Notes on Covariant Loop Quantum Gravity

Contents

List of Figures

Preface

Acknowledgments

Introduction

These are companion notes for the book [1]

Chapter 1

Spacetime as a quantum object

1.1 Geometry

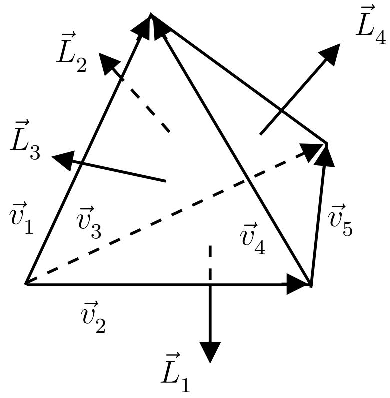

Figure 1.1: Four normals $\vec{L}_a$ to the faces.

The vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ define the tetrahedron. The vectors $\vec{v}_4$ and $\vec{v}_5$ can be given by

$$\begin{aligned} \vec{v}_4 &= \vec{v}_3 - \vec{v}_1 \\ \vec{v}_5 &= \vec{v}_2 - \vec{v}_1 \end{aligned} \tag{1.1}$$

$$
\begin{aligned}
\vec{L}_1 &= -\frac{1}{2}\vec{v}_2 \times \vec{v}_3 \\
\vec{L}_2 &= -\frac{1}{2}\vec{v}_3 \times \vec{v}_1 \\
\vec{L}_3 &= -\frac{1}{2}\vec{v}_1 \times \vec{v}_2 \\
\vec{L}_4 &= -\frac{1}{2}\vec{v}_4 \times \vec{v}_5 \\
&= -\frac{1}{2}(\vec{v}_3 - \vec{v}_1) \times (\vec{v}_2 - \vec{v}_1)
\end{aligned} \tag{1.2}
$$

(a) Closure.

$$
\begin{aligned}
\vec{C} &:= \sum_{a=1}^{4} \vec{L}_a \\
&= -\frac{1}{2}\Big(\vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1 + \vec{v}_1 \times \vec{v}_2 \\
&\qquad + \vec{v}_3 \times \vec{v}_2 - \vec{v}_3 \times \vec{v}_1 - \vec{v}_1 \times \vec{v}_2\Big) \\
&= 0.
\end{aligned} \tag{1.3}
$$

(b) (i) Areas.

The area of a triangle is

$$A = \frac{1}{2}|\vec{A} \times \vec{B}|$$

so the area of the $i-th$ face is

$$A_i = |\vec{L}_i|.$$

(ii) Volume.

In terms of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ the volume is

$$V = \frac{1}{6}\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \tag{1.4}$$

Consider the cross product

$$
\begin{aligned}
\vec{L}_1 \times \vec{L}_2 &= \frac{1}{4}(\vec{v}_2 \times \vec{v}_3) \times (\vec{v}_3 \times \vec{v}_1) \\
&= \frac{1}{4}[\vec{v}_3(\vec{v}_2 \times \vec{v}_3) \cdot \vec{v}_1 - \vec{v}_1(\vec{v}_2 \times \vec{v}_3) \cdot \vec{v}_3] \\
&= -\frac{3}{2} V \vec{v}_3 \qquad (1.5)
\end{aligned}
$$

where we used the identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{C}(\vec{A} \cdot \vec{B}) - \vec{B}(\vec{A} \cdot \vec{C})$ and (1.4). Now

$$
\begin{aligned}
(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 &= \frac{3}{4} V \vec{v}_3 \cdot (\vec{v}_1 \times \vec{v}_2) \\
&= \frac{9}{2} V \frac{1}{6} \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \\
&= \frac{9}{2} V^2. \qquad (1.6)
\end{aligned}
$$

So that

$$
V^2 = \frac{2}{9}(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 = \frac{2}{9}\epsilon_{ijk} L_1^i L_2^j L_3^k = \frac{2}{9} \det L. \qquad (1.7)
$$

(iii) Angles between edges.

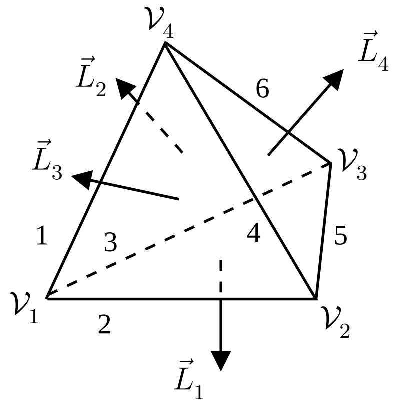

Figure 1.2: Edges.

As with (1.5) we can derive

$$
\begin{aligned}
\vec{L}_1 \times \vec{L}_2 &= -\frac{3}{2} V \vec{v}_3 \\
\vec{L}_3 \times \vec{L}_1 &= -\frac{3}{2} V \vec{v}_2 \\
\vec{L}_2 \times \vec{L}_3 &= -\frac{3}{2} V \vec{v}_1
\end{aligned} \tag{1.8}
$$

Therefore we can write for the angles at vertex $\mathcal{V}_1$

$$
\begin{aligned}
\cos\theta_{12} &= \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{(\vec{L}_2 \times \vec{L}_3) \cdot (\vec{L}_3 \times \vec{L}_1)}{|\vec{L}_2 \times \vec{L}_3||\vec{L}_3 \times \vec{L}_1|} \\
\cos\theta_{23} &= \frac{\vec{v}_2 \cdot \vec{v}_3}{|\vec{v}_2||\vec{v}_3|} = \frac{(\vec{L}_3 \times \vec{L}_1) \cdot (\vec{L}_1 \times \vec{L}_2)}{|\vec{L}_3 \times \vec{L}_1||\vec{L}_1 \times \vec{L}_2|} \\
\cos\theta_{31} &= \frac{\vec{v}_3 \cdot \vec{v}_1}{|\vec{v}_3||\vec{v}_1|} = \frac{(\vec{L}_1 \times \vec{L}_2) \cdot (\vec{L}_2 \times \vec{L}_3)}{|\vec{L}_1 \times \vec{L}_2||\vec{L}_2 \times \vec{L}_3|}
\end{aligned} \tag{1.9}
$$

By symmetry at vertex $\mathcal{V}_2$ $(123 \rightarrow 254)$ we have

$$
\begin{aligned}
\cos\theta_{25} &= \frac{(\vec{L}_5 \times \vec{L}_4) \cdot (\vec{L}_4 \times \vec{L}_2)}{|\vec{L}_5 \times \vec{L}_4||\vec{L}_4 \times \vec{L}_2|} \\
\cos\theta_{54} &= \frac{(\vec{L}_4 \times \vec{L}_2) \cdot (\vec{L}_2 \times \vec{L}_5)}{|\vec{L}_4 \times \vec{L}_2||\vec{L}_2 \times \vec{L}_5|} \\
\cos\theta_{42} &= \frac{(\vec{L}_2 \times \vec{L}_5) \cdot (\vec{L}_5 \times \vec{L}_4)}{|\vec{L}_2 \times \vec{L}_5||\vec{L}_5 \times \vec{L}_4|}
\end{aligned} \tag{1.10}
$$

Similar results hold for vertices $\mathcal{V}_3$ and $\mathcal{V}_4$.

1.2 Complements

1.2.1 Vector Spaces

A vector space over a field F (for example real or complex numbers) is a set V together with two operations that satisfy the requirements listed below.

Vectors obey the following rules:

(i) Associativty of addition: $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$;

(ii) Commutativity of addition: $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$;

(iii) There exist an identity element of addition: $\mathbf{0}\in V$, called the zero vector, such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in V$;

(iv) Inverse element of addition: For every $\mathbf{v}\in V$, there exists an element $-\mathbf{v}\in V$, called the addative inverse of $\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$;

(v) Identity element of scalar multiplication: $1\mathbf{v}=\mathbf{v}$, where 1 denotes the multiplicative identity in the field F;

(vi) Distributivity of scalar multiplication with respect to vector addition: $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+b\mathbf{v}$;

(vii) Distributivity of scalar multiplication with respect to field addition: $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$;

Examples

First example

The vector space $v^{A_1A_2}$ has vectors v^{00}, v^{01}, v^{10} and v^{11}. Then the vector space $z^{A_1A_2}$ of 2−index spinors have "vectors" $z^{00}=v^{00}$, $z^{01}=\frac{1}{2}(v^{01}+v^{10})$ and $z^{11}=v^{11}$.

(i)

(ii)

(iii) Define $0^{A_1A_2}$ as $0^{00}=0$, $0^{01}=\frac{1}{2}(0+0)$, and $0^{11}=0$.

(iv)

Second example

Consider the space of completely symmetric $n-$index spinors

$$z^{A_1\dots A_n}=z^{(A_1\dots A_n)} \tag{1.11}$$

(i) $(z_1^{(A_1\dots A_n)}+z_2^{(A_1\dots A_n)})+z_3^{(A_1\dots A_n)}=z_1^{(A_1\dots A_n)}+(z_2^{(A_1\dots A_n)}+z_3^{(A_1\dots A_n)})$;

(ii) $z_1^{(A_1\dots A_n)}+z_2^{(A_1\dots A_n)}=z_2^{(A_1\dots A_n)}+z_1^{(A_1\dots A_n)}$

(iii) Define $0^{(A_1\dots A_n)}=0$ for all values of A_1 to A_n, then $z^{(A_1\dots A_n)}+0^{(A_1\dots A_n)}=z^{(A_1\dots A_n)}$

(iv)

1.2.2 Matrix representation from a Hilbert space

The number of linearly independent vectors of a Hilbert space is the dimensionality of the space, which in general is infinite.

Let us assume there exists a set of linearly independent vectors $\{\psi_\nu(x)\}$ in a Hilbert space such that any vector $\phi(x)$ can be expressed as a linear combination of the set $\{\psi_\nu(x)\}$:

$$\phi(x) = \sum_\nu \psi_\nu(x)\phi_\nu. \tag{1.12}$$

Hilbert space is "compact"?

Orthogonalisation:

If the complete set of vectors $\{\psi_\nu(x)\}$ is not orthogonal, it may be repaced by an orthogonl set $\{\varphi_\nu(x)\}$:

From this point on we assume we have complete set of orthogonal vectors in a Hilbert space. If one takes a comlete orthogonal and normalised set $\{\psi_\nu(x)\}$ as a basis of the space.

Let $\hat{A}$ be an operator on a Hilbert space

$$\hat{A}\psi_\nu(x) = \sum_\mu \psi_\mu(x)M(\hat{A})_{\mu\nu}. \tag{1.13}$$

or

$$\hat{A}\phi(x) = \sum_\mu \phi(x)M(\hat{A}). \tag{1.14}$$

This form is convenient for the consruction of the matrix representative $M(hatA)$.

From (1.13) it follows thta the matrix representaive of the product of two operators A and B is given by the corresponding matrix representatives,

$$M(\hat{A}\hat{B}) = M(\hat{A})M(\hat{B}). \tag{1.15}$$

which follows from

$$\hat{A}\hat{B}\psi_\nu(x) = \sum_\mu \psi_\mu M(\hat{A}\hat{B})_{\mu\nu}$$

and

$$\begin{aligned}
\hat{A}\hat{B}\psi_\nu &= \hat{A}\sum_\sigma \psi_\sigma(x) M(\hat{B})_{\sigma\nu} \\
&= \sum_\sigma \left(\hat{A}\psi_\sigma(x)\right) M(\hat{B})_{\sigma\nu} \\
&= \sum_\sigma \left(\sum_\mu \psi_\mu(x) M(\hat{A})_{\mu\sigma}\right) M(\hat{B})_{\sigma\nu} \\
&= \sum_\mu \psi_\mu(x) \left(\sum_\sigma M(\hat{A})_{\mu\sigma} M(\hat{B})_{\sigma\nu}\right) \qquad (1.16)
\end{aligned}$$

or

$$\hat{A}\hat{B}\phi = \phi M(\hat{A}\hat{B})$$

and

$$\begin{aligned}
\hat{A}\hat{B}\phi &= \hat{A}(\hat{B}\phi) \\
&= \hat{A}\phi M(\hat{B}) \\
&= \phi M(\hat{A}) M(\hat{B}). \qquad (1.17)
\end{aligned}$$

Obviously the identity operator $\hat{1}$ is represnted by the unit matrix $\mathbf{1}$.

If there exists the inverse operator $\hat{A}^{-1}$ such that

$$\hat{A}\hat{A}^{-1} = \hat{A}^{-1}\hat{A} = \hat{1} \qquad (1.18)$$

we have the matrix representatives of these are

$$M(\hat{A})M(\hat{A}^{-1}) = M(\hat{A}^{-1})M(\hat{A}) = \mathbf{1} \qquad (1.19)$$

If the basis is orthonormalised,

$$< \phi, \phi >= \mathbf{1} \qquad < \psi_\mu, \psi_\nu >= \delta_{\mu\nu} \qquad \text{for all } \mu, \nu \qquad (1.20)$$

then the matrix representation is given directly by the scalar product

$$\begin{aligned} <\psi_\mu, \hat{A}\psi_\nu> &= <\psi_\mu, \sum_\mu \psi_\sigma M(\hat{A})_{\sigma\nu}> \\ &= \sum_\mu <\psi_\mu, \psi_\sigma> M(\hat{A})_{\sigma\nu} \\ &= M(\hat{A})_{\mu\nu} \end{aligned} \tag{1.21}$$

or rewritten

$$<\phi, \hat{A}\phi>= M(\hat{A}). \tag{1.22}$$

The adjoint of an operator is defined by

$$<\phi, \hat{A}^\dagger\varphi> = <\hat{A}\phi, \varphi> \tag{1.23}$$

then

$$\begin{aligned} <\psi_\mu, \hat{A}^\dagger\psi_\nu> &= <\hat{A}\psi_\mu, \psi_\nu> \\ &= <\psi_\nu, \hat{A}\psi_\mu>^* \\ &= M(\hat{A})^*_{\nu\mu} \end{aligned} \tag{1.24}$$

From this and (1.21), we see that if $\hat{A}$ is Hermitian then so is its matrix representative:

$$M(\hat{A}^\dagger) = M(\hat{A}). \tag{1.25}$$

Also, if $\hat{A}$ is unitary, that is it satisfies $\hat{A}^{-1} = \hat{A}^\dagger$, then so is $M(\hat{A})$:

$$M(\hat{A})^{-1} = M(\hat{A})^\dagger. \tag{1.26}$$

1.2.3 Unitary group representation

The operators $\hat{A}, \hat{B}, \hat{C}, \ldots$ acting on a Hilbert space form a group if the following four properties hold:

i) Closure: if $\hat{A}$ and $\hat{B}$ belong to G, their product $\hat{A}\hat{B}$ also belongs to G.

ii) The associative law holds: $(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})$ for every $\hat{A}, \hat{B}$, and $\hat{C}$ in G.

iii) There is an identity element $\hat{1}$ in G such that

$$\hat{1}\hat{A} = \hat{A}\hat{1} = \hat{A}.$$

iv) For every $\hat{A} \in G$, there is an inverse $\hat{A}^{-1} \in G$ such that

$$\hat{A}\hat{A}^{-1} = \hat{A}^{-1}\hat{A} = \hat{1}.$$

Furthermore, the group is unitary if $\hat{A}^{\dagger} = \hat{A}^{-1}$.

There matrix representatives form, a possible infinite dimensional, (unitary) representaion of the group.

1.2.4 Sesquilinear form

A bilinear form such that

(i) $F(\alpha u + \beta v, w) = \alpha^* F(u, w) + \beta^* F(u, w)$

(ii) $F(u, \alpha v + \beta w) = \alpha F(u, v) + \beta F(u, w)$.

An inner product is a sesquilinear form

1.2.5 $SU(2)$ representations and spinors

$$U = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \tag{1.27}$$

where

$$|a|^2 + |b|^2 = 1. \tag{1.28}$$

The matrix elements are written as

$$U^A{}_B$$

with indices $A, B, C, \ldots$ taking the values $0, 1$.

Measure

Equation (1.28) defines a sphere of unit radius in $\mathbb{C}^2 \sim \mathbb{R}^4$.

Representations of a group

If g_i is a member of a group G, then the object $D(g_i)$ is called a representation of G if it obeys:

$$D(g_i)D(g_j) = D(g_i g_j) \tag{1.29}$$

for all elements of the group.

A representation is called reducible if $D(g_i)$ can be brought into block diagonal form; for example the following matrix is reducible representation:

$$D(g_i) = \begin{pmatrix} D_1(g_i) & 0 & 0 \\ 0 & D_2(g_i) & 0 \\ 0 & 0 & D_3(g_i) \end{pmatrix} \tag{1.30}$$

One simple way of generating higher representations of a group is simply to multily several vectors (or spinors) together. The product $u^i v^j$, for example transforms as follows:

$$(u^{A'_1} v^{B'_1}) = [U^{A'_1}{}_{A_1}(g_j) U^{B'_1}{}_{B_1}(g_j)](u^{A_1} v^{B_1}) \tag{1.31}$$

We have the multiplication rule:

$$\begin{aligned}
(u^{A''_1} v^{A''_2}) &= [U^{A''_1}{}_{A'_1}(g') U^{A''_2}{}_{A'_2}(g')]\ [U^{A'_1}{}_{A_1}(g) U^{A'_2}{}_{A_2}(g)](u^{A_1} v^{A_2}) \\
&= \left(U^{A''_1}{}_{A'_1}(g') U^{A'_1}{}_{A_1}(g) u^{A_1}\right) \left(U^{A''_2}{}_{A'_2}(g') U^{A'_2}{}_{A_2}(g) v^{B_1}\right) \\
&= \left(U^{A''_1}{}_{A_1}(g'g) u^{A_1}\right) \left(U^{A'_2}{}_{A_2}(g'g) v^{A_2}\right) \\
&= [U^{A''_1}{}_{A_1}(g'g) U^{A'_2}{}_{A_2}(g'g)](u^{A_1} v^{A_2})
\end{aligned} \tag{1.32}$$

so the matrix $U^{A'_1}{}_{A_1}(g) U^{B'_1}{}_{B_1}(g)$ forms a representation of the group.

We can consider arbitrary product of vectors (or spinors). Then the matrix

$$U^{A'_1, A'_2, \cdots, A'_n}{}_{A_1, A_2, \cdots, A_n}(g) := U^{A'_1}{}_{A_1}(g) U^{A'_2}{}_{A_2}(g) \cdots U^{A'_n}{}_{A_n}(g) \tag{1.33}$$

forms a representation of the group as it has the same multiplication rule as the representation $U^{A_1'}_{\ A_1}(g)$.

These products of vector or spinors are, in general reducible; that is, here are subsets of the product that transform among themselves. By taking appropriate combinations, for example the symmetric combinations of the indices, we can extract irreducible representations.

Representations and spin

The vector space of the completely symmetric $n-$index spinors

$$z^{A_1 \dots A_n} = z^{(A_1 \dots A_n)} \tag{1.34}$$

transform under the action of $SU(2)$,

$$z^{A_1' \dots A_n'} = U^{A_1'}_{\ A_1} \dots U^{A_n'}_{\ A_n} z^{A_1 \dots A_n} \tag{1.35}$$

This space transforms into itself under the action of $SU(2)$ on all of the indices and therefore defines a representation of the group. This space is denoted $\mathcal{H}_j$, where $j = n/2$ (so that $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$) and the representation it defines is called the spin$-j$ representation. Thie representation is irreduciable.

$$U^{A_1'}_{\ A_1} U^{A_2'}_{\ A_2} \dots U^{A_n'}_{\ A_n} z^{A_1 A_2 \dots A_n} \quad = \quad U^{A_2'}_{\ A_2} \dots U^{A_n'}_{\ A_n} (U^{A_1'}_{\ 0} z^{0 A_2 \dots A_n} + U^{A_1'}_{\ 1} z^{1 A_2 \dots A_n}), \tag{1.36}$$

Exercise

Show that $\mathcal{H}_j$ has dimension $2j + 1$.

Proof:

The independent vectors of an $n-$index symmetric spinors are

$$z^{(000 \dots 0)}, \ z^{(100 \dots 0)} \ z^{(110 \dots 0)}, \ \dots \ , z^{(111 \dots 1)} \tag{1.37}$$

of which there are obviously $n + 1$ or $2j + 1$.

□

Part (1)

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.38}$$

Show that

(1) $\epsilon_{AC}\epsilon^{CB} = -\delta^B_A$, $\epsilon_{BA}\epsilon^{AB} = -2$, $\epsilon_{AB}\epsilon^{AB} = 2$.

(2) ϵ^{AB} is invariant under the action of $SU(2)$, i.e.

$$U^A{}_C U^B{}_D \epsilon^{CD} = \epsilon^{AB} \tag{1.39}$$

(3) $\det U = \epsilon^{BD} U^0{}_B U^1{}_D = \frac{1}{2}\epsilon_{AC}\epsilon^{BD} U^A{}_B U^C{}_D = 1$.

(4) $U^{-1} = -\epsilon U \epsilon$; that is, $(U^{-1})^A{}_B = -\epsilon_{BD} U^D{}_C \epsilon^{CA}$.

Proof:

Part (1.1)

$$\begin{aligned} \epsilon_{AC}\epsilon^{CB} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -\delta^B_A. \end{aligned} \tag{1.40}$$

Put $A = B$ and sum,

$$\epsilon_{BC}\epsilon^{CB} = -\delta^B_B = -2. \tag{1.41}$$

As $\epsilon_{AB} = -\epsilon_{BA}$ we obviously have $\epsilon_{AB}\epsilon^{AB} = 2$.

Part (1.2)

$$\begin{aligned} U^A{}_C U^B{}_D \epsilon^{CD} &= -U^A{}_C U^B{}_D \epsilon^{DC} \\ &= -U^A{}_C U^B{}_D \epsilon^{CD} \end{aligned} \tag{1.42}$$

As $U^A{}_C U^B{}_D \epsilon^{CD}$ is anisymmetric in A and B, it has only two non-zero components, and one is the negative of the other, so it must be of the form $Const.\ \epsilon^{AB}$. The Const. can be found from $Const. = U^0{}_C U^1{}_D \epsilon^{CD}$ which from part (1.3) is just the determinant which for $SU(2)$ is 1.

Part (1.3)

$$
\begin{aligned}
\epsilon^{BD}U^0{}_B U^1{}_D &= \epsilon^{01}U^0{}_0 U^1{}_1 + \epsilon^{10}U^0{}_1 U^1{}_0 \\
&= U^0{}_0 U^1{}_1 - U^0{}_1 U^1{}_0 \\
&:= \det U \qquad (1.43)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\epsilon_{AC}\epsilon^{BD}U^A{}_B U^C{}_D &= \frac{1}{2}\epsilon^{BD}(U^0{}_B U^1{}_D - U^1{}_B U^0{}_D) \\
&= \frac{1}{2}(\epsilon^{BD}U^0{}_B U^1{}_D + \epsilon^{DB}U^0{}_D U^1{}_B) \\
&= \det U \qquad (1.44)
\end{aligned}
$$

Part (1.4)

$$
\begin{aligned}
-\epsilon U \epsilon &= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{b} & a \\ -\overline{a} & b \end{pmatrix} \\
&= \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix} = U^\dagger = U^{-1}. \qquad (1.45)
\end{aligned}
$$

$\square$

Part 2

$\square$

Part 3

A two component spinor z^{AB} can be decomposed into its symmetric and antisymmetric parts

$$
z^{AB} = z_0\epsilon^{AB} + z_1^{AB}, \qquad z_0 = \frac{1}{2}z^A{}_A, \quad z_1^{AB} = z^{(AB)}, \qquad (1.46)
$$

which are invariant under the action of $SU(2)$.

Proof:

We write

$$
\begin{aligned}
z^{AB} &= \frac{1}{2}(z^{AB} - z^{BA}) + \frac{1}{2}(z^{AB} + z^{BA}) \\
&= z^{[AB]} + z^{(AB)} \qquad (1.47)
\end{aligned}
$$

As $z^{[AB]}$ has only two non-zero components, and one is the negative of the other, we have $z^{[AB]} = z_0\epsilon^{AB}$ where z_0 is a constant. To find the constant write,

$$
\begin{aligned}
z^{A}{}_{A} &:= z^{AB}\epsilon_{AB} \\
&= (z_0\epsilon^{AB} + z^{(AB)})\epsilon_{AB} \\
&= 2z_0. \qquad (1.48)
\end{aligned}
$$

□

We now prove invariance under the action of $SU(2)$. For $z^{[AB]}$ is a straightforward consequence of (1.39)

$$
\begin{aligned}
U^{A}{}_{C}U^{B}{}_{D}z^{[CD]} &= z_0 U^{A}{}_{C}U^{B}{}_{D}\epsilon^{CD} \\
&= z_0\epsilon^{AB} \\
&= z^{[AB]}. \qquad (1.49)
\end{aligned}
$$

Now the symmetric part.Now the symmetric part.Recall that an invariant subspace of a linear mapping

$$T : V \to V$$

from a vector space V to itself is a subspace W of V such that $T(W)$ is contained in W.

$$
\begin{aligned}
U^{0}{}_{C}U^{0}{}_{D}z^{(CD)} &= U^{0}{}_{C}U^{0}{}_{D}z^{(CD)} \\
&= \frac{1}{2}(U^{0}{}_{C}U^{0}{}_{D} + U^{0}{}_{D}U^{0}{}_{C})z^{CD} \\
&= a^2 z^{00} - 2a\overline{b}z^{01} - 2a\overline{b}z^{10} + \overline{a}^2 z^{11} \\
&= a^2 z^{(00)} - 4a\overline{b}z^{(01)} + \overline{a}^2 z^{(11)}. \qquad (1.50)
\end{aligned}
$$

and so $z^{(AB)}$ is an invariant subspace

Because of the invariance of ϵ^{AB}, this decomposition is $SU(2)$ invariant:

$$\begin{aligned} z^{A'B'} &= U^{A'}_{\ A} U^{B'}_{\ B} \left(z_0 \epsilon^{AB} + z^{(AB)} \right) \\ &= z_0 \epsilon^{A'B'} + U^{A'}_{\ A} U^{B'}_{\ B} z^{(AB)} \end{aligned} \tag{1.51}$$

the one-dimensional invariant subspace formed by the scalars z_0 defines the trivial representation $j = 0$. The three-dimensional invariant subspace formed by the symmetric spinors z_1^{AB} defines the adjoint representation $j = 1$.

$\square$

Part 4

If we tensor the two representations of spin j_1 and j_2, we obtain a space of spinors with $(2j_1 + 2j_2)$ indices, symmetric in the first $2j_1$ and the last $2j_2$ indices:

$$(\mathbf{z} \otimes \mathbf{y})^{A_1 \ldots A_{2j_1} B_1 \ldots B_{2j_2}} = z^{A_1 \ldots A_{2j_1}} y^{B_1 \ldots B_{2j_2}}. \tag{1.52}$$

By symmetrizing all the indices

$$z^{(A_1 \ldots A_{2j_1}} y^{B_1 \ldots B_{2j_2})} \tag{1.53}$$

Alternatively, we can contract k indices of the first group with k indices of the second group using k ϵ_{AB}s, and then symmetrizing the $2(j_1 + j_2 - k)$ remaining indices to obtain the spin-j_3 representation,

$$z^{A_1 \ldots A_k (A_{k+1} \ldots A_{2j_1}} y^{|B_1 \ldots B_k| B_{k+1} \ldots B_{2j_2})} \epsilon_{A_1 B_1} \cdots \epsilon_{A_k B_k} \tag{1.54}$$

where the pair of $|$'s indiates indices between them are *not* to be symmetrised over.

Prove (1.55).

Proof

If we make no contractions with ϵ_{AB}, the number of free indices is $2j_1 + 2j_2$. This corresponds to the maximum number of free indices possible to symmetrise over. Thus

$$j_3 \leq (j_1 + j_2).$$

Say $2j_2 > 2j_1$, then we can contract at most $2j_1$ indices of the first group with $2j_1$ indices of the second group, leaving $2j_2 - 2j_1$ free indices in the second group. This corresponds to the minimum number of free indices possible and so is the the smallest value $2j_3$ can take. Now, say $2j_1 > 2j_2$, then we can contract at most $2j_2$ indices of the second group with $2j_2$ indices of the first group, leaving $-(2j_2 - 2j_1)$ free indices in the first group. Thus

$$|j_1 - j_2| \leq j_3.$$

As

$$2(j_1 + j_2 - k) = 2j_3$$

we have

$$j_1 + j_2 + j_3 = k + 2j_3 \in \mathbb{N}.$$

□

$$j_1 + j_2 + j_3 \in \mathbb{N}, \qquad |j_1 - j_2| \leq j_3 \leq (j_1 + j_2). \tag{1.55}$$

These two conditions are called the *Clebsch-Gordan conditions*. They are equivalent to the fact that there exist three non-negative integers a, b, c such that

$$2j_1 = b + c, \qquad 2j_2 = c + a, \qquad 2j_3 = a + b \tag{1.56}$$

Show that (1.56) implies (1.55).

Proof:

Note: $j_1 + j_2 + j_3 = \frac{1}{2}(b + c) + \frac{1}{2}(c + a) + \frac{1}{2}(a + b) = a + b + c \in \mathbb{N}$.

Now we focus on the second condition of (1.55). First

$$j_1 + j_2 = \frac{1}{2}(a + b + 2c) \tag{1.57}$$

which obviously implies $j_3 \leq (j_1 + j_2)$. Now

$$|j_1 - j_2| \quad = \quad \frac{1}{2}|a - b| \tag{1.58}$$

Say $a > b$, then $\frac{1}{2}(a - b) \leq \frac{1}{2}(a + b)$ or $j_1 - j_2 \leq j_3$. Say $b > a$. Set $\frac{1}{2}(b - a) \leq \frac{1}{2}(a + b)$, in both of these cases:

$$|j_1 - j_2| \leq j_3 \tag{1.59}$$

Say $a = b$, then $j_1 - j_2 = 0$ Obviously $j_3 \geq 0$, so again we have $|j_1 - j_2| \leq j_3$.

□

The second condition in (1.55) is actually a more succint form of the system of triangle inequalities:

$$j_1 + j_2 \geq j_3 \tag{1.60}$$
$$j_1 + j_3 \geq j_2 \tag{1.61}$$
$$j_2 + j_3 \geq j_1. \tag{1.62}$$

Proof:

Inequality (1.60) is obviously the $j_3 \leq j_1 + j_2$ part of (1.55). Inequalities (1.61) and (1.62) rewritten are

$$-(j_1 - j_2) \leq j_3 \qquad \text{and} \qquad j_1 - j_2 \leq j_3$$

and we obviously have $|j_1 - j_2| \leq j_3$. Proving the relation in the opposite direction is trivial.

□

For

$$(j_1, j_2, j_3) \tag{1.63}$$

draw diagrams for: (i) $(1/2, 1/2, 1)$, (ii) $(5/2, 5/2, 4)$, (iii) $(1, 3/2, 5/2)$, (iv) $(5/2, 2, 5)$, (v) $(5/2, 5/2, 7/2)$, (vi) $(1, 1, 1)$.

Proof:

We easily see that:

$$\begin{aligned} a &= j_2 - j_1 + j_3 \\ b &= j_3 - j_2 + j_1 \\ c &= j_1 - j_3 + j_2 \end{aligned} \tag{1.64}$$

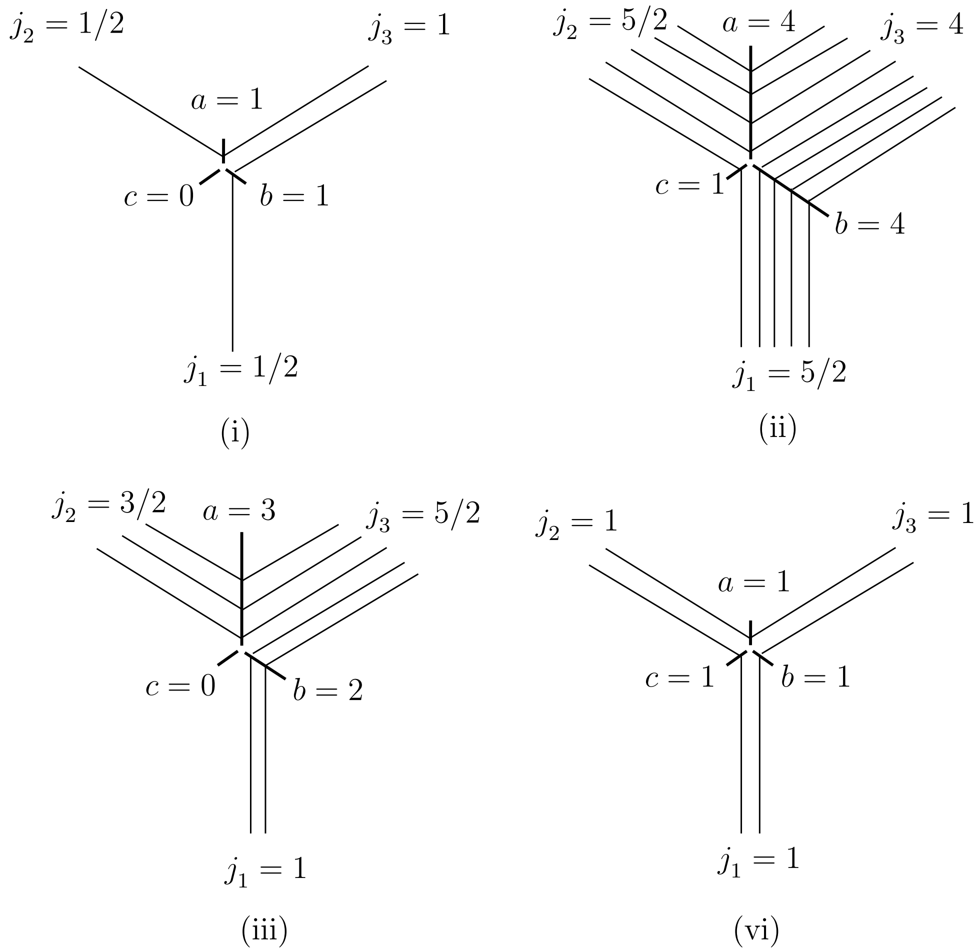

Figure 1.3: (i) $a = 1$, $b = 1$, $c = 0$. (ii) $a = 4$, $b = 4$, $c = 1$. (iii) $a = 3$, $b = 2$, $c = 0$. (vi) $a = 1$, $b = 1$, $c = 1$.

we check the examples:

(i) $(1/2, 1/2, 1)$: $a = 1/2 - 1/2 + 1 = 1$, $b = 1 - 1/2 + 1/2 = 1$, $c = 1/2 - 1 + 1/2 = 0$.

(ii) $(5/2, 5/2, 4)$: $a = 5/2 - 5/2 + 4 = 4$, $b = 4 - 5/2 + 5/2 = 4$, $c = 5/2 - 4 + 5/2 = 1$

(iii) $(1, 3/2, 5/2)$: $a = 3/2 - 1 + 5/2 = 3$, $b = 5/2 - 3/2 + 1 = 2$, $c = 1 - 5/2 + 3/2 = 0$

(iv) $(5/2, 2, 5)$: $a = 2 - 5/2 + 5 = 9/2$, $b = 5 - 2 + 5/2 = 11/2$, $c = 5/2 - 5 + 2 = -1/2$. None admissible as a, b, and c are not positive integers. This means that $j_1 + j_2 + j_3 \notin \mathbb{N}$. Also note that the triangle inequality (1.60) is not satisfied as $5/2 + 2 \ngeq 5$.

(v) $(5/2, 5/2, 7/2)$: $a = 5/2 - 5/2 + 7/2 = 7/2$, $b = 7/2 - 5/2 + 5/2 = 7/2$, $c = 5/2 - 7/2 + 5/2 = 3/2$. None admissible as a, b, and c are not positive integers. This means that $j_1 + j_2 + j_3 \notin \mathbb{N}$. Note however that the triangle inequalities *are* satified.

(vi) $(1, 1, 1)$: $a = 1 - 1 + 1 = 1$, $b = 1 - 1 + 1 = 1$, $c = 1 - 1 + 1 = 1$.

$\square$

1.2.6 Pauli Matrices

The Pauli matrices σ_i with $i = 1, 2, 3$ are

$$\sigma_i{}^A{}_B = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{1.65}$$

The Pauli matrices satisfy

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{1.66}$$

and

$$\epsilon \sigma_i \epsilon = \sigma_i^T = \sigma_i^* \tag{1.67}$$

Proof:

Equation (1.66) is equivalent to

$$\sigma_i^2 = I, \qquad \sigma_i \sigma_j = i\sigma_k \tag{1.68}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. We check explicitly, first

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.69}$$

which gives $\sigma_i^2 = I$. Next

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} &= i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{1.70}$$

which gives the second part of (1.68).

We prove (1.67) by direct calculation:

$$
\begin{aligned}
\epsilon\sigma_1\epsilon &= \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}^T = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}^* = \sigma_1^*\\
\epsilon\sigma_2\epsilon &= \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}i & 0\\0 & i\end{pmatrix} = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}^T = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}^* = \sigma_2^*\\
\epsilon\sigma_3\epsilon &= \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^T = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^* = \sigma_3^*
\end{aligned}
\tag{1.71}
$$

$\square$

We have that

$$e^{i\alpha\hat{n}\cdot\hat{\sigma}} = I\cos\alpha + i\hat{n}\cdot\hat{\sigma}\sin\alpha. \tag{1.72}$$

Proof:

From (1.66)

$$(A_i\sigma_i)(B_j\sigma_j) = \delta_{ij} + i\epsilon_{ijk}A_iB_j\sigma_k \tag{1.73}$$

or

$$(\hat{A}\cdot\hat{\sigma})(\hat{B}\cdot\hat{\sigma}) = \hat{A}\cdot\hat{B} + i\hat{\sigma}\cdot\hat{A}\times\hat{B} \tag{1.74}$$

so that $(\hat{n}\cdot\hat{\sigma})(\hat{n}\cdot\hat{\sigma}) = \hat{n}\cdot\hat{n} = 1$. Therfore $(\hat{n}\cdot\hat{\sigma})^n = \hat{n}\cdot\hat{\sigma}$ for n odd and $(\hat{n}\cdot\hat{\sigma})^n = 1$ for n even. Therefore

$$
\begin{aligned}
e^{i\alpha\hat{n}\cdot\hat{\sigma}} &= \sum_{n=0}^{\infty}\frac{1}{n!}(i\alpha)^n(\hat{n}\cdot\hat{\sigma})^n\\
&= I(1-\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}-\ldots) + i\hat{n}\cdot\hat{\sigma}(\alpha-\frac{\alpha^3}{3!}+\frac{\alpha^5}{5!}-\ldots)\\
&= I\cos\alpha + i\hat{n}\cdot\hat{\sigma}\sin\alpha.
\end{aligned}
\tag{1.75}
$$

$\square$

The matrices $\tau_i = -i\sigma_i/2$ satisfy the relation

$$[\tau_i, \tau_j] = \epsilon_{ij}^{\ \ k} \tau_k, \tag{1.76}$$

(this defines the algebra of $su(2)$)

Proof:

Using (1.66) we have

$$\begin{aligned} [\sigma_i, \sigma_j] &= \sigma_i \sigma_j - \sigma_j \sigma_i \\ &= [\delta_{ij} + i\epsilon_{ijk}\sigma_k] - [\delta_{ij} + i\epsilon_{jik}\sigma_k] \\ &= 2i\epsilon_{ijk}\sigma_k. \end{aligned} \tag{1.77}$$

or in terms if the τ's:

$$(2i\tau_i)(2i\tau_j) - (2i\tau_j)(2i\tau_i) = 2i\epsilon_{ij}^{\ \ k}(2i\tau_k), \tag{1.78}$$

which is (1.76).

□

1.2.7 Euler angles

We can view $SU(2)$ as a geometric object. In fact it is the simplest non-trivial case of a Lie group as a geometric object.

A general 2×2 unitary matrix can be written

$$U = \begin{pmatrix} a^0 + ia^3 & a^2 + ia^1 \\ -a^2 + ia^1 & a^0 - ia^3 \end{pmatrix}. \tag{1.79}$$

The determinant of this matrix in unity provided

$$(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 = 1. \tag{1.80}$$

Imposing this and if also a^i are real, we have $U^\dagger = U^{-1}$. The group manifold is therefore a three-sphere, S^3.

Euler angles are often used to parametrise $SU(2)$. We write an arbitrary $SU(2)$ unitary matrix U as

$$
\begin{aligned}
U &= \exp(-i\psi\sigma_3/2)\exp(-i\theta\sigma_2/2)\exp(-i\phi\sigma_3/2) \\
&= \exp(\psi\tau_3)\exp(\theta\tau_2)\exp(\phi\tau_3) \\
&= \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \qquad (1.81)
\end{aligned}
$$

where

$$
\psi \in [0, 2\pi[, \qquad \theta \in [0, \pi[, \qquad \phi \in [0, 4\pi[. \qquad (1.82)
$$

We obtain the Euler-parametrisation of the three-sphere from

$$
\begin{aligned}
U = &= \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 & -e^{i\phi/2}\sin\theta/2 \\ e^{-i\phi/2}\sin\theta/2 & e^{i\phi/2}\cos\theta/2 \end{pmatrix} \\
&= \begin{pmatrix} e^{-i(\phi+\psi)/2}\cos\theta/2 & -e^{-i(\psi-\phi)/2}\sin\theta/2 \\ e^{-i(\psi-\phi)/2}\sin\theta/2 & e^{i(\phi+\psi)/2}\cos\theta/2 \end{pmatrix} \\
&= \begin{pmatrix} a^0 + ia^3 & a^2 + ia^1 \\ -a^2 + ia^1 & a^0 - ia^3 \end{pmatrix} \qquad (1.83)
\end{aligned}
$$

and so

$$
\begin{aligned}
a^0 &= \cos(\theta/2)\cos((\phi+\psi)/2) \\
a^1 &= \sin(\theta/2)\sin((\psi-\phi)/2) \\
a^2 &= -\sin(\theta/2)\cos((\psi-\phi)/2) \\
a^3 &= -\cos(\theta/2)\sin((\phi+\psi)/2). \qquad (1.84)
\end{aligned}
$$

We have the invariant measure on a three-sphere

$$
\int da^0 da^1 da^2 da^3 \delta\left(\sum_{i=0}^{3} a_i^2 - 1\right) \qquad (1.85)
$$

Or we can use the usual metric on a three-sphere which can be obtained from

$$
\begin{aligned}
ds^2 &= (da^0)^2 + (da^1)^2 + (da^2)^2 + (da^3)^2 \\
&= \left(-\frac{1}{2}\sin(\theta/2)\cos((\phi+\psi)/2)d\theta - \frac{1}{2}\cos(\theta/2)\sin((\phi+\psi)/2)(d\phi+d\psi)\right)^2 \\
&+ \left(\frac{1}{2}\cos(\theta/2)\sin((\psi-\phi)/2)d\theta + \frac{1}{2}\sin(\theta/2)\cos((\psi-\phi)/2)(d\psi-d\phi)\right)^2 \\
&+ \left(-\frac{1}{2}\cos(\theta/2)\cos((\psi-\phi)/2)d\theta + \frac{1}{2}\sin(\theta/2)\sin((\psi-\phi)/2)(d\psi-d\phi)\right)^2 \\
&+ \left(\frac{1}{2}\sin(\theta/2)\sin((\phi+\psi)/2)d\theta - \frac{1}{2}\cos(\theta/2)\cos((\phi+\psi)/2)(d\psi+d\phi)\right)^2 \\
&= \frac{1}{4}\left(d\theta^2 + \cos^2(\theta/2)(d\phi+d\psi^2)^2 + \sin^2(\theta/2)(d\phi-d\psi)^2\right) \\
&= \frac{1}{4}\left(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\psi d\phi\right). \qquad (1.86)
\end{aligned}
$$

From this we read off that

$$
g_{ij} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{pmatrix} \qquad (1.87)
$$

and find

$$
\begin{aligned}
g &= \det(g_{ij}) \\
&= \frac{1}{4^3}\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{pmatrix} \\
&= \frac{1}{64}(1-\cos^2\theta) \\
&= \frac{1}{64}\sin^2\theta. \qquad (1.88)
\end{aligned}
$$

The volume element, $\sqrt{g}d\theta d\psi d\phi$, is therefore

$$
dV = \frac{1}{8}d\psi\sin\theta d\theta d\phi \qquad (1.89)
$$

and using that the volume of the sphere is $2\pi^2$ we can normalise so that the volume of the group manifold is unity, we have for the Haar measure

$$\int dU = \frac{1}{16\pi^2}\int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta \int_0^{4\pi} d\phi. \tag{1.90}$$

Proof that $2\pi^2$ is the volume of a three-sphere

We can easily derive the volume of a three-sphere by writing the integral

$$I = \int_{\mathbb{R}^4} \exp\left(-\frac{1}{2}\sum_{i=0}^{4} x_i^2\right) dV. \tag{1.91}$$

two different ways. First of all, we use that the integral decomposes as

$$I = \prod_{i=1}^{4} \int_{\mathbb{R}^4} \exp\left(-\frac{1}{2}x_i^2\right) dx_i \tag{1.92}$$

and using

$$\int_0^{\infty} \exp\left(-\frac{1}{2}x^2\right) dx = (2\pi)^{1/2}. \tag{1.93}$$

we find

$$I = (2\pi)^2 \tag{1.94}$$

Now we write the integral in spherical polar coordinates and use that the function

$$\exp\left(-\frac{1}{2}\sum_{i=0}^{4} x_i^2\right) \tag{1.95}$$

is rotationally invariant:

$$I = \int_0^{\infty}\int_{S^3(r)} \exp\left(-\frac{1}{2}r^2\right) r^3 d\Omega dr = \left(\int_0^{\infty} \exp\left(-\frac{1}{2}r^2\right) r^3 dr\right)\left(\int_{S^3(1)} d\Omega\right). \tag{1.96}$$

Notice that on the LHS that the intgration is over a three-spheres of radius r, but on the RHS the integration is over a three-sphere of radius $r = 1$. The integral

$$\int_{S^3(1)} d\Omega \tag{1.97}$$

is the volume of a three-sphere of unit radius.

We can easily perform the integration over r,

$$\begin{aligned}
\int_0^\infty \exp\left(-\frac{1}{2}r^2\right) r^3 dr &= 2\int_0^\infty \exp(-u)u^2 du \\
&= 2\left[-\exp(-u)u^2\right]_0^\infty + \frac{1}{2}\int_0^\infty \exp(-u)udu \\
&= 2\left[-\exp(-u)u\right]_0^\infty + \frac{1}{2}\int_0^\infty \exp(-u)du \\
&= 2\left[-\exp(-u)\right]_0^\infty = 2.
\end{aligned} \tag{1.98}$$

So that

$$\int_{S^3(1)} d\Omega = \frac{I}{2} = 2\pi^2. \tag{1.99}$$

□

1.2.8 Peter-Weyl theorem

It allows us to make a change of variables $g \to g_1 g$, identical to that which plays an important role in deriving results that hold for finite groups.

$$\int dg \left(D_{ij}^{(\alpha)}(g)\right)^* D_{kl}^{(\beta)}(g) = \frac{1}{dim\ \alpha}\delta^{\alpha\beta}\delta_{ik}\delta_{jl}. \tag{1.100}$$

1.2.9 Wigner matrices

Here the representation matrices $D^j_{nm}(U)$ are called the Wigner matrices.

$$\begin{aligned}
D^j_{nm}(\psi,\theta,\phi) &= < j, m|e^{\psi\tau_3}e^{\theta\tau_2}e^{\phi\tau_3}|j, n > \\
&= e^{-im\phi}D^j(\psi,\theta,0)e^{-in\phi}
\end{aligned} \tag{1.101}$$

are a complete set of fucntions on the three-sphere with

$$\frac{1}{16\pi^2}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\psi\int_0^{4\pi}d\phi\left(D^j_{mn}(\psi,\theta,\phi)\right)^* D^{j'}_{m'n'}(\psi,\theta,\phi)=\frac{1}{dim\ j}\delta^{jj'}\delta_{mm'}\delta_{nn'}. \tag{1.102}$$

1.2.10 Relations

We have the relations

$$\begin{aligned}
Tr(\tau_i\tau_j) &= -\frac{1}{2}\delta_{ij}, \qquad Tr(\tau_i\tau_j\tau_k)=-\frac{1}{4}\epsilon_{ijk} && (1.103)\\
\delta^{ij}\tau_i{}^A{}_B\tau_j{}^C{}_D &= -\frac{1}{4}(\delta^A_D\delta^C_B-\epsilon^{AC}\epsilon_{BD}) && (1.104)\\
\delta^A_B\delta^D_C &= \delta^A_C\delta^D_B+\epsilon^{AD}\epsilon_{BC} && (1.105)\\
\delta^{ij}Tr(A\tau_i)Tr(B\tau_j) &= -\frac{1}{4}\left[Tr(AB)-Tr(AB^{-1})\right] && (1.106)\\
Tr(A)Tr(B) &= Tr(AB)+Tr(AB^{-1}) && (1.107)
\end{aligned}$$

for A and B $SL(2,\mathbb{C})$ matrices.

Proof:

Proof of equation (1.103)

$$\begin{aligned}
Tr(\sigma_i\sigma_j) &= Tr(I\delta_{ij}+i\epsilon_{ijk}\sigma_k)\\
&= 2\delta_{ij}+i\epsilon_{ijk}Tr\sigma_k\\
&= 2\delta_{ij} && (1.108)
\end{aligned}$$

where we used $Tr\sigma_k=0$. In terms of the τ's the above equation reads

$$Tr(2i\tau_i\, 2i\tau_j)=2\delta_{ij}$$

proving the first part of (1.103).

Multiplying (1.66) on the right by σ_k gives

$$\sigma_i\sigma_j\sigma_k=\delta_{ij}\sigma_k+i\epsilon_{ijk'}\sigma_{k'}\sigma_k$$

taking the trace gives

$$\begin{aligned}
Tr(\sigma_i\sigma_j\sigma_k) &= \delta_{ij}Tr\sigma_k + i\epsilon_{ijk'}Tr(\sigma_{k'}\sigma_k) \\
&= i\epsilon_{ijk'}\delta^k_{k'} \\
&= 2i\epsilon_{ijk}
\end{aligned} \tag{1.109}$$

In terms of the τ's the above equation reads

$$Tr(2i\tau_i\ 2i\tau_j\ 2i\tau_k) = 2i\epsilon_{ijk}$$

proving the second part of (1.103).

Proof of equation (1.104)

Equation (1.104) is equivalent to

$$\delta^{ij}\sigma_i{}^A{}_B\sigma_j{}^C{}_D = \delta^A_D\delta^C_B - \epsilon^{AC}\epsilon_{BD} \tag{1.110}$$

We check by direct calculation, in reference to (1.65). There are 16 possibilities in total.

$A = C$, $B \neq D$:

$$\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^0{}_1 &= 0 \\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^0{}_0 &= 0 \\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^1{}_1 &= 0 \\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^1{}_0 &= 0
\end{aligned} \tag{1.111}$$

$B = D$, $A \neq C$:

$$\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^1{}_0 &= 0 \\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^0{}_0 &= 0 \\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^1{}_1 &= 0 \\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^0{}_1 &= 0
\end{aligned} \tag{1.112}$$

$A = C$ and $B = D$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^0{}_0 &= 1 \\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^0{}_1 &= 0 \\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^1{}_0 &= 0 \\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^1{}_1 &= 1
\end{aligned} \tag{1.113}
$$

$A \neq C$ and $B \neq D$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^1{}_1 &= -1 \\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^1{}_0 &= 2 \\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^0{}_1 &= 2 \\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^0{}_0 &= -1
\end{aligned} \tag{1.114}
$$

All 16 possible answers have been shown to be in accordance with (1.110).

Proof of equation (1.105)

As the indices take two values, we have the identity

$$\epsilon_{[EB}\epsilon_{C]F} = 0 \tag{1.115}$$

which reduces to

$$\epsilon_{EB}\epsilon_{CF} + \epsilon_{BC}\epsilon_{EF} + \epsilon_{CE}\epsilon_{BF} = 0 \tag{1.116}$$

Contracting this with ϵ^{EA} and ϵ^{FD}, then using $\epsilon_{EB}\epsilon^{EA} = \delta^A_B$, $\epsilon_{CF}\epsilon^{FD} = -\delta^D_C$ and $\epsilon_{EF}\epsilon^{EA}\epsilon^{FD} = \epsilon^{AD}$ etc we obtain the binor identity:

$$-\delta^A_B\delta^D_C + \epsilon_{BC}\epsilon^{AD} + \delta^A_C\delta^D_B = 0. \tag{1.117}$$

Proof of equation (1.106)

$$(\delta^{ij}\tau_i{}^A{}_B\tau_j{}^C{}_D)A^B{}_AB^D{}_C = -\frac{1}{4}(\delta^A_D\delta^C_B - \epsilon^{AC}\epsilon_{BD})A^B{}_AB^D{}_C \tag{1.118}$$

Say $U \in SL(2,\mathbb{C})$. We write for U

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{1.119}$$

where $\det U = 1$. Then

$$\begin{aligned}
-\epsilon U \epsilon &= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -c & a \\ -d & b \end{pmatrix} \\
&= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= (U^T)^{-1}
\end{aligned} \tag{1.120}$$

$$\begin{aligned}
\delta^{ij} Tr(A\tau_i) Tr(B\tau_j) &= -\frac{1}{4}(\delta^A_D \delta^C_B - \epsilon^{AC}\epsilon_{BD}) A^B{}_A B^D{}_C \\
&= -\frac{1}{4} Tr(AB) + \frac{1}{4} A^B{}_A (\epsilon_{BD} B^D{}_C \epsilon^{AC}) \\
&= -\frac{1}{4}\left[Tr(AB) - A^B{}_A (B^{-1})^A{}_B\right] \\
&= -\frac{1}{4}\left[Tr(AB) - Tr(AB^{-1})\right]
\end{aligned} \tag{1.121}$$

Where we have used $(B^{-1})^A{}_B = \epsilon_{BD} B^D{}_C \epsilon^{AC}$.

Proof of equation (1.107)

$$(\delta^A_B \delta^D_C) A^B{}_A B^C{}_D = (\delta^A_C \delta^D_B + \epsilon^{AD}\epsilon_{BC}) A^B{}_A B^C{}_D \tag{1.122}$$

$$\begin{aligned}
Tr(A) Tr(B) &= (\delta^A_C \delta^D_B + \epsilon^{AD}\epsilon_{BC}) A^B{}_A B^C{}_D \\
&= Tr(AB) + A^B{}_A (\epsilon_{BC} B^C{}_D \epsilon^{AD}) \\
&= Tr(AB) + Tr(AB^{-1}).
\end{aligned} \tag{1.123}$$

$\square$

Chapter 2

Physics without time

2.0.11 Hamilton principal function

An important solution of the Hamilton-Jacobi equations, which gives the reason for S to be denoted by S, the Hamilton principal function. It is defined

$S = \text{action} = \int^t dt' \mathcal{L}$.

We will see that $S(q, q_0, t)$ is the generating function where the time-indepndent canonical variables (α, β) coincide with the initial data (q_0, p_0).

Exercise

Prove that the Hamilton function of a harmonic oscillator is

$$S(q, t, q', t') = m\omega \frac{(q^2 + q'^2) cos\omega(t - t') - 2qq'}{2 \sin \omega(t' - t)}$$

for motion that starts at (q, t) and ends at (q', t').

Solution

First we need the solution of

$$\ddot{q} = -\frac{k}{m} q(\tilde{t}) = -\omega^2 q(\tilde{t})$$

with boundary conditions $q = q(\tilde{t} = t)$ and $q' = q(\tilde{t} = t')$. It is well know the general solution is of the form $A \cos(\omega\tilde{t}) + B \sin(\omega\tilde{t})$. The solution with correct boundary conditions is easily seen to be

$$q_{qtq't'}(\tilde{t}) = \frac{q\sin(\omega t') - q'\sin(\omega t)}{\sin\omega(t'-t)}\cos(\omega\tilde{t}) + \frac{q'\cos(\omega t) - q\cos(\omega t')}{\sin\omega(t'-t)}\sin(\omega\tilde{t}) \tag{2.1}$$

This simplifies

$$\begin{aligned} q_{qtq't'}(\tilde{t}) &= \frac{q[\sin(\omega t')\cos(\omega\tilde{t}) - \cos(\omega t')\sin(\omega\tilde{t})] + q'[\sin(\omega\tilde{t})\cos(\omega t) - \cos(\omega\tilde{t})\sin(\omega t)]}{\sin\omega(t'-t)} \\ &= \frac{q\sin\omega(t'-\tilde{t}) + q'\sin\omega(\tilde{t}-t)}{\sin\omega(t'-t)}. \end{aligned} \tag{2.2}$$

The velocity is then

$$\dot{q}_{qtq't'}(\tilde{t}) = \frac{-q\cos\omega(t'-\tilde{t}) + q'\cos\omega(\tilde{t}-t)}{\sin\omega(t'-t)}. \tag{2.3}$$

The Hamilton function is

$$
\begin{aligned}
S(q,t,q',t') &= \int_t^{t'} d\tilde{t}\left(\frac{1}{2}m\dot{q}^2_{qtq't'} - \frac{1}{2}kq^2_{qtq't'}\right) \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\int_t^{t'} d\tilde{t}\left[\{-q\cos\omega(t'-\tilde{t}) + q'\cos\omega(\tilde{t}-t)\}^2\right. \\
&\qquad\qquad \left.-\{q\sin\omega(t'-\tilde{t}) + q'\sin\omega(\tilde{t}-t)\}^2\right] \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\int_t^{t'} d\tilde{t}\Big[q^2(\cos^2\omega(t'-\tilde{t}) - \sin^2\omega(t'-\tilde{t})) \\
&\qquad +{q'}^2(\cos^2\omega(\tilde{t}-t) - \sin^2\omega(\tilde{t}-t)) \\
&\qquad -2qq'\big(\cos\omega(t'-\tilde{t})\cos\omega(\tilde{t}-t) + \sin\omega(t'-\tilde{t})\sin\omega(\tilde{t}-t)\big)\Big] \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\int_t^{t'} d\tilde{t}\Big[q^2\cos 2\omega(t'-\tilde{t}) + {q'}^2\cos 2\omega(\tilde{t}-t) - 2qq'\cos\omega(2\tilde{t}-t-t')\Big] \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\Big(q^2\frac{1}{2\omega}\Big[-\sin 2\omega(t'-\tilde{t})\Big]_t^{t'} + {q'}^2\frac{1}{2\omega}\Big[\sin 2\omega(\tilde{t}-t)\Big]_t^{t'} \\
&\quad -2qq'\frac{1}{2\omega}\Big[\sin\omega(2\tilde{t}-t-t')\Big]_t^{t'}\Big) \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\left((q^2+{q'}^2)\frac{1}{2\omega}\sin 2\omega(t'-t) - 2qq'\frac{1}{2\omega}(2\sin\omega(t'-t))\right) \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\left((q^2+{q'}^2)\frac{1}{2\omega}2\sin\omega(t-t')\cos\omega(t-t') - 2qq'\frac{1}{2\omega}(2\sin\omega(t'-t))\right) \\
&= m\omega\frac{(q^2+{q'}^2)cos\omega(t-t') - 2qq'}{2\sin\omega(t'-t)}. \qquad (2.4)
\end{aligned}
$$

□

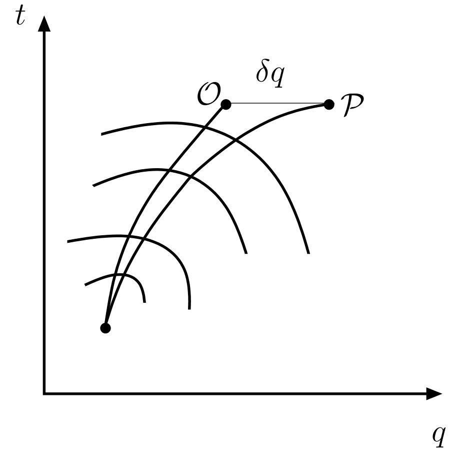

Figure 2.1: Hamilton.

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1} \mathcal{L}(q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t), t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t), \dot{q}(t), t)dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q}\right) dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \delta q dt + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q(t)\right]_{t_0}^{t_1} \\
&= \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q \qquad (2.5)
\end{aligned}$$

so

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q$$

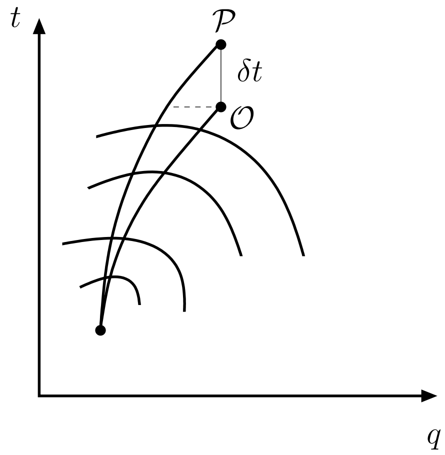

Figure 2.2: Hamilton.

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1+\delta t} \mathcal{L}(q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t), t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t), \dot{q}(t), t)dt \\
&= \mathcal{L}\delta t + \int_{t_0}^{t_1} \mathcal{L}(q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t), t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t), \dot{q}(t), t)dt \\
&= \mathcal{L}\delta t + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q \qquad (2.6)
\end{aligned}$$

$$\delta q = -\dot{q}\delta t$$

so

$$\delta S = \left(\mathcal{L} - \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta t$$

In general we have

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q - \left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \right) \delta t \tag{2.7}$$

we have

$$\frac{\partial S}{\partial q} = \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}} = p \tag{2.8}$$

$$\frac{\partial S}{\partial t} = \mathcal{L} - \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} = -\mathcal{H} \tag{2.9}$$

solve the first equation for $\dot{q}$ so that we have

$$\dot{q} = \dot{q}(q, \frac{\partial S}{\partial q}, t)$$

and substitute the value of $\dot{q}$ into the second giving

$$\frac{\partial S}{\partial t} + \mathcal{H} \left(\frac{\partial S}{\partial q}, q, t \right) = 0, \tag{2.10}$$

that is, Hamilton principal function solves the Hamiltonian-Jacobi equation.

Chapter 3

Gravity

3.1 Einstein-Hilbert action

$$S[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) \tag{3.1}$$

3.2 Tetrads

3.2.1 Spin connection

The spin connection ${\omega_\mu}^{IJ}$ defines a covariant derivative D_μ on generalised tensors. For example its action on V^I_ν is

$$D_\mu V^I_\nu = \partial_\mu V^I_\nu + {\omega_\mu}^I{}_J V^J_\nu - \Gamma^\sigma_{\mu\nu} V^I_\sigma \tag{3.2}$$

We expect D_μ will annihilate the Minkowski metric

$$D_\mu \eta_{IJ} = 0 \tag{3.3}$$

implies

$$D_\mu \eta_{IJ} = \partial_\mu \eta_{IJ} + {\omega_\mu}^{IK}\eta_{KJ} + {\omega_\mu}^{JK}\eta_{IK} = 0 \tag{3.4}$$

which implies

$$\omega^{IJ}_\mu = -\omega^{JI}_\mu . \tag{3.5}$$

Torsion

$$T^I_{\mu\nu} = \partial_{[\mu} e^I_{\nu]} + e^J_{[\nu} \omega_{\mu]}{}^I{}_J \tag{3.6}$$

General covariant Dirac equation

Specifically there are problems coupling gravity to spinor fields: there are no finite-dimensional spinor representations of the general covaraince group. However, there are of course spinorial representations of the Lorentz group. This fact is utilised by employing tetrad fields descfribing a flat tangent space at every point of spacetime. The Dirac matrices are contracted on tetrads,

$$\gamma^I e^\mu_I(x) = \gamma^\mu(x). \tag{3.7}$$

We wish to construct a generally covariant Dirac equation

$$i\gamma^I e^\mu_I D_\mu \psi - m\psi = 0 \tag{3.8}$$

3.2.2 Cartan structure equations

First Cartan equation

$$\begin{aligned} T^I_{\mu\nu} &= D_{[\mu} e^I_{\nu]} \\ &= \partial_{[\mu} e^I_{\nu]} + e^J_{[\nu} \omega^I_{\mu]J} \end{aligned} \tag{3.9}$$

Second Cartan equation

$$F^I_{\mu\nu J} = \partial_{[\mu} \omega^I_{\nu]}{}_J + \omega^I_{[\mu\,|K|} \omega^K_{\nu]}{}_J \tag{3.10}$$

Vanishing of the torsion

The connection is said to be compatible with the tetrad if it satifies

$$D_\mu e^I_\nu = 0 \tag{3.11}$$

or

$$D_\mu e^I_\nu = \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu - \Gamma^\sigma_{\mu\nu} e^I_\sigma = 0. \tag{3.12}$$

It implies the vanishing of the torsion

$$T^I_{\mu\nu} = D_{[\mu} e^I_{\nu]} = 0 \tag{3.13}$$

To directly solve the compatibility condition for the spin connection ω_μ^{IJ}, one can use the same trick that was used to solve $\nabla_\rho g_{\mu\nu} = 0$ for the affine connection $\Gamma^\gamma_{\mu\nu}$. First contract the compatibility condition to give

$$e^\mu_J e^\nu_K (\partial_{[\mu} e_{\nu]I} + \omega_{[\mu I}{}^L e_{\nu]L}) = 0 \tag{3.14}$$

Then, do a cyclic permutation of the free indices I, J, and K, and add and subtract the three resultig equations:

$$\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} + 2e^\mu_J\, \omega_{\mu IK} = 0 \tag{3.15}$$

where we have used the definition $\Omega_{JKI} := e^\mu_J e^\nu_K \partial_{[\mu} e_{\nu]I}$. The solution for the spin connection is

$$\omega_{\mu KI} = \frac{1}{2} e^J_\mu (\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ}) \tag{3.16}$$

so that the connection in terms of the tetrad reads

$$\omega_\mu^{IJ} = \frac{1}{2} e^{\nu[I} (\partial_\mu e^{J]}_\nu - \partial_\nu e^{J]}_\mu + e^{J]\sigma} e^K_\mu \partial_\sigma e_{\nu K}) \tag{3.17}$$

Tetrad Riemann tensor

The tetrads encode the information about the space-time metric and will be taken as one of the independent variables in the action principle.

Now if one is going to operate on objects that have internal indices one needs to introduce an appropriate derivative (covariant derivative). We introduce an arbitrary covariant derivative via

$$D_\alpha V_I = \partial_\alpha V_I + \omega_{\alpha I}{}^J V_J. \tag{3.18}$$

Where $\omega_{\alpha I}{}^{J}$ is a Lorentz connection (the derivative annihilates the Minkowski metric η_{IJ}). We define a curvature via

$$\Omega_{\alpha\beta I}{}^{J} V_J = (D_\alpha D_\beta - D_\beta D_\alpha) V_I \tag{3.19}$$

We obtain

$$\Omega_{\alpha\beta}{}^{IJ} = 2\partial_{[\alpha}\omega_{\beta]}{}^{IJ} + \omega_{\alpha I}{}^{K}\omega_{\beta K}{}^{J} - \omega_{\beta I}{}^{K}\omega_{\alpha K}{}^{J}. \tag{3.20}$$

We use the covariant derivative which annihilates the tetrad,

$$D_\mu e^I_\nu = 0. \tag{3.21}$$

The connection is completely determined by the tetrad. The action of this on the generalized tensor V^I_ν is given by

$$D_\mu V^I_\nu = \partial_\alpha V^I_\nu - \omega_\mu{}^I{}_J V^J_\nu - \Gamma^\gamma_{\alpha\nu} V^I_\gamma. \tag{3.22}$$

We define a curvature $R_{\mu\nu}{}^{IJ}$ by

$$R_{\mu\nu I}{}^{J} V_J = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_I. \tag{3.23}$$

This is easily related to the usual curvature defined by

$$R_{\mu\nu\gamma}{}^{\delta} V_\delta = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\gamma \tag{3.24}$$

via substituting $V_\gamma = V_I e^I_\gamma$ into this expression (see below for details). One obtains,

$$R_{\mu\nu\gamma}{}^{\delta} = e^I_\gamma R_{\mu\nu I}{}^{J} e^\delta_J, \quad R_{\mu\nu} = R_{\mu\gamma I}{}^{J} e^I_\nu e^\gamma_J \text{ and } R = R_{\mu\nu}{}^{IJ} e^\mu_I e^\nu_J \tag{3.25}$$

for the Riemann tensor, Ricci tensor and Ricci scalar respectively.

Relating usual curvature to the mixed index curvature

The usual Riemann curvature tensor ${R_{\alpha\beta\gamma}}^{\delta}$ is defined by

$$ {R_{\alpha\beta\gamma}}^{\delta}V_{\delta} = (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})V_{\gamma}. \tag{3.26} $$

To find the relation to the mixed index curvature tensor let us substitute $V_{\gamma} = e^I_{\gamma}V_I$

$$ {R_{\alpha\beta\gamma}}^{\delta}V_{\delta} = (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})V_{\gamma} \tag{3.27} $$

$$ = (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})(e^I_{\gamma}V_I) \tag{3.28} $$

$$ = e^I_{\gamma}(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})V_I \tag{3.29} $$

$$ = e^I_{\gamma}{R_{\alpha\beta I}}^{J}e^{\delta}_{J}V_{\delta} \tag{3.30} $$

where we have used $\nabla_{\alpha}e^I_{\beta} = 0$. Since this is true for all V_{δ} we obtain

$$ {R_{\alpha\beta\gamma}}^{\delta} = e^I_{\gamma}{R_{\alpha\beta I}}^{J}e^{\delta}_{J}. \tag{3.31} $$

Using this expression we find

$$ R_{\alpha\beta} = {R_{\alpha\gamma\beta}}^{\gamma} = {R_{\alpha\gamma I}}^{J}e^I_{\beta}e^{\gamma}_{J}. \tag{3.32} $$

Contracting over α and β allows us write the Ricci scalar

$$ R = {R_{\alpha\beta}}^{IJ}e^{\alpha}_{I}e^{\beta}_{J}. \tag{3.33} $$

3.3 Einstein-Hilbert action in tetradic form

$$ \begin{aligned} S[e] &= \frac{1}{16\pi G}\int dx^4 e e^I_{\mu}e^J_{\nu}F^{KL}\epsilon_{IJKL} \\ &= \frac{1}{16\pi G}\int \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}dx^{\mu}dx^{\nu}dx^{\rho}dx^{\sigma}e^I_{\mu}e^J_{\nu}F^{KL}\epsilon_{IJKL} \end{aligned} \tag{3.34} $$

$$S[e] = \frac{1}{16\pi G}\int e^I \wedge e^J \wedge F^{KL}\epsilon_{IJKL} \tag{3.35}$$

Fermion action

$$S_f[\psi, e] \;\; = \;\; \int d^4x\epsilon^{\mu\nu\sigma\rho}\overline{\psi}\gamma^I(D_\mu\psi)e^J_\nu e^K_\sigma e^L_\rho\epsilon_{IJKL} \tag{3.36}$$

3.4 First order formulism

$$\begin{aligned} S[e,\omega] &= \frac{1}{16\pi G}\int dx^4 e e^I_\mu e^J_\nu F^{KL}[\omega]\epsilon_{IJKL} \\ &= \frac{1}{16\pi G}\int \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma e^I_\mu e^J_\nu F^{KL}[\omega]\epsilon_{IJKL} \end{aligned} \tag{3.37}$$

in differential geometry form

$$S[e,\omega] = \frac{1}{16\pi G}\int e^I \wedge e^J \wedge F^{KL}[\omega]\epsilon_{IJKL} \tag{3.38}$$

3.4.1 Preliminaries

Definition of the spacetime epsilon, $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{0123} = +1 \tag{3.39}$$

as with othe spacetime tensor indices are lowered using the metric tensor,

$$\begin{aligned} \epsilon_{\mu\nu\sigma\rho} &= g_{\mu\mu'}g_{\nu\nu'}g_{\sigma\sigma'}g_{\rho\rho'}\epsilon^{\mu'\nu'\sigma'\rho'} \\ &= \det(g)\epsilon^{\mu\nu\sigma\rho}. \end{aligned} \tag{3.40}$$

Definition of the internal indice epsilon, ϵ_{IJKL}:

$$\epsilon^{0123} = +1 \tag{3.41}$$

and plus or minus 1 for even and odd permutations of the set $\{0,1,2,3\}$.

The internal indices of ϵ_{IJKL} are raised with the Minkowski metric η^{IJ}. Since η_{IJ} has signature $(-,+,+,+)$, it follows that

$$\epsilon^{IJKL} = -\epsilon_{IJKL}.$$

to see this consider,

$$\begin{aligned}
\epsilon^{0123} &= \eta^{0I}\eta^{1J}\eta^{2K}\eta^{3L}\epsilon_{IJKL} \\
&= (-1)(+1)(+1)(+1)\epsilon_{0123} = -\epsilon_{0123}.
\end{aligned}$$

Consider the total antisymmetrisation of $T^{IJKL}_{I'J'K'L'}$ in its higher indices:

$$A^{IJKL}_{MNOP} := T^{[IJKL]}_{MNOP}$$

The tensor A^{IJKL}_{MNOP} will be zero unless each upper indice takes a different value, and then (for fixed M,N,O,P) it will take the same value as A^{0123}_{MNOP} with a plus sign when (I,J,K,L) is an even permutation of $(0,1,2,3)$ and a minus sign for an odd permutation of $(0,1,2,3)$. This implies

$$T^{[IJKL]}_{MNOP} = T^{[0123]}_{MNOP}\,\epsilon^{IJKL}. \tag{3.42}$$

So that we can write

$$\begin{aligned}
\delta^{[I}_M\delta^J_N\delta^K_O\delta^{L]}_P &= \delta^{[0}_M\delta^1_N\delta^2_O\delta^{3]}_P\,\epsilon^{IJKL} \\
&=: -\frac{1}{4!}\epsilon_{MNOP}\epsilon^{IJKL}.
\end{aligned} \tag{3.43}$$

Put this to

$$\epsilon^{IJKL}\epsilon_{PMNO} = -4!\delta^{[I}_P\delta^J_M\delta^K_N\delta^{L]}_O$$

and consider summing over $I = P$

$$
\begin{aligned}
\sum_I \epsilon^{IJKL}\epsilon_{IMNO} &= -4! \sum_I \delta_I^{[I}\delta_M^J\delta_N^K\delta_O^{L]} \\
&= -4! \left(\delta_0^{[0}\delta_M^J\delta_N^K\delta_O^{L]} + \delta_1^{[1}\delta_M^J\delta_N^K\delta_O^{L]} + \delta_2^{[2}\delta_M^J\delta_N^K\delta_O^{L]} + \delta_3^{[3}\delta_M^J\delta_N^K\delta_O^{L]}\right)
\end{aligned}
\tag{3.44}
$$

Consider $J = 1, K = 2, L = 3$ then

$$
\begin{aligned}
\sum_I \epsilon^{I123}\epsilon_{IMNO} &= -4!\delta_0^{[0}\delta_M^1\delta_N^2\delta_O^{3]} \\
&= -3! \times 4 \left(\delta_M^{[1}\delta_N^2\delta_O^{3]} + 0 + 0 + 0\right) \\
&= -3!\delta_M^{[1}\delta_N^2\delta_O^{3]}
\end{aligned}
\tag{3.45}
$$

Consider $J = 0, K = 2, L = 3$ then

$$
\begin{aligned}
\sum_I \epsilon^{I023}\epsilon_{IMNO} &= -4!\delta_1^{[1}\delta_M^0\delta_N^2\delta_O^{3]} \\
&= -3!\delta_M^{[0}\delta_N^2\delta_O^{3]}
\end{aligned}
\tag{3.46}
$$

etc. So that we have

$$
\sum_I \epsilon^{IJKL}\epsilon_{IMNO} = -3!\delta_M^{[J}\delta_N^K\delta_O^{L]}. \tag{3.47}
$$

Put (3.47) to

$$
\sum_I \epsilon^{IJKL}\epsilon_{IOMN} = -3!\delta_O^{[J}\delta_M^K\delta_N^{L]}
$$

Consider $J = O$ and sum over J:

$$
\begin{aligned}
\sum_{I,J} \epsilon^{IJKL}\epsilon_{IJMN} &= -3!\sum_{J} \delta^{[J}_{J}\delta^{K}_{M}\delta^{L]}_{N} \\
&= -2!\sum_{J}\left(\delta^{J}_{J}\delta^{[K}_{M}\delta^{L]}_{N} - \delta^{K}_{J}\delta^{[J}_{M}\delta^{L]}_{N} + \delta^{L}_{J}\delta^{[J}_{M}\delta^{K]}_{N}\right) \\
&= -\sum_{J}\left(\delta^{J}_{J}\delta^{K}_{M}\delta^{L}_{N} - \delta^{J}_{J}\delta^{L}_{M}\delta^{K}_{N} - \delta^{K}_{J}\delta^{J}_{M}\delta^{L}_{N} + \delta^{K}_{J}\delta^{L}_{M}\delta^{J}_{N} + \delta^{L}_{J}\delta^{J}_{M}\delta^{K}_{N} - \delta^{L}_{J}\delta^{K}_{M}\delta^{J}_{N}\right) \\
&= -\left(4\delta^{K}_{M}\delta^{L}_{N} - 4\delta^{L}_{M}\delta^{K}_{N} - \delta^{K}_{M}\delta^{L}_{N} + \delta^{L}_{M}\delta^{K}_{N} + \delta^{L}_{M}\delta^{K}_{N} - \delta^{K}_{M}\delta^{L}_{N}\right) \\
&= -2!\left(\delta^{K}_{M}\delta^{L}_{N} - \delta^{K}_{N}\delta^{L}_{M}\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\sum_{I,J,K} \epsilon^{IJKL}\epsilon_{IJKN} &= -2!\sum_{K}\left(\delta^{K}_{K}\delta^{L}_{N} - \delta^{K}_{N}\delta^{L}_{K}\right) \\
&= -2!\left(4\delta^{L}_{N} - \delta^{L}_{N}\right) \\
&= -6\delta^{L}_{N}.
\end{aligned}
$$

Lastly, we obviously have

$$
\sum_{I,J,K,J} \epsilon^{IJKL}\epsilon_{IJKL} = -6\sum_{I}\delta^{I}_{I} = -24.
$$

In summary, we have the identities,

$$
\begin{aligned}
\epsilon^{IJKL}\epsilon_{MNOP} &= -24\delta^{[I}_{M}\delta^{J}_{N}\delta^{K}_{O}\delta^{L]}_{P} && (3.48) \\
\epsilon^{IJKL}\epsilon_{IMNO} &= -6\delta^{J}_{[M}\delta^{K}_{N}\delta^{L}_{O]} \\
&= -\left(\delta^{J}_{M}\delta^{K}_{N}\delta^{L}_{O} + \delta^{L}_{M}\delta^{J}_{N}\delta^{K}_{O} + \delta^{K}_{M}\delta^{L}_{N}\delta^{J}_{O} - \delta^{K}_{M}\delta^{J}_{N}\delta^{L}_{O} - \delta^{L}_{M}\delta^{K}_{N}\delta^{J}_{O} - \delta^{J}_{M}\delta^{L}_{N}\delta^{K}_{O}\right) && (3.49) \\
\epsilon^{IJKL}\epsilon_{IJMN} &= -4\delta^{K}_{[M}\delta^{L}_{N]} \\
&= -2(\delta^{K}_{M}\delta^{L}_{N} - \delta^{K}_{N}\delta^{L}_{M}) && (3.50) \\
\epsilon^{IJKL}\epsilon_{IJKN} &= -6\delta^{L}_{N} && (3.51) \\
\epsilon^{IJKL}\epsilon_{IJKL} &= -24 && (3.52)
\end{aligned}
$$

An important formula is

$$
\begin{aligned}
e^0_{\ \mu} e^1_{\ \nu} e^2_{\ \sigma} e^3_{\ \rho}\ \epsilon^{\mu\nu\sigma\rho} d^4x &= \det(e) d^4x \\
&= s|\det(e)| d^4x \\
&= s\sqrt{-g} d^4x
\end{aligned}
\tag{3.53}
$$

where $s = sign(\det(e))$.

3.4.2 Derivation of equations

Varying we get

$$
\begin{aligned}
-32G\delta S &= \int d^4x \epsilon_{IJKL} [\epsilon^{\mu\nu\sigma\rho} (\delta e^I_\mu) e^J_\nu F^{KL}_{\sigma\rho}] + \int d^4x \epsilon_{IJKL} [\epsilon^{\mu\nu\sigma\rho} e^I_\mu (\delta e^J_\nu) F^{KL}_{\sigma\rho} \\
&\quad + \int d^4x [\epsilon^{\mu\nu\sigma\rho} e^I_\mu e^J_\nu \delta F^{KL}_{\sigma\rho}] \\
&= 2\int d^4x \epsilon_{IJKL} [\epsilon^{\mu\nu\sigma\rho} (\delta e^I_\mu) e^J_\nu F^{KL}_{\sigma\rho}] + \int d^4x [\epsilon^{\mu\nu\sigma\rho} e^I_\mu e^J_\nu D_\sigma \delta\omega^{KL}_\rho]
\end{aligned}
\tag{3.54}
$$

Using the identity

$$
\begin{aligned}
& \int d^4x \epsilon^{\rho\mu\nu\sigma} \epsilon_{IJKL} \frac{\partial}{\partial x^\rho} [e^I_\mu e^J_\nu \delta\omega^{KL}_\sigma] \\
= & \int d^4x \epsilon^{\rho\mu\nu\sigma} \epsilon_{IJKL} D_\rho [e^I_\mu e^J_\nu \delta\omega^{KL}_\sigma] \\
= & \int d^4x \epsilon^{\rho\mu\nu\sigma} \epsilon_{IJKL} [(D_\rho e^I_\mu) e^J_\nu \delta\omega^{KL}_\sigma + e^I_\mu (D_\rho e^J_\nu) \delta\omega^{KL}_\sigma + e^I_\mu e^J_\nu D_\rho \delta\omega^{KL}_\sigma] \\
= & \int d^4x \epsilon^{\rho\mu\nu\sigma} \epsilon_{IJKL} [2(D_\rho e^I_\mu) e^J_\nu \delta\omega^{KL}_\sigma + e^I_\mu e^J_\nu D_\rho \delta\omega^{KL}_\sigma]
\end{aligned}
\tag{3.55}
$$

The fact that the term on the L.H.S is a total derivative and we are ignoring surface terms,

$$
\delta S = -\frac{1}{16G} [\int d^4x \epsilon^{\mu\nu\sigma\rho} \epsilon_{IJKL} (e^I_\mu F^{JK}_{\nu\sigma} \delta e^L_\rho) - \int d^4x \epsilon^{\mu\nu\sigma\rho} \epsilon_{IJKL} ((D_\mu e^I_\nu) e^J_\sigma \delta\omega^{KL}_\rho)] \tag{3.56}
$$

Varying just e but keeping ω fixed implies the vanishing of the first term:

$$
\epsilon^{\mu\nu\sigma\rho} \epsilon_{IJKL} e^I_\mu F^{JK}_{\nu\sigma} = 0 \tag{3.57}
$$

2nd order formulism

Assume $T^I_{\mu\nu} = 0$

1st order formulism

Variation with respect to ω gives

$$\epsilon^{\mu\nu\sigma\rho}\epsilon_{IJKL}(D_\mu e^I_\nu)e^J_\sigma = 0$$

which implies with use of (3.50)

$$\epsilon^{\mu\nu\sigma\rho}(D_\mu e^{[I}_\nu)e^{J]}_\sigma = \epsilon^{\mu\nu\sigma\rho}(D_\mu e^I_\nu)e^J_\sigma = 0$$

then contracting with $e^{\sigma'}_J$ implies

$$\epsilon^{\mu\nu\sigma\rho}(D_\mu e^I_\nu)\delta^{\sigma'}_\sigma = \epsilon^{\mu\nu\sigma'\rho}(D_\mu e^I_\nu) = 0$$

which implies

$$D_{[\mu}e^I_{\nu]} = 0. \tag{3.58}$$

1.5 order formulism

3.4.3 Alternative derivation of equations

Variation

The Ricci scalar of this curvature can be expressed as $e^\alpha_I e^\beta_J \Omega_{\alpha\beta}{}^{IJ}$. The action can be written

$$S_{H-P} = \int d^4x \; e \; e^\alpha_I e^\beta_J \Omega_{\alpha\beta}{}^{IJ} \tag{3.59}$$

where $e = \sqrt{-g}$ but now g is a function of the frame field.

We will derive the Einstein equations by varying this action with respect to the tetrad and spin connection as independent quantities.

As a shortcut to performing the calculation we introduce a connection compatible with the tetrad, $\nabla_\alpha e^I_\beta = 0$. ref: A. Ashtekar "Lectures on non-perturbative canonical gravity" (with invited contributions), Bibliopolis, Naples 19988. The connection associated with this covariant derivative is completely determined by the tetrad. The difference between the two connections we have introduced is a field $C_{\alpha I}{}^J$ defined by

$$C_{\alpha I}{}^J V_J = (D_\alpha - \nabla_\alpha) V_I. \tag{3.60}$$

We can compute the difference between the curvatures of these two covariant derivatives (see below for details),

$$\Omega_{\alpha\beta}{}^{IJ} - R_{\alpha\beta}{}^{IJ} = \nabla_{[\alpha} C_{\beta]}{}^{IJ} + C_{[\alpha}{}^{IM} C_{\beta]M}{}^{J} \tag{3.61}$$

The reason for this intermediate calculation is that it is easier to compute the variation by reexpressing the action in terms of ∇ and $C_\alpha{}^{IJ}$ and noting that the variation with respect to $\omega_\alpha{}^{IJ}$ is the same as the variation with respect to $C_\alpha{}^{IJ}$ (when keeping the tetrad fixed). The action becomes

$$S_{H-P} = \int d^4x \; e \; e^\alpha_I e^\beta_J (R_{\alpha\beta}{}^{IJ} + \nabla_{[\alpha} C_{\beta]}{}^{IJ} + C_{[\alpha}{}^{IM} C_{\beta]M}{}^{J}) \tag{3.62}$$

We first vary with respect to $C_\alpha{}^{IJ}$. The first term does not depend on $C_\alpha{}^{IJ}$ so it does not contribute. The second term is a total derivative. The last term yields $e^{[a}_M e^{b]}_N \delta^M_{[I} \delta^K_{J]} C_{bK}{}^N = 0$. We show below that this implies that $C_\alpha{}^{IJ} = 0$ as the prefactor $e^{[a}_M e^{b]}_N \delta^M_{[I} \delta^K_{J]}$ is non-degenerate. This tells us that ∇ coincides with D when acting on objects with only internal indices. Thus the connection D is completely determined by the tetrad and Ω coincides with R. To compute the variation with respect to the tetrad we need the variation of $e = \det e^I_\alpha$. From the standard formula

$$\delta \det(a) = \det(a) (a^{-1})_{ji} \delta a_{ij} \tag{3.63}$$

we have $\delta e = e e^\alpha_I \delta e^I_\alpha$. Or upon using $\delta(e^I_\alpha e^\alpha_I) = 0$, this becomes $\delta e = -e e^I_\alpha \delta e^\alpha_I$. We compute the second equation by varying with respect to the tetrad,

$$\begin{aligned} \delta S_{H-P} &= \int d^4x \; e \left((\delta e^\alpha_I) e^\beta_J \Omega_{\alpha\beta}{}^{IJ} + e^\alpha_I (\delta e^\beta_J) \Omega_{\alpha\beta}{}^{IJ} - e^K_\gamma (\delta e^\gamma_K) e^\alpha_I e^\beta_J \Omega_{\alpha\beta}{}^{IJ} \right) \\ &= 2 \int d^4x \; e \left(e^\beta_J \Omega_{\alpha\beta}{}^{IJ} - \frac{1}{2} e^\gamma_M e^\delta_N e^I_\alpha \Omega_{\gamma\delta}{}^{MN} \right) (\delta e^\alpha_I) \end{aligned} \tag{3.64}$$

One gets, after substituting $\Omega_{\alpha\beta}{}^{IJ}$ for $R_{\alpha\beta}{}^{IJ}$ as given by the previous equation of motion,

$$e^\gamma_J R_{\alpha\gamma}{}^{IJ} - \frac{1}{2} R_{\gamma\delta}{}^{MN} e^\gamma_M e^\delta_N e^I_\alpha = 0 \tag{3.65}$$

which, after multiplication by $e_{I\beta}$ just tells us that the Einstein tensor $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$ of the metric defined by the tetrads vanishes. We have therefore proved that the Palatini variation of the action in tetradic form yields the usual Einstein equations.

Details of proof

The derivative defined by $D_\alpha V_I$ only knows how to act on internal indices. However, we find it convenient to consider a torsion-free extension to spacetime indices. All calculations will be independent of this choice of extension. Applying $\mathcal{D}_a$ twice on V_I,

$$\begin{aligned} \mathcal{D}_\alpha \mathcal{D}_\beta V_I &= \mathcal{D}_\alpha (\nabla_\beta V_I + C_{\beta I}{}^J V_J) \\ &= \nabla_a (\nabla_b V_I + C_{\beta I}{}^J V_J) + C_{\alpha I}{}^K (\nabla_b V_K + C_{\beta K}{}^J V_J) + \overline{\Gamma}^\gamma_{\alpha\beta} (\nabla_\gamma V_I + C_{\gamma I}{}^J V_J) \end{aligned} \tag{3.66}$$

where $\overline{\Gamma}^\gamma_{\alpha\beta}$ is unimportant, we need only note that it is symmetric in α and β as it is torsion-free. Then

$$\begin{aligned} \Omega_{\alpha\beta I}{}^J V_J &= (\mathcal{D}_\alpha \mathcal{D}_\beta - \mathcal{D}_\beta \mathcal{D}_\alpha) V_I \\ &= (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V_I + \nabla_\alpha (C_{\beta I}{}^J V_J) - \nabla_\beta (C_{\alpha I}{}^J V_J) \\ &\quad + C_{\alpha I}{}^K \nabla_\beta V_K - C_{\beta I}{}^K \nabla_\alpha V_K + C_{\alpha I}{}^K C_{\beta K}{}^J V_J - C_{\beta I}{}^K C_{\alpha K}{}^J V_J \\ &= R_{\alpha\beta I}{}^J V_J + (\nabla_\alpha C_{\beta I}{}^J - \nabla_\beta C_{\alpha I}{}^J + C_{\alpha I}{}^K C_{\beta K}{}^J - C_{\beta_I}{}^K C_{\alpha K}{}^J) V_J \end{aligned} \tag{3.67}$$

Hence

$$\Omega_{ab}{}^{IJ} - R_{ab}{}^{IJ} = 2\nabla_{[a} C_{b]}{}^{IJ} + 2C_{[a}{}^{IK} C_{b]K}{}^J \tag{3.68}$$

=== Varying the action with respect to the field $C_\alpha{}^{IJ}$ ===

We would expect ∇_a to also annihilate the Minkowski metric $\eta_{IJ} = e_{\beta I} e^\beta_J$. If we also assume that the covariant derivative $\mathcal{D}_\alpha$ annihilates the Minkowski metric (then said to be torsion-free) we have,

$$\begin{aligned} 0 &= (\mathcal{D}_\alpha - \nabla_\alpha) \eta_{IJ} \\ &= C_{\alpha I}{}^K \eta_{KJ} + C_{aJ}{}^K \eta_{IK} \\ &= C_{\alpha IJ} + C_{\alpha JI}. \end{aligned} \tag{3.69}$$

Implying $C_{\alpha IJ} = C_{\alpha[IJ]}$.

From the last term of the action we have from varying with respect to $C_{\alpha I}{}^{J}$,

$$
\begin{aligned}
\delta S_{EH} &= \delta \int d^4x\; e\; e^{\gamma}_{M} e^{\beta}_{N} C_{[\gamma}{}^{MK} C_{\beta]K}{}^{N} \\
&= \delta \int d^4x\; e\; e^{[\gamma}_{M} e^{\beta]}_{N} C_{\gamma}{}^{MK} C_{\beta K}{}^{N} \\
&= \delta \int d^4x\; e\; e^{M[\gamma} e^{\beta]}_{N} C_{\gamma M}{}^{K} C_{\beta K}{}^{N} \\
&= \int d^4x\; e e^{M[\gamma} e^{\beta]}_{N} \big(\delta^{\alpha}_{\gamma} \delta^{I}_{M} \delta^{K}_{J} C_{\beta K}{}^{N} + C_{\gamma M}{}^{K} \delta^{\alpha}_{\beta} \delta^{I}_{K} \delta^{N}_{J} \big) \delta C_{\alpha I}{}^{J} \\
&= \int d^4x\; e (e^{I[\alpha} e^{\beta]}_{N} C_{\beta J}{}^{N} + e^{M[\beta} e^{\alpha]}_{J} C_{\beta M}{}^{I}) \delta C_{\alpha I}{}^{J}
\end{aligned}
\tag{3.70}
$$

or

$$
e^{[\alpha}_{I} e^{\beta]}_{K} C_{\beta J}{}^{K} + e^{K[\beta} e^{\alpha]}_{J} C_{\beta KI} = 0 \tag{3.71}
$$

or

$$
C_{\beta I}{}^{K} e^{[\alpha}_{K} e^{\beta]}_{J} + C_{\beta J}{}^{K} e^{[\alpha}_{I} e^{\beta]}_{K} = 0. \tag{3.72}
$$

where we have used $C_{\beta KI} = -C_{\beta IK}$. This can be writtem more compactly as

$$
e^{[\alpha}_{M} e^{\beta]}_{N} \delta^{M}_{[I} \delta^{K}_{J]} C_{\beta K}{}^{N} = 0. \tag{3.73}
$$

=== Vanishing of $C_{\alpha}{}^{IJ}$ ===

We will show following the reference "Geometrodynamics vs. Connection Dynamics" ref: "Geometrodynamics vs. Connection Dynamics", Joseph D. Romano, Gen.Rel.Grav. 25 (1993) 759-854 that

$$
C_{\beta I}{}^{K} e^{[\alpha}_{K} e^{\beta]}_{J} + C_{\beta J}{}^{K} e^{[\alpha}_{I} e^{\beta]}_{K} = 0 \quad Eq.1 \tag{3.74}
$$

implies $C_{\alpha I}{}^{J} = 0$. First we define the spacetime tensor field by

$$
S_{\alpha\beta\gamma} := C_{\alpha IJ} e^{I}_{\beta} e^{J}_{\gamma}. \tag{3.75}
$$

Then the condition $C_{\alpha IJ} = C_{\alpha[IJ]}$ is equivalent to $S_{\alpha\beta\gamma} = S_{\alpha[\beta\gamma]}$. Contracting Eq. 1 with $e^I_\alpha e^J_\gamma$ one calculates that

$$C_{\beta J}{}^{I} e^J_\gamma e^\beta_I = 0. \tag{3.76}$$

As $S_{\alpha\beta}{}^{\gamma} = C_{\alpha I}{}^{J} e^I_\beta e^\gamma_J$, we have $S_{\beta\gamma}{}^{\beta} = 0$. We write it as

$$(C_{\beta I}{}^{J} e^\beta_J) e^I_\gamma = 0, \tag{3.77}$$

and as e^I_α are invertible this implies

$$C_{\beta I}{}^{J} e^\beta_J = 0. \tag{3.78}$$

Thus the terms $C_{\beta I}{}^{K} e^\beta_K e^\alpha_J$, and $C_{\beta J}{}^{K} e^\alpha_I e^\beta_K$ of Eq. 1 both vanish and Eq. 1 reduces to

$$C_{\beta I}{}^{K} e^\alpha_K e^\beta_J - C_{\beta J}{}^{K} e^\beta_I e^\alpha_K = 0. \tag{3.79}$$

If we now contract this with $e^I_\gamma e^J_\delta$, we get

$$\begin{aligned} 0 &= (C_{\beta I}{}^{K} e^\alpha_K e^\beta_J - C_{\beta J}{}^{K} e^\beta_I e^\alpha_K) e^I_\gamma e^J_\delta \\ &= C_{\beta I}{}^{K} e^\alpha_K e^I_\gamma \delta^\beta_\delta - C_{\beta J}{}^{K} \delta^\beta_\gamma e^\alpha_K e^J_\delta \\ &= C_{\delta I}{}^{K} e^I_\gamma e^\alpha_K - C_{\gamma J}{}^{K} e^J_\delta e^\alpha_K \end{aligned} \tag{3.80}$$

or

$$S_{\gamma\delta}{}^{\alpha} = S_{(\gamma\delta)}{}^{\alpha}. \tag{3.81}$$

Since we have $S_{\alpha\beta\gamma} = S_{\alpha[\beta\gamma]}$ and $S_{\alpha\beta\gamma} = S_{(\alpha\beta)\gamma}$, we can successively interchange the first two and then last two indices with appropriate sign change each time to obtain,

$$S_{\alpha\beta\gamma} = S_{\beta\alpha\gamma} = -S_{\beta\gamma\alpha} = -S_{\gamma\beta\alpha} = S_{\gamma\alpha\beta} = S_{\alpha\gamma\beta} = -S_{\alpha\beta\gamma} \tag{3.82}$$

Implying $S_{\alpha\beta\gamma} = 0$, or

$$C_{\alpha IJ} e^I_\beta e^J_\gamma = 0, \tag{3.83}$$

and since the e^I_α are invertible, we get $C_{\alpha IJ} = 0$. This is the desired result.

3.4.4 GR minimally coupled to Fermion action

Second-order formulation

$$S[\psi, e] = \frac{1}{16\pi G}\int d^4x\epsilon^{\mu\nu\sigma\rho}e^I_\mu e^J_\nu F^{KL}_{\sigma\rho}[\omega(e)]\epsilon_{IJKL} + \int d^4x\epsilon^{\mu\nu\sigma\rho}\overline{\psi}\gamma^I(D_\mu[\omega[e]]\psi)e^J_\nu F^{KL}_{\sigma\rho}[\omega]\epsilon_{IJKL} \tag{3.84}$$

$$\begin{aligned} -32G\delta S[\psi, e, \omega] &= \\ &+ \int d^4x\epsilon^{\mu\nu\sigma\rho}(\delta\overline{\psi})\gamma^I(D_\mu\psi)e^J_\nu F^{KL}_{\sigma\rho}[\omega[e]]\epsilon_{IJKL} \\ &+ \int d^4x\epsilon^{\mu\nu\sigma\rho}\overline{\psi}\gamma^I(D_\mu\delta\psi)e^J_\nu F^{KL}_{\sigma\rho}[\omega[e]]\epsilon_{IJKL} \end{aligned} \tag{3.85}$$

Variation with respect to the conjugate field $\overline{\psi}$ gives the curved spacetime Dirac equation:

$$\epsilon^{\mu\nu\sigma\rho}\gamma^I(D_\mu\psi)e^J_\nu F^{KL}_{\sigma\rho}[\omega[e]]\epsilon_{IJKL} = 0 \tag{3.86}$$

First-order formulation

$$S[\psi, e, \omega] = \frac{1}{16\pi G}\int d^4x\epsilon^{\mu\nu\sigma\rho}e^I_\mu e^J_\nu F^{KL}_{\sigma\rho}[\omega(e)]\epsilon_{IJKL} = \int d^4x\epsilon^{\mu\nu\sigma\rho}\overline{\psi}\gamma^I(D_\mu[\omega]\psi)e^J_\nu e^K_\sigma e^L_\rho\epsilon_{IJKL} \tag{3.87}$$

$$\begin{aligned} -32G\delta S[\psi, e, \omega] &= 2\int d^4x\epsilon_{IJKL}[\epsilon^{\mu\nu\sigma\rho}(\delta e^I_\mu)e^J_\nu F^{KL}_{\sigma\rho}] + \int d^4x[\epsilon^{\mu\nu\sigma\rho}e^I_\mu e^J_\nu D_\sigma\delta\omega^{KL}_\rho] + \\ &+ \int d^4x\epsilon^{\mu\nu\sigma\rho}\overline{\psi}\gamma^I((\delta D_\mu[\omega])\psi)e^J_\nu e^K_\sigma e^L_\rho\epsilon_{IJKL} \\ &+ \int d^4x\epsilon^{\mu\nu\sigma\rho}(\delta\overline{\psi})\gamma^I(D_\mu\psi)e^J_\nu F^{KL}_{\sigma\rho}[\omega]\epsilon_{IJKL} \\ &+ \int d^4x\epsilon^{\mu\nu\sigma\rho}\overline{\psi}\gamma^I(D_\mu\delta\psi)e^J_\nu F^{KL}_{\sigma\rho}[\omega]\epsilon_{IJKL} \end{aligned} \tag{3.88}$$

The variation of ω does not give the vanishing of the torsion.

3.5 Boundary term

$$S_{EH\ boundary} = \int_\Sigma k^{ab}\ q_{ab}\sqrt{q}d^3\sigma \tag{3.89}$$

where k^{ab} is the extrinsic curvature of the boundary.

In the case of pure gravity without cosmological constant, since the Ricci scalar vanishes on solutions of the Einstein equations, the bulk action vanishes and the Hamilton function is just given by the boundary term

$$S_{EH}[q] = \int_\Sigma k^{ab}[q]\ q_{ab}\sqrt{q}d^3\sigma. \tag{3.90}$$

3.5.1 Derivation of Action where there is a boundary

In the event that the manifold has a boundary $\partial\mathcal{V}$, the action should be supplemented by a boundary term so that the variational principle is well defined. The appropriate action is

$$S_{EH} + S_{GHY} = \int_\mathcal{V} dx^4\sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi}\int_{\partial\mathcal{V}} d^3y\epsilon\sqrt{h}K, \tag{3.91}$$

where S_{EH} is the usual Einstein-Hilbert action, S_{GHY} is the Gibbons-Hawking-York boundary term, h_{ab} is the inuced metric on the boundary, h is the determinant, K is the trace of the second fundamental form, ϵ is equal to $+1$ where $\partial\mathcal{V}$ is timelike and -1 where $\partial\mathcal{V}$ is spacelike, and y^a are the coordinates intrinsic to the boundary $\partial\mathcal{V}$. Varing this action with respect to the metric $g_{\alpha\beta}$, subject to the condition

$$\delta g_{\alpha\beta}\big|_{\partial\mathcal{V}} = 0, \tag{3.92}$$

gives the Einstein equations; the addition of the boundary term means that in performing the variation, the geometry intrinsic to the boundary encoded in the induced metric h_{ab} is held fixed.

That boundary term needed in the gravitation case is due to the fact that R, the gravitatioal Lagrangian density, contains second derivatives of the metric tensor. This non-typical feature of field theories, which are usually formulated in terms of Lagrangians that involve first derivatives of fields varied over.

The GHY term is essential. When one passes to the Hamiltonian formulism, it is necessary to include the GHY term in order to reproduce the correct Arnowitt-Deser-Minner (ADM enery). The term is required to ensue the old path integrl (a la Hawking et al) for quantum gravity has the correct composition properties. When calculating black hole entropy using the Euclidean semiclassical approach, the entire contribution comes from the GHY term. This term has been has had more recent applications in loop quantum gravity in calculating transition amplitudes and background-independent scattering amplitudes.

In order to have a finite value for the action, we may have to subtract off a surface term for flat spacetime:

$$S_{EH} + S_{GHY,0} = \frac{1}{16\pi}\int_V \sqrt{-g}(R-2\Lambda)dx^4 + \frac{1}{8\pi}\int_{\partial V} d^3y\epsilon\sqrt{h}Kd^3y - \frac{1}{8\pi}\int_{\partial V} d^3y\epsilon\sqrt{h}K_0 d^3y \tag{3.93}$$

where K_0 is the extrinsic curvature of the boundary imbedded in flat space. As $\sqrt{h}$ is invariant under variations of $g_{\alpha\beta}$ this additional term does not effect the field equtions, as such that this is referred to as the non-dynamical term. This term is irrelevant if spacetime is compact or if we are considering compact regions of spacetime.

3.5.2 Introduction to hypersurfaces

Defining hypersurfaces

In a four-dimensional spacetime manifold, a hypersurface is a three-dimensional submanifold that can be either timelike, spacelike, or null.

A particular hyper-surface Σ can be selected either by imposing a constraint on the coordinates

$$f(x^\alpha) = 0, \tag{3.94}$$

or by giving parametric equations,

$$x^\alpha = x^\alpha(y^a), \tag{3.95}$$

where y^a $(a = 1, 2, 3)$ are coordinates intrinsic to the hyper-surfaace.

For example, a two-sphere in three-dimensional Euclidean space can be described either by

$$f(x^\alpha) = x^2 + y^2 + z^2 - r^2 = 0,$$

where r is the radius of the sphere, or by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad and\ z = r\cos\theta,$$

where θ and ϕ are intrinsic coordinates.

For example, a two-sphere in three-dimensional Euclidean space can be described either by

$$f(x^\alpha) = x^2 + y^2 + z^2 - r^2 = 0,$$

where r is the radius of the sphere, or by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad and\ z = r\cos\theta,$$

where θ and ϕ are intrinsic coordinates.

Hypersurface orthogonal vector fields

We start with the family of hyper-surfaces given by

$$f(x^\alpha) = C \tag{3.96}$$

where different members of the family correspond to different values of the constant C. Consider two neighbouring points P and Q with coordinates x^α and $x^\alpha + dx^\alpha$, respectively, lying in the same hyper-surface. We then have to first order

$$C = f(x^\alpha + dx^\alpha) = f(x^\alpha) + \frac{\partial f}{\partial x^\alpha} dx^\alpha. \tag{3.97}$$

Subtracting off $C = f(x^\alpha)$ from this equation gives

$$\frac{\partial f}{\partial x^\alpha} dx^\alpha = 0 \tag{3.98}$$

at P. This implies that $f_{,\alpha}$ is normal to the hyper-surface. A unit normal n_α can be introduced in the case where the hyper-surface is not null. This is defined by

$$n^\alpha n_\alpha \equiv \epsilon = \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike,} \\ +1 & \text{if } \Sigma \text{ is timelike,} \end{cases}, \tag{3.99}$$

and we require that n^α point in the direction of increasing $f : n^\alpha f_{,\alpha} > 0$. It can then easily be checked that n_α is given by

$$n_\alpha = \frac{\epsilon f_{,\alpha}}{|g^{\alpha\beta} f_{,\alpha} f_{,\beta}|^{\frac{1}{2}}} \tag{3.100}$$

if the hypersurface either spacelike or timelke.

Induced and transverse metricInduced and transverse metric

The three vectors

$$e^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_{\partial \mathcal{M}} \qquad a = 1, 2, 3 \tag{3.101}$$

are tangential to the hyper-surface.

The induced metric is the three-tensor h_{ab} defined by

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b. \tag{3.102}$$

This acts as a metric tensor on the hyper-surface in the y^a coordinates. For displacements confined to the hyper-surface (so that $x^\alpha = x^\alpha(y^a)$)

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} dy^a\right) \left(\frac{\partial x^\beta}{\partial y^b} dy^b\right) \\ &= \left(g_{\alpha\beta} e^\alpha_a e^\beta_b\right) dy^a dy^b \\ &= h_{ab} dy^a dy^b. \end{aligned} \tag{3.103}$$

Because the three vectors $e^\alpha_1, e^\alpha_2, e^\alpha_3$ are tangential to the hyper-surface,

$$n_\alpha e^\alpha_a = 0 \tag{3.104}$$

where n_α is the unit vector ($n_\alpha n^\alpha = \pm 1$) normal to the hyper-surface.

We introduce what is called the transverse metric

$$h_{\alpha\beta} = g_{\alpha\beta} - \epsilon n_\alpha n_\beta. \tag{3.105}$$

It isolates the part of the metric that is transverse to the normal n^α.

Is is easily seen that this four-tensor

$$h^\alpha{}_\beta = \delta^\alpha{}_\beta - \epsilon n^\alpha n_\beta \tag{3.106}$$

projects out the part of a four-vector transverse to the normal n^α as

$$h^\alpha{}_\beta n^\beta = (\delta^\alpha{}_\beta - \epsilon n^\alpha n_\beta) n^\beta = (n^\alpha - \epsilon^2 n^\alpha) = 0 \quad \text{and if} \quad w^\alpha n_\alpha = 0 \quad \text{then} \quad h^\alpha{}_\beta w^\beta = w^\beta. \tag{3.107}$$

We have

$$h_{ab} = h_{\alpha\beta} e^\alpha_a e^\beta_b. \tag{3.108}$$

If we define h^{ab} to be the inverse of h_{ab}, it is easy to check

$$h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b \tag{3.109}$$

where

$$h^{\alpha\beta} = g^{\alpha\beta} - \epsilon n^\alpha n^\beta. \tag{3.110}$$

Note that variation subject to the condition

$$\delta g_{\alpha\beta}\big|_{\partial\mathcal{M}} = 0, \tag{3.111}$$

implies that $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$, the induced metric on $\partial\mathcal{M}$, is held fixed during the variation.

3.5.3 Proof of main result

Variation of the Einstein-Hilbert term has an additional term that converts into a boundary term that does not vanish when one has boundary. We will add in the variation of the GHY-boundary term, and show that their sum results in:

$$\delta S_{TOTAL} = \delta S_{EH} + \delta S_{GHY} = \frac{1}{16\pi}\int_{\mathcal{M}} \left(G_{\alpha\beta} + \Lambda g_{\alpha\beta}\right)\delta g^{\alpha\beta}\sqrt{-g}d^4x \tag{3.112}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is the Einstein tensor, which together with the cosmological term, produces the correct left-hand side to the Einstein field equations.

Variation of Einstein-Hilbert term

We will use the identity

$$\delta\sqrt{-g} \equiv -\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}$$

and the Palatini identity:

$$\delta R_{\alpha\beta} \equiv \nabla_\mu(\delta\Gamma^\mu_{\alpha\beta}) - \nabla_\beta(\delta\Gamma^\mu_{\alpha\mu}),$$

$$\int_{\mathcal{V}} g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}d^4x \tag{3.113}$$

We consider the variation of the Einstein-Hilbert term:

$$\begin{aligned}
(16\pi)\delta S_{EH} &= \int_{\mathcal{V}} \delta(g^{\alpha\beta}R_{\alpha\beta}\sqrt{-g})d^4x \\
&= \int_{\mathcal{V}} \left(R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} + g^{\alpha\beta}R_{\alpha\beta}\delta\sqrt{-g} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}\right)d^4x \\
&= \int_{\mathcal{V}} \left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)d^4x + \int_{\mathcal{V}} g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}d^4x
\end{aligned} \tag{3.114}$$

The first term gives us what we need for the left-hand side of the Einstein fields equations. We must account for the second term.

By the Palatini identity

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\mu \delta V^\mu, \qquad \delta V^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\Gamma^\beta_{\alpha\beta}. \tag{3.115}$$

We will need Stokes theorem in the form:

$$
\begin{aligned}
\int_{\mathcal{V}} \nabla_\mu A^\mu \sqrt{-g} d^4x &= \int_{\mathcal{V}} \partial_\mu \left(\sqrt{-g} A^\mu\right) d^4x \\
&= \oint_{\partial\mathcal{V}} A^\mu d\Sigma_\mu \\
&= \oint_{\partial\mathcal{V}} \epsilon A^\mu n_\mu \sqrt{|h|} d^3y \qquad (3.116)
\end{aligned}
$$

where n_μ is the unit normal to $\partial\mathcal{V}$, and $\epsilon \equiv n_\mu n^\mu = \pm 1$ and y^a are cordinates intrinsic to the boundary. And $d\Sigma_\mu = \epsilon n_\mu d\Sigma$ where $d\Sigma = |h|^{\frac{1}{2}} d^3y$ where $h = \det[h_{ab}]$, is an invariant three-dimensional volume element on the hypersurface. In our particulr case we take $A^\mu = \delta V^\mu$.

We now evaluate $\delta V^\mu n_\mu$ on the boundary $\partial\mathcal{M}$, keeping in mind that on $\partial\mathcal{M}$, $\delta g_{\alpha\beta} = 0 = \delta g^{\alpha\beta}$. Taking this into account we have

$$
\delta\Gamma^\mu_{\alpha\beta}\big|_{\partial\mathcal{M}} = \frac{1}{2} g^{\mu\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}). \qquad (3.117)
$$

It is useful to note that

$$
\begin{aligned}
g^{\alpha\mu}\delta\Gamma^\beta_{\alpha\beta}\big|_{\partial\mathcal{M}} &= \frac{1}{2} g^{\alpha\mu} g^{\beta\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}) \\
&= \frac{1}{2} g^{\mu\nu} g^{\alpha\beta}(\delta g_{\nu\alpha,\beta} + \delta g_{\alpha\beta,\nu} - \delta g_{\nu\beta,\alpha}) \qquad (3.118)
\end{aligned}
$$

where in the second line we have swapped around α and ν and used that the metric is symmetric. It is then not difficult to work out $\delta V^\mu = g^{\mu\nu} g^{\alpha\beta}(\delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu})$.

So now

$$
\begin{aligned}
\delta V^\mu n_\mu\big|_{\partial\mathcal{M}} &= n^\mu g^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\
&= n^\mu(\epsilon n^\alpha n^\beta + h^{\alpha\beta})(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\
&= n^\mu h^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \qquad (3.119)
\end{aligned}
$$

where in the second line we used the identity $g^{\alpha\beta} = \epsilon n^\alpha n^\beta + h^{\alpha\beta}$, and in the third line we have used the anti-symmetry in α and μ. As $\delta g_{\alpha\beta}$ vanishes everywhere on the boundary, $\partial\mathcal{M}$, its tangential derivatives must also vanish: $\delta g_{\alpha\beta,\gamma} e^\gamma_c = 0$. It follows that $h^{\alpha\beta}\delta g_{\mu\beta,\alpha} = h^{ab} e^\alpha_a e^\beta_b \delta g_{\mu\beta,\alpha} = 0$. So finally we have

$$n^\mu \delta V_\mu\big|_{\partial\mathcal{M}} = -h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^\mu. \tag{3.120}$$

Gathering the results we obtain

$$(16\pi)\delta S_{EH} = \int_{\mathcal{V}} G_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}d^4x - \oint_{\partial\mathcal{V}} \epsilon h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^\mu\sqrt{h}d^3y. \tag{3.121}$$

We next show that the above boundary term will be cancelled by the variation of S_{GHY}.

Variation of the boundary term

We now turn to the variation of the S_{GHY} term. Because the induced metric is fixed on $\partial\mathcal{V}$, the only quantity to be varied is K is the trace of the extrinsic curvature.

We have

$$\begin{aligned}
K &= n^\alpha{}_{;\alpha} \\
&= g^{\alpha\beta}n_{\alpha;\beta} \\
&= (\epsilon n^\alpha n^\beta + h^{\alpha\beta})n_{\alpha;\beta} \\
&= h^{\alpha\beta}n_{\alpha;\beta} \\
&= h^{\alpha\beta}(n_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta}n_\gamma)
\end{aligned} \tag{3.122}$$

where we have used that $0 = (n^\alpha n_\alpha)_{;\beta}$ implies $n^\alpha n_{\alpha;\beta} = 0$. So the variation of K is

$$\begin{aligned}
\delta K &= -h^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta}n_\gamma \\
&= -h^{\alpha\beta}n^\mu g_{\mu\gamma}\delta\Gamma^\gamma_{\alpha\beta} \\
&= -h^{\alpha\beta}n^\mu g_{\mu\gamma}\frac{1}{2}g^{\gamma\sigma}\big(\delta g_{\sigma\alpha;\beta} + \delta g_{\sigma\beta;\alpha} - \delta g_{\alpha\beta;\sigma}\big) \\
&= -\frac{1}{2}h^{\alpha\beta}\big(\delta g_{\mu\alpha;\beta} + \delta g_{\mu\beta;\alpha} - \delta g_{\alpha\beta;\mu}\big)n^\mu \\
&= \frac{1}{2}h^{\alpha\beta}\delta g_{\alpha\beta;\mu}n^\mu
\end{aligned} \tag{3.123}$$

where we have use the fact that the tangential derivatives of $\delta g_{\alpha\beta}$ vanish on $\partial\mathcal{V}$. We have obtained

$$(16\pi)\delta S_{GHY} = \oint_{\partial\mathcal{V}} \epsilon h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^\mu\sqrt{h}d^3y \tag{3.124}$$

which cancels the second integral on the right-hand side of (3.121). The total variation of the gravitational action is:

$$\delta S_{TOTAL} = \frac{1}{16\pi}\int_{\mathcal{V}}\left(G_{\alpha\beta}+\Lambda g_{\alpha\beta}\right)\delta g^{\alpha\beta}\sqrt{-g}d^4x. \tag{3.125}$$

This produces the correct left-hand side of the Einstein equations.

3.5.4 Variation of the Matter action

Matter actions only depend on the field and its first derivatives,

$$S_M = \int_{\mathcal{V}}\mathcal{L}[\phi,\phi_{,a};g_{\alpha\beta}]\sqrt{-g}d^4x \tag{3.126}$$

Variation of $S_M[\phi;g_{\alpha\beta}]$ yields

$$\begin{aligned}\delta S_M &= \int_{\mathcal{V}}\delta\left(\mathcal{L}\sqrt{-g}\right)d^4x\\ &= \int_{\mathcal{V}}\left(\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}\sqrt{-g}+\mathcal{L}\delta\sqrt{-g}\right)d^4x\\ &= \int_{\mathcal{V}}\left(\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}}-\frac{1}{2}\mathcal{L}g_{\alpha\beta}\right)\delta g^{\alpha\beta}\sqrt{-g}d^4x.\end{aligned} \tag{3.127}$$

If we define the stress-energy tensor by

$$T_{\alpha\beta} := -2\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}}+\mathcal{L}g_{\alpha\beta} \tag{3.128}$$

then

$$\delta S_M = -\frac{1}{2}\int_{\mathcal{V}}T_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}d^4x, \tag{3.129}$$

and this produces the correct right-hand side of the Einstein field equations, so that $\delta(S_G+S_M)=0$ implies

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{3.130}$$

because the variation $\delta g^{\alpha\beta}$ is arbitrary within $\mathcal{V}$.

3.6 Hamiltonian general relativity

3.6.1 ADM variables

$$
\begin{aligned}
\pi^{ab} &= \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}} = \frac{\partial}{\partial \dot{q}_{ab}} \frac{\sqrt{q}(q^{a'c'}q^{b'd'} - q^{a'b'}q^{c'd'})(\dot{q}_{a'b'} - 2\mathcal{D}_{(a'}N_{b')})(\dot{q}_{c'd'} - 2\mathcal{D}_{(c'}N_{d')})}{4N} \\
&= \frac{\sqrt{q}(q^{ac'}q^{bd'} - q^{ab}q^{c'd'})K_{c'd'}}{2} + \frac{\sqrt{q}(q^{a'a}q^{b'b} - q^{a'b'}q^{ab})K_{a'b'}}{2} \\
&= \frac{\sqrt{q}}{2}\left[(q^{ac}q^{bd} - q^{ab}q^{cd}) + (q^{ca}q^{db} - q^{cd}q^{ab})\right] K_{cd} \\
&= \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{cd} \\
&= \sqrt{q}G^{abcd}K_{cd} \\
&= \sqrt{q}(K^{ab} - Kq^{ab}) \qquad (3.131)
\end{aligned}
$$

$$
\begin{aligned}
\pi^{ab} &= \sqrt{q}G^{abcd}K_{cd} \\
&= \sqrt{q}(K^{ab} - Kq^{ab}). \qquad (3.132)
\end{aligned}
$$

From this we can write $\pi^{ab}\dot{q}_{ab} - \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{cd}\dot{q}_{ab} = 0$. Then sunstituiting this into the action we find,

$$
\begin{aligned}
S[N,\vec{N},q,\pi] &= \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{cd}\dot{q}_{ab} \right.\\
&\qquad \left. + \sqrt{q}N(q^{ac}q^{bd} - q^{ab}q^{cd})K_{ab}K_{cd} + \sqrt{q}NR[q]\right)\\
&= \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} + \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})(\dot{q}_{ab} - NK_{ab})K_{cd} + \sqrt{q}NR[q]\right)\\
&= \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} + \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})(2\mathcal{D}_{(a}N_{b)} + NK_{ab})K_{cd} + \sqrt{q}NR[q]\right)\\
&= \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - N\left\{\sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{ab}K_{cd} - \sqrt{q}R[q]\right\} \right.\\
&\qquad \left. + \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})2\mathcal{D}_{(a}N_{b)}K_{cd}\right)\\
&= \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - N\left\{\sqrt{q}(K_{ab}K^{ab} - K^2) - \sqrt{q}R[q]\right\} \right.\\
&\qquad \left. - 2N_{(a}\mathcal{D}_{b)}\sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{cd}\right)\\
&= \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - N\left\{\sqrt{q}(K_{ab}K^{ab} - K^2) - \sqrt{q}R[q]\right\} \right.\\
&\qquad \left. - 2N_{(a}\mathcal{D}_{b)}\sqrt{q}(K^{ab} - q^{ab}K)\right)\\
&= \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - N\sqrt{q}\{K_{ab}K^{ab} - K^2 - R[q]\} - 2N_b\{\mathcal{D}_a\pi^{ab}\}\right). \qquad (3.133)
\end{aligned}
$$

where in the last line we have used (3.132). With the use of the DeWitt super metric

$$
G_{abcd} = \frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}) \qquad (3.134)
$$

With the use of (3.132), $K^{ab}K_{ab} - K^2$ can be written as $G_{abcd}\pi^{ab}\pi^{cd}$:

$$
\begin{aligned}
G_{abcd}\pi^{ab}\pi^{cd} &= \frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})\sqrt{q}(K^{ab} - q^{ab}K)\sqrt{q}(K^{cd} - q^{cd}K)\\
&= \frac{\sqrt{q}}{2}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})(K^{ab}K^{cd} - q^{ab}K^{cd}K - q^{cd}K^{ab}K + q^{ab}q^{cd}K^2)\\
&= \frac{\sqrt{q}}{2}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})(K^{ab}K^{cd} - 2q^{ab}K^{cd}K + q^{ab}q^{cd}K^2)\\
&= \frac{\sqrt{q}}{2}\left((2K^{ab}K_{ab} - K^2) - 2(2K - 3K)K + (3 + 3 - 9)K^2\right)\\
&= \sqrt{q}\left(K^{ab}K_{ab} - K^2\right). \qquad (3.135)
\end{aligned}
$$

We substitute this result into the last line of (3.133). The action written in Hamiltonian form then reads

$$S[N, \vec{N}, q, \pi] = \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - NC(\pi, q) - 2N^a C_a(\pi, q)\right) \tag{3.136}$$

where

$$C = G_{abcd}\pi^{ab}\pi^{cd} - \sqrt{q}R[q] \tag{3.137}$$

is the scalar constraint, or Hamiltonian constraint, and where

$$C^b = \mathcal{D}_a\pi^{ab} \tag{3.138}$$

is the vector, or diffeomorphism constraint.

3.7 Ashtekar formulism

3.8 Self-dual Connection Formulation

3.8.1 Self-dual Curvature

We will need the totally antisymmetry tensor or Levi-Civita symbol, ϵ_{IJKL}. Recall that this is equal to either +1 or -1 depending on whether $IJKL$ is either an even or odd permutation of 0123, respectively, and zero if any two indices take the same value. The internal indices of ϵ_{IJKL} are raised with the Minkowski metric η^{IJ}.

Given any anti-symmetric tensor T^{IJ}, we define its dual as

$$*T^{IJ} = \frac{1}{2}\epsilon_{KL}{}^{IJ}T^{\ KL} \tag{3.139}$$

The self-dual part of any tensor T^{IJ} is defined as

$${}^{+}T^{IJ} := \frac{1}{2}\Big(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}\Big) \tag{3.140}$$

with the anti-self-dual part defined as

$${}^{-}T^{IJ} := \frac{1}{2}\Big(T^{IJ} + \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}\Big) \tag{3.141}$$

(the appearance of the imaginary unit i is related to the Minkowski signature as we will see below).

Tensor decomposition

Now given any anti-symmetric tensor T^{IJ}, we can decompose it as

$$T^{IJ}=\frac{1}{2}(T^{IJ}-\frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL})+\frac{1}{2}(T^{IJ}+\frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL})=\ {}^{+}T^{IJ}+\ {}^{-}T^{IJ} \tag{3.142}$$

where ${}^{+}T^{IJ}$ and ${}^{-}T^{IJ}$ are the self-dual and anti-self-dual parts of T^{IJ} respectively. Define the projector onto (anti-)self-dual part of any tensor as

$$P^{(\pm)}=\frac{1}{2}(1\mp i*). \tag{3.143}$$

The meaning of these projectors can be made explicit. Let us concentrate of P^{+},

$$(P^{+}T)^{IJ}=(\frac{1}{2}(1-i*)T)^{IJ}=\frac{1}{2}(\delta^{I}_{K}\delta^{J}_{L}-i\frac{1}{2}\epsilon_{KL}{}^{IJ})T^{KL}=\frac{1}{2}(T^{IJ}-\frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL})=\ {}^{+}T^{IJ}. \tag{3.144}$$

Then

${}^{\pm}T^{IJ}=(P^{(\pm)}T)^{IJ}$.

The Lie bracket

An important object is the Lie bracket defined by

$$[F,G]^{IJ}:=F^{IK}G_{K}^{\ J}-G^{IK}F_{K}^{\ J}, \tag{3.145}$$

it appears in the curvature tensor (see the last two terms of ()), it also defines the algebraic structure. We have the results (proved below):

$$P^{(\pm)}[F,G]^{IJ}=[P^{(\pm)}F,G]^{IJ}=[F,P^{(\pm)}G]^{IJ}=[P^{(\pm)}F,P^{(\pm)}G]^{IJ} \tag{3.146}$$

and

$$[F,G]=[P^{+}F,P^{+}G]+[P^{-}F,P^{-}G]. \tag{3.147}$$

That is the Lie bracket, which defines an algebra, decomposes into two separate independent parts.

Identities for the totally anti-symmetric tensor

The internal indices of ϵ_{IJKL} are raised with the Minkowski metric η^{IJ}. Since η_{IJ} has signature $(-,+,+,+)$, it follows that

$$\epsilon^{IJKL} = -\epsilon_{IJKL}.$$

to see this consider,

$$\begin{aligned} \epsilon^{0123} &= \eta^{0I}\eta^{1J}\eta^{2K}\eta^{3L}\epsilon_{IJKL} \\ &= (-1)(+1)(+1)(+1)\epsilon_{0123} = -\epsilon_{0123}. \end{aligned}$$

We then have the identities,

$$\epsilon^{IJKO}\epsilon_{LMNO} = -6\delta^I_{[L}\delta^J_M\delta^K_{N]} \tag{3.148}$$

$$\begin{aligned} \epsilon^{IJMN}\epsilon_{KLMN} &= -4\delta^I_{[K}\delta^J_{L]} \\ &= -2(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K). \end{aligned} \tag{3.149}$$

Definition of self-dual tensor

Therefore the square of the duality operator is minus the identity,

$$**T^{IJ} = \frac{1}{4}\epsilon_{KL}{}^{IJ}\epsilon_{MN}{}^{KL}T^{\ MN} = -T^{IJ} \tag{3.150}$$

The minus sign here is due to the minus sign in (3.149), which is in turn due to the Minkowski signature. Had we used Euclidean signature, i.e. $(+,+,+,+)$, instead there would have been a positive sign. We define S^{IJ} to be self-dual if and only if

$$*S^{IJ} = iS^{IJ}. \tag{3.151}$$

(with Euclidean signature the self-duality condition would have been $*S^{IJ} = S^{IJ}$). Say S^{IJ} is self-dual, write it as a real and imaginary part,

$$S^{IJ} = \frac{1}{2}U^{IJ} + i\frac{1}{2}V^{IJ}.$$

Write the self-dual condition in terms of U and V,

$$*(U^{IJ} + iV^{IJ}) = \frac{1}{2}\epsilon_{KL}{}^{IJ}(U^{KL} + iV^{KL}) = i(U^{IJ} + iV^{IJ}).$$

Equating real parts we read off

$$V^{IJ} = -\frac{1}{2}\epsilon_{KL}{}^{IJ}U^{KL}$$

and so

$$S^{IJ} = \frac{1}{2}(U^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}U^{KL}) \tag{3.152}$$

where U^{IJ} is the real part of $2T^{IJ}$. Now given any tensor T^{IJ}, we can decompose it as

$$\begin{aligned} T^{IJ} &= \frac{1}{2}(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{IJ}) + \frac{1}{2}(T^{IJ} + \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{IJ}) \\ &= {}^{+}T^{IJ} + {}^{-}T^{IJ} \end{aligned} \tag{3.153}$$

since $*({}^{+}T^{IJ}) = i\,{}^{+}T^{IJ}$ and $*({}^{-}T^{IJ}) = -i\,{}^{-}T^{IJ}$, ${}^{+}T^{IJ}$ and ${}^{-}T^{IJ}$ are the self-dual and anti-self-dual parts of T^{IJ}.

Important lengthy calculation

The following lengthy calculation is important as all the other important formula can easily be derived from it. From the definition of the Lie bracket and with the use of (3.148) we have

$$
\begin{aligned}
*[F,*G]^{IJ} &= \frac{1}{2}\epsilon_{MN}{}^{IJ}(F^{MK}(*G)_K{}^N - (*G)^{MK}F_K{}^N) \\
&= \frac{1}{2}\epsilon_{MN}{}^{IJ}(F^{MK}\frac{1}{2}\epsilon_{OPK}{}^N G^{OP} - \frac{1}{2}\epsilon_{OP}{}^{MK}G^{OP}F_K{}^N) \\
&= \frac{1}{4}(\epsilon_{MN}{}^{IJ}\epsilon_{OP}{}^{KN} + \epsilon_{NM}{}^{IJ}\epsilon_{OP}{}^{NK})F^M{}_K G^{OP} \\
&= \frac{1}{2}\epsilon_{MN}{}^{IJ}\epsilon_{OP}{}^{KN}F^M{}_K G^{OP} \\
&= \frac{1}{2}\epsilon^{MIJN}\epsilon_{OPKN}F_M{}^K G^{OP} \\
&= -\frac{1}{2}\epsilon^{KIJN}\epsilon_{OPMN}F^M{}_K G^{OP} \\
&= \frac{1}{2}(\delta^K_O\delta^I_P\delta^J_M + \delta^K_M\delta^I_O\delta^J_P + \delta^K_P\delta^I_M\delta^J_O \\
&\quad - \delta^K_P\delta^I_O\delta^J_M - \delta^K_M\delta^I_P\delta^J_O - \delta^K_O\delta^I_M\delta^J_P)F^M{}_K G^{OP} \\
&= \frac{1}{2}(F^J{}_K G^{KI} + F^K{}_K G^{IJ} + F^I{}_K G^{JK} \\
&\quad - F^J{}_K G^{IK} - F^K{}_K G^{JI} - F^I{}_K G^{KJ}) \\
&= -F^{IK}G_K{}^J + G^{IK}F_K{}^J \\
&= -[F,G]^{IJ}
\end{aligned}
\tag{3.154}
$$

That gives the formula

$$
*[F,*G]^{IJ} = -[F,G]^{IJ}. \tag{3.155}
$$

from which everything else is then easy to derive.

Derivation of important results

Now using (3.155) in conjunction with $** = -1$

$$
*\left(-[F,G]^{IJ}\right) = *\left(*[F,*G]^{IJ}\right) = **[F,*G]^{IJ} = -[F,*G]^{IJ}.
$$

So we have

$$
*[F,G]^{IJ} = -[F,*G]^{IJ}. \tag{3.156}
$$

Consider

$$*[F,G]^{IJ} = - * [G,F]^{IJ} = -[G,*F]^{IJ} = [*F,G]^{IJ}$$

where in the first step we have used the anti-symmetry of the Lie bracket to swap F and G, in the second step we used (3.156) and in the last step we used the anti-symmetry of the Lie bracket again. So we have

$$*[F,G]^{IJ} = [*F,G]^{IJ}. \tag{3.157}$$

Then

$$\begin{aligned} \left(P^{(\pm)}[F,G]\right)^{IJ} &= \frac{1}{2}\left([F,G]^{IJ} \mp i * [F,G]\right) \\ &= \frac{1}{2}\left([F,G]^{IJ} + [F, \mp i * G]\right) \\ &= [F, P^{(\pm)}G]^{IJ}. \end{aligned} \tag{3.158}$$

where we used (3.156). Similarly we have

$$\left(P^{(\pm)}[F,G]\right)^{IJ} = [P^{(\pm)}F, G]^{IJ} \tag{3.159}$$

where we have used (3.157). Now as $P^{(\pm)}$ are projection operators they satisfy $\left(P^{(\pm)}\right)^2 = P^{(\pm)}$, as can be verified by direct computation:

$$\begin{aligned} \left(P^{(\pm)}\right)^2 &= \frac{1}{4}\left(1 \mp i*\right)\left(1 \mp i*\right) \\ &= \frac{1}{4}\left(1 - ** \mp 2i*\right) \\ &= \frac{1}{4}\left(2 \mp 2i*\right) \\ &= P^{(\pm)}. \end{aligned} \tag{3.160}$$

Applying this in cojunction with (3.158)

$$\begin{aligned} \left(P^{(\pm)}[F,G]\right)^{IJ} &= \left(\left(P^{(\pm)}\right)^2 [F,G]\right)^{IJ} \\ &= \left(P^{(\pm)}[F, P^{(\pm)}G]\right)^{IJ} \\ &= [P^{(\pm)}F, P^{(\pm)}G]^{IJ} \end{aligned} \tag{3.161}$$

From (3.161) and (3.159) we have

$$\left[P^{(\pm)}F, P^{(\pm)}G\right]^{IJ} = \left[P^{(\pm)}F, G\right]^{IJ} = \left[P^{(\pm)}F, P^{(\pm)}G + P^{(\mp)}G\right]^{IJ}$$

where we used that any G can be written as a sum of its self-dual and anti-self-dual parts, i.e. $G = P^{(\pm)}G + P^{(\mp)}G$. This implies:

$$\begin{aligned} [P^+F, P^-G]^{IJ} &= 0 \\ [P^-F, P^+G]^{IJ} &= 0. \end{aligned} \tag{3.162}$$

Summary of main results

Altogether we have,

$$P^{(\pm)}[F,G]^{IJ} = [P^{(\pm)}F, G]^{IJ} = [F, P^{(\pm)}G]^{IJ} = [P^{(\pm)}F, P^{(\pm)}G]^{IJ}. \tag{3.163}$$

We then have that any bracket splits as

$$\begin{aligned} [F,G]^{IJ} &= [P^+F + P^-F, P^+G + P^-F]^{IJ} \\ &= [P^+F, P^+G]^{IJ} + [P^-F, P^-G]^{IJ}. \end{aligned} \tag{3.164}$$

into a part that depends only on self-dual Lorentzian tensors and is itself the self-dual part of $[F,G]^{IJ}$ by (3.163), and a part that depndends only on anti-self-dual Lorentzian tensors and is the anit-self-dual part of $[F,G]^{IJ}$ again by (3.163).

We can write

$$so(1,3)_{\mathbb{C}} = so(1,3)^+_{\mathbb{C}} + so(1,3)^-_{\mathbb{C}} \tag{3.165}$$

where $so(1,3)^{\pm}_{\mathbb{C}}$ contains only the self-dual (anti-self-dual) elements of $so(1,3)_{\mathbb{C}}$.

Self-dual curvature

Instead of considering the connection $\omega_a{}^{IJ}$ we will consider its self-dual part, ${}^+A_a{}^{IJ}$, with respect to the internal indices, that is,

$$i\,{}^{+}A_a{}^{IJ} = \frac{1}{2}\epsilon_{MN}{}^{IJ}\,{}^{+}A_a{}^{MN}$$

${}^{+}A_a{}^{IJ}$ is related to $\omega_a{}^{IJ}$ by

$$ {}^{+}A_a{}^{IJ} = \frac{1}{2}\omega_a{}^{IJ} - \frac{i}{4}\epsilon_{MN}{}^{IJ}\omega_a{}^{MN} \tag{3.166}$$

Define $F_{ab}{}^{JK}$ as the curvature of the self-dual connection,

$$F_{ab}{}^{IJ} = \nabla_a\,{}^{+}A_b{}^{IJ} - \nabla_b\,{}^{+}A_a{}^{IJ} + {}^{+}A_a{}^{IK}\,{}^{+}A_{bK}{}^{J} - {}^{+}A_b{}^{IK}\,{}^{+}A_{aK}{}^{J} \tag{3.167}$$

We use the above results to show this corresponds to the self-dual part of the curvature of the usual connection:

$$\begin{aligned} F_{ab}{}^{IJ} &= \nabla_a\,{}^{+}A_b{}^{IJ} - \nabla_b\,{}^{+}A_a{}^{IJ} + {}^{+}A_a{}^{IK}\,{}^{+}A_{bK}{}^{J} - {}^{+}A_b{}^{IK}\,{}^{+}A_{aK}{}^{J} \\ &= (\nabla_a(P^+\omega_b)^{IJ} - \nabla_b(P^+\omega_a)^{IJ}) + [P^+\omega_a, P^+\omega_b]^{IJ} \\ &= (P^+2\nabla_{[a}\omega_{b]})^{IJ} + (P^+[\omega_a,\omega_b])^{IJ} \\ &= (P^+\Omega_{ab})^{IJ} \end{aligned} \tag{3.168}$$

Which was the desired result. Thus we have,

$$\begin{aligned} F_{ab}{}^{IJ} &= \frac{1}{2}\Big(\Omega_{ab}{}^{IJ} - \frac{i}{2}\epsilon_{MN}{}^{IJ}\Omega_{ab}{}^{MN}\Big) \\ &= \nabla_{[a}\omega_{b]}{}^{IJ} + \omega_{[a}{}^{IK}\omega_{b]K}{}^{J} - \frac{i}{2}\epsilon_{MN}{}^{IJ}(\nabla_{[a}\omega_{b]}{}^{MN} + \omega_{[a}{}^{MK}\omega_{b]K}{}^{N}). \end{aligned} \tag{3.169}$$

Using

$$\begin{aligned} 2\nabla_{[a}\omega_{b]}{}^{IJ} + [\omega_a,\omega_b]^{IJ} &= (P^+2\nabla_{[a}\omega_{b]})^{IJ} + (P^-2\nabla_{[a}\omega_{b]})^{IJ} + [P^+\omega_a + P^-\omega_a, P^+\omega_b + P^-\omega_b]^{IJ} \\ &= \Big((P^+2\nabla_{[a}\omega_{b]})^{IJ} + [P^+\omega_a, P^+\omega_b]^{IJ}\Big) + \Big((P^-2\nabla_{[a}\omega_{b]})^{IJ} + [P^-\omega_a, P^-\omega_b]^{IJ}\Big). \end{aligned} \tag{3.170}$$

we see that the Palatini curvature decomposes as,

$$\Omega_{ab}{}^{IJ}[\omega] = \Omega_{ab}{}^{IJ}[{}^{+}A] + \Omega_{ab}{}^{IJ}[{}^{-}A] \tag{3.171}$$

and therefore the Palatini action can be written in terms of a self-dual and anti-self-dual part which depend respectively only on the self-dual and anti-self-dual connections.

3.8.2 Self-dual Action

In the self-dual formulation of general relativity, the variables are a self-dual Lorentz connection.

The other variable is a complex tetrad. The action in the self-dual formulation is built using the curvature of the self-dual Lorentz connection.

As in the Palatini formulism, one can use the tetrad to define a metric g on M by

$$g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta$$

However, because the tetrad is complex, the metric is now complex.

The self-dual action can be written

$$S_{EH} = \int d^4x \; e \; e^a_I e^b_J F_{ab}{}^{IJ} \tag{3.172}$$

We introduce a connection compatible with the tetrad via $\nabla_a e^b_I = 0$. We will like to proceed analogously as with the Palatini case and show that one of the equations of motion implies that ${}^+\mathcal{D}_a$ is the self-dual part of the unique, torsion-free covariant derivative ∇_a compatible with e^I_a.

If $\Gamma_{aI}{}^J$ denotes the Chrristoffel symbol of ∇_a, we define the self-dual part ${}^+\nabla_a$ of ∇_a by

$${}^+\nabla_a v_I := \partial_a v_I + {}^+\Gamma_{aI}{}^J v_J, \tag{3.173}$$

where ${}^+\Gamma_{aI}{}^J$ is the self-dual part of $\Gamma_{aI}{}^J$. The difference between these two connections when applied to a tensor with purely internal indices is

$${}^+C_{aI}{}^J V_J = ({}^+\mathcal{D}_a - {}^+\nabla_a) V_I. \tag{3.174}$$

Note that ${}^+C_{aI}{}^J = A_{aI}{}^J - {}^+\Gamma_{aI}{}^J$ is indeed the self-dual part of $C_{aI}{}^J$.

$$F_{ab}{}^{IJ} - {}^+R_{ab}{}^{IJ} = 2\nabla_{[a} {}^+C_{b]}{}^{IJ} + 2\, {}^+C_{[a}{}^{IK}\, {}^+C_{b]K}{}^J \tag{3.175}$$

The self-dual action is written as

$$S_{EH} = \int d^4x \; e \; e^a_I e^b_J ({}^+R_{ab}{}^{IJ} + 2\nabla_{[a} {}^+C_{b]}{}^{IJ} + 2\, {}^+C_{[a}{}^{IK}\, {}^+C_{b]K}{}^J) \tag{3.176}$$

Variation with respect to ${}^+C_{b]}{}^{IJ}$ produces

$$ {}^+C_{bI}{}^{K}e_K^{[a}e_J^{b]} + {}^+C_{bJ}{}^{K}e_I^{[a}e_K^{b]} = 0. \tag{3.177} $$

where we have used ${}^+C_{bKI} = -{}^+C_{bIK}$. This can be writtem more compactly as

$$ e_M^{[a}e_N^{b]}\delta_{[I}^M\delta_{J]}^K \, {}^+C_{bK}{}^N = 0. \tag{3.178} $$

The variation with respect to the tetrad goes along the similarly except $\Omega_{ab}{}^{IJ}$ replaced everywhere by $F_{ab}{}^{IJ}$.

3.8.3 Derivation of Ashtekar's Formalism from the Self-dual Action

The Palatini action

$$ S(e,A) = \int d^4x e e_I^a e_J^b \Omega_{ab}{}^{IJ}[\omega] \tag{3.179} $$

where the Ricci tensor, $\Omega_{ab}{}^{IJ}$, is thought of as constructed purely from the connection ω_a^{IJ}, not using the frame field. Variation with espect to the tetrad gives Einstein's equations written in terms of the tetrads, but for a Ricci tentor constructed from the connection that has no a priori relationship with the tetrad. Variation with respect to the connection tells us the connection satifies the usual compatitibility condition

$$ D_b e_a^I = 0. $$

This determines the connection in terms of the teterad and we recover the usual Ricci tensor.

The self-dual action for general relativity is given by (see appendix E)

$$ S(e,A) = \int d^4x e e_I^a e_J^b F_{ab}{}^{IJ}[A] \tag{3.180} $$

where F is the curvature of the A, the self-dual part of ω,

$$ A_a^{IJ} = \frac{1}{2}(\omega_a^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{MN}\omega_a^{MN}). \tag{3.181} $$

It can be shown that $F[A]$ is the self-dual part of $\Omega[\omega]$ (see appendix E).

Define vector fields orthogonal to n^α,

$$E_I^b = q_a^b e_b^a = (\delta_b^a + n_a n^b) e_I^a$$

Writing $E_I^a = (\delta_b^a + n_b n^a) e_I^b$ then

$$\begin{aligned}
& \int d^4x (e E_I^a E_J^b F_{ab}{}^{IJ} - 2e E_I^a e_J^d n_d n^b F_{ab}{}^{IJ}) \\
= & \int d^4x (e(\delta_c^a + n_c n^a) e_I^c (\delta_d^b + n_d n^b) e_J^d F_{ab}{}^{IJ} - 2e(\delta_c^a + n_c n^a) e_I^c e_J^d n_d n^b F_{ab}{}^{IJ}) \\
= & \int d^4x (e e_I^a e_J^b F_{ab}{}^{IJ} + e n_c n^a e_I^c e_J^b F_{ab}{}^{IJ} + e e_I^a n_d n^b e_J^d F_{ab}{}^{IJ} + e n_c n^a n_d n^b E_I^c E_J^d F_{ab}{}^{IJ} \\
& - 2e e_I^a e_J^d n_d n^b F_{ab}{}^{IJ} - 2 n_c n^a e_I^c e_J^d n_d n^b F_{ab}{}^{IJ}) \\
= & \int d^4x e e_I^a e_J^b F_{ab}{}^{IJ} = S(E, A)
\end{aligned} \tag{3.182}$$

where we used $F_{ab}{}^{IJ} = F_{ba}{}^{JI}$ and $n^a n^b F_{ab}^i = 0$. So the action can be written

$$S(E, A) = \int d^4x (e E_I^a E_J^b F_{ab}{}^{IJ} - 2e E_I^a e_J^d n_d n^b F_{ab}{}^{IJ}) \tag{3.183}$$

We have $e = N\sqrt{q}$. We now define

$$\tilde{E}_I^a = \sqrt{q} E_I^a$$

An internan tensor S^{IJ} is self-dual if and only if

$$*S^{IJ} := \frac{1}{2} \epsilon^{IJ}{}_{MN} S^{MN} = i S^{IJ}$$

and given the curvature $F_{ab}{}^{IJ}$ is self-dual we have

$$F_{ab}{}^{IJ} = -i \frac{1}{2} \epsilon^{IJ}{}_{MN} F_{ab}{}^{MN}$$

Substituting this into the action (3.183) we have,

$$S(E, A) = \int d^4x (-i \frac{1}{2} (\frac{N}{\sqrt{q}}) \tilde{E}_I^a \tilde{E}_J^b \epsilon^{IJ}{}_{MN} F_{ab}{}^{MN} - 2N n^b \tilde{E}_I^a n_J F_{ab}{}^{IJ}) \tag{3.184}$$

where we denoted $n_J = e^d_J n_d$. We pick the gauge $\tilde{E}^a_0 = 0$ and $n^I = \delta^I_0$ (this means $n_I = \eta_{IJ} n^J = \eta_{00}\delta^I_0 = -\delta^I_0$). Writing $\epsilon_{IJKL} n^L = \epsilon_{IJK}$, which in this gauge $\epsilon_{IJK0} = \epsilon_{IJK}$. Therefore,

$$
\begin{aligned}
S(E,A) &= \int d^4x(-i\frac{1}{2}(\frac{N}{\sqrt{q}})\tilde{E}^a_I \tilde{E}^b_J(\epsilon^{IJ}{}_{M0} F_{ab}{}^{M0} + \epsilon^{IJ}{}_{0M} F_{ab}{}^{0M}) - 2Nn^b \tilde{E}^a_I n_J F_{ab}{}^{IJ}) \\
&= \int d^4x(-i(\frac{N}{\sqrt{q}})\tilde{E}^a_I \tilde{E}^b_J \epsilon^{IJ}{}_M F_{ab}{}^{M0} + 2Nn^b \tilde{E}^a_I F_{ab}{}^{I0}) \qquad (3.185)
\end{aligned}
$$

The indices I, J, M range over $1, 2, 3$ and we denote them with lower case letters in a moment. By the self-duality of A^{IJ}_a,

$$
\begin{aligned}
A^{i0}_a &= -i\frac{1}{2}\epsilon^{i0}{}_{jk} A^{jk}_a \\
&= i\frac{1}{2}\epsilon^i{}_{jk} A^{jk}_a \\
&= iA^i_a. \qquad (3.186)
\end{aligned}
$$

where we used $\epsilon^{i0}{}_{jk} = -\epsilon^i{}_{0jk} = -\epsilon^i{}_{jk0} = -\epsilon^i{}_{jk}$. This implies

$$
\begin{aligned}
F_{ab}{}^{i0} &= \partial_a A^{i0}_b - \partial_a A^{i0}_b - A^{ik}_a A_{bk}{}^0 - A^{ik}_b A_{ak}{}^0 \\
&= i(\partial_a A^i_b - \partial_a A^i_b - A^{ik}_a A_{bk} - A^{ik}_b A_{ak}) \\
&= i(\partial_a A^i_b - \partial_a A^i_b - \frac{1}{2}\epsilon^{ik}{}_j A^j_a A_{bk} + \frac{1}{2}\epsilon^{ik}{}_j A^j_b A_{ak} \\
&= i(\partial_a A^i_b - \partial_a A^i_b + \epsilon_{ijk} A^j_a A^k_b) \\
&= iF^i_{ab}. \qquad (3.187)
\end{aligned}
$$

We replace in the second term Nn^b by $t^b - n^b$. We need

$$
\mathcal{L}_t A^i_b = t^a \partial_a A^i_b + A^i_a \partial_b t^a
$$

and

$$
\mathcal{D}_b(t^a A^i_a) = \partial_b(t^a A^i_a) + \epsilon_{ijk} A^j_b (t^a A^k_a)
$$

to obtain

$$
\begin{aligned}
\mathcal{L}_t A^i_b - \mathcal{D}_b(t^a A^i_a) &= t^a(\partial_a A^i_b - \partial_b A^i_a + \epsilon_{ijk} A^j_a A^k_b) \\
&= t^a F^i_{ab}.
\end{aligned}
$$

The action bcomes

$$
\begin{aligned}
S &= \int d^4x(-i(\frac{N}{\sqrt{q}})\tilde{E}^a_I \tilde{E}^b_J \epsilon^{IJ}{}_M F_{ab}{}^{M0} - 2(t^a - N^a)\tilde{E}^b_I F_{ab}{}^{I0}) \\
&= \int d^4x(-2i\tilde{E}^b_i \mathcal{L}_t A^i_b + 2i\tilde{E}^b_i \mathcal{D}_b(t^a A^i_a) + 2iN^a \tilde{E}^b_i F^i_{ab} - (\frac{N}{\sqrt{q}})\epsilon_{ijk}\tilde{E}^a_i \tilde{E}^b_j F^k_{ab}) \qquad (3.188)
\end{aligned}
$$

where we swapped the dummy variables a and b in the second term of the first line. Integrating by parts on the second term,

$$
\begin{aligned}
\int d^4x \tilde{E}^b_i \mathcal{D}_b(t^a A^i_a) &= \int dt d^3x \tilde{E}^b_i(\partial_b(t^a A^i_a) + \epsilon_{ijk} A^j_b(t^a A^k_a)) \\
&= -\int dt d^3x t^a A^i_a(\partial_b \tilde{E}^b_i + \epsilon_{ijk} A^j_b \tilde{E}^b_k) \\
&= -\int d^4x t^a A^i_a \mathcal{D}_b \tilde{E}^b_i \qquad (3.189)
\end{aligned}
$$

where we have thrown away the boundary term and where we used the formula for the covariant derivative on a vector density $\tilde{V}^b_i$:

$$
\mathcal{D}_b \tilde{V}^b_i = \partial_b \tilde{V}^b_i + \epsilon_{ijk} A^j_b \tilde{V}^b_k.
$$

The final form of the action we require is

$$
S = \int d^4x(-2i\tilde{E}^b_i \mathcal{L}_t A^i_b - 2i(t^a A^i_a)\mathcal{D}_b \tilde{E}^b_i + 2iN^a \tilde{E}^b_i F^i_{ab} + (\frac{N}{\sqrt{q}})\epsilon_{ijk}\tilde{E}^a_i \tilde{E}^b_j F^k_{ab}) \qquad (3.190)
$$

There is a term of the form "$p\dot{q}$" thus the quantity $\tilde{E}^a_i$ is the conjugate momentum to A^i_a. Hence, we can immediately write

$$
\{A^i_a(x), \tilde{E}^b_j(y)\} = \frac{i}{2}\delta^b_a \delta^i_j \delta^3(x, y). \qquad (3.191)
$$

Variation of action with respect to the non-dynamical quantities $(t^a A_a^i)$, that is the time component of the four-connection, the shift fucntion N^b, and lapse function N give the constraints

$$\begin{aligned} \mathcal{D}_a \tilde{E}_i^a &= 0, \\ F_{ab}^i \tilde{E}_i^b &= 0, \\ \epsilon_{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{ab}^k &= 0. \end{aligned} \tag{3.192}$$

Varying with respect to N actually gives the last constraint in (3.192) divided by $\sqrt{q}$, it has been rescaled to make the constraint polynomial in the fundamental variables. The connection A_a^i can be written

$$\begin{aligned} A_a^i &= \frac{1}{2}\epsilon^i{}_{jk} A_a^{jk} \\ &= \frac{1}{2}\epsilon^i{}_{jk}\big(\omega_a^{jk} - i\frac{1}{2}(\epsilon^{jk}{}_{m0}\omega_a^{m0} + \epsilon^{jk}{}_{0m}\omega_a^{0m})\big) \\ &= \Gamma_a^i - i\omega_a^{0i} \end{aligned} \tag{3.193}$$

$$\begin{aligned} E_{ci}\omega_a^{0i} &= -q_a^b E_{ci}\omega_b^{i0} \\ &= -q_a^b E_{ci} e^{di} \nabla_b e_d^0 \\ &= q_a^b q_c^d \nabla_b n_d = K_{ac} \end{aligned} \tag{3.194}$$

where we used $e_d^0 = \eta^{0I} g_{dc} e_I^c = -g_{dc} e_0^c = -n_d$, therefore $\omega_a^{0i} = K_a^i$. So the connection reads

$$A_a^i = \Gamma_a^i - iK_a^i. \tag{3.195}$$

3.9 Differential forms - aint they great

3.9.1 Defition

$$d\Phi = \frac{\partial \Phi}{\partial x^\mu} dx^\mu \tag{3.196}$$

$$dX_\nu = \partial_\mu X_\nu dx^\mu \wedge dx^\mu \tag{3.197}$$

For a $p-$form ϕ with Lorentz indices one writes, for example,

$$X^I \wedge Y^J = X^I_\mu Y^J_{\nu\sigma} \, dx^\mu \wedge dx^\nu \wedge dx^\sigma \tag{3.198}$$

Exterior derivative D acts on a $p-$form ϕ with Lorentz indice

We consider the gauge exterior derivative D acts on a $p-$form ϕ with Lorentz indices. For the spacetime scalar,

$$D\phi = d\phi = \frac{\partial \phi}{\partial x^\mu} dx^\mu \tag{3.199}$$

which can be written

$$D\phi = d\phi \tag{3.200}$$

We now consider 1-forms in Lorentz indicies. For example

$$D\phi^I = (\partial_\mu \phi^I + \omega_\mu{}^I{}_J \phi^J) dx^\mu \tag{3.201}$$

or for example we could have

$$\begin{aligned} D(\phi^I_\nu dx^\nu) &= (\partial_\mu \phi^I_\nu + \omega_\mu{}^I{}_J \phi^J_\nu - \overline{\Gamma}^\sigma_{\mu\nu} \phi^I_\sigma) dx^\mu \wedge dx^\nu \\ &= \partial_\mu \phi^I_\nu dx^\mu \wedge dx^\nu + (\omega_\mu{}^I{}_J dx^\mu) \wedge (\phi^J_\nu dx^\nu) \\ &= d\phi^I + \omega^I{}_J \wedge \phi^J \end{aligned} \tag{3.202}$$

or for example we could have

$$\begin{aligned} D(\phi^I_{\nu\sigma} dx^\nu \wedge dx^\sigma) &= (\partial_\mu \phi^I_{\nu\sigma} + \omega_\mu{}^I{}_J \phi^J_{\nu\sigma} - \overline{\Gamma}^\rho_{\mu\nu} \phi^I_{\rho\sigma} - \overline{\Gamma}^\rho_{\mu\sigma} \phi^I_{\nu\rho}) dx^\mu \wedge dx^\nu \wedge dx^\sigma \\ &= (\partial_\mu \phi^I_{\nu\sigma} + \omega_\mu{}^I{}_J \phi^J_{\nu\sigma}) dx^\mu \wedge dx^\nu \wedge dx^\sigma \\ &= d\phi^I + \omega^I{}_J \wedge \phi^J \end{aligned} \tag{3.203}$$

etc, which can universally be written:

$$D\phi^I = d\phi^I + \omega^I{}_J \wedge \phi^J \tag{3.204}$$

We now consider 2-forms in Lorentz indicies. For example

$$D\phi^{IJ} = (\partial_\mu \phi^{IJ} + \omega_{\mu\ K}^{\ I} \phi^{KJ} + \omega_{\mu\ K}^{\ J} \phi^{IK}) dx^\mu \tag{3.205}$$

or for example we could have

$$\begin{aligned} D(\phi_\nu^{IJ} dx^\nu) &= (\partial_\mu \phi_\nu^{IJ} + \omega_{\mu\ K}^{\ I} \phi_\nu^{KJ} + \omega_{\mu\ K}^{\ J} \phi_\nu^{IK} - \overline{\Gamma}_{\mu\nu}^\sigma \phi_\sigma^{IJ}) dx^\mu \wedge dx^\nu \\ &= \partial_\mu \phi_\nu^{IJ} dx^\mu \wedge dx^\nu + (\omega_{\mu\ K}^{\ I} dx^\mu) \wedge (\phi_\nu^{KJ} dx^\nu) + (\omega_{\mu\ K}^{\ I} dx^\mu) \wedge (\phi_\nu^{IK} dx^\nu) \\ &= d\phi^{IJ} + \omega_{\ K}^I \wedge \phi^{KJ} + \omega_{\ K}^J \wedge \phi^{IK} \end{aligned} \tag{3.206}$$

or for example we could have

$$\begin{aligned} D(\phi_{\nu\sigma}^{IJ} dx^\nu \wedge dx^\sigma) &= (\partial_\mu \phi_{\nu\sigma}^{IJ} + \omega_{\mu\ J}^{\ I} \phi_{\nu\sigma}^J - \overline{\Gamma}_{\mu\nu}^\rho \phi_{\rho\sigma}^I - \overline{\Gamma}_{\mu\sigma}^\rho \phi_{\nu\rho}^I) dx^\mu \wedge dx^\nu \wedge dx^\sigma \\ &= (\partial_\mu \phi_{\nu\sigma}^I + \omega_{\mu\ K}^{\ I} \phi_{\nu\sigma}^{KJ} + \omega_{\mu\ K}^{\ J} \phi_{\nu\sigma}^{IK}) dx^\mu \wedge dx^\nu \wedge dx^\sigma \\ &= d\phi^{IJ} + \omega_{\ K}^I \wedge \phi^{KJ} + \omega_{\ K}^J \wedge \phi^{IK} \end{aligned} \tag{3.207}$$

etc, which can universally be written:

$$D\phi^I = d\phi^{IJ} + \omega_{\ K}^I \wedge \phi^{KJ} + \omega_{\ K}^J \wedge \phi^{IK} \tag{3.208}$$

In this way Cartan's equation can be written

$$\begin{aligned} De^I &= (\partial_\mu e_\nu^I + \omega_{\mu\ J}^{\ I} e_\nu^J) dx^\mu \wedge dx^\nu \\ &= (\partial_{[\mu} e_{\nu]}^I dx^\mu \wedge dx^\nu) + (\omega_{\mu\ J}^{\ I} dx^\mu) \wedge (e_\nu^J dx^\nu) \\ &= de^I + \omega_{\ J}^I \wedge e^J \\ &= T_{\mu\nu}^I dx^\mu \wedge dx^\nu \\ &= T^I, \end{aligned} \tag{3.209}$$

or more compactly

$$De = T. \tag{3.210}$$

3.9.2 Usefull results

$$\begin{aligned} e^0 \wedge e^1 \wedge e^3 \wedge e^4 &= e^0{}_\mu e^1{}_\nu e^2{}_\sigma e^3{}_\rho dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\rho \\ &= e^0{}_\mu e^1{}_\nu e^2{}_\sigma e^3{}_\rho \epsilon^{\mu\nu\sigma\rho} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= \det(e) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= s|\det(e)| d^4x = s\sqrt{-g} d^4x = s dV, \end{aligned} \tag{3.211}$$

where $s = sign(\det(e))$

$$\begin{aligned} e^I \wedge e^J \wedge e^K \wedge e^L &= e^I{}_\mu e^J{}_\nu e^K{}_\sigma e^L{}_\rho dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\rho \\ &= Const.\ \epsilon^{IJKL} d^4x \end{aligned} \tag{3.212}$$

To determine Const. put $A = 0, B = 1, C = 2, D = 3$ in (3.212) and compare with (3.211)

$$\begin{aligned} e^0 \wedge e^1 \wedge e^2 \wedge e^3 &= Const.\ \epsilon^{0123} d^4x \\ &= |\det(e)| d^4x \end{aligned} \tag{3.213}$$

so that we have:

$$e^I \wedge e^J \wedge e^K \wedge e^L = \epsilon^{IJKL} |\det(e)| d^4x. \tag{3.214}$$

We compute

$$\begin{aligned} Tr(e \wedge e \wedge e \wedge e) &= \epsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L \\ &= \epsilon_{IJKL} \epsilon^{IJKL} |\det(e)| d^4x \\ &= -24 |\det(e)| d^4x, \end{aligned} \tag{3.215}$$

where we have used (3.214) and relations (). This result relates to adding a cosmological term. Similarly, we get

$$\begin{aligned} Tr(e \wedge e \wedge F) &= \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} \\ &= \frac{1}{2} \epsilon_{IJKL} F^{KL}{}_{MN} e^I \wedge e^J \wedge e^M \wedge e^N \\ &= \frac{1}{2} \epsilon_{IJKL} \epsilon^{IJKL} F^{KL}{}_{MN} |e| d^4x \\ &= -\left(\delta^M_K \delta^N_L - \delta^M_L \delta^N_K\right) F^{KL}{}_{MN} |e| d^4x \\ &= -2 |\det(e)| R d^4x, \end{aligned} \tag{3.216}$$

Hence the gravitational action () can be written as

$$\begin{aligned} S[g[e]] &= -\frac{1}{32\pi G}\int Tr[e\wedge e\wedge F] \\ &= -\frac{1}{32\pi G}\int \epsilon_{IJKL} e^I\wedge e^J\wedge F^{KL} \\ &= -\frac{1}{16\pi G}\int e^I\wedge e^J\wedge\left(\frac{1}{2}\epsilon_{IJKL}F^{KL}\right) \\ &= -\frac{1}{16\pi G}\int e\wedge e\wedge F^* \end{aligned} \tag{3.217}$$

Holst term

$$\frac{1}{2}\epsilon_{IJKL}F^{KL} = \frac{1}{2}\epsilon_{IJKL}F^{KL}$$

$$\begin{aligned} S_\gamma &\sim \frac{1}{\gamma}\int Tr(e\wedge e\wedge *F) \\ &= \frac{1}{\gamma}\int \epsilon_{IJKL}e^I\wedge e^J\wedge\left(-\frac{1}{2}\epsilon^{KLMN}F_{MN}\right) \\ &= -\frac{1}{\gamma}\int \frac{1}{2}\epsilon_{IJKL}\epsilon^{KLMN}e^I\wedge e^J\wedge F^{MN} \\ &= -\frac{1}{\gamma}\int \frac{1}{2}\cdot -2\left(\delta_I^M\delta_J^N-\delta_I^M\delta_J^N\right)e^I\wedge e^J\wedge F^{MN} \\ &= \frac{2}{\gamma}\int e^I\wedge e^J\wedge F_{IJ} \end{aligned} \tag{3.218}$$

Varying with respect to e gives

$$\delta_e S_\gamma \sim \frac{1}{\gamma}\int Tr(e\wedge *F\wedge\delta e), \tag{3.219}$$

which implies

$$\epsilon_{IJKL}e^J\wedge(*F)^{KL}=0 \tag{3.220}$$

we get the following equations of motion:

$$
\begin{aligned}
0 &= \epsilon^{KLMN}\epsilon_{KLIJ}e^I \wedge F_{MN} \\
&= -2\left(\delta^M_I\delta^N_J - \delta^N_I\delta^M_J\right)e^I \wedge F_{MN} \\
&= -4e^I \wedge F_{IJ} \qquad (3.221)
\end{aligned}
$$

3.9.3 Derivation of the Cartan structure equations

We define

$$
\begin{aligned}
\omega^I{}_J &= \omega^I{}_{JK}e^K \qquad (3.222)\\
\mathcal{R}^I{}_J &= R^I{}_{JKL}e^K \wedge e^L. \qquad (3.223)
\end{aligned}
$$

The Cartan structure equations are

$$
\begin{aligned}
de^I &= e^J \wedge \omega^I{}_J = \omega^I{}_{JK}e^J \wedge e^K \qquad (3.224)\\
\frac{1}{2}\mathcal{R}^I{}_J &= d\omega^I{}_J + \omega^I{}_N \wedge \omega^N{}_J \qquad (3.225)
\end{aligned}
$$

Proof:

First equation

$$
\begin{aligned}
de^I &= d(e^I{}_\nu dx^\nu) \\
&= \partial_\mu e^I{}_\nu dx^\mu \wedge dx^\nu \\
&= (\partial_\mu e^I{}_\nu - \Gamma^\rho_{\mu\nu}e^I{}_\rho)dx^\mu \wedge dx^\nu \\
&= \nabla_\mu e^I{}_\nu dx^\mu \wedge dx^\nu \\
&= (\nabla_\mu e^I{}_\nu)\delta^\mu_\sigma\delta^\nu_\rho\, dx^\sigma \wedge dx^\rho \\
&= (\nabla_\mu e^I{}_\nu)e_J^\mu e_K^\nu e^J{}_\sigma e^K{}_\rho dx^\sigma \wedge dx^\rho \\
&= (\nabla_\mu e^I{}_\nu)e_J^\mu e_K^\nu e^J \wedge e^K \\
&= \omega^I{}_{JK}e^J \wedge e^K. \qquad (3.226)
\end{aligned}
$$

Second equation:

$$
\begin{aligned}
R^I{}_{JKL} e^K \wedge e^L &= e^I{}_\mu e_J{}^\nu R^a{}_{\nu KL} e^K{}_\sigma e^L{}_\rho dx^\sigma \wedge dx^\rho \\
&= e^I{}_\mu e_J{}^\nu R^\mu{}_{\nu\sigma\rho} dx^\sigma \wedge dx^\rho \\
&= 2e^I{}_\mu e_J{}^\nu (\partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\delta_{\nu\rho}\Gamma^\mu_{\sigma\delta}) dx^\sigma \wedge dx^\rho
\end{aligned} \tag{3.227}
$$

Consider $d\omega^I{}_J$

$$
\begin{aligned}
d\omega^I{}_J &= d(\Gamma^I{}_{JK} e^K{}_\nu dx^\nu) \\
&= d[(e^I{}_\sigma e_K^\rho \nabla_\rho e_J{}^\sigma) e^K{}_\nu dx^\nu] \\
&= d(e^I{}_\sigma \nabla_\nu e_J{}^\sigma dx^\nu) \\
&= d([e^I{}_\sigma \partial_\nu e_J{}^\sigma + e^I{}_\sigma \Gamma^\sigma_{\nu\rho} e_J{}^\rho] dx^\nu) \\
&= [(\partial_a e^I{}_\sigma)(\partial_\nu e_J{}^\sigma) + e^I{}_\sigma \partial_\mu \partial_\nu e_J{}^\sigma + (\partial_\mu e^I{}_\sigma)\Gamma^\sigma_{\nu\rho} e_J{}^\rho \\
&\quad + e^I{}_\sigma \Gamma^\sigma_{\nu\rho}(\partial_\mu e_J{}^\rho) + e^I{}_\sigma(\partial_\mu \Gamma^c_{\nu\rho}) e_J{}^\rho] dx^\mu \wedge dx^\nu.
\end{aligned} \tag{3.228}
$$

The term $e^I{}_\sigma \partial_\mu \partial_\nu e_J{}^\sigma dx^\mu \wedge dx^\nu$ vanishes. Now

$$
\begin{aligned}
\omega^I{}_K \wedge \omega^K{}_J &= (e^I{}_\sigma \nabla_\mu e_K^\sigma dx^\mu) \wedge (e^K{}_\rho \nabla_\nu e_J{}^\rho dx^\nu) \\
&= [e^I{}_\sigma \partial_\mu e_K^\sigma + e^I{}_\sigma \Gamma^\sigma_{\mu\delta} e_K^\delta] e^K{}_\rho [\partial_\nu e_J{}^\rho + \Gamma^\rho_{\nu\epsilon} e_J{}^\epsilon] dx^\mu \wedge dx^\nu \\
&= [-e_K^\sigma(\partial_\mu e^I{}_\sigma) + e^I{}_\sigma \Gamma^\sigma_{\mu\delta} e_K^\delta] e^K{}_\rho [\partial_\nu e_J{}^\rho + \Gamma^\rho_{\nu\epsilon} e_J{}^\epsilon] dx^\mu \wedge dx^\nu \\
&= [-(\partial_\mu e^I{}_\rho) + e^I{}_\sigma \Gamma^\nu_{\mu\rho}][\partial_\nu e_J{}^\rho + \Gamma^\rho_{\nu\epsilon} e_J{}^\epsilon] dx^\mu \wedge dx^\nu \\
&= -(\partial_\mu e^I{}_\sigma)(\partial_\nu e_J{}^\sigma) - (\partial_\mu e^I{}_\sigma)\Gamma^\sigma_{\nu\rho} e_J{}^\rho + e^I{}_\sigma \Gamma^\sigma_{\mu\rho}(\partial_\nu e_J{}^\rho) \\
&\quad + e^I{}_\sigma \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\epsilon} e_J{}^\epsilon] dx^\mu \wedge dx^\nu
\end{aligned} \tag{3.229}
$$

Combining (3.228) and (3.229)

$$
\begin{aligned}
d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J &= [e^I{}_\sigma \Gamma^\sigma_{\nu\rho}(\partial_\mu e_J{}^\rho) + e^I{}_\sigma \Gamma^\sigma_{\mu\rho}(\partial_\nu e_J{}^\rho) \\
&\quad e^I{}_\sigma(\partial_\mu \Gamma^\sigma_{\nu\rho}) e_J{}^\rho + e^I{}_\sigma \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\epsilon} e_J{}^\epsilon] dx^\mu \wedge dx^\nu \\
&= [e^I{}_\sigma(\partial_\mu \Gamma^\sigma_{\nu\rho}) e_J{}^\rho + e^I{}_\sigma \Gamma^c_{\mu\rho} \Gamma^\rho_{\nu\epsilon} e_J{}^\epsilon] dx^\mu \wedge dx^\nu \\
&= e^I{}_\mu e_J{}^\nu (\partial_\sigma \Gamma^a_{\rho\nu} + \Gamma^\mu_{\sigma\delta} \Gamma^\delta_{\rho\nu}) dx^\sigma \wedge dx^\rho \\
&= \frac{1}{2} R^I{}_{JKL} e^K \wedge e^L.
\end{aligned} \tag{3.230}
$$

□

Chapter 4

Classical discretization

4.1 Lattice QCD

We consider for simplicity an $SU(2)$ Yang-Mills theory in four dimensions.

$$A(x) = A^i_\mu(x)\tau_i dx^\mu, \tag{4.1}$$

where τ_i are the $su(2)$ generators, namely, the Pauli matrices multiplied by $i/2$.

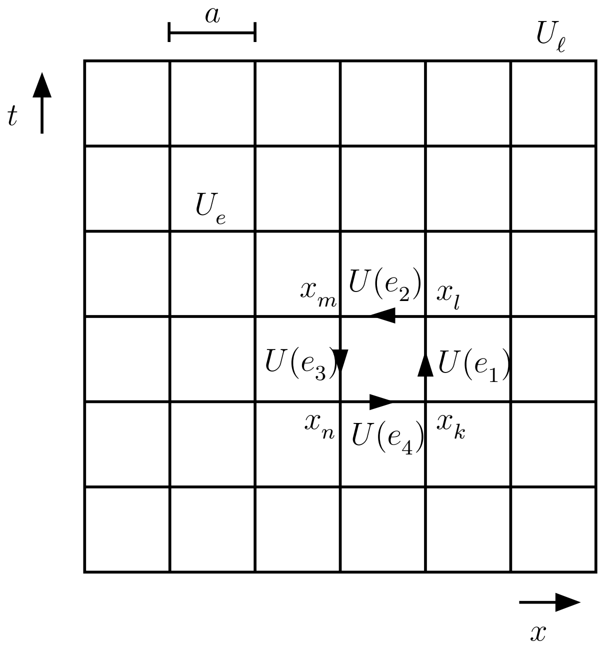

Figure 4.1: .

$$U_e = Pe^{\int_e A} \tag{4.2}$$

along and edge e

$$U_e = e^{gaA_\mu(s_e)\dot{e}^\mu} \tag{4.3}$$

$$U_e = I + aA_\mu(s_e)\dot{e}^\mu \tag{4.4}$$

4.1.1 Derivation of Wilson action for Lattice QCD

The ordered product of four group elements around a plaquette f, an elementary square in the lattice

$$U_f = U_{e_1}U_{e_2}U_{e_3}U_{e_4} \tag{4.5}$$

The Wilson operator

$$W_\square = Tr[U_{e_1}U_{e_2}U_{e_3}U_{e_4}] \tag{4.6}$$

We use the Campbell-Hausdorff theorem to combine each of the Us in a plaquette into a single exponential...

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots} \tag{4.7}$$

In turn

$$\begin{aligned} e^A e^B e^C &= e^{A+B+\frac{1}{2}[A,B]+\cdots} e^C \\ &= e^{A+B+\frac{1}{2}[A,B]+\cdots+C+\frac{1}{2}[A+B+\frac{1}{2}[A,B]+\cdots,C]+\ldots} \\ &= e^{A+B+C+\frac{1}{2}([A,B]+[A,C]+[B,C])+\ldots} \end{aligned} \tag{4.8}$$

In turn

$$\begin{aligned} e^A e^B e^C e^D &= e^{A+B+C+\frac{1}{2}([A,B]+[A,C]+[B,C])+\ldots} e^D \\ &= e^{A+B+C+\frac{1}{2}([A,B]+[A,C]+[B,C])+\cdots+D+\frac{1}{2}[A+B+C+\frac{1}{2}([A,B]+[A,C]+[B,C])+\ldots,D]} \\ &= e^{A+B+C+D+\frac{1}{2}([A,B]+[A,C]+[A,D]+[B,C]+[B,D]+[C,D])+\ldots} \end{aligned} \tag{4.9}$$

We use this final result on:

$$\begin{aligned}
W_\Box &= Tr[U_{e_1}U_{e_2}U_{e_3}U_{e_4}] \\
&= Tr[e^{gaA_\mu(s_{e_1})\dot{e}_1^\mu}e^{gaA_\mu(s_{e_2})\dot{e}_2^\mu}e^{gaA_\mu(s_{e_3})\dot{e}_3^\mu}e^{gaA_\mu(s_{e_4})\dot{e}_4^\mu}] \\
&= Tr\Big\{\exp\Big[ga(A(e_1)+A(e_2)+A(e_3)+A(e_4)) \\
&\quad +\frac{g^2a^2}{2}\big([A(e_1),A(e_2)]+[A(e_1),A(e_3)]+[A(e_1),A(e_4)] \\
&\quad +[A(e_2),A(e_3)]+[A(e_2),A(e_4)]+[A(e_3),A(e_4)]\big)+\mathcal{O}(g^3a^3)\Big]\Big\} \qquad (4.10)
\end{aligned}$$

$$U(x_i,x_j) = U^{-1}(x_j,x_i) \qquad (4.11)$$

$$\int_{e_i} A_\mu dl^\mu = aA(e_i). \qquad (4.12)$$

In the continuum limit ($a \to 0$):

$$\begin{aligned}
A(e_1) &= A_0(x_k) \\
A(e_2) &= A_1(x_k) \\
A(e_3) &= -A_0(x_n) \\
A(e_4) &= -A_1(x_l) \qquad (4.13)
\end{aligned}$$

where a denotes the lattice constant.

$$U(e_1) = e^{gaA(e_i)} \qquad (4.14)$$

From (4.11) and (4.12) it follows that $A(e_i)$ changes it sign if the starting and the end points are exchanged:

$$A(e_i) = -A(-e_i). \qquad (4.15)$$

From (4.13) the sum

$$\begin{aligned}
A(e_1)+A(e_3) &= A_0(x_k)-A_0(x_n) = a\frac{\partial A_0}{\partial x^1}\bigg|_{x=x_k}+\mathcal{O}(a^2) \\
A(e_2)+A(e_4) &= A_1(x_k)-A_1(x_l) = -a\frac{\partial A_1}{\partial x^0}\bigg|_{x=x_k}+\mathcal{O}(a^2) \qquad (4.16)
\end{aligned}$$

From (4.13) and the fact

$$
\begin{aligned}
A_1(x_n) &= A_1(x_k) - a\frac{\partial}{\partial x^1}A_1(x_k) + \ldots \\
&= A_1(x_k) + \mathcal{O}(a),
\end{aligned}
$$

we then have the additional results:

$$
\begin{aligned}
[A(e_1), A(e_2)] &= [A_0(x_k), A_1(x_k)] \\
[A(e_1), A(e_3)] &= [A_0(x_k), -A_0(x_n)] = [A_0(x_k), -A_0(x_k)] + \mathcal{O}(a) = \mathcal{O}(a) \\
[A(e_1), A(e_4)] &= [A_0(x_k), -A_1(x_l)] = -[A_0(x_k), A_1(x_k)] + \mathcal{O}(a) \\
[A(e_2), A(e_3)] &= [A_1(x_k), -A_0(x_n)] = -[A_1(x_k), A_0(x_k)] + \mathcal{O}(a) \\
[A(e_2), A(e_4)] &= [A_1(x_k), -A_1(x_l)] = \mathcal{O}(a) \\
[A(e_3), A(e_4)] &= [-A_0(x_n), -A_1(x_l)] = [A_0(x_n), A_1(x_l)] + \mathcal{O}(a)
\end{aligned} \tag{4.17}
$$

The Wilson operator can then be expressed as

$$
\begin{aligned}
W_\square &= Tr\left\{\exp\left[-ga^2\left(\frac{\partial A_0}{\partial x^1} - \frac{\partial A_1}{\partial x^0} - g\left[A_0, A_1\right]\right) + \mathcal{O}(a^3)\right]\right\} \\
&= Tr\left\{\exp\left[-ga^2 F_{10} + \mathcal{O}(a^3)\right]\right\}
\end{aligned} \tag{4.18}
$$

We expand the exponential:

$$
\begin{aligned}
W_\square &= Tr\left\{I - ga^2 F^a_{10}\tau_a + \frac{1}{2}g^2 a^4 F^a_{10} F^b_{10}\tau_a\tau_b + \mathcal{O}(a^5)\right\} \\
&= Tr(I) - ga^2 F^a_{10} Tr(\tau_a) + \frac{1}{2}g^2 a^4 F^a_{10} F^b_{10} Tr(\tau_a\tau_b) + \mathcal{O}(a^5) \\
&= 2 - \frac{1}{4}g^2 a^4 F^a_{10} F^a_{10} + \mathcal{O}(a^5).
\end{aligned} \tag{4.19}
$$

Which implies

$$
\frac{2}{g^2}(2 - W_\square) = \frac{1}{2}a^4 F^a_{10} F^a_{10} + \mathcal{O}(a^5). \tag{4.20}
$$

If this is summed over all plaquettes, we approximate the well know expression for the action

$$\sum_{\square} \frac{2}{g^2}(2 - W_{\square}) = \frac{1}{4} \int d^4x F^a_{\mu\nu} F^{a\mu\nu} + \mathcal{O}(a^5). \tag{4.21}$$

The additional factor 1/2 is due to the fact that $F^a_{\mu\nu} F^{a\mu\nu}$ contributes $F^a_{10} F^{a10}$ and $F^a_{01} F^{a01} = F^a_{10} F^{a10}$.

Since $Tr(U_{\square})$ is complex for any finite lattice constant, we must take the real part of the trace, making the Wilson action a real positive number.

$$S_W = \sum_{\square} \frac{2}{g^2}(2 - W_{\square}) = \sum_{\square} \frac{2}{g^2} \left(2 - \mathrm{Re}[Tr(U_{\square})]\right) \tag{4.22}$$

$$S_W \sim \frac{1}{4} \int d^4x F^a_{\mu\nu} F^{a\mu\nu} + \mathcal{O}(a^5) \tag{4.23}$$

Wilson's result that the discrete action

$$S = \beta \sum_f TrU_f + c.c. \tag{4.24}$$

approximates the continuous action in the limit in which a is small. The coupling constant β is a number that depends on a and goes to zero when a goes to zero.

Consider now a boundary of the lattice. Let l be the boundary edges or "links". If we fix the value of these and integrate the exponential of the action over the bulk groups elements,

$$W(U_\ell) = \int dU_e e^{\frac{i}{\hbar} S[U]}, \tag{4.25}$$

we obtain the transition amplitude of the truncated theory.

4.1.2 Hamiltonian lattice theory

4.2 Discretization of general relativity on a two-complex

4.2.1 Boundary

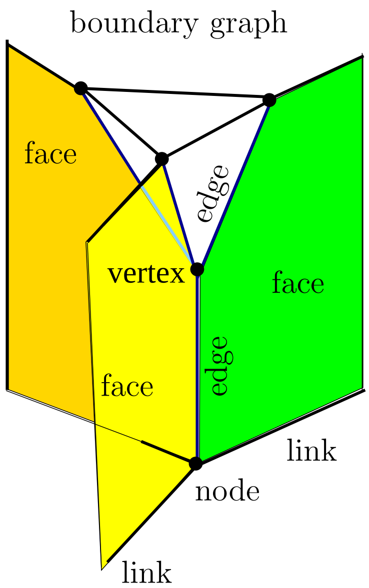

Figure 4.2: Two-complex terminology.

Table 4.1: Bulk terminology and notation

Bulk triangulation $\triangle$	Two-complex $\triangle^*$
Tetrahedron (τ)	Vertex (v)
Triangle (t)	Edge (e)
Segment (s)	Face (f)
Point (p)	

Table 4.2: Boundary terminology and notation

Bulk triangulation $\partial\triangle$	Boundary graph Γ
Triangle (t)	Node, (n) (boundary vertex)
Segment (s)	Link, (ℓ) (boundary edge)

Chapter 5

Three-dimensional eculidean theory

5.0.2 Commutation relation

Prove that

$$(L^i_\ell \psi)(U) := -i\frac{d}{dt}\psi(Ue^{t\tau_i})\Big|_{t=0} \tag{5.1}$$

satisfies:

$$[U_\ell, L^i_{\ell'}] = i\ (8\pi\hbar G)\delta_{\ell\ell'}U_\ell \tau^i. \tag{5.2}$$

first consider $\ell \neq \ell'$:

$$\begin{aligned}
[U_\ell, L^i_{\ell'}]\psi(U) &= (8\pi\hbar G)\left(-iU_\ell \frac{d}{dt}\psi(U_{\ell'}e^{t\tau_i})\Big|_{t=0} + i\,\frac{d}{dt}\left[U_\ell \psi(U_{\ell'}e^{t\tau_i})\right]\Big|_{t=0}\right) \\
&= 0
\end{aligned} \tag{5.3}$$

Second consider $\ell = \ell'$:

$$\begin{aligned}
[U_\ell, L'_\ell]\psi(U) &= (8\pi\hbar G)\left(-iU_\ell \frac{d}{dt}\psi(U_\ell e^{t\tau_i})\Big|_{t=0} + i\,\frac{d}{dt}\left[U_\ell e^{t\tau_i}\psi(U_\ell e^{t\tau_i})\right]\Big|_{t=0}\right) \\
&= i(8\pi\hbar G)\,\frac{d}{dt}\left(U_\ell e^{t\tau_i}\right)\Big|_{t=0}\psi(U) \\
&= i\ (8\pi\hbar G)U_\ell \tau^i \psi(U).
\end{aligned} \tag{5.4}$$

5.0.3 Laplacian operator

$$\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u_i} \left(\sqrt{g} g^{ik} \frac{\partial}{\partial u_k} \right) \tag{5.5}$$

5.0.4 Spin networks

Gauge

Gauge invariant states must satisfy

$$\psi(U_\ell) = \psi(\Lambda_{s_\ell} U_\ell \Lambda_{t_\ell}^{-1}) \tag{5.6}$$

Prove that this equivalent to

$$\vec{C}_n \psi = 0 \tag{5.7}$$

for every node n of the boundary graph.

$$\vec{C}_n = \vec{L}_{\ell_1} + \vec{L}_{\ell_2} + \vec{L}_{\ell_3} = \tag{5.8}$$

5.0.5 Spin-network bases

The Peter-Wweyl theorem states that the matrix elements $D^j_{mn}(U)$ of the Wigner matrices seen as functions of $SU(2)$, are orthogonal with respect to the scalar product defined by the Haar measure. Secifically:

$$\int dU \, \overline{D^{j'}_{m'n'}(U)} D^j_{mn}(U) = \frac{1}{d_j} \delta^{jj'} \delta_{mm'} \delta_{nn'}, \tag{5.9}$$

where $d_j = 2j + 1$ is the dimension of the representaion.

Chapter 6

Bubbles and the cosmological constant

Chapter 7

The real world: four-dimensional lorentzian theory

7.1 Introduction

7.1.1 A Crash Course in Simplexes and Complexes

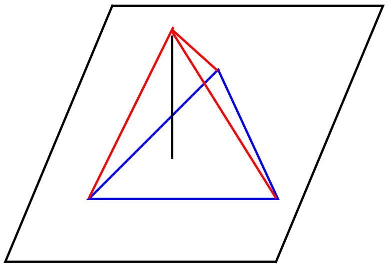

Figure 7.1: We cunstruct a 3-simplex by lifting the center of the 2-simplex.

Dimension of simplex	0	1	2	3	4
Number of vertices	1	2	3	4	5
Number of edges	0	1	3	6	10
Number of triangles	0	0	1	4	10
Number of tetrahedra	0	0	0	1	5

We cunstruct a 3-simplex by lifting the center of the 2-simplex - see (7.1). This implies that the number of edges of the n-simplex, $\#e_n$, goes as $0 + 1 + \cdots + (n-2) + (n-1) + n$ so is $(n+1)n/2$ or

$$\#e_n = \frac{(n+1)n}{2}. \tag{7.1}$$

There are $n+1$ choices for the start point of an edge and n choices for the end point. Multiplying these numbers gives $(n+1)n$, but since this counts each edge twice, the total number of edges is $(n+1)n/2$, just as we found above.

Let us denote the number of triangles in an $n-$simplex as $\#t_n$. Then it is fairly easy to see that $\#t_n = \#t_{n-1}+$ [number of edges of the $(n-1)-$ simplex] or $\#t_n = \#t_{n-1} + \frac{n(n-1)}{2}$.So we need to solve:

$$\#t_n - \#t_{n-1} = \frac{n(n-1)}{2}.$$

We can use the trial solution $\#t_n = A(n+1)n(n-1)$,

$$\begin{aligned} A(n+1)n(n-1) - An(n-1)(n-2) &= An(n-1)[(n+1)-(n-2)] \\ &= 3An(n-1), \end{aligned}$$

so works if we choose $A = 1/6$. The number of triangles is

$$\#t_n = \frac{(n+1)n(n-1)}{6}, \tag{7.2}$$

(note this obviously works for $n=0$ and $n=1$).

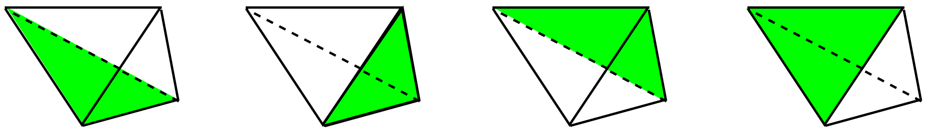

Figure 7.2: Structure of interaction vertex.

Let us denote the number of tetrahedra in an $n-$simplex as $\#tet_n$. Then it is fairly easy to see that $\#tet_n = \#tet_{n-1}+$ [number of triangles of the $(n-1)-$ simplex] or $\#tet_n = \#tet_{n-1} + \frac{n(n-1)(n-2)}{6}$. So we need to solve:

$$\#tet_n - \#tet_{n-1} = \frac{n(n-1)(n-2)}{6}.$$

We can use the trial solution $\#t_n = A(n+1)n(n-1)(n-2)$,

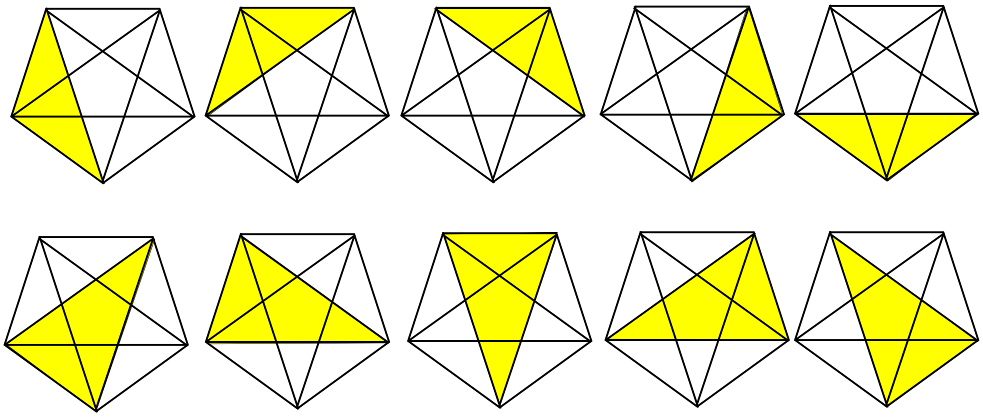

Figure 7.3: 4-simplex. All 10 faces of a 4-simplex.

$$\begin{aligned} &A(n+1)n(n-1)(n-2) - An(n-1)(n-2)(n-3) \\ &\qquad = An(n-1)(n-2)[(n+1)-(n-3)] \\ &\qquad = 4An(n-1)(n-2), \end{aligned}$$

so works if we choose $A = 1/24$. The number of tetrahedra is

$$\#tet_n = \frac{(n+1)n(n-1)(n-2)}{24}, \tag{7.3}$$

(note this obviously works for $n = 0$, $n = 1$ and $n = 1$).

In an n-dimensional simplex there are

$$C(n,k) = \frac{(n+1)!}{k!(n+1-k)!} \tag{7.4}$$

$k-$simplicies.

7.2 Classical discretization

7.3 The Lorentz Group

7.3.1 Irreducible representations of the rotation group

$$[L_i, L_j] = i\epsilon_{ijk} L_k \tag{7.5}$$

Table 7.1: Bulk terminology and notation

Bulk triangulation $\triangle$	Two-complex $\triangle^*$
4-simplex (v)	Vertex (v)
Tetrahedron (τ)	Edge (e)
Triangle (t)	Face (f)
Segment (s)	
Point (p)	

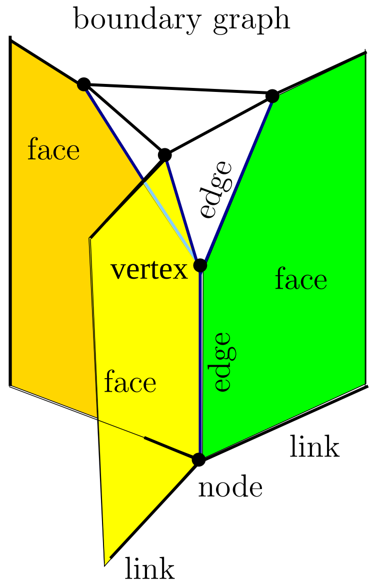

Figure 7.4: Two-complex terminology.

The square of $\vec{L}$ commutes with all components, that is

$$[\vec{L}, \vec{L}^2] = 0, \qquad \vec{L}^2 = L_x^2 + L_y^2 + L_z^2 \tag{7.6}$$

For example:

$$\begin{aligned}
[L_x, L_x^2 + L_y^2 + L_z^2] &= L_x(L_y^2 + L_z^2) - (L_y^2 + L_z^2)L_x \\
&= (L_x L_y^2 - L_y L_x L_y + L_y L_x L_y - L_y^2 L_x) \\
&\qquad + (L_x L_z^2 - L_z L_x L_z + L_z L_x L_z - L_z^2 L_x) \\
&= ([L_x, L_y]L_y + L_y[L_x, L_y]) + ([L_x, L_z]L_z + L_z[L_x, L_z]) \\
&= (iL_z L_y + iL_y L_z) + (-iL_z L_y - iL_y L_z) \\
&= 0.
\end{aligned}$$

Table 7.2: Boundary terminology and notation

Boundary triangulation $\partial\triangle$	Boundary graph Γ
Tetrahedron (τ)	Node (boundary vertex) (n)
Triangle (t)	Link (boundary edge) (ℓ)

To obtain the spectrum of these operators we introduce the following Hermitian conjugate operators:

$$L^+ := L_x + iL_y, \qquad L^- := L_x - iL_y \tag{7.7}$$

These operators, which are not Hermitian, are called ladder operators. From (7.6) we have that

$$[L^+, \vec{L}^2] = [L^-, \vec{L}^2] = 0. \tag{7.8}$$

The three operators L^+, L^-, L_z determine the operator $\vec{L}$ but are more convenient for algebraic transformations. Let us calculate their commutation relations:

$$\begin{aligned} [L_z, L^+] &= [L_z, L_x + iL_y] \\ &= iL_y + L_x \\ &= L^+, \end{aligned}$$

and

$$\begin{aligned} [L_z, L^-] &= [L_z, L_x - iL_y] \\ &= iL_y - L_x \\ &= -L^-, \end{aligned}$$

and

$$\begin{aligned} [L^+, L^-] &= [L_x + iL_y, L_x - iL_y] \\ &= -2i[L_x, L_y] \\ &= 2L_z. \end{aligned}$$

In summary:

$$[L_z, L^+] = L^+, \qquad [L_z, L^-] = -L^-, \qquad [L^+, L^-] = 2L_z. \tag{7.9}$$

We express $\vec{L}^2$ in terms of the ladder operators to get

$$\vec{L}^2 = \frac{1}{2}(L^+L^- + L^-L^+) + L_z^2 \tag{7.10}$$

as well as

$$\begin{aligned} L^+L^- &= (L_x + iL_y)(L_x - iL_y) \\ &= L_x^2 + L_y^2 - i[L_x, L_y] \\ &= \vec{L}^2 - L_z^2 + L_z. \end{aligned}$$

Similarly for L^-L^+, so:

$$L^+L^- = \vec{L}^2 - L_z(L_z - 1), \qquad L^+L^- = \vec{L}^2 - L_z(L_z + 1). \tag{7.11}$$

As $\vec{L}^2$ commutes with all components of $\vec{L}$ we can find a set of common eigenvectors for one componenet (as is usual we take L_z). As the eigenvalue of $\vec{L}^2$ must be positive, we can without loss of generality written it as $j(j+1)$ for positive j. For L_z we write the corresponding eigenvalue m. That is,

$$\vec{L}^2|j,m> = j(j+1)|j,m>, \qquad L_z|j,m> = m|j,m> . \tag{7.12}$$

We obtain the spectrum of these operators by algebraic methods. Using (7.11) we get

$$\begin{aligned} L^+L^-|j,m> &= [j(j+1) - m(m-1)]\ |j,m> \\ L^-L^+|j,m> &= [j(j+1) - m(m+1)]\ |j,m> \end{aligned} \tag{7.13}$$

The ladder operators are Hermitian conjugate to each other, $(L^-)^\dagger = L^+$; and thus,

$$0 \leq < j,m|(L^+)^\dagger L^+|j,m> = < j,m|L^-L^+|j,m> \tag{7.14}$$

holds. Using (7.13) we obtain

$$< j,m|L^-L^+|j,m> = [j(j+1) - m(m+1)]\ < j,m|j,m> \geq 0 \tag{7.15}$$

and analogously,

$$< j,m|L^+L^-|j,m> = [j(j+1) - m(m-1)] \ < j,m|j,m> \geq 0 \tag{7.16}$$

Hence, we infer the following relations for the quantum numbers j and m:

$$\begin{aligned} j(j+1) - m(m+1) &= (j-m)(j+m+1) \geq 0, \\ j(j+1) - m(m-1) &= (j+m)(j-m+1) \geq 0. \end{aligned} \tag{7.17}$$

For the quantum number m they set the bounds

$$-j \leq m \leq j. \tag{7.18}$$

For $m = j$ we have

$$L^+|j,m>= 0.$$

For $m = -j$ we have

$$L^-|j,m>= 0.$$

We now look at the effect of the ladder operators on the states $|j,m>$, we have

$$\vec{L}^2(L^+|j,m>) = L^+\vec{L}^2|j,m>= j(j+1)(L^+|j,m>). \tag{7.19}$$

Similarly,

$$\vec{L}^2(L^-|j,m>) = L^-\vec{L}^2|j,m>= j(j+1)(L^-|j,m>). \tag{7.20}$$

This says that $L^+|j,m>$ and $L^-|j,m>$ are eigenvectors of $\vec{L}^2$ with eigenvalue $j(j+1)$. From (7.9) we have

$$L_zL^+ = L^+(L_z + 1)$$

implying

$$L_z(L^+|j,m>) = (m+1)(L^+|j,m>) \tag{7.21}$$

From (7.9) we have

$$L_zL^- = L^-(L_z - 1)$$

implying

$$L_z(L^-|j,m>) = (m-1)(L^-|j,m>) \tag{7.22}$$

The ladder operators L^+ and L^- raise and lower, respectively, the eigenvalue m by one unit. Repeated application of L^+ on $|j,m>$, we obtain a sequence of eigenvectors of $\vec{L}^2$ and L_z, with increasing m:

$$|j,m>, L^+|j,m>, (L^+)^2|j,m>, \ldots, (L^+)^p|j,m>, \tag{7.23}$$

and the corresponding eigenvalues of L_z are

$$m, m+1, m+2, \ldots, m+p = j.$$

The number p is a positive integer and the maximal value of $m+p$ must be equal to j, as $L^+|j,j>= 0$

correspondingly, we get for the operator L^- the sequence

$$|j,m>, L^-|j,m>, (L^-)^2|j,m>, \ldots, (L^-)^q|j,m>, \tag{7.24}$$

with eigenvalues

$$m, m-1, \ldots, m-q = -j.$$

We have

$$p + q = 2j.$$

As p and q are positive integers we get the following values for the quantum number j:

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots \quad .$$

The quantum numbers

$$m = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots \quad .$$

The eigenvalues of the operator $\vec{L}^2$ are $j(j+1)$, and for each fixed value j there are $2j+1$ different eigenvalues m of L_z. They are

$$-j, -j+1, \dots, 0, 1, \dots, +j.$$

$$L^+|j,m>= a_m|j,m+1>$$

and from (7.15)

$$|a_m|^2 < j,m+1|j,m+1> = < j,m|(L^+)^\dagger L^+|j,m>= [j(j+1) - m(m+1)] < j,m|j,m>$$

and so:

$$|a_m|^2 = [j(j+1) - m(m+1)].$$

We obtain $2j+1$ orthonormal eigenvectors

$$|j,-j>, |j,-j+1>, \dots, |j,m>, \dots, |j,j>$$

satisfying the eigenvalue equations

$$\vec{L}^2|j,m>= j(j+1)|j,m>, \qquad L_z|j,m>= m|j,m> . \tag{7.25}$$

$$\begin{aligned}
L^+|j,m> &= \sqrt{j(j+1) - m(m+1)}|j,m+1> = \sqrt{(j+m+1)(j-m)}|j,m+1> \\
L^-|j,m> &= \sqrt{j(j+1) - m(m-1)}|j,m-1> = \sqrt{(j+m)(j-m+1)}|j,m-1> \\
L^+|j,j> &= L^-|j,-j>= 0.
\end{aligned} \tag{7.26}$$

7.3.2 Finite dimensional non-unitary representations of the Lorentz group

Vectors and spinors.

Vector transformations

Now we know that a boost in the $x-$direction results in

$$t' = \beta(t - vx), \qquad x' = \beta(x - vt), \qquad y' = y, \qquad z' = z. \tag{7.27}$$

where

$$\beta = \frac{1}{(1 - v^2)^{\frac{1}{2}}} \tag{7.28}$$

$$\exp(-i\omega \cdot \hat{\mathbf{S}} - i\xi \cdot \hat{\mathbf{K}}) \tag{7.29}$$

Here $\hat{\mathbf{S}}$ and $\hat{\mathbf{K}}$

For infintesimal transformations this becomes

$$I - i\delta\omega \cdot \hat{\mathbf{S}} - i\delta\xi \cdot \hat{\mathbf{K}} \tag{7.30}$$

The transformation matrix for a rotation of the coordinate system around the x axis $(\delta\omega_1 \neq 0, \delta\omega_2 = \delta\omega_3 = 0)$, in the positive sense of rotation is given by

$$\begin{pmatrix} 0\, | & 0 & 0 & 0 \\ - & - & - & - \\ 0\, | & 0 & 0 & 0 \\ 0\, | & 0 & 0 & +\omega_1 \\ 0\, | & 0 & -\delta\omega_1 & 0 \end{pmatrix} \tag{7.31}$$

Therefore

$$\hat{S}_1 = -i \begin{pmatrix} 0\, | & 0 & 0 & 0 \\ - & - & - & - \\ 0\, | & 0 & 0 & 0 \\ 0\, | & 0 & 0 & -1 \\ 0\, | & 0 & +1 & 0 \end{pmatrix}. \tag{7.32}$$

Similarly, we obtain

$$\hat{S}_2 = -i \begin{pmatrix} 0\,| & 0 & 0 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & +1 \\ 0\,| & 0 & 0 & 0 \\ 0\,| & -1 & 0 & 0 \end{pmatrix}, \tag{7.33}$$

$$\hat{S}_3 = -i \begin{pmatrix} 0\,| & 0 & 0 & 0 \\ - & - & - & - \\ 0\,| & 0 & -1 & 0 \\ 0\,| & 1 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix}. \tag{7.34}$$

Concerning the Lorentz boosts, we first recall the definition of the rapidity variable ξ, for which

$$\xi = \frac{\beta}{|\beta|}\mathrm{artanh}|\beta|, \quad \beta = \frac{\mathbf{v}}{c} \tag{7.35}$$

Then the matrix describing a boost in the x direction, is

$$\begin{pmatrix} \cosh\xi & | & -\sinh\xi & 0 & 0 \\ --- & - & --- & - & - \\ -\sinh\xi & | & \cosh\xi & 0 & 0 \\ 0 & | & 0 & 1 & 0 \\ 0 & | & 0 & 0 & 1 \end{pmatrix} \tag{7.36}$$

There is a certain similarity between the boost and the rotation matrices, the differences originating in the asymmetry of temporal and spatial directions in Minkowski space.

Rapidities in the same spatial direction are additive just as rotation angles. For an infintesimal boost ($\xi \to \delta\xi_1, \delta\xi_2 = \delta\xi_3 = 0$) we obtain

$$\begin{pmatrix} 1 & | & -\delta\xi_1 & 0 & 0 \\ -- & - & -- & - & - \\ -\delta\xi_1 & | & 1 & 0 & 0 \\ 0 & | & 0 & 1 & 0 \\ 0 & | & 0 & 0 & 1 \end{pmatrix} \tag{7.37}$$

Therefore

$$\hat{K}_1 = -i \begin{pmatrix} 0 \,| & 1 & 0 & 0 \\ - & - & - & - \\ 1 \,| & 0 & 0 & 0 \\ 0 \,| & 0 & 0 & 0 \\ 0 \,| & 0 & 0 & 0 \end{pmatrix}. \tag{7.38}$$

Similarly, we obtain

$$\hat{K}_2 = -i \begin{pmatrix} 0 \,| & 0 & 1 & 0 \\ - & - & - & - \\ 0 \,| & 0 & 0 & 0 \\ 1 \,| & 0 & 0 & 0 \\ 0 \,| & 0 & 0 & 0 \end{pmatrix}, \tag{7.39}$$

$$\hat{K}_3 = -i \begin{pmatrix} 0 \,| & 0 & 0 & 1 \\ - & - & - & - \\ 0 \,| & 0 & 0 & 0 \\ 0 \,| & 0 & 0 & 0 \\ 1 \,| & 0 & 0 & 0 \end{pmatrix}. \tag{7.40}$$

These are the generators of the Lorentz group.

If we enumarate the generators $\hat{S}_i$, $\hat{K}_i$ $(i = 1, 2, 3)$ combining them to form a six-component vector $\hat{L}_i$ $(i = 1, 2, 3, 4, 5, 6)$, then the structure constants C_{ijk} are defined by

$$[\hat{L}_i, \hat{L}_j] = C_{ijk}\hat{L}_k. \tag{7.41}$$

We have to calculate all commutators between generators. As we already know,

$$[\hat{S}_i, \hat{S}_j] = -i\epsilon_{ijk}\hat{S}_k. \tag{7.42}$$

A commutator between boost generators is, for example,

$$
\begin{aligned}
[\hat{K}_1, \hat{K}_2] &= -\left\{ \begin{pmatrix} 0\,| & 1 & 0 & 0 \\ - & - & - & - \\ 1\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0\,| & 0 & 1 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 1\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix} \right. \\
&\qquad \left. - \begin{pmatrix} 0\,| & 0 & 1 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 1\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0\,| & 1 & 0 & 0 \\ - & - & - & - \\ 1\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix} \right\} \qquad (7.43)
\end{aligned}
$$

So that

$$[\hat{K}_1, \hat{K}_2] = i\hat{S}_3, \qquad (7.44)$$

altogether we obtain

$$[\hat{K}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{S}_k, \qquad (7.45)$$

Finally we must calculate $[\hat{S}_i, \hat{K}_j]$. We have for example,

$$
\begin{aligned}
[\hat{S}_1, \hat{K}_2] &= -\left\{ \begin{pmatrix} 0\,| & 0 & 0 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & -1 \\ 0\,| & 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} 0\,| & 0 & 1 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 1\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix} \right. \\
&\qquad \left. - \begin{pmatrix} 0\,| & 0 & 1 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 1\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0\,| & 0 & 0 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & -1 \\ 0\,| & 0 & +1 & 0 \end{pmatrix} \right\} \\
&= -\begin{pmatrix} 0\,| & 0 & 0 & 0 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \\ +1| & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0\,| & 0 & 0 & -1 \\ - & - & - & - \\ 0\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \\ 0\,| & 0 & 0 & 0 \end{pmatrix} \\
&= -i\hat{K}_3 \qquad (7.46)
\end{aligned}
$$

and in general

$$[\hat{S}_i, \hat{K}_j] = -i\epsilon_{ijk}\hat{K}_k. \tag{7.47}$$

Spinor transformations

Here we introduve a spinor in flat minkowsky spaetime. We introduce the matrix

$$X = tI + (\vec{x} \cdot \vec{\sigma}) \tag{7.48}$$

or

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{7.49}$$

note that the determinant of X is

$$\begin{aligned} \det(X) &= \det \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \\ &= (t+z)(t-z) - (x-iy)(x+iy) \\ &= t^2 - x^2 - y^2 - z^2. \end{aligned} \tag{7.50}$$

It is required that this matrix is Hermitian

$$X^\dagger = X \tag{7.51}$$

as then and only then are the coordinates (t, x, y, z) real. A Lorent transformation is defined by the requirement that it leaves the interval invariant:

$$t^{'2} - x^{'2} - y^{'2} - z^{'2} = t^2 - x^2 - y^2 - z^2 \tag{7.52}$$

Introduce the matrix X':

$$X' := LXL^\dagger \tag{7.53}$$

via the 'transformation' matrix L. The requirement of preservation of the interval is then

$$\det X' = \det X \tag{7.54}$$

or

$$\det X' = (\det L)(\det X)(\det L^\dagger) \tag{7.55}$$

This is satisfied if

$$\det L = e^{i\lambda}. \tag{7.56}$$

We first recrict ourselves to the case $\lambda = 0$. The matrix X' is guaranteed to be Hermitian as

$$\begin{aligned} (X')^\dagger &= (LXL^\dagger)^\dagger \\ &= (L^\dagger)^\dagger X^\dagger L^\dagger \\ &= LXL^\dagger \\ &= X'. \end{aligned} \tag{7.57}$$

Thus the only requirment of the transformation matrix L is that $\det L = 1$ therfore $L \in SL(2, \mathbb{C})$. Since the action of members of $SL(2, \mathbb{C})$ preserves the Lorentz invaraint quantity, we can associate a 4-vector $(t, \mathbf{x})$ with the matrix X, and we can associate a Lorentz transformation with any member of $SL(2, \mathbb{C})$. It is seen that the more general case of $\lambda \neq 0$ can be included. It is seen as an additional phase factor that has no effect on the 4-vector obtained.

We can read off t', x', y', z' from

$$X' = \begin{pmatrix} t' + z' & x' - iy' \\ x' + iy' & t' - z' \end{pmatrix}. \tag{7.58}$$

Now we know that a boost in the $x-$direction results in

$$t' = \beta(t - vx), \qquad x' = \beta(x - vt), \qquad y' = y, \qquad z' = z. \tag{7.59}$$

where

$$\beta = \frac{1}{(1 - v^2)^{\frac{1}{2}}} \tag{7.60}$$

Consider the matrix

$$
\begin{aligned}
e^{(\rho/2)\sigma_x} &= \exp\left[(\rho/2)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right]\\
&= cosh(\rho/2)I - sinh(\rho/2)\sigma_x\\
&= \begin{pmatrix}cosh(\rho/2) & -sinh(\rho/2)\\ -sinh(\rho/2) & cosh(\rho/2)\end{pmatrix} \qquad (7.61)
\end{aligned}
$$

It is easily seen that the determinant is 1:

$$
\begin{aligned}
\det e^{(\rho/2)\sigma_x} &= \det\begin{pmatrix}cosh(\rho/2) & -sinh(\rho/2)\\ -sinh(\rho/2) & cosh(\rho/2)\end{pmatrix}\\
&= cosh^2(\rho/2) - sinh^2(\rho/2) = 1. \qquad (7.62)
\end{aligned}
$$

One finds the effect on X is such that

$$
\begin{aligned}
X' &= \begin{pmatrix}t'+z' & x'-iy'\\ x'+iy' & t'-z'\end{pmatrix}\\
&= e^{-(\rho/2)\sigma_x}(tI + x^i\sigma_i)e^{-(\rho/2)\sigma_x}\\
&= e^{-\rho\sigma_x}tI + e^{-\rho\sigma_x}x\sigma_x + ye^{-(\rho/2)\sigma_x}\sigma_y e^{-(\rho/2)\sigma_x} + ze^{-(\rho/2)\sigma_x}\sigma_z e^{-(\rho/2)\sigma_x} \qquad (7.63)
\end{aligned}
$$

Compute

$$
\begin{aligned}
\sigma_y e^{-(\rho/2)\sigma_x} &= \sigma_y[I - (\rho/2)\sigma_x + \frac{1}{2!}(\rho/2)^2 I - \frac{1}{3!}(\rho/2)^3\sigma_x + \frac{1}{4!}(\rho/2)^4 I - \ldots]\\
&= \sigma_y[1 + \frac{1}{2!}(\rho/2)^2 + \frac{1}{4!}(\rho/2)^4 + \ldots] + i\sigma_z[(\rho/2) + \frac{1}{3!}(\rho/2)^3 + \ldots]\\
&= \sigma_y cosh(\rho/2) + i\sigma_z sinh(\rho/2) \qquad (7.64)
\end{aligned}
$$

$$
\begin{aligned}
e^{-(\rho/2)\sigma_x}\sigma_y e^{-(\rho/2)\sigma_x} &= e^{-(\rho/2)\sigma_x}[\sigma_y cosh(\rho/2) + i\sigma_z sinh(\rho/2)]\\
&= [cosh(\rho/2)I - sinh(\rho/2)\sigma_x][cosh(\rho/2)\sigma_y + isinh(\rho/2)\sigma_z]\\
&= cosh^2(\rho/2)\sigma_y - sinh^2(\rho/2)\sigma_y + icosh(\rho/2)sinh(\rho/2)(\sigma_z - \sigma_z)\\
&= \sigma_y \qquad (7.65)
\end{aligned}
$$

Similarly

$$e^{-(\rho/2)\sigma_x}\sigma_z e^{-(\rho/2)\sigma_x} = \sigma_z. \tag{7.66}$$

Thus

$$\begin{aligned}
X' &= \begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix} \\
&= te^{-\rho\sigma_x} + xe^{-\rho\sigma_x}\sigma_x + y\sigma_y + z\sigma_z \\
&= t[cosh\rho I - sinh\rho\sigma_x] + x[-sinh\rho I + cosh\rho\sigma_x] + y\sigma_y + z\sigma_z \\
&= \begin{pmatrix} tcosh\rho - xsinh\rho & xcosh\rho - tsinh\rho \\ xcosh\rho - tsinh\rho & tcosh\rho - xsinh\rho \end{pmatrix} + \begin{pmatrix} z & -iy \\ iy & -z \end{pmatrix}
\end{aligned} \tag{7.67}$$

Therefore

$$\begin{aligned}
t' &= tcosh\rho - xsinh\rho \\
x' &= xcosh\rho - tsinh\rho \\
y' &= y \\
z' &= z
\end{aligned} \tag{7.68}$$

If we write

$$cosh\rho = \frac{1}{(1-v^2)^{\frac{1}{2}}} \tag{7.69}$$

then

$$\begin{aligned}
sinh\rho &= \sqrt{cosh^2\rho - 1} \\
&= \sqrt{\frac{1}{(1-v^2)} - 1} \\
&= \frac{v}{(1-v^2)^{\frac{1}{2}}}
\end{aligned} \tag{7.70}$$

And so we arrive at the Lorentz transformation for a boost in the $x-$direction.

$$\begin{aligned} t' &= \frac{t - vx}{(1-v^2)^{\frac{1}{2}}} \\ x' &= \frac{x - vt}{(1-v^2)^{\frac{1}{2}}} \\ y' &= y \\ z' &= z \end{aligned} \tag{7.71}$$

It can be checked that $e^{-(\rho/2)\sigma_y}$ and $e^{-(\rho/2)\sigma_z}$ give Lorentz boosts along the y and $z-$directions respectively. An infintesimal Lorentz transformation in the general direction $\vec{n}$ is

$$L(d\rho) = I - (d\rho/2)\vec{n} \cdot \vec{\sigma}. \tag{7.72}$$

This can also be written as

$$L(\rho + d\rho) = [I - (d\rho/2)\vec{n} \cdot \vec{\sigma}]L(\rho) \tag{7.73}$$

That is

$$dL(\rho) = L(\rho + d\rho) - L(\rho) = -(d\rho/2)\vec{n} \cdot \vec{\sigma} L(\rho) \tag{7.74}$$

which can be integrated and so the general Lorentz transformation is implimented by (for $\vec{\rho} = \rho\vec{n}$)

$$e^{-(\vec{\rho}/2)\cdot\vec{\sigma}} = cosh(\rho/2)I - sinh(\rho/2)\vec{n} \cdot \vec{\sigma}. \tag{7.75}$$

It is not unitary as

$$\begin{aligned} [e^{-(\vec{\rho}/2)\cdot\vec{\sigma}}]^\dagger e^{-(\vec{\rho}/2)\cdot\vec{\sigma}} &= e^{-(\vec{\rho}/2)\cdot\vec{\sigma}^\dagger} e^{-(\vec{\rho}/2)\cdot\vec{\sigma}} \\ &= e^{-\vec{\rho}\cdot\vec{\sigma}} \neq I. \end{aligned} \tag{7.76}$$

The relationship is two-to-one mapping since a given Lorentz transformation (in the general case, including rotations) can be represented by either $+L$ or $-L$, for $L \in SL(2, \mathbb{C})$.

We can write

$$X = \begin{pmatrix} X^{00} & X^{01} \\ X^{10} & X^{11} \end{pmatrix} \tag{7.77}$$

Take this matrix to be the outer-product

$$\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \otimes (z^0, z^1) = \begin{pmatrix} w^0 z^0 & w^0 z^1 \\ w^1 z^0 & w^1 z^1 \end{pmatrix} \tag{7.78}$$

or in index notation

$$X^{AB} = w^A z^B \tag{7.79}$$

From (7.53)

$$X'^{AB} = \overline{L}^A{}_D L^B{}_C w^D z^C \tag{7.80}$$

Then we have

$$\begin{pmatrix} w'^0 z'^0 & w'^0 z'^1 \\ w'^0 z'^1 & w'^1 z'^1 \end{pmatrix} = \left[L \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \right]^\dagger \otimes \left[L \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \right]$$

So that

$$z' = Lz. \tag{7.81}$$

this defines a spinor, just as the transormation law for vectors and tensors are defined by the way they transform. One has pulled out half of the transformation law - it is in a sense the square-root of a vector.

It is seen as an additional phase factor that has no effect on the 4-vector obtained from a given spinor, but it rotates a flag through an angle λ.

Correspondance bewtween vectos and spinors

$$X^{AB} = x^\mu \sigma_\mu{}^{AB} \tag{7.82}$$

Using (1.104)

$$\sum_{i=1}^{3} \sigma_i{}^A{}_B \sigma_i{}^C{}_D = \delta^A_D \delta^C_B - \epsilon^{AC} \epsilon_{BD} \tag{7.83}$$

$$\begin{aligned}
\sum_{i=0}^{3} \sigma_i{}^A{}_B \sigma_i{}^C{}_D &= \delta^A{}_B \delta^C{}_D + (\delta^A_D \delta^C_B - \epsilon^{AC}\epsilon_{BD}) \\
&= \delta^A{}_B \delta^C{}_D + [\delta^A_D \delta^C_B - (\delta^A_B \delta^C_D - \delta^A_D \delta^C_B)] \\
&= \qquad (7.84)
\end{aligned}$$

7.3.3 Infinite dimensional unitary representation

We introduce the operator L^{ab} in order to define the action of the Lorentz group on fields:

$$J^{IJ} = x^I p^J - x^J x^I = i(x^I \partial^J - x^J \partial^I) \qquad (7.85)$$

This generates the algebra of the Lorentz group:

$$\begin{aligned}
[J^{IJ}, J^{KL}]\phi(x) &= i(x^I\partial^J - x^J\partial^I)i(x^K\partial^L - x^L\partial^K)\phi(x) - i(x^K\partial^L - x^L\partial^K)i(x^I\partial^J - x^J\partial^I)\phi(x) \\
&= -x^I\partial^J[x^K\partial^L\phi(x)] + x^I\partial^J[x^L\partial^K\phi(x)] + x^J\partial^I[x^K\partial^L\phi(x)] - x^J\partial^I[x^L\partial^K\phi(x)] \\
&\quad + x^K\partial^L[x^I\partial^J\phi(x)] - x^K\partial^L[x^J\partial^I\phi(x)] - x^L\partial^K[x^I\partial^J\phi(x)] + x^L\partial^I[x^J\partial^I\phi(x)] \\
&= i\left\{(\partial^J x^K)x^I i\partial^L - (\partial^J x^L)x^I i\partial^K - (\partial^I x^K)x^J i\partial^L + (\partial^I x^L)x^J i\partial^K\right\}\phi(x) \\
&- i\left\{(\partial^L x^I)x^K i\partial^J - (\partial^L x^J)x^K i\partial^I - (\partial^K x^I)x^L i\partial^J + (\partial^K x^J)x^L i\partial^I\right\}\phi(x) \\
&= i\eta^{JK}(x^I i\partial^L - x^L i\partial^I) - i\eta^{JL}(x^I i\partial^K - x^K i\partial^I)\phi(x) \\
&\quad - i\eta^{IK}(x^J i\partial^L - x^L i\partial^J) + i\eta^{IL}(x^J i\partial^K - x^K i\partial^J)\phi(x) \\
&= (i\eta^{JK}J^{IL} - i\eta^{JL}J^{IK} - i\eta^{IK}J^{JL} + i\eta^{IL}J^{JK})\phi(x) \qquad (7.86)
\end{aligned}$$

That is

$$[J^{IJ}, J^{KL}] = i\eta^{JK}J^{IL} - i\eta^{IK}J^{JL} - i\eta^{JL}J^{IK} + i\eta^{IL}J^{JK} \qquad (7.87)$$

Let us define

$$U(\Lambda) = \exp\left(i\epsilon^{IJ}J_{IJ}\right) \qquad (7.88)$$

The action of the Lorentz group on a scalar field

$$U(\Lambda)\phi(x^I) = \phi\left((\Lambda^{-1})^I{}_J x^J\right) \qquad (7.89)$$

It is important to notice that the operator acts upon the coordinates, and not upon the argument of ϕ. That is

$$U(\Lambda)\phi(Ax) = \phi\left(A\Lambda^{-1}x\right) \tag{7.90}$$

This provides an infinite-dimensional unitary representation of the Lorentz group:

$$\begin{aligned}
[U(\Lambda')U(\Lambda)\phi](x) &= [U(\Lambda')(U(\Lambda)\phi)](x) \\
&= [U(\Lambda')\phi](\Lambda^{-1}x) \\
&= \phi(\Lambda^{-1}\Lambda'^{-1}x) \\
&= \phi((\Lambda'\Lambda)^{-1}x) \\
&= [U(\Lambda'\Lambda)\phi](x)
\end{aligned} \tag{7.91}$$

This representation is reducible.

7.3.4 Unitary irreducible representations of $SL(2, \mathbb{C})$

Casimir opertors

An essential characteristic of a Lie group is it *rank*, which is defined as the largest number of genertors cummuting with each other.

Lie groups which have special importance in physics are semisimple.

Let us investigate a group $G = \{G_\nu\}$ with an abelian subgroup $A = \{A_i\}$. Then if

$$G_\nu A_i G_\nu^{-1} = A_j, \tag{7.92}$$

$$A_i A_k = A_k A_i \tag{7.93}$$

hold for every A_i aand for every element G_ν of the full group, i.e. $G_\nu A_i G_\nu^{-1}$ is again an element of the abelian subgroup, the group A is called an abelian invariant subgroup. If the subgroup A is invariant but not abelian, i.e. (7.92) holds, but not (7.93), then it is simply called an invariant subgroup.

We can now define simple and semisimple Lie groups: a Lie group is called *simple* if it does not possess a *continuous* (hence Lie type) invariant subgroup. A lie group is called *semisimple* if it does not possess a continuous abelian invariant subgroup. Note that a semisimple Lie group can well contain a continuous invariant subgroup, although it is not allowed to have a continuous abelian invariant subgroup.

We want to study Lie algebras of groups with continuous invariant subgroups.

The lie algebra is

$$\begin{aligned}
[L^i, L^j] &= \epsilon^{ij}{}_k L^k \\
[L^i, K^j] &= \epsilon^{ij}{}_k K^k \\
[K^i, K^j] &= \epsilon^{ij}{}_k L^k
\end{aligned} \tag{7.94}$$

Theorem of Rach

For any semisimple Lie group of rank ℓ, there exist a set of ℓ Casimir opertors. These are functions of $C_\lambda(L_1, L_2, \ldots, L_n)$ $(\lambda = 1, 2, \ldots, \ell)$ of the generators L_i, and commute with every operator in the group and therefore among themselves: $[C_\lambda, C'_\lambda] = 0$. The eigenvalues of the C_λ uniquely characterise the multiplets of the group.

From (7.94)the only vanishing commutators between distinct generators are: $[L^1, K^1] = [L^2, K^2] = [L^3, K^3] = 0$. However, L^1 and K^1 do not commute with L^2, K^2, L^3, K^3, also L_2 and K_2 do not commute with L^1, K^1, L^3, K^3, and also L^3 and K^3 do not commute with L^1, K^1, L^2, K^2. Therefore, for the Lorentz group, the largest number of generators commutating with each other is 2, and hence the rank of the group is 2.

So $SL(2\mathbb{C})$ has rank 2 and therefore has two Casimir opertors. These can be taken to be

$$\begin{aligned}
C_1 &= \frac{1}{2} J_{IJ} J^{IJ} = \vec{K}^2 - \vec{L}^2, \\
C_2 &= \frac{1}{8} \epsilon_{IJKL} J^{IJ} J^{KL} = \vec{K} \cdot \vec{L}.
\end{aligned} \tag{7.95}$$

7.3.5 Construction of a Casimir operator

The simplest operator that commutes with the infintesimal operators $\{X_i\}$ must be at least second order in these operators because a linear combination of X_i will not commute with $\{X_i\}$ due to

$$[X_i, X_j] = C^k_{ij} X_k. \tag{7.96}$$

Thus we considre the most general linear combination of second order operators

$$C = \sum_{i,j} M^{ij} X_i X_j. \tag{7.97}$$

The matrix M^{ij} must be symmetric since every matrix can be written as a sum of a symmetric and antisymmetric matrices and from (7.96) the antisymmetric part reduces to a linear combination of X_i.

Consider

$$\begin{aligned}
[C, X_k] &= \sum_{i,j} M^{ij} [X_i X_j, X_k] \\
&= \sum_{i,j} M^{ij} \Big(X_i [X_j, X_k] + [X_i, X_k] X_j \Big) \\
&= \sum_{i,j} M^{ij} \Big(X_i X_l C^l_{jk} + C^l_{ik} X_l X_j \Big) \\
&= \sum_{i,j,l} M^{ij} C^l_{jk} \left(X_i X_l + X_l X_i \right)
\end{aligned} \tag{7.98}$$

where we have used that M^{ij} is symmetric. If we find $\sum_j M^{ij} C^l_{jk}$ that is antisymmetric with respect to the interchange of i and l.

To this end we introduce the so-called metric tensor

$$g_{ij} = \sum_{kl} C^k_{il} C^l_{jk} \tag{7.99}$$

which is symmetric in i and j. We will show that one of the Casimir operators is always given by

$$C = \sum_{i,j} g^{ij} X_i X_j \tag{7.100}$$

where g^{ij} is the inverse metric tensor. We show that the tensor

$$a^{il}_k = \sum_j g^{ij} C^l_{jk} \tag{7.101}$$

is antisymmetric in i and l. We introduce the tensor

$$
\begin{aligned}
b_{jmn} &= \sum_{l} g_{jl} C^{l}_{mn} \\
&= \sum_{k,l} C^{k}_{ji} C^{i}_{lk} C^{l}_{mn} \\
&= \sum_{k,l} C^{k}_{ji} C^{l}_{mn} C^{i}_{lk}.
\end{aligned} \tag{7.102}
$$

Next we apply the identity

$$
\sum_{l} \left[C^{l}_{mn} C^{i}_{lk} + C^{l}_{nk} C^{i}_{lm} + C^{l}_{km} C^{i}_{ln} \right] = 0 \tag{7.103}
$$

which follows from the Jacobi identity

$$
[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \tag{7.104}
$$

Substituting (7.103) into (7.102)

$$
\begin{aligned}
b_{jmn} &= -\sum_{k,l} C^{k}_{ji} \left(C^{l}_{nk} C^{i}_{lm} + C^{l}_{km} C^{i}_{ln} \right) \\
&= \sum_{k,l} \left(C^{k}_{ji} C^{l}_{nk} C^{i}_{ml} + C^{k}_{ij} C^{l}_{km} C^{i}_{ln} \right).
\end{aligned} \tag{7.105}
$$

The RHS is invariant under any cyclic permutation of the indices j,m and n. We demonstrate this for the cyclic permutation

$$
(jmn) \to (mnj), \tag{7.106}
$$

other cases can be readily verified by the reader. Thus, we consider

$$
\begin{aligned}
b_{mnj} &= \sum_{k,l} \left(C^{k}_{mi} C^{l}_{jk} C^{i}_{nl} + C^{k}_{im} C^{l}_{kn} C^{i}_{lj} \right) \\
&= \sum_{k,l} \left(C^{i}_{ml} C^{k}_{ji} C^{l}_{nk} + C^{l}_{km} C^{i}_{ln} C^{k}_{ij} \right) \\
&= b_{jmn}
\end{aligned} \tag{7.107}
$$

where we renamed the summation indices in the first term by $(ikl) \to (lik)$, and in the second term by $(ikl) \to (kli)$.

As C^l_{mn} is antisymmetric in m and n the tensor b_{jmn} is antisymmetric in m and n, i.e. the last two indices. Because of (7.107) b_{jmn} is antisymmetric in j and n as well, and due to the general cyclic invariance of b_{jmn} it has to be antisymmetric with respect to the interchange of any two indices.

Now, by (7.98)

$$
\begin{aligned}
[C, X_k] &= \sum_{i,l} \sum_j g^{ij} C^l_{jk} \left(X_i X_l + X_l X_i \right) \\
&= \sum_{i,l} a^{il}_k \left(X_i X_l + X_l X_i \right) \qquad (7.108)
\end{aligned}
$$

Using $C^l_{jk} = g^{nl} b_{njk}$,

$$
\begin{aligned}
a^{il}_k &= \sum_j g^{ij} C^l_{jk} \\
&= \sum_{j,p} g^{ij} \delta^l_p C^p_{jk} \\
&= \sum_{j,p,n} g^{ij} g^{ln} g_{np} C^p_{jk} \\
&= \sum_{j,n} g^{ij} g^{ln} b_{njk} \qquad (7.109)
\end{aligned}
$$

thus

$$
\begin{aligned}
a^{li}_k &= \sum_{j,n} g^{lj} g^{in} b_{njk} \\
&= -\sum_{j,n} g^{lj} g^{in} b_{jnk} \\
&= -\sum_{j,n} g^{ln} g^{ij} b_{njk} \\
&= -a^{il}_k. \qquad (7.110)
\end{aligned}
$$

and hence by (7.108)

$$[C, X_k] = 0. \tag{7.111}$$

$\square$

First Casimir operator

We write

$$\begin{aligned} X_i &= L^i \quad \text{for } i = 1,2,3; \\ X_i &= K^{i-3} \quad \text{for } i = 4,5,6. \end{aligned} \tag{7.112}$$

We prove $\vec{K}^2 - \vec{L}^2$ is a Casimir operator by proving it is proportional to $g^{ij} X_i X_j$.

The commutation relations (7.94)

$$\begin{aligned} [X_i, X_j] &= \sum_{k=1}^{3} \epsilon_{ijk} X_k \quad \text{for} \quad i,j,k = 1,2,3; \\ [X_i, X_j] &= \sum_{k=4}^{6} \epsilon_{i(j-3)(k-3)} X_k \quad \text{for} \quad i = 1,2,3\ ,\ j,k = 4,5,6; \\ [X_i, X_j] &= \sum_{k=1}^{3} \epsilon_{(i-3)(j-3)k} X_k \quad \text{for} \quad i,j = 4,5,6\ ,\ k = 1,2,3 \end{aligned} \tag{7.113}$$

Thus we can then write for the structure constants (we use notation $C^k_{ij} \equiv C_{ijk}$)

$$\begin{aligned} C_{ijk} &= \epsilon_{ijk} \quad \text{for} \quad i,j,k = 1,2,3; \\ C_{ijk} &= \epsilon_{i(j-3)(k-3)} \quad \text{for} \quad i = 1,2,3\ ,\ j = 4,5,6\ ,\ k = 4,5,6; \\ C_{ijk} &= -\epsilon_{(i-3)j(k-3)} \quad \text{for} \quad i = 4,5,6\ ,\ j = 1,2,3\ ,\ k = 4,5,6; \\ C_{ijk} &= \epsilon_{(i-3)(j-3)k} \quad \text{for} \quad i,j = 4,5,6\ ,\ k = 1,2,3 \end{aligned} \tag{7.114}$$

with all other values of C_{ijk} being zero. We now proceed to calculate the components of g_{ij} as defined in (7.99),

Consider g_{ij} for $i,j = 1,2,3$:

$$
\begin{aligned}
g_{ij} &= \sum_{\alpha=1}^{6}\sum_{\beta=1}^{6} C_{i\alpha\beta}C_{j\beta\alpha} \\
&= \sum_{\alpha=1,2,3}\sum_{\beta=1,2,3} \epsilon_{i\alpha\beta}\epsilon_{j\beta\alpha} + \sum_{\alpha=1,2,3}\sum_{\beta=4,5,6} \underbrace{C_{i\alpha\beta}}_{=0}\underbrace{C_{j\beta\alpha}}_{=0} \\
&= \sum_{\alpha=4,5,6}\sum_{\beta=1,2,3} \underbrace{C_{i\alpha\beta}}_{=0}\underbrace{C_{j\beta\alpha}}_{=0} + \sum_{\alpha=4,5,6}\sum_{\beta=4,5,6} \epsilon_{i(\alpha-3)(\beta-3)}\epsilon_{j(\beta-3)(\alpha-3)} \\
&= -4\delta_{ij}
\end{aligned} \tag{7.115}
$$

Consider g_{ij} for $i, j = 4, 5, 6$:

$$
\begin{aligned}
g_{ij} &= \sum_{\alpha=1,2,3}\sum_{\beta=1,2,3} \underbrace{C_{i\alpha\beta}}_{=0}\underbrace{C_{j\beta\alpha}}_{=0} + \sum_{\alpha=1,2,3}\sum_{\beta=4,5,6} (-\epsilon_{(i-3)\alpha(\beta-3)})\epsilon_{(j-3)(\beta-3)\alpha} \\
&= \sum_{\alpha=4,5,6}\sum_{\beta=1,2,3} \epsilon_{(i-3)(\alpha-3)\beta}(-\epsilon_{(j-3)\beta(\alpha-3)}) + \sum_{\alpha=4,5,6}\sum_{\beta=4,5,6} \underbrace{C_{i\alpha\beta}}_{=0}\underbrace{C_{j\beta\alpha}}_{=0} \\
&= 4\delta_{ij}
\end{aligned} \tag{7.116}
$$

Consider g_{ij} for $i = 1, 2, 3$ and $j = 4, 5, 6$:

$$
\begin{aligned}
g_{ij} &= \sum_{\alpha=1,2,3}\sum_{\beta=1,2,3} C_{i\alpha\beta}\underbrace{C_{j\beta\alpha}}_{=0} + \sum_{\alpha=1,2,3}\sum_{\beta=4,5,6} \underbrace{C_{i\alpha\beta}}_{=0}C_{j\beta\alpha} \\
&= \sum_{\alpha=4,5,6}\sum_{\beta=1,2,3} \underbrace{C_{i\alpha\beta}}_{=0}C_{j\beta\alpha} + \sum_{\alpha=4,5,6}\sum_{\beta=4,5,6} C_{i\alpha\beta}\underbrace{C_{j\beta\alpha}}_{=0} \\
&= 0.
\end{aligned} \tag{7.117}
$$

Using that g_{ij} is symmetric, we finally have altogether

$$
(g_{ij}) = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \tag{7.118}
$$

So that

$$(g^{ij}) = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.119}$$

And so we have the Casimir operator

$$C_1 = 4g^{ij}X_iX_j = \vec{K}^2 - \vec{L}^2. \tag{7.120}$$

$\square$

Second Casimir operator

Now

$$\begin{aligned} [C_2, L^j] &= [\vec{K}\cdot\vec{L}, L^j] \\ &= [K^1L^1 + K^2L^2 + K^3L^3, L^j] \\ &= K^1[L^1, L^j] + [K^1, L^j]L^1 + K^2[L^2, L^j] + [K^2, L^j]L^2 + K^3[L^3, L^j] + [K^3, L^j]L^3 \\ &= \epsilon^{1j}{}_k(K^1L^k + K^kL^1) + \epsilon^{2j}{}_k(K^2L^k + K^kL^2) + \epsilon^{3j}{}_k(K^3L^k + K^kL^3) \end{aligned} \tag{7.121}$$

For $j = 1$ we easily have

$$\begin{aligned} [C_2, L^1] &= [\vec{K}\cdot\vec{L}, L^1] \\ &= \epsilon^{21}{}_k(K^2L^k + K^kL^2) + \epsilon^{31}{}_k(K^3L^k + K^kL^3) \\ &= -(K^2L^3 + K^3L^2) + (K^3L^2 + K^2L^3) \\ &= 0. \end{aligned} \tag{7.122}$$

Similarly,

$$[C_2, L^2] = [C_2, L^3] = 0. \tag{7.123}$$

Now

$$
\begin{aligned}
[C_2, K^j] &= [\vec{K}\cdot\vec{L}, K^j] \\
&= [K^1L^1 + K^2L^2 + K^3L^3, K^j] \\
&= K^1[L^1, K^j] + [K^1, K^j]L^1 + K^2[L^2, K^j] + [K^2, K^j]L^2 + K^3[L^3, K^j] + [K^3, K^j]L^3 \\
&= \epsilon^{1j}{}_k(K^1K^k + L^kL^1) + \epsilon^{2j}{}_k(K^2K^k + L^kL^2) + \epsilon^{3j}{}_k(K^3K^k + L^kL^3) \qquad (7.124)
\end{aligned}
$$

For $j = 1$ we easily have

$$
\begin{aligned}
[C_2, K^1] &= [\vec{K}\cdot\vec{L}, K^1] \\
&= \epsilon^{21}{}_k(K^2K^k + L^kL^2) + \epsilon^{31}{}_k(K^3K^k + L^kL^3) \\
&= -(K^2K^3 + L^3L^2) + (K^3K^2 + L^2L^3) \\
&= [K^3, K^2] + [L^2, L^3] \\
&= \epsilon^{32}{}_kL^k + \epsilon^{23}{}_kL^k \\
&= 0. \qquad (7.125)
\end{aligned}
$$

Similarly,

$$[C_2, K^2] = [C_2, K^3] = 0. \qquad (7.126)$$

□

As C_1 and C_2 commute with every generator of the group, it follows that they will commute with each other:

$$[C_1, C_2] = 0, \qquad (7.127)$$

explicitly

$$
\begin{aligned}
[C_1, C_2] &= [C_1, \vec{K}\cdot\vec{L}] \\
&= [C_1, K^1L^1 + K^2L^2 + K^3L^3] \\
&= [C_1, K^1]L^1 + K^1[C_1, L^1] + [C_1, K^2]L^2 + K^2[C_1, L^2] + K^3[C_1, L^3] \\
&= 0. \qquad (7.128)
\end{aligned}
$$

By Rahc's theorem C_1 and C_2 uniquely characterise the multiplets of the group.

□

Details from Ruhl

Details from Gel'fand

These go under the name of the principal series.

$$V^{(p,k)} \tag{7.129}$$

p is real and k is a half-integer.

These representations are infinite dimensional.

$V^{(p,k)}$ have hermitian generators $J^{IJ} = -J^{JI}$.

timelike vector

t^I with $t^I = (1, 0, 0, 0)$

The generators J^{IJ} of $SL(2, \mathbb{C})$ then split into rotations and boosts

$L^I = \frac{1}{2}\epsilon^I{}_{JKL} J^{JK} t^L$ so $L^I = (0, L^i)$

$K^I = J^{IJ} t_J$ so $K^I = (0, K^i)$

with the lie algebra

$$\begin{aligned} [L^i, L^j] &= \epsilon^{ij}{}_k L^k \\ [L^i, K^j] &= \epsilon^{ij}{}_k K^k \\ [K^i, K^j] &= \epsilon^{ij}{}_k L^k \end{aligned} \tag{7.130}$$

Casimirs.

$$\begin{aligned} C_1 &= \frac{1}{2} J_{IJ} J^{IJ} = \vec{K}^2 - \vec{L}^2, \\ C_2 &= \frac{1}{8}\epsilon_{IJKL} J^{IJ} J^{KL} = \vec{K} \cdot \vec{L}. \end{aligned} \tag{7.131}$$

For $V^{(p,k)}$

$$\begin{aligned} C_1 &= p^2 - k^2 + 1, \\ C_2 &= pk. \end{aligned} \tag{7.132}$$

The little group that prefers the vector t^I.

$$SU(2) \subset SL(2,\mathbb{C}) \tag{7.133}$$

$V^{(p,k)}$ are also representations of $SU(2)$ but they are reducible. The decomposition is

$$V^{(p,k)} = \mathcal{H}_k \oplus \mathcal{H}_{j+1} \oplus \cdots = \bigoplus_{j=k}^{\infty} \mathcal{H}_j \tag{7.134}$$

ortho-normal basis diagalise $\mathcal{C}_1, \mathcal{C}_2$, $\vec{L}^2, L_z$

$$|(p,k); j, m> \tag{7.135}$$

Comment:

K^i sends $\mathcal{H}_j \to \mathcal{H}_{j-1} \oplus \mathcal{H}_j \oplus \mathcal{H}_{j+1}$

Formula:

$$\begin{aligned} [L^i, L^j] &= \epsilon^{ij}{}_k L^k \\ [L^i, K^j] &= \epsilon^{ij}{}_k K^k \\ [K^i, K^j] &= \epsilon^{ij}{}_k L^k \end{aligned} \tag{7.136}$$

$$[L^3, L^2] = -L^1, \quad [L^3, L^1] = L^2, \quad [L^2, L^1] = -L^3 \tag{7.137}$$

$$\begin{aligned} [L^3, K^1] &= K^2, \quad [L^2, K^1] = -K^3, \quad [L^1, K^1] = 0 \\ [L^3, K^2] &= -K^1, \quad [L^2, K^1] = 0, \quad [L^1, K^2] = K^3 \\ [L^3, K^3] &= 0, \quad [L^2, K^3] = -K^1, \quad [L^1, K^3] = -K^2 \end{aligned} \tag{7.138}$$

$$[K^1, K^2] = -L^3, \quad [K^1, K^3] = L^2, \quad [K^2, K^3] = -L^1 \tag{7.139}$$

We introduce the following combinations:

$$\begin{aligned} L^+ &= iL^1 - L^2, \quad H^- = iL^1 + L^2, \quad L_3 = iL^3 \\ K^+ &= iK^1 - K^2, \quad K^- = iK^1 + K^2, \quad K_3 = iK^3. \end{aligned} \tag{7.140}$$

It is easy to obtain the commutation relations between these operators:

$$[L^+, L_3] = -L^+, \quad [L^-, L_3] = L^-, \quad [L^+, L^-] = 2L_3 \tag{7.141}$$

$$\begin{aligned} [K^+, L^+] &= [L^-, K^-] = [L_3, K_3] = 0 \\ [L^+, K_3] &= -K_3, \quad [L^-, K_3] = -K^- \\ [L^+, K^-] &= -[L^-, K^+] = 2K_3 \\ [K^+, L_3] &= -K^+, \qquad [K^-, L_3] = K^- \end{aligned} \tag{7.142}$$

$$[K^+, K_3] = L^+, \quad [K^-, K_3] = -L^-, \quad [K^+, K^-] = -2L_3 \tag{7.143}$$

$$\ldots$$

$$\begin{aligned} L_3|j,m> &= m|j,m> \\ L^-|j,m> &= \sqrt{(j+m)(j-m+1)}|j,m-1> \\ L^+|j,m> &= \sqrt{(j+m+1)(j-m)}|j,m+1> \\ m &= -j,-j+1,\ldots,j-1,j. \end{aligned} \tag{7.144}$$

We write out operators K_3, K^+, K^- in the basis $\{|j,m>\}$:

$$\begin{aligned} K_3|j,m> &= C_j\sqrt{j^2-m^2}|j-1,m> -A_j m|j,m> -C_{j+1}\sqrt{(j+1)^2-m^2}|j+1,m> \\ K^+|j,m> &= C_j\sqrt{(j-m)(j-m-1)}|j-1,m+1> -A_j\sqrt{(j-m)(j+m+1)}|j,m+1> \\ &\quad + C_{j+1}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1> \\ K^-|j,m> &= -C_j\sqrt{(j+m)(j+m-1)}|j-1,m-1> -A_j\sqrt{(j+m)(j-m+1)}|j,m-1> \\ &\quad - C_{j+1}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1> \end{aligned} \tag{7.145}$$

$$\begin{gathered} A_j = \frac{ij_0j_1}{j(j+1)}, \qquad C_j = \frac{i}{j}\sqrt{\frac{(j^2-j_0^2)(j^2-j_1^2)}{4j^2-1}} \\ m = -j,-j+1,\ldots,j-1,j; \quad j = j_0, j_0+1,\ldots \end{gathered} \tag{7.146}$$

Injection map Y

The decomposition (7.134) gives a way of starting with the representation space $\mathcal{H}_j$ of $SU(2)$ and getting a representation of $V^{(p,k)}$.

$Y : \mathcal{H}_j \to$ minimal block $V^{(p,j)}$ $(k = j)$

$$Y : \mathcal{H}_j \to V^{(p,j)} \tag{7.147}$$

$$|j, m> \mapsto \ |(p, k); j, m> . \tag{7.148}$$

Chapter 8

Classical limit

8.1 Intrinsic coherent states

8.1.1 Tetrahedron geometry

$$h_{ab} = \vec{e}_a \cdot \vec{e}_b \tag{8.1}$$

is the metric in these coordinates. The three vectors

$$\vec{E}^a = \frac{1}{2}\epsilon^{abc}\vec{e}_b \times \vec{e}_c \tag{8.2}$$

We use the identities

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}), \qquad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

and the standard formula for the inverse matrix

$$h^{ab} = 3\frac{\epsilon^{acd}\epsilon^{bef}h_{ce}h_{df}}{3!det(h)}$$

to compute

$$
\begin{aligned}
\vec{E}^a \cdot \vec{E}^b &= \frac{1}{4}\epsilon^{acd}\epsilon^{bef}(\vec{e}_c \times \vec{e}_d) \cdot (\vec{e}_e \times \vec{e}_f) \\
&= \frac{1}{4}\epsilon^{acd}\epsilon^{bef}\vec{e}_c \cdot [\vec{e}_d \times (\vec{e}_e \times \vec{e}_f)] \\
&= \frac{1}{4}\epsilon^{acd}\epsilon^{bef}\vec{e}_c \cdot [(\vec{e}_d \cdot \vec{e}_f)\vec{e}_e - (\vec{e}_d \cdot \vec{e}_e)\vec{e}_f] \\
&= \frac{1}{4}\epsilon^{acd}\epsilon^{bef}[h_{ce}h_{df} - h_{de}h_{cf}] \\
&= \frac{1}{2}\epsilon^{acd}\epsilon^{bef}h_{ce}h_{df} \\
&= (deth)h^{ab} \qquad (8.3)
\end{aligned}
$$

The dihedral angle between two triangles is given by

$$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}. \qquad (8.4)$$

We now want to move to the quantum theory. For this we first establish the 'genaralised uncertainty relation'.

8.1.2 The generalised uncertainty relation

Consider any two arbitrary operators A and B. Recall that in a given state $|\psi>$, the mean value of an operator $\mathcal{O}$ is

$$< \mathcal{O} > = < \psi|\mathcal{O}|\psi > . \qquad (8.5)$$

The uncertainty, defined as the standard deviation, for two operators A and B is:

$$(\Delta A)^2 = < (A- < A >)^2 >, \qquad (\Delta B)^2 = < (B- < B >)^2 > \qquad (8.6)$$

Proof:

Consider the zero mean operators $\tilde{A} := A- < A >$ and $\tilde{B} := B- < B >$ then

$$(\Delta A)^2 = < \tilde{A}^2 >, \qquad (\Delta B)^2 = < \tilde{B}^2 > \qquad (8.7)$$

Note $(\Delta A)^2(\Delta B)^2 =< \tilde{A}^2 >< \tilde{B}^2 >$. From the Schwartz inequality

$$< \tilde{A}^2 >< \tilde{B}^2 > \geq | < \tilde{A}\tilde{B} > |^2 = \overline{< \tilde{A}\tilde{B} >} < \tilde{A}\tilde{B} > \tag{8.8}$$

We have

$$\begin{aligned} \overline{< \tilde{A}\tilde{B} >} < \tilde{A}\tilde{B} > &= Re\left(< \tilde{A}\tilde{B} >\right)^2 + Im\left(< \tilde{A}\tilde{B} >\right)^2 \\ &\geq Im\left(< \tilde{A}\tilde{B} >\right)^2 \\ &= \left(\frac{< \tilde{A}\tilde{B} > - \overline{< \tilde{A}\tilde{B} >}}{2i}\right)^2 \end{aligned} \tag{8.9}$$

and also

$$\begin{aligned} < \tilde{A}\tilde{B} > &= < (A- < A >)(B- < B >) > \\ &= < AB > - < A >< B > \end{aligned} \tag{8.10}$$

Swapping A and B around gives $< \tilde{A}\tilde{B} > = < AB > - < A >< B >$. Putting this altogather:

$$\begin{aligned} (\Delta A)^2(\Delta B)^2 &\geq \left(\frac{< \tilde{A}\tilde{B} > - \overline{< \tilde{A}\tilde{B} >}}{2i}\right)^2 \\ &= \left(\frac{< AB > - \overline{< AB >}}{2i}\right)^2 \\ &= \left(\frac{< AB > - < BA >}{2i}\right)^2 \\ &= \left(\frac{< [A, B] >}{2i}\right)^2 . \end{aligned} \tag{8.11}$$

We have

$$\Delta A \Delta B \geq \frac{1}{2}\Big| < [A, B] > \Big|. \tag{8.12}$$

8.1.3 Quantum geometry

Now we move to the quantum theory. Here the quantities $\vec{E}^a$ are quantised as

$$\vec{E}_a = 8\pi G\hbar\gamma\vec{L}_a \tag{8.13}$$

$$[\vec{L}^i_a, \vec{L}^j_b] = i\delta_{ab}\epsilon^{ij}{}_k L^k_a. \tag{8.14}$$

Let us find the commutator of two angles

$$\begin{aligned}
[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] &= (8\pi G\hbar\gamma)^4 [L_{1i}L^i_2, L_{1j}L^j_3] \\
&= (8\pi G\hbar\gamma)^4 [L_{1i}, L_{1j}]L^i_2 L^j_3 \\
&= (8\pi G\hbar\gamma)^4 i\epsilon_{ijk}L^i_1 L^j_2 L^k_3 \\
&= i(8\pi G\hbar\gamma)\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3).
\end{aligned} \tag{8.15}$$

where we have used that only L_{1i} and L_{1j} don't commut. Using the (8.12), from the commutation relation, the Heisenberg relation follows:

$$\Delta(\vec{E}_1 \cdot \vec{E}_2) \cdot \Delta(\vec{E}_1 \cdot \vec{E}_3) \geq \frac{1}{2} 8\pi G\hbar\gamma \big| < \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3) > \big| \tag{8.16}$$

8.1.4 $SU(2)$ coherent states

Deriviving coherent states

We have

$$[L_x, L_y] = iL_z, \quad [L_y, L_z] = iL_x, \quad [L_z, L_x] = iL_y \tag{8.17}$$

which implies knowing one means one cannot know the others with certainty. We have that for $L^2 = L^2_x + L^2_y + L^2_z$,

$$[L^2, L_x] = 0, \quad [L^2, L_y] = 0, \quad [L^2, L_z] = 0. \tag{8.18}$$

Therefore we can simultaneously know L^2 and one of the components L_x, L_y, and L_z. We deonte $|j, m >$ the eigenstate of L_z and L^2. The eigenstates have the following spectral properties

$$
\begin{aligned}
L^2|j,m> &= j(j+1)|j,m> \\
L_z|j,m> &= m|j,m>.
\end{aligned} \tag{8.19}
$$

We introduce the operators:

$$
L_+ = L_x + iL_y \quad \text{and} \quad L_- = L_x - iL_y. \tag{8.20}
$$

Note that

$$
L_x = \frac{1}{2}(L_+ + L_-) \quad \text{and} \quad L_y = \frac{1}{2i}(L_+ - L_-). \tag{8.21}
$$

We have

$$
\begin{aligned}
[L_z, L_+] &= [L_z, L_x] + i[L_z, L_y] \\
&= iL_y + L_x = L_+.
\end{aligned} \tag{8.22}
$$

Using (8.22) we have

$$
L_+|j,m> = [L_z, L_+]|j,m> = L_z(L_+|j,m>) - mL_+|j,m>
$$

or

$$
L_z(L_+|j,m>) = (m+1)(L_+|j,m>). \tag{8.23}
$$

Similarly we have

$$
L_z(L_-|j,m>) = (m-1)(L_-|j,m>) \tag{8.24}
$$

Implying that $L_+|j,m>$ is proportional to $|j,m+1>$ and $L_-|j,m>$ is proportional to $|j,m-1>$, and so by (8.21) we have

$$
<j,m|L_x|j,m> = 0 = <j,m|L_y|j,m>. \tag{8.25}
$$

Moreover, we have

$$\begin{aligned}
< j,m|L_x^2|j,m > &= \frac{1}{4} < j,m|(L_+ + L_-)^2|j,m > \\
&= \frac{1}{4} < j,m|L_+^2 + L_-^2 + L_+L_- + L_-L_+|j,m > \\
&= \frac{1}{2} < j,m|L^2 - L_z^2|j,m > \\
&= \frac{1}{2}(j(j+1) - m^2) \qquad (8.26)
\end{aligned}$$

Similarly we have $< j,m|L_y^2|j,m >= \frac{1}{2}(j(j+1)-m^2)$. Therefore,

$$\Delta L_z = \Delta L_y = \sqrt{\frac{1}{2}\left(j(j+1)-m^2\right)} \qquad (8.27)$$

So the uncertainty is minimised for $m = j$. 'in this case, we have

$$\Delta L_x = \Delta L_y = \sqrt{\frac{j}{2}}. \qquad (8.28)$$

The state $|j,j>$ staturates the Heisenberg relations $\Delta L_x \Delta L_y \geq \frac{1}{2}| < L_z > |$,

$$\Delta L_x \Delta L_y = \frac{j}{2}. \qquad (8.29)$$

Rotated coherent state

In terms of Euler angles (θ, ϕ) the representation of a rotation

$$R = e^{-i\phi L_z} e^{-i\theta L_y} \qquad (8.30)$$

Introduce the unit vector $\vec{m}$, $(m_x^2 + m_y^2 + m_z^2 = 1)$, then

$$
\begin{aligned}
\Delta(\vec{L}\cdot\vec{m}) &= \sqrt{<j,j|(\vec{L}\cdot\vec{m})^2|j,j> -(<j,j|\vec{L}\cdot\vec{m}|j,j>)^2} \\
&= \sqrt{<j,j|m_x^2L_x^2+m_y^2L_y^2+m_z^2L_z^2+\dots|j,j> -(<j,j|\cdots+m_zL_z|j,j>)^2} \\
&= \sqrt{<j,j|m_x^2L_x^2+m_y^2L_y^2|j,j>} \\
&= \sqrt{(m_x^2+m_y^2)<j,j|L_x^2|j,j>} \\
&= \sqrt{1-m_z^2}\sqrt{\frac{j}{2}} \\
&= \sqrt{1-(\vec{z}\cdot\vec{m})^2}\sqrt{\frac{j}{2}}
\end{aligned}
\tag{8.31}
$$

where we used

$$
<j,j|L_xL_z|j,j>= j<j,j|L_x|j,j>= 0
$$

So

$$
\Delta(\vec{L}\cdot\vec{m}) = \sqrt{1-(\vec{n}\cdot\vec{m})^2}\sqrt{\frac{j}{2}} \tag{8.32}
$$

This implies that for $\vec{m}$ orthogoanal to $\vec{n}$ we have

$$
\Delta(\vec{L}\cdot\vec{m}) = \sqrt{\frac{j}{2}} \tag{8.33}
$$

This implies the relation

$$
\Delta(\vec{L}\cdot\vec{m}_x)\Delta(\vec{L}\cdot\vec{m}_y) = \frac{j}{2} \tag{8.34}
$$

where $\vec{m}_x$ and $\vec{m}_y$ are orthogonal to $\vec{n}$ and each other. Thus these states also saturate the Heisenburg relations, and are also coherent states. Some properties are the folowing

8.1.5 Resolution of identity

$$
I_j = \frac{2j+1}{4\pi}\int_{S^2} d^2\vec{n}\ |j,\vec{n}><j,\vec{n}| \tag{8.35}
$$

Where $d^2\vec{n}$ is the standard $\mathbb{R}^3$ measure restricted to the unit sphere:

$$d^2\vec{n} = dxdydz\delta\left(\sum_{i=1}^{3} x_i^2 - 1\right)$$

The measure can be written

$$\frac{1}{4\pi}\int_{S^2} d^2\vec{n} = \frac{1}{4\pi}\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \tag{8.36}$$

The $SU(2)$ Haar measure in Euler coordinates is

$$\begin{aligned}\int_{SU(2)} d\mu(g) &= \frac{1}{2\pi}\int_0^{2\pi} d\gamma \frac{1}{4\pi}\int_{S^2} d^2\vec{n} \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\gamma \frac{1}{4\pi}\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi\end{aligned} \tag{8.37}$$

The definition of coherent states in terms of Wigner matrices is:

$$|j,\vec{n}>= \sum_m D^{(j)}(R)^j{}_m |j,m> . \tag{8.38}$$

Consider

$$\int_{S^2} d^2\vec{n} |j,\vec{n}>< j,\vec{n}| = \int_{S^2} d^2\vec{n} \sum_{m,m'} D^{(j)}(\theta,\phi,0)^j{}_m \overline{D^{(j)}(\theta,\phi,0)^j{}_{m'}} |j,m>< j,m'| \tag{8.39}$$

Invariance under $U(1)$,

$$D^{(j)}(\theta,\phi,0)^j{}_m \mapsto D^{(j)}(\theta,\phi,0)^j{}_m e^{-i\chi j} \tag{8.40}$$

means we can write (8.39) as

$$\int_{S^2} d^2\vec{n} \frac{1}{2\pi}\int_0^{2\pi} d\chi \sum_{m,m'} D^{(j)}(\theta,\phi,0)^j{}_m |j,m> e^{-i\chi j} e^{i\chi j} \overline{D^{(j)}(\theta,\phi,0)^j{}_{m'}} < j,m'| \tag{8.41}$$

As $D^{(j)}(\theta,\phi,0)^j{}_m e^{-i\chi j} = D^{(j)}(\theta,\phi,\gamma)^j{}_m$, (8.39) becomes:

$$\int_{S^2} d^2\vec{n}\, \frac{1}{2\pi} \int_0^{2\pi} d\chi \sum_{m,m'} D^{(j)}(\theta,\phi,\chi)^j{}_m \overline{D^{(j)}(\theta,\phi,\chi)^j{}_{m'}}\, |j,m><j,m'| \tag{8.42}$$

We have promoted the S^2 integral to an $SU(2)$ integral using (8.37). The Peter-Weyl theorem:

$$\begin{aligned} 4\pi \sum_{mm'} \int_{SU(2)} d\mu(g)\, \overline{D^{(j)}(g)^j{}_m} D^{(j)}(g)^j{}_{m'} |j,m><j,m'| &= \frac{4\pi}{2j+1} \sum_{mm'} \delta_{mm'} |j,m><j,m'| \\ &= \frac{4\pi}{2j+1} |j,m><j,m| \end{aligned} \tag{8.43}$$

Therefore we have

$$\int_{S^2} d^2\vec{n}\, |j,\vec{n}><j,\vec{n}| = \frac{4\pi}{2j+1} \sum_m |j,m><j,m|. \tag{8.44}$$

By the resolution of identity,

$$I_j = \sum_m |j,m><j,m| \tag{8.45}$$

we have

$$I_j = \frac{2j+1}{4\pi} \int_{S^2} d^2\vec{n}\, |j,\vec{n}><j,\vec{n}|. \tag{8.46}$$

$\square$

Tensor products of coherent states

$$\begin{array}{lll} L^{\otimes}_{x_1} := L_{x_1} \otimes 1 \otimes 1 \otimes 1, & L^{\otimes}_{y_1} := L_{y_1} \otimes 1 \otimes 1 \otimes 1, & L^{\otimes}_{z_1} := L_{z_1} \otimes 1 \otimes 1 \otimes 1 \\ L^{\otimes}_{z_2} := 1 \otimes L_{z_2} \otimes 1 \otimes 1, & L^{\otimes}_{y_2} := L_{y_2} \otimes 1 \otimes 1 \otimes 1, & L^{\otimes}_{z_2} := L_{z_2} \otimes 1 \otimes 1 \otimes 1 \\ L^{\otimes}_{z_3} := 1 \otimes 1 \otimes L_{z_3} \otimes 1, & L^{\otimes}_{y_3} := L_{y_3} \otimes 1 \otimes 1 \otimes 1, & L^{\otimes}_{z_3} := L_{z_3} \otimes 1 \otimes 1 \otimes 1 \\ L^{\otimes}_{z_4} := 1 \otimes 1 \otimes 1 \otimes L_{z_4}, & L^{\otimes}_{y_4} := L_{y_4} \otimes 1 \otimes 1 \otimes 1, & L^{\otimes}_{z_4} := L_{z_4} \otimes 1 \otimes 1 \otimes 1 \end{array} \tag{8.47}$$

$$(L_1^{\otimes})^2 = (L_{x_1} \otimes 1 \otimes 1 \otimes 1)^2 + (L_{y_1} \otimes 1 \otimes 1 \otimes 1)^2 + (L_{z_1} \otimes 1 \otimes 1 \otimes 1)^2 \tag{8.48}$$

For $a = 1, 2, 3, 4$ we have

$$\begin{array}{lll} [L_{x_a}^{\otimes}, L_{y_a}^{\otimes}] = iL_{x_a}^{\otimes} & [L_{y_a}^{4\otimes}, L_{z_a}^{4\otimes}] = iL_{x_a}^{4\otimes} & [L_{z_a}^{4\otimes}, L_{x_a}^{4\otimes}] = iL_{y_a}^{4\otimes} \\ [(L_a^{\otimes})^2, L_{x_a}^{\otimes}] = 0 & [(L_a^{4\otimes})^2, L_{y_a}^{4\otimes}] = 0 & [(L_a^{4\otimes}), L_{z_a}^{4\otimes}] = 0 \end{array} \tag{8.49}$$

For $a \neq b$ we have

$$[L_{x_a}^{\otimes}, L_{y_b}^{\otimes}] = 0 \qquad [L_{y_a}^{4\otimes}, L_{z_b}^{4\otimes}] = 0 \qquad [L_{z_a}^{4\otimes}, L_{x_b}^{4\otimes}] = 0 \tag{8.50}$$

The states $|j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 >$ are eigenstates of $L_{z_1}^{\otimes}$, $(L_1^{\otimes})^2$, $L_{z_1}^{\otimes}$, $(L_2^{\otimes})^2$, $L_{z_3}^{\otimes}$, $(L_3^{\otimes})^2$, $L_{z_4}^{\otimes}$, and $(L_4^{\otimes})^2$:

$$\begin{array}{rl} & (L_{z_1} \otimes 1 \otimes 1 \otimes 1)(|j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 >) \\ = & (j_1)\ |j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 > \end{array} \tag{8.51}$$

and

$$\begin{array}{rl} & (L_1^{\otimes})^2(|j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 >) \\ = & (j_1(j_1 + 1))\ |j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 > \end{array} \tag{8.52}$$

etc.

We have the Heisenberg relation

$$\Delta L_{x_a}^{\otimes} \Delta L_{y_a}^{\otimes} \geq \frac{1}{2} | < L_{z_a}^{\otimes} > |$$

Let us calculate the expectation value $< (L_{x_a}^{\otimes})^2 >$

$$
\begin{aligned}
& < j_1, j_1| \otimes < j_2, j_2| \otimes < j_3, j_3| \otimes < j_4, j_4|(L^{\otimes}_{x_a})|j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 > \\
= & \frac{1}{2} < j_1, j_1| \otimes < j_2, j_2| \otimes < j_3, j_3| \otimes < j_4, j_4|((L^{\otimes}_{x_a})^2 + (L^{\otimes}_{y_a})^2) \\
& \quad |j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 > \\
= & \frac{1}{2} < j_1, j_1| \otimes < j_2, j_2| \otimes < j_3, j_3| \otimes < j_4, j_4| \left(L_a^2 - (L^{\otimes}_{z_a})^2)\right) \\
& \quad |j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 > \\
= & \frac{1}{2}[j_a(j_a + 1) - (j_a)^2] \\
= & \frac{1}{2} j_a
\end{aligned}
\tag{8.53}
$$

Define

$$
\begin{aligned}
& |j_1, \vec{n}_1 > \otimes |j_2, \vec{n}_2 > \otimes |j_3, \vec{n}_3 > \otimes |j_4, \vec{n}_4 > \\
= & D_{\vec{n}_1}(R_1)|j_1, j_1 > \otimes D_{\vec{n}_2}(R_1)|j_2, j_2 > \otimes D_{\vec{n}_3}(R_3)|j_3, j_3 > \otimes D_{\vec{n}_4}(R_4)|j_4, j_4 >
\end{aligned}
\tag{8.54}
$$

The expansion of these states in terms of $L^{\otimes}_{z_a}$ eigenstates

$$
\begin{aligned}
& |j_1, \vec{n}_1 > \otimes |j_2, \vec{n}_2 > \otimes |j_3, \vec{n}_3 > \otimes |j_4, \vec{n}_4 > \\
= & \sum_{m_1, m_2, m_3, m_4} \phi_{m_1, m_2, m_3, m_4}(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4)|j_1, j_1 > \otimes |j_2, j_2 > \otimes |j_3, j_3 > \otimes |j_4, j_4 >
\end{aligned}
\tag{8.55}
$$

where

$$
\begin{aligned}
& \phi_{m_1, m_2, m_3, m_4}(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4) \\
= & < j_1, m_1| \otimes < j_2, m_2| \otimes < j_3, m_3| \otimes < j_4, m_4| \\
& D_{\vec{n}_1}(R_1)|j_1, j_1 > \otimes D_{\vec{n}_2}(R_1)|j_2, j_2 > \otimes D_{\vec{n}_3}(R_3)|j_3, j_3 > \otimes D_{\vec{n}_4}(R_4)|j_4, j_4 > \\
= & < j_1, m_1|D_{\vec{n}_1}(R_1)|j_1, j_1 >< j_2, m_2|D_{\vec{n}_2}(R_1)|j_2, j_2 > \\
& < j_3, m_3|D_{\vec{n}_3}(R_3)|j_3, j_3 >< j_4, m_4|D_{\vec{n}_4}(R_4)|j_4, j_4 > \\
= & \phi_{m_1}(\vec{n}_1)\phi_{m_2}(\vec{n}_2)\phi_{m_3}(\vec{n}_3)\phi_{m_4}(\vec{n}_4)
\end{aligned}
\tag{8.56}
$$

We have a resolution of identity

$$1_{j_1,j_2,j_3,j_4} = \frac{2j_1+1}{4\pi}\frac{2j_2+1}{4\pi}\frac{2j_3+1}{4\pi}\frac{2j_4+1}{4\pi}\int_{S^2} d\vec{n}_1 \int_{S^2} d\vec{n}_2 \int_{S^2} d\vec{n}_3 \int_{S^2} d\vec{n}_4$$
$$|j_1,\vec{n}_1> \otimes |j_2,\vec{n}_2> \otimes |j_3,\vec{n}_3> \otimes |j_4,\vec{n}_4>< j_1,\vec{n}_1| \otimes < j_2,\vec{n}_2| \otimes < j_3,\vec{n}_3| \otimes < j_4,\vec{n}_4| \tag{8.57}$$

8.1.6 Livine-Speziale interwiners

If we note

$$\sum_{a=1}^{4} j_a \vec{n}_a = 0 \tag{8.58}$$

and associate with the node the coherent state

$$|j_1,\vec{n}_1> \otimes |j_2,\vec{n}_2> \otimes |j_3,\vec{n}_3> \otimes |j_4,\vec{n}_4> \tag{8.59}$$

In order to get a coherent interwiner, we project down to the invariant subspace

$$||j_a,\vec{n}_a>= \int_{SU(2)} dh \left(D^{j_1}(h)|j_1,\vec{n}_1> \otimes D^{j_2}(h)|j_2,\vec{n}_2> \otimes D^{j_3}(h)|j_3,\vec{n}_3> \otimes D^{j_4}(h)|j_4,\vec{n}_4>\right) \tag{8.60}$$

These are the Livine-Speziale interwiners.

8.1.7 Thin and thick wedges and time-oriented tetrahedra

Then, the inner product $N_a \cdot N_b$ reads

$$N_a \cdot N_b = -\cosh \xi \tag{8.61}$$

The inner product between two timelike normals in the same light-cone is always negative. In particular

$$N_a \cdot N_b \in (-1, -\infty) \tag{8.62}$$

with -1 being the case when they are parallel and negative infinity reached in the limit where one of them becomes null. Thus, the rapidity ξ at a thick wedge is always positive

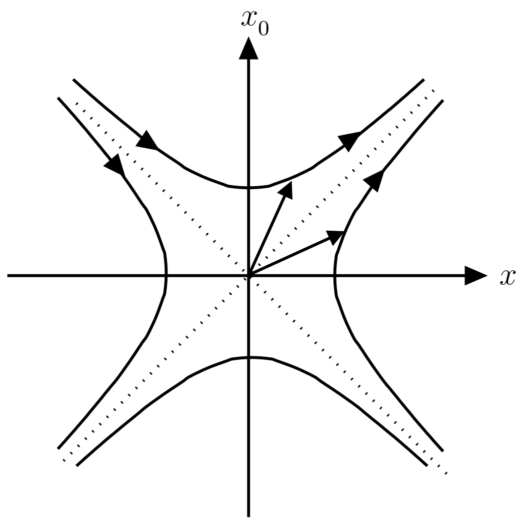

Figure 8.1: Proper orthchronous Lorentz transformations in $SO(1,1)^{+\uparrow}$ Minkowski space-time.

$$\xi \in (0, \infty) \tag{8.63}$$

We then define the dihedral anangle on a thick-wedge to be negative by

$$\Theta_{thin} \equiv -\xi = -\text{arccosh}\left(-N_a \cdot N_b\right) \tag{8.64}$$

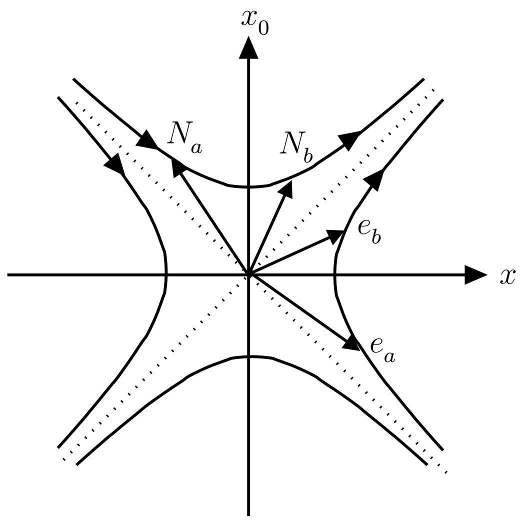

Figure 8.2: A thick wedge.

We now take the case of a thin wedge. We consider the inner product between one of the normals and the PT transformation of the other. Notice that

$$PT\ N = -N. \tag{8.65}$$

We then define the dihedral angle for a thin wedge to be positive, by

$$\begin{aligned} \Theta_{thin} \equiv -\xi &= \operatorname{arccosh}\left(-PT\ N_a \cdot N_b\right) \\ &= \operatorname{arccosh}\left(-N_a \cdot PT\ N_b\right) \\ &= \operatorname{arccosh}\left(N_a \cdot N_b\right) \end{aligned} \tag{8.66}$$

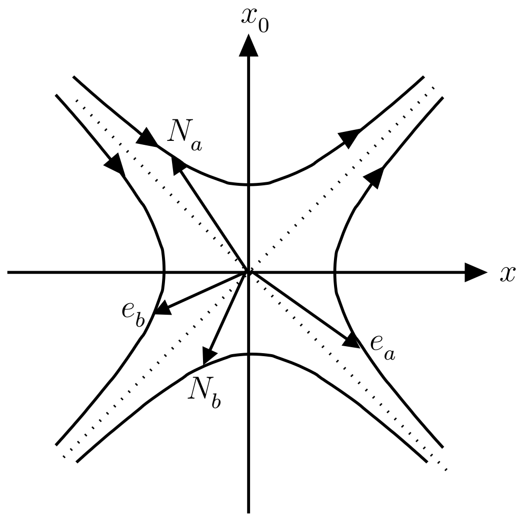

Figure 8.3: A thin wedge.

8.1.8 Spinors

Recall that a spinor $\mathbf{z} \in \mathbb{C}^2$ is a two-diemnsional complex vector

$$\mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix}, \qquad z^0, z^1 \in \mathbb{C}. \tag{8.67}$$

$$g\mathbf{z} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix}, \qquad g \in SL(2, \mathbb{C}) \tag{8.68}$$

where $ad - bd = 1$, and of course a representation of the subgroup $SU(2)$

$$h\mathbf{z} = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix}, \qquad h \in SU(2). \tag{8.69}$$

The bilinear form

$$(\mathbf{w}, \mathbf{z}) = w^A z^B \epsilon_{AB} = (w^0, w^1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix}, = w^0 z^1 - w^1 z^0 \tag{8.70}$$

is invariant under $SL(2, \mathbb{C})$:

$$\begin{aligned}
(g\mathbf{w}, g\mathbf{z}) &= \left[\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}\right]^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\
&= (w^0, w^1) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\
&= (w^0, w^1) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} b & d \\ -a & -c \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\
&= (w^0, w^1) \begin{pmatrix} 0 & ad - bc \\ -(ad - bc) & d \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\
&= (w^0, w^1) \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\
&= (\mathbf{w}, \mathbf{z}).
\end{aligned} \tag{8.71}$$

The bilinear form is the scalar product of $\mathcal{H}_{\frac{1}{2}}$

$$< \mathbf{w}, \mathbf{z} >= \overline{w^A} z^B \delta_{AB} = (\overline{w^0}, \overline{w^1}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix}, = \overline{w^0} z^0 + \overline{w^1} z^1 \tag{8.72}$$

This is invariant under $SU(2)$:

$$\begin{aligned} < h\mathbf{w}, h\mathbf{z} > &= \left[\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}\right]^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\ &= (\overline{w^0}, \overline{w^1}) \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\ &= (\overline{w^0}, \overline{w^1}) \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\ &= (\overline{w^0}, \overline{w^1}) \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \\ &= < \mathbf{w}, \mathbf{z} > \end{aligned} \tag{8.73}$$

but not under $SL(2, \mathbb{C})$, it is easy to see

$$< g\mathbf{w}, g\mathbf{z} > = < \mathbf{w}, (g^\dagger g)\mathbf{z} > \tag{8.74}$$

and note that $g^\dagger g$ is equal to the identity only when $g^\dagger = g^{-1}$, i.e. when g is unitary.

8.2 Spinors, vectors, and bivectors

8.2.1 Spinors and 3-vectors

$$\begin{aligned} \vec{n} &= < \mathbf{n} | \vec{\sigma} | \mathbf{n} > \\ &= \overline{n^A} \vec{\sigma}_{AB} n^B \end{aligned} \tag{8.75}$$

$$\begin{aligned} n_x &= (\overline{n}^0, \overline{n}^1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n^0 \\ n^1 \end{pmatrix} \\ &= n^0 \overline{n}^1 + n^1 \overline{n}^0 \\ &= 2Re(n^0 \overline{n}^1) \end{aligned} \tag{8.76}$$

$$\begin{aligned} n_y &= (\overline{n}^0, \overline{n}^1) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} n^0 \\ n^1 \end{pmatrix} \\ &= i n^0 \overline{n}^1 - i n^1 \overline{n}^0 \\ &= 2Im(n^0 \overline{n}^1) \end{aligned} \tag{8.77}$$

$$
\begin{aligned}
n_z &= (\overline{n}^0, \overline{n}^1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} n^0 \\ n^1 \end{pmatrix} \\
&= |n^0|^2 - |n^1|^2
\end{aligned} \tag{8.78}
$$

so that

$$
n_x = 2Re(n^0\overline{n}^1), \qquad n_y = 2Im(n^0\overline{n}^1), \qquad n_z = (|n^0|^2 - |n^1|^2).
$$

So

$$
\vec{n} = \begin{pmatrix} 2Re(n^0\overline{n}^1) \\ 2Im(n^0\overline{n}^1) \\ |n^0|^2 - |n^1|^2 \end{pmatrix} \tag{8.79}
$$

$$
\begin{aligned}
\vec{n} \cdot \vec{\sigma} &= \vec{n}^i \sigma_i \\
&= \begin{pmatrix} n^3 & n^1 - in^2 \\ n^1 + in^2 & -n^3 \end{pmatrix} \\
&= \begin{pmatrix} \overline{n}^0 n^0 & \overline{n}^0 n^1 \\ \overline{n}^1 n^0 & \overline{n}^1 n^1 \end{pmatrix}
\end{aligned} \tag{8.80}
$$

with a norm

$$
|\vec{n}|^2 = (Re(n^0\overline{n}^1))^2 + (Im(n^0\overline{n}^1))^2 + (|n^0|^2 - |n^1|^2)^2 \tag{8.81}
$$

8.2.2 Spinors and 4-vectors

8.2.3 Spinors and bivectors

8.3 Extrinsic coherent states

In configuration space and momentum space representation:

$$
\psi_z(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\omega}{2\hbar}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, \tag{8.82}
$$

$$
\psi_z(p) = \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\hbar}{2\omega} \frac{(p - P_0)^2}{\hbar^2} + \frac{i}{\hbar} p X_0 \right\}. \tag{8.83}
$$

substituting $t = \hbar/\omega$ this is rewritten

$$\psi_z(x) = \frac{1}{(\pi t)^{\frac{1}{4}}} \exp - \left\{ \frac{1}{2t}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, \tag{8.84}$$

$$\psi_z(p) = \left(\frac{t}{\pi}\right)^{\frac{1}{4}} \exp - \left\{ \frac{t}{2}(p - P_0)^2 + \frac{i}{\hbar} p X_0 \right\}. \tag{8.85}$$

From these formulae we can read off the most important properties of coherent states: In both configuration and momentum representation the wavefunctions are Gaussian distributions, centered at

$$< \hat{X} >_{\Psi_z} = X_0, \quad \text{and} \quad < \hat{P} >_{\Psi_z} = P_0, \tag{8.86}$$

respectively. Furthermore we can see that the width of the distribution in the configuration representtion is inversely proportional to that on the momentum representation.

We come to an important property. However, we must first derive the result

$$e^{t\Delta/2}\delta_y(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{1}{2t}(x-y)^2}. \tag{8.87}$$

To do this we use the Dirac delta-function representation

$$\delta_y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip\cdot(x-y)}.$$

Then note

$$
\begin{aligned}
e^{t\Delta/2}\delta_y(x) &= e^{t\Delta/2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip\cdot y}\exp\left\{\frac{t}{2}\frac{d^2}{dx^2}\right\}e^{ip\cdot x} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip\cdot y}\left[1-\frac{t}{2}p^2+\frac{1}{2!}\left(\frac{t}{2}\right)^2 p^4-\ldots\right]e^{ip\cdot x} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)}e^{-tp^2/2} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp-\frac{t}{2}\left\{p^2-\frac{2ip}{t}(x-y)\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp-\frac{t}{2}\left\{\left(p-\frac{i}{t}(x-y)\right)^2+\frac{1}{t^2}(x-y)^2\right\} \\
&= e^{-\frac{1}{2t}(x-y)^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tp^2/2} \\
&= \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(x-y)^2}. \qquad (8.88)
\end{aligned}
$$

Now the coherent state can be obtained as an *analytic continuation* of the heat kernel:

$$
\begin{aligned}
\left[\frac{1}{\sqrt{2\pi t}}e^{\frac{1}{2t}(x-y)^2}\right]_{y\to z}(x) &\propto \exp\left\{-\frac{1}{2t}\left(x-X_0-\frac{it}{\hbar}P_0\right)^2\right\} \\
&= \exp\left\{-\frac{1}{2t}\left[(x-X_0)^2-2\frac{it}{\hbar}(x-X_0)P_0-\frac{t^2}{\hbar^2}P_0^2\right]\right\} \\
&\propto \exp\left\{-\left[\frac{1}{2t}(x-X_0)^2-\frac{i}{\hbar}xP_0\right]\right\}
\end{aligned}
$$

that is,

$$
\psi_z(x) \sim \left[e^{-t\Delta/2}\delta_y(x)\right]_{y\to z}(x). \qquad (8.89)
$$

Complexification of a Lie Group

The complexification of a Lie group $\mathbf{g}$ is denoted $\mathbf{g}_{\mathbb{C}}$. Recall the defining properties of a Lie algebra: antisymmetric $[X,Y]=-[Y,X]$, bilinearity $[\alpha X,Y]=\alpha[X,Y]=[X,\alpha Y]$ for any real number α, and satisfies the Jacobi identity $[[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X]\equiv 0$. X_1+iX_2 where $X_1,X_2\in\mathbf{g}$

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1]). \tag{8.90}$$

It is clear that (8.90) is *real* bilnear and skew-symmetric. If we prove that it is *complex* linear in the first factor, it will be complex linear in the second because of the skew-symmetry. As we already know it is real linear in the first factor, it suffices to show that it is *imaginary* linear. This is not difficult to prove, all we need to do is verify that

$$[i(X_1 + iX_2), Y_1 + iY_2] = i[X_1 + iX_2, Y_1 + iY_2] \tag{8.91}$$

is true. This is easily done by expanding each side and seeing they are indeed equal.

It remains to check the Jacobi identity. First consider Y and Z to be in $\mathbf{g}$ but take X to be in $\mathbf{g}_{\mathbb{C}}$. Now, $X = X_1 + iX_2$ is linear in the Jacobi identity and the Jacobi identity holds separately for X_1 and X_2,

$$[[X_1, Y], Z] + [[Z, X_1], Y] + [[Y, Z], X_1] + i([[X_2, Y], Z] + [[Z, X_2], Y] + [[Y, Z], X_2]) \equiv 0, \tag{8.92}$$

and so the Jacobi identity holds for $X \in \mathbf{g}_{\mathbb{C}}$ and $Y, Z \in \mathbf{g}$. Similarly for Y and Z. Therefore we have shown that the elements of the complexification $\mathbf{g}_{\mathbb{C}}$ satisfy the Jacobi identity.

Chapter 9

Matter

9.1 The Dirac Equation and the Standard Representation

The Dirac equation for particles of res mass m_0 can be written

$$i\hbar\gamma^\mu \frac{\partial\Psi(x)}{\partial x^\mu} - m_0 c\Psi(x) = 0 \tag{9.1}$$

The adjoint field $\overline{\Psi}(x)$ is defined by

$$\overline{\Psi}(x) = \Psi^\dagger\gamma^0$$

and satisfies the adjoint Dirac equation

$$i\hbar\gamma^\mu \frac{\partial\overline{\Psi}(x)}{\partial x^\mu} + m_0 c\overline{\Psi}(x) = 0 \tag{9.2}$$

where the *gamma* matrices satisfy the common commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{9.3}$$

The Dirac equations (J.68) and (9.2) can be obtained from the Lagrangian density

$$\mathscr{L} = c\overline{\Psi}(x)\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\Psi(x) \tag{9.4}$$

by varying the action $S = \int d^4x\mathscr{L}$ independently with respect to the fields Ψ_α and Ψ_α. The standard representation the γ^μ can be written explicitly

$$\gamma^i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{9.5}$$

With γ^5

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{9.6}$$

As follows from:

$$\begin{aligned}
\gamma^5 &= i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \hat{\sigma}_1 \\ -\hat{\sigma}_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \hat{\sigma}_2 \\ -\hat{\sigma}_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \hat{\sigma}_3 \\ -\hat{\sigma}_3 & 0 \end{pmatrix} \\
&= i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \hat{\sigma}_1 \\ -\hat{\sigma}_1 & 0 \end{pmatrix}\begin{pmatrix} -\hat{\sigma}_2\hat{\sigma}_3 & 0 \\ 0 & -\hat{\sigma}_2\hat{\sigma}_3 \end{pmatrix} \\
&= i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & -\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 \\ \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 & 0 \end{pmatrix} \\
&= -i\begin{pmatrix} 0 & \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 \\ \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 & 0 \end{pmatrix}
\end{aligned}$$

and

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

9.2 Massless Fermion

We begin with the (massfree) Dirac action in symmetric form

$$S = i\int d^4x \left[\overline{\Psi}(x)\gamma^\mu\frac{\partial}{\partial x^\mu}\Psi(x) - \frac{\partial}{\partial x^\mu}\overline{\Psi}(x)\gamma^\mu\Psi(x)\right] \tag{9.7}$$

which can be expressed as

$$S = i\int d^4x\overline{\Psi}(x)\gamma^\mu\frac{\partial}{\partial x^\mu}\Psi(x) + c.c. \tag{9.8}$$

where *c.c.* stands for complex conjugate.

9.2.1 Chiral Spinor

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

where

$$\psi = -\chi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

The chiral spinor $\psi \in \mathbb{C}^2$ with 2-complex components ψ^A, $A = 1, 2$.

The Dirac equation for massfree particles (which are the only kind of particle that can have the left-hand spinor)

The standard representation (J.70) the Euclidean action is

9.2.2 Fermion Doubling on a Lattice

If you try to apply a regular lattice formulation discussed in section 4.1.1, for gauge fields, one encounters the problem of *fermion doubling problem.* In flat spacetime Dirac action, (see sectionJ.2.3), is

$$S = \int \overline{\psi} \left(i\hbar\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} - m_0 \mathbb{1} \right) \psi d^4x \tag{9.9}$$

Fermion fields are defined on the lattice points x_l and flat spacetime Dirac Lagrangian is discretised accoeding to, for example,

$$\begin{aligned} \frac{1}{a^4} S &= \sum_{x_i} \mathcal{L}_{lattice}(x_i) \\ &= \sum_{x_i} \left[\sum_{\mu} \frac{-\hbar}{2ai} \left\{ \overline{\psi}(x_i)\gamma_{\mu}\psi(x_i + a_{\mu}) - \psi(x_i + a_{\mu})\gamma_{\mu}\psi(x_i) \right\} - m_0 \overline{\psi}(x_i)\psi(x_i) \right] \end{aligned} \tag{9.10}$$

with lattice parameter a. Here a_{μ} denotes a vector of length a in the μ direction. In the continuum limit (9.10) becomes the symmetric Lagrangian

$$\mathcal{L} = \overline{\psi}\left[\frac{1}{2}i\hbar\left(\gamma_\mu \overset{\rightarrow\mu}{\partial} - \gamma^\mu \overset{\leftarrow\mu}{\partial}\right) - m_0\mathbb{1}\right]\psi \tag{9.11}$$

Now we introduce the translation operators into (9.10) (note here that no summation is made over μ)

$$\psi(x_i + a_\mu) = \exp\left(a_\mu \partial^\mu\right)\psi(x_i). \tag{9.12}$$

In position space the Dirac operator then becomes

$$\frac{\hbar}{2ai}\gamma_\mu\left[\exp\left(a_\mu \overset{\rightarrow\mu}{\partial}\right) - \exp\left(a_\mu \overset{\leftarrow\mu}{\partial}\right)\right] - m_0\mathbb{1} \tag{9.13}$$

and the corresponding propagator in momentum space is

$$\begin{aligned} S(q) &= \left[-\frac{\hbar}{2ai}\gamma_\mu\left(e^{-ia_\mu p^\mu} - e^{ia_\mu p^\mu}\right) - m_0\right]^{-1} \\ &= \left(\frac{\hbar}{a}\gamma_\mu \sin a_\mu p^\mu - m_o\right)^{-1} \end{aligned} \tag{9.14}$$

There are two limiting cases to consider $p_\mu \to 0$ and $a_\mu p^\mu \to \pi$. The former yields

$$\lim_{p_\mu \to 0} S(p) = \left[\gamma_\mu p^\mu - m_0\right]^{-1}, \tag{9.15}$$

which is the usual continuum expression for the propagator. In the latter case we have

$$\lim_{a_\mu p^\mu \to \pi} S(p) = \left[\frac{1}{a}\gamma_\mu(\pi - a_\mu p^\mu) - m_0\right]^{-1}, \tag{9.16}$$

Hence there are two free solutions to for every given lattice constant a for every fermion.

The energy of the second solution becomes infinitely large in the limit $a \to 0$, therefore these solutions decouple from all processes and becomes unobservable.

9.2.3 Berezin Integral

We wish to prove

$$\int d\psi d\overline{\psi}\ \psi^a \overline{\psi}^b \psi^c \overline{\psi}^d = \epsilon^{ac}\epsilon^{bd} = (\delta^{ab}\delta^{cd} - \delta^{ad}\delta^{bc}). \tag{9.17}$$

Note that when the indices take two values: 1 and 2, then we have the identity: $\epsilon^{ac}\epsilon^{bd} = (\delta^{ab}\delta^{cd} - \delta^{ad}\delta^{bc})$. We evaluate of the integral (9.17),

$$\begin{aligned}\int d\psi^1 d\psi^2 d\overline{\psi}^1 d\overline{\psi}^2\ \psi^a \overline{\psi}^b \psi^c \overline{\psi}^d &= \left(\int d\psi^1 d\psi^2\ \psi^a \psi^c\right)\left(\int d\overline{\psi}^1 d\overline{\psi}^2 \overline{\psi}^b \overline{\psi}^d\right) \\ &= \epsilon^{ac}\epsilon^{bd} \\ &= (\delta^{ab}\delta^{cd} - \delta^{ad}\delta^{bc}).\end{aligned} \tag{9.18}$$

We evalute the integrations:

$$\begin{aligned}\int \mathcal{D}\psi_e\ \psi_e \overline{\psi}_e &= \int \frac{1}{v_e} d\psi_e d\overline{\psi}_e e^{-v_e \overline{\psi}_e \psi_e}\ \overline{\psi}_e \psi_e \\ &= \end{aligned} \tag{9.19}$$

Chapter 10

Black holes

Chapter 11

Cosmology

Chapter 12

Scattering

Appendix A

Represention theory for groups

In this appedix we introduce some basic facts about the representation theory of groups.

A.1 Basic concepts

A.2 Matrix coefficients

If we choose a basis for V, then any linear transformation of V can be represented as a matrix. Let (σ, V) be a unitary representation of G, and let $\{v_1, \dots, v_n\}$ be an orthonormal basis. Then the *matrix coefficients* associated with this basis are given by

$$U_{ij}(g) = \langle v_j, \sigma(g)v_i\rangle_V, \qquad g \in G, \quad i,j = 1,\dots,n \tag{A.1}$$

For $g \in G$, let $U(g) = \Big(U_{ij}(g)\Big)_{i,j=1}^{n} \in M_{n,n}(\mathbb{C})$ be the matrix whose entries are the matrix coefficients of (σ, V). Recall that if $M \in M_{n,n}(\mathbb{C})$, then its *conjugate transpose* is the matrix $M^\dagger$ with entries given by

$$(M^\dagger)_{ij} = \overline{M_{ji}}. \tag{A.2}$$

A matrix M is *unitary* if it is invertable and $M^{-1} = M^\dagger$.

Lemma A.2.1 *(a)* $U(g_1g_2) = U(g_1)U(g_2)$*,*
(b) $U(g^{-1}) = U(g)^\dagger$*,*
(c) $U(g)$ *is unitary,*

(d) the matrix coefficients of the adjoint representation σ' with respect to the dual basis $\theta_{v_1}, \ldots, \theta_{v_n}$ are

$$\langle \theta_{v_j}, \sigma'(g)\theta_{v_i} \rangle_{V'} = \overline{U_{ij}(g)}.$$

Proof:

(a) We have

$$\begin{aligned} U_{ij}(g_1 g_2) &= \langle v_j, \sigma(g_1 g_2) v_i \rangle_V \\ &= \langle v_j, \sigma(g_1)\sigma(g_2) v_i \rangle_V && (\sigma \textit{ is a group homomorphism}) \\ &= \left\langle v_j, \sigma(g_1)\left(\sum_{k=1}^{n} \langle v_k, \sigma(g_2) v_i \rangle_V v_k\right)\right\rangle_V && w = \sum_{k=1}^{n} \langle v_k, w \rangle_V v_k \textit{ for all } w \in V \\ &= \sum_{k=1}^{n} \langle v_k, \sigma(g_1) v_i \rangle_V \langle v_j, \sigma(g_2) v_k \rangle_V \\ &= \sum_{k=1}^{n} U_{ik}(g_1) U_{kj}(g_2). \end{aligned} \tag{A.3}$$

(b) We have

$$\begin{aligned} U_{ij}(g^{-1}) &= \langle v_j, \sigma(g^{-1}) v_i \rangle_V \\ &= \langle \sigma(g) v_j, v_i \rangle_V && (\sigma(g^{-1}) = \sigma(g)^\dagger) \\ &= \overline{\langle v_i, \sigma(g) v_j \rangle_V} \\ &= \overline{U_{ij}(g)}. \end{aligned} \tag{A.4}$$

(c) It follows from part (a) that $U(g^{-1}) = U(g)^{-1}$. Then it follows from part (b) that $U(g)$ is unitary.

(d) We have

$$\begin{aligned} \langle \theta_{v_j}, \sigma'(g)\theta_{v_i} \rangle_{V'} &= \langle \theta_{v_j}, \theta_{\sigma(g) v_j} \rangle_{V'} \\ &= \langle \sigma(g) v_i, v_j \rangle_V \\ &= \overline{\langle v_j, \sigma(g) v_i \rangle_V} \\ &= \overline{U_{ij}(g)}. \end{aligned} \tag{A.5}$$

It follows from lemma A.2.1 that we have a group homomorphism

$$U : G \to U(n), \qquad g \mapsto U(g), \tag{A.6}$$

where $U(n)$ is the group of $n \times n$ unitary matrices (under multiplication). This group homomorphism is called a unitary matrix realisation of σ. Note that is depends on the choice of basis.

Suppose $B = \{v_1, \ldots, v_n\}$ and $B' = \{w_1, \ldots, w_n\}$ are two orthonormal bases for V. Then we have a change of basis matrix Λ whose (i, j) entry is given by

$$\Lambda_{ij} := \langle w_j, v_i \rangle_V. \tag{A.7}$$

Let U and U' be the unitary matrix realisations of the representation σ on V in terms of bases B and B', respectively. We have

$$U'(g) = \Lambda U(g) \Lambda^\dagger \tag{A.8}$$

or in terms of matrix components

$$U'_{ij}(g) = \overline{\Lambda_{ki}} U_{kl}(g) \Lambda_{lj}. \tag{A.9}$$

Proof:

$$\begin{aligned}
U'_{ij}(g) &= \langle w_j, \sigma(g) w_i \rangle_V \\
&= \left\langle \sum_{l=1}^{n} \langle v_l, w_j \rangle_V v_l, \sum_{k=1}^{n} \langle v_k, w_i \rangle \sigma(g) v_k \right\rangle_V \\
&= \sum_{k,l=1}^{n} \overline{\langle v_l, w_j \rangle_V} \langle v_l, \sigma(g) v_k \rangle_V \langle v_k, w_i \rangle_V \\
&= \sum_{k,l=1}^{n} \overline{\langle w_i, v_k \rangle_V} U_{kl}(g) \langle w_j, v_l \rangle_V \\
&= \sum_{k,l=1}^{n} \overline{\Lambda_{ki}} U_{kl}(g) \Lambda_{lj}.
\end{aligned} \tag{A.10}$$

We also have that Λ is unitary:

$$
\begin{aligned}
\delta_{ij} &= \langle w_j, w_i \rangle_V \\
&= \left\langle w_j, \sum_{k=1}^{n} \langle v_k, w_i \rangle_V v_k \right\rangle_V \\
&= \sum_{k=1}^{n} \langle w_j, v_k \rangle_V \langle v_k, w_i \rangle_V \\
&= \sum_{k=1}^{n} \overline{\langle w_i, v_k \rangle_V} \langle w_j, v_k \rangle_V \\
&= \sum_{k=1}^{n} \overline{\Lambda_{ki}} \Lambda_{kj}. \qquad \text{(A.11)}
\end{aligned}
$$

A.3 Sequilinear forms and matrix representations

Appendix B

Results for finite groups

B.0.1 Unitary Representations of Groups

R	Group elements.
$[G]$	The collection of group elements
R_a	An indexed group element.
$\alpha, \beta, \ldots$	Labels irreducible representations.
D	Matrix representation.
$D^{(\alpha)}$	Indexed irriduceable matrix representation.
$i, j, \ldots$	Labels components of matrix representations
h	Dimension of group.
χ	Character

Proof Let $\{A_a\}$ be a representation of the group G. Construct the Hermitian matrix $H = \sum_{a=1}^{h} A_a A_a^\dagger$

$$H^\dagger = [\sum_{a}^{h} (A_a A_a^\dagger)]^\dagger = \sum_{a}^{h} (A_a A_a^\dagger)^\dagger = \sum_{a}^{h} A_a A_a^\dagger = H \tag{B.1}$$

From the earlier discussion of matrix algebra, any Hermitian matrix can be diagonalized by a similarity transformation. Let

$$d = \sum_{a}^{h} U^\dagger A_a A_a^\dagger U = \sum_{\alpha} (U^\dagger A_a U)(U^\dagger A_a^\dagger U) = \sum_{\alpha} (U^\dagger A_a U)(U^\dagger A_a U)^\dagger \tag{B.2}$$

Hence, $d = \sum_a \tilde{A}_a \tilde{A}_a^\dagger$ where $\tilde{A}_a = U^\dagger A_a U = U^{-1} A_a U$. The elements of the diagonal d matrix are real and positive.

$$d_{jj} = \sum_a^h \sum_k (\tilde{A}_a)_{jk} (\tilde{A}_a^\dagger)_{kj} = \sum_a^h \sum_k (\tilde{A}_a)_{jk} (\tilde{A}_a)^*_{jk} = \sum_a^h \sum_k |(\tilde{A})_{jk}|^2 \tag{B.3}$$

for all $j = 1, \ldots, h$. Since d is diagonal, we can define its square-root $d^{1/2}$

$$d^{1/2} = \begin{pmatrix} d_{11}^{1/2} & 0 & \cdots & 0 \\ 0 & d_{11}^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{hh}^{1/2} \end{pmatrix} \tag{B.4}$$

and $d^{-1/2}$, which is given by an analogous expression. Evidently, $(d^{1/2})^2$ and $d^{1/2}d^{-1/2} = 1$, where I is the identity matrix. Diagonal matrices commute with each other, and we can write for the identity I

$$I = d^{-1/2} d d^{-1/2} = d^{-1/2} \sum_a^h \tilde{A}_a \tilde{A}_a^\dagger \, d^{-1/2}. \tag{B.5}$$

By the rearangement theorem, $\{\tilde{A}_b \tilde{A}_a\}$ all a, and any one b is equal to $\{\tilde{A}\}$ for all a. Hence,

$$\begin{aligned} I &= d^{-1/2} \sum_a (\tilde{A}_b \tilde{A}_a) \, (\tilde{A}_b \tilde{A}_a)^\dagger d^{-1/2} \\ &= d^{-1/2} \sum_a (\tilde{A}_b \tilde{A}_a \, \tilde{A}_a^\dagger \tilde{A}_b^\dagger) d^{-1/2} \\ &= d^{-1/2} \tilde{A}_b d^{1/2} \left(d^{-1/2} \sum_a \tilde{A}_a \tilde{A}_a^\dagger d^{-1/2} \right) d^{1/2} \tilde{A}_b^\dagger d^{-1/2} \end{aligned} \tag{B.6}$$

using (B.5) in this gives

$$\begin{aligned} I &= (d^{-1/2} \tilde{A}_b d^{1/2})(d^{-1/2} \tilde{A}_b d^{1/2})^\dagger \\ &= (d^{-1/2} U^{-1} A_b U d^{1/2})(d^{-1/2} U^{-1} A_b U d^{1/2})^\dagger. \end{aligned} \tag{B.7}$$

Let us define the matrix $B_a := d^{-1/2} U^{-1} A_a U d^{1/2}$. It is easy to see that B_a has the same multiplication table as A_α. If we have $A_a A_b = A_c$, then

$$
\begin{aligned}
B_a B_b &= (d^{-1/2}U^{-1}A_\alpha U d^{1/2})(d^{-1/2}U^{-1}A_b U d^{1/2}) \\
&= (d^{-1/2}U^{-1}A_a)\,(U d^{1/2} d^{-1/2} U^{-1})\,(A_b d^{1/2} U) \\
&= (d^{-1/2}U^{-1}\,A_a A_b\,U d^{1/2}) \\
&= (d^{-1/2}U^{-1}A_c U d^{1/2}) = \ B_c
\end{aligned} \tag{B.8}
$$

Therefore $\{B_a\}$ is a unitary reprsentation of the group G.

Representation of finite groups can always be taken to be unitary. It is essential that the sum over $g \in G$ converge. This is guaranteed for a finite group, but may not work for infinite groups. In particular, non-compact Lie groups, such as the Lorentz group, have no finite dimensional unitary representations.

B.0.2 Schur's First Lemma

Theorem B.0.1 *A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix.*

Proof:

We can take these matrices to be unitary without loss of generality. Suppose there is a matrix M that commutes with all of the A_a but which is not a constant multiple of the unit matrix:

$$
MA_a = A_a M \tag{B.9}
$$

for all $a = 1, 2, \ldots, |G|$. By taking the edjoint of each of these equations, we obtain

$$
A_a^\dagger M^\dagger = M^\dagger A_a^\dagger . \tag{B.10}
$$

Since A_a are unitary

$$
A_a^{-1} M^\dagger = M^\dagger A_a^{-1} . \tag{B.11}
$$

Multiplying both this from left and right we obtain

$$
M^\dagger A_a = A_a M^\dagger \tag{B.12}
$$

So that, if M commutes with every matrix of the representation, then so does $M^{\dagger}$. As such the Hermitian matries

$$H_1 = M + M^{\dagger}, \quad H_2 = i(M - M^{\dagger}) \tag{B.13}$$

also commute with every matrix of the representation. We prove the statement of the theorem for Hermitician matrices. We start with

$$HA_a = A_a H. \tag{B.14}$$

Since a hermitian matrix H can be diagonalized by some U, $D = U^{-1}HU$.

$$U^{-1}HUU^{-1}A_a U = U^{-1}A_a UU^{-1}HU, \tag{B.15}$$

Let $\tilde{A}_a = U^{-1}A_a U$

$$D\tilde{A}_a = \tilde{A}_a D. \tag{B.16}$$

A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix.

$$(\tilde{A}_a)_{ij}(D_{ii} - D_{jj}) = 0. \tag{B.17}$$

(i) Suppose all the diagonal elements of D are distinct: $D_{ii} \neq D_{jj}$ if $i \neq j$. Then (B.17) implies that

$$(\tilde{A}_a)_{ij} = 0, \qquad i \neq j,$$

i.e., the off-diagonal elements of $\tilde{A}_\alpha$ must vanish. They form a reducible representatoin composed of d one-dimentional representations. Since the $\tilde{A}_a$ are obtained from the A_a by a similarity transformation, the A_a are a reducible representaion.

(ii) Suppose p of the diagonal elements of D are equal, but the remaining entries are distinct from these and from each other. Using similarity transfromations, (for example (B.18)), we can arrange the diagonal elements so that the first p are equal.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{B.18}$$

Hence, we can assume this is the case without loss of generality, i.e. $D_{11} = D_{22} = \cdots = D_{pp}$, $D_{mm} \neq D_{nn}$, otherwise. The $(\tilde{A}_i)_{mn}$ must vanish for any pair of unequal diagonal entries.

$$\tilde{A}_i = \begin{pmatrix} B_1 & 0 \\ 0 & B_2, \end{pmatrix} \tag{B.19}$$

where B_1 is a $p \times p$ matrix and B_2 is a $(p-d) \times (p-d)$ diagonal matrix.

We have shown that if a Hermtian matrix is not a multiple of the unit matrix and commutes with all the matrices of a representation, then that representation is necessarily reducible. Thus, if a non-zero Hermtian matrix which commutes with all the matrices of an irreducible representation that matrix must be a multiple of the unit matrix.

Given that $M + M^\dagger$ and $i(M - M^\dagger)$ are both Hermitician matrices that commute with all of the matrices of an irreducible representation they are a constant multiple of the unit matrix. we have $H_1 = M + M^\dagger = c_1 I$ and $H_2 = i(M - M^\dagger) = c_2 I$. We have that

$$M = H_1 - iH_2 = (c_1 - ic_2)I$$

and hence must be proportional to the unit marix.

□

B.0.3 Schur's Second Lemma

Theorem B.0.2 *Let $\{A_1, A_2, \ldots, A_{|G|}\}$ and $\{B_1, B_2, \ldots, B_{|G|}\}$ be two irreducible representations of a group G of dimensionalities d an d' respectively. If there is a matrix M such that*

$$MA_a = B_a M$$

for $a = 1, 2, \ldots, |G|$, then if $d = d'$, either $M = 0$ or the two representations are related by a similarity transformation. If $d \neq d'$, then $M = 0$.

As

$$MA_a = B_a M$$

then

$$A_a^\dagger M^\dagger = M^\dagger B_a^\dagger$$

using unitary

$$A_a^{-1} M^\dagger = M^\dagger B_a^{-1}. \tag{B.20}$$

Multiplying on the right by M, we get

$$A_a^{-1} M^\dagger M = M^\dagger B_a^{-1} M. \tag{B.21}$$

By the group properties B_a^{-1} is some B_b and A_a^{-1} is the corresponding A_b, and so by the posulate of the theorem, $MA_b = B_b M$, or

$$MA_b^{-1} = B_b^{-1} M$$

substituting this into (B.21), gives

$$A_a^{-1} M^\dagger M = M^\dagger M A_a^{-1}. \tag{B.22}$$

Thus, the $d' \times d'$ matrix $MM^\dagger$ commutes with all the matrices of an irreducible representation. By Schur's first lemma, $MM^\dagger$ must therefore be a constant multiple of the unit matrix,

$$MM^\dagger = cI. \tag{B.23}$$

(i) Say $d = d'$. If $c \neq 0$, equation (B.23) implies that

$$M^{-1} = \frac{1}{c} M^\dagger.$$

plus (B.20) can be rearranged

$$A_a = M^{-1} B_a M,$$

so the two representations are related by a similarity transfromation and are, therefore, equivalent and since $M^\dagger M = c^d$,it follows that $c^{-d'/2} M$ is a unitary matrix. If $c = 0$ then

$$(MM^\dagger)_{ij} = \sum_k M_{ik} (M^\dagger)_{kj} = \sum_k M_{ik} M_{jk}^* = 0. \tag{B.24}$$

By setting $i = j$, we obtain

$$\sum_k M_{ik} {M^*}_{ik} = \sum_k |M_{ik}|^2 = 0 \tag{B.25}$$

which implies that $M_{ik} = 0$ for all i and k, i.e., that M is the zero matrix.

(ii) Say $d \neq d'$. We take $d < d'$. Them M is a rectangular matrix with d columns and d' rows,

$$\begin{pmatrix} M_{11} & \cdots & M_{1d} \\ M_{21} & \cdots & M_{2d} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ M_{d'1} & \cdots & M_{d'd}. \end{pmatrix} \tag{B.26}$$

We add $d' - d$ columns of zeros

$$N = \begin{pmatrix} M_{11} & \cdots & M_{1d} & 0 & \cdots & 0 \\ M_{21} & \cdots & M_{2d} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & & \vdots & & \vdots \\ M_{d'1} & \cdots & M_{d'd} & 0 & \cdots & 0 \end{pmatrix} \tag{B.27}$$

Taking the adjoint of this matrix yields

$$N^\dagger = \begin{pmatrix} M^*_{11} & M^*_{21} & \cdots & M_{d'1} \\ M^*_{12} & M^*_{22} & \cdots & M_{d'2} \\ \vdots & & \cdots & \vdots \\ M^*_{d1} & M^*_{2d} & \cdots & M^*_{d'd} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{B.28}$$

We have

$$NN^\dagger = MM^\dagger = cI.$$

Taking the determinant

$$\det(NN^\dagger) = \det(N)\det(N^\dagger) = c^{d'} = 0.$$

so $c = 0$, so that, as before we arrive at

$$\sum_k |N_{ik}|^2 = 0. \tag{B.29}$$

which implies that $N_{ik} = 0$ for all i and k, i.e. N is the zero matrix. Hence, $M = 0$.

□

B.0.4 Orthogonality relations

Let $D^{(\alpha)}(R)_{ij}$ be the (i,j) matrix elements of the α^{th} irreducible unitary representation of the element R of G. The range of R will be the h elements of G. The range of α will be the number of inequivalent irreducible representations. The range of i and j will be the dimension d_α of the α^{th} representation.

Theorem:

$$\sum_R D^{(\alpha)}(R)^*_{ij} D^{(\beta)}(R)_{i'j'} = \frac{h}{d_\alpha}\delta_{\alpha\beta}\delta_{ii'}\delta_{jj'} \tag{B.30}$$

Proof:

Let $D^{(1)}(R)$ not be equivalent to $D^{(2)}(R)$. We define the matrix M,

$$M = \sum_R D^{(2)}(R) X D^{(1)}(R^{-1}). \tag{B.31}$$

Then

$$\begin{aligned} D^{(2)}(S)M &= \sum_R D^{(2)}(S)\left[D^{(2)}(R)XD^{(1)}(R^{-1})\right]D^{(1)}(S^{-1})D^{(1)}(S) \\ &= \sum_R \left[D^{(2)}(S)D^{(2)}(R)\right]X\left[D^{(1)}(R^{-1})D^{(1)}(S^{-1})\right]D^{(1)}(S) \end{aligned} \tag{B.32}$$

If $SR = T$, then

$$D^{(\alpha)}(S)D^{(\alpha)}(R) = D^{(\alpha)}(SR) = D^{(\alpha)}(T)$$

and

$$D^{(\alpha)}(R^{-1})D^{(\alpha)}(S^{-1}) = D^{(\alpha)}(R^{-1}S^{-1}) = D^{(\alpha)}(T^{-1})$$

So by the rearrangment theorem, (B.32) becomes

$$\begin{aligned} D^{(2)}(S)M &= \sum_T D^{(2)}(T)XD^{(1)}(T^{-1})D^{(1)}(S) \\ &= MD^{(1)}(S) \end{aligned} \tag{B.33}$$

for all S.

Hence by Schur's second lemma, $M = 0$. Now choose X to be a matrix every element of which is zero except one, which we shall call $X_{j'i}$, and $X_{j'i} = 1$.

$$M = 0 = \sum_R D^{(2)}(R)XD^{(1)}(R^{-1}) \tag{B.34}$$

or

$$\begin{aligned} 0 &= \sum_{kl}\sum_R D^{(2)}(R)_{i'l}X_{kl}D^{(1)}(R^{-1})_{lj} \\ &= \sum_R D^{(2)}(R)_{i'j'}D^{(1)}(R^{-1})_{ij} \\ &= \sum_R D^{(1)}(R)^*_{ji}D^{(2)}(R^{-1})_{i'j'} \end{aligned} \tag{B.35}$$

The last step follows from the unitarity of the representation.

$$D^{(\alpha)}(R^{-1})_{ij} = [D^{(\alpha)}(R^{-1})]^{-1}_{ij} = D^{(\alpha)}(R)^*_{ji} \tag{B.36}$$

This proves the theorem for $\alpha \neq \beta$

$$\sum_i D^{(\alpha)}_{ii}(R)\sum_j D^{(\beta)}_{jj}(R)^* = g\delta_{\alpha\beta} \tag{B.37}$$

case(b): $\alpha = \beta = 1$

We define $M = \sum_R D^{(1)}(R)D^{(1)}(R^{-1})$. Then just as in case (a),

$$MD^{(1)}(S) = D^{(1)}(S)M \tag{B.38}$$

for all S. Hence $M = cI$. Let $X_{jj'} = 1$ for a particular j and j' and choose all other elements of X equal to zero.

$$M = cI \sum_R D^{(1)}(R)XD^{(1)}(R^{-1})$$

or in component form

$$c\delta_{ii'} = \sum_R D^{(1)}(R)_{ij} D^{(1)}(R^{-1})_{j'i'}. \tag{B.39}$$

Let $j = j'$ and sum over j:

$$c\sum_{i=1}^{d_1} 1 = \sum_R \sum_i D^{(1)}(R^{-1})_{j'i} D^{(1)}(R)_{ij}. \tag{B.40}$$

$$\begin{aligned} cd_1 &= \sum_R \sum_i D^{(1)}(R^{-1}R)_{jj'} \\ &= \sum_R \sum_i D^{(1)}(I)_{jj'} \\ &= \sum_R \delta_{j'j} = h\delta_{j'j} \end{aligned} \tag{B.41}$$

Therefore

$$c = \frac{h}{d_1}\delta_{jj'}. \tag{B.42}$$

We replace c in (B.39) by this value and rewrite $D^{(1)}(R^{-1})_{j'i'}$ as $D^{(1)}(R)^*_{i'j'}$ to get

$$\sum_R D^{(1)}(R)^*_{i'j'} D^{(1)}(R)_{ij} = \frac{h}{d_1}\delta_{ii'}\delta_{jj'}. \tag{B.43}$$

This establishes the theorem for general $\alpha = \beta$.

□

B.0.5 The Characters of a Representation

The characters are invariant under similarity transformations, all elements of one class have the same character.

First Orthogonal Relation

If α and β are irreducible and N_k is the number of elements in c_k, then

$$\sum_k \chi^{(\alpha)}(c_k)^* \chi^{(\beta)}(c_k) N_k = h\delta_{\alpha\beta} \tag{B.44}$$

Proof:

Starting with the orthogonality relatioins (B.30), theorem and setting $i = j$ and $i' = j'$ summing over i and i', we get

$$\sum_R \chi^{(\alpha)*}(R)\chi^{(\beta)}(R) = \frac{h}{d_\alpha}\delta_{\alpha\beta} \underbrace{\sum_{i,i'} \delta_{ii'}\delta_{ii'}}_{=d_\alpha}$$

or

$$\sum_R \chi^{(\alpha)}(c_k)^* \chi^{(\beta)}(c_k) N_k = h\delta_{\alpha\beta}. \tag{B.45}$$

$\square$

The Second Orthogonality Relation If $D^{(\alpha)}$ is irreducible, then

$$\sum_\alpha \chi^{(\alpha)}(c_k)^* \chi^{(\alpha)}(c_l) = \frac{h}{N_l}\delta_{kl} \tag{B.46}$$

Proof:

Let

$$\mathbf{y}^{(\alpha)} = h^{-1/2} \begin{pmatrix} \chi^{(\alpha)}(C_1) \\ \vdots \\ \chi^{(\alpha)}(C_{K(G)}) \end{pmatrix} \tag{B.47}$$

$$\mathbf{Y} = \begin{pmatrix} \chi^1(C_1)|C_1|^{1/2} & \cdots & \chi^{K(G)}(C_1)|C_1|^{1/2} \\ \vdots & \ddots & \vdots \\ \chi^1(C_{K(G)})|C_{K(G)}|^{1/2} & \cdots & \chi^{K(G)}(C_{K(G)})|C_{K(G)}|^{1/2} \end{pmatrix} \tag{B.48}$$

These relations can be summarised by the matrix equation

$$\mathbf{Y}^\dagger \mathbf{Y} = \mathbb{1} \tag{B.49}$$

Taking the determinant of $\mathbf{Y}^\dagger \mathbf{Y} = \mathbb{1}$ gives

$$|\det \mathbf{Y}|^2 = 1 \tag{B.50}$$

or

$$\det \mathbf{Y} = e^{i\alpha} \tag{B.51}$$

And so the matrix $\mathbf{Y}$ has an inverse, $\mathbf{Y}^{-1}$. Hence we have from (B.49) that $\mathbf{Y}\mathbf{Y}^\dagger = \mathbb{1}$. Written out explicitly this reads

$$\begin{aligned} \mathbf{Y}\mathbf{Y}^\dagger &= \begin{pmatrix} \chi^1(C_1)|C_1|^{1/2} & \cdots & \chi^{K(G)}(C_1)|C_1|^{1/2} \\ \vdots & \ddots & \vdots \\ \chi^1(C_{K(G)})|C_{K(G)}|^{1/2} & \cdots & \chi^{K(G)}(C_{K(G)})|C_{K(G)}|^{1/2} \end{pmatrix} \times \\ &\quad \begin{pmatrix} \chi^{1*}(C_1)|C_1|^{1/2} & \cdots & \chi^{K(G)*}(C_{K(G)})|C_{K(G)}|^{1/2} \\ \vdots & \ddots & \vdots \\ \chi^{1*}(C_1)|C_1|^{1/2} & \cdots & \chi^{K(G)*}(C_{K(G)})|C_{K(G)}|^{1/2} \end{pmatrix} \\ &= \mathbb{1} \end{aligned} \tag{B.52}$$

implying the charcter orthogonality relations

$$h^{-1} \sum_{\alpha=1}^{K(G)} \chi^{(\alpha)*}(C_m)\chi^{(\alpha)}(C_n)[|C_m||C_n|]^{1/2} = \delta_{mn} \tag{B.53}$$

or

$$\sum_{\alpha=1}^{K(G)} \chi^{(\alpha)*}(C_m)\chi^{(\alpha)}(C_n) = \frac{h}{[|C_m||C_n|]^{1/2}}\delta_{mn}$$

which easily becomes

$$\sum_{\alpha=1}^{K(G)} \chi^{(\alpha)*}(C_m)\chi^{(\alpha)}(C_n) = \frac{h}{|C_n|}\delta_{mn} \tag{B.54}$$

which is the desired result.

□

Appendix C

$SU(2)$ Holonomies

The transformation:

$$\tau \cdot \vec{A} \rightarrow \tau \cdot \vec{A}' = \exp\left(\frac{i}{2}\vec{a} \cdot \tau\right)(\tau \cdot \vec{A}) \exp\left(-\frac{i}{2}\vec{a} \cdot \tau\right) \tag{C.1}$$

effects a rotation of the field $\vec{A}_\alpha$ around the axis $\vec{n} = \vec{a}/|\vec{a}|$ by an angle $|\vec{a}|$.

Proof:

First calculate

$$\begin{aligned} (\tau \cdot \vec{a})^2 &= \sum_{i,j=1}^{3} a_i a_j \tau_i \tau_j \\ &= \frac{1}{2} \sum_{i,j=1}^{3} a_i a_j (\tau_i \tau_j + \tau_j \tau_i) \\ &= \sum_{i,j=1}^{3} a_i a_j \delta_{ij} = |\vec{a}|. \end{aligned} \tag{C.2}$$

since the Pauli matrices obey the relation

$$\tau_i \tau_j + \tau_j \tau_i = 2\delta_{ij}.$$

We can now calculate the Taylor series expansion of the exponential:

$$
\begin{aligned}
\exp\left(\pm\frac{i}{2}\vec{a}\cdot\tau\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\pm\frac{i}{2}\vec{a}\cdot\tau\right) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!2^{2n}}(\vec{a}\cdot\tau)^{2n} + \pm i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!2^{2n+1}}(\vec{a}\cdot\tau)^{2n+1} \\
&= \cos\frac{|\vec{a}|}{2} \pm i\frac{\vec{a}\cdot\tau}{|\vec{a}|}\sin\frac{|\vec{a}|}{2} \\
&= \cos\frac{|\vec{a}|}{2} \pm i\vec{n}\cdot\tau\sin\frac{|\vec{a}|}{2}
\end{aligned} \tag{C.3}
$$

Will we need the relation

$$
(\vec{a}\cdot\tau)(\vec{b}\cdot\tau) = \vec{a}\cdot\vec{b} + i\tau\cdot(\vec{a}\times\vec{b}), \tag{C.4}
$$

which can be easily derived from the commutation relation

$$
\tau_i\tau_j - \tau_j\tau_i = 2i\sum_{k=1}^{3}\epsilon_{ijk}\tau_k. \tag{C.5}
$$

You simply rewrite it as

$$
\begin{aligned}
2\tau_i\tau_j &= \tau_j\tau_i + \tau_i\tau_j + 2i\sum_{k=1}^{3}\epsilon_{ijk}\tau_k \\
&= 2\delta_{ij} + 2i\sum_{k=1}^{3}\epsilon_{kij}\tau_k
\end{aligned}
$$

and contract with $a_i b_j$. We use the above relation to obtain ($\theta \equiv |\vec{a}|/2$)

$$
\begin{aligned}
&\tau\cdot\vec{A}' = \\
= \;& [\cos\theta + i(\vec{n}\cdot\tau)\sin\theta](\tau\cdot\vec{A})[\cos\theta - i(\vec{n}\cdot\tau)\sin\theta] \\
= \;& (\tau\cdot\vec{A})\cos^2\theta + i\sin\theta\cos\theta[(\vec{n}\cdot\tau)(\tau\cdot\vec{A}) - (\tau\cdot\vec{A})(\vec{n}\cdot\tau)] + \sin^2\theta(\vec{n}\cdot\tau)(\tau\cdot\vec{A})(\vec{n}\cdot\tau) \\
= \;& (\tau\cdot\vec{A})\cos^2\theta - 2\sin\theta\cos\theta\tau\cdot(\vec{n}\times\vec{A}) + \sin^2\theta[(\vec{n}\cdot\vec{A}) + i\tau\cdot(\vec{n}\times\vec{A})](\vec{n}\cdot\tau) \\
= \;& (\tau\cdot\vec{A})\cos^2\theta - \sin 2\theta[\tau\cdot(\vec{n}\times\vec{A})] + \sin^2\theta\left\{[(\vec{n}\cdot\vec{A})(\vec{n}\cdot\tau) - \tau\cdot[(\vec{n}\times\vec{A})\times\vec{n}]\right\} \\
= \;& \tau\cdot\left[\vec{A}\cos^2\theta - (\vec{n}\times\vec{A})\sin 2\theta + \sin^2\theta\left\{\vec{n}(\vec{n}\cdot\vec{A}) - (\vec{n}\times\vec{A})\times\vec{n}\right\}\right] \\
= \;& \tau\cdot\left[\left\{\vec{n}(\vec{n}\cdot\vec{A}) - \vec{n}\times(\vec{n}\times\vec{A})\right\}\cos^2\theta - (\vec{n}\times\vec{A})\sin 2\theta + \sin^2\theta\left\{\vec{n}(\vec{n}\cdot\vec{A}) + \vec{n}\times(\vec{n}\times\vec{A})\right\}\right] \\
= \;& \tau\cdot\left[\vec{n}(\vec{n}\cdot\vec{A}) - \vec{n}\times(\vec{n}\times\vec{A})\cos 2\theta - (\vec{n}\times\vec{A})\sin 2\theta\right] \qquad \text{(C.6)}
\end{aligned}
$$

where we have used

$$\vec{n}\times(\vec{n}\times\vec{A}) = \vec{n}(\vec{n}\cdot\vec{A}) - \vec{A} \qquad \text{(C.7)}$$

which is easily derived by contracting the identity

$$\sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{i'j'k} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}.$$

with $n_j n_{i'} A_{j'}$.

We have obtained the formula

$$\vec{A}' = \vec{n}(\vec{n}\cdot\vec{A}) - \vec{n}\times(\vec{n}\times\vec{a})\cos 2\theta - (\vec{n}\times\vec{A})\sin 2\theta. \qquad \text{(C.8)}$$

The expression for $\vec{A}'$ on the RHS is just the vector $\vec{A}$ rotated by an angle $|\vec{a}| \equiv 2\theta$ around the $\vec{n}$ axis. For example for $\vec{n} = \vec{e}_z$ we get

$$
\begin{aligned}
A'_x &= A_x\cos 2\theta + A_y\sin 2\theta \\
A'_y &= -A_x\sin 2\theta + A_y\cos 2\theta \\
A'_z &= A_z. \qquad \text{(C.9)}
\end{aligned}
$$

Appendix D

Harmonic analysis

D.1 Connection between theory of finite groups and theory of compact groups

The volume element constructed in section 1.2.7 had the property that it is invariant. What this means that if a subset Ω of the group manifold with volume V_Ω, then the image set $g\Omega$ under left multiplication has exactly the same volume.

If a group manifold is a compact set, both the left- and right-invariant volume elements coincide. The resulting measure on the group is the so-called Haar measure.

For a compact group, we can replace the sums over group elements that we in the representation theory of finite groups (the previous appendix) using the invariant Haar measure. the invariance property is expressed by

$$d[g_1 g] = d[g]$$

for any constant element g_1. This allows us to make a change of variables transformation

$$g \to g_1 g$$

identical to that which played such an important role in deriving the finite group theorems - see...

Consequently we will be able to derive all the results from finite group, such as the existence of an invariant inner product and the othogonality theorems for compact groups. As we shall see we just need to take over the results by simple replacement of sums by integration.

D.2 Group integration analogies

We list some useful analogies:

$$\int dx\delta(x)f(x) = f(0) \iff \int_G dU\delta(U)f(U) = f(I) \tag{D.1}$$

$$\int dy\delta(y-a)f(y) = f(a) \iff \int_G dU\delta(U\gamma^{-1})f(U) = f(\gamma) \tag{D.2}$$

$$f(x) = \sum_n a_n e_n(x) \iff f(g) = \sum_j a_j\chi_j(g) \quad (f(g) = f(\gamma g\gamma^{-1}))$$

$$a_n = \int \overline{e_n(x)}f(x)dx \iff a_j = \int_G \overline{\chi_j(g)}f(g)dg \tag{D.3}$$

$$\delta(x-y) = \sum_n \overline{e_n(x)}e_n(y) \iff \delta(gx^{-1}) = \sum_j \overline{\chi_j(g)}\chi_j(x) \tag{D.4}$$

$$\delta(a+x-a) \iff \delta(xgx^{-1}) = \delta(g) \tag{D.5}$$

$$\int dx\overline{e_m(x)}e_n(x) = \delta_{mn} \iff \int D^j_{mn}(g)\overline{D^{j'}_{m'n'}(g)}dg = \frac{1}{d_j}\delta^{jj'}\delta_{mm'}\delta_{nn'} \tag{D.6}$$

$$\int dy\overline{e_m(y)}e_n(x-y) = \delta_{mn}e_n(x) \iff \int_G \overline{\chi_{j'}(g)}\chi_j(gx)dg = V_G\delta_{jj'}\frac{\chi_j(x)}{d_j} \tag{D.7}$$

Appendix E

Intertwiners

E.1 Introduction

E.1.1 The Angular Momentum Operator

In quantum mechanics, (any) Hermitian vector operator $\vec{J}$ whose components satisfy the commutation relation

$$[J_i, J_j] = i\epsilon_{ij}{}^k J_k \tag{E.1}$$

is called the angular momentum operator. All components of $\vec{J}$ commute with the squared angular momentum

$$J^2 = J_x^2 + J_y^2 + J_z^2. \tag{E.2}$$

Therefore one can simultaneously diagonalise J^2 and one of the components, conventionally chosen as J_z. Let us write the eigenvalue equations

$$J^2|\lambda, \mu\rangle \quad = \quad \lambda|\lambda, \mu\rangle, \tag{E.3}$$

$$J_z|\lambda, \mu\rangle \quad = \quad \mu|\lambda, \mu\rangle. \tag{E.4}$$

To derive the solution of the eigenvalue problem, it is appropriate to use the following raising and lowering operators

$$J_\pm = J_x \pm J_y. \tag{E.5}$$

There commutators with J^2 and J_z are given by

$$[J^2, J_\pm] = 0, \qquad [J_z, J_\pm] = \pm J_\pm. \tag{E.6}$$

These realtions imply that the raising and lowering operatorsraise and lower the eigenvalue of J_z by one, while leaving the eigenvalue of J^2 unchanged:

$$\begin{aligned} J^2(J_\pm|\lambda,\mu\rangle) &= \lambda J_\pm|\lambda,\mu\rangle, \\ J_z(J_\pm|\lambda,\mu\rangle &= (\mu \pm 1)J_\pm|\lambda,\mu\rangle. \end{aligned} \tag{E.7}$$

This means that the states $J_\pm|\lambda,\mu\rangle$ must be proportional to $|\lambda,\mu\pm\rangle$:

$$J_\pm|\lambda,\mu\rangle = C_\pm(\lambda,\mu)|\lambda,\mu\pm\rangle. \tag{E.8}$$

Thus if any one of the eigenstates $|\lambda,\mu\rangle$ is known, then all the eigenstates for that of λ can be derived by repeatedly applying J_+ and J_-.

On the other hand, the inequality

$$\langle\lambda,\mu|J^2 - J_z^2|\lambda,\mu\pm\rangle = \langle\lambda,\mu|J_x^2 + J_y^2|\lambda,\mu\pm\rangle \geq 0 \tag{E.9}$$

implies that the eigenvalue μ is restrcited by

$$\mu^2 \leq \lambda. \tag{E.10}$$

This condition is satisfied only if there is a maximal eigenvalue μ_{max}, such that the action of the raising operator on the state $|\lambda,\mu_{max}\rangle$ doesnot give a new eigenvalue, but instead

$$J_+|\lambda,\mu_{max}\rangle = 0. \tag{E.11}$$

Similarly, there must exist a minimal eigenvalue μ_{min}, such that

$$J_-|\lambda,\mu_{min}\rangle = 0. \tag{E.12}$$

Using the identities

$$\begin{aligned} J_-J_+ &= J^2 - J_z^2 - J_z, \\ J_+J_- &= J^2 - J_z^2 + J_z. \end{aligned} \tag{E.13}$$

in (E.11) and (E.12), one finds the relation

$$\lambda = \mu_{max}(\mu_{max} + 1) = -\mu_{min}(-\mu_{min} + 1), \tag{E.14}$$

showing that μ_{max} and μ_{min} are related to each other by

$$\mu_{max} = -\mu_{min}. \tag{E.15}$$

Consider now acting repeatedly with J_- on the state $|\lambda, \mu_{max}\rangle$. After a certain numbe of actions, say N, one must arrive at a state $|\lambda, \mu_{min}\rangle$. Since the eigenvalue μ is lowered by one at each step, the difference $\mu_{max} - \mu_{min}$ must be equal to N, giving

$$\mu_{max} = \frac{N}{2}. \tag{E.16}$$

The eigenvalue λ is then

$$\lambda = \mu_{max}(\mu_{max} + 1) = \frac{N}{2}\left(\frac{N}{2} + 1\right). \tag{E.17}$$

The possible eigenvalues of the operators J^2 and J_z have therefore been determined. The eigenvalue equations (E.3) and (E.4) can be written as

$$\begin{aligned} J^2|jm\rangle &= j(j+1)|jm\rangle \\ J_z|jm\rangle &= m|jm\rangle, \end{aligned} \tag{E.18}$$

where j may be any integer or half integer, and m ranges from $-j$ to j in steps of 1.

To complete the solution of the eigenvalue problem, it remains to find the coefficients $C_\pm(\lambda, \mu) \equiv C_\pm(j, m)$ in (E.8), since this will show how the angular momentumoperator acts on the eigenstates $|jm\rangle$. By multiplying the equation with its adjoint, we find

$$\begin{aligned} |C_\pm(j,m)|^2 &= \langle J_\mp J_\pm|jm\rangle \\ &= \langle jm|J^2 - J_z^2 \mp J_z|jm\rangle \\ &= j(j+1) - m(m \pm 1), \end{aligned} \tag{E.19}$$

which determines $C_\pm(j,m)$ up to a phase. The usual convention is to choose the phase such that $C_\pm(j,m)$ is real and positive. This implies

$$J_\pm|jm\rangle = \sqrt{j(j+1) - m(m \pm 1)}|jm \pm 1\rangle. \tag{E.20}$$

E.1.2 Spin-j Representation

The states $|jm\rangle$ with fixed value of j span the $(2j+1)$-dimensional vector space $\mathcal{H}_j$. A general element of $\mathcal{H}_j$ has the form

$$|v\rangle = \sum_{m=-j}^{j} v^m |jm\rangle \tag{E.21}$$

the index taking values $-j, -j+1, \ldots, j$. The defintion

$$\langle u|v\rangle = \sum_{m=-j}^{j} \overline{u}^m v^m \tag{E.22}$$

for a scalar product between elements of $\mathcal{H}_j$ promotes $\mathcal{H}_j$ to a Hilbert space.

The relevance of the space $\mathcal{H}_j$ to $SU(2)$ representation theory is that the matrices representing the operator $g = e^{-i\alpha\vec{n}\cdot\vec{J}}$ on $\mathcal{H}_j$ define an irreducible representation of $SU(2)$. The matrices whose elements are given by

$$D^{(j)m}{}_n(g) := \langle jm|e^{-i\alpha\vec{n}\cdot\vec{J}}|jn\rangle \tag{E.23}$$

are known as the Wigner matrices. The great orthogonality theroem of group theory implies that the Wigner matrices satisfy

$$\int dg \overline{D^{(j)m}{}_n(g)} D^{(j')m'}{}_{n'}(g) = \frac{1}{d_j}\delta_{jj'}\delta^{m'}_m\delta^{n'}_n, \tag{E.24}$$

where dg is the normalised Haar measure of $SU(2)$, and

$$d_j = 2j+1$$

denotes the dimension of $\mathcal{H}_j$.

E.2 $\mathcal{H}_j \otimes \mathcal{H}_j$

E.2.1 Tensor Product Space and Angular Momentum Operators

A general state of $\mathcal{H}_j \otimes \mathcal{H}_j$ has the form

$$|\Psi\rangle = \sum_{mn=-j}^{j} a_{mn}|jm\rangle \otimes |jn\rangle \tag{E.25}$$

We define angular momentum operators on the product space as follows:

$$\begin{aligned}
\mathbf{J}_x^{(1)} &= J_x^{(1)} \otimes \mathbb{1}^{(2)}, \quad \mathbf{J}_y^{(1)} = J_y^{(1)} \otimes \mathbb{1}^{(2)}, \quad \mathbf{J}_z^{(1)} = J_z^{(1)} \otimes \mathbb{1}^{(2)}, \\
\mathbf{J}_x^{(2)} &= \mathbb{1}^{(1)} \otimes J_x^{(2)}, \quad \mathbf{J}_y^{(2)} = \mathbb{1}^{(1)} \otimes J_y^{(2)}, \quad \mathbf{J}_z^{(2)} = \mathbb{1}^{(1)} \otimes J_z^{(2)}
\end{aligned} \tag{E.26}$$

where each angular momentum operators acts upon the corresponding factor of the tensor product.

The total angular momentum operators satisfy the very same commutation realtions,

$$[\mathbf{J}_i^{(1)}, \mathbf{J}_j^{(1)}] = i\epsilon_{ij}{}^k \mathbf{J}_k^{(1)}, \qquad [\mathbf{J}_i^{(2)}, \mathbf{J}_j^{(2)}] = i\epsilon_{ij}{}^k \mathbf{J}_k^{(2)}, \qquad [\mathbf{J}_i^{(1)}, \mathbf{J}_j^{(2)}] = 0. \tag{E.27}$$

We have the commutator

$$\begin{aligned}
[\vec{n}\cdot\vec{\mathbf{J}}^{(1)}, \vec{n}\cdot\vec{\mathbf{J}}^{(2)}] &= n_i n_j [J_i^{(1)} \otimes \mathbb{1}^{(2)}, \mathbb{1}^{(1)} \otimes J_j^{(2)}] \\
&= n_i n_j (J_i^{(1)} \otimes J_j^{(2)} - J_i^{(1)} \otimes J_j^{(2)}) \\
&= 0.
\end{aligned} \tag{E.28}$$

A set of coupled eigenstates for the total angular momentum operator,

$$\begin{aligned}
\mathbf{J}_z^{(1)}|jm\rangle &= m|jm\rangle \\
\left(\mathbf{J}^{(1)}\right)^2|jm\rangle &= j(j+1)|jm\rangle
\end{aligned} \tag{E.29}$$

Consider the total angular momentum operator squared,

$$\begin{aligned}
\left(\mathbf{J}^{(1)} + \mathbf{J}^{(2)}\right)^2 &= \left(J^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J^{(2)}\right)^2 \\
&= \left(J^{(1)}\right)^2 \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes \left(J^{(2)}\right)^2 + 2\left(J^{(1)} \otimes \mathbb{1}^{(2)}\right)\cdot\left(\mathbb{1}^{(1)} \otimes J^{(2)}\right).
\end{aligned} \tag{E.30}$$

Consider

$$
\begin{aligned}
\mathbf{J}_-\mathbf{J}_+ &= \left(J^{(1)}\otimes\mathbb{1}^{(2)}+\mathbb{1}^{(1)}\otimes J^{(2)}\right)_-\left(J^{(1)}\otimes\mathbb{1}^{(2)}+\mathbb{1}^{(1)}\otimes J^{(2)}\right)_+ \\
&= \left\{\left(J^{(1)}\otimes J^{(2)}\right)_x - i\left(J^{(1)}\otimes J^{(2)}\right)_y\right\}\left\{\left(J^{(1)}\otimes J^{(2)}\right)_x + i\left(J^{(1)}\otimes J^{(2)}\right)_y\right\} \\
&= \left(J_x^{(1)}\otimes J_x^{(2)}\right)^2 + \left(J_y^{(1)}\otimes J_y^{(2)}\right)^2 + i\left[\left(J^{(1)}\otimes J^{(2)}\right)_x, \left(J^{(1)}\otimes J^{(2)}\right)_y\right] \\
&= \left(J^{(1)}\right)^2\otimes\mathbb{1}^{(2)} + \mathbb{1}^{(1)}\otimes\left(J^{(2)}\right)^2 - \left(J_z^{(1)}\otimes J_z^{(2)}\right)^2 - J_z^{(1)}\otimes J_z^{(2)} \qquad \text{(E.31)}
\end{aligned}
$$

Now

$$
\begin{aligned}
2\left(J^{(1)}\otimes\mathbb{1}^{(2)}\right)\cdot\left(\mathbb{1}^{(1)}\otimes J^{(2)}\right) &= 2\left(J^{(1)}\otimes J^{(2)}\right)_x + 2\left(J^{(1)}\otimes J^{(2)}\right)_y + 2\left(J^{(1)}\otimes J^{(2)}\right)_z \\
&= 2\left(J^{(1)}\otimes J^{(2)}\right)_z + \left(\left(J^{(1)}\otimes J^{(2)}\right)_+ + \left(J^{(1)}\otimes J^{(2)}\right)_-\right) \\
&\qquad + i\left(\left(J^{(1)}\otimes J^{(2)}\right)_- - \left(J^{(1)}\otimes J^{(2)}\right)_+\right) \\
&= 2\left(J_z^{(1)}\otimes J_z^{(2)}\right) + (1-i)\mathbf{J}_+ + (1+i)\mathbf{J}_-. \qquad \text{(E.32)}
\end{aligned}
$$

Substituting (E.31) and (E.32) into (E.30) gives

$$
\left(\mathbf{J}^{(1)}+\mathbf{J}^{(2)}\right)^2 = \mathbf{J}_-\mathbf{J}_+ + \left(J_z^{(1)}\otimes J_z^{(2)}\right)^2 + 3\left(J_z^{(1)}\otimes J_z^{(2)}\right) + (1-i)\mathbf{J}_+ + (1+i)\mathbf{J}_- \qquad \text{(E.33)}
$$

E.2.2 State of Zero Angular Momentum

We consider the problem of constructing a state of zero angular momentum on the tensor product space $\mathcal{H}_j\otimes\mathcal{H}_j$. Suppose that the state

$$
|\Psi_0\rangle = \sum_{mn=-j}^{j} c_{mn}|jm\rangle\otimes|jn\rangle \qquad \text{(E.34)}
$$

satisfies

$$
\begin{aligned}
\left(\mathbf{J}^{(1)}+\mathbf{J}^{(2)}\right)^2|\Psi_0\rangle &= 0 \\
\left(\mathbf{J}_z^{(1)}+\mathbf{J}_z^{(2)}\right)|\Psi_0\rangle &= 0. \qquad \text{(E.35)}
\end{aligned}
$$

This is equivalent to the state being invariant under rotations generated by the total angular momentum $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$, which in turn implies that the tensor defined by the coefficients c_{mn} is invariant under the action of $SU(2)$ by the matrices $D^{(j)m}{}_{n}(g)$. To see this note that

$$e^{+i\alpha\vec{n}\cdot\vec{\mathbf{J}}}|\Psi_0\rangle = |\Psi_0\rangle \tag{E.36}$$

or

$$\langle\Psi_0|e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}} = \langle\Psi_0|. \tag{E.37}$$

Now as $[\vec{n}\cdot\vec{\mathbf{J}}^{(1)}, \vec{n}\cdot\vec{\mathbf{J}}^{(2)}] = 0$ (see (E.28)) we have $e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}}$ factorises as $e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(1)}} \otimes e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(2)}}$. Inserting this into (E.37),

$$\sum_{mm'=-j}^{j} \overline{c_{mm'}}\langle jm| \otimes \langle jm'| \left(e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(1)}} \otimes e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(2)}}\right) = \sum_{ll'=-j}^{j} \overline{c_{ll'}}\langle jl| \otimes \langle jl'|.$$

As we will see $c_{mm'}$ can be choosen to be real and so we will drop the compex conjugation. Now contract on the right by $|jn\rangle \otimes |jn'\rangle$,

$$\sum_{mm'=-j}^{j} c_{mm'}\langle jm| \otimes \langle jm'| \left(e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(1)}} \otimes e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(2)}}\right) |jn\rangle \otimes |jn'\rangle$$
$$= \sum_{ll'=-j}^{j} c_{ll'}\left(\left(\langle jl| \otimes \langle jl'\right)\left(|jn\rangle \otimes |jn'\rangle\right)\right).$$

which becomes

$$\sum_{mm'=-j}^{j} c_{mm'}\langle jm|e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(1)}}|jn\rangle\langle jm'|e^{-i\alpha\vec{n}\cdot\vec{\mathbf{J}}^{(2)}}|jn'\rangle = c_{nn'}$$

or

$$\sum_{mm'=-j}^{j} c_{mm'}D^{(j)m}{}_{n}(g)D^{(j)m'}{}_{n'}(g) = c_{nn'}. \tag{E.38}$$

Now let us return to finding the form of the zero angular momentum state. The eigenvalue equation for the z-component requires that the coefficient $c_{mn} = 0$ unless $m = -n$:

$$\begin{aligned}\left(\mathbf{J}_z^{(1)}+\mathbf{J}_z^{(2)}\right)|\Psi_0\rangle &= \left(J_z^{(1)}\otimes \mathbb{1}^{(2)}+\mathbb{1}^{(1)}\otimes J_z^{(2)}\right)\sum_{mn=-j}^{j} c_{mn}|jm\rangle\otimes|jn\rangle \\ &= \sum_{mn=-j}^{j} c_{mn}(m+n)|jm\rangle\otimes|jn\rangle = 0.\end{aligned}$$

We have that the state be written

$$|\Psi_0\rangle = \sum_{m=-j}^{j} c_m|jm\rangle\otimes|j-m\rangle. \tag{E.39}$$

The requirement that this state has zero total angular momentum reduces to the requirement that it be annihilated by either $\mathbf{J}_+$ or $\mathbf{J}_-$. We write

$$\begin{aligned}\mathbf{J}_+|\Psi_0\rangle &= \left(\mathbf{J}_+^{(1)}+\mathbf{J}_+^{(2)}\right)|\Psi_0\rangle \\ &= \left(J_+^{(1)}\otimes \mathbb{1}^{(2)}+\mathbb{1}^{(1)}\otimes J_+^{(2)}\right)\sum_{m=-j}^{j} c_m|jm\rangle\otimes|j-m\rangle \\ &= \sum_{m=-j}^{j} c_m\Big(\sqrt{j(j+1)-m(m+1)}|jm+1\rangle\otimes|j-m\rangle \\ &\quad +\sqrt{j(j+1)-m(m-1)}|jm\rangle\otimes|j-m-1\rangle\Big) \\ &= \sum_{m=-j}^{j-1} c_m\sqrt{j(j+1)-m(m+1)}|jm+1\rangle\otimes|j-m\rangle \\ &+ \sum_{m=-j}^{j-1} c_{m+1}\sqrt{j(j+1)-m(m+1)}|jm+1\rangle\otimes|j-m\rangle = 0\end{aligned} \tag{E.40}$$

(where we have used $|j,j+1\rangle = |j,-j-1\rangle = 0$). this is satisfied if we impose the condition $c_{m+1} = -c_m$. We have altogether that c_{mn} be proportional to $(-1)^{j-m}\delta_{m,-n}$. Defining the tensor $\epsilon^{(j)}_{mn}$ so that $\epsilon^{(j)}_{j,-j} = +1$, we therefore have

$$\epsilon^{(j)}_{mn} = (-1)^{j-m}\delta_{m,-n}, \qquad \epsilon^{(j)mn} = (-1)^{j-m}\delta_{m.-n}. \tag{E.41}$$

We see that $\epsilon^{(j)}_{mn}$ satisfies

$$
\begin{aligned}
\epsilon^{(j)}_{nm} &= (-1)^{j-n}\delta_{n,-m} \\
&= (-1)^{j+m}\delta_{m,-n} \\
&= (-1)^{j+m}(-1)^{j-m}(-1)^{j-m}\delta_{m,-n} \\
&= (-1)^{2j}\epsilon^{(j)}_{mn}
\end{aligned} \tag{E.42}
$$

and

$$
\begin{aligned}
\sum_{m'=-j}^{j} \epsilon^{(j)}_{nm'}\epsilon^{(j)m'n} &= \sum_{m'=-j}^{j}(-1)^{2j}(-1)^{-m-m'}\delta_{m,-m'}\delta_{m',-n} \\
&= (-1)^{2j}\sum_{m'=-j}^{j}\delta_{-m,m'}\delta_{m',-n} \\
&= (-1)^{2j}\delta^{n}_{m}.
\end{aligned} \tag{E.43}
$$

The remaining properties of the epsilon tensor are analogous to those of the fundamental representation. For the matrix elements of the inverse matrix, there holds the relation

$$
D^{(j)m}{}_{n}(g^{-1}) = \epsilon^{(j)mm'}\epsilon^{(j)}_{nn'}D^{(j)n'}{}_{m'}(g). \tag{E.44}
$$

This can be derived from (E.38), which reads

$$
\epsilon^{(j)}_{ln'}D^{(j)l}{}_{m}(g)D^{(j)n'}{}_{m'}(g) = \epsilon^{(j)}_{mm'}
$$

Contracting this with $D^{(j)m}{}_{n}(g^{-1})$ gives

$$
D^{(j)m}{}_{n}(g^{-1})D^{(j)l}{}_{m}(g)\epsilon^{(j)}_{ln'}D^{(j)n'}{}_{m'}(g) = D^{(j)m}{}_{n}(g^{-1})\epsilon^{(j)}_{mm'}
$$

or

$$
\epsilon^{(j)}_{nn'}D^{(j)n'}{}_{m'}(g) = D^{(j)l}{}_{n}(g^{-1})\epsilon^{(j)}_{lm'}
$$

and contracting this with $\epsilon^{(j)mm'}$ gives

$$\begin{aligned}
\epsilon^{(j)mm'}\epsilon^{(j)}_{nn'}D^{(j)n'}{}_{m'}(g) &= D^{(j)l}{}_{n}(g^{-1})\epsilon^{(j)mm'}\epsilon^{(j)}_{lm'} \\
&= D^{(j)l}{}_{n}(g^{-1})(-1)^{2j}\epsilon^{(j)}_{lm'}\epsilon^{(j)m'm} \\
&= D^{(j)l}{}_{n}(g^{-1})(-1)^{4j}\delta^{m}_{l} \\
&= D^{(j)m}{}_{n}(g^{-1})
\end{aligned} \tag{E.45}$$

as required.

Indices are raised and lowered according to the convention

$$v^m = \epsilon^{(j)mn}v_n, \qquad v_m = v^n\epsilon^{(j)}_{nm}. \tag{E.46}$$

An antisymmetric, $SU(2)$-invariant product on $\mathcal{H}_j$ can be defined as

$$(u|v) = \epsilon^{(j)}_{mm'}u^m v^n = \sum_{m=-j}^{j}(-1)^{j-m}u^m v^{-m}. \tag{E.47}$$

E.2.3 Relating the Space $\mathcal{H}_j$ to the Fundamental Representation

We now explain how the spin-j representation is related to the fundamental representation. We will show how the states $|jm\rangle$ can be constructed from the states $|+\rangle$ and $|-\rangle$ which spane the space $\mathcal{H}_{1/2}$, and are eigenstates of the angular momentum operators on $\mathcal{H}_{1/2}$ with eigenvalues $j = \frac{1}{2}$ and $m = \pm\frac{1}{2}$. To this end, let us consider the state

$$|\Psi_j\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle, \tag{E.48}$$

which is an element of the $2j$-fold tensor product space $\mathcal{H}_{1/2} \otimes \cdots \otimes \mathcal{H}_{1/2}$. We show that the state $|\Psi_j\rangle$ is an eigenstate of the total angular momentum operator

$$\mathbf{J}^{(tot)} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)} + \cdots + \mathbf{J}^{(2j)}, \tag{E.49}$$

where each operator $\mathbf{J}^{(i)}$ acts on the i-th factor in the tensor product $\otimes_{i=1}^{2j}\mathcal{H}_{1/2}$, i.e.,

$$\mathbf{J}^{(i)} \equiv \mathbb{1}^{(1)} \otimes \cdots \otimes \mathbb{1}^{(i-1)} \otimes J^{(i)} \otimes \mathbb{1}^{(i+1)} \otimes \cdots \mathbb{1}^{(2j)}. \tag{E.50}$$

For the z-component of $\mathbf{J}^{(tot)}$, we immediately have

$$\mathbf{J}^{(tot)}|\Psi_j\rangle = j|\Psi_j\rangle, \tag{E.51}$$

since each state $|+\rangle$ is an eigenstate of the corresponding operator J_z with eigenvalue $+\frac{1}{2}$. For the square of the total angular momentum, we have

$$\left(\mathbf{J}^{(tot)}\right)^2|\Psi_j\rangle = \left(\sum_i \left(\mathbf{J}^{(i)}\right)^2 + \sum_{i\neq k} \vec{\mathbf{J}}^{(i)}\cdot\vec{\mathbf{J}}^{(k)}\right)|\Psi_j\rangle. \tag{E.52}$$

The first sum contains $2j$ terms, and in each of them J^2 acts on the corresponding state $|+\rangle$, producing an eigenvalue $\frac{3}{4}$. For $i < k$, a cross term explicitly reads as

$$\begin{aligned}
\vec{\mathbf{J}}^{(i)}\cdot\vec{\mathbf{J}}^{(k)} &= \mathbb{1}^{(1)}\otimes\cdots J_z^{(i)}\otimes\cdots\otimes J_z^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ \mathbb{1}^{(1)}\otimes\cdots J_y^{(i)}\otimes\cdots\otimes J_y^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ \mathbb{1}^{(1)}\otimes\cdots J_z^{(i)}\otimes\cdots\otimes J_z^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)}
\end{aligned} \tag{E.53}$$

To evaluate the cross terms, we note that each of the $2j(2j-1)$ terms in the sum can be written as

$$\vec{\mathbf{J}}^{(i)}\cdot\vec{\mathbf{J}}^{(k)} = \mathbf{J}_z^{(i)}\mathbf{J}_z^{(k)} + \frac{1}{2}\left(\mathbf{J}_+^{(i)}\mathbf{J}_-^{(k)} + \mathbf{J}_-^{(i)}\mathbf{J}_+^{(k)}\right). \tag{E.54}$$

as is easily shown for $i < k$,

$$\begin{aligned}
\mathbf{J}_+^{(i)}\mathbf{J}_-^{(k)} + \mathbf{J}_-^{(i)}\mathbf{J}_+^{(k)} &= \mathbb{1}^{(1)}\otimes\cdots J_+^{(i)}\otimes\cdots\otimes J_-^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ \mathbb{1}^{(1)}\otimes\cdots J_-^{(i)}\otimes\cdots\otimes J_+^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&= \mathbb{1}^{(1)}\otimes\cdots(J_x^{(i)}+iJ_y^{(i)})\otimes\cdots\otimes(J_x^{(i)}-iJ_y^{(i)})\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ \mathbb{1}^{(1)}\otimes\cdots(J_x^{(i)}-iJ_y^{(i)})\otimes\cdots\otimes(J_x^{(i)}+iJ_y^{(i)})\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&= \mathbb{1}^{(1)}\otimes\cdots J_x^{(i)}\otimes\cdots\otimes J_x^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&- i\mathbb{1}^{(1)}\otimes\cdots J_x^{(i)}\otimes\cdots\otimes J_y^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ i\mathbb{1}^{(1)}\otimes\cdots J_y^{(i)}\otimes\cdots\otimes J_x^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ \mathbb{1}^{(1)}\otimes\cdots J_y^{(i)}\otimes\cdots\otimes J_y^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ c.c. \\
&= 2\mathbb{1}^{(1)}\otimes\cdots J_x^{(i)}\otimes\cdots\otimes J_x^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)} \\
&+ 2\mathbb{1}^{(1)}\otimes\cdots J_y^{(i)}\otimes\cdots\otimes J_y^{(k)}\otimes\cdots\otimes\mathbb{1}^{(2j)}.
\end{aligned} \tag{E.55}$$

Let us now consider the action of (E.54) on the state $|\Psi_j\rangle$. The action of the first term on the state $\mathbb{1}^{(1)} \otimes \cdots \otimes |+\rangle_{(i)} \otimes \cdots \otimes |+\rangle_{(k)} \otimes \cdots \otimes \mathbb{1}^{(2j)}$ gives the eigenvalue $m_i m_k = 1/4$. The action of the other two terms gives zero, because the state is anihilated by the raising operator J_+. Hence (E.52) reads,

$$\left(\mathbf{J}^{(tot)}\right)^2 |\Psi_j\rangle = j(j+1)|\Psi_j\rangle. \tag{E.56}$$

Together, (E.51) and (E.56) show that $|\Psi_j\rangle$ can be identified with the state $|jj\rangle$. That is,

$$|jj\rangle = \underbrace{|+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle}_{2j \text{ times}}. \tag{E.57}$$

The remaining states $|jm\rangle$ can now be constructed by repeatedly acting on (E.57) with the lowering operator

$$J_-^{(tot)} = J_-^{(1)} + J_-^{(2)} + \cdots + J_-^{(2j)}. \tag{E.58}$$

Recalling (E.20), we obtain

$$|jm\rangle = \sqrt{\frac{(j+m)!(j-m)!}{(2j)!}}\Big(\underbrace{|+\rangle \otimes \cdots \otimes |+\rangle}_{j+m \text{ terms}} \otimes \underbrace{|-\rangle \otimes \cdots \otimes |-\rangle}_{j-m \text{ terms}} + \text{all permutations}\Big). \tag{E.59}$$

The factor $\sqrt{\frac{(j+m)!(j-m)!}{(2j)!}}$ obviously normalises the state.

E.3 Intertwiners

Intertwiners, or invariant tensors of $SU(2)$, play a crucial role in loop quantum gravity.

E.3.1 Clebsch-Gordan coefficients

Consider the tensor product space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. An obvious basis on this space is provided by the tensor product states $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$, which are eigenstates of the mutually commuting operators

$$\left(J^{(1)}\right)^2, \quad \left(J^{(2)}\right)^2, \quad J_z^{(1)}, \quad J_z^{(2)}. \tag{E.60}$$

Another complete set of commuting operators on $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ is formed by the operators

$$\left(J^{(1)}\right)^2, \quad \left(J^{(2)}\right)^2, \quad \left(J^{(1)} + J^{(2)}\right)^2, \quad J_z^{(1)} + J_z^{(2)}. \tag{E.61}$$

Let us check that $\left[\left(J^{(1)}\right)^2, \left(J^{(1)} + J^{(2)}\right)^2\right] = 0$:

$$\begin{aligned}\left[\left(J^{(1)}\right)^2, \left(J^{(1)} + J^{(2)}\right)^2\right] &= 2\left[\left(J^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}, J_x^{(1)} \otimes J_x^{(2)} + J_y^{(1)} \otimes J_y^{(2)} + J_z^{(1)} \otimes J_z^{(2)}\right] \\ &= 0. \end{aligned} \tag{E.62}$$

Let us check that $\left[\left(J^{(1)} + J^{(2)}\right)^2, J_z^{(1)} + J_z^{(2)}\right] = 0$:

$$\begin{aligned} & \left[\left(J^{(1)} + J^{(2)}\right)^2, J_z^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J_z^{(2)}\right] \\ = & \left[J_x^{(1)} \otimes J_x^{(2)} + J_y^{(1)} \otimes J_y^{(2)} + J_z^{(1)} \otimes J_z^{(2)}, J_z^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J_z^{(2)}\right] \\ = & [J_x^{(1)}, J_z^{(1)}] \otimes J_x^{(2)} + [J_y^{(1)}, J_z^{(1)}] \otimes J_y^{(2)} + J_x^{(1)} \otimes [J_x^{(2)}, J_z^{(2)}] + J_y^{(1)} \otimes [J_y^{(2)}, J_z^{(2)}] \\ = & -iJ_y^{(1)} \otimes J_x^{(2)} + iJ_x^{(1)} \otimes J_y^{(2)} - iJ_x^{(1)} \otimes J_y^{(2)} + iJ_y^{(1)} \otimes J_x^{(2)} \\ = & \ 0. \end{aligned} \tag{E.63}$$

Let us denote their eigenstates by $|j_1 j_2; jm\rangle$. Since both sets of staes span the space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$, they must be related to each other by unitary transformation of the form

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} |j_1 j_2; jm\rangle \tag{E.64}$$

and

$$|j_1 j_2; jm\rangle = \sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \tag{E.65}$$

The coefficients in these expansions are known as the Clebsch-Gordon coefficients.

We can derive orthogonality relations. First

$$
\begin{aligned}
|j_1 m_1\rangle \otimes |j_2 m_2\rangle &= \sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} |j_1 j_2; jm\rangle \\
&= \sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} \sum_{m_1' m_2'} C^{(j_1 j_2 j) m_1' m_2'}{}_{m} |j_1 m_1'\rangle \otimes |j_2 m_2'\rangle \\
&= \sum_{m_1' m_2'} \left(\sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} C^{(j_1 j_2 j) m_1' m_2'}{}_{m} \right) |j_1 m_1'\rangle \otimes |j_2 m_2'\rangle \\
&= \sum_{m_1' m_2'} \delta^{m_1'}_{m1} \delta^{m_2'}_{m2}\ |j_1 m_1'\rangle \otimes |j_2 m_2'\rangle, \qquad \text{(E.66)}
\end{aligned}
$$

implying

$$
\sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} C^{(j_1 j_2 j) m_1' m_2'}{}_{m} = \delta^{m_1'}_{m1}\ \delta^{m_2'}_{m2}. \qquad \text{(E.67)}
$$

Second

$$
\begin{aligned}
|j_1 j_2; jm\rangle &= \sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \\
&= \sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} \sum_{j' m'} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m'} |j_1 j_2; j' m'\rangle \\
&= \sum_{j' m'} \left(\sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m'} \right) |j_1 j_2; j' m'\rangle \\
&= \sum_{j' m'} \delta_{jj'} \delta^{m}_{m'}\ |j_1 j_2; j' m'\rangle, \qquad \text{(E.68)}
\end{aligned}
$$

implying

$$
\sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m'} = \delta_{jj'} \delta^{m}_{m'}. \qquad \text{(E.69)}
$$

We can work out $bf J_z |j_1 j_2; jm\rangle$ two different ways:

$$
\begin{aligned}
\mathbf{J}_z |j_1 j_2; jm\rangle &= \left(\mathbf{J}^{(1)}_z + \mathbf{J}^{(2)}_z \right) \sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \\
&= \sum_{m_1 m_2} (m_1 + m_2) C^{(j_1 j_2 j) m_1 m_2}{}_{m} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \qquad \text{(E.70)}
\end{aligned}
$$

and

$$\begin{aligned}\mathbf{J}_z|j_1j_2;jm\rangle &= m|j_1j_2;jm\rangle \\ &= \sum_{m_1m_2} mC^{(j_1j_2j)m_1m_2}{}_m|j_1m_1\rangle\otimes|j_2m_2\rangle. \end{aligned} \tag{E.71}$$

By identifying the above realations we obtain, because of linear independence, the condition

$$(m-m_1-m_2)C^{(j_1j_2j)m_1m_2}{}_m = 0. \tag{E.72}$$

Thus it follows that the Clebsch-gordon coefficients vanish if $m \neq m_1 + m_2$. Thus the double sum in (E.65) reduces to a single sum

$$|j_1j_2;jm\rangle = \sum_{m_1} C^{(j_1j_2j)m_1m_2}{}_m|j_1m_1\rangle\otimes|j_2m-m_1\rangle \tag{E.73}$$

The conservation of angualr momentum (more precisely the projection onto the quantisation direction) is expressed by the relation $m = m_1 + m_2$.

Next we want to work out the possible values of j defined by

$$\mathbf{J}^2|j_1j_2;jm\rangle = j(j+1)|j_1j_2;jm\rangle.$$

Since $\mathbf{J}$ is an anugular momentum operator we have

$$-j \leq m \leq j.$$

From the relation $m = m_1 + m_2$ it follows that the greatest value of m, which we denote m_{max} is

$$m_{max} = j_1 + j_2. \tag{E.74}$$

This values appears onlly once in (E.73), namely if $m_1 = j_1$ and $m_2 = j_2$. This shows that the largest eigenvalue$j$, which we denote by j_{max}, must be

$$j_{max} = j_1 + j_2. \tag{E.75}$$

The second largest value of m is $m_{max} - 1$. It occurs twice, for

$$m_1 = j_1, \quad m_2 = j_2 - 1 \qquad \text{and} \qquad m_1 = j_1 - 1, \quad m_2 = j_2 \tag{E.76}$$

corresponding to the states

$$|j_1 j_1\rangle \otimes |j_2 j_2 - 1\rangle \qquad \text{and} \qquad |j_1 j_1 - 1\rangle \otimes |j_2 j_2\rangle \tag{E.77}$$

respectively. One of the two linearly independent combinations of the two states must belong to $|j_1 j_2; j = j_1 + j_2 m\rangle$, since m assumes all values $j_1 + j_2 \geq m \geq -j_1 - j_2$ in integer steps. The other linear combination necessarily belongs to the state

$$|j_1 j_2; jm\rangle \quad \text{with} \quad j = j_1 + j_2 - 1 \tag{E.78}$$

because there are no states with $j > j_{max} = j_1 + j_2$.

Continuing the argument one finds that for j, all values

$$j = \begin{cases} j_1 + j_2 \\ j_1 + j_2 - 1 \\ \vdots \\ |j_1 - j_2| \end{cases} \tag{E.79}$$

appears precisely once. This is the so called trianglar conditions.

Total number of total angular momumentum eigenstates

An easy way of checking that $|j_1 - j_2|$ is the lower limit is by counting the number of states $|j_1 j_2; jm\rangle$. This is done by calculating the sum $\sum_{J=|j_1-j_2|}^{j_1+j_2}(2J+1)$.

First consider the case where $J = j_1 + j_2$ is an integer, (say $j_1 > j_2$), then

$$\begin{aligned} \sum_{J=|j_1-j_2|}^{j_1+j_2}(2J+1) &= \sum_{J=|j_1-j_2|}^{j_1+j_2} 2J + [(j_1+j_2) - |j_1 - j_2| + 1] \\ &= \sum_{J=0}^{j_1+j_2} 2J - \sum_{J=0}^{|j_1-j_2|-1} 2J + [(j_1+j_2) - |j_1 - j_2| + 1] \\ &= (j_1+j_2)(j_1+j_2+1) - (j_1 - j_2 - 1)(j_1 - j_2) + 2j_2 + 1 \\ &= (j_1+j_2)^2 - (j_1-j_2)^2 + (j_1+j_2) + (j_1 - j_2) + 2j_2 + 1 \\ &= 4j_1 j_2 + 2j_1 + 2j_2 + 1 \\ &= (2j_1+1)(2j_2+1). \end{aligned} \tag{E.80}$$

Now consider the case where $J = j_1 + j_2$ is a half-integer. We will need to know how to perform the summation $\sum_{n=1/2}^{N/2} 2n$ where increments are in 1. This is easily,

$$
\begin{aligned}
\sum_{n=1/2}^{N/2} 2n &= 1 + 3 + \cdots + (N-2) + N \\
&= (1 + 2 + 3 + \cdots + (N-1) + N) - 2(1 + 2 + \cdots (N-1)/2) \\
&= \frac{1}{2}N(N+1) - \left(\frac{N-1}{2}\right)\left(\frac{N+1}{2}\right) \\
&= \frac{1}{4}2(N^2 + N) - \frac{1}{4}(N^2 - 1) \\
&= \frac{1}{4}(N^2 + 2N + 1) \\
&= \frac{1}{4}(N+1)^2. \qquad \text{(E.81)}
\end{aligned}
$$

Assuming $j_1 > j_2$, then

$$
\begin{aligned}
\sum_{J=j_1-j_2}^{j_1+j_2} (2J+1) &= \sum_{J=|j_1-j_2|}^{j_1+j_2} 2J + [(j_1 + j_2) - |j_1 - j_2| + 1] \\
&= \sum_{J=1/2}^{2(j_1+j_2)/2} 2J - \sum_{J=1/2}^{2(j_1-j_2-1)/2} 2J + [(j_1 + j_2) - (j_1 - j_2) + 1] \\
&= \frac{1}{4}[2(j_1 + j_2) + 1]^2 - \frac{1}{4}[2(j_1 - j_2 - 1) + 1]^2 + 4j_2 + 1 \\
&= (j_1 + j_2)^2 + (j_1 + j_2) + \frac{1}{4} - (j_1 - j_2)^2 + (j_1 - j_2) - \frac{1}{4} + 4j_2 + 1 \\
&= 4j_1 j_2 + 2j_1 + 2j_2 + 1 \\
&= (2j_1 + 1)(2j_2 + 1). \qquad \text{(E.82)}
\end{aligned}
$$

Summarising the properties of the Clebsch-Gordon coefficients:

• The coefficient $C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m}$ is trivally zero unless the conditions

$$
|j_2 - j_1| \leq j \leq j_1 + j_2 \qquad \text{and} \qquad j_1 + j_2 + j = \text{ integer} \qquad \text{(E.83)}
$$

are met. These conditions are called the Clebsch-Gordon conditions, or the triangular conditions, for the spins j_1, j_2, j.

• Moreover, $C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} = 0$ whenever $m \neq m_1 + m_2$.

• The orthogonality relations of the Clebsch-Gordon coeeficients reads

$$\sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} C^{(j_1 j_2 j) m_1' m_2'}{}_{m} = \delta^{m_1'}_{m1}\, \delta^{m_2'}_{m2}. \tag{E.84}$$

and

$$\sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m'} = \delta_{jj'} \delta^{m}_{m'}. \tag{E.85}$$

The effect of a $SU(2)$ rotation on (E.64)

$$D^{(j_1)}(g)|j_1 m_1\rangle \otimes D^{(j_2)}(g)|j_2 m_2\rangle = \sum_{jm} C^{(j_1 j_2 j) m_1 m_2}{}_{m} D^{(j)}(g)|j_1 j_2; jm\rangle. \tag{E.86}$$

Contracting with

$$\langle j_1 n_1| \otimes \langle j_2 n_2| = \sum_{j'n} C^{(j_1 j_2 j')}{}_{n_1 n_2}{}^{n} \langle j_1 j_2; j' n| \tag{E.87}$$

gives

$$D^{(j_1) m_1}{}_{n_1}(g) D^{(j_2) m_2}{}_{n_2}(g) = \sum_{jmn} C^{(j_1 j_2 j) m_1 m_2}{}_{m} C^{(j_1 j_2 j)}{}_{n_1 n_2}{}^{n} D^{(j) m}{}_{n}(g) \tag{E.88}$$

Contracting this with a Clebsch-Gordan coefficient and using the orthogonailty relation (E.85), we further find

$$D^{(j_1) m_1}{}_{n_1}(g) D^{(j_2) m_2}{}_{n_2}(g) C^{(j_1 j_2 j)}{}_{n_1 n_2}{}^{m} = C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{n} D^{(j) m}{}_{n}(g), \tag{E.89}$$

showing how the Clebsch-Gordan coefficient itself behaves under $SU(2)$ transformations, and justifying the index structure used in the notation $C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m}$.

E.3.2 The $3j$-symbol

The Wigner $3j$-symbol is defined by lowering the upper index of the Clebsch-Gordan coefficient using the epsilon tensor, and multiplying by a numerical factor (which is inserted to optimise the symmetry properties of the resulting object):

$$
\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \frac{1}{\sqrt{d_{j_3}}}(-1)^{j_1-j_2+j_3} C^{(j_1 j_2 j_3)}{}_{m_1 m_2}{}^{n} \epsilon^{(j_3)}_{n m_3} \\
&= \frac{1}{\sqrt{d_{j_3}}}(-1)^{j_1-j_2-m_3} C^{(j_1 j_2 j_3)}{}_{m_1 m_2}{}^{-m_3}.
\end{aligned} \tag{E.90}
$$

While the $3j$-symbol is often introduced as a more symmetric version of the Clebsch-Gordan coefficient, its relevance to loop quantum gravity follows from its behaviour under $SU(2)$ transformations. Starting from () and recalling () for the elements of an inverse Wigner matrix, one can show that the $3j$-symbol is invariant under the action of $SU(2)$:

$$
D^{(j_1)m_1}{}_{n_1}(g) D^{(j_2)m_2}{}_{n_2}(g) D^{(j_3)m_3}{}_{n_3}(g) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}. \tag{E.91}
$$

In the language of angular momentum, the Clebsch-Gordan coefficient couples two angular momenta j_1 and j_2 to a total angular momentum j, whereas the $3j$-symbol couples the three angular momenta j_1, j_2, and j_3 to total angular momentum zero. The $SU(2)$ invariance of the $3j$-symbol implies that the state

$$
|\Psi_0\rangle = \sum_{m_1 m_2 m_3} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle \tag{E.92}
$$

is rotationally invariant, and hence is an eigenstate of the total angular momentum operator $J^{(1)} + J^{(2)} + J^{(3)}$ with zero.

Basic properties of the $3j$-symbol

The basic properties of the $3j$-symbol follow from the corresponding properties of the clebsch-Gordon coefficients:

• The value of the $3j$-symbol can be non-zero only if the triangular conditions

$$
|j_1 - j_2| \leq j \leq j_1 + j_2 \qquad \text{and} \qquad j_1 + j_2 + j = \text{ integer} \tag{E.93}
$$

as well as the condition

$$
m_1 + m_2 + m_3 = 0 \tag{E.94}
$$

are satisfied.

• The $3j$-symbolsatisifies the orthogonality relations

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} = \frac{1}{d_j} \delta_{jj'} \delta_m^{m'} \tag{E.95}$$

and

$$\sum_{jm} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix} = \delta_{m_1'}^{m_1} \delta_{m_2}^{m_2'}. \tag{E.96}$$

When the Condon-Shortley phase convention is followed, the $3j$-symbol is real-valued, and possesses several convenient symmetry properties. Interchanging any two columns in te symbol produces the factor

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{E.97}$$

This in particular implies that the symbol is invariant under cyclic permutations of its columns. The same phase factor results from reversing the sign of all the magnetic numbers:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{E.98}$$

the $3j$-symbol reduces to the epsilon tensor:

$$\begin{pmatrix} j & j' & 0 \\ m & n & 0 \end{pmatrix} = \delta_{jj'} \frac{1}{\sqrt{d_j}} \epsilon^{(j)}_{mn}. \tag{E.99}$$

E.3.3 Three-valent intertwiners

$$\iota_{m_1 m_2 m_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{E.100}$$

E.4 General case

We consider the direct product

$$\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_L} \cong \mathcal{H}_0 \oplus_\ell \mathcal{H}_\ell \tag{E.101}$$

A basis for $\mathcal{H}_{j_i}$ is $\{e^{j_i}_{k_i}\}$.

Group action

$$D^{(j_i)}_{\alpha_i\beta_i}(g) = \left\langle e^{(j_i)}_{\beta_i}, \sigma(g)\left(e^{(j_i)}_{\alpha_i}\right)\right\rangle \tag{E.102}$$

Group action on direct product

$$\begin{aligned} D^{(j_1)\dots(j_L)}_{\alpha_1\beta_1;\dots;\alpha_L\beta_L}(g) &= \left\langle e^{(j_1)}_{\beta_1}\otimes\cdots\otimes e^{(j_L)}_{\beta_L}, \sigma(g)\left(e^{(j_1)}_{\alpha_1}\otimes\cdots\otimes e^{(j_L)}_{\alpha_L}\right)\right\rangle \\ &= \left\langle e^{(j_1)}_{\beta_1}, \sigma(g) e^{(j_1)}_{\alpha_1}\right\rangle\cdots\left\langle e^{(j_L)}_{\beta_L}, \sigma(g) e^{(j_L)}_{\alpha_L}\right\rangle \\ &= \prod_i^L D^{(j_i)}_{\alpha_i\beta_i}(g) \end{aligned} \tag{E.103}$$

Invariant vector in the direct product

$$\sigma(g)v_i = v_i \qquad \text{for all } g\in G \tag{E.104}$$

Computing matrix elements

$$\begin{aligned} \left\langle v_j, \sigma(g)v_i\right\rangle_{\mathcal{H}^\otimes} &= \left\langle v_j, v_i\right\rangle_{\mathcal{H}^\otimes} \\ &= \delta_{ij} \end{aligned} \tag{E.105}$$

i.e. they transform under the trivial representation in the direct sum (if it is non-zero).

Let us now expand these invariant vectors in terms of the basis vectors of the direct product:

$$v_i = \sum_{\alpha_1\dots\alpha_L} v_i^{\alpha_1\dots\alpha_L} e^{(j_1)}_{\alpha_1}\otimes\cdots\otimes e^{(j_L)}_{\alpha_L} \tag{E.106}$$

We compute

$$\begin{aligned} &\sigma(g)\left(e^{(j_1)}_{\alpha_1}\otimes\cdots\otimes e^{(j_L)}_{\alpha_L}\right) \\ =& \sum_{\beta_1,\dots,\beta_L}\left\langle e^{(j_1)}_{\beta_1}\otimes\cdots\otimes e^{(j_L)}_{\beta_L}, \sigma(g)\left(e^{(j_1)}_{\alpha_1}\otimes\cdots\otimes e^{(j_L)}_{\alpha_L}\right)\right\rangle e^{(j_1)}_{\beta_1}\otimes\cdots\otimes e^{(j_L)}_{\beta_L} \\ =& \prod_i^L D^{(j_i)}_{\alpha_i\beta_i}(g)\left(e^{(j_1)}_{\beta_1}\otimes\cdots\otimes e^{(j_L)}_{\beta_L}\right). \end{aligned} \tag{E.107}$$

Now from

$$\begin{aligned}
\sigma(g)v_i &= \sum_{\alpha_1\ldots\alpha_L} v_i^{\alpha_1\ldots\alpha_L}\sigma(g)\left(e_{\alpha_1}^{(j_1)}\otimes\cdots\otimes e_{\alpha_L}^{(j_L)}\right) \\
&= \sum_{\alpha_1\ldots\alpha_L} v_i^{\alpha_1\ldots\alpha_L} D_{\alpha_1\beta_1}^{(j_1)}(g)\cdots D_{\alpha_L\beta_L}^{(j_L)}(g)\left(e_{\beta_1}^{(j_1)}\otimes\cdots\otimes e_{\beta_L}^{(j_L)}\right) \\
&= v_i \\
&= \sum_{\alpha_1\ldots\alpha_L} v_i^{\beta_1\ldots\beta_L}\left(e_{\beta_1}^{(j_1)}\otimes\cdots\otimes e_{\beta_L}^{(j_L)}\right) \qquad \text{(E.108)}
\end{aligned}$$

we deduce

$$\sum_{\alpha_1\ldots\alpha_L} v_i^{\alpha_1\ldots\alpha_L} D_{\alpha_1\beta_1}^{(j_1)}(g)\cdots D_{\alpha_L\beta_L}^{(j_L)}(g) = v_i^{\beta_1\ldots\beta_L}. \qquad \text{(E.109)}$$

E.5 Tensor products for a link

$$\begin{aligned}
&(L_z\otimes 1\otimes 1\otimes 1+1\otimes L_z\otimes 1\otimes 1+1\otimes 1\otimes L_z\otimes 1+1\otimes 1\otimes 1\otimes L_z) \\
&(|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4>) \\
= \ &(j_1+j_2+j_3+j_4)\ |j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \qquad \text{(E.110)}
\end{aligned}$$

Taking the square

$$\begin{aligned}
&(L_z\otimes 1\otimes 1\otimes 1+1\otimes L_z\otimes 1\otimes 1+1\otimes 1\otimes L_z\otimes 1+1\otimes 1\otimes 1\otimes L_z)^2 \\
&(|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4>) \\
= \ &(j_1+j_2+j_3+j_4)^2\ |j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \qquad \text{(E.111)}
\end{aligned}$$

The commutation realtion is:

$$\begin{aligned}
[L_x^{4\otimes},L_y^{4\otimes}] &= (L_x\otimes 1\otimes 1\otimes 1+1\otimes L_x\otimes 1\otimes 1+1\otimes 1\otimes L_x\otimes 1+1\otimes 1\otimes 1\otimes L_x) \\
&\quad (L_y\otimes 1\otimes 1\otimes 1+1\otimes L_y\otimes 1\otimes 1+1\otimes 1\otimes L_y\otimes 1+1\otimes 1\otimes 1\otimes L_y)- \\
&\quad (L_y\otimes 1\otimes 1\otimes 1+1\otimes L_y\otimes 1\otimes 1+1\otimes 1\otimes L_y\otimes 1+1\otimes 1\otimes 1\otimes L_y) \\
&\quad (L_x\otimes 1\otimes 1\otimes 1+1\otimes L_x\otimes 1\otimes 1+1\otimes 1\otimes L_x\otimes 1+1\otimes 1\otimes 1\otimes L_x) \\
&= (L_xL_y-L_yL_x)\otimes 1\otimes 1\otimes 1+1\otimes(L_xL_y-L_yL_x)\otimes 1\otimes 1 \\
&+ 1\otimes 1\otimes(L_xL_y-L_yL_x)\otimes 1+1\otimes 1\otimes 1\otimes(L_xL_y-L_yL_x) \\
&= i(L_z\otimes 1\otimes 1\otimes 1+1\otimes L_z\otimes 1\otimes 1+1\otimes 1\otimes L_z\otimes 1+1\otimes 1\otimes 1\otimes L_z) \\
&= iL_z^{4\otimes} \qquad \text{(E.112)}
\end{aligned}$$

We have the Heisenberg relation

$$\Delta L_x^{4\otimes} \Delta L_y^{4\otimes} \geq \frac{1}{2} | < L_z^{4\otimes} > |$$

Similarly we have

$$[L_y^{4\otimes}, L_z^{4\otimes}] = i L_x^{4\otimes} \qquad [L_z^{4\otimes}, L_x^{4\otimes}] = i L_y^{4\otimes}. \tag{E.113}$$

$$\begin{aligned} & (L_x \otimes 1 \otimes 1 \otimes 1 + 1 \otimes L_x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes L_x \otimes 1 + 1 \otimes 1 \otimes 1 \otimes L_x)^2 \\ = & \; (L_x^2 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes L_x^2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes L_x^2 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes L_x^2) \\ + & \; 2(L_x \otimes L_x \otimes 1 \otimes 1 + L_x \otimes 1 \otimes L_x \otimes 1 + L_x \otimes 1 \otimes 1 \otimes L_x \\ + & \; 1 \otimes L_x \otimes L_x \otimes 1 + 1 \otimes L_x \otimes 1 \otimes L_x + 1 \otimes 1 \otimes L_x \otimes L_x) \end{aligned} \tag{E.114}$$

$$L^2 = (L_x^{4\otimes})^2 + (L_y^{4\otimes})^2 + (L_z^{4\otimes})^2 \tag{E.115}$$

We calculate the commutation relation

$$\begin{aligned} [L^2, L_x^{4\otimes}] &= [L_x^{4\otimes})^2 + (L_y^{4\otimes})^2 + (L_z^{4\otimes})^2, L_x^{4\otimes}] \\ &= [(L_y^{4\otimes})^2 + (L_z^{4\otimes})^2, L_x^{4\otimes}] \\ &= (L_y^{4\otimes})^2 L_x^{4\otimes} - L_x^{4\otimes}(L_y^{4\otimes})^2 + (L_z^{4\otimes})^2 L_x^{4\otimes} - L_x^{4\otimes}(L_z^{4\otimes})^2 \\ &= (L_y^{4\otimes})^2 L_x^{4\otimes} - (L_y^{4\otimes} L_x^{4\otimes} L_y^{4\otimes}) + (L_y^{4\otimes} L_x^{4\otimes} L_y^{4\otimes}) - L_x^{4\otimes}(L_y^{4\otimes})^2 + \\ &+ (L_z^{4\otimes})^2 L_x^{4\otimes} - (L_z^{4\otimes} L_x^{4\otimes} L_z^{4\otimes}) + (L_z^{4\otimes} L_x^{4\otimes} L_z^{4\otimes}) - L_x^{4\otimes}(L_z^{4\otimes})^2 \\ &= L_y^{4\otimes}[L_y^{4\otimes}, L_x^{4\otimes}] + [L_y^{4\otimes}, L_x^{4\otimes}] L_y^{4\otimes} + L_z^{4\otimes}[L_z^{4\otimes}, L_x^{4\otimes}] + [L_z^{4\otimes}, L_x^{4\otimes}] L_z^{4\otimes} \\ &= -i L_y^{4\otimes} L_z^{4\otimes} + i L_z^{4\otimes} L_y^{4\otimes} + i L_y^{4\otimes} L_z^{4\otimes} - i L_z^{4\otimes} L_y^{4\otimes} \\ &= 0. \end{aligned} \tag{E.116}$$

Similarly we have

$$[L^2, L_y^{4\otimes}] = 0 \qquad [L^2, L_z^{4\otimes}] = 0. \tag{E.117}$$

From books on quantum mechanics we know how to derive formulism from the commutation relations. We have

$$\begin{aligned}
& L^2|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \\
= & \left((L_x^{4\otimes})^2+(L_y^{4\otimes})^2+(L_z^{4\otimes})^2\right)|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \\
= & (j_1+j_2+j_3+j_4)(j_1+j_2+j_3+j_4+1)|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4>
\end{aligned} \tag{E.118}$$

Let us calculate the expectation value $<(L_x^{4\otimes})^2>$

$$\begin{aligned}
& <j_1,j_1|\otimes<j_2,j_2|\otimes<j_3,j_3|\otimes<j_4,j_4|(L_x^{4\otimes})^2|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \\
= & \frac{1}{2}<j_1,j_1|\otimes<j_2,j_2|\otimes<j_3,j_3|\otimes<j_4,j_4|((L_x^{4\otimes})^2+(L_y^{4\otimes})^2) \\
& |j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \\
= & \frac{1}{2}<j_1,j_1|\otimes<j_2,j_2|\otimes<j_3,j_3|\otimes<j_4,j_4|\left(L^2-(L_z^{4\otimes})^2)\right) \\
& \quad |j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \\
= & \frac{1}{2}[(j_1+j_2+j_3+j_4)(j_1+j_2+j_3+j_4+1)-(j_1+j_2+j_3+j_4)^2] \\
= & \frac{1}{2}(j_1+j_2+j_3+j_4)
\end{aligned} \tag{E.119}$$

Therefore the tensor product

$$|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \tag{E.120}$$

saturates the Heiseinberg uncertainty and is a coherent state. We then consider

$$\begin{aligned}
& |j_1,\vec{n}_1>\otimes|j_2,\vec{n}_2>\otimes|j_3,\vec{n}_3>\otimes|j_4,\vec{n}_4> \\
= & \left(D(R_1)\otimes D(R_2)\otimes D(R_3)\otimes D(R_3)\otimes D(R_4)\right)|j_1,j_1>\otimes|j_2,j_2>\otimes|j_3,j_3>\otimes|j_4,j_4> \\
= & (D(R_1)|j_1,j_1>)\otimes(D(R_2)|j_2,j_2>)\otimes(D(R_3)|j_3,j_3>)\otimes(D(R_4)|j_4,j_4>)
\end{aligned} \tag{E.121}$$

The expansion of these states in terms of L_z eigenstates is

$$\begin{aligned}
& |j_1,\vec{n}_1>\otimes|j_2,\vec{n}_2>\otimes|j_3,\vec{n}_3>\otimes|j_4,\vec{n}_4> \\
= & \sum_{m_1m_2m_3m_4}\phi_{m_1,m_2,m_3,m_4}(\vec{n}_1,\vec{n}_2,\vec{n}_3,\vec{n}_4)|j_1,m_1>\otimes|j_2,m_2>\otimes|j_3,m_3>\otimes|j_4,m_4>
\end{aligned} \tag{E.122}$$

where

$$
\begin{aligned}
& \phi_{m_1,m_2,m_3,m_4}(\vec{n}_1,\vec{n}_2,\vec{n}_3,\vec{n}_4) \\
= \; & < j_1,m_1| \otimes < j_2,m_2| \otimes < j_3,m_3| \otimes < j_4,m_4| \\
& (D(R_1)|j_1,j_1>) \otimes (D(R_2)|j_2,j_2>) \otimes (D(R_3)|j_3,j_3>) \otimes (D(R_4)|j_4,j_4>) \\
= \; & D^{(j_1)}(R_1)^{j_1}_{m_1} D^{(j_2)}(R_2)^{j_2}_{m_2} D^{(j_3)}(R_3)^{j_3}_{m_3} D^{(j_4)}(R_4)^{j_4}_{m_4} \qquad \text{(E.123)}
\end{aligned}
$$

Appendix F

Inverse matrix

Say we have a three-dimensional matrix

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

and want to form the inverse matrix. First we form the matrix who's elements are the cofactors

$$\mathbf{M}_{Cofactors} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

we then take the transpose of this, forming what is known as the adjoint matrix,

$$adj\ \mathbf{M} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix},$$

dividing this by the determinant of $\mathbf{M}$ then gives the inverse matrix,

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} adj\ \mathbf{M}.$$

We have

$$
\begin{array}{lll}
A_{11} = +\det\begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix} & A_{21} = -\det\begin{pmatrix} M_{12} & M_{13} \\ M_{32} & M_{33} \end{pmatrix} & A_{31} = +\det\begin{pmatrix} M_{12} & M_{13} \\ M_{22} & M_{23} \end{pmatrix} \\
A_{12} = -\det\begin{pmatrix} M_{21} & M_{23} \\ M_{31} & M_{33} \end{pmatrix} & A_{22} = +\det\begin{pmatrix} M_{11} & M_{13} \\ M_{31} & M_{33} \end{pmatrix} & A_{32} = -\det\begin{pmatrix} M_{11} & M_{13} \\ M_{21} & M_{23} \end{pmatrix} \\
A_{13} = +\det\begin{pmatrix} M_{21} & M_{22} \\ M_{31} & M_{32} \end{pmatrix} & A_{23} = -\det\begin{pmatrix} M_{11} & M_{12} \\ M_{31} & M_{32} \end{pmatrix} & A_{33} = +\det\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.
\end{array}
$$

In index form the inverse matrix can be written

$$
\left(\mathbf{M}^{-1}\right)_{ab} = \frac{1}{2!\det\mathbf{M}}\epsilon^{bcd}\epsilon^{aef}M_{ce}M_{df} \tag{F.1}
$$

as can be verified.

Proof:

$$
\begin{aligned}
\det\mathbf{M}\,(\mathbf{M}^{-1})_{11} &= \frac{1}{2}\left(\epsilon^{1cd}\epsilon^{1ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{123}\epsilon^{123}M_{22}M_{33} + \epsilon^{123}\epsilon^{132}M_{23}M_{32} + \epsilon^{132}\epsilon^{123}M_{32}M_{23} + \epsilon^{132}\epsilon^{132}M_{33}M_{22}\right) \\
&= (M_{22}M_{33} - M_{23}M_{32}) = A_{11}
\end{aligned}
$$

$$
\begin{aligned}
\det\mathbf{M}\,(\mathbf{M}^{-1})_{12} &= \frac{1}{2}\left(\epsilon^{2cd}\epsilon^{1ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{213}\epsilon^{123}M_{12}M_{33} + \epsilon^{213}\epsilon^{132}M_{13}M_{32} + \epsilon^{231}\epsilon^{123}M_{32}M_{13} + \epsilon^{231}\epsilon^{132}M_{33}M_{12}\right) \\
&= -(M_{12}M_{33} - M_{13}M_{32}) = A_{21}
\end{aligned}
$$

$$
\begin{aligned}
\det\mathbf{M}\,(\mathbf{M}^{-1})_{13} &= \frac{1}{2}\left(\epsilon^{3cd}\epsilon^{1ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{312}\epsilon^{123}M_{12}M_{23} + \epsilon^{312}\epsilon^{132}M_{13}M_{22} + \epsilon^{321}\epsilon^{123}M_{22}M_{13} + \epsilon^{321}\epsilon^{132}M_{23}M_{12}\right) \\
&= M_{12}M_{23} - M_{13}M_{22} = A_{31}
\end{aligned}
$$

$$
\begin{aligned}
\det\mathbf{M}\,(\mathbf{M}^{-1})_{21} &= \frac{1}{2}\left(\epsilon^{1cd}\epsilon^{2ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{123}\epsilon^{213}M_{21}M_{33} + \epsilon^{123}\epsilon^{231}M_{23}M_{31} + \epsilon^{132}\epsilon^{213}M_{31}M_{23} + \epsilon^{132}\epsilon^{231}M_{33}M_{21}\right) \\
&= -(M_{21}M_{33} - M_{23}M_{31}) = A_{12}
\end{aligned}
$$

$$\begin{aligned}
\det \mathbf{M}\ (\mathbf{M}^{-1})_{22} &= \frac{1}{2}\left(\epsilon^{2cd}\epsilon^{2ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{213}\epsilon^{213}M_{11}M_{33}\epsilon^{213}\epsilon^{231}M_{13}M_{31}+\epsilon^{231}\epsilon^{213}M_{31}M_{13}+\epsilon^{231}\epsilon^{231}M_{33}M_{11}\right) \\
&= M_{11}M_{33}-M_{13}M_{31}=A_{22}
\end{aligned}$$

$$\begin{aligned}
\det \mathbf{M}\ (\mathbf{M}^{-1})_{23} &= \frac{1}{2}\left(\epsilon^{3cd}\epsilon^{2ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{312}\epsilon^{213}M_{11}M_{23}+\epsilon^{312}\epsilon^{231}M_{13}M_{21}+\epsilon^{321}\epsilon^{213}M_{21}M_{13}+\epsilon^{321}\epsilon^{231}M_{23}M_{11}\right) \\
&= -(M_{11}M_{23}-M_{13}M_{21})=A_{32}
\end{aligned}$$

$$\begin{aligned}
\det \mathbf{M}\ (\mathbf{M}^{-1})_{31} &= \frac{1}{2}\left(\epsilon^{1cd}\epsilon^{3ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{123}\epsilon^{312}M_{21}M_{32}+\epsilon^{123}\epsilon^{321}M_{22}M_{31}+\epsilon^{132}\epsilon^{312}M_{31}M_{22}+\epsilon^{132}\epsilon^{321}M_{32}M_{21}\right) \\
&= M_{21}M_{32}-M_{22}M_{31}=A_{13}
\end{aligned}$$

$$\begin{aligned}
\det \mathbf{M}\ (\mathbf{M}^{-1})_{32} &= \frac{1}{2}\left(\epsilon^{2cd}\epsilon^{3ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{213}\epsilon^{312}M_{11}M_{32}+\epsilon^{213}\epsilon^{321}M_{12}M_{31}+\epsilon^{231}\epsilon^{312}M_{31}M_{12}+\epsilon^{231}\epsilon^{321}M_{32}M_{11}\right) \\
&= -(M_{11}M_{32}-M_{12}M_{31})=A_{23}
\end{aligned}$$

$$\begin{aligned}
\det \mathbf{M}\ (\mathbf{M}^{-1})_{33} &= \frac{1}{2}\left(\epsilon^{3cd}\epsilon^{3ef}M_{ce}M_{df}\right) \\
&= \frac{1}{2}\left(\epsilon^{312}\epsilon^{312}M_{11}M_{22}+\epsilon^{312}\epsilon^{321}M_{12}M_{21}+\epsilon^{321}\epsilon^{312}M_{21}M_{12}+\epsilon^{321}\epsilon^{321}M_{22}M_{11}\right) \\
&= M_{11}M_{22}-M_{12}M_{21}=A_{33}
\end{aligned}$$

□

Equation (F.1) can be rewritten

$$\left(\mathbf{M}^{-1}\right)^{b}{}_{a}=\frac{1}{2!\det \mathbf{M}}\epsilon_{acd}\epsilon^{bef}M^{c}_{e}M^{d}_{f} \tag{F.2}$$

Triad and co-triad

The triad is E^a_j

E^a_j is the inverse of E^i_a, that is:

$$E^a_j E^j_b = \delta^a_b, \qquad E^a_j E^k_a = \delta^k_j. \tag{F.3}$$

E^a_j is the co-triad.

Thiemann's Identity I

It follows from (F.2) that the co-triad, E^a_i, is given by

$$E^a_i = \frac{1}{2!} \frac{\epsilon^{abc}\epsilon_{ijk} E^j_b E^k_c}{\det(E^i_a)}$$

and we have the identity

$$\tilde{E}^a_i = \frac{1}{2!} \epsilon^{abc}\epsilon_{ijk} E^i_b E^j_c. \tag{F.4}$$

Appendix G

The Lorentz Group

Recall the Lorentz transformation equations.

$$
\begin{aligned}
t' &= \gamma(t - vx/c^2) \\
x' &= \gamma(x - vt), \\
y' &= y, \\
z' &= z.
\end{aligned}
\tag{G.1}
$$

where

$$
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{G.2}
$$

This can be written in matrix form,

$$
\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c^2 & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \tag{G.3}
$$

These Lorentz transformation correspond to the special case of velocity parallel to the $x - x'$ axis, and hence are easy.

Here we shall demonstrate that these Lorentz transformations form a group. The axioms of a group are:

1. Closure: If Λ_1 and Λ_2 are transformations then their composition is also a transformation of the group.

2. Associativity: $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$.

3. Identity: There is an element of the group Λ_e of the group such that for any element Λ of the group

$$\Lambda\Lambda_e = \Lambda_e\Lambda = \Lambda.$$

4. Inverse: For any element Λ of the group there is an element Λ^{-1} such that

$$\Lambda\Lambda^{-1} = \Lambda_e$$

and

$$\Lambda^{-1}\Lambda = \Lambda_e.$$

We want consider the composition of two Lorentz transformations. Recall the formula for addition of velocities,

$$v_2' = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}. \tag{G.4}$$

We have

$$\begin{aligned}
\gamma_2' &:= \frac{1}{\sqrt{1 - v_2'^2/c^2}} \\
&= \frac{1}{\sqrt{\left[1 - \left(\frac{v_1+v_2}{1+v_1v_2/c^2}\right)^2 /c^2\right]}} \\
&= \frac{1 + v_1v_2/c^2}{\sqrt{\left[(1 + v_1v_2/c^2)^2 - (v_1 + v_2)^2 /c^2\right]}} \\
&= \frac{1 + v_1v_2/c^2}{\sqrt{[(1 + v_1^2/c^2)(1 + v_2^2/c^2)]}} \\
&= \gamma_1\gamma_2(1 + v_1v_2/c^2)
\end{aligned} \tag{G.5}$$

and

$$
\begin{aligned}
\gamma_2' v_2' &= \gamma_1\gamma_2(1+v_1v_2/c^2)\frac{v_1+v_2}{1+v_1v_2/c^2} \\
&= \gamma_1\gamma_2(v_1+v_2) \qquad \text{(G.6)}
\end{aligned}
$$

Employing these results, we can write,

$$
\begin{aligned}
\begin{pmatrix} t'' \\ x'' \\ y'' \\ z'' \end{pmatrix} &= \begin{pmatrix} \gamma_2 & -\gamma_2 v_2/c^2 & 0 & 0 \\ -\gamma_2 v_2 & \gamma_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\gamma_1 v_1/c^2 & 0 & 0 \\ -\gamma_1 v_1 & \gamma_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} \gamma_1\gamma_2(1+v_1v_2/c^2) & -\gamma_1\gamma_2(v_1+v_2)/c^2 & 0 & 0 \\ -\gamma_1\gamma_2(v_1+v_2) & \gamma_1\gamma_2(1+v_1v_2/c^2) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} \gamma_2' & -\gamma_2' v_2'/c^2 & 0 & 0 \\ -\gamma_2' v_2' & \gamma_2' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad \text{(G.7)}
\end{aligned}
$$

and so composition also gives a Lorentz transformation and is consistent.

We have:

Closure property follows from the fact that composition of two Lorentz transformations is also a Lorentz transformation.

The identity element is given by $v = 0$, if which case the matrix is the identity matrix

$$
\Lambda_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

The inverse element is given by $v_2 = -v_1$ as by (G.4) $v_2' = 0$

Associativity follows from the associativity of matrix multiplication.

The velocity v in (G.1) is a parameter of this group. Since the velocity varies as $0 \leq v < c$, where v is strictly less than the speed of light c, the Lorentz group is noncompact.

Appendix H

Primer on Scalar Matter

H.1 Klein-Gordon equation

Appendix I

Primer on Yang-Mills Matter

I.1 Gauge invariance of Maxwell's theory

The idea of gauge symmetries was introduced in Maxwell's theory. Maxwell's equations are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \nabla \cdot \vec{B} = 0$$

where ρ is the charge density and $\vec{J}$ the current density. The last two equations can be solved by writing fields in terms of a scalar potential, ϕ, and a vector potential, $\vec{A}$:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}.$$

The potentials uniquely determine the fields, but the fields do not uniquely determine the potentials - we can make the changes:

$$\phi' = \phi + \frac{\partial \Lambda}{\partial t} \qquad \vec{A}' = \vec{A} - \nabla\Lambda$$

without effecting the electric and magnetic fields, where $\Lambda(\vec{x}, t)$ is an arbitrary function of space-time. These are called gauge transformations. There is an elegant relativistic notation: the gauge field is

$$A^\mu = (\phi, \vec{A})$$

and the above gauge transformations read,

$$A^{\mu\prime} = A^{\mu} + \partial^{\mu}\Lambda.$$

The so-called field strength tensor is introduced,

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$

which is easily shown to be invariant under gauge transformations. In components,

$$F^{0i} = E^{i}, \qquad \epsilon^{ijk}F^{jk} = B^{i}.$$

Maxwell's source-free action is given by:

$$S = -\frac{1}{2}\int d^4x \Big(F_{\mu\nu}F^{\mu\nu}\Big).$$

The ability to varying the gauge potential at different points in space and time without changing the physics is called a local invariance. Electromagnetic theory possess the simplest kind of local gauge symmetry called $U(1)$. A theory that displays local gauge invariance is called a gauge theory. In order to formulate other gauge theories we turn the above reasoning inside out. This is the subject of the next section.

I.2 The connection and gauges theories

I.2.1 The connection and Maxwell's theory

We know from quantum mechanics that if we replace the wave-function, $\psi(x)$, describing the electron field by

$$\psi'(x) = \exp(i\theta)\psi(x),$$

where θ is constant, that it leaves physical predictions unchanged. We consider the imposition of local invariance on the phase of the electron field,

$$\psi'(x) = \Omega\psi(x) = \exp(i\theta(x))\psi(x)$$

The problem is that derivatives of $\psi(x)$ are not covariant under this transformation:

$$\partial_\mu(\exp(i\theta(x))\psi(x)) = \Omega\partial_\mu\psi(x) + \partial_\mu\Omega\psi(x).$$

In order to cancel out the second unwanted term, one introduces a new derivative operator $\mathcal{D}_\mu$ that is covariant. To construct $\mathcal{D}_\mu$, one introduces a new field, the connection A_μ:

$$\mathcal{D}_\mu = \partial_\mu + igA_\mu(x).$$

Then

$$(\mathcal{D}_\mu\psi)' = \partial_\mu\psi' + igA'_\mu\psi' = \Omega\partial_\mu\psi + (\partial\Omega)\psi + igA'_\mu\Omega\psi$$

The term $\partial_\mu\Omega$ is precisely cancelled out by requiring the connection field transforms as

$$A'_\mu(x) = A_\mu(x) + \frac{i}{g}[\partial_\mu\Omega(x)]\Omega^{-1}(x) \quad Eq1.$$

We then have that

$$(\mathcal{D}_\mu\psi)' = \Omega\mathcal{D}_\mu\psi.$$

Note that Eq 1 is equivalent to

$$A'_\mu(x) = A_\mu(x) + \frac{1}{g}\partial_\mu\theta(x)$$

which looks the same as a gauge transformation of the gauge potential of Maxwell's theory. It is possible to construct an invariant action for the connection field itself. We want an action that only has two derivatives (since actions with higher derivatives are not unitary). Define the quantity:

$$F_{\mu\nu} = \frac{-i}{g}[\mathcal{D}_\mu, \mathcal{D}_\mu] = \frac{-i}{g}[\partial_\mu + igA_\mu(x), \partial_\nu + igA_\nu(x)]$$

$$= \frac{-i}{g}\Big([\partial_\mu, \partial_\nu] + ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]\Big)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\nu.$$

The unique action with only two derivatives is given by:

$$S = -\frac{1}{2}\int d^4x\Big(F_{\mu\nu}F^{\mu\nu}\Big).$$

Therefore one can derive electromagnetic theory from arguments based solely on symmetry.

I.2.2 The connection and Yang-Mills gauge theory

We now generalize the above reasoning to general gauge groups. One begins with the generators of some Lie algebra:

$$[T_i, T_j] = if^{ijk}T^k$$

Let there be a fermion field that transforms as

$$\mathbf{\Psi'} \mapsto \hat{\Omega}(x)\mathbf{\Psi}(x) = \exp(i\theta^i(x)T^i)\mathbf{\Psi}(x)$$

Again the derivatives of $\mathbf{\Psi}(x)$ are not covariant under this transformation. We introduce a covariant derivative

$$\mathcal{D}_\mu = \mathbf{I}\partial_\mu + ig\mathbf{A}_\mu(x)$$

with connection field given by

$$\mathbf{A}_\mu(x) = A^i_\mu(x)T^i$$

We require that $\mathbf{A}_\mu(x)$ transforms as:

$$\mathbf{A}'_\mu(x) = \hat{\Omega}\mathbf{A}_\mu(x)\hat{\Omega}^{-1} + \frac{i}{g}\hat{\Omega}(\partial_\mu\hat{\Omega}^{-1}).$$

We define the field strength operator

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + ig[\mathbf{A}_\mu, \mathbf{A}_\nu] = (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gf^{ijk}A^j_\mu A^k_\nu)T^i.$$

As $\mathcal{D}_\mu$ is covariant, this means that the $F^i_{\mu\nu}$ tensor is also covariant:

$$\mathbf{F}_{\mu\nu} \mapsto \mathbf{F}'_{\mu\nu} = \hat{\Omega}\mathbf{F}_{\mu\nu}\hat{\Omega}^{-1}$$

Note that $\mathbf{F}_{\mu\nu}$ is only invariant under gauge transformations if $\hat{\Omega}$ is a scalar, that is, only in the case of electromagnetism.

We can now construct an invariant action out of this tensor. Again we want an action that only has two derivatives. The simplest choice is the trace of the commutator:

$$Tr(\hat{\Omega}\mathbf{F}_{\mu\nu}\hat{\Omega}^{-1}\hat{\Omega}\mathbf{F}^{\mu\nu}\hat{\Omega}^{-1}) = Tr(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu})$$

The unique action with only two derivatives is given by:

$$S = -\frac{1}{2}\int d^4x Tr(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) = -\frac{1}{2}\int d^4x Tr\Big(F^i_{\mu\nu}T^j F^{\mu\nu}_j T^j\Big)$$

This is the action for Yang-mills theory.

I.2.3 The connection and Yang-Mills gauge theory

We now generalize the above reasoning to general gauge groups. One begins with the generators of some Lie algebra:

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$$\mathcal{D}_\mu = \mathbf{I}\partial_\mu + ig\mathbf{A}_\mu(x)$$

with connection field given by

$$\mathbf{A}_\mu(x) = A^i_\mu(x)T^i$$

We require that $\mathbf{A}_\mu(x)$ transforms as:

$$\mathbf{A}'_\mu(x) = \hat{\Omega}\mathbf{A}_\mu(x)\hat{\Omega}^{-1} + \frac{i}{g}\hat{\Omega}(\partial_\mu\hat{\Omega}^{-1}).$$

We define the field strength operator

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + ig[\mathbf{A}_\mu, \mathbf{A}_\nu] = (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gf^{ijk}A^j_\mu A^k_\nu)T^i.$$

As $\mathcal{D}_\mu$ is covariant, this means that the $F^i_{\mu\nu}$ tensor is also covariant:

$$\mathbf{F}_{\mu\nu} \mapsto \mathbf{F}'_{\mu\nu} = \hat{\Omega}\mathbf{F}_{\mu\nu}\hat{\Omega}^{-1}$$

Note that $\mathbf{F}_{\mu\nu}$ is only invariant under gauge transformations if $\hat{\Omega}$ is a scalar, that is, only in the case of electromagnetism.

We can now construct an invariant action out of this tensor. Again we want an action that only has two derivatives. The simplest choice is the trace of the commutator:

$$Tr(\hat{\Omega}\mathbf{F}_{\mu\nu}\hat{\Omega}^{-1}\hat{\Omega}\mathbf{F}^{\mu\nu}\hat{\Omega}^{-1}) = Tr(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu})$$

The unique action with only two derivatives is given by:

$$S = -\frac{1}{2}\int d^4x Tr(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) = -\frac{1}{2}\int d^4x Tr\Big(F^i_{\mu\nu}T^j F^{\mu\nu}_j T^j\Big)$$

This is the action for Yang-mills theory.

Appendix J

Primer on Dirac's equation

J.1 Special relativistic formulation of Dirac's eqtaion

We start with the special relativistic formulation of Dirac's eqtaion.

Dirac wanted to construct a Hamiltonian that is linear in spatial derivatives, not just in time derivatives so that time and space are put of the same footing. He postulated an equation of the form

$$i\hbar\frac{\partial\Psi}{\partial t} = [-i\hbar(\hat{\alpha}_1\frac{\partial}{\partial x^1} + \hat{\alpha}_2\frac{\partial}{\partial x^2} + \hat{\alpha}_3\frac{\partial}{\partial x^3}) + \hat{\beta}m_0]\Psi \tag{J.1}$$

where $\hat{\alpha}_,\hat{\beta}$ are $N \times N$ matrices and ψ is a column vector

$$\begin{pmatrix} \psi_1(\mathbf{x},t) \\ \psi_2(\mathbf{x},t) \\ \vdots \\ \psi_N(\mathbf{x},t) \end{pmatrix} \tag{J.2}$$

To find the concrete form of this equation we follow the natural requirements:

• Energy-momentum relation for relativistic free particle

$$E^2 = \mathbf{p}^2 + m_0^2, \tag{J.3}$$

• continuity equation for the density

• Lorentz covariance

J.1.1 Energy-momentum relation for relativistic free particle

Every component ψ_σ of the spinor must satify the Klein-gordon equation

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left(-\hbar^2 \nabla^2 + m_0^2\right) \psi_\sigma \tag{J.4}$$

On the other hand, iterating (J.1) gives

$$\begin{aligned}
-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} &= i\hbar \frac{\partial}{\partial t}(i\hbar \frac{\partial}{\partial t} \Psi) = i\hbar \frac{\partial}{\partial t} \hat{\mathcal{H}} \Psi = \hat{\mathcal{H}}^2 \Psi \\
&= \left[-i\hbar \hat{\alpha}_i \frac{\partial}{\partial x^i} + \hat{\beta} m_0\right] \left[-i\hbar \hat{\alpha}_j \frac{\partial}{\partial x^j} + \hat{\beta} m_0\right] \Psi \\
&= -\hbar^2 \sum_{i,j=1}^{3} \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \Psi}{\partial x^i \partial x^j} - i\hbar m_0 \sum_{i=1}^{3} (\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i) \frac{\partial \Psi}{\partial x^i} + \hat{\beta}^2 m_0^2 \Psi
\end{aligned} \tag{J.5}$$

Comparison with (J.4) implies the following requirements

$$\begin{aligned}
\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= 2\delta_{ij} \mathbb{1}, \\
\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= 0, \\
\hat{\alpha}_i^2 = \hat{\beta}^2 &= \mathbb{1}.
\end{aligned} \tag{J.6}$$

For the Hamiltonian to be Hermitian, the matrices $\hat{\alpha}_i, \hat{\beta}$ have to be Hermitian

$$\hat{\alpha}_i^\dagger = \hat{\alpha}_i, \quad \hat{\beta}^\dagger = \hat{\beta}. \tag{J.7}$$

Therefore the eigenvalues are real. Since $\hat{\alpha}_i^2 = \mathbb{1}$ and $\hat{\beta}^2 = \mathbb{1}$, it follows that the eigenvalues can only take the values ± 1. The eigenvalues are independent of the representation. Consider the diagonal representation of $\hat{\alpha}_i$, for example, we have

$$\hat{\alpha}_i = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & A_N \end{pmatrix} \tag{J.8}$$

with eigenvalues $A_1, \ldots, A_N$, and $\hat{\alpha}_i^2 = \mathbb{1}$ yields

$$\hat{\alpha}_i^2 = \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0\cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & 0 & \cdots & 0 \\ 0 & A_2^2 & 0 & \cdots & 0 \\ 0 & 0 & A_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0\cdots & 0 & A_N^2 \end{pmatrix} \tag{J.9}$$

from which

$$A_k^2 = 1, \quad i.e. \quad A_k = \pm 1. \tag{J.10}$$

Now from the anticommutation relations we have

$$\hat{\alpha}_i = -\hat{\beta}\hat{\alpha}_i\hat{\beta}$$

and the identity

$$Tr\hat{A}\hat{B} = Tr\hat{B}\hat{A}$$

we conclude

$$Tr\hat{\alpha}_i = -Tr\hat{\beta}\hat{\alpha}_i\hat{\beta} = -Tr\hat{\alpha}_i\hat{\beta}^2 = -Tr\hat{\alpha}_i = 0. \tag{J.11}$$

We see that the matrices $\hat{\alpha}_i, \beta$ must even dimensional.

The smallest even dimension for which the (J.6) can be fulfilled is $N = 4$. In fact it is easily shown that the following is a representation

$$\hat{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\hat{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{J.12}$$

To see this note that

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \tag{J.13}$$

where $\hat{\sigma}_i$ are Pauli's 2×2 matrices which satify the relation

$$\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2\delta_{ij} \mathbb{1}, \tag{J.14}$$

as this means

$$\begin{aligned}
\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} \hat{\sigma}_i \hat{\sigma}_j & 0 \\ 0 & \hat{\sigma}_i \hat{\sigma}_j \end{pmatrix} + \begin{pmatrix} \hat{\sigma}_j \hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_j \hat{\sigma}_i \end{pmatrix} \\
&= \begin{pmatrix} \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i \end{pmatrix} \\
&= 2\delta_{ij} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix},
\end{aligned} \tag{J.15}$$

and also

$$\begin{aligned}
\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} = 0.
\end{aligned} \tag{J.16}$$

J.1.2 Continuity equation for the density

Next we want to construct the four-current density and the equation of continuity.

First multiply (J.1) from the left by $\Psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ to obtain

$$i\hbar \Psi^\dagger \frac{\partial}{\partial t} \Psi = \frac{\hbar c}{i} \sum_{k=1}^{3} \Psi^\dagger \hat{\alpha}_k \frac{\partial}{\partial x^k} \Psi + m_0 c^2 \Psi^\dagger \hat{\beta} \Psi. \tag{J.17}$$

Next form the Hermitian conjugate of (J.1), i.e.

$$-i\hbar\frac{\partial\Psi^\dagger}{\partial t} = -\frac{\hbar c}{i}\sum_{k=1}^{3}\frac{\partial\Psi^\dagger}{\partial x^k}\hat{\alpha}_k^\dagger + m_0 c^2\Psi^\dagger\hat{\beta}^\dagger.$$

and multiply this equation from the right by Ψ. Taking into account the Hermiticity of the matrices $\hat{\alpha}_i$ and $\hat{\beta}$ (i.e. $\hat{\alpha}_i^\dagger = \hat{\alpha}_i$ and $\hat{\beta}^\dagger = \hat{\beta}$), we obtain

$$-i\hbar\frac{\partial\Psi^\dagger}{\partial t}\Psi = -\frac{\hbar c}{i}\sum_{k=1}^{3}\frac{\partial\Psi^\dagger}{\partial x^k}\hat{\alpha}_k^\dagger\Psi + m_0 c^2\Psi^\dagger\hat{\beta}^\dagger\Psi. \tag{J.18}$$

Then, (J.17) minus (J.18) yields

$$i\hbar\frac{\partial}{\partial t}(\Psi^\dagger\Psi) = \frac{\hbar c}{i}\sum_{k=1}^{3}(\Psi^\dagger\hat{\alpha}_k\Psi) \tag{J.19}$$

or

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{J.20}$$

where

$$\rho = \Psi^\dagger\Psi = \sum_{i=1}^{4}\psi_i^*\psi_i \tag{J.21}$$

is the positive definite density and

$$j^k = c\Psi^\dagger\hat{\alpha}^k\Psi \quad \text{or} \quad \mathbf{j} = c\Psi^\dagger\hat{\alpha}\Psi. \tag{J.22}$$

The conservation law follows from (J.20)

$$\frac{\partial}{\partial t}\int_V d^3x\Psi^\dagger\Psi = -\int_V d^3x\nabla\cdot\mathbf{j} = -\int_S \mathbf{j}\cdot\mathbf{dS}, \tag{J.23}$$

where V denotes a particular volume and S its surface.

J.1.3 Lorentz covariance

$$\frac{\hat{\beta}}{c}\left(i\hbar\frac{\partial}{\partial t}+i\hbar c\sum_{k=1}^{3}\hat{\alpha}_k\frac{\partial}{\partial x^k}-\hat{\beta}m_0c\right)\psi(x)=0$$

or

$$\left(\hat{\beta}i\hbar\frac{\partial}{\partial ct}+i\hbar\sum_{k=1}^{3}\hat{\beta}\hat{\alpha}_k\frac{\partial}{\partial x^k}-m_0c\right)\psi(x)=0$$

With the definitions

$$\gamma^0,\quad \gamma^i=\hat{\beta}\hat{\alpha}_i,\quad i=1,2,3 \tag{J.24}$$

this can be finally written in the form

$$i\hbar\left(\gamma^0\frac{\partial}{\partial x^0}+\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}+-m_0c\right)\psi(x)=0. \tag{J.25}$$

A more elegant formulation of the anti-commutation relations (J.6) is possible with the γ matrices

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}I \tag{J.26}$$

I being the unit 4×4 unit matrix.

In the representation we have been using so far

$$\gamma^i=\begin{pmatrix}0 & \hat{\sigma}^i\\ -\hat{\sigma}^i & 0\end{pmatrix},\quad \gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \tag{J.27}$$

How do we construct the wave function $\psi'(x')$ in one inertial frame if we know the wave function $\psi(x)$ in another frame, where the two frames are related by the Lorentz transformation a^ν_μ? Here we construct the Lorentz transformation law between $\psi(x)$ and $\psi'(x')$.

We start from the principle of special relativity which states that the laws of physics should take the same form in all inertial systems, $\psi'(x')$ must be a solution of a Dirac equation which has the form

$$\left(i\hbar\gamma^{\mu'}\frac{\partial}{\partial x'^\mu}-m_0\right)\psi'(x')=0 \tag{J.28}$$

in the primed system, where $\gamma^{\mu'}$ satisfy the same anti-commutation relations as γ^μ:

$$\gamma^{\mu'}\gamma^{\nu'} + \gamma^{\nu'}\gamma^{\mu'} = 2\eta^{\mu\nu} I \tag{J.29}$$

and

$$\gamma'^{0\dagger} = \gamma'^{0} \tag{J.30}$$

$$\gamma'^{i\dagger} = -\gamma'^{i} \quad i = 1, 2, 3. \tag{J.31}$$

It can be shown that $\gamma^{\mu'}$ that satify the above relations are identical to γ^μ up to a unitary transformation $\hat{U}$, i.e.

$$\gamma^{\mu'} = \hat{U}^\dagger \gamma^\mu \hat{U}, \quad \hat{U}^\dagger = \hat{U}^{-1}. \tag{J.32}$$

Since unitary transformations do not change the physics, we may use the same γ matrices in both Lorentz systems. From now on we just take $\gamma^\mu = \gamma^{\mu'}$.

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - m_0 \right) \psi(x) = 0 \quad \text{and} \quad \left(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - m_0 \right) \psi'(x') = 0$$

Let $\hat{a}$ denote the matrix of the Lorentz transformation a^ν_μ. We write

$$\psi'(x') = \psi'(\hat{a}x') = \hat{S}(\hat{a})\psi(x) = \hat{S}(\hat{a}^{-1}x') \tag{J.33}$$

We must have an inverse transformation

$$\psi(x) = \hat{S}^{-1}(\hat{a})\psi'(x') = \hat{S}^{-1}(\hat{a})\psi'(\hat{a}x)$$

Start with Dirac equation

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - m_0 \right) \psi(x) = 0$$

expressing $\psi(x)$ by $\hat{S}^{-1}(\hat{a})\psi'(x')$ yields

$$\left(i\hbar\gamma^\mu \hat{S}^{-1}(\hat{a}) \frac{\partial}{\partial x^\mu} - m_0 \hat{S}^{-1}(\hat{a}) \right) \psi'(x') = 0.$$

We the multiply by $\hat{S}(\hat{a})$

$$\left(i\hbar\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})\frac{\partial}{\partial x^{\mu}} - m_0\right)\psi'(x') = 0 \tag{J.34}$$

Now we transform $\partial/\partial x^{\mu}$ to x' coordinates. Using

$$x'^{\nu} = a^{\nu}{}_{\mu}x^{\mu} \qquad \Rightarrow \qquad \frac{\partial x'^{\nu}}{\partial x^{\mu}} = a^{\nu}{}_{\mu}$$

the transformation for the partial derivative is given by

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\nu}} = a^{\nu}{}_{\mu}\frac{\partial}{\partial x'^{\nu}} \tag{J.35}$$

So that (J.34) becomes

$$\left(i\hbar(\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})a^{\nu}{}_{\mu})\frac{\partial}{\partial x'^{\nu}} - m_0\right)\psi'(x') = 0. \tag{J.36}$$

Comparing this to Dirac's equation in the x' coordinates we see that $\hat{S}(\hat{a})$ must satisfy

$$\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})a^{\nu}{}_{\mu} = \gamma^{\nu} \tag{J.37}$$

or equivalently

$$\hat{S}(\hat{a})\gamma^{\nu}\hat{S}^{-1}(\hat{a}) = a_{\mu}{}^{\nu}\gamma^{\mu} \tag{J.38}$$

J.2 Action Principle

An action principle involves a quantity (the action) to be mininised (or maximised) that is an integral.

J.2.1 Action Principle for Several Dependent variables and Several Independent variables

We wish to mininise (or maximise) integrals of the following form:

$$S = \int\int\int \mathscr{L}(y_1, y_{1x_1}, \ldots, y_{1x_m}, \ldots, y_n, y_{nx_1}, \ldots y_{nx_m} \ldots, x_1, x_2, \ldots, x_m) dx_1 dx_2 \ldots dx_m. \tag{J.39}$$

The deviation, $y_i(x_1, \ldots, x_m, \alpha)$, from the configuration yeilding an extremal action is

$$y_i(x_1, \ldots, x_m, \alpha) = y_i(x_1, \ldots, x_m, 0) + \alpha\eta_i(x_1, \ldots, x_m), \qquad i = 1, 2, \ldots, n. \tag{J.40}$$

The $\eta_i(x_1, \ldots, x_m)$ are the arbitrary deviation. This deviation, $\eta_i(x_1, \ldots, x_m)$ is required to differentiable and to vanish at the end points (i.e. the limits of the integrations).

From (J.40) we have

$$\frac{\partial y_i}{\partial x_j}(x_1, \ldots, x_m, \alpha) = \frac{\partial y_i}{\partial x_j}(x_1, \ldots, x_m, 0) + \alpha\frac{\partial \eta_i}{\partial x_j}(x_1, \ldots, x_m), \quad i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m. \tag{J.41}$$

Substituting (J.40) into (J.39) makes the action a function of α, i.e. $S = S(\alpha)$. This action has a minimum at $\alpha = 0$, so that

$$\frac{\partial S}{\partial \alpha}(\alpha = 0) = 0. \tag{J.42}$$

Differentiating the integral (J.39) with respect to α and then setting $\alpha = 0$, we obtain

$$\frac{\partial S}{\partial \alpha}(\alpha = 0) = \int_{x_{int}}^{x_{fin}} \sum_i \left[\frac{\partial \mathscr{L}}{\partial y_i}\eta_i(x) + \sum_j \frac{\partial \mathscr{L}}{\partial y_{ix_j}} \frac{\partial \eta_i(x)}{\partial x_j}\right] dx_1 \ldots dx_m = 0. \tag{J.43}$$

Integrating by parts results in

$$\int_{x_{int}}^{x_{fin}} \sum_i \left[\frac{\partial \mathscr{L}}{\partial y_i} - \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial \mathscr{L}}{\partial y_{ix_j}}\right)\right] \eta_i(x_1, \ldots, x_m) dx_1 \ldots dx_m = 0. \tag{J.44}$$

As the $\eta_i(x_1, \ldots, x_m)$ are arbitrary and independent of each other the integrand vanishes

$$\frac{\partial \mathscr{L}}{\partial y_i} - \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial \mathscr{L}}{\partial(\partial y_i/\partial x_j)}\right) = 0, \qquad i = 1, 2, \ldots, n. \tag{J.45}$$

J.2.2 Complex dependent variables

We now let $\mathscr{L}$ be a function of a complex valued dependent variables $y(x) = y_R(x) + iy_I(x)$ and $\overline{y}(x) = y_R(x) - iy_I(x)$ which depends on x, the independent variable. The action is of the form

$$S = \int_{x_1}^{x_2} \mathscr{L}(\overline{y}, y, \overline{y}_x, y_x)dx. \tag{J.46}$$

We compare neighbouring paths by writing

$$y_R(x,\alpha) = y(x,0)_R + \alpha\eta_R(x), \qquad y_I(x,\alpha) = y(x,0)_I + \alpha\eta_I(x) \tag{J.47}$$

where the η_I and η_R are independent, but subject them to the condition

$$\eta_{R,I}(x_1) = \eta_{R,I}(x_2) = 0. \tag{J.48}$$

$$\begin{aligned} S(\alpha) \;&=\; \int_{x_1}^{x_2} \mathscr{L}(\overline{y}, y)dx + \int_{x_1}^{x_2} \Big[\frac{\partial \mathscr{L}}{\partial \overline{y}}(\alpha\eta_R - i\alpha\eta_I) + \frac{\partial \mathscr{L}}{\partial \overline{y}_x}(\alpha\frac{d\eta_R}{dx} - i\alpha\frac{d\eta_R}{d\eta_I}) \\ &\quad + \frac{\partial \mathscr{L}}{\partial y}(\alpha\eta_R + i\alpha\eta_I) + \frac{\partial \mathscr{L}}{\partial y_x}(\alpha\frac{d\eta_R}{dx} + i\alpha\frac{d\eta_R}{d\eta_I})\Big] + \mathcal{O}(\alpha^2) \end{aligned} \tag{J.49}$$

Differentiating with respect to α and setting $\alpha = 0$ we obtain

$$\begin{aligned} &\int_{x_1}^{x_2} \Big[\frac{\partial \mathscr{L}}{\partial \overline{y}}(\eta_R(x) - i\eta_I(x)) + \frac{\partial \mathscr{L}}{\partial \overline{y}_x}(\frac{d\eta_R(x)}{dx} - i\frac{d\eta_R(x)}{d\eta_I}) \\ &\quad + \frac{\partial \mathscr{L}}{\partial y}(\eta_R(x) + i\eta_I(x)) + \frac{\partial \mathscr{L}}{\partial y_x}(\frac{d\eta_R(x)}{dx} + i\frac{d\eta_R(x)}{d\eta_I})\Big]\, dx = 0. \end{aligned} \tag{J.50}$$

Integrating by parts

$$\int_{x_1}^{x_2} \left\{ \left[\frac{\partial \mathscr{L}}{\partial \overline{y}} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \overline{y}_x}\right)\right] \left(\eta_R(x) - i\eta_I(x)\right) + \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y_x}\right)\right] \left(\eta_R(x) + i\eta_I(x)\right) \right\} dx \tag{J.51}$$

Which can be written

$$\begin{aligned} \int_{x_1}^{x_2} \Bigg\{ &\left(\left[\frac{\partial \mathscr{L}}{\partial \overline{y}} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \overline{y}_x}\right)\right] + \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y_x}\right)\right]\right) \eta_R(x) \\ &+ i\left(-\left[\frac{\partial \mathscr{L}}{\partial \overline{y}} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \overline{y}_x}\right)\right] + \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y_x}\right)\right]\right) \eta_I(x) \Bigg\} dx = 0. \end{aligned} \tag{J.52}$$

Since the η_R and η_I are arbitrary and independent of each other, each of the terms vanishes independently and we get

$$\left[\frac{\partial \mathscr{L}}{\partial \overline{y}} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \overline{y}_x}\right)\right] + \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y_x}\right)\right] = 0 \tag{J.53}$$

$$-\left[\frac{\partial \mathscr{L}}{\partial \overline{y}} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \overline{y}_x}\right)\right] + \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y_x}\right)\right] = 0 \tag{J.54}$$

or, finally,

$$\frac{\partial \mathscr{L}}{\partial \overline{y}} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \overline{y}_x}\right) = 0 \qquad \text{and} \qquad \frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y_x}\right) = 0. \tag{J.55}$$

J.2.3 Action for Dirac's equation

This is an example of an action with several dependent complex variables and several independent variables. Dirac's equation can be written

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - m_0 \mathbb{1} \right) \Psi(x, y, z, t) = 0 \tag{J.56}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{J.57}$$

and γ^μ are the 4×4 matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} \tag{J.58}$$

where $\hat{\sigma}_i$ are Pauli's 2×2 matrices and I is the 2×2 unit matrix. With the explicit form of the Pauli matrices we have in detail.

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{J.59}$$

The action can be written

$$S = \int \overline{\Psi} \left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - m_0 \mathbb{1} \right) \Psi d^4x \tag{J.60}$$

where $\overline{\Psi} = \Psi^\dagger \gamma^0$.

Derivation of Dirac's equation

We write the Lagrangian density in component form:

$$\mathscr{L} = \Psi^*_\alpha \left(i\hbar(\gamma^0\gamma^\mu)_{\alpha\beta} \frac{\partial}{\partial x^\mu} - m_0 \gamma^0_{\alpha\beta} \right) \Psi_\beta \tag{J.61}$$

Using the following Euler-Lagrage equations

$$\frac{\partial \mathscr{L}}{\partial \Psi^*_\alpha} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathscr{L}}{\partial(\partial \Psi^*_\alpha / \partial x^\mu)} \right) = 0, \qquad \alpha = 1, 2, 3, 4. \tag{J.62}$$

we easily obtain

$$\left(i\hbar(\gamma^0\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu} - m_0\gamma^0_{\alpha\beta}\right)\Psi_\beta = 0$$

or

$$\gamma^0\left(i\hbar\gamma^\mu\frac{\partial}{\partial x^\mu} - m_0\mathbb{1}\right)\Psi = 0 \tag{J.63}$$

and upon multiplying from the left by γ^0 and using $(\gamma^0)^2 = I$ we obtain Dirac's equation (J.56).

Derivation of the adjoint Dirac equation

The adjoint Dirac equation is given by

$$i\hbar\frac{\partial\overline{\Psi}}{\partial x^\mu}(x,y,z,t)\gamma^\mu + m_0\overline{\Psi}(x,y,z,t) = 0. \tag{J.64}$$

It is derived from the following Euler-Lagrange equations

$$\frac{\partial\mathscr{L}}{\partial\Psi_\beta} - \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial\Psi_\beta/\partial x^\mu)}\right) = 0, \qquad \beta = 1,2,3,4. \tag{J.65}$$

Substituting (J.61) into this equation gives

$$\left[-m_0\Psi^*_\alpha\gamma^0_{\alpha\beta}\right] - \frac{\partial}{\partial x^\mu}\left[i\hbar\Psi^*_\alpha(\gamma^0\gamma^\mu)_{\alpha\beta}\right] = 0$$

or

$$i\hbar\frac{\partial\overline{\Psi}_\alpha}{\partial x^\mu}\gamma^\mu_{\alpha\beta} + m_0\overline{\Psi}_\beta = 0 \tag{J.66}$$

and we have obtained the component form of the adjoint Dirac's equation (J.64).

An alternative action

An alternative way of expressing the action is in the more symmetric form

$$S = \int \left\{ \frac{i\hbar}{2} \left(\overline{\Psi} \gamma^\mu \frac{\partial \Psi}{\partial x^\mu} - \frac{\partial \overline{\Psi}}{\partial x^\mu} \gamma^\mu \Psi \right) - m_0 \overline{\Psi} \Psi \right\} d^4 x \tag{J.67}$$

which can be obtained from (J.60) by an integration by parts and vanishing of the boundary terms.

J.3 Weyl Spinors

Dirac's equation is

$$i\hbar \gamma^\mu \partial_\mu \Psi - m_0 c \Psi = 0 \tag{J.68}$$

where

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \tag{J.69}$$

The standard representation the γ^μ can be written explicitly

$$\gamma^i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{J.70}$$

With γ^5

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{J.71}$$

As follows from:

$$
\begin{aligned}
\gamma^5 &= i\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}0 & \hat{\sigma}_1\\ -\hat{\sigma}_1 & 0\end{pmatrix}\begin{pmatrix}0 & \hat{\sigma}_2\\ -\hat{\sigma}_2 & 0\end{pmatrix}\begin{pmatrix}0 & \hat{\sigma}_3\\ -\hat{\sigma}_3 & 0\end{pmatrix}\\
&= i\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}0 & \hat{\sigma}_1\\ -\hat{\sigma}_1 & 0\end{pmatrix}\begin{pmatrix}-\hat{\sigma}_2\hat{\sigma}_3 & 0\\ 0 & -\hat{\sigma}_2\hat{\sigma}_3\end{pmatrix}\\
&= i\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}0 & -\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3\\ \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 & 0\end{pmatrix}\\
&= -i\begin{pmatrix}0 & \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3\\ \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 & 0\end{pmatrix}
\end{aligned}
$$

and

$$
\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}.
$$

J.4 Chiralilty and the Weyl Equation

J.4.1 Chirality

We need $(\gamma_5)^2 = \mathbb{1}$

$$
\hat{P}_+ := \frac{1}{2}(\mathbb{1} + \gamma_5) \tag{J.72}
$$

$$
\hat{P}_- := \frac{1}{2}(\mathbb{1} - \gamma_5) \tag{J.73}
$$

These are projection operators.

Proof:

$$
\begin{aligned}
\hat{P}_+\hat{P}_- &= \frac{1}{4}(\mathbb{1} + \gamma_5)(\mathbb{1} - \gamma_5) = 0,\\
\hat{P}_-\hat{P}_+ &= 0,\\
(\hat{P}_+)^2 &= \frac{1}{4}(\mathbb{1} + \gamma_5)^2 = \frac{1}{2}(\mathbb{1} + \gamma_5) = \hat{P}_+,\\
(\hat{P}_-)^2 &= \hat{P}_-.
\end{aligned} \tag{J.74}
$$

□

We show that the four-spinors $\Psi = \hat{P}_+\Psi$ and $\Psi = \hat{P}_-\Psi$ each have only two linearly independent components.

Proof:

Positive chirality

We look at the eigenvalues and eigenvectors of the matrix $\hat{P}_+ = (\mathbb{1} + \gamma_5)/2$:

$$\begin{aligned}
0 &= \det(\hat{P}_+ - \lambda\mathbb{1}) \\
&= \det\left(\frac{\mathbb{1}+\gamma_5}{2} - \mathbb{1}\right) \\
&= \det\begin{pmatrix} \frac{1}{2}-\lambda & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}-\lambda & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2}-\lambda & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2}-\lambda \end{pmatrix} \\
&= \left(\frac{1}{2}-\lambda\right)^4 - \frac{1}{4}\left(\frac{1}{2}-\lambda\right)^2 + \left(\frac{1}{4}\right)^2 - \frac{1}{4}\left(\frac{1}{2}-\lambda\right)^2 \\
&= \left[\left(\frac{1}{2}-\lambda\right)^2 - \frac{1}{4}\right]^2 = \lambda^2(\lambda-1)^2
\end{aligned} \tag{J.75}$$

The eigenvalues are thus $\lambda = 0$ and $\lambda = 1$, each occuring twice. The eigenvectors of

$$\frac{1}{2}(\mathbb{1} + \gamma_5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{J.76}$$

for $\lambda = 1$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{J.77}$$

and for $\lambda = 0$

$$e_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{J.78}$$

If we expand Ψ into e_1 to e_4, we obtain

$$\begin{aligned}
\Psi^{(+)} &= \frac{1}{2}(\mathbb{1} + \gamma_5)\Psi(x) \\
&= \frac{1}{2}(\mathbb{1} + \gamma_5)\sum_{i=1}^{4} \psi_i(x) e_i \\
&= \sum_{i=1}^{4} \psi_i(x)\lambda_i e_i \\
&= \psi_1(x)e_1 + \psi_2(x)e_2 \\
&= \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_1(x) \\ \psi_2(x). \end{pmatrix}
\end{aligned} \tag{J.79}$$

If we decompose the Dirac spinor into two two-spinors ψ and χ,

$$\Psi^{(+)} = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{J.80}$$

then obvioulsy

$$\psi = +\chi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{J.81}$$

Negative chirality

It turns out that the eigenvalues of $\hat{P}_- = (1 - \gamma_5)/2$ are the same as for the positive projection operator. The eigenvectors of

$$\frac{1}{2}(\mathbb{1} - \gamma_5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{J.82}$$

for $\lambda = 1$ are

$$\tilde{e}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \tag{J.83}$$

and for $\lambda = 0$

$$\tilde{e}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{e}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{J.84}$$

We then have

$$\begin{aligned} \Psi^{(-)} &= \frac{1}{2}(\mathbb{1} - \gamma_5)\Psi(x) \\ &= \frac{1}{2}(\mathbb{1} - \gamma_5)\sum_{i=1}^{4} \psi_i(x)\tilde{e}_i \\ &= \sum_{i=1}^{4} \psi_i(x)\lambda_i\tilde{e}_i \\ &= \psi_1(x)\tilde{e}_1 + \psi_2(x)\tilde{e}_2 \\ &= \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ -\psi_1(x) \\ -\psi_2(x). \end{pmatrix} \end{aligned} \tag{J.85}$$

If we decompose the Dirac spinor into two two-spinors ψ and χ,

$$\Psi^{(-)} = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{J.86}$$

then obvioulsy

$$\psi = -\chi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{J.87}$$

□

J.4.2 Chirality and massless Dirac particles

Multiplying Dirac'sequation (J.68) from the left by γ^0 and rearanging, we obtain

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-i\hbar\gamma^0\gamma^i\partial_i + m_0 c\gamma^0\right)\Psi = \hat{H}_f\Psi. \tag{J.88}$$

We wish to consider the commutator between $\hat{H}_f$ and $\hat{P}_+$ ($\hat{P}_-$). We need only consider $[\hat{H}_f, \gamma^5]_-$,

$$\begin{aligned}
[\hat{H}_f, \gamma^5]_-\Psi &= [\left(-i\hbar\gamma^0\gamma^i\partial_i + m_0 c\gamma^0\right), \gamma^5]_-\Psi \\
&= -i\hbar[\gamma^0\gamma^i, \gamma^5]_-\partial_i\Psi + m_0 c[\gamma^0, \gamma^5]_-\Psi.
\end{aligned} \tag{J.89}$$

Now by (J.69) $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$, $(\gamma^0)^2 = \mathbb{1}$ and $(\gamma^i)^2 = -\mathbb{1}$. First we try

$$\begin{aligned}
[\gamma^0\gamma^2, \gamma^5]_- &= i\gamma^0\gamma^2(\gamma^0\gamma^1\gamma^2\gamma^3) - i(\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^0\gamma^2 \\
&= i\gamma^0\gamma^2(\gamma^0\gamma^1\gamma^2\gamma^3) - i(\gamma^0\gamma^1\gamma^2(\gamma^0\gamma^2)\gamma^3) \\
&= i\gamma^0\gamma^2(\gamma^0\gamma^1\gamma^2\gamma^3) + i(\gamma^0\gamma^1(\gamma^0\gamma^2)\gamma^2\gamma^3) \\
&= i\gamma^0\gamma^2(\gamma^0\gamma^1\gamma^2\gamma^3) + i(\gamma^0(\gamma^0\gamma^2)\gamma^1\gamma^2\gamma^3) \\
&= i\gamma^0\gamma^2(\gamma^0\gamma^1\gamma^2\gamma^3) + i(\gamma^0\gamma^2)(\gamma^0\gamma^1\gamma^2\gamma^3) = 0.
\end{aligned}$$

In general we have $[\gamma^0\gamma^i, \gamma^5]_- = 0$. Now we turn to $[\gamma^0, \gamma^5]_-$,

$$\begin{aligned}
[\gamma^0, \gamma^5]_- &= i\gamma^0(\gamma^0\gamma^1\gamma^2\gamma^3) - i(\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^0 \\
&= i\gamma^0(\gamma^0\gamma^1\gamma^2\gamma^3) + i\gamma^0(\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^0 = 2\gamma^0\gamma^5.
\end{aligned}$$

Hence (J.89) becomes

$$[\hat{H}_f, \gamma^5]_-\Psi = 2m_0 c\gamma^0\gamma^5\Psi. \tag{J.90}$$

Therefore, the projection operators $\hat{P}_+$ and $\hat{P}_-$ commute with the Hamiltonian operator and have simultaneous eigenstates only for massless particles (i.e. when $m_0 = 0$).

J.4.3 Dirac equation as a pair of coupled equations for two-component spinors

Consider the free Dirac equation written in the form

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}_f\Psi = \left(c\hat{\alpha}\cdot\hat{\mathbf{p}} + m_0c^2\hat{\beta}\right)\Psi \tag{J.91}$$

Substituting $\hat{\mathbf{p}} = -i\hbar\nabla$, we have

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-ic\hbar\hat{\alpha}\cdot\nabla + m_0c^2\hat{\beta}\right)\Psi \tag{J.92}$$

We split the four-component spinor into two two-component spinors ψ and χ, i.e.

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\psi\\ \chi\end{pmatrix} \tag{J.93}$$

with

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \chi = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} \tag{J.94}$$

Using the explicit form for the $\hat{\alpha}$ and $\hat{\beta}$ matrices

$$\hat{\alpha}_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}, \qquad \hat{\beta} = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$$

(J.92) can be written as

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi\\ \chi\end{pmatrix} = -ic\hbar\begin{pmatrix}0 & \hat{\sigma}\\ \hat{\sigma} & 0\end{pmatrix}\cdot\nabla\begin{pmatrix}\psi\\ \chi\end{pmatrix} + m_0c^2\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\psi\\ \chi\end{pmatrix} \tag{J.95}$$

or

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial(ct)} &= -i\hbar\hat{\sigma}\cdot\nabla\chi + m_0c\psi \\ i\hbar\frac{\partial\chi}{\partial(ct)} &= -i\hbar\hat{\sigma}\cdot\nabla\psi - m_0c\chi \end{aligned} \tag{J.96}$$

J.4.4 The Weyl equation

The massless equations are then

$$\begin{aligned}\frac{\partial\psi}{\partial(ct)} &= -\hat{\sigma}\cdot\nabla\chi \\ \frac{\partial\chi}{\partial(ct)} &= -\hat{\sigma}\cdot\nabla\psi\end{aligned} \tag{J.97}$$

Right-handed chiral spinor

For positive chirality, $\psi = +\chi$. Substituting this into (J.97), the two equations reduce to the single equation:

$$\dot{\psi}_R = -\hat{\sigma}\cdot\nabla\psi_R. \tag{J.98}$$

or

$$\sigma^\mu\partial_\mu\psi_R = 0. \tag{J.99}$$

explicitly

$$\mathbb{1}_2\frac{1}{c}\frac{\partial\psi_R}{\partial t} + \sigma_x\frac{\partial\psi_R}{\partial x} + \sigma_y\frac{\partial\psi_R}{\partial y} + \sigma_z\frac{\partial\psi_R}{\partial z} = 0. \tag{J.100}$$

where $\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3) = (\mathbb{1}_2, \sigma_x, \sigma_y, \sigma_z)$.

Left-handed chiral spinor

For negative chirality, $\psi = -\chi$. Substituting this into (J.97), the two equations reduce to the single equation:

$$\dot{\psi}_L = +\hat{\sigma}\cdot\nabla\psi_L. \tag{J.101}$$

or

$$\tilde{\sigma}^\mu\partial_\mu\psi_L = 0. \tag{J.102}$$

where $\tilde{\sigma}^\mu = (\mathbb{1}_2, -\sigma_x, -\sigma_y, -\sigma_z)$.

Equation (J.99) can be obtained from the Lagrangian density

$$\mathscr{L} = i\overline{\psi}_R \sigma^\mu \partial_\mu \psi_R \tag{J.103}$$

alternatively

$$\mathscr{L} = \frac{i}{2}(\overline{\psi}_R \sigma^\mu \partial_\mu \psi_R - \overline{\partial_\mu \psi_R} \sigma^\mu \psi_R). \tag{J.104}$$

Equation (J.102) can be obtained from the Lagrangian density

$$\mathscr{L} = i\overline{\psi}_L \tilde{\sigma}^\mu \partial_\mu \psi_L \tag{J.105}$$

alternatively

$$\mathscr{L} = \frac{i}{2}(\overline{\psi}_L \tilde{\sigma}^\mu \partial_\mu \psi_L - \overline{\partial_\mu \psi_L} \tilde{\sigma}^\mu \psi_L). \tag{J.106}$$

J.5 General Relativistic Dirac's Equation

J.6 Action

Appendix K

Fermionic Integration

K.1 Determinant as a Fermionic Integral

In this method a variational equation obeyed by the Gaussian intgral is obtained. Since $\zeta^* M \zeta$ contains the product of two anticommuting variables and thus is itself a commuting quantity,

$$\delta_M (\zeta^* M \zeta)^n = n(\zeta^* \delta M \zeta)(\zeta^* M \zeta)^{n-1} \tag{K.1}$$

Hence varying M gives

$$\delta I = -\int [d\zeta^* d\zeta] \zeta^* \delta M \zeta e^{-\zeta^* M \zeta} \tag{K.2}$$

Again since $\zeta^* M \zeta$ commutes, the derivative is given by

$$\frac{\partial}{\partial \zeta_k^*} e^{-\zeta^* M \zeta} = -\sum_{l=1}^{N} M_{kl} \zeta_l e^{-\zeta^* M \zeta}. \tag{K.3}$$

Hence

$$\delta I = -\int [d\zeta^* d\zeta] \zeta^* \delta M M^{-1} \frac{\partial}{\partial \zeta^*} \zeta e^{-\zeta^* M \zeta} \tag{K.4}$$

This integral may be evaluated by "integration by parts" by using the following relation, where F is an arbitrary function of the Grassmann variables

$$\frac{\partial}{\partial \zeta_k^*}\zeta_m^* F = \delta_{km} F - \zeta_m^* \frac{\partial}{\partial \zeta_k^*} F. \tag{K.5}$$

To prove the above equation we need only consider three cases

(i) k=m and F is a fuction of ζ_k^* as well. The r.h.s. vanishes as it should.

(ii) k=m and F is not a function of ζ_k^*. Then we should just be left with F.

(iii) $k \neq m$. And for this case (K.5) gives the correct answer again. The minus sign is there as the derivative has to pass through the first grassmann variable.

Now, using (K.5) in (K.4) we obtain,

$$\delta I = -\int [d\zeta^* d\zeta] \sum_{km} (\delta M M^{-1})_{mk} \left\{ \delta_{km} - \frac{\partial}{\partial \zeta_k^*} \zeta_m^* \right\} e^{-\zeta^* M \zeta} \tag{K.6}$$

Since the Grassmann integral of a derivative vanishes,

$$\delta = (tr \delta M M^{-1}) I, \tag{K.7}$$

which gives

$$\delta \ln I = \frac{\delta I}{I} tr \delta M M^{-1} = \delta \ln det M, \tag{K.8}$$

and the solution is

$$I = (const.) \exp(Tr \ln M) = (const.) \det M. \tag{K.9}$$

To determine the constant, we set M to 1, in which limit

$$I = \left[\int d\zeta^* d\zeta e^{-\zeta^* \zeta} \right]^N = 1^N = 1 \tag{K.10}$$

Hence the constant is unity and the complete evaluation gives

$$I = \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) e^{-\zeta^* M \zeta} = det M \tag{K.11}$$

K.2 Grassmann Integration with Fermionic Sources

It is not surprising that the Gaussian integral formula (K.11) can be generalised to the case of general bilinear forms in the exponent:

$$\int \prod_{k=1}^{n} (d\zeta_k^* d\zeta_k) \exp -\left(\zeta^* M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho\right) = \det M \exp(\rho^\dagger M^{-1} \rho). \tag{K.12}$$

Here ρ is an n-component vector of Grassmann variables. Equation (K.12) is obtained by translating the integration variable, $\zeta' = \zeta + M^{-1}\rho$,

$$\begin{aligned}
\det M &= \int \prod_{k=1}^{n} (d\zeta'^*_k d\zeta'_k) \exp -\zeta'^* M \zeta' \\
&= \int \prod_{k=1}^{n} (d\zeta_k^* d\zeta_k) \exp -(\zeta^\dagger + \rho^\dagger M^{-1}) M (\zeta + M^{-1}\rho) \\
&= \int \prod_{k=1}^{n} (d\zeta_k^* d\zeta_k) \exp -\left(\zeta^\dagger M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho + \rho^\dagger M^{-1} \rho\right) \\
&= \left[\int \prod_{k=1}^{n} (d\zeta_k^* d\zeta_k) \exp -\left(\zeta^\dagger M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho\right)\right] \exp -(\rho^\dagger M^{-1} \rho)
\end{aligned} \tag{K.13}$$

K.3 Qunatum fermion field on a 2-complex

Appendix L

Coherent states in ordinary quantum mechanics

L.1 Prove coherent states saturate uncerntainty

Prove the states

$$|z>$$

which is the eigenvalue of $\hat{a}$ with

$$\hat{a}|z>= z|z>$$

saturate uncerntainty, i.e. we have

$$\Delta X \Delta P = \frac{\hbar}{2}.$$

For z we will have

$$z = \sqrt{\frac{\omega}{2\hbar}} X_0 + i \frac{1}{\sqrt{2\hbar\omega}} P_0$$

Proof:

Note, for example,

$$
\begin{aligned}
<z|\hat{X}|z> &= \int dX <z|X> X <X|z> \\
&= \int dX \psi_z^*(X) X \psi_z(X) \\
<X>.
\end{aligned}
$$

Therefore we can avoid integrals, instead just use $\hat{a}$ and $\hat{a}^\dagger$.

Recall that

$$
\begin{aligned}
(\Delta x)^2 &= <(x-<x>)^2> = <x^2> - <x>^2 \\
(\Delta p)^2 &= <(p-<p>)> = <p^2> - <p>^2 .
\end{aligned}
$$

It is easier verified via the definitions

$$(\Delta X)^2 = <\hat{X}^2> - <\hat{X}>^2, \qquad (\Delta P)^2 = <\hat{P}^2> - <\hat{P}>^2$$

As

$$\hat{a} = \sqrt{\frac{\omega}{2\hbar}}\hat{X} + i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}, \qquad \hat{a}^\dagger = \sqrt{\frac{\omega}{2\hbar}}\hat{X} - i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}$$

we obtain

$$
\begin{aligned}
\hat{X} &= \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger) \\
\hat{P} &= -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger).
\end{aligned}
$$

Recall that

$$\hat{a}|z> = z|z>, \qquad <z|\hat{a}^\dagger = \overline{z} <z|.$$

First we find $<z|\hat{X}|z>$,

$$
\begin{aligned}
< z|\hat{X}|z > &= \sqrt{\frac{\hbar}{2\omega}} < z|(\hat{a}+\hat{a}^\dagger)|z > \\
&= \sqrt{\frac{\hbar}{2\omega}}(z < z|z > + \overline{z} < z|z >) \\
&= \sqrt{\frac{\hbar}{2\omega}}(z+\overline{z}) \\
&= \sqrt{\frac{\hbar}{2\omega}} 2 \sqrt{\frac{\omega}{2\hbar}} X_0 = X_0.
\end{aligned}
$$

Now consider $< z|\hat{P}|z >$,

$$
\begin{aligned}
< z|\hat{P}|z > &= -i\sqrt{\frac{\hbar\omega}{2}} < z|(\hat{a}-\hat{a}^\dagger)|z > \\
&= -i\sqrt{\frac{\hbar\omega}{2}}(z < z|z > - \overline{z} < z|z >) \\
&= -i\sqrt{\frac{\hbar\omega}{2}}(z-\overline{z}) \\
&= -i\sqrt{\frac{\hbar\omega}{2}} 2(i\frac{1}{\sqrt{2\hbar\omega}}P_0) \\
&= P_0.
\end{aligned}
$$

Now let us compute $< \hat{X}^2 >$,

$$
\begin{aligned}
< z|\hat{X}^2|z > &= \frac{\hbar}{2\omega} < z|(\hat{a}+\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega} < z|(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega} < z|(\hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1 + (\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega}\left(z^2 + 2\overline{z}z + 1\overline{z}^2\right) \\
&= \frac{\hbar}{2\omega}\left((z+\overline{z})^2 + 1\right) \\
&= < z|\hat{X}|z >^2 + \frac{\hbar}{2\omega}
\end{aligned}
$$

where we have used $[\hat{a},\hat{a}^\dagger] = 1$. We can now write

$$\begin{aligned} \Delta X &= \sqrt{< z|\hat{X}^2|z > - < z|\hat{X}|z >^2} \\ &= \sqrt{\frac{\hbar}{2\omega}}. \end{aligned}$$

Now let us compute $< \hat{P}^2 >$,

$$\begin{aligned} < z|\hat{P}^2|z > &= -\frac{\hbar\omega}{2} < z|(\hat{a} - \hat{a}^\dagger)^2|z > \\ &= -\frac{\hbar\omega}{2} < z|(\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + (\hat{a})^\dagger)^2|z > \\ &= -\frac{\hbar\omega}{2} < z|(\hat{a}^2 - 2\hat{a}^\dagger\hat{a} - 1 + (\hat{a}^\dagger)^2|z > \\ &= -\frac{\hbar\omega}{2} \left(z^2 - 2\overline{z}z + \overline{z}^2 - 1\right) \\ &= -\frac{\hbar\omega}{2} \left((z - \overline{z})^2 - 1\right) \\ &= < z|\hat{P}|z >^2 + \frac{\hbar\omega}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} \Delta P &= \sqrt{< z|\hat{P}^2|z > - < z|\hat{P}|z >^2} \\ &= \sqrt{\frac{\hbar\omega}{2}}. \end{aligned}$$

For coherent states

$$\Delta X \Delta P = \sqrt{\frac{\hbar}{2\omega}}\sqrt{\frac{\hbar\omega}{2}} = \frac{\hbar}{2}.$$

therefore we have equality in the Heisenberg uncertainty relation.

□

L.2 Coherent states in position representation

Use

$$\hat{a}|z>= z|z>$$

to find the wavefunction in the position represenation.

Proof:

Let $|X>$ and $|P>$ be the eigenstates of $\hat{X}$ and $\hat{P}$. We write

$$\hat{a}|z>= \left(\sqrt{\frac{\omega}{2\hbar}}\hat{X} + i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}\right)|z>= z|z>$$

where

$$z = \sqrt{\frac{\omega}{2\hbar}}X_0 + i\frac{1}{\sqrt{2\hbar\omega}}P_0 \tag{L.1}$$

and act on the right with a state $< X|$ and obtain

$$< X|\left(\sqrt{\frac{\omega}{2\hbar}}\hat{X} + i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}\right)|z>= z < X|z>$$

Since $\hat{X}|X>= X|X>$ and $\hat{P} = -i\hbar\frac{\partial}{\partial X}$, we have $(\hat{X}|X>)^\dagger =< X|\hat{X}^\dagger = < X|(\hat{X} = (X|X>)^\dagger =< X|X$ and $(i\hat{P}|X>)^\dagger = -i < X|\hat{P}^\dagger = -i < X|\hat{P} = (\hbar\frac{\partial}{\partial X}|X>)^\dagger = -\hbar\frac{\partial}{\partial X} < X|$. That is,

$$< X|\hat{X} = < X|X \qquad \text{and} \qquad < X|i\hat{P} = \hbar\frac{\partial}{\partial X} < X|,$$

from which we obtain a differential equation

$$\sqrt{\frac{\omega}{2\hbar}} < X|z> X + \frac{1}{\sqrt{2\hbar\omega}}\hbar\frac{\partial}{\partial X} < X|z>= z < X|z>$$

or

$$\frac{\partial}{\partial X}\psi_z(X) = -\frac{\omega}{\hbar}\left(X - \sqrt{\frac{2\hbar}{\omega}}z\right)\psi_z(X)$$

This has the solution

$$\psi_z(X) = \mathcal{N} \exp - \left\{ \frac{\omega}{2\hbar} \left(X - \sqrt{\frac{2\hbar}{\omega}} z \right)^2 \right\}$$

or upon subsituting (L.1),

$$\begin{aligned}
\psi_z(X) &= \mathcal{N} \exp - \left\{ \frac{\omega}{2\hbar} \left(X - X_0 - i\frac{P_0}{\omega} \right)^2 \right\} \\
&= \mathcal{N} \exp - \frac{\omega}{2\hbar} \left\{ (X - X_0)^2 - 2i\,(X - X_0)\,\frac{P_0}{\omega} - \frac{P_0^2}{\omega^2} \right\} \\
&= \mathcal{N} \exp - \left\{ \frac{\omega}{2\hbar} (X - X_0)^2 - i\,(X - X_0)\,\frac{P_0}{\hbar} - \frac{P_0^2}{2\hbar\omega} \right\} \\
&= \mathcal{N} e^{P_0^2/2\hbar\omega} e^{-i\frac{X_0 P_0}{\hbar}} \exp - \left\{ \frac{\omega}{2\hbar} (X - X_0)^2 - \frac{i}{\hbar} X P_0 \right\}.
\end{aligned}$$

Normalisation requires

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi_z(X)|^2 dX \\
&= |\mathcal{N}|^2 e^{P_0^2/\hbar\omega} \int_{-\infty}^{\infty} \exp - \left\{ \frac{\omega}{\hbar} (X - X_0)^2 \right\} dX \\
&= |\mathcal{N}|^2 e^{P_0^2/\hbar\omega} \sqrt{\frac{\pi\hbar}{\omega}}.
\end{aligned}$$

We can choose $\mathcal{N}$ to be

$$\mathcal{N} = \left(\frac{\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-P_0^2/2\hbar\omega} e^{i\frac{X_0 P_0}{\hbar}}$$

so that

$$\psi_z(X) = \left(\frac{\omega}{\pi\hbar} \right)^{\frac{1}{4}} \exp - \left\{ \frac{\omega}{2\hbar} (X - X_0)^2 - \frac{i}{\hbar} X P_0 \right\}.$$

□

L.3 Direct derivation

We prove that the wavefunctions minimises the Heisenberg uncertainty relations, in configuration and momentum space respectively, are

$$
\begin{aligned}
\psi_z(x) &= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\omega}{2\hbar}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, & \text{(L.2)} \\
\psi_z(p) &= \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp - \left\{ \frac{\hbar}{2\omega} \frac{(p - P_0)^2}{\hbar^2} + \frac{i}{\hbar} p X_0 \right\}. & \text{(L.3)}
\end{aligned}
$$

where

$$
< \hat{X} >_{\Psi_z} = X_0, \quad \text{and} \quad < \hat{P} >_{\Psi_z} = P_0. \tag{L.4}
$$

Denote the fluctuation of an observable $\hat{O}$ by

$$
\Delta_{\Psi_z}(\hat{O}) = \left(< \hat{O}^2 >_\Psi - < \hat{O} >_\Psi^2 \right)^{\frac{1}{2}},
$$

and find that

$$
\Delta_{\Psi_z}(\hat{X}) \Delta_{\Psi_z}(\hat{P}) = \frac{\hbar}{2}, \qquad \frac{\Delta_{\Psi_z}(\hat{P})}{\Delta_{\Psi_z}(\hat{X})} = \omega. \tag{L.5}
$$

Proof:

$$
\begin{aligned}
(\Delta x)^2 &= < (x - < x >)^2 > = < x^2 > - < x >^2 \\
(\Delta p)^2 &= < (p - < p >) > = < p^2 > - < p >^2
\end{aligned}
$$

Define

$$
\alpha = x - < x >, \qquad \beta = -i\hbar \left(\frac{d}{dx} - \left\langle \frac{d}{dx} \right\rangle \right) \tag{L.6}
$$

then

$$
\begin{aligned}
(\Delta x)^2(\Delta p)^2 &= \int_{-\infty}^{\infty}\psi^*\alpha^2\psi dx\int_{-\infty}^{\infty}\psi^*\beta^2\psi dx \\
&= \int_{-\infty}^{\infty}(\alpha^\dagger\psi^*)(\alpha\psi)dx\int_{-\infty}^{\infty}(\beta^\dagger\psi^*)(\beta\psi)dx
\end{aligned}
\tag{L.7}
$$

where for β we have performed integration by parts and used that the normalised wave-function vanishes at $x = \pm\infty$.

We have the iequality

$$
\int\left|f - g\frac{\int fg^*dx}{\int |g|^2dx}\right|^2 dx \geq 0.
$$

Equality holds only if $f = \gamma g$. Expanding gives

$$
\begin{aligned}
\int\left|f - g\frac{\int fg^*dx}{\int |g|^2dx}\right|^2 &= \int\left(f^* - g^*\frac{(\int fg^*dx)^*}{\int |g|^2dx}\right)\left(f - g\frac{\int fg^*dx}{\int |g|^2dx}\right)dx \\
&= \int |f|^2dx + \int |g|^2dx\frac{\left|\int fg^*dx\right|^2}{\left(\int |g|^2dx\right)^2} - 2\frac{\left|\int fg^*dx\right|^2}{\int |g|^2dx} \geq 0
\end{aligned}
$$

so

$$
\int |f|^2dx\int |g|^2dx \geq |\int fg^*dx|^2
$$

If we know replace f by $\alpha\psi$ and g by $\beta\psi$, (L.7) becomes

$$
(\Delta x)^2(\Delta p)^2 \geq |\int(\alpha^\dagger\psi^*)(\beta^\dagger\psi)dx|^2 = |\int\psi^*\alpha\beta\psi|^2. \tag{L.8}
$$

The last term can be wrtten

$$
\begin{aligned}
&\left|\int\psi^*\left[\frac{1}{2}(\alpha\beta - \beta\alpha) + \frac{1}{2}(\alpha\beta + \beta\alpha)\right]\psi dx\right|^2 \\
&= \frac{1}{4}\left|\int\psi^*(\alpha\beta - \beta\alpha)\psi dx\right|^2 + \frac{1}{4}\left|\int\psi^*(\alpha\beta + \beta\alpha)\psi dx\right|^2 \\
&\qquad + \frac{1}{2}Re\left(\int\psi^*(\alpha\beta - \beta\alpha)\psi dx\int\psi^*(\alpha\beta + \beta\alpha)\psi dx\right).
\end{aligned}
$$

The third term vanishes because $\int \psi^*(\alpha\beta - \beta\alpha)\psi dx \int \psi^*(\alpha\beta + \beta\alpha)\psi dx$ is purely imaginary. In deriving the Heisenberg uncertainty principle we note that the second term $\frac{1}{4}\left|\int \psi^*(\alpha\beta + \beta\alpha)\psi dx\right|^2$ is positive and write

$$\begin{aligned}(\Delta x)^2(\Delta p)^2 \geq \left|\int \psi^* \alpha\beta\psi dx\right|^2 &\geq \frac{1}{4}\left|\int \psi^*(\alpha\beta - \beta\alpha)\psi dx\right|^2 \\ &= \frac{1}{4}\left|\int \psi^*(i\hbar)\psi dx\right|^2 \\ &= \frac{\hbar^2}{4}.\end{aligned}$$

The minimum uncertainty product is obtained onlt when the two conditions are fulfilled:

$$\alpha\psi = \gamma\beta\psi \tag{L.9}$$

$$\int \psi^*(\alpha\beta + \beta\alpha)\psi dx = 0. \tag{L.10}$$

Equations (L.9) and (L.6) give the differential equation

$$(x- < x >)\psi_0 = \gamma(-i\hbar\frac{d\psi_0}{dx} - < p > \psi_0)$$

or

$$\frac{d\psi_0}{dx} = \left[\frac{i}{\gamma\hbar}(x- < x >) + \frac{i < p >}{\hbar}\right]\psi_0$$

which is readily integrated

$$\int \frac{d\psi_0}{\psi_0} = \left[\frac{i}{\gamma\hbar}(x- < x >) + \frac{i < p >}{\hbar}\right] dx$$

so that

$$\ln\psi_0 = \frac{i}{2\gamma\hbar}(x- < x >)^2 + \frac{i < p > x}{\hbar} - \ln\mathcal{N}$$

(where $\mathcal{N}$ is an arbitrary constant) or

$$\psi_0(x) = \mathcal{N} \exp\left[\frac{i}{2\gamma\hbar}(x- < x >)^2 + \frac{i < p > x}{\hbar}\right] \tag{L.11}$$

Equation (L.10), using (L.9), becomes

$$\begin{aligned}
\int \psi_0^*(\alpha\beta + \beta\alpha)\psi dx &= \int \left(\psi_0^* \frac{1}{\gamma}\alpha^2 + (\beta^\dagger \psi_0^*)(\alpha\psi_0)\right) dx \\
&= \int \left(\psi_0^* \frac{1}{\gamma}\alpha^2 + (\frac{1}{\gamma^*}\alpha^\dagger \psi_0^*)(\alpha\psi_0)\right) dx \\
&= \left(\frac{1}{\gamma} + \frac{1}{\gamma^*}\right) \int \psi_0^* \alpha^2 \psi_0 dx = 0 \qquad \text{(L.12)}
\end{aligned}$$

which implies that γ is purely imaginary. Requiring the wavefunction (L.11) to be normalisable means γ is negative imaginary. We now obtain $\mathcal{N}$ from the mormalisation condition

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = 1,$$

that is

$$\begin{aligned}
1 &= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[\left(\frac{i}{2\gamma\hbar} - \frac{i}{2\gamma^*\hbar}\right)(x- < x >)^2\right] dx \\
&= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[-\frac{1}{|\gamma|\hbar}(x- < x >)^2\right] dx \\
&= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[-\frac{1}{|\gamma|\hbar}x^2\right] dx \\
&= \mathcal{N}^2 \sqrt{|\gamma|\hbar\pi}
\end{aligned}$$

so

$$\mathcal{N} = (|\gamma|\hbar\pi)^{-\frac{1}{4}}. \tag{L.13}$$

The value of γ can be found from

$$\int_{-\infty}^{\infty} (x- < x >)^2 |\psi_0(x)|^2 dx = (\Delta x)^2$$

that is

$$
\begin{aligned}
(\Delta x)^2 &= \frac{1}{\sqrt{|\gamma|\hbar\pi}} \int_{-\infty}^{\infty} (x- < x >)^2 \exp\left[-\frac{1}{|\gamma|\hbar}(x- < x >)^2\right] dx \\
&= \frac{1}{\sqrt{|\gamma|\hbar\pi}} \int_{-\infty}^{\infty} x^2 \exp\left[-\frac{1}{|\gamma|\hbar}x^2\right] dx \\
&= \frac{1}{\sqrt{|\gamma|\hbar\pi}} \frac{1}{2}\sqrt{(|\gamma|\hbar)^3\pi} \\
&= \frac{1}{2}|\gamma|\hbar. \qquad \text{(L.14)}
\end{aligned}
$$

So the wavefunction is

$$
\psi_0(x) = [2\pi(\Delta x)^2]^{-\frac{1}{4}} \exp\left[-\frac{(x- < x >)^2}{4(\Delta x)^2} + \frac{i < p > x}{\hbar}\right] \qquad \text{(L.15)}
$$

As the uncertainty priniple is saturated we have equality: $(\Delta x)(\Delta p) = \hbar/2$. As ω is defined by $\omega = \Delta p/\Delta x$,

$$
\left(\Delta_{\Psi_z}(\hat{X})\right)^2 = \frac{\hbar}{2\omega}, \qquad \left(\Delta_{\Psi_z}(\hat{P})\right)^2 = \frac{\hbar\omega}{2}. \qquad \text{(L.16)}
$$

and the wavefunction can be written

$$
\psi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{\omega}{2\hbar}(x- < x >)^2 + \frac{i < p > x}{\hbar}\right] \qquad \text{(L.17)}
$$

We now transform to the momentum representation

$$
\begin{aligned}
\psi_0(p) &:= \frac{1}{\sqrt{2\pi}\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi_0(x)dx \\
&= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}\hbar} \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}(x- < x >)^2 - \frac{i(p- < p >)x}{\hbar}\right] dx \\
&= \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}\left\{(x- < x >)^2 + \frac{i2(p- < p >)x}{\omega}\right\}\right] dx
\end{aligned}
\qquad \text{(L.18)}
$$

The contents of the curly brackets rearranged becomes

$$
\begin{aligned}
&(x- < x >)^2 + \frac{i2(p- < p >)x}{\omega} \\
&\quad = x^2 + 2\frac{- < x > \omega + i(p- < p >)}{\omega} x + < x >^2 \\
&\quad = \left(x + \frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 - \left(\frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 + < x >^2 \\
&\quad \mapsto x^2 - \left(\frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 + < x >^2 \\
&\quad = x^2 + \frac{(p- < p >)^2}{\omega^2} + 2\frac{i(p- < p >)}{\omega} < x > \qquad \text{(L.19)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\psi_0(p) \\
= \ & \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}\left\{x^2 + \frac{(p- < p >)^2}{\omega^2} + 2\frac{i(p- < p >)}{\omega} < x >\right\}\right] dx \\
= \ & e^{\frac{i<x><p>}{\hbar}} \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp\left[-\frac{\omega}{2\hbar}\frac{(p- < p >)^2}{\omega^2} - \frac{ip < x >}{\hbar}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}x^2\right] dx \\
= \ & e^{\frac{i<x><p>}{\hbar}} \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \sqrt{\left(\frac{2\pi\hbar}{\omega}\right)} \exp\left[-\frac{1}{2\hbar\omega}(p- < p >)^2 - \frac{ip < x >}{\hbar}\right] \\
= \ & e^{\frac{i<x><p>}{\hbar}} \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp\left[-\frac{\hbar}{2\omega}\frac{(p- < p >)^2}{\hbar^2} - \frac{ip < x >}{\hbar}\right] \qquad \text{(L.20)}
\end{aligned}
$$

The norm in the momentum representation is defined by

$$
\int_{-\infty}^{\infty} |\psi_0(p)|^2 \frac{dp}{\hbar}
$$

Normalisation is guaranted by

$$
\begin{aligned}
1 = \int_{-\infty}^{\infty} |\psi_0(x)|^2 dx &= \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}\hbar} \int_{-\infty}^{\infty} e^{ipx/\hbar} \psi_0(p) dp \right|^2 dx \\
&= \frac{1}{2\pi\hbar^2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi_0^*(p) dp \right) \left(\int_{-\infty}^{\infty} e^{i\tilde{p}x/\hbar} \psi_0(\tilde{p}) d\tilde{p} \right) dx \\
&= \frac{1}{\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\psi_0^*(p) \psi_0(\tilde{p}) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\tilde{p}-p)x/\hbar} \frac{dx}{\hbar} \right) \right] dp d\tilde{p} \\
&= \frac{1}{\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_0^*(p) \psi_0(\tilde{p}) \delta(p - \tilde{p}) dp d\tilde{p} \\
&= \int_{-\infty}^{\infty} |\psi_0(p)|^2 \frac{dp}{\hbar}.
\end{aligned}
$$

Bibliography

[1] Covariant Loop Quantum Gravity (an elementary introduction toquantum gravity and spinfoam theory) Carlo Rovelli and Francesca Vidotto.