Deriving the Frobenius Deterninant Formula and Hook Length Formula for the number of Standard Young Tablbleaux

Chapter 1

Hook Formula from Vandermonde Determinant

1.1 Primer on Standard Young Tableaux and Definitions

For any natural number n, we say λ is a partion of n, and write $\lambda \vdash n$, if λ is a sequence $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers satisfying:

1. $\sum_{i=1}^{k} \lambda_i = n$ and
2. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

Figure 1.1: Standard Young tableaux for $\lambda = (3, 2)$.

Let f_λ denote the number of standard Young tableaux of shape λ. Given a stadard Young tableau of shape λ with n cells, n must appear in one of the bottom-right corner boxes, it is immediate that removing the cell with n gives a standard tableaux of shape $\mu \to \lambda$. It is obvious that all standard tableaux of a shape proceeding λ can be obtained in this manner. Thus the number of standard tableaux satidfies the recurrence relation

$$f_\lambda = \sum_{\mu \to \lambda} f_\mu. \tag{1.1}$$

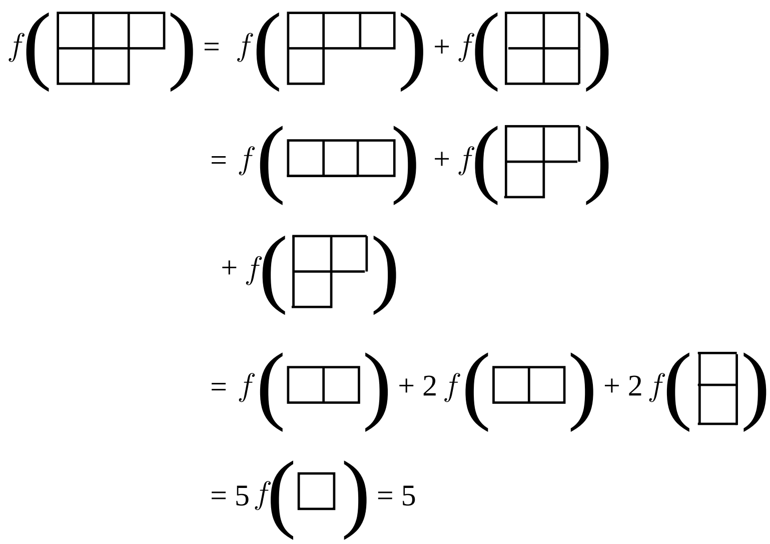

Figure 1.2: Using the recurrence relation to compute the number of standard Young tableaux.

1.2 The Vandermonde Determinant

The following determinant

$$\Delta_n(x_1,\ldots,x_n) = \det \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 \\ x_n & \cdots & x_3 & x_2 & x_1 \\ x_n^2 & \cdots & x_3^2 & x_2^2 & x_1^2 \\ x_n^3 & \cdots & x_3^3 & x_2^3 & x_1^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_n^{n-1} & \cdots & x_3^{n-1} & x_2^{n-1} & x_1^{n-1} \end{bmatrix} \tag{1.2}$$

is called the Vandermonde Determinant.

Let us consider this to be a polynomial in x_1 with $x_2, x_3 \ldots, x_n$, for the moment, treated as constants. We then have an $(n-1)-th$ order polynomial in x_1 and so has $n-1$ roots. The fact that the determinant vanishes when any two columns are equal it follows that $x_2, x_3, \ldots, x_n$ are the roots of this polynomial and implies

$$\Delta_n = (x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)\; q_{n-1}(x_2, x_3, \ldots, x_n). \tag{1.3}$$

We then consider $q_{n-1}(x_2, x_3, \ldots, x_n)$ as an $(n-2)-th$ polynomial in x_2. This has the $n-2$ roots $x_3, x_4, \ldots, x_n$ again following from the fact that the determinant vanishes when any two columns are equal (note that x_1 cannot be a root of $q_{n-1}(x_2, x_3, \ldots, x_n)$ as that would make Δ_n an $n+1-$th order polynomial in x_1). So that

$$\begin{aligned}\Delta_n &= [(x_1-x_2)(x_1-x_3)\dots(x_1-x_n]) \times \\ &\times[(x_2-x_3)(x_2-x_4)\dots(x_2-x_n)] \times q_{n-2}(x_3,x_4,\dots,x_n)\end{aligned} \tag{1.4}$$

Continuing in this way we obtain

$$\Delta_n(x_1,\dots,x_n) = \det\begin{bmatrix} 1 & \cdots & 1 & 1 & 1 \\ x_n & \cdots & x_3 & x_2 & x_1 \\ x_n^2 & \cdots & x_3^2 & x_2^2 & x_1^2 \\ x_n^3 & \cdots & x_3^3 & x_2^3 & x_1^3 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ x_n^{n-1} & \cdots & x_3^{n-1} & x_2^{n-1} & x_1^{n-1} \end{bmatrix} = K\prod_{i<j}(x_i-x_j) \tag{1.5}$$

It is easy to see that Δ_n is a polynomial of degree $n(n-1)/2$, and as the product $\prod_{i>j}(x_i-x_j)$ has the same degree means that $\Delta_n = K\prod_{i>j}(x_i-x_j)$ where K is a constant. We can determine K in the following way: First, note the in the expansion of the determinant we have the term $x_{n-1}\cdots x_2^{n-2}x_1^{n-1}$ with coefficient 1. Second, from the product

$$\begin{aligned} K\prod_{i<j}(x_i-x_j) &= K(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)\times \\ &\quad (x_2-x_3)\dots(x_2-x_n)\times \\ &\quad \vdots \\ &\quad (x_{n-1}-x_n) \end{aligned} \tag{1.6}$$

it is easy to see we have the term $Kx_1^{n-1}x_2^{n-2}\cdots x_{n-1}$. Thus $K=1$.

It has the following properties:

(a) When $n=1$ we have

$$\Delta_1(x_1) = 1. \tag{1.7}$$

(b) We have that

$$\begin{aligned} \Delta_n(\mu x_1,\dots,\mu x_n) &= \prod_{i<j}(\mu x_i - \mu x_j) \\ &= \mu^{n(n-1)/2}\prod_{i<j}(x_i-x_j) \\ &= \mu^{\deg \Delta_n}\Delta_n(x_1,\dots,x_n) \end{aligned} \tag{1.8}$$

(c) We have that

$$
\begin{aligned}
\Delta_{n+1}(x_1,\ldots,x_n,x_{n+1}=0) &= \det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & x_n & \cdots & x_3 & x_2 & x_1 \\ 0 & x_n^2 & \cdots & x_3^2 & x_2^2 & x_1^2 \\ 0 & x_n^3 & \cdots & x_3^3 & x_2^3 & x_1^3 \\ . & . & . & \cdots & . & \\ . & . & . & \cdots & . & \\ 0 & x_n^{n-1} & \cdots & x_3^{n-1} & x_2^{n-1} & x_1^{n-1} \\ 0 & x_n^n & \cdots & x_3^n & x_2^n & x_1^n \end{bmatrix} \\
&= \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 \\ x_n & \cdots & x_3 & x_2 & x_1 \\ x_n^2 & \cdots & x_3^2 & x_2^2 & x_1^2 \\ x_n^3 & \cdots & x_3^3 & x_2^3 & x_1^3 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ x_n^{n-1} & \cdots & x_3^{n-1} & x_2^{n-1} & x_1^{n-1} \end{bmatrix} \\
&= x_1x_2\cdots x_n \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 \\ x_n & \cdots & x_3 & x_2 & x_1 \\ x_n^2 & \cdots & x_3^2 & x_2^2 & x_1^2 \\ x_n^3 & \cdots & x_3^3 & x_2^3 & x_1^3 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ x_n^{n-1} & \cdots & x_3^{n-1} & x_2^{n-1} & x_1^{n-1} \end{bmatrix} \\
&= x_1x_2\cdots x_n\Delta_n(x_1,\ldots,x_n). \qquad (1.9)
\end{aligned}
$$

1.3 A Lemma

Lemma 1.3.1 *Let*

$$
\begin{aligned}
g(x_1,\ldots,x_m;y) &= x_1\Delta_m(x_1+y,x_2,\ldots,x_m)+ \\
&+ x_2\Delta_m(x_1,x_2+y,\ldots,x_n)+\cdots+x_m\Delta_m(x_1,x_2,\ldots,x_m+y)
\end{aligned} \qquad (1.10)
$$

Then

$$
g(x_1,\ldots,x_m;y) = \left(x_1+\cdots+x_m+\binom{m}{2}y\right)\ \Delta_n(x_1,\ldots,x_n). \qquad (1.11)
$$

Proof:

The function g is a homogeneous polynomial of degree $1 + \deg \Delta_m(x_1, \ldots, x_m)$ in the variables $x_1, \ldots, x_m, y$, that is:

$$g(\mu x_1, \ldots, \mu x_m; \mu y) = \mu^{1+\deg \Delta_m(x_1,\ldots,x_m)} g(x_1, \ldots, x_m; y).$$

If we interchange x_i and x_j, then the g changes sign. For example, if we interchange x_1 and x_2, then

$$\begin{aligned}
g(x_2, x_1, \ldots, x_m; y) &= x_2\Delta_m(x_2 + y, x_1, \ldots, x_m) + \\
&+ x_1\Delta_m(x_2, x_1 + y, \ldots, x_n) + \cdots + x_m\Delta_m(x_2, x_1, \ldots, x_m + y) \\
&= -x_2\Delta_m(x_1, x_2 + y, \ldots, x_m) + \\
&- x_1\Delta_m(x_1 + y, x_2, \ldots, x_n) + \cdots - x_m\Delta_m(x_1, x_2, \ldots, x_m + y) \\
&= -g(x_2, x_1, \ldots, x_m; y)
\end{aligned}$$

So if $x_i = x_j$, then g vanishes, i.e. g is divisible by $x_i - x_j$ and hence divisible by $\Delta_m(x_2, \ldots, x_m)$. If $y = 0$, the assertion is obvious. In order for the homogeneity condition to hold we must have:

$$g(x_1, \ldots, x_m; y) = (x_1 + \cdots + x_m + Cy)\ \Delta_n(x_1, \ldots, x_n)$$

where C is a constant. Therefore, we only have to prove that $C = \binom{m}{2}$. Let us consider

$$\begin{aligned}
&= x_1\Delta_m(x_1 + y, x_2, \ldots, x_m) \\
&= x_1(x_1 + y - x_2)(x_1 + y - x_3)\ldots(x_1 + y - x_n) \times \\
&\qquad (x_2 - x_3)\ldots(x_2 - x_n) \times \\
&\qquad\qquad \vdots \\
&\qquad (x_{n-1} - x_n) \\
&= \frac{x_1 y}{x_1 - x_2}\Delta_m(x_1, x_2, \ldots, x_m) + \frac{x_1 y}{x_1 - x_3}\Delta_m(x_1, x_2, \ldots, x_m) \\
&\quad + \cdots + \frac{x_1 y}{x_1 - x_n}\Delta_m(x_1, x_2, \ldots, x_m)
\end{aligned} \tag{1.12}$$

Next, let us consider

$$
\begin{aligned}
&= x_2\Delta_m(x_1, x_2+y, \ldots, x_m) \\
&= x_1(x_1-x_2-y)(x_1-x_3)\ldots(x_1+y-x_n)\times \\
&\quad (x_2+y-x_3)\ldots(x_2+y-x_n)\times \\
&\qquad\qquad \vdots \\
&\quad (x_{n-1}-x_n) \\
&= \frac{-x_2y}{x_1-x_2}\Delta_m(x_1,x_2,\ldots,x_m) + \frac{x_2y}{x_2-x_3}\Delta_m(x_1,x_2,\ldots,x_m) \\
&\quad +\cdots+\frac{x_2y}{x_2-x_n}\Delta_m(x_1,x_2,\ldots,x_m) \qquad (1.13)
\end{aligned}
$$

and so on. If we expand g, then the terms of degree 1 in y are $\frac{x_iy}{x_i-x_j}\Delta_m(x_1,\ldots,x_m)$ and $\frac{-x_jy}{x_i-x_j}\Delta_m(x_1,\ldots,x_m)$ for all pairs (i,j) with $1 \le i < j \le m$. The sum of these is obviously $\binom{m}{2}\Delta_n(x_1,\ldots,x_n)$.

$\square$

1.4 Satisfying Recursive Relations

We introduce a function f defined on the $m-$tuples $(\lambda_1,\ldots,\lambda_m)$, $m \ge 1$, with the following properties:

$$
\begin{aligned}
f(\lambda_1,\ldots,\lambda_m) &= 0 \qquad \text{unless } \lambda_1 \ge \lambda_2 \ge \cdots \ge 0; & (1.14)\\
f(\lambda_1,\ldots,\lambda_m,0) &= f(\lambda_1,\ldots,\lambda_m); & (1.15)\\
f(\lambda_1,\ldots,\lambda_m) &= f(\lambda_1-1,\lambda_2,\ldots,\lambda_m)+ \\
&\quad + f(\lambda_1,\lambda_2-1,\ldots,\lambda_m)+\cdots+f(\lambda_1,\ldots,\lambda_m-1) \\
&\qquad\qquad \text{if } \lambda_1 \ge \lambda_2 \ge \cdots \ge 0; & (1.16)\\
f(n) &= 1 \qquad \text{if } n \ge 0. & (1.17)
\end{aligned}
$$

We claim that $f(\lambda_1,\ldots,\lambda_m)$ counts the number of Young tableaux of shape $(\lambda_1,\ldots,\lambda_m)$. Condition (1.14) is obvious and so are (1.15) and (1.17).

To see that the number of Young tableaux of shape $(\lambda_1,\ldots,\lambda_m)$ satisfies (1.16), we consider the entry n. As already explained in the first section it must be the last entry in one of the rows, and if we remove the square with n, then we obtain a Young tableaux for $n-1$. If two rows are the same length, then n can only be the last entry in thhe lowest of these; buut including all terms in (1.16) does not matter since if, say, $n_1 = n_2$, then $f(\lambda_1-1,\lambda_2,\ldots,\lambda_m) = 0$.

Theorem 1.4.1 *The number of Young tableaux that have shape* $(\lambda_1, \ldots, \lambda_m)$ *satisfies*

$$f(\lambda_1, \ldots, \lambda_m) = \frac{\Delta(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m) n!}{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!}, \tag{1.18}$$

and in fact this formula for f *is correct if* $\lambda_1 + m - 1 \geq \lambda_2 + m - 2 \geq \cdots \geq \lambda_m$.

Proof:

We first observe that, if for some i we have

$$\lambda_i + m - i = \lambda_{i+1} + m - i - 1,$$

then the expression Δ on the RHS of (1.18) is 0, in accordance with the fact that the shape is not allowed.

Condition (1.17) follows from $\Delta(n) = 1$.

Condition (1.15) follows from the use of the identity (1.9):

$$\begin{aligned}
&f(\lambda_1, \ldots, \lambda_m, \lambda_{m+1} = 0) \\
&= \frac{\Delta(\lambda_1 + m + 1 - 1, \lambda_2 + m + 1 - 2, \ldots, \lambda_m + 1, 0) n!}{(\lambda_1 + m + 1 - 1)!(\lambda_2 + m + 1 - 2)! \cdots (\lambda_m + 1)!} \\
&= \prod_{i=1}^{m} (\lambda_i + m + 1 - i) \frac{\Delta(\lambda_1 + m + 1 - 1, \lambda_2 + m + 1 - 2, \ldots, \lambda_m + 1) n!}{(\lambda_1 + m + 1 - 1)!(\lambda_2 + m + 1 - 2)! \cdots (\lambda_m + 1)!} \\
&= \frac{\Delta(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m) n!}{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!} \\
&= f(\lambda_1, \ldots, \lambda_m).
\end{aligned} \tag{1.19}$$

where we used $\Delta(\lambda_1{+}m{+}1{-}1, \lambda_2{+}m{+}1{-}2, \ldots, \lambda_m{+}1) = \Delta(\lambda_1{+}m{-}1, \lambda_2{+}m{-}2, \ldots, \lambda_m)$, which follows from the fact the Δ depends on the differences between its arguments.

Finally, we come to condition (1.16). Substituting $x_i = \lambda_i + m - i$ and $y = -1$ into the definition of g

$$g(\lambda_1{+}m{-}1, \lambda_2{+}m{-}2, \ldots, \lambda_m; -1) = \sum_{i=1}^{m} (\lambda_i{+}m{-}i) \Delta_n(\lambda_1{+}m{-}1, \ldots, \lambda_i{+}m{-}i{-}1, \ldots, \lambda_m). \tag{1.20}$$

Substituting $x_i = \lambda_i + m - i$ and $y = -1$ into (1.11), and using $\sum_{i=1}^m (\lambda_i + m - i) - \frac{1}{2}m(m-1) = n$, gives:

$$g(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m; -1) = n\Delta_n(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m). \quad (1.21)$$

From (1.20) and (1.21) we have that

$$\frac{\Delta(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m)n!}{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!}$$
$$= \sum_{i=1}^{m} \frac{(\lambda_i + m - i)\Delta_n(\lambda_1 + m - 1, \ldots, \lambda_i + m - i - 1, \ldots, x_n)(n-1)!}{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!} \quad (1.22)$$

which is condition (1.16).

□

Formula (1.18) is known as the Frobenius Determinant Formula

1.5 Obtaining the Hooklength Formula from the Frobenius Determinant Formula

In the last section we obtained the following formula for the number of Young tableaux of shape $(\lambda_1, \ldots, \lambda_m)$:

$$f(\lambda_1, \ldots, \lambda_m) = \frac{n!}{\frac{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!}{\Delta(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m)}}. \quad (1.23)$$

Definition The hook corresponding to a cell is the union of the cell and all the cells to the right in the same row and all the cells below it in the same column. The hooklength is the number of cells in the hook.

We write

$$\frac{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!}{\Delta(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m)} = \prod_{i=1}^{m} \frac{(\lambda_i + m - i)!}{\prod_{j=i+1}^{m}[(\lambda_i + m - i) - (\lambda_j + m - j)]} \quad (1.24)$$

Let us look at some examples.

Example 1

We will find that the ith term in (1.24) is equal to the product of the hooklenghs corresponding to the ith row.

12	11	10	9	8	7	6

Figure 1.3: Hooklengths of the top row of a Young tableaux.

Let us consider the $i = 1$ term in (1.24) corresponding to the rectangular Young tableaux (fig 1.3).

$$\begin{aligned}
&\frac{(\lambda_1 + m - 1)!}{\prod_{j=2}^{m}[(\lambda_1 + m - 1) - (\lambda_j + m - j)]} \equiv \frac{(\lambda_1 + m - 1)!}{\prod_{j=2}^{m}[(\lambda_1 - \lambda_j + j - 1)]} \\
&= \frac{(\lambda_1 + m - 1)!}{(\lambda_1 - \lambda_6 + 6 - 1)(\lambda_1 - \lambda_5 + 5 - 1)\cdots(\lambda_1 - \lambda_2 + 2 - 1)} \\
&= \frac{(7 + 6 - 1)!}{(7 - 7 + 6 - 1)(7 - 7 + 5 - 1)(7 - 7 + 4 - 1)(7 - 7 + 3 - 1)(7 - 7 + 2 - 1)} \\
&= \frac{12!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}
\end{aligned} \tag{1.25}$$

This is the product of the hooklengths corresponding to the top row of the Young tableaux fig 1.3.

Let us consider the ith tem in the product (1.24). We have that

$$\begin{aligned}
&\frac{(\lambda_i + m - i)!}{\prod_{j=i+1}^{m}[(\lambda_i + m - i) - (\lambda_j + m - j)]} \equiv \frac{(\lambda_i + m - i)!}{\prod_{j=i+1}^{m}[\lambda_i - \lambda_j + j - i]} \\
&= \frac{(\lambda_i + m - i)!}{(\lambda_i - \lambda_m + m - i)(\lambda_i - \lambda_{m-1} + (m - 1) - i)\cdots(\lambda_i - \lambda_{i+1} + 1)} \\
&= \frac{(7 + 6 - i)!}{(m - 1)(m - 2)\cdots(7 - 7 + 2 - 1)} \\
&= (13 - i)(12 - i)\cdots(7 - i)
\end{aligned} \tag{1.26}$$

which is the product of hooklengths for the ith row. So we have for rectangular Young tableaux that

$$\frac{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!}{\Delta(\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m)} \tag{1.27}$$

is the product of all hooklengths.

Example 2

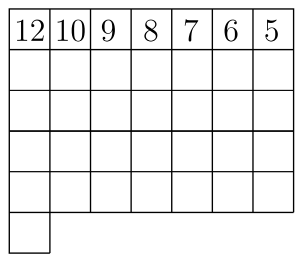

Figure 1.4: Hooklengths of the top row of a Young tableaux.

Let us consider the top row of the rectangular Young tableaux (fig 1.4).

$$\begin{aligned}
&\frac{(\lambda_1 + m - 1)!}{\prod_{j=2}^{m}[(\lambda_1 + m - 1) - (\lambda_j + m - j)]} \\
&= \frac{(7 + 6 - 1)!}{(7 - 1 + 6 - 1)(7 - 7 + 5 - 1)(7 - 7 + 4 - 1)(7 - 7 + 3 - 1)(7 - 7 + 2 - 1)} \\
&= \frac{12!}{11 \cdot 4 \cdot \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{12 \times 11}{11} \frac{10!}{4 \cdot 3 \cdot 2 \cdot 1}
\end{aligned} \tag{1.28}$$

Note how we have factorized the result in the last line. We will be doing a similar factorisation with other examples. This will be important in proving the result for a general Young tableaux.

Example 3

Let us consider the top row of the rectangular Young tableaux (fig 1.5).

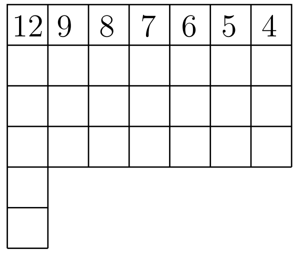

Figure 1.5: Hooklengths of the top row of a Young tableaux.

$$
\begin{aligned}
&\frac{(\lambda_1+m-1)!}{\prod_{j=2}^{m}[(\lambda_1+m-1)-(\lambda_j+m-j)]} \\
&= \frac{(7+6-1)!}{(7-1+6-1)(7-1+5-1)(7-7+4-1)(7-7+3-1)(7-7+2-1)} \\
&= \frac{12!}{11\cdot 10\cdot 3\cdot 2\cdot 1} \\
&= \frac{12\times 11\times 10}{11\cdot 10}\frac{9!}{3\cdot 2\cdot 1}
\end{aligned} \tag{1.29}
$$

Example 4

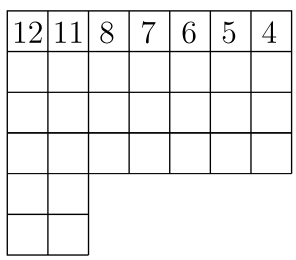

Figure 1.6: Hooklengths of the top row of a Young tableaux.

Let us consider the top row of the rectangular Young tableaux (fig **??**).

$$
\begin{aligned}
&\frac{(\lambda_1+m-1)!}{\prod_{j=2}^{m}[(\lambda_1+m-1)-(\lambda_j+m-j)]} \\
&= \frac{(7+6-1)!}{(7-2+6-1)(7-2+5-1)(7-7+4-1)(7-7+3-1)(7-7+2-1)} \\
&= \frac{12!}{10\cdot 9\cdot 3\cdot 2\cdot 1} \\
&= \frac{12\times 11\times 10\times 9}{10\times 9}\frac{8!}{3\cdot 2\cdot 1}
\end{aligned} \tag{1.30}
$$

1.5.1 General case

We will now move onto the general case. Will we be considering the ith term of the product on the RHS of (1.24):

$$\begin{aligned} & \frac{(\lambda_i + m - i)!}{\prod_{j=i+1}^{m}[(\lambda_i + m - i) - (\lambda_j + m - j)]} \\ &= \frac{(\lambda_i + m - i)!}{\prod_{j=i+i}^{m}[(\lambda_i - \lambda_j + j - i)]} \\ &= \frac{(\lambda_i + m - i)!}{(\lambda_i - \lambda_m + m - i)(\lambda_i - \lambda_{m-1} + (m-1) - i)\cdots(\lambda_i - \lambda_{i+1} + 1)} \end{aligned} \tag{1.31}$$

A rectangular Young tableaux

First say we have a rectangular Young tableaux with, then (1.31) is equal to

$$(\lambda_i + m - i)(\lambda_i + m - 1 - i)\cdots(m + 1 - i),$$

which is the product of hooklengths for the ith row.

A general Young tableaux

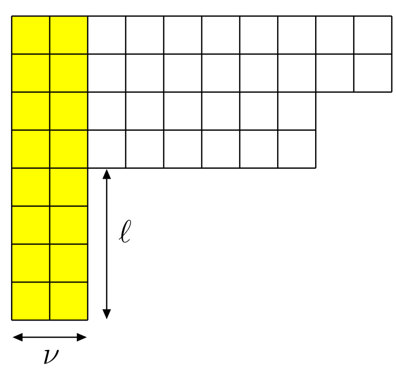

Figure 1.7: General Young tableaux.

Next we consider the case where for $\lambda_m = \lambda_{m-1} = \cdots \lambda_{m-\ell} = \nu$.

Then the ith term (1.31), for $1 \geq i \geq m - \ell$, is equal to

$$
\begin{aligned}
&\frac{(\lambda_i+m-i)!}{\prod_{j=i+1}^{m}[(\lambda_i+m-i)-(\lambda_j+m-j)]} = \frac{(\lambda_i+m-i)!}{\prod_{j=i+1}^{m}[(\lambda_i-\lambda_j+j-i)]}\\
&= \frac{(\lambda_i+m-i)!}{(\lambda_i-\lambda_m+m-i)\cdots(\lambda_i-\lambda_{m-\ell+1}+m-\ell+1-i)(\lambda_i-\lambda_{m-\ell}+m-\ell-i)\cdots(\lambda_i-\lambda_{i+1}+1)}\\
&= \frac{(\lambda_i+m-i)!}{(\lambda_i-\nu+m-i)\cdots(\lambda_i-\nu+m-i-\ell+1)(\lambda_i-\lambda_{m-\ell}+m-\ell-i)\cdots(\lambda_i-\lambda_{i+1}+1)}\\
&= (\lambda_i+m-i)(\lambda_i+m-i-1)\cdots(\lambda_i+m-i-\nu+1)\times\\
&\prod_{i=1}^{m-\ell}\frac{(\lambda_i-\nu+m-\ell-i)!}{\prod_{j=i+1}^{m-\ell}[(\lambda_i-\lambda_j+j-i)]}\\
&(\lambda_i+m-i)(\lambda_i+m-i-1)\cdots(\lambda_i+m-i-\nu+1)\times\\
&\prod_{i=1}^{m-\ell}\frac{(\lambda_i-\nu+m-\ell-i)!}{\prod_{j=i+1}^{m-\ell}[(\lambda_i-\nu+m-\ell-i)-(\lambda_j-\nu+m-\ell-j)]}
\end{aligned}
\tag{1.32}
$$

which is the product of hooklengths for first ν cells in th ith row times the equivalent of (1.31) for the Young tableaux obtained by removing the first ν columns, depicted in fig 1.8 (b).

For $m-\ell+1 \geq i \geq m$:

$$
\begin{aligned}
&\frac{(\lambda_i+m-i)!}{\prod_{j=i+1}^{m}[(\lambda_i+m-i)-(\lambda_j+m-j)]} = \frac{(\lambda_i+m-i)!}{\prod_{j=i+1}^{m}[(\lambda_i-\lambda_j+j-i)]}\\
&= \frac{(\lambda_i+m-i)!}{(\lambda_i-\lambda_m+m-i)(\lambda_i-\lambda_{m-1}+m-i-1)\cdots(\lambda_i-\lambda_{i+1}+1)}\\
&= \frac{(\lambda_i+m-i)!}{(m-i)(m-i-1)\cdots(m-i-\ell+1)\cdots(\lambda_i-\lambda_{i+1}+1)}\\
&= (\lambda_i+m-i)\cdots(m+1-i)
\end{aligned}
\tag{1.33}
$$

Which is the product of hooklengths for the ith row, $m-\ell+1 \geq i \geq m$.

Substituting (1.32) and (1.33) into (1.24) we obtain

$$
\begin{aligned}
&\frac{(\lambda_1+m-1)!(\lambda_2+m-2)!\cdots\lambda_m!}{\Delta(\lambda_1+m-1,\lambda_2+m-2,\ldots,\lambda_m)}\\
&=\prod_{i=1}^{m}\frac{(\lambda_i+m-i)!}{\prod_{j=i+1}^{m}[(\lambda_i+m-i)-(\lambda_j+m-j)]}\\
&=\prod_{i=m-\ell+1}^{m}[(\lambda_i+m-i)\cdots(m+1-i)]\times\prod_{i=1}^{m-\ell}[(\lambda_i+m-i)\cdots(\lambda_i+m-i-\nu+1)]\\
&\times\prod_{i=1}^{m-\ell}\frac{(\lambda_i-\nu+m-\ell-i)!}{\prod_{j=i+1}^{m-\ell}[(\lambda_i-\nu+m-\ell-i)-(\lambda_j-\nu+m-\ell-j)]}
\end{aligned}
\tag{1.34}
$$

The third line is the product of hooklengths of the yellow cells of fig 1.7. The fourth line is the equivalent of the second line but for the Young tableaux obtained by removing the first ν columns, depicted in fig 1.8 (b).

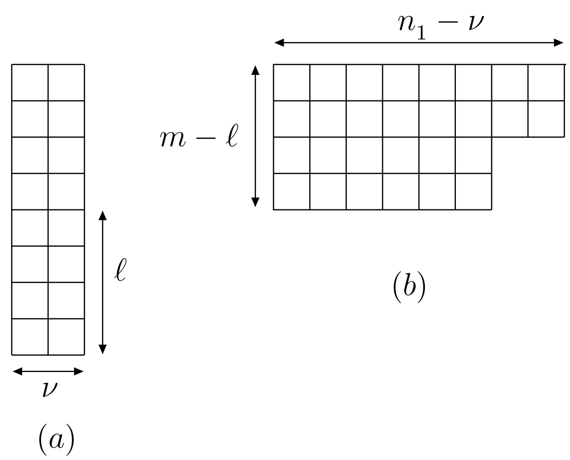

Figure 1.8: General Young tableaux.

We can continue the above process until we are left with a rectangular Young tableaux; in the case of a rectangular Young tableaux we already know that () is equal to the product of all hooklengths. We have proved that for the original Young tableaux that

$$
\frac{(\lambda_1+m-1)!(\lambda_2+m-2)!\cdots\lambda_m!}{\Delta(\lambda_1+m-1,\lambda_2+m-2,\ldots,\lambda_m)}=\prod h_\lambda(p,q) \tag{1.35}
$$

where $h_\lambda(p,q)$ denotes the hooklength of the cell (p,q) and where the product is over all cells in the Young tableaux.

Theorem 1.5.1 *The number of Young tableaux of a given shape and a total of n cells is $n!$ divided by the product of all hooklengths.*

$$f^{\lambda} = \frac{n!}{\prod h_{\lambda}(p,q)} \tag{1.36}$$

where the product is over all cells in the Young tableaux.

Proof:

Substitute (1.35) into (1.23).

$\square$