{6.127}$$

$$\begin{aligned} \nabla\gamma &= [(\beta^2 - \alpha^2)\lambda - \nabla\beta \times \lambda] + i[-2\alpha\beta\lambda + (\nabla\alpha \times \lambda)] \\ &= [(\beta^2 - \alpha^2) + i(-2\alpha\beta)]\lambda + i[\nabla\alpha + i\nabla\beta] \times \lambda \\ &= -\gamma^2\lambda + i(\nabla\gamma \times \lambda). \end{aligned} \tag{6.128}$$

We will obtain a pair of simple differential equations that determine γ and show in that γ in turn determines the metric via l_0 and λ_j.

The first differential equation is found by forming the Laplacian for γ from (6.128) and using (6.115), (6.127), and (6.116) we obtain

$$
\begin{aligned}
\nabla^2\gamma &= \nabla\cdot\nabla\gamma \\
&= \nabla\cdot[-\gamma^2\lambda + i(\nabla\gamma\times\lambda)] \\
&= -\gamma^2\nabla\cdot\lambda - 2\gamma\nabla\gamma\cdot\lambda + i\nabla\cdot(\nabla\gamma\times\lambda) \\
&= -2\alpha\gamma^2 + 2\gamma^3 + i\partial_i[\epsilon_{ijk}(\partial_j\gamma)\lambda_k] \\
&= -2\alpha\gamma^2 + 2\gamma^3 - i\partial_i\gamma(\epsilon_{ijk}\partial_j\lambda_k) \\
&= -2\alpha\gamma^2 + 2\gamma^3 - i\nabla\gamma\cdot(\nabla\times\lambda) \\
&= -2\alpha\gamma^2 + 2\gamma^3 - i\nabla\gamma\cdot(-2\beta\lambda) \\
&= -2\alpha\gamma^2 + 2\gamma^3 - 2i\beta\gamma^2 \\
&= -2\gamma^2(\alpha + i\beta - \gamma) \\
&= 0
\end{aligned}
\tag{6.129}
$$

Thus γ is a complex harmonic function. The second differential equation is obtained by squaring (6.128) and using (6.127)

$$
\begin{aligned}
(\nabla\gamma)^2 &= [-\gamma^2\lambda + i(\nabla\gamma\times\lambda)]^2 \\
&= \gamma^4 - (\nabla\gamma\times\lambda)\cdot(\nabla\gamma\times\lambda) \\
&= \gamma^4 + i(\nabla\gamma\times\lambda)\cdot(\nabla\gamma + \gamma^2\lambda) \\
&= \gamma^4
\end{aligned}
\tag{6.130}
$$

we get

$$
\nabla^2\gamma = 0, \quad (\nabla\gamma)^2 = \gamma^4 \tag{6.131}
$$

Let us introduce $\omega \equiv 1/\gamma$, then

$$
\nabla\omega = \nabla\left(\frac{1}{\gamma}\right) = -\frac{1}{\gamma^2}\nabla\gamma \tag{6.132}
$$

Then the equation $(\nabla\gamma)^2 = \gamma^4$ becomes

$$
(\nabla\omega)^2 = \frac{1}{\gamma^4}(\nabla\gamma)^2 = 1.
$$

where we used (6.130). Repeating (6.129), altogether we have

$$
\nabla^2\gamma = 0, \quad (\nabla\omega)^2 = 1, \quad \omega \equiv \frac{1}{\gamma} \tag{6.133}
$$

These equations determine the function γ completely, dependent on consistent boundary conditions.

As we shall now show, these equations completely replace the field equations since the metric functions l_0 and λ are determined by γ.

6.1.9 Expression for λ

It is more convenient to express (6.127) and (6.128) in terms of ω. For equation (6.127) we have

$$\nabla\left(\frac{1}{\omega}\right)\cdot\lambda = -\frac{\lambda\cdot\nabla\omega}{\omega^2} = -\frac{1}{\omega^2} \tag{6.134}$$

where we have used (6.132) and (6.127). For equation (6.128) we have

$$\nabla\left(\frac{1}{\omega}\right) = -\frac{1}{\omega^2}\lambda + i(\nabla\left(\frac{1}{\omega}\right)\times\lambda) \tag{6.135}$$

or

$$-\frac{1}{\omega^2}\nabla\omega = -\frac{1}{\omega^2}\lambda - \frac{i}{\omega^2}(\nabla\omega\times\lambda)$$

Substituting (6.135) into (6.134) gives

$$\lambda\cdot\nabla\omega = 1.$$

Altogether we have

$$\lambda\cdot\nabla\omega = \lambda\cdot\nabla\omega^* = 1, \qquad \nabla\omega = \lambda + i(\nabla\omega\times\lambda) \tag{6.136}$$

Thus

$$\begin{aligned}
\nabla\omega\times\nabla\omega^* &= [\lambda + i(\nabla\omega\times\lambda)]\times[\lambda - i(\nabla\omega^*\times\lambda)] \\
&= -i\lambda\times(\nabla\omega^*\times\lambda) + i(\nabla\omega\times\lambda)\times\lambda + (\nabla\omega\times\lambda)\times(\nabla\omega^*\times\lambda) \\
&= -i[\nabla\omega^* - \lambda(\lambda\cdot\nabla\omega^*)] - i[\nabla\omega - \lambda(\lambda\cdot\nabla\omega)] + (\nabla\omega\times\lambda)\times(\nabla\omega^*\times\lambda) \\
&= -i(\nabla\omega^* - \lambda) - i(\nabla\omega - \lambda) + (\nabla\omega\times\lambda)\times(\nabla\omega^*\times\lambda) \\
&= -i[\nabla\omega^* + \nabla\omega] + 2i\lambda + (\nabla\omega\times\lambda)\times(\nabla\omega^*\times\lambda)
\end{aligned} \tag{6.137}$$

where we used the vector identity $\lambda\times(\nabla f\times\lambda) = \lambda\cdot\lambda\nabla f - \lambda(\lambda\cdot\nabla f)$ and that $\lambda\cdot\nabla\omega = \lambda\cdot\nabla\omega^* = 1$. Consider the last term,

$$\begin{aligned}
[(\nabla\omega\times\lambda)\times(\nabla\omega^*\times\lambda)]_p &= \epsilon_{pil}(\nabla\omega\times\lambda)_i(\nabla\omega^*\times\lambda)_l \\
&= \epsilon_{pil}(\epsilon_{ijk}\lambda_k\partial_j\omega)(\epsilon_{lmn}\lambda_n\partial_m\omega^*) \\
&= \epsilon_{pil}\epsilon_{lmn}(\epsilon_{ijk}\lambda_k\lambda_n\partial_j\omega\partial_m\omega^*) \\
&= (\delta_{pm}\delta_{in} - \delta_{pn}\delta_{im})(\epsilon_{ijk}\lambda_k\lambda_n\partial_j\omega\partial_m\omega^*) \\
&= \epsilon_{ijk}\lambda_k\lambda_i\partial_j\omega\partial_p\omega^* - \lambda_p\epsilon_{ijk}\lambda_k\partial_j\omega\partial_i\omega^* \\
&= \lambda_p\epsilon_{ijk}\lambda_i\partial_j\omega\partial_k\omega^* \\
&= \lambda_p[\lambda\cdot(\nabla\omega\times\nabla\omega^*)] \qquad (6.138)
\end{aligned}$$

So

$$\begin{aligned}
\nabla\omega\times\nabla\omega^* &= -i[\nabla\omega^* + \nabla\omega] + 2i\lambda + \lambda[\lambda\cdot(\nabla\omega\times\nabla\omega^*)] \\
&= -i[\nabla\omega^* + \nabla\omega] + B\lambda \qquad (6.139)
\end{aligned}$$

Now as

$$(\partial_i\omega)\varepsilon_{ijk}\partial_j\omega\partial_k\omega^* = 0$$

or

$$\nabla\omega\cdot(\nabla\omega\times\nabla\omega^*) = 0$$

dotting (6.139) with $\nabla\omega$ and using $(\nabla\omega)^2 = 1$ and $\lambda\cdot\nabla\omega = 1$ we obtain for B

$$B = i[1 + \nabla\omega\cdot\nabla\omega^*] \qquad (6.140)$$

So we get for λ

$$\lambda = \frac{\nabla\omega + \nabla\omega^* - i(\nabla\omega\times\nabla\omega^*)}{1 + \nabla\omega\cdot\nabla\omega^*} \qquad (6.141)$$

and we have our expression for λ_j in terms of ω.

6.1.10 Expression for l_0

We prove that the choice

$$l_0^2 = Re\,(\gamma) = \alpha. \tag{6.142}$$

leads to a solution of the first two field equations (6.59) and (6.60) as we will now show. We have first from the first equation in (6.133) that

$$0 = \nabla^2\alpha = \nabla^2(l_0^2). \tag{6.143}$$

which solves (6.59). Next consider

$$\nabla^2(\alpha\lambda_j) = \alpha\nabla^2\lambda_j + 2(\partial_k\alpha)(\partial_k\lambda_j) \tag{6.144}$$

By using (6.119), (6.114), and (6.121) we get

$$\begin{aligned}
\nabla^2(\alpha\lambda_j) &= \alpha\nabla^2\lambda_j + 2(\partial_k\alpha)(\partial_k\lambda_j) \\
&= \alpha[2\partial_j\alpha + 2(\nabla\beta\times\lambda)_j - 4\beta^2\lambda_j] \\
&\quad + 2(\partial_k\alpha)[\alpha(\delta_{jk} - \lambda_j\lambda_k) + \beta\epsilon_{jkl}\lambda_l] \\
&= 4\alpha\partial_j\alpha - 4\alpha\beta^2\lambda_j - 2\lambda_j\alpha(\lambda\cdot\nabla\alpha) + 2\alpha(\nabla\beta\times\lambda)_j + 2\beta(\nabla\alpha\times\lambda)_j \\
&= 4\alpha\partial_j\alpha - 4\alpha\beta^2\lambda_j - 2\lambda_j\alpha(\beta^2 - \alpha^2) + 2\alpha(\nabla\beta\times\lambda)_j + 2\beta(\nabla\alpha\times\lambda)_j \\
&= 4\alpha\partial_j\alpha + 2\alpha(\alpha^2 - 3\beta^2)\lambda_j + 2\alpha(\nabla\beta\times\lambda)_j + 2\beta(\nabla\alpha\times\lambda)_j
\end{aligned} \tag{6.145}$$

We simplify further using (6.122) and (6.126),

$$\begin{aligned}
\nabla^2(\alpha\lambda_j) &= 4\alpha\partial_j\alpha + 2\alpha(\alpha^2 - 3\beta^2)\lambda_j + 2\alpha[(\beta^2 - \alpha^2)\lambda_j - \partial_j\alpha] \\
&\quad + 2\beta[2\alpha\beta\lambda_j + \partial_j\beta] \\
&= 2\alpha\partial_j\alpha + 2\beta\partial_j\beta \\
&= \partial_j(\alpha^2 + \beta^2).
\end{aligned} \tag{6.146}$$

Or

$$\nabla^2(\alpha\lambda) = \nabla(\alpha^2 + \beta^2). \tag{6.147}$$

We calculate the RHS of (6.147). From $\alpha = p(1 - \cos\theta)$ and $\beta = p\sin\theta$ we have

$$\alpha^2 + \beta^2 = 2p^2(1 - \cos\theta) = 2\alpha p \tag{6.148}$$

From the definition of p,

$$p = \frac{L + A}{2l_0}. \tag{6.149}$$

We then have

$$L + A = \frac{l_0}{\alpha}(\alpha^2 + \beta^2). \tag{6.150}$$

Thus with $l_0^2 = \alpha$ the RHS of (6.147) is

$$\partial_j[(L + A)l_0] = \partial_j(\alpha^2 + \beta^2). \tag{6.151}$$

We see that $l_0^2 = \alpha$ is indeed a solution.

6.2 The Kerr Solution

6.2.1 A solution to the field equations

Consider the function

$$\gamma = [(x - a)^2 + (y - b)^2 + (z - c)^2]^{-1/2} \tag{6.152}$$

We will calculate $\nabla^2\gamma$ and $(\nabla\omega)^2$, $(\omega = 1/\gamma)$. First

$$\frac{\partial}{\partial x}[(x - a)^2 + (y - b)^2 + (z - c)^2]^{-1/2} = -(x - a)[(x - a)^2 + (y - b)^2 + (z - c)^2]^{-3/2}$$

and

$$\begin{aligned} & \frac{\partial^2}{\partial x^2}[(x - a)^2 + (y - b)^2 + (z - c)^2]^{-1/2} \\ = \; & -[(x - a)^2 + (y - b)^2 + (z - c)^2]^{-3/2} \\ & + 3(x - a)^2[(x - a)^2 + (y - b)^2 + (z - c)^2]^{-5/2} \end{aligned} \tag{6.153}$$

so that

$$\begin{aligned}\nabla^2\gamma &= \nabla^2[(x-a)^2+(y-b)^2+(z-c)^2]^{-1/2} \\ &= -3[(x-a)^2+(y-b)^2+(z-c)^2]^{-3/2} \\ &\quad + 3[(x-a)^2+(y-b)^2+(z-c)^2][(x-a)^2+(y-b)^2+(z-c)^2]^{-5/2} \\ &= 0. \end{aligned} \tag{6.154}$$

Now consider $\nabla\omega$

$$\begin{aligned}\nabla\omega &= \nabla[(x-a)^2+(y-b)^2+(z-c)^2]^{1/2} \\ &= \frac{[(x-a)\hat{i}+(y-b)\hat{j}+(z-c)\hat{k}]}{[(x-a)^2+(y-b)^2+(z-c)^2]^{1/2}} \end{aligned} \tag{6.155}$$

so obviously we have

$$(\nabla\omega)^2 = 1. \tag{6.156}$$

Therefore (6.152) is a solution of the field equations (6.133).

6.2.2 The Schwarzschild solution

Consider the choice

$$\gamma = (x^2+y^2+z^2)^{-1/2} \tag{6.157}$$

We obviously have

$$l_0^2 = \frac{1}{r}. \tag{6.158}$$

Next as $\omega = 1/\gamma$, $\omega = r$ and

$$\begin{aligned}\nabla\omega &= \nabla(x^2+y^2+z^2)^{1/2} \\ &= \frac{x\hat{i}+y\hat{j}+z\hat{i}}{(x^2+y^2+z^2)^{1/2}} \\ &= \frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{i} \end{aligned} \tag{6.159}$$

Obviously

$$\nabla\omega \times \nabla\omega^* = \nabla\omega \times \nabla\omega = 0 \tag{6.160}$$

Next

$$\nabla\omega \cdot \nabla\omega^* = \nabla\omega \cdot \nabla\omega = 1 \tag{6.161}$$

Recall λ is given by

$$\begin{aligned} \lambda &= \frac{\nabla\omega + \nabla\omega^* - i(\nabla\omega \times \nabla\omega^*)}{1 + \nabla\omega \cdot \nabla\omega^*} \\ &= \nabla\omega \\ &= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{i} + \frac{z}{r}\hat{i} \end{aligned} \tag{6.162}$$

Reading off the components of λ we get

$$\lambda_1 = \frac{x}{r}, \quad \lambda_2 = \frac{y}{r}, \quad \lambda_3 = \frac{z}{r}. \tag{6.163}$$

Combining this with (6.158) gives for l_a

$$l_a = l_0(1, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{\sqrt{r}}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{6.164}$$

in agreement with (6.3).

6.2.3 The Kerr solution

Consider the complex valued choice,

$$\gamma = (x^2 + y^2 + (z - ia)^2)^{-1/2} \tag{6.165}$$

Let us calculate the functions l_0 and λ. We first split ω into real and imaginary parts

$$\omega = 1/\gamma = \rho + i\sigma = (x^2 + y^2 + (z - ia)^2)^{1/2} = (r^2 - a^2 - 2iaz)^{1/2} \tag{6.166}$$

Squaring and equating real and imaginary parts gives

$$\rho^2 - \sigma^2 = r^2 - a^2, \qquad \sigma = -\frac{az}{\rho} \tag{6.167}$$

substituting for σ in the real part gives

$$\rho^2 - \frac{a^2 z^2}{\rho^2} = r^2 - a^2$$

resulting in the quadratic equation in ρ^2:

$$\rho^4 - \rho^2(r^2 - a^2) - a^2 z^2 = 0. \tag{6.168}$$

Completeing the square we obatain

$$\rho^2 = \frac{r^2 - a^2}{2} + \left[\frac{(r^2 - a^2)^2}{4} + a^2 z^2\right]^{1/2} \tag{6.169}$$

where we have choosen the plus sign so that for $r \gg a, \rho \sim r$. Now $l_0^2 = \text{Re}\,(\gamma)$. From

$$\gamma = \frac{1}{\rho + i\sigma} = \frac{\rho}{\rho^2 + \sigma^2} - \frac{i\sigma}{\rho^2 + \sigma^2}$$

we have

$$l_0^2 = \frac{\rho}{\rho^2 + \sigma^2} = \frac{\rho^3}{\rho^4 + a^2 z^2}. \tag{6.170}$$

Now let us move onto the calculation of λ. First

$$\begin{aligned} \nabla\omega &= \nabla\left(\frac{1}{\gamma}\right) \\ &= \nabla(r^2 - 2iaz - a^2)^{1/2} \\ &= \frac{1}{2\omega}(2x\hat{i} + 2y\hat{j} + 2z\hat{k} - 2ia\hat{k}) \\ &= \frac{\hat{r} - ia\hat{k}}{\omega} \end{aligned} \tag{6.171}$$

then

$$\begin{aligned}\nabla\omega + \nabla\omega^* &= \frac{\hat{r} - ia\hat{k}}{\omega} + \frac{\hat{r} + ia\hat{k}}{\omega^*} \\ &= \frac{\omega^*(\hat{r} - ia\hat{k}) + \omega(\hat{r} + ia\hat{k})}{|\omega|^2} \\ &= \frac{(\rho - i\sigma)(\hat{r} - ia\hat{k}) + (\rho + i\sigma)(\hat{r} + ia\hat{k})}{|\omega|^2} \\ &= 2\frac{\rho\hat{r} - a\sigma\hat{k}}{|\omega|^2} \end{aligned} \tag{6.172}$$

and

$$\nabla\omega \cdot \nabla\omega^* = \frac{\hat{r} - ia\hat{k}}{\omega} \cdot \frac{\hat{r} + ia\hat{k}}{\omega^*} = \frac{r^2 + a^2}{|\omega|^2}$$

and

$$\nabla\omega \times \nabla\omega^* = \frac{\hat{r} - ia\hat{k}}{\omega} \times \frac{\hat{r} + ia\hat{k}}{\omega^*} = 2ia\frac{\mathbf{r} \times \mathbf{k}}{|\omega|^2}$$

so λ is

$$\begin{aligned}\lambda &= \frac{\nabla\omega + \nabla\omega^* - i(\nabla\omega \times \nabla\omega^*)}{1 + \nabla\omega \cdot \nabla\omega^*} \\ &= \frac{2[\rho\hat{r} - a\sigma\hat{k} + a(\mathbf{r} \times \mathbf{k})]}{|\omega|^2 + r^2 + a^2}. \end{aligned} \tag{6.173}$$

We can simplify this by noting

$$\begin{aligned}|\omega|^2 &= [(r^2 - a^2)^2 + 4a^2z^2]^{1/2} \\ &= 2[\frac{(r^2 - a^2)^2}{4} + a^2z^2]^{1/2} \\ &= 2\rho^2 - (r^2 - a^2) \end{aligned}$$

because then

$$|\omega|^2 + r^2 + a^2 = 2(\rho^2 + a^2).$$

Substituting this into (6.173) we then have for λ

$$\lambda = \frac{[\rho\hat{r} - a\sigma\hat{k} + a(\mathbf{r}\times\mathbf{k})]}{\rho^2 + a^2} \tag{6.174}$$

As

$$\mathbf{r}\times\mathbf{k} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\times\hat{\mathbf{k}} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$$

and as $\sigma = -az/\rho$, λ is

$$\begin{aligned}\lambda &= \frac{1}{\rho^2 + a^2}\left[\rho\hat{r} + \frac{a^2 z}{\rho}\hat{k} + a(y\hat{\mathbf{i}} - x\hat{\mathbf{j}})\right] \\ &= \frac{1}{\rho^2 + a^2}\left[(\rho x + ay)\hat{i} + (\rho y - ax)\hat{j} + \frac{(\rho^2 + a^2)z}{\rho}\hat{k}\right].\end{aligned} \tag{6.175}$$

We can read off the components

$$\lambda_1 = \frac{\rho x + ay}{a^2 + \rho^2},$$

$$\lambda_2 = \frac{\rho y - ax}{a^2 + \rho^2},$$

and

$$\lambda_3 = \frac{z}{\rho}.$$

Then

$$\begin{aligned}l_\mu dx^\mu &= l_0\left[dx^0 + \frac{\rho x + ay}{a^2 + \rho^2}dx + \frac{\rho y - ax}{a^2 + \rho^2}dy + \frac{z}{\rho}dz\right] \\ &= l_0\left[dx^0 + \frac{\rho}{a^2 + \rho^2}(xdx + ydy) + \frac{a}{a^2 + \rho^2}(ydx - xdy) + \frac{z}{\rho}dz\right]\end{aligned}$$

Finally, as

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu - 2ml_\mu l_\nu dx^\mu dx^\nu$$

we have

$$\begin{aligned} ds^2 &= (dx^0)^2 - (d\mathbf{x})^2 - \frac{2m\rho^3}{\rho^4 + a^2z^2}\left[dx^0 + \frac{\rho x}{a^2+\rho^2}(xdx + ydy)\right. \\ &\qquad \left. + \frac{a}{a^2+\rho^2}(ydx - xdy) + \frac{z}{\rho}dz\right]^2 \end{aligned} \tag{6.176}$$

This is the form obtained by Kerr in 1963.

6.3 Independence of Metric on Angular Variable φ

We now make axially symmetry of this solution manifest. Consider

$$\begin{aligned} u &= x^0 + u \\ x &= \rho\sin\theta\cos\varphi - a\rho\sin\theta\sin\varphi \\ y &= \rho\sin\theta\cos\varphi + a\rho\sin\theta\sin\varphi \\ z &= \rho\cos\theta. \end{aligned} \tag{6.177}$$

The middle two transformations can be written as a complex expression

$$(\rho + ia)e^{i\varphi}\sin\theta = x + iy \tag{6.178}$$

Now let us find the differential dz^2

$$dz^2 = (\cos\theta d\rho - \rho\sin\theta d\theta)^2 \tag{6.179}$$

Now let us find the differential $dx^2 + dy^2$

$$\begin{aligned} dx^2 + dy^2 &= |d(x+iy)|^2 \\ &= |d[(\rho+ia)e^{i\varphi}\sin\theta]|^2 \\ &= |e^{i\varphi}\sin\theta d\rho + i(\rho+ia)e^{i\varphi}\sin\theta d\varphi + (\rho+ia)e^{i\varphi}\cos\theta d\theta|^2 \\ &= |\sin\theta d\rho + i(\rho+ia)\sin\theta d\varphi + (\rho+ia)\cos\theta d\theta|^2 \\ &= |\sin\theta d\rho - a\sin\theta d\varphi + \rho\cos\theta d\theta + i[\rho\sin\theta d\varphi + a\cos\theta d\theta]|^2 \\ &= (\sin\theta d\rho - a\sin\theta d\varphi + \rho\cos\theta d\theta)^2 + (\rho\sin\theta d\varphi + a\cos\theta d\theta)^2 \end{aligned} \tag{6.180}$$

Now let us find the differential $xdx + ydy$

$$\begin{aligned} xdx + ydy &= \frac{1}{2}d|x + iy|^2 \\ &= \frac{1}{2}d|(\rho + ia)e^{i\varphi}\sin\theta|^2 \\ &= \frac{1}{2}d|(\rho + ia)\sin\theta|^2 \\ &= \frac{1}{2}d[(\rho^2 + a^2)\sin^2\theta] \\ &= (\rho^2 + a^2)\sin\theta\cos\theta d\theta + \rho\sin^2\theta d\rho \end{aligned} \tag{6.181}$$

Now let us find the differential $xdy - ydx$

$$\begin{aligned} xdy - ydx &= -Im[(x + iy)d(x - iy)] \\ &= -Im\{[(\rho + ia)e^{i\varphi}\sin\theta]d[(\rho - ia)e^{-i\varphi}\sin\theta]\} \\ &= -Im\{[(\rho + ia)e^{i\varphi}\sin\theta][e^{-i\varphi}\sin\theta d\rho - i(\rho - ia)e^{-i\varphi}\sin\theta d\varphi + \\ &\quad + (\rho - ia)e^{i\varphi}\cos\theta d\theta]\} \\ &= -Im\{(\rho + ia)\sin^2\theta d\rho - i(\rho^2 + a^2)\sin^2\theta d\varphi + (\rho^2 + a^2)\sin\theta\cos\theta d\theta\} \\ &= (\rho^2 + a^2)\sin\theta^2 d\varphi - a\sin^2\theta d\rho \end{aligned} \tag{6.182}$$

Now let us find the differential zdz

$$zdz = \rho\cos^2\theta d\rho - \rho^2\sin\theta\cos\theta d\theta \tag{6.183}$$

$$\frac{2m\rho^3}{\rho^4 + a^2z^2} = \frac{2m\rho}{\rho^2 + a^2\cos^2\theta} \tag{6.184}$$

Substituting these results into (6.176)

$$\begin{aligned} ds^2 &= (du - d\rho)^2 - (\sin\theta d\rho - a\sin\theta d\varphi + \rho\cos\theta d\theta)^2 - (\rho\sin\theta d\varphi + a\cos\theta d\theta)^2 \\ &\quad -(\cos\theta d\rho - \rho\sin\theta d\theta)^2 \\ &\quad - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du - d\rho + \frac{\rho}{a^2 + \rho^2}\left\{(\rho^2 + a^2)\sin\theta\cos\theta d\theta + \rho\sin^2\theta d\rho\right\} \\ &\quad - \frac{a}{a^2 + \rho^2}\left\{(\rho^2 + a^2)\sin^2\theta d\varphi - a\sin^2\theta d\rho\right\} + \cos^2\theta d\rho - \rho\sin\theta\cos\theta d\theta\Big]^2 \end{aligned} \tag{6.185}$$

We multiply out the first and second line and get

$$
\begin{aligned}
& du^2 + (1 - \sin^2\theta - \cos^2)\theta d\rho^2 - [(\rho^2 + a^2)\cos^2\theta + \rho^2 \sin^2\theta]d\theta^2 - (\rho^2 + a^2)\sin^2\theta d\varphi^2 \\
& - 2dud\rho - (2\rho \sin\theta\cos\theta - 2\rho\sin\theta\cos\theta)d\rho d\theta + 2a\sin^2\theta d\rho d\varphi \\
& + (a\rho\sin\theta\cos\theta - \rho a\sin\theta\cos\theta)d\theta d\varphi \\
= \;& du^2 - (\rho^2 + a^2\cos^2\theta)d\theta^2 - (\rho^2 + a^2)\sin^2\theta d\varphi^2 - 2dud\rho + 2a\sin^2\theta d\rho d\varphi \qquad (6.186)
\end{aligned}
$$

The last two lines can be simplified,

$$
\begin{aligned}
& - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du + \left(-1 + \frac{\rho^2}{a^2+\rho^2}\sin^2\theta + \frac{a^2}{a^2+\rho^2}\sin^2\theta + \cos^2\theta\right)d\rho \\
& + (\rho\sin\theta\cos\theta - \rho\sin\theta\cos\theta)\,d\theta + \left(-a\sin^2\theta\right)d\varphi\Big]^2 \\
= \;& - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du + \left(\frac{\rho^2}{a^2+\rho^2} + \frac{a^2}{a^2+\rho^2} - 1\right)\sin^2\theta d\rho - a\sin^2\theta d\varphi\Big]^2 \\
= \;& - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du - a\sin^2\theta d\varphi\Big]^2 \qquad (6.187)
\end{aligned}
$$

Putting together (6.186) and (6.187) we get

$$
\begin{aligned}
ds^2 \;=\;& du^2 - (\rho^2 + a^2\cos^2\theta)d\theta^2 - (\rho^2 + a^2)\sin^2\theta d\varphi^2 - 2dud\rho + 2a\sin^2\theta d\rho d\varphi \\
& - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\Big[du - a\sin^2\theta d\varphi\Big]^2. \qquad (6.188)
\end{aligned}
$$

Simplifying, we finally obtain the line element

$$
\begin{aligned}
ds^2 \;=\;& \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right)du^2 - 2dud\rho + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(2a\sin^2\theta)dud\varphi \\
& + 2a\sin^2\theta d\rho d\varphi - (\rho^2 + a^2\cos^2\theta)d\theta^2 - \left((\rho^2 + a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right)d\varphi^2. \qquad (6.189)
\end{aligned}
$$

There is no dependence of the line element on the angular coordinate φ, so that the solution (6.189) is manifestly axially symmetric. This is the advanced Eddington-Finkelstein form of Kerr's solution.

6.4 Boyer-Lindquist Coordinates

We wish to make a change of coordinates that puts the line element in a form suh that the only cross-term is $d\varphi dt$.

write the element (6.189) as

$$ds^2 = g_{00}du^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 + 2g_{03}dud\varphi + 2g_{01}dud\rho + 2g_{13}d\rho d\varphi \tag{6.190}$$

We guess the form of the desired transformation

$$\begin{aligned} t &= u - A(\rho) & du &= dt + A'd\rho \\ \tilde{\varphi} &= \varphi - B(\rho) & d\varphi &= d\tilde{\varphi} + B'd\rho \end{aligned} \tag{6.191}$$

(6.190) is then

$$\begin{aligned} ds^2 &= g_{00}(dt + A'd\rho)^2 + g_{22}d\theta^2 + g_{33}(d\tilde{\varphi} + B'd\rho)^2 + 2g_{03}(dt + A'd\rho)(d\tilde{\varphi} + B'd\rho) \\ &\quad + 2g_{01}(dt + A'd\rho)d\rho + 2g_{13}d\rho(d\tilde{\varphi} + B'd\rho) \\ &= g_{00}dt^2 + (g_{00}A'^2 + g_{33}B'^2 + 2g_{01}A' + 2g_{13}B' + 2g_{03}A'B')d\rho^2 \\ &\quad + g_{22}d\theta^2 + g_{33}d\tilde{\varphi}^2 + 2g_{03}dtd\tilde{\varphi} + 2(g_{03}A' + g_{33}B' + g_{13})d\tilde{\varphi}d\rho \\ &\quad + 2(g_{00}A' + g_{03}B' + g_{01})dtd\rho \end{aligned} \tag{6.192}$$

We now demand that the coefficients of $d\tilde{\varphi}d\rho$ and $dtd\rho$ zanish, we must have

$$g_{03}A' + g_{33}B' + g_{13} = 0 \tag{6.193}$$

and

$$g_{00}A' + g_{03}B' + g_{01} = 0 \tag{6.194}$$

respectively. Multiplying (6.193) by g_{03} and (6.194) by g_{33} and subtracting implies

$$A' = \frac{g_{33}g_{01} - g_{03}g_{13}}{g_{03}^2 - g_{00}g_{03}}. \tag{6.195}$$

Multiplying (6.193) by g_{00} and (6.194) by g_{03} and subtracting implies

$$B' = \frac{g_{00}g_{13} - g_{03}g_{01}}{g_{03}^2 - g_{00}g_{03}}. \tag{6.196}$$

$$\begin{aligned} g_{33}g_{01} - g_{03}g_{13} &= -\left((\rho^2 + a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right)(-1) \\ &\quad - \left(\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right)a\sin^2\theta \\ &= (\rho^2 + a^2)\sin^2\theta \end{aligned} \tag{6.197}$$

$$\begin{aligned} g_{00}g_{13} - g_{03}g_{01} &= \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right)a\sin^2\theta - \left(\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right)(-1) \\ &= a\sin^2\theta \end{aligned} \tag{6.198}$$

$$\begin{aligned} g_{03}^2 - g_{00}g_{33} &= \left(\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right)^2 - \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right)\times \\ &\quad \times -\left((\rho^2 + a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right) \\ &= \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta}[\frac{4m^2\rho^2a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta} + \\ &\quad + (\rho^2 + a^2\cos^2\theta - 2m\rho)[(\rho^2 + a^2) + \frac{2m\rho a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}] \\ &= \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta}[(\rho^2 + a^2\cos^2\theta)(\rho^2 + a^2) + 2m\rho a^2\sin^2\theta - 2m\rho(\rho^2 + a^2)] \\ &= \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta}[(\rho^2 + a^2\cos^2\theta)(\rho^2 + a^2) - 2m\rho(\rho^2 + a^2\cos^2\theta)] \\ &= (\rho^2 + a^2 - 2m\rho)\sin^2\theta \end{aligned} \tag{6.199}$$

Therefore

$$A' = \frac{\rho^2 + a^2}{\rho^2 + a^2 - 2m\rho} \tag{6.200}$$

and

$$B' = \frac{a}{\rho^2 + a^2 - 2m\rho} \tag{6.201}$$

The line element is now given by (6.192) with the last two terms vanishing by construction. The calculation of the coefficient of $d\rho^2$ can be simplified

$$\begin{aligned} & g_{00}A'^2 + g_{33}B'^2 + 2g_{01}A' + 2g_{13}B' + 2g_{03}A'B' \\ = \; & A'(A'g_{00} + B'g_{03} + g_{01}) + B'(A'g_{03} + B'g_{33} + g_{13}) + g_{01}A' + g_{13}B' \\ = \; & g_{01}A' + g_{13}B' \\ = \; & (-1)\frac{\rho^2 + a^2}{\rho^2 + a^2 - 2m\rho} + a\sin^2\theta\frac{a}{\rho^2 + a^2 - 2m\rho} \\ = \; & -\frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho} \end{aligned} \tag{6.202}$$

The line element is then

$$ds^2 = g_{00}dt^2 + (g_{01}A' + g_{13}B')d\rho^2 + g_{22}d\theta^2 + g_{33}d\tilde{\varphi}^2 + 2g_{03}dtd\tilde{\varphi} \tag{6.203}$$

$$\begin{aligned} ds^2 \; = \; & \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right)dt^2 - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho}d\rho^2 - (\rho^2 + a^2\cos^2\theta)d\theta^2 \\ & - \left((\rho^2 + a^2)\sin^2\theta + \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}(a^2\sin^4\theta)\right)d\tilde{\varphi}^2 + \frac{4ma\rho\sin^2\theta}{\rho^2 + a^2\cos^2\theta}dtd\tilde{\varphi} \end{aligned} \tag{6.204}$$

This is the Kerr solution in Boyer-Lindquist coordinates and is analogous to the Schwarzschild coordinates for a non-rotating black hole.

6.5 Interptetation as Rotating Body

$$\begin{aligned} ds^2 \; = \; & \left(1 - \frac{2mr}{\rho^2}\right)dt^2 + \frac{2mar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) \\ & -\frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2 \end{aligned} \tag{6.205}$$

where

$$\Delta(r) = r^2 - 2mr + a^2 \tag{6.206}$$

and

$$\rho^2 = r^2 + a^2 \cos^2 \theta \tag{6.207}$$

$$a = J/M \tag{6.208}$$

where J is the Komar anagular momentum.

6.6 Basic Propertire of the Kerr Solution

6.6.1 Boyer-Lindquist cordinates

If we let $a \to 0$ we obtain

$$\begin{aligned} ds^2 &= \left(1 - \frac{2mr}{r^2}\right) dt^2 - \frac{r^2}{r^2 - 2GMr} dr^2 \\ &\quad -r^2 d\theta^2 - \frac{\sin^2 \theta}{r^2} r^4 d\phi^2 \end{aligned} \tag{6.209}$$

i.e. as $a \to 0$ Boyer-Lindquist cordinates reduce to Schwarschild coordinates.

The Boyer-Lindquist form is the most useful one for investigating the properties of the Kerr solution.

• If we set $a = 0$, the regain the Schwarschild solution in Schwarschild coordinates and so m is identified with the geometric mass;

• The metric coefficients are independent of t and hence the solution is stationary;

• The metric coefficients are independent of ϕ and hence the solution is axially symmetric, i.e. there is an axis such that the solution is invariant under rotation about this axis.

• As for discrete symmetries, the solution is invariant under the symultaneous inversion of t and ϕ, that is under the transformation

$$t \to -t, \qquad \phi \to -\phi. \tag{6.210}$$

This suggests that the Kerr solution may correspond to a spinning source, since running time backwards with negative spin direction is equivalent to unning time forward with positive spin directin.

6.6.2 Cartesian Coordinates

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{6.211}$$

6.7 Singularities and Event Horizons

Event horizons are null surfaces beyond which it is impossible to return to a certain region of space.

The stataionary limit surface is timelike exery where except where it s tangent to the event horizons at the poles. It represents the place past which it is impossible to remail stationay.

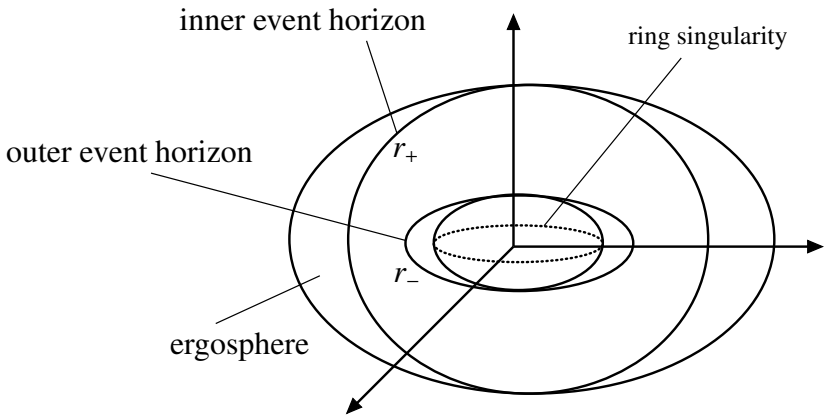

Figure 6.1: Rotating blackhole

$$\Delta = r^2 - 2mr + a^2 = 0, \tag{6.212}$$

two null event horizons

$$r_\pm = m \pm (m^2 - a^2)^{1/2}. \tag{6.213}$$

Then in a similarly way in which the Reissner-Nordstrom solution is regular in three regions:

I. $r_+ < r < \infty$

II. $r_- < r < r_+$

III. $0 < r < r_-$

In the limit $a \to 0$

Appendix A

Derivation of the Full Field Equations of General Relativity

We must have the condition that physical laws reduce to those of special relativity in an infintesimal small inertial frame.There is a simply procedure for writing down the equations of matter or radiation in a grtavitational field. They are obtained from equations of matter or radiation in special relativity by replacing partial derivatives ∂_a by covariant derivatives ∇_a and the Minkowski metric η_{ab} with the gravitational metric g_{ab}, and replacing the volume element d^4x by the invariant volume element $\sqrt{-g}d^4x$, where g is the dertminant of g_{ab} (where, however, the metric no longer flat but is a solution to the full field equations). The resulting equations will automatically satisfy the principle of general covariance and the equivalence principle. This procedure is called "minimal coupling". These do not represent the only choice as there are in fact infinetly many different generally covariant equations that reduce to the same special relativistic equations. The principle of "minimal coupling" is a simplicity principle. It asserts that no terms explicitly containing the curvature tensor (which vanishes in special relativity) should be added in making the transition from special relativity to general relativity.

Here we first derive Einstein's vacuum field equations for general relativity. To obtain the full fields equations, we take the action to be

$$S = \int_{\mathcal{M}} (\mathfrak{L}_G + \kappa \mathfrak{L}_M)\, d^4x \tag{A.1}$$

where $\mathfrak{L}_G$ is the Lagrangian density for the gravitational field, $\mathfrak{L}_M$ is the Lagragian density of the matter or radiation field, and κ is a coupling constant. Both Lagrangians are functionals of the metric and its derivatives.

The method we use to obtain the full field equations is precisely analogous to the way a source term for the Maxwell's equations, a current J^a, arises from a variation of the coupled matter-Maxwell action with respect to the gauge potential, see section A.5.3.

The energy-momentum tensor T^{ab} of matter or radiation content can be computed, in general, as the variation of the action with respect to the metric. If the matter/radiation action is $S_M[\varphi, g_{ab}]$, the energy-momentum tensor can be computed as

$$T^{ab} = \frac{1}{(-g)^{\frac{1}{2}}} \frac{\delta S_M}{\delta g_{ab}(x)}. \tag{A.2}$$

We will obtain the full field equations of general relativity:

$$G^{ab} = \kappa T^{ab} \tag{A.3}$$

where G^{ab} is the Einstein tensor.

We then consider the most important energy-momentum tensors. We first obtain the energy-momentum tensor for dust, and then for a perfect fluid in Minkowski spacetime. We do this not via an action principle, but by physical reasoning (it is possible to write down an a dust Lagrangian, producing all the required field equations - which we touch upon). We make the tansition to general relativity using the princple of minimal coupling. We then develop the tensorial formulation of Maxwell's theory governing the electromagnetic field. We pass to the generalivistic theory via the minimal coupling procedure, and obtain the energy momentum tensor in source free regions via (A.2). Using this write down the the full field equations for gravity coupled to the electromanetic field in source free regions, known as the Einstein-Maxwell equations.

A.1 Mathematical Preliminaries

A.1.1 The Metric Connection

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}) \tag{A.4}$$

$$\begin{aligned}
\nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{da} \\
&= \partial_c g_{ab} - \frac{1}{2} g^{de}(\partial_a g_{ec} + \partial_c g_{ea} - \partial_e g_{ac}) g_{db} - \frac{1}{2} g^{de}(\partial_b g_{ec} + \partial_c g_{eb} - \partial_e g_{bc}) g_{da} \\
&= \partial_c g_{ab} - \frac{1}{2}(\partial_a g_{bc} + \partial_c g_{ba} - \partial_b g_{ac}) - \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}) \\
&= 0.
\end{aligned}$$

$$\partial_c g_{ab} = \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad} \tag{A.5}$$

A.1.2 Tensor Densities

A tensor density of weighht W, usually denoted by a gothic letter, $\mathfrak{T}^{a\cdots}_{b\cdots}$, transforms like an ordinary tensor, excep that J^W appears as a factor where

$$J = \left|\frac{\partial x^a}{\partial x'^b}\right|, \tag{A.6}$$

that is, tensor density of weighht W transforms as

$$\mathfrak{T}'^{a\cdots}_{b\cdots} = J^W \frac{\partial x'^a}{\partial x^c} \cdots \frac{\partial x^d}{\partial x'^b} \cdots \mathfrak{T}^{a\cdots}_{b\cdots} \tag{A.7}$$

We adhere to the definition of the covariant derivative given by

$$\nabla_c \mathfrak{T}^{a\cdots}_{b\cdots} = \partial_c \mathfrak{T}^{a\cdots}_{b\cdots} + \Gamma^a_{dc}\mathfrak{T}^{d\cdots}_{b\cdots} + \cdots - \Gamma^d_{bc}\mathfrak{T}^{a\cdots}_{d\cdots} - W\Gamma^d_{dc}\mathfrak{T}^{a\cdots}_{b\cdots}. \tag{A.8}$$

For example, the covariant derivative of a vector density of weight W is

$$\nabla_c \mathfrak{T}^a = \partial_c \mathfrak{T}^a + \Gamma^a_{bc}\mathfrak{T}^b - W\mathfrak{T}^a.$$

In the special case where $W = 1$ and we contract over a and c, we get the important result

$$\nabla_a \mathfrak{T}^a = \partial_a \mathfrak{T}^a. \tag{A.9}$$

A.1.3 Matrix Identities

Consider the matrix $A = (A_{ij})$. Denote by a the determinant of A and denote by $\mathscr{C}^{ij}$ the cofactor of A_{ij}. Let us fix i, and expand the determinant a by the ith row. Then

$$a = \sum_{j=1}^{n} A_{ij}\mathscr{C}^{ij} \tag{A.10}$$

where we have included summation for clarity. Note we would have

$$0 = \sum_{j=1}^{n} A_{ij}\mathscr{C}^{kj} \tag{A.11}$$

if $k \neq i$ as the resulting determinant would have two rows in common. It follows that the inverse (A^{ij}) is given by

$$(A^{ij}) = \frac{1}{a}(\mathscr{C}^{ij})^T = \frac{1}{a}(\mathscr{C}^{ji}) \tag{A.12}$$

If we partially differentiate both sides of (A.10) with respect to A_{ij}, we get

$$\frac{\partial a}{\partial A_{ij}} = \mathscr{C}^{ij}, \tag{A.13}$$

since A_{ij} does not occur in any of the cofactors $\mathscr{C}^{ji}$ (for fixed i, while j runs from 1 to n). This works for any i and so the formula (A.13) holds for all i and j. Using (A.12) in (A.13)

$$\begin{aligned} \frac{\partial a}{\partial A_{ij}} &= \mathscr{C}^{ij} \\ &= aA^{ji}. \end{aligned} \tag{A.14}$$

Now let us supose that the A_{ij} are all functions of the coordinates x^k. Then the determinant depends on A_{ij}, which in turn are functions of the coordinates x^k, that is,

$$a = a\left(A_{ij}(x^k)\right).$$

Differentiating with respect to x^k, we obtain

$$\begin{aligned} \frac{\partial a}{\partial x^k} &= \frac{\partial a}{\partial A_{ji}}\frac{\partial A_{ij}}{\partial x^k} \\ &= A^{ji}\frac{\partial A_{ij}}{\partial x^k} \\ &= aA^{ji}\frac{\partial A_{ij}}{\partial x^k} \end{aligned} \tag{A.15}$$

by (A.12).

A.1.4 Metric Identities

Applying (A.15) to the metric determinant g and using that g^{ab} is stmmetric, we obtain the equation

$$\partial_c g = g g^{ab} \partial_c g_{ab}. \tag{A.16}$$

We combine this with (A.5) and find

$$\begin{aligned} \partial_c g &= g g^{ab} \partial_c g_{ab} (\Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad}) \\ &= 2 g \Gamma^a_{ac}. \end{aligned} \tag{A.17}$$

Let us compute the covariant derivative of g using (A.8). Then since g is a scalkar density of weight +2, we have

$$\nabla_c g = \partial_c g - 2 g \Gamma^a_{ac}$$

and so by equation (A.17) it follows that

$$\nabla_c g \equiv 0. \tag{A.18}$$

Similarly, we find from equation (A.17) that

$$\begin{aligned} \partial_c (-g)^{\frac{1}{2}} &= \frac{1}{2} (-g)^{-\frac{1}{2}} \partial_c (-g) \\ &= (-g)^{\frac{1}{2}} \Gamma^a_{ac}, \end{aligned}$$

that is, by (A.8)

$$\nabla_c (-g)^{\frac{1}{2}} \equiv 0.. \tag{A.19}$$

In particular, for the tensor $T^{a\cdots}_{b\cdots}$, this leads to the identity

$$\nabla_c \left[(-g)^{\frac{1}{2}} T^{a\cdots}_{b\cdots} \right] = (-g)^{\frac{1}{2}} (\nabla_c T^{a\cdots}_{b\cdots}), \tag{A.20}$$

that is, one can move factors of $(-g)^{\frac{1}{2}}$ and g through covariant derivatives in the same way that one can with factors involving the covariant or contravariant metric.

A.1.5 Palatini Equation

Recall the Riemann tensor

$$R^a_{\ bcd} = \partial_c \Gamma^a_{\ bd} + \Gamma^e_{\ bd}\Gamma^a_{\ ec} - (c \leftrightarrow d) \tag{A.21}$$

The variation of the Riemann tensor with respect to the the connection (due to a variation in the connection due to a variation in the metric). Let us consider arbitrary variations of the connection to a new connection $\overline{\Gamma}^a_{\ bc}$:

$$\Gamma^a_{\ bc} \mapsto \overline{\Gamma}^a_{\ bc} = \Gamma^a_{\ bc} + \delta\Gamma^a_{\ bc}. \tag{A.22}$$

The variation $\delta\Gamma^a_{\ bc}$ is the difference of two connections, and is therefore itself a tensor. We can take its covariant derivative,

$$\nabla_c(\delta\Gamma^a_{\ bd}) = \partial_c(\delta\Gamma^a_{\ bd}) + \Gamma^a_{\ ec}(\delta\Gamma^e_{\ bd}) - \Gamma^e_{\ cb}(\delta\Gamma^a_{\ ed}) - \Gamma^e_{\ cd}(\delta\Gamma^a_{\ eb}) \tag{A.23}$$

We use this expression to calculate the first order variation in the Riemann tensor

$$\begin{aligned}
\delta R^a_{\ bcd} &= \partial_c(\delta\Gamma^a_{\ bd}) + \Gamma^a_{\ ec}(\delta\Gamma^e_{\ bd}) + \Gamma^e_{\ bd}(\delta\Gamma^a_{\ ec}) - (c \leftrightarrow d) \\
&= \partial_c(\delta\Gamma^a_{\ bd}) + \Gamma^a_{\ ec}(\delta\Gamma^e_{\ bd}) - \Gamma^e_{\ bc}(\delta\Gamma^a_{\ ed}) - (c \leftrightarrow d) \\
&= \partial_c(\delta\Gamma^a_{\ bd}) + \Gamma^a_{\ ec}(\delta\Gamma^e_{\ bd}) - \Gamma^e_{\ cb}(\delta\Gamma^a_{\ ed}) - \Gamma^e_{\ cd}(\delta\Gamma^a_{\ eb}) - (c \leftrightarrow d) \\
&= \nabla_c(\delta\Gamma^a_{\ db}) - \nabla_d(\delta\Gamma^a_{\ cb}).
\end{aligned}$$

This result,

$$\delta R^a_{\ bcd} = \nabla_c(\delta\Gamma^a_{\ db}) - \nabla_d(\delta\Gamma^a_{\ cb}). \tag{A.24}$$

is called the Palantini equation.

Contracting over a and c gives the result

$$\delta R_{bd} = \nabla_a\left(\delta\Gamma^a_{\ bd}\right) - \nabla_d\left(\delta\Gamma^a_{\ ba}\right) \tag{A.25}$$

A.2 The Einstein Lagrangian

The Einsten Lagrangian that gives the vacuum field equations for General Relativity are

$$\mathcal{L}_G = (-g)^{\frac{1}{2}} R \tag{A.26}$$

where the G denotes that it is the Lagrangian gravitational.

The variation

$$g_{ab} \mapsto g_{ab} + \delta g_{ab}$$

induces a variation in g^{ab}, which we write

$$g^{ab} \mapsto g^{ab} + \delta g^{ab}. \tag{A.27}$$

$$\begin{aligned} \delta^a_c &= g^{ab} g_{bc} \\ &\mapsto (g^{ab} + \delta g^{ab})(g_{bc} + \delta g_{bc}) \\ &= \delta^a_c + \delta g^{ab} g_{bc} + g^{ab} \delta g_{bc} + O(\delta^2) \end{aligned} \tag{A.28}$$

But since δ^a_c is a constant tensor, it remains unaltered by any variations and therefore

$$\delta g^{ab} g_{bc} + g^{ab} \delta g_{bc} = 0 \tag{A.29}$$

to first order. Multiplying through by g^{cd} we have

$$\delta g^{ad} = -g^{ab} g^{cd} \delta g_{bc}. \tag{A.30}$$

We will need the variation of g under the variation δg_{ab}. Consider, for an arbitrary matrix $\mathbf{M}$, the variation of $\ln|\det M|$ induced by a variation of $\mathbf{M}$'s elements,

$$\begin{aligned} \delta \ln|\det \mathbf{M}| &= \ln|\det(\mathbf{M} + \delta\mathbf{M})| - \ln|\det \mathbf{M}| \\ &= \ln \frac{\det(\mathbf{M} + \delta\mathbf{M})}{\det \mathbf{M}} \\ &= \ln \det \mathbf{M}^{-1}(\mathbf{M} + \delta\mathbf{M})| \\ &= \ln \det(\mathbf{1} + \mathbf{M}^{-1}\delta\mathbf{M})| \end{aligned} \tag{A.31}$$

Write $\mathbf{N} = \mathbf{1} + \epsilon$, then

$$\begin{aligned}
\det \mathbf{N} &= \epsilon^{ijkl} N_{0i} N_{1j} N_{2k} N_{3l} \\
&= \epsilon^{0123}(1+\varepsilon_{00})(1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33}) + \epsilon^{1023}\varepsilon_{01}\varepsilon_{10}(1+\varepsilon_{22})(1+\varepsilon_{33}) + \cdots \\
&= 1 + \mathrm{Tr}\varepsilon + O(\varepsilon^2)
\end{aligned}$$

Using this in (A.31)

$$\begin{aligned}
\delta \ln |det\mathbf{M}| &= \ln(1 + \mathrm{Tr}\ \mathbf{M}^{-1}\delta\mathbf{M}) \\
&= \mathrm{Tr}\ \mathbf{M}^{-1}\delta\mathbf{M}.
\end{aligned}$$

Substituting the metric tensor in place of $\mathbf{M}$ gives

$$\delta g = g g^{ab} \delta g_{ab}.$$

This is used to obtain the result

$$\begin{aligned}
\delta(-g)^{\frac{1}{2}} &= -\frac{1}{2(-g)^{\frac{1}{2}}}\delta g \\
&= -\frac{1}{2}\frac{g}{(-g)^{\frac{1}{2}}} g^{ab}\delta g_{ab} \\
&= \frac{1}{2}(-g)^{\frac{1}{2}} g^{ab}\delta g_{ab}.
\end{aligned} \tag{A.32}$$

We write the action as

$$S = \int_{\mathcal{M}} \mathfrak{g}^{ab} R_{ab}\ d^4x \tag{A.33}$$

A variation in the metric results in a variation in the Ricci tensor, so that

$$\delta S = \int_{\mathcal{M}} \left(\delta\mathfrak{g}^{ab} R_{ab} + \mathfrak{g}^{ab}\delta R_{ab}\right)\ d^4x \tag{A.34}$$

We now use the Palantini equation in the form (A.25), so that the second term on the RHS becomes

$$\begin{aligned}\int_{\mathcal{M}} \mathfrak{g}^{ab}\delta R_{ab}\, d^4x &= \int_{\mathcal{M}} \mathfrak{g}^{ab}\left[\nabla_c\left(\delta\Gamma^c_{ab}\right) - \nabla_b\left(\delta\Gamma^c_{ac}\right)\right]\, d^4x \\ &= \int_{\mathcal{M}} \left[\nabla_c\left(\mathfrak{g}^{ab}\delta\Gamma^c_{ab}\right) - \nabla_b\left(\mathfrak{g}^{ab}\delta\Gamma^c_{ac}\right)\right]\, d^4x \\ &= \int_{\mathcal{M}} \partial_c\left[\mathfrak{g}^{ab}\delta\Gamma^c_{ab} - \mathfrak{g}^{ac}\delta\Gamma^b_{ab}\right]\, d^4x \end{aligned} \tag{A.35}$$

Where we have used (A.9). This can be converted into a surface integral using the divergence theorem, which vanishes as we assume variations vanish there.

Hence (A.35) reduces to

$$\begin{aligned}\delta S &= \int_{\mathcal{M}} \delta\mathfrak{g}^{ab} R_{ab}\, d^4x \\ &= \int_{\mathcal{M}} R_{ab}\delta\left[(-g)^{\frac{1}{2}} g^{ab}\right] d^4x \\ &= \int_{\mathcal{M}} \left[R_{ab} g^{ab}\delta(-g)^{\frac{1}{2}} + R_{ab}(-g)^{\frac{1}{2}}\delta g^{ab}\right]\, d^4x \\ &= \int_{\mathcal{M}} (-g)^{\frac{1}{2}}\left(\frac{1}{2}R g^{cd} - R_{ab}g^{ac}g^{bd}\right)\delta g_{cd}\, d^4x \\ &= -\int_{\mathcal{M}} (-g)^{\frac{1}{2}}\left(R^{cd} - \frac{1}{2}R g^{cd}\right)\delta g_{cd}\, d^4x \\ &= -\int_{\mathcal{M}} \left[-(-g)^{\frac{1}{2}} G^{ab}\right]\delta g_{ab}\, d^4x \end{aligned} \tag{A.36}$$

where we have used (A.30) and (A.32).

A.3 The Full Field Equations - Inclusion of Matter and Radiation

The full action is the sum of the gravitational action plus the matter action

$$S\left[g_{ab}, \phi\right] = \int_{\mathscr{M}} \left(\mathscr{L}_G + \kappa\mathscr{L}_M\right) d^4x \tag{A.37}$$

where ϕ denotes the matter fields. Variation of this total action with respect to the matter fields ϕ is equivalent to variation of the matter action S_M alone with respect to the matter fiekds,

$$\frac{S[g_{ab},\phi]}{\delta\phi} = 0 \qquad \Leftrightarrow \qquad \frac{S_M[g_{ab},\phi]}{\delta\phi} = 0 \tag{A.38}$$

and will simply give rise to the equations of motion of the matter fields in the gravitational field.

Variation of the gravitational action with respect to the gravitational field will give rise to the gravitational part of the field equations, we obtain

$$\frac{\delta S_G}{\delta g_{ab}} = -(-g)^{\frac{1}{2}} G^{ab}. \tag{A.39}$$

Variation of the matter action with respect to the gravitational field will give us the source term for the graviational field equations provided by the matter fields

$$\frac{\delta S_M}{\delta g_{ab}} = (-g)^{\frac{1}{2}} T^{ab}, \tag{A.40}$$

where the latter equation defines the energy-momentum tensor for the fields present. Dividing through by $(-g)^{\frac{1}{2}}$, we have the full field equations

$$G^{ab} = \kappa T^{ab} \tag{A.41}$$

The above is precisely analogous to the way a source term for the Maxwell's equations, a current J^a, arises from a variation of the coupled matter-Maxwell action with respect to the gauge potential, see section A.5.3.

A.4 The Energy-Momentum Tensor for Dust, a Perfect Fluid

A.4.1 Source term: From Newton's Theory to Special Relativity

Let us first recall the non-relativist Newtonian theory for gravity in the pressence of matter. In terms of the gravitational potential defined by $\mathbf{F} = -\vec{\nabla}\varphi$ Newton's law of gravity in its differential form is

$$\vec{\nabla}^2\varphi = 4\pi G\rho \tag{A.42}$$

where ρ is the rest local density of matter and φ is the local gravitational potential (gravitational potential energy per unit mass). In order to see how ρ can be generalised in special relativity consider a dust cloud. A cloud in its rest frame has energy density

$$\rho_0 = m_0 n_0$$

where m_0 is the rest mass of a dust particle and n_0 the number of dust particles per unit volume. In special relativity, the mass of a body is greater than its rest mass by a factor of γ and the volume decreases by a factor of γ because of Lorentz contraction. Hence, the energy density as viewed by an observer with velocity $v = \beta$ with respect to the cloud is

$$\rho = \rho_0 \gamma^2.$$

Obviously ρ is neither a scalar nor the component of a four-vector. The behaviour of ρ is exactly that of the time-time component of a second-rank tensor T^{ab}:

$$T^{ab} = \rho_0 u^a u^b, \tag{A.43}$$

where

$$u^a = \gamma(1, \mathbf{u})$$

is the four-vector velocity of the cloud. In the rest frame of the cloud only the time-time component of the tensor is non-zero.

A.4.2 Dust

The simplest form of matter field, that of dust. Let us formulate the again the energy-momentum tensor, and interpret all its componenets. A dust field may be characterised two quantities, the 4-velocity vector field

$$u^a = \frac{dx^a}{d\tau}, \tag{A.44}$$

where τ is the proper time along the world-line of the dust particle and a scalar field

$$\rho_0 = \rho_0(x) \tag{A.45}$$

describing the proper density of the flow, that is, the density which would be measured by an observer moving with the flow. The simplest second-rank tensor we can construct from these two quantities is

$$T^{ab} = \rho_0 u^a u^b, \tag{A.46}$$

where u^a is the 4-velocity

$$u^a = \gamma(1, \mathbf{u}), \tag{A.47}$$

where $\gamma = (1 - u^2)^{-\frac{1}{2}}$. Recall that the proper time is defined by

$$\begin{aligned} d\tau^2 &= \eta_{ab} dx^a dx^b \\ &= dt^2 - dx^2 - dy^2 - dz^2 \\ &= dt^2(1 - u^2) \\ &= \gamma^2 dt^2 \end{aligned} \tag{A.48}$$

The zero-zero component of T^{ab}

$$\begin{aligned} T^{00} &= \rho_0 \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} \\ &= \rho_o \frac{dt^2}{d\tau^2} \\ &= \gamma^2 \rho_o, \end{aligned} \tag{A.49}$$

by (A.48). Therefore, the relativistic energy density of the matter field is the component T^{00} of the second rank tensor T^{ab}.

$$\rho = \gamma^2 \rho_0 \tag{A.50}$$

Using (A.47) and (A.50), the components of T^{ab} can be written in the form

$$T^{ab} = \rho \begin{pmatrix} 1 & u_x & u_y & u_z \\ u_x & u_x^2 & u_x u_y & u_x u_z \\ u_y & u_x u_y & u_y^2 & u_y u_z \\ u_z & u_x u_z & u_y u_z & u_z^2 \end{pmatrix} \tag{A.51}$$

This tensor has the interpretation of the flow of the a component of the four-momentum along the b direction. Note the following

(i) T^{00} is the energy density.

(ii) T^{0i} is the energy flux in the i direction.

(iii) T^{ii} is the flux of the i component of momentum in the i direction.

(iv) T^{ij} is the flux of the i component of momentum in the j direction.

(v) T^{i0} is the density of the i component of momentum.

This is summarised by:

$$T^{ab} = \left(\begin{array}{c|c} \text{Energy density} & \text{Energy flux} \\ \hline \text{Density of momentum} & \text{"stress"} \end{array}\right) \tag{A.52}$$

Equations of motion

We now show that the equations that govern the force free motion of a matter field of dust can be written in the following compat way:

$$\partial_b T^{ab} = 0. \tag{A.53}$$

Using (A.51), in the case of $a = 0$, this equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0.$$

This is the equation of continuity

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \mathbf{u}) = 0. \tag{A.54}$$

In fluid dynamics, this expresses the conservation of matter with density ρ moving with velocity $\mathbf{u}$. Since matter is the same as energy in special relativity, it follows that the conservation of energy equation for dust is $\partial_b T^{0b} = 0$. The equation corresponding to $a = \alpha$ ($\alpha = 1, 2, 3$) are

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \frac{\partial}{\partial x}(\rho u_x \mathbf{u}) + \frac{\partial}{\partial y}(\rho u_y \mathbf{u}) + \frac{\partial}{\partial z}(\rho u_z \mathbf{u}) = \mathbf{0}.$$

or

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \mathbf{u}) + \rho \frac{\partial \mathbf{u}}{\partial t} + \rho u_x \frac{\partial \mathbf{u}}{\partial x} + \rho u_y \frac{\partial \mathbf{u}}{\partial y} + \rho u_z \frac{\partial \mathbf{u}}{\partial z} = \mathbf{0}.$$

Combining this with (A.54), the above equation can be written

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u} \right] = \mathbf{0}. \tag{A.55}$$

Comparing this with the Navier-Stokes equation of motion for a perfect fluid in fluid dynamics, namely,

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u} \right] = -\text{grad}\ p + \rho\, \mathbf{f}, \tag{A.56}$$

where p is the pressure in the fluid and $\mathbf{f}$ is the body force per unit mass, we see that (A.55) is simply the Navier-Stokes equation in the absence of pressure or external forces.

The covariant counterpart of (A.53) is

$$\nabla_b T^{ab} = 0. \tag{A.57}$$

A.4.3 Dust Lagrangian

Dust cannot be considered a fundamental field. Nevertheless, it often provides an intersting matter source for model systems; developing its canonical formulation is thus worthwhile.

One can in fact introduce a dust Lagrangian

$$\mathfrak{L}_{dust} = (-g)^{\frac{1}{2}} \rho (g^{ab} u_a u_b - 1) \tag{A.58}$$

Variang with respect to ρ yields

$$\frac{1}{(-g)^{\frac{1}{2}}} \frac{\partial \mathfrak{L}_{dust}}{\partial \rho} = g^{ab} u_a u_b - 1 = 0 \tag{A.59}$$

so that $g^{ab} u_a u_b = 1$. Recall

$$\frac{\partial (-g)^{\frac{1}{2}}}{\partial g^{ab}} = -\frac{1}{2} (-g)^{\frac{1}{2}} g_{ab}$$

so that

$$\begin{aligned} T_{ab} &= \frac{1}{(-g)^{\frac{1}{2}}}\frac{\partial \mathfrak{L}_{dust}}{\partial g^{ab}} \\ &= -\frac{1}{2}g_{ab}\rho(g^{cd}u_c u_d - 1) + \rho u_a u_b \\ &= \rho u_a u_b \end{aligned} \tag{A.60}$$

where we have used $g^{ab}u_a u_b = 1$.

A.4.4 Perfect Fluid

A perfect fluid is characterised by three quantities:

i) a 4-velocity $u^a = dx^a/d\tau$,

ii) a proper density $\rho_0 = \rho_0(x)$,

iii) a scalar pressure field $p = p(x)$.

In the limit as p vanishes, a perfect fluid reduces to dust. This suggests that one takes the energy-momentum tensor for a perfect fluid to be of the form

$$T^{ab} = \rho_0 u^a u^b + pS^{ab} \tag{A.61}$$

for some symmetric tensor S^{ab}. The only second-rank tensors which are associated with the fluid are $u^a u^b$ and the metric g^{ab}, and so the simplest assumption is that

$$S^{ab} = \lambda u^a u^b + \mu g^{ab} \tag{A.62}$$

where λ and μ are constants.

We investigate the conservation law $\partial_b T^{ab} = 0$ in special relativity in Minkowski coordinates and require that it reduces in an appropriate limit to the continuity equation (A.54) and the Navier-Stokes equation (A.56) in the ansence of body forces.

It follows directly from the definition of the 4-velocity that

$$u^a u_a = 1$$

This implies

$$u_a \partial_b u^a = \frac{1}{2} \partial_b (u_a u^a) = 0. \tag{A.63}$$

In the non-relativistic limit,

$$\begin{aligned} p &\ll \rho, \\ u^a &= (1, \mathbf{u}), \\ |\mathbf{u}| \frac{dp}{dt} &\ll |\text{grad} p|, \end{aligned} \tag{A.64}$$

these equations become

$$\begin{aligned} 0 &= \nabla_b T^{ab} \\ &= \partial_b T^{ab} \\ &= \partial_b [\rho u^a u^b + p(\lambda u^a u^b + \mu \eta^{ab})] \\ &= u^a u^b \partial_b \rho + (\rho_0 + \lambda p) \partial_b (u^a u^b) + (\mu \eta^{ab} + \lambda u^a u^b) \partial_b p \end{aligned} \tag{A.65}$$

Projecting parallel to u_a:

$$\begin{aligned} u_a \partial_b T^{ab} &= u^a \partial_a \rho_0 + (\rho_0 + \lambda p) u_a \partial_b (u^a u^b) + u_a (\mu \eta^{ab} + \lambda u^a u^b) \partial_b p \\ &= u^a \partial_a \rho_0 + (\mu + \lambda) u^a \partial_a p + (\rho_0 + \lambda p)(\partial_a u^a + u^b u_a \partial_b u^a) \\ &= u^a \partial_a \rho_0 + (\mu + \lambda) u^a \partial_a p + (\rho_0 + \lambda p) \partial_a u^a \\ &= 0 \end{aligned} \tag{A.66}$$

This becomes

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \text{grad}) \rho + (\mu + \lambda) \left(\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \text{grad}) p \right) + (\rho_0 + \lambda p) \text{div } \mathbf{u} = 0. \tag{A.67}$$

Projecting perpendicular to u_a:

$$\begin{aligned} (\eta_{ac} - u_a u_c) \partial_b T^{cb} &= (\rho_0 + \lambda p)(\eta_{ac} - u_a u_c) \partial_b (u^c u^b) + \mu (\eta_{ac} - u_a u_c) \eta^{cb} \partial_b p \\ &= (\rho_0 + \lambda p)(\eta_{ac} - u_a u_c) u^b \partial_b u^c + (\delta^b_a - u_a u^b) \partial_b p \\ &= (\rho_0 + \lambda p) u^b \partial_b u_a + \mu \partial_a p - \mu u_a u^b \partial_b p \\ &= 0 \end{aligned} \tag{A.68}$$

For $a \neq 0$ equation (A.68) becomes

$$-(\rho_0 + \lambda p)\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u}\right) + \mu \text{grad} p + \mu \mathbf{u}\left(\frac{\partial p}{\partial t} + \mathbf{u} \cdot \text{grad} p\right) = 0 \tag{A.69}$$

In the non-relativistic limit given by (A.64), we have that (A.67) and (A.69) become, respectively,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) + (\mu + \lambda)\left(\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \text{grad}) p\right) &= 0, & \text{(A.70)} \\ \rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u}\right] &= \mu \text{grad}\ p. & \text{(A.71)} \end{aligned}$$

These reduce to the continuity equation and the Navier-Stokes equation in the absence of body forces if we take $\mu + \lambda = 0$ and $\mu = -1$. Using this in (A.62), then substituting this into (A.61) we have

$$T^{ab} = (\rho_0 + p)u^a u^b - p g^{ab} \tag{A.72}$$

A.5 The Energy-Momentum Tensor for the Electro-Magnetic Field

A.5.1 Reminder of Maxwell's Equations

The source equations are

$$\begin{aligned} \text{div}\ \mathbf{E} &= \rho & \text{(A.73)} \\ \text{curl}\ \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j}, & \text{(A.74)} \end{aligned}$$

and the internal equations

$$\begin{aligned} \text{div}\ \mathbf{B} &= \rho & \text{(A.75)} \\ \text{curl}\ \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0} & \text{(A.76)} \end{aligned}$$

where **E** is the electric field, **B** is the magnetic field. ρ is the charge density, and **j**. The quantities ρ and **j** cannot be prescribed independently because, differentiating (A.73) with respect to t, we get

$$\operatorname{div} \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \rho}{\partial t}, \tag{A.77}$$

and taking the divergence of (A.74) gives

$$\operatorname{div} \frac{\partial \mathbf{E}}{\partial t} = \operatorname{div} \mathbf{j}. \tag{A.78}$$

Thus, ρ and **j** must satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0. \tag{A.79}$$

In order to write these equations in tensorial form, one defines the anti-symmetric tensor F^{ab}, called the electromagnetic field strength tensor, by

$$\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{A.80}$$

and the current density or source 4-vector by

$$j^a = (\rho, \mathbf{j}). \tag{A.81}$$

Then the source equations and internal equations can be written in the form

$$\partial_b F^{ab} = j^a, \tag{A.82}$$

$$\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0. \tag{A.83}$$

The anti-symmetry of F_{ab} means that (A.83) can be written in the more compact form as

$$\partial_{[a} F_{bc]} = 0. \tag{A.84}$$

The continuity equation (A.79) becomes

$$\partial_a j^a = 0. \tag{A.85}$$

This obviously follows from (A.82) as $\partial_a \partial_b F^{ab} = \partial_a j^a = 0$

A.5.2 Potential Formulation of Maxwell's Equations

Instead of working with **E** and **B** directly, it is often more convenient to work with potentials. The scalar potential ϕ and the vector potential **A** are related to **E** and **B** by the equations

$$\mathbf{E} = -\text{grad}\ \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{A.86}$$

$$\mathbf{B} = \text{curl}\ \mathbf{A}. \tag{A.87}$$

The internal equation are autometically solved.

If we define the 4-potential by

$$A^a = (\phi, \mathbf{A}), \tag{A.88}$$

then (A.86) and (A.87) are equivalent to

$$F_{ab} = \partial_b A_a - \partial_a A_b \tag{A.89}$$

as we now verify. First

$$\begin{aligned} F_{01} &= \frac{\partial A_0}{\partial x} + \frac{\partial A_1}{\partial t} = -E_x \\ F_{02} &= \frac{\partial A_0}{\partial y} + \frac{\partial A_2}{\partial t} = -E_y \\ F_{03} &= \frac{\partial A_0}{\partial y} + \frac{\partial A_3}{\partial t} = -E_z \end{aligned}$$

then

$$\begin{aligned} F_{23} &= \frac{\partial(-A_2)}{\partial z} - \frac{\partial(-A_3)}{\partial y} = B_x \\ F_{13} &= \frac{\partial(-A_1)}{\partial z} - \frac{\partial(-A_3)}{\partial x} = -B_y \\ F_{12} &= \frac{\partial(-A_1)}{\partial y} - \frac{\partial(-A_2)}{\partial x} = B_z. \end{aligned}$$

The 4-potential is not uniquely defined since one can perform a gauge transformation

$$A_a \mapsto \tilde{A}_a = A_a + \partial_a \psi \tag{A.90}$$

where ψ is an arbitrary scalar field. The gauge transformations change the potentials, it leaves F_{ab}, and hence **E** and **B**, unaltered.

Transition to Covariant Formulism

To obtain the covariant formulation, we simply replace ordinary derivatives by covariant derivatives. However (A.84) and (A.89)

$$\nabla_{[a} F_{bc]} = \partial_{[a} F_{bc]} \tag{A.91}$$

and

$$\nabla_{[b} A_{a]} = \partial_{[b} A_{a]} \tag{A.92}$$

as

$$\begin{aligned}
\nabla_{[a} F_{bc]} &= \nabla_a F_{bc} + \nabla_c F_{ab} + \nabla_b F_{ca} \\
&= \partial_a F_{bc} - \Gamma^d_{ba} F_{dc} - \Gamma^d_{ca} F_{bd} + \\
&+ \partial_c F_{ab} - \Gamma^d_{ac} F_{db} - \Gamma^d_{bc} F_{ad} \\
&+ \partial_b F_{ca} - \Gamma^d_{cb} F_{da} - \Gamma^d_{ab} F_{cd} \\
&= \partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} \\
&= \partial_{[a} F_{bc]},
\end{aligned}$$

and

$$\begin{aligned}
2\nabla_{[b} A_{a]} &= \partial_b A_a - \Gamma^c_{ab} A_c - (\partial_a A_b - \Gamma^c_{ba} A_c) \\
&= 2\partial_{[b} A_{a]}.
\end{aligned} \tag{A.93}$$

The covariant formulation of Maxwell's equations in special relativity are

$$\nabla_b F^{ab} = j^a \tag{A.94}$$

$$\partial_{[a} F_{bc]} = 0 \tag{A.95}$$

subject to

$$\nabla_a j^a = 0. \tag{A.96}$$

In terms of the 4-potetial, we still have

$$F_{ab} = \partial_b A_a - \partial_a A_b. \tag{A.97}$$

By the principle of minimal gravitational coupling, one adopts equations (A.94) and (A.95) in general relativity, where, however, the metric is no longer flat but a solution of the full field equations $G^{ab} = \kappa T^{ab}$ where T^{ab} is the energy-momentum tensor of the electromagnetic field.

A.5.3 Action for Maxwell Equations and Minimal Coupling to Charged Matter

Consider the Lagragian for the electrodynamical field defined by

$$\mathscr{L}_{Max}[A_a, F_{ab}] = \frac{1}{4\pi}\left[-\frac{1}{2}F_{ab}F^{ab} + (A_{a,b} - A_{b,a})F^{ab}\right] \tag{A.98}$$

(Max stands for Maxwell, obviously). Then

$$\begin{aligned} \frac{\delta \mathscr{L}_{Max}}{\delta A_a} &= \frac{\partial \mathscr{L}_E}{\partial A_a} - \left(\frac{\partial \mathscr{L}_E}{\partial A_{a,b}}\right)_{,b} \\ &= 0 - \frac{1}{4\pi}(F^{ab} - F^{ba})_{,b} \end{aligned} \tag{A.99}$$

and the field equations corresponding to a variation with respect to ϕ_a become

$$(F^{ab} - F^{ba})_{,b} = 0. \tag{A.100}$$

Similarly,

$$\begin{aligned}\frac{\delta \mathscr{L}_{Max}}{\delta F_{ab}} &= \frac{\partial \mathscr{L}_{Max}}{\partial F_{ab}} \\ &= \frac{\partial}{\partial F_{ab}} \frac{1}{4\pi}\left[-\frac{1}{2}\eta^{ce}\eta^{df}F_{cd}F_{ef} + \eta^{ce}\eta^{df}(A_{c,d} - A_{d,c})F_{ef}\right] \\ &= \frac{1}{4\pi}\left[-\frac{1}{2}\eta^{ae}\eta^{bf}F_{ef} - \frac{1}{2}\eta^{ca}\eta^{db}F_{cd} + \eta^{ca}\eta^{db}(A_{c,d} - A_{d,c})\right] \\ &= \frac{\eta^{ac}\eta^{bd}}{4\pi}\left[-F_{cd} + (A_{c,d} - A_{d,c})\right]\end{aligned} \tag{A.101}$$

and so the field equations corresponding to a variation with respect to F_{ab} are

$$F_{ab} = \partial_b A_a - \partial_a A_b. \tag{A.102}$$

This equation defines F_{ab} in terms of the 4-potential and implies that F_{ab} is anti-symmetric. The definition also means that the internal equations are automatically satisfied and (A.100) reduces to

$$\partial_a F^{ab} = \partial^2 A^b - \partial^b (A_a A^a) = 0, \tag{A.103}$$

namely, the source equations (for source-free regions). Equation (A.102) also allows one to re-express the Lagrangian as

$$\mathscr{L}_{Max} = \frac{1}{8\pi}\eta^{ac}\eta^{bd}F_{ab}F_{cd} \tag{A.104}$$

In order to add sources, one can add $\int (-g)^{\frac{1}{2}} A_a j^a d^4x$ to the Maxwell action, $S_{Max}[A^a]$, thus coupling the charged matter currect to the Maxwell gauge field. What is producing the current j^a? A possibility the presence of a charged scalar matter field, which we denote by ϕ

If we replace the action for the matter field in the absence of an electrmagnetic field with the action for the matter field coupled to the gaug potential,

$$S_M[\phi] \rightarrow S_M[\phi, A^a] \tag{A.105}$$

the combined Maxwell and matter action

$$S[A^a, \phi] = S_{Max}[A^a] + S_M[\phi, A^a] \tag{A.106}$$

will then give rise to the Maxwell equations with source:

$$\partial_a F^{ab} = \partial^2(A^b) - \partial^b(A_a A^a) = j^b$$

provided that we define the current j^a as the variation of the matter action with respect to the gauge field,

$$j^a \propto \frac{\delta S_M[\phi, A^a]}{\delta A_a}. \tag{A.107}$$

Variation of the total action with respect to the scalar field ϕ is equivalent to variation of the matter action S_M alone with respect to the matter fields

$$\frac{S[A^a, \phi]}{\delta\phi} = 0 \qquad \Leftrightarrow \qquad \frac{S_M[A^a, \phi]}{\delta\phi} = 0 \tag{A.108}$$

and simply give rise to the equations of motion of the matter field in the Maxwell field

This is the promised analogue to the procedure we followed when obtaining the full field equations of general relativity in section A.3.

A.5.4 The Maxwell Energy-momentum Tensor

To make the transition to the full theory we ussume that

$$\mathscr{L}_E = \frac{(-g)^{\frac{1}{2}}}{8\pi} g^{ac} g^{bd} F_{ab} F_{cd} \tag{A.109}$$

together with the definition (A.102) of F_{ab} in terms of ϕ_a. The factor $(-g)^{\frac{1}{2}}$ is included to ensure that $\mathscr{L}_E$ is a scalar density.

We will need

$$\frac{\partial g}{\partial g^{ab}} = -g g_{ab} \quad \Rightarrow \quad \frac{\partial(-g)^{\frac{1}{2}}}{\partial g^{ab}} = -\frac{1}{2}(-g)^{\frac{1}{2}} g_{ab}$$

Using this we obtain

$$\begin{aligned}\frac{\partial \mathscr{L}_E}{\partial g^{ab}} &= \frac{\partial}{\partial g^{ab}}\left[\frac{(-g)^{\frac{1}{2}}}{8\pi} g^{ce} g^{df} F_{cd} F_{ef}\right] \\ &= \frac{1}{8\pi}\left[\frac{1}{2}(-g)^{\frac{1}{2}} g_{ab} g^{ce} g^{df} F_{cd} F_{ef} + (-g)^{\frac{1}{2}} g^{cd} F_{ac} F_{bd} + (-g)^{\frac{1}{2}} g^{cd} F_{ca} F_{db}\right] \\ &= -\frac{(-g)^{\frac{1}{2}}}{4\pi}\left[-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}\right]\end{aligned}$$

so that

$$\frac{\partial \mathscr{L}_E}{\partial g^{ab}} = -\frac{(-g)^{\frac{1}{2}}}{4\pi}\left[-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}\right] \tag{A.110}$$

The analogue of (A.40) for the contravariant metric is

$$\frac{\delta \mathscr{L}_E}{\delta g^{ab}} = -(-g)^{\frac{1}{2}} T_{ab}. \tag{A.111}$$

These last two equations lead to the definition of the Maxwell energy-momentum tensor T_{ab} in source-free regions

$$T_{ab} = \frac{1}{4\pi}(-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}) \tag{A.112}$$

Let us look at the interpretation of the components of T^{ab} in special relativity. Note that the T^{00} componentis

$$T^{ab} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2), \tag{A.113}$$

which is the usual expressionfor the energt density. The components

$$(T_{01}, T_{0,2}, T_{0,3}) = -\frac{1}{4\pi}\mathbf{E} \times \mathbf{B}, \tag{A.114}$$

where the vector $\mathbf{E} \times \mathbf{B}$ is the Poynting vector of electrodynamics and represents the momentum density of the electrodynamic field.

We have that

$$T_{ab} = \left(\begin{array}{c|c} \text{Energy density} & \text{Energy flux} \\ \hline \text{Energy flux} & \text{"stress"} \end{array}\right) \tag{A.115}$$

We don't compute the "stress" part of the tensor. We will just mention that it can be thouhgt of as the pressure of the electrodynamical field.

A.6 Einstein-Maxwell Equations

The full field equations in source-free regions are called the Einstein-Maxwell equations which read

$$G_{ab} = -2g^{cd}F_{ac}F_{bd} + \frac{1}{2}g_{ab}F_{cd}F^{cd}. \tag{A.116}$$