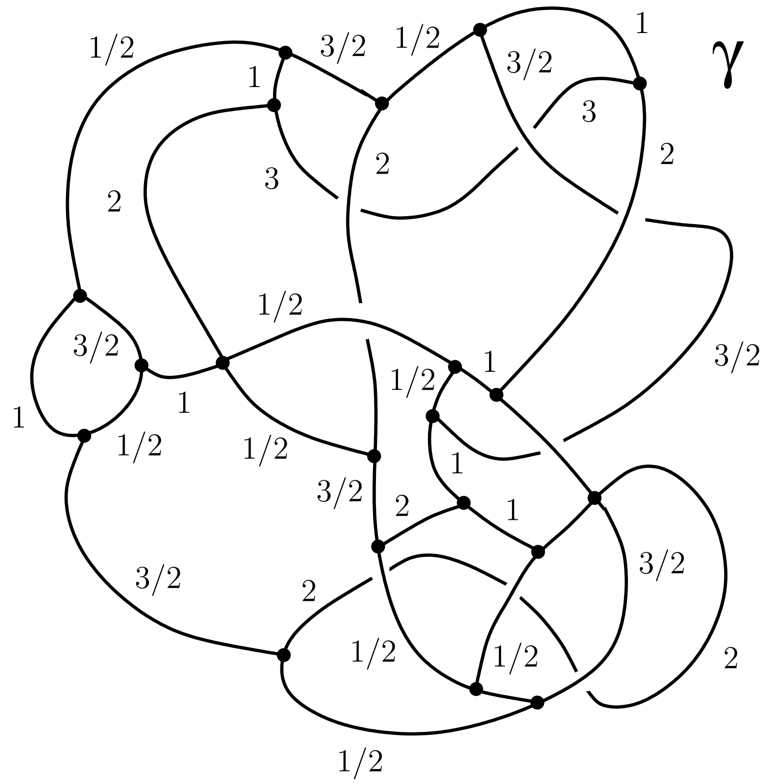
Volume II
Loop Quantum Gravity



Draft version

By

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- Physics Glossary
- Maths Glossary

- Many Detailed Appendices with Worked Exercises

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• Conceptual issues.

• Relational mechanics.

• Partial, complete and Dirac observables.

• The problem of quantizing gravity.

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• Canonical Quatization Of GR (Breif Sketch).

• Ashtekar's New Variables.

• Loops and the Loop Representation.

• Difficulties.

Chapter 3: Formal Developments (1992-2005)

• Spin networks: how to generate independent Wilson loops

• Area and volume operators and the quantum geometry of space.

• Functional integration and the inner product.

• Spatial Invariant Hilbert Space and spin network states.

• Uniqueness of kinematic representation.

Chapter 4: Dynamics: The Hamiltonian constraint (1994-1997) and Spin foams (1994-present)

• Real Formalism and the Hamiltonian constraint.

• Quantization of the Hamiltonian constraint.

• Introduction to Spin foams - the quantum geometry of spacetime.

• Unsettled concerns.

• Consistent discrete quantum gravity.

• Proposal for a reduced phase space quantization of gravity coupled to matter.

Chapter 5: Physical Applications of LQG: Black Hole Entropy and Loop Quantum Cosmology

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• Microscopic Source of Black Hole Entropy.

• Loop Quantum Cosmology.

• Inflation From Loop Quantum Cosmology.

• A Quantum Black hole.

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• Possibility of having:
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(ii) Control over Quantum Dirac Observables of LQG,
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• Quantization of the Master Constraint Operator for GR.

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• 3D Quantum Gravity

• 4D Lorentzian Spinfoams

• Semi-Classical Limit

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• The Problem of Time.

• Covariant quantum mechanics.

• Multiple-Event Probability.

• General boundaries formulation of quantum mechanics.

• Emergence of Temporal Phenomena.

• Consistent discrete quantum gravity.

• Bearing of Matters of Quantum Gravity on Interpretations of QM

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Terminology and Notation

Here is a list of symbols.

$[\ ,\ ]$	commutator
$\{\ ,\ \}$	Poisson bracket
$\dagger$	Hermitian conjugation
$:=$	definition
$\equiv$	identity
$\stackrel{*}{=}$	only true in a special coordinate system
iff	If and only if
η_{ab}	Minkowski metric
$\eta(x)$	test function of a variation of action
$\mathcal{A}$	space of gauge fields or area
$A_\mu(x)$	Yang-Mills connection
D_μ	covariant derivative
$\mathcal{M}$	spacetime manifold
$\mathbf{M}$	The Master consraint
$\hat{\mathbf{M}}$	The Master consraint operator
$\omega^\alpha_{\mu\beta}$	spin connection
$\mathcal{C}$	constraint surface in phase space
S	labells spin-network
s	equivalent class of spin-networks under the action of Diff denoted $s-$ knots
$s(S)$	denotes equivalent class S to which belongs
g_{ab}	spacetime metric
K_{ab}	extrinsic curvature of Σ
G_{ab}	Einstein tensor
T_{ab}	The energy-momentum tensor
e^a_I, E^a_i	tetrad and triad
$\mathcal{L}_t$	Lie derivative with respect to t
n_a	unit normal to Σ_t
N, $(\tilde{N})$	lapse function (density)
N^a	shift vector on Σ
$\Omega_{\alpha\beta}$	sympletic form
$\mathcal{A}/\mathcal{G}$	space of gauge fields moduli gauge transformations
$[A]$	gauge equivalence classe of the connection A
$\mathcal{HA}$	the holonomy algebra
$\overline{\mathcal{HA}}$	the completion of the holonomy algebra in the norm $\\|f\\| := \sup_{[A]\in\mathcal{A}/\mathcal{G}} \mid f([A]) \mid$
$\overline{\mathcal{A}/\mathcal{G}}$	spectrum of $\overline{\mathcal{HA}}$

Preface

> So how do you go about teaching them something new? By mixing what they know with what they don't know. Then, when they see in their fog something they recognize they think, "Ah, I know that!" And then it's just one more step to "Ah, I know the whole thing." And their mind thrusts forward into the unknown and they begin to recognize what they didn't know before and they increase their powers of understanding"
>
> PICASSO

Warning: We are sure there are lots of mistakes in these notes. Use at your own risk! Corrections and other feedback would be very appreciated.

These notes are meant to be supplementary to Loop quantum gravity literarture. It is explicit (fills in all the gaps normally left out of the calculations);It is a self-contained and bringing together of relavant maths. Blow-by-blow account. Present material in lauguage more familiar to a broarder community of theoretical physists. To which modern coordinate-free differential geometry is unfamiliar and possibly daunting.

Style of the Book:

The report is mainly aimed at theoretic physicists who are non-experts. fill in calculations an details that are skipped over in th litrature. people who we have tried to make it accessible to people who are not familiar with modern methods and will have no need to apply these methods in their field of study. for example coordinate-free diffential gometry

I should stress at the beginning that I am a physicist and not a mathematician will not always be at a level of rigour that would satisfy a proper mathematical physicist.

This report assumes the reader is somewhat familiar with special relativity and it would help if he/she has knowledge with general realtivity at the undergraduate level. The treatment is self-contained in that an a priori knowledge is not assumed. Try to bring together many useful results. Many proofs are added and others expanded on to make them more accessible.

The many excellent expositions of Loop-quantum gravity

The initial idea was to try to present the subject at a level lower than in most of the litrature. It may take the reader rather long time to make his/her way through the report.

the report offer something of value for the mathematically inclined and the not so alike.

[32] T. Thiemann, Introduction to Modern Canonical Quantum General Relativity, [gr-qc/0110034]

I'm not really an expert and I will try to indicate where my understanding is a bit more iffy. At the moment it is an enthusiastic if not entirely reliable account, which I hope to improve on this through critisms, suggestions...

The purpose of these notes, not to replace any book or paper. You should turn to other source for other explainations.

Nevertheless, these notes written to be reasonably self-contained and comprehensible.

I make no claim to originality in these notes.

Often follows quite closely the treatments of

I have used a number of sources.

though it is not indispensable and understanding for the report.

This discussion is intended to provide only a passing familiarity with the scheme to allow the unfamiliar reader to follow certain calculations and to have a general understanding of the results.

Acknowledgments

Dedicated

Introduction

the beginning of the revolutionary contributions to physics by Einstein,

An attempt to apply the principles of quantum mechanics to general relativity.

The problem of merging gravitation and quantum mechanics is extrememly difficult - it has defied solution for over seventy years. Classical theory developed by Einstein took 10 years. the principles of quantum mrchanics and general relativity.

Canonical quantization is the oldest non-perturbative approach to quantization of general relativity.

"Is it just me or is quantum gravity easier nowadays?"

Didn't think he would see the completion of quantum gravity in his own life time. dead and buried. There is now hope (not a word usually assocciated with quantum gravity for a lot of people).

In spite of these compelling features... Quantizing gravity equivalent to finding a general solution to the classical field equations. The techniqual difficulties seemed even worse than non-renormalization of perturbative quantum gravity. Seem hopelessly difficult.

A school of thought has a long history in studies of quantum gravity. The viewpoint here is that it may well be possible to quantise pure general relativity consistently, and in a way that respects the geometrical framework of the classical theory, but to do so requires the use of techniques that are quite different from the weak-field perturbative methods that, for example, have dominated most particle-physics based approaches to quantum gravity. Much effort has been devoted to finding such nonperturbative schemes, and in this paper we will be concerned with a particular one that has evolved from the introduction of a new set of canonical variables to describe the phase space of classical general relativity.

A matter of getting the right perspective on the problem. General relativists have argued for decades that gravity must be quantized non-perturbativly - easily said that done!!

List of acheivements:

(1) Predcition of quantized strectrum for area and volume operators.

(2) Black hole entropy from first principles for physical black holes in that

(i) non-exremal black holes non-rotating black holes with Maxwell charges and dilatonic charges (1998), and recently has even been extended to rotating black holes and black holes with arbitrary disortions, also

(ii) definition of black hole applies restrictions to horizon geometry only and not to the bulk spacetime outside the black hole. As a consequence, we can have time varying dynamics outside black hole; also means results extend to cosmological horizons too.

(3) Resolution of the big bang singularity.

(4) initial boundary conditions for the universe are determined, rather than guessed at.

(5) New mechansim for inflation from quantum geometry.

(6) A mathematically consistent quantum theory of gravity coupled to the standard model - explicitly demonstrated to be finite (1997). However serious worries about whether it is physically consistent - in particular whether it has the correct classical limit.

Progress in conceptual no less dramatic

(1) Relational quantum mechanincs.

If it ultimently the case that it does not lead to the correct theory, we will have leared many things of value along the way.

The book is primary on loop quantum gravity, however: Brian Greene in The Fabric of the Cosmos he says

"If I were to hazard a guess on future developments, I'd imagine that the background-indpendent techniques developed by the loop quantum gravity community will be adapted to string theory, paving the way for a string formulation that is background indepndent. And that spark, I suspect, that will ignite a third susperstring revolution in which, I'm optimistic, many of the remaining deep mysteries will be solved".

Paths through the report

Introductory book on general relativity [**?**]

T. Thiemann, Introduction to Modern Canonical Quantum General Relativity, [gr-qc/0110034] [32]

Chapter 1

Classical GR, Einstein's hole argument and Physical Geometry

A warning is given to the reader that the material covered in this chaprter is not usually taught in GR courses or appear in many books. Do not skip this chapter as it is essential before progressing to loop quantum gravity.

Gravity and spacetime are the same entity. Spacetime is best described by a metric field. The field equations related to the energy-momentum tensor. It turns out to require even more sweeping revisions of our spacetime concepts. Wont have been taught in lecture courses in GR and many may not be awear of its . This amazing property hidden in GR has not been fully absorbed by the (theoretical) pysics community. The significance of this final step may be unlikely to be familiar to the reader and he/she may be shocked by the outcome.

The distinction between a vector and its vector components with respect to a coordinate induced basis vectors. It is importantant that in the following discourse, the two terms should not be confused.

spacetime is still understood as a *non-dynamical* entity which provides an arena for the laws of physics but does not itself take part.

Rubber-Sheet Analogy of Curved Space Time

One way to think of General Relativity is to use the idea of a "rubber sheet geometry" where in the absence of gravitation the sheet is flat, but a central massive body curves up the sheet in its vicinity so that a free body (which would otherwise have moved in a straight line) is forced to orbit the central body curved space is simila in nature to the curved surface of a rubber sheet. Helpfull in proving some insight into the idea of a curved spacetime.

Claims that GR can be If the marble doesn't weight too much it will move in a straight line when progected across the sheet. This

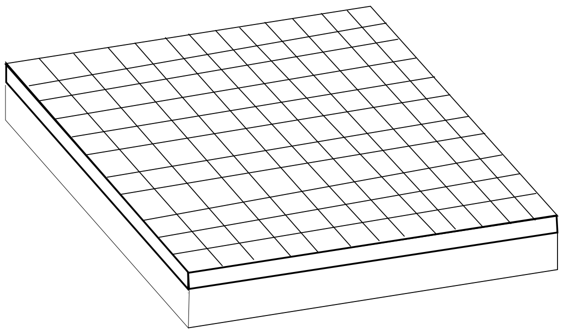


Figure 1.1: Rubbersheet simulation of geodesic moton in special relativity.

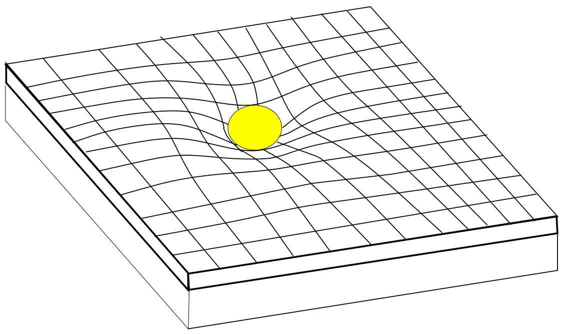


Figure 1.2: rubbersheet. It doesnot matter that the coordinates are time-dependent - it still serves as a physical refference system.

In general relativity any change in mass of the central sourse will spread out like a ripple in the rubber-sheet geometry

Geometry, Coordinates and Metrics

Before we go any further we need to clarify a few things.

The components of the field in the basis induced by the coordinate system.

In the general theory of relativity, the square of the interval is given by:

$$ds^2 = g_{ab}\, dx^a\, dx^b, \tag{1.1}$$

where g_{ab} are the components of the metric tensor, and dx^a is the differentai of the coordinate x^a. his struggle to understand "the meaning of the coordinates".

A coordinate sytem in space time $\mathcal{M}$ permits us to associate, with each space-time event of $\mathcal{M}$ four real numbers, the values of x, y, z, and t. To locate a point in space time requires specification of these four numbers.

The description will depend on the identificatoin of each point - the choice of coordinate system, and also on the directions chosen for the coordinate axes.

Let us denote the distance between two spacetime points P and Q as described by the metrics $g_{ab}(x)$ as $d_g(P, Q)$.

$$d\tau^2 = g_{ab}(x)\, dx^a\, dx^b \tag{1.2}$$

We have

$$d_g(P, Q) = \int_P^Q d\tau = \int_P^Q \left(g_{ab}(x) dx^a dx^b\right)^{1/2} \tag{1.3}$$

The description of a particular spacetime geometry will vary with the choice of coordinate system, however the distance between two spacetime points should be the same in every coordinate system. The interval

$$d\tau^2 = g_{ab}(x)\, dx^a\, dx^b = g'_{ab}(y)\, dy^a\, dy^b \tag{1.4}$$

is invariant. The coordinate differentials are different in the new coordinate system, so that must change in such a way that $d\tau^2$ is invariant. $g'_{ab}(y)$ will in general be *functionally different* from $g_{ab}(x)$ (i.e the components will involve different functions of its coordinates compared with g). It gives the same distance between space-time events and yeilds the same angle between the same two vectors. it represents the same physical solution, veiwed from a different coordinate system.

1.1 Einstein's Hole Argument

General Covariance

In special relativity there are special coordinate frames, called **inertial frames** i.e. moving with constant velocity (we'll not go into anymore detail with regards to this). The description in one frame x and another frame x' are related to each other via a Lorentz transformation

Maxwell's equations have the same form in all inertial frames. Written as equations whose form is invariant under Lorentz transformations. General covariance is the idea that the laws of nature must be the same in all reference frames, and hence all coordinate systems.

Since gravity distorts spacetime geometry there are no global inertial reference frames. That there should not be any privileged reference systems in which the laws of nature should be written. The laws of nature must therefore take the same form in all coordinate systems. :

General covariance is the idea that the laws of nature must be the same in all reference frames, and hence all coordinate systems.

It is always possible to write laws in a coordinate-independent way. A physical system acting in a certain way doesn't know which coordinate system you are using to describe it. There is a more subtle point of the assertion of general covariance,

Special coordinates should not have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates.

This requirement is **NOT** the same as the more generally quoted 'principle of general covariance' stated above!!

which is that there should be **no** preffered coordinate system that has a **physical role** to play in the formulation of the equations of motion. Let us illustrae this with a simple analagy: what is the difference between Newtonian theory and special relativity? - Newtons equations of motion can be written in a Lorentz covariant form - do we conclude that Lorenzt covariance has no physical content? No, the point is that in Newton's theory there is a special frame that plays a **physical role** in formulating the equations of motion! The frame that is at rest with respect to this notion that we call the ether. If we were to insist on modifying the equations of motion in special relativity, that there should be a special inertial systems that play a role in formulating the EQM, then we would be reintroducing the Ether at least to some observers. Similarly in GR there should be **no preffered coordinate system** that plays a physical role to play in formulating the field equatios, for this would reintroduce gravity as a force, at least to some observers. - we would have lost the equivalence principle.

These arguments . In the next section we will by using very basic ideas of GR, introduced in the previous section and which should be familiar to anyone who knows anything about GR, together with some fairly straightforward maths to demonstrate a striking and, at first sight, rather alarming consequence of general covariance for GR.

General Covariance (Pure Gravity - No Matter)

a trival mathematical observation. Take the two differential equations (1.5) and (1.6). They both have the same mathematical form.

$$\frac{d^2 f(x)}{dx^2} + \omega^2 f(x) = 0, \tag{1.5}$$

$$\frac{d^2 g(y)}{dy^2} + \omega^2 g(y) = 0. \tag{1.6}$$

Say we find out that one solution to Eq(1.5) is

$$f(x) = \cos \omega x, \tag{1.7}$$

we immediatly know that

$$g(y) = \cos \omega y \tag{1.8}$$

solves Eq.(1.6). Eq. (1.8) is the same function as Eq (1.7), written as a function of y instead. It is always possible to write laws in a coordinate-independent way. A physical system acting in a certain way doesn't know which coordinate system you are using to describe it.

However, general covariance also implies a *distinct solution relating to the y-coordinate system*, which is pretty obvious once you think about it. single coordinate system this). We take these one metric function other metric is of its Now, general covariance say the laws of physics should be the same for all reference systems, demanding that the equations of motion have the same form in both our coordinate systems. So you have exactly *the same differential equation to solve in both these coordinate systems, execpt in one the independent variables are the x-coordinates and in the other the independent variables are the y-coordinates.*

$$R_{ab}(x) = 0 \tag{1.9}$$
$$R_{ab}(y) = 0 \tag{1.10}$$

Once you find out that a solution to the EOM in the x-coordinates (1.22) is, say

$$g_{00}(x) = \cos t, \quad g_{11} = x^2, \quad g_{11}, \quad \text{etc} \tag{1.11}$$

then one imeddiately knows that

$$g_{00}(y) = \cos t, \quad g_{11}(y) = y^2, \quad g_{11}(y), \quad \text{etc} \tag{1.12}$$

solves the EOM in the y-coordinates (1.23).

Once this observation is noted, it then becomes evident that if one of our metric functions is a solution then, at the same time, so is the other! Let us denote this metric function as $\tilde{g}_{ab}(y)$.

However, general covariance also implies a *distinct solution relating to the y-coordinate system*, which is pretty obvious once you think about it: General covariance demands that the equations of motion have the same form in all coordinate systems - so we have exactly the same diffrential equation to solve in both coordinate systems, (except, of course, in one the independent variable is x and in the other it is y). Say we have a metric function in the x-coordinates that is a solution, then at the same time, so is the metric function in the y-coordinates sytsem that has the same functional form! (The same functional form meaning it is the same function except that it is a function of y instead of x). Let us denote this metric function as $\tilde{g}_{ab}(y)$.

$$\boxed{g_{ab}(x = u) = \tilde{g}_{ab}(y = u).} \tag{1.13}$$

To see this more clearly, perform a coordinate transformation on the metric $\tilde{g}_{ab}(y)$.

$$\tilde{g}'_{ab}(x) = \Lambda_a^c \Lambda_b^d \tilde{g}_{cd} \tag{1.14}$$

Where $\Lambda_a^c = \frac{\partial x^c}{\partial y^a}$ is the (inverse) Jacobian matrix for the coordinate transformation $y^a = y^a(x^b)$. $\Lambda_a^c = \delta_a^c$.

$$d_{\tilde{g}}(P, Q) = \sum_P^Q \left(\tilde{g}_{ab}(y) dy^a dy^b \right)^{1/2} = \sum_P^Q \left(\tilde{g}'_{ab}(x) dx^a dx^b \right)^{1/2}. \tag{1.15}$$

Unless $\Lambda_a^c = \delta_a^c$.

$$\tilde{g}'_{ab}(x) \neq g_{ab}(x) \tag{1.16}$$

$$d_g(P, Q) \neq d_{\tilde{g}}(P, Q) \tag{1.17}$$

General covariance implies that, once we find one solution to the equations of motion of GR, it immediately gives rise to other **distinct solutions*****, one for each conceivable coordinate system!!!***

Illustrative examples

(a)

from blind mathematical point of view

$$\frac{d^2 f(x)}{dx^2} + \omega^2 f(x) = 0, \tag{1.18}$$

$$\frac{d^2g(y)}{dy^2} + \omega^2 g(y) = 0. \tag{1.19}$$

Say we find out that one solution to Eq(1.18) is

$$f(x) = \cos \omega x, \tag{1.20}$$

we immediatly know that

$$g(y) = \cos \omega y \tag{1.21}$$

solves Eq.(1.19).

$$R_{ab}(x) = 0 \tag{1.22}$$
$$R_{ab}(y) = 0 \tag{1.23}$$

Once you find out that a solution to the EOM in the x-coordinates (1.22) is, say

$$g_{00}(x) = \cos t, \;\; g_{11} = x^2, \;\; g_{11}, \quad \text{etc} \tag{1.24}$$

then one imeddiately knows that

$$g_{00}(y) = \cos t, \;\; g_{11}(y) = y^2, \;\; g_{11}(y), \quad \text{etc} \tag{1.25}$$

solves the EOM in the y-coordinates (1.23).

I should include what was explained in [401]: Higher-dimensional Algebra and Topological Quantum Field Theory

Intuitive Explanation

The rubber sheet sits in space - this ambiant space that asigns a length to the curve. Also, there is absolute time which clocks exemplify - Newton's laws tell you the position of the rubber sheet in the ambient space at such and such a time.

However, in GR spacetime is not embeded in some further container - hence GR does not predict the proper time. The action principle describing the dynamics of the rubber sheet refers to things external to the dynamical system the theory describes i.e. absolute time and the ambient space, whereas the Einstein-Hilbert action does not.

The information contained in the solution is the value of the matter field at the place where the gravitational field takes such and such a value, for every spacetime point. This information is preserved if we actively drag the gravitational field simultaniously over the spacetime manifold. Seeing as no localization over spacetime should be preffered the two configurations are both solutions.

More subtle point - is that Pre-Maxwell there was a priveledged inertal frame in which the EOM take a particular simple form - the frame at rest with respect to the eather. If we were to drop the assumption that there is a favered frame would reintroduce the eather to at least some observers. Similarly, if there was natures priledged frame in which the fields takes a particularly simple form then we would introduce gravity as a force at least to some observers. Should note that this is what Einstein was attempty for 4 years between 1912 and 1915.

Penrose a leading person in the field of GR [17] (section 19.6):

> *" ... Gravitation is not to be regarded as a force; for, to an observer who is falling freely (such as our astrnaut A), there is no gravitational force to be felt. Instead, gravitation manifests itself in the form of spacetime curvature. Now, it is important, if this idea is to work, that there be no 'preffered coordinates' in the theory. For, if a certain limited class of coordinate systems were taken to be Nature's preffered choices, then these would define "natural observer systems" with respect to which the notion of a "gravitational force could be reintroduced, and the central role of the principle of equivalence would be lost."*

> *"The requirement, in the text, of 'no preffered coordinates' is not only vague, but also something that might be regarded as somewhat too strong. In flat space, for example, it could be reasonably said that the choices of 'Cartesian coordinates' (here the Minkowski coordinates (t, x, y, z) of [], for which the metric takes the particularly simple form $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$) are 'preffered' over all other coordinate systems, and cosmological models also have special coordinate systems in which the metric form looks particularly simple (). The point is, rather, the more subtle one that such special coordinates should not have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates."*

> ***Special coordinates should not have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates.***

Active Diffeomorphisms

There is a simple geometric view of how these solutions are related.

A **congruence** is a space-filling family of curves such that every point of $\mathcal{M}$ lies on just one curve.

The reader may find it helpful to note that if two functions have the same functional form is equivalent to

$$\boxed{\phi(x = u) = \tilde{\phi}(y = u), \quad \text{for all values of } u. \qquad (1.26)}$$

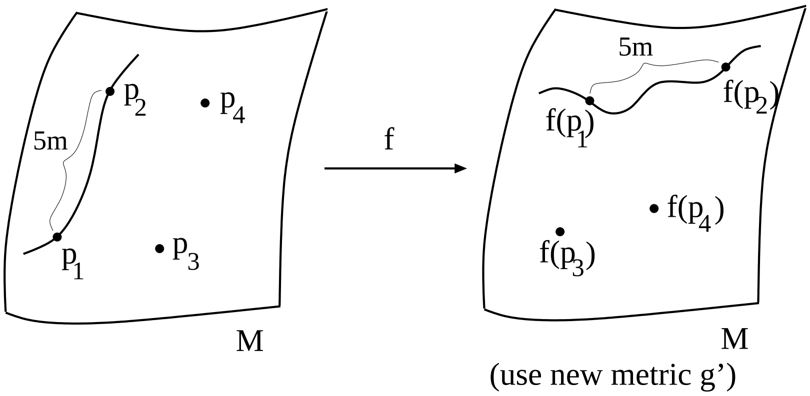


Figure 1.3: Passive spatial difeomorphism $f : M \to M$ refers to invariance under change of coordinates. The same object in a different coordinate system. Any theory of nature is invariant under passive difeomorphisms.

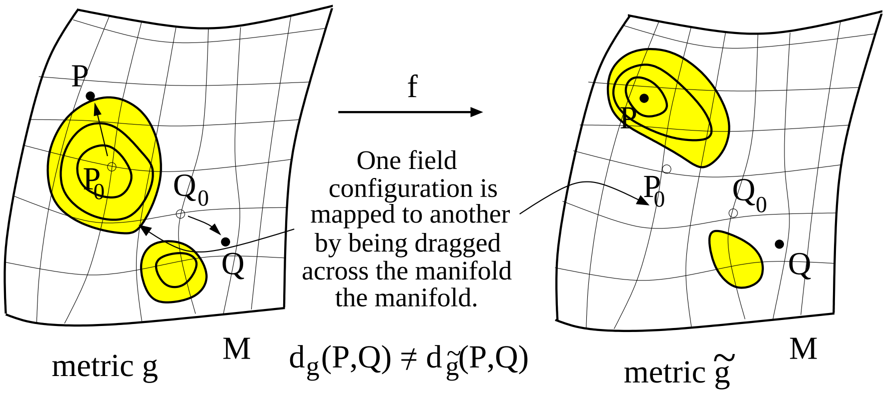


Figure 1.4: An active difeomorphism $f : M \to M$ draggs fields on the manifold while remaining in the same coordinate system. f is viewed as a map that associates one point in the manifold to another one.

If two fields $\tilde{X}(x)$ and $X(x)$ are related to each other through an active diffeomorphism then there is always a coordinate system, which we denote $y^\mu(x)$, in which $\tilde{X}(x)$ has the same functional form as $X(x)$. That is the tensor functions $X(x)$ and $\tilde{X}(y)$ are the same function but they correspond to two different coordinate systems.

$$\boxed{X^{ab...d}_{ef...g}\,(x^a = u^{(a)}) = \tilde{X}^{ab...d}_{ef...g}\,(y^a = u^{(a)}), \qquad (1.27)}$$

where $u^{(a)}$ are four numbers.

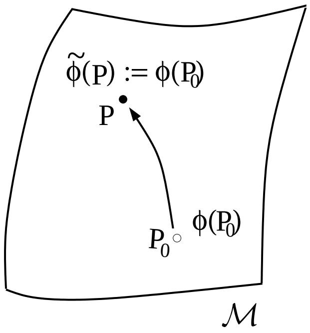


Figure 1.5: The value of $\tilde{\phi}(P)$ at P is equated to the value of $\phi(P_0)$ at P_0, i.e. $\tilde{\phi}(P) = \phi(P_0)$. Under this transformation f we indentify one point of the manifold P_0 to another point P $f : P_o \to P$.

$$g_{ab}(x^a = u^{(a)}) = \tilde{g}_{ab}(y^a = u^{(a)}), \tag{1.28}$$

see Fig(activediff5). We are dragging the tensor function at P_0 over to the point P keeping the coordinate lines "attached".

We then use a coordinate system that asigns the newly identified points the origonal coordinate values (note we are not doing a coordinate transform here - there are no Jacobian matrices involved)

As a consequence of this, if two metrics $g_{\mu\nu}(x)$ and $\tilde{g}(x)$ are related by an active diff transformation and one of them is a solution to Einstein's equations then so is the other. These are distinct metrics in that they describe different geometry.

Generalization to Gravity Coupled to Matter Fields

Expressible as tensor equations that reduce to laws consistent with special relativity in a frame in free-fall.

$$g_{ab}(x), \ \frac{\partial g_{ab}(x)}{\partial x^c}, \ \frac{\partial^2 g_{ab}(x)}{\partial x^d \partial x^c}, \ \text{and } T_{ab}(x) \tag{1.29}$$

exactly the same differential equation but now involves:

$$\tilde{g}_{ab}(y), \ \frac{\partial \tilde{g}_{ab}(y)}{\partial y^c}, \ \frac{\partial^2 \tilde{g}_{ab}(y)}{\partial y^d \partial y^c}, \ \text{and } \tilde{T}_{ab}(y). \tag{1.30}$$

We have an equation of motion for the matter fields. An equation of motion for the gravitational field. Each of these equations has the same mathematical form in any

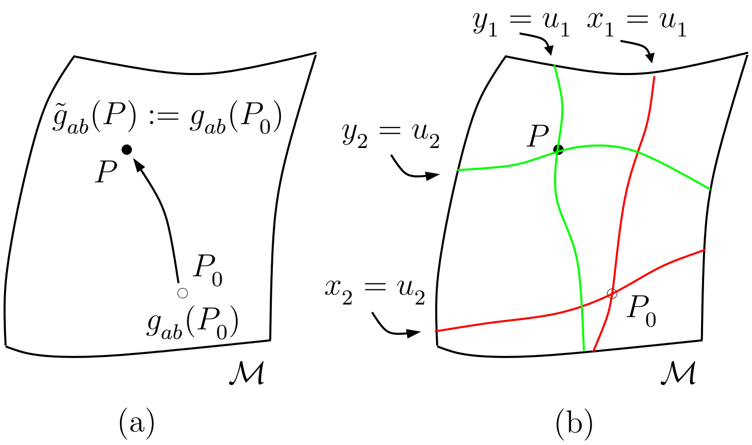


Figure 1.6: The value of the metric function $\tilde{g}_{ab}$ at P is defined by the value of the metric function g_{ab} at P_0, i.e. $\tilde{g}_{ab}(P) = g_{ab}(P_0)$. We go to a new coordinate system which assigns P the same coordinate values that P_0 has in the x-coordinates, so that $\tilde{g}_{ab}(y_1 = u_1, y_1 = u_2) = g_{ab}(x_1 = u_1, x_1 = u_2)$, compare to (1.28).

coordinate system so the arguememnts given before apply equally here. So that once we find one solution it gives rise to distinct solution, one for each concievable coordinate sytstem.

Then so are the set of fields in the y-coordinate system, that have the same functional form.

$$g_{ab}(x = u) = \tilde{g}_{ab}(y = u), \quad \text{for all values of } u. \tag{1.31}$$

$$E_a(x = u) = \tilde{E}_a(y = u), \quad \text{for all values of } u \tag{1.32}$$

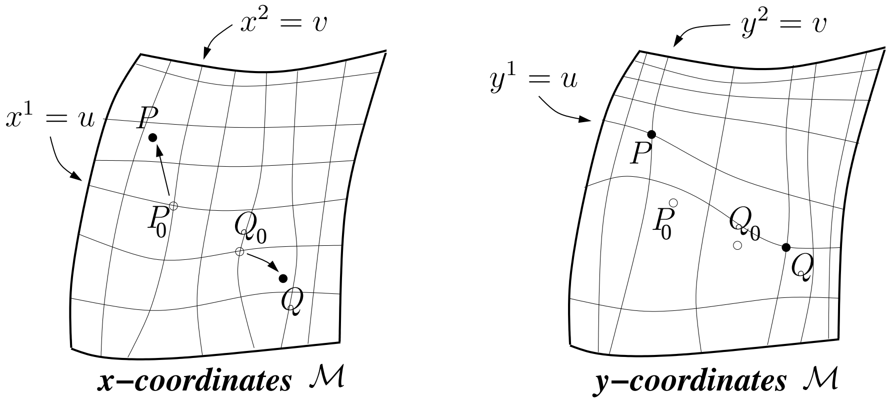
General covariance implies that once we find one solution to the EOM, it immediately gives rise to other *distinct solutions*, all related to the original solution by an active diffeomorphism!

General Covariance When Spacetime is Non-Dymanical

$$\triangle A_\mu = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left(\sqrt{g} g^{ij} \frac{\partial A_\mu}{\partial q_j} \right) \tag{1.33}$$

The new metric $g_{\mu\nu}(y)$ does not represent Minkowskian spacetime, the approach fails if spacetime is fixed and non-dynamical.

The precedure for generating new distinct solutions fails when spacetime is fixed and non-dynamical.



(a) (b)

Figure 1.7: (a) An active diffeomorphism in which we indentify one point of the manifold to another point. (b) We then go to a coordinate system that asigns the newly identified points the origonal coordinate values.

It is definitly true that general covariance has no content when spacetime is non-dynamical - the physical system does not care which coordinate system we use to view it.

Einstein's Problem with General Covariance (1912)

Einstein very rapidly worked out physical implications of this that couldn't be acceptable. He demonstrates these troubles with the hole argument to which we now turn.

Consider the situation depicted by fig (1.1). Initially the universe is filled by matter but the a hole forms whcih later goes away. Now consider the effect of an active diffeomorthism that reduces to identity outside the hole.

Let us denote the distance between two spacetime points P and Q as described by the metrics $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x)$ respectively as $d_g(P,Q)$ and $d_{\tilde{g}}(P,Q)$. They will not be equal to each other

$$\boxed{d_g(P,Q) \neq d_{\tilde{g}}(P,Q).} \tag{1.34}$$

Now the problem comes: the distance d between two distinct points P and Q which are both inside the hole although the metrics have the same initial data.

If we demand general covariance, GR doesn't determine the distance between spacetime events!

Einstein could not accept this and spent the next three years frantically, (frantically because Hilbert - maybe the best mathemation in the world at that time had also thrown

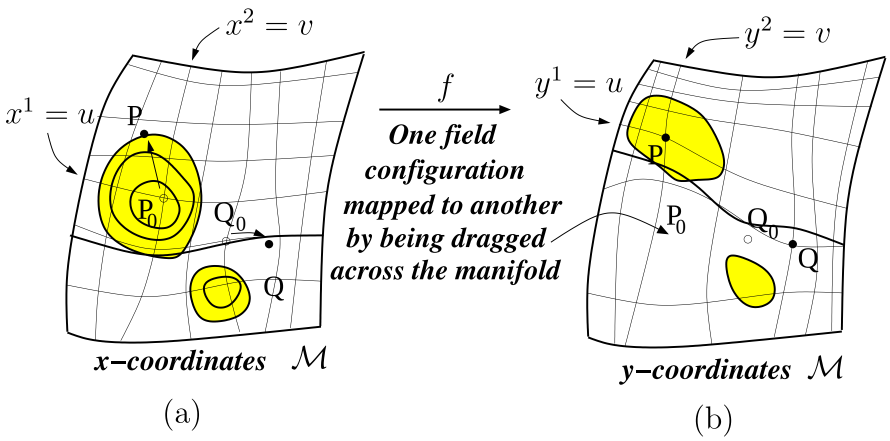


Figure 1.8: (a) An active diffeomorphism in which we actively drag the tensor function over the, in doing so indentify one point of the manifold to another. (b) We then go to a coordinate system which asigns the newly identified points their origonal coordinate values. That is to say - we carry the tensor function over the manifold, keeping the coordinate lines 'attached'.

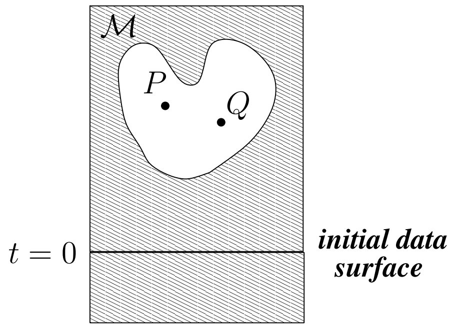


Figure 1.9: Einstein's hole argument.

himself into the problem), looking for *non-genrally covariant* field equations. If you were to look up Einstein's 1912 paper *Outline of a Theory of Gravity* you would find that he claims that the theory of gravity must be expressed in terms of the metric $g_{\mu\nu}(x,t)$, that the right-hand side must depend on the energy-momentum tensor, that the left-hand side must depend on $g_{\mu\nu}(x,t)$ and its derivatives up to second order, and that these *equations must not be generally covariant!*

The Resolution of the Hole Argument (1915)

In 1915 Einstein formulated his final field equations, however Einstein had changed his mind - they were generally covariant. What had happend? Well, he had realized that there was a mistaken assumption about the nature of spacetime and in dropping this assumption there would no longer be any incompatibility between general covariance and

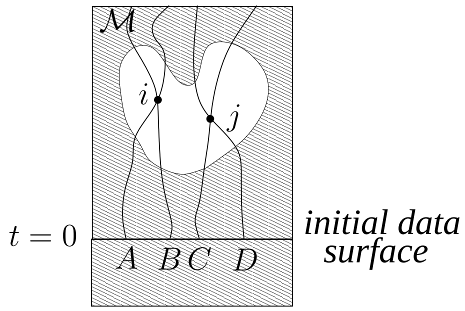


Figure 1.10: Resolution of Einstein's hole argument.

determinicity. To understand this let us see how the hole argument was resolved. The idea was to define locations using physical objects, for example particles. [1] Consider the arrangement in fig (1.1). We have four particles labeled by A, B, C and D. The particles A and B intersect in i and similarly the particles C and D intersect in j. These particles start at the initial surface and their geodesics are found by solving the equations of motion. After we have performed the active diffeomorphism we need to solve to find the geodesics for the new metric $\tilde{g}_{\alpha\beta}$. The distances between such defined locations *is deterministic*. This is because the tragectories are dragged across together with the metric by the active diff transformation. This is because we solve to find the geodesics for the transformed metric. A deterministic quantity is the distance between the two particles in fig **??**. So physical geometry is invariably define with respect to matter degrees of freedom (or in principle using degrees of freedom of the gravitational field itself).

What Einstein construed from the solution of the hole argument is that it is meaningful to refer to a location as a place where two freely falling particles intersect; however it is not meanigful to refer to a location as a point in spacetime (a spcetime event) because the distance from one such point to another is in undetermined in GR. That is spacetime points have, in themselves, no physical significance.

Remark
It is important to realize that the details of the Einstein-Hilbert action the fact that the fundemental physical theory is background invariant. You can add new terms to the Einstein-Hilbert action to change the theories high energy behaviour, but if these terms are invariant under coordinate transformations the resulting theory is also background independent. The failure to find such a theory indicates that background independence is an essetial ingredient and must be faced squarely.

[1] The lesson to be learned from the hole argument doesn't depend on whether or not the physical objects affect the gravitaion field or not. The important point is that physical objects move along geodesics. So for simpicity we consider only test particles.

1.2 Background Independence - A Farewell to Spacetime

1.2.1 Comparision of GR with the Rubber Sheet Analogy

There is another limitation of the analogy that is not so widely recognised and is of much more physic importance: the rubber sheet dynamic's is played out in space and evolves with respect to absolute time. Spacetime geometry however is not embedded in a further container and does not evolve with respect to some externaly provided time - there is no a priori given areana in which the dynamics of spacetime geometry is played out. In a sense the gravitation is its own areana! As such it is only meaningful to talk about relations between some degrees of freedom and other degrees of freedom of the gravitational field, or if matter is present, relations between matter degrees and the degrees of freedom of the gravitatoin degrees of freedom. What general relativity actually predicts is correlations between mearsurable quantities.

Spacetime has no character independent of observation - the way we experiance the world is only through observation and measurment. When the dynamics of gravity can be ignored the gravitational field gives rise to , through the observations that we make,the appearance a background spacetime.

How can field theory not be defined on a spacetime. Spacetime may be curved and change with time but in GR things still move on spacetime; fields and particles have dynamics on a curved spacetime. Physics on a curved spacetime is not GR! A dynamical theory of spacetime which is also generaly covariant is background independent! (the reader should note this conculsion of background independence that in the argument we didn't need to specify the exact form of the field equations).

Not taught so as to not overburden or confuse the student with conceptual difficulties raised by this - or teachers lecturers are not aware of this .

1.2.2 The View of the World that Emerges

The View of the World That Emerges

It is meaningless to talk of the geometry of spactime (in the absence of dynamical entities) as if it were an entity having independent existence. When one makes a measurement of *physical "geometry"* one is making a measurement of a certain aspect of the relationships that exists between physical objects that live in the world - there is no "geometry" without matter! This idea is not new and goes back to the times of Aristotle and Decartes; they advocated that space is an abstraction of the fact that some parts of matter can be in touch with others (see Descartes [81]). Of course, Aristotle and Decartes were knew nothing of relativity and the bringing together of space and time.

Einstein's modern day version is not motivated by mere philosophical considerations but lies at the basis of a physical theory, for which there has been obtained spectacular empirical support - binary pulsar's period decay due to gravitational radiation, discovery of black holes in the sky, the existence of an exanding universe.

Einstein's modern day version goes a step further; what the reader might find more disturbing is that, GR is also telling us that there is no time either! Which begs the question how do we describe dynamics without reference to time or evolution!?

This picture of spacetime is fundementally different from Newton's (and Minkowski's spacetime) notion. Newton introduced the idea of physical space as an idependent entity because he needed it for his dynamical theory. The Newtonian picture of the world is a background space on which on which matter moves. Points exist irrespective of whether there is matter present or not.

Not just point particles. It generalizes to fields on the spacetime manifold. Any two sytems related by an active diffeomorphism are physically indistinguishable. A physical system is not described by a field configuration (or by the location of the particles), but rather by the equivalence class of field configurations (and particle locations), related by all diffeomorphisms.

In GR a point in spacetime becomes an abstract notion which in itself has no physical meaning. This is the resolution of Einstein's struggle to understand "the meaning of the coordinates".

When spacetime is dynamical, general covariance is formally equivalent to background-independence.

How the field is localized over the spacetime manifold has no physical significance. In general relativistic physics, the "location" of physical objects and physical fields is not determined with respect to a pre-existing space. Physical are only "located" with respect to each other. Physical meaning lies in the relationships between fields. The spacetime manifold has turned out to be a conveinent mathematical device, devoid of any physical meaning.

A spacetime point and the electromagnetic field at Andromeda and drag it to the room your are in - amazingly it should represent the same physical situation!! No longer can we think of a point of the spacetime manifold as a place "where" things happen.

It is meaningless to talk about one spacetime point being causally related to another!! No background causal structure.

It was only at this point, in 1915, that GR was born. It was in making this final step that Einstein remarked ***"beyond my wildest expectations"***. So as the reader doesn't think this is all being made up we quote Einstein's own words

> *"All our spacetime verifications invariably amount to a determination of spacetime coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meeting of two or more of these points."*

Spacetime geometry has no meaning independent of observations.

Spacetime measutements and gravitational experiments are made by using objects, matter fields or particles and their mutal relationships.

mutal relationships between gravitational field and objects, matter fields or particles are preserved under active diffeomorphisms.

quantum mechanics must undergo the same deep transformation that classical special relatvistic mechanics had to undergo in jumping to generel relativistic physics: In general realtivisic physics, the "location" of physical objects and physical fields is not determined with respect to a preexisting space.

Einstein's Hole Argument: Compact Form

One point gets associated with another point. Go to a coordinate system which asigns this newly identified point the same coordinate values that the old point had in the x-coordinates. That way $\tilde{g}$ in the y-coordinates, i.e. $\tilde{g}_{ab}(y)$, is the same function g is in the x-coordinates , i.e. $g_{ab}(x)$. That is $g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x)$. Now, general covariance demands that the EOM be the same in both coordinae systems. So we have exactly the same differentiation equation in both coordinate systems. Therefore, if one of the metric functions is a solution then so is the other. Now consider the situation in fig() with initial data and metric solution $g_{ab}(x)$. Let us perform an active diff transformation which reduces to identity outside the hole. We obtain another distinct solution $\tilde{g}_{ab}(x)$ with the same initial conditions. The upshot is if we demand physical theories to be general covariant, GR does not determine the distance between to spacetime points.

The resolution to the

"Geometry up to Diffeomorphisms"

— — — — — — — — —

1.2.3 Common Misunderstandings

Often the general relativist will use terms which have a different meaning to many people in the rest of the physics community, leading to much confusion.

When a general relatvist referes to diffeomorphisms they are most likely referring to active diffeommorphisms and not passive diffeomrophisms (if they are using the coordinate-free geometry formulism then the only diffeomorphisms are active diffeomorphisms!)

When it is said that GR is invariant under diffeomorphisms, it is meant that the theory is invaraiant under active diffeomorphisms. These are the gauge transformations of GR and they should not be confused with the freedom of chosing coordinates on the space-time M. Invariance under coordinate transformations is not a special feature of GR, all physical theories are invaraint under coordinate transformations!

It is sometimes stated that an active diffeomorphism is just a coordinate transformation viewed differently. This is misleading, consider a non-uniform translation in Minkowski spacetime. Under a passive transformation the resulting spacetime is, of course, still Minkowski but under the active transformation the resulting spacetime is no longer Minkowski. (Under a uniform translation the active transformation results in Minkowski spacetime but this is only because of the homogeneity of Minkowski spacetime).

People should be aware of the differing use of the term general covariance. The principle is defined as the condition that the equations of motion should take the same form in all coordinate systems. However, when a general relatvist says that GR is a generally covariant theory they are not emphasing that it is invaraint under general coordinate transformations but rather that the theory is background independent as a direct consequence of coordinate invariance.

Even though the full content of GR is that it discards the very notion of space-time, a general relatvist may continue to use the terms "space" and "time" but it should be understood that they do so only to indicate certain aspects of the gravitational field.

1.2.4 The Blessing of background independence - Non-Peturbative Quantum Gravity Finite and Requires No Renormalization!

The reason is fairly simple:

There is *no* background metric to provide scales so there can be no UV-divergences in observable quantities arising from indefinitely small distances as happens in ordinary, *background-dependent* field theories!!!

or put another way

The absence of the divergences that usually plague interacting field theories in a Minkowskian background spacetime can be understood intuitively from the diffeomorphism invariance of the theory "short and long distances are gauge equivalent".

quote a more precise argument

???? built from the product of a pair of field operators evaluated at a single point, it is not well-defined. In this scheme, one introduces an artificial separation of the single point to a pair of closely separated points and . The problematic terms involving field products such as $\hat{\phi}^2(x)$ becomes $\hat{\phi}(x)\hat{\phi}(x')$, whose expectation value is well defined. If one is interested in the low energy behavior captured by the point-defined quantum field theory one takes the coincidence limit. Once the divergences present are identified, they may be removed (regularization) or moved (by renormalizing the coupling constants), to produce a well-defined, finite stress tensor at a single point.????

" A background independent operator must always be finite. This is because the regulator scale and the background metric are always introduced together in the regularization procedure. This is necessary, because the scale that the regularization parameter refers to must be described in terms of a background metric or coordinate chart introduced in the construction of the regulated operator. Because of this the dependence of the regulated operator on the cuttoff, or regulator parameter, is related to its dependence on the background metric. When one takes the limit of the regulator parameter going to zero one isolates the non-vanishing terms. If these have any dependence on the regulator parameter (which would be the case if the term is blowing up) then it must also have dependence on the background metric. Conversely, if the terms that are nonvanishing in the limit the regulator is removed have no dependence on the background metric, it must be finite. "

The field is specified against a backgroand spacetime, assigning for the components of the electric and magnetic fields, E_i, B_i. The uncertainity relations apply to physical, observable quantities, such as the position and momentum of a particle, or the values of the magnetic and electric fields.

The proper length between two abstractly defined spacetime points, and nor are areas of a surface or the volume of a region specified using coordinates.

The idea of quantum fluctuations of spacetime points, a naive yet popular conception of what a quantum theory of gravity might entail.

When the dynamics of gravitational field can be neglected, we recover the background dependent matter field theories of particle physics.

The conventional mathemaical formulism of quantum field theory relies very much on the existence of a background space time. take general relativity seriously reconstruct quantum field theory from scratch in a form that does not require background space.

Observables of Quantum Gravity are Finite

The regularization parameter can often be derictly interpreted as the distance between spacetime points.

Einstein's theory can be stated as a variational problem: one takes a manifold (the manifold is simply a blank background upon which we place the metric to put the familiar features of space and time into it. We try all possible assignments of metrics to the manifold to find those that minmise the Einstein-Hilbert action - these are the solutions to

Einstein's equations. Nowhere in the action does there appear a fixed background metric, or any fixed geometric structute at all.

Einstein's theory can be recast as a Yang-Mills theory. This Yang-Mills theory exists without reference to a background metric. Such a theory makes no disticntion between small and large distances, as described by a background metric; take the same coordinate system but introduce two distinct metrics. According to one metric the proper distance betwen two points might be small and the other the proper distance between these two points might be large. The Yang-Mills theory is blind to either metric and as a result, one can argue that, such a *background independent* quantum theory will not suffer from UV divergences.

By introducing a point-splitting regularization. If the end result has no memory of which particular metric we used we wont have broken spatial diff invariance. In General relativity the proper distance between two abstractly defined points, has in itself, no physical meaning and so we should not expect UV-divergences to occur in physical quantities as we remove the regulator by sending $\epsilon \rightarrow 0$.

By replacing point particles with strings, a graviton interactions are "smeared" such that there is now no localized interaction point anymore. However this simple argument there is no localized interaction point for the simple reason that (according to classical GR) spacetime points have no independent physical reality! Say we have some object at some position in the spacial manifold. According to GR its position can have no physical meaning. Background independent theories smeare themselves! Rather singular functions 1.2.4

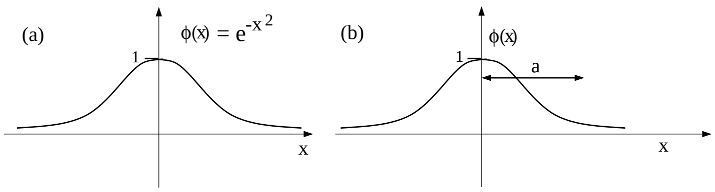


Figure 1.11: Illustration of smearing. operator valued distributions.

(operator counterparts of distributions reffered to as **operator valued distributions**, but we wont get into the details of this here, see apendix ??). The location of this object in space has no physical meaning and we can remove this dependency be averaging the position of the object over the whole spacial manifold; in doing so no position is favoured over anyother. This has the effect of smearing the oject over the whole spacial manifold - the resulting mathematical object is much more regular and does not give rise to UV-divergencies. These vague argunents have been made precise in [**?**] and our hopes confermed to be true??. We do not need the smearing affects of strings over point particles, back-ground independent field theories *smear themselves*! In [] the UV-divergencies come back when background dependence is reintroduced.

It is a "topological field theory " but with "local degrees of freedom" and as such knows nothing about rulers or clocks.

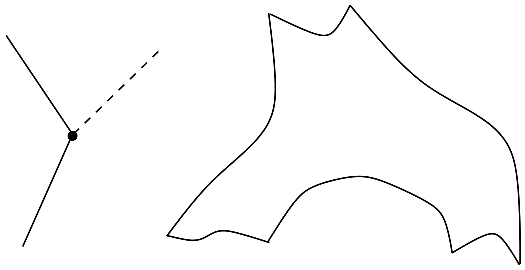


Figure 1.12: Regime where gravity is very strong so that the non-perturbative and background independence of GR must be taken into account. That spacetime points have no independent physical reality casts doubt on the hand-wavey argument I gave above.

There is no longer a clear cut distinction between physical objects that form the reference system (laboratory walls) and physical objects whose dynamics we are describing.

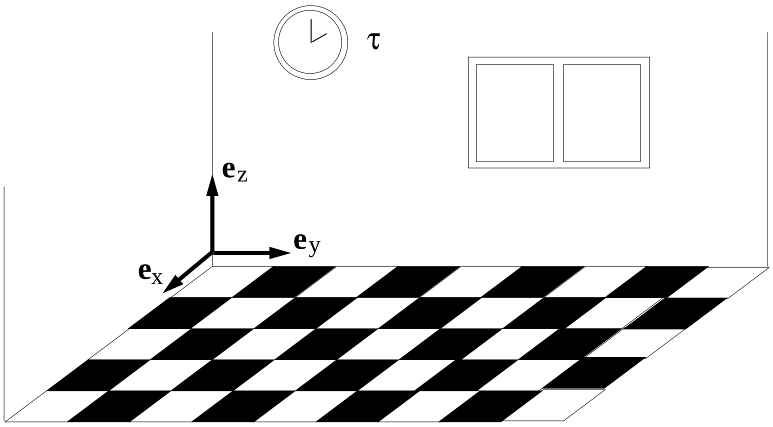


Figure 1.13: Labortary walls exemplify Newton's absolute space and the clock absolute time. We can define positions relative to the wall.

1.3 Physical Geometry

The kinematic phase space expresses all potential outcomes of measuments of partial observables. Dynamics is a restriction on on these configurations and expresses the existence of correlations among measurements of partial observables.

[280]

> To measure t we use clocks. A clock is a system with a variable, for instance the position of a hand, which has a simple behaviour in t. In this

paper, we shall denote a clock variable (the position of the hand) as T; we shall denote varaibles of different clocks as $T, T', T'', \ldots$. Good clocks may have, for instance, linear behaviour in t:

$$T(t) = \alpha t. \tag{1.35}$$

It is an elementary-physics-course observation that we never really measure t; rather, we always measure $T's$. The value of a physical quantity Q, measured at time t, denoted $Q(t)$. Since time is determined by measuring a clock varaible T, what is actually measured is not $Q(t)$, but only the combined quantity $Q(T)$. Thus, t does not ever appear in laboratory measurements.

What we learn from GR is that coordinates do not have any meaning independent of observations, invloving clocks and light signals. According to GR, a coordinate system is defined only by explicity carrying out space-time measurements.

In the theoretical analysis of an experiment, one (arbitrary) coordinate system $\vec{x}, x^o$, and then equations of motion, as (N.-19) with the data in the following way. First we have to locally solve the coordinates $\vec{x}, x^o$ with respect to quantities $f_1 \ldots f_4$ that represent the physical objects used as clocks and as spatial reference system

$$f_1(\vec{x}, x^o) \ldots f_4(\vec{x}, x^o) \to \vec{x}(f_1, \ldots \ldots, f_4), x^o(f_1, \ldots, f_4) \tag{1.36}$$

and then express the rest of the remaining fields (f_i $i = 5 \ldots N$) as functions of $f_1 \ldots f_4$

$$f_i(f_1, \ldots, f_4) = f_i(\vec{x}(f_1, \ldots, f_4), x^o(f_1, \ldots, f_4)). \tag{1.37}$$

If, for instance, $F(\vec{x}, t)$ is a scalar, then for every quadruplet of numbers $f_1 \ldots f_4$, the quantity $F_{(f_1 \ldots f_4)} = F(f_1, \ldots, f_4)$ can be compared with experimental data. This procedure is routinely performed in any analysis of experimental gravitational data - the physical time f_4 representing the the reading of the laboratory

1.3.1 Physical GPS Coordinates

It is difficult to discuus observables without some pressuposed notion of time; we pressume, the undeniable, existense of clocks. We must at some point verify such variables emerge from the timeless, fundemental description in the suitable physical regime.

$$\begin{aligned} ct_P &= \frac{1}{2}[c(t_A + t_B) + (x_B - x_A)], \\ x_P &= \frac{1}{2}[c(t_A - t_B) + (x_B + x_A)] \end{aligned} \tag{1.38}$$

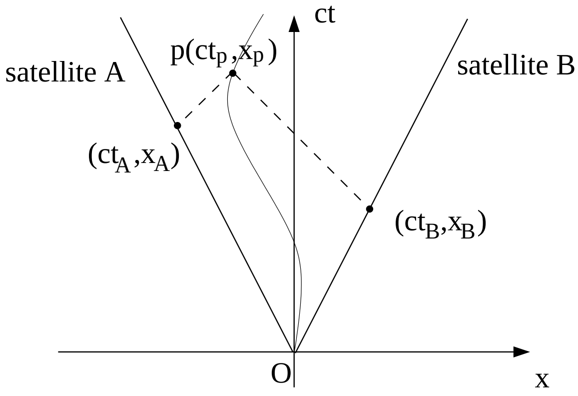


Figure 1.14: GPS.

1.3.2 Physical Area

Geometric quantities are the observables of the gravitational field. When the gravitational field is quantized, geometric observables will be represented by operators. We then try to compute their spectrum and eigenstates.

Naive formular:

$$
\begin{aligned}
||\,d\xi^1 \times\, d\xi^2|| &= ||\,d\xi^1||\;||\,d\xi^2||\sin\theta \\
&= \sqrt{||\,d\xi^1||^2||\,d\xi^2||^2(1-\cos^2\theta)} \\
&= \sqrt{||\,d\xi^1||^2||\,d\xi^2||^2-(||\,d\xi^1\cdot\,d\xi^2||)^2} \\
&= \sqrt{|q_{11}q_{22}-(q_{12})^2|}\,d\sigma_1\,d\sigma_2 \\
&= \sqrt{\det q}\,d^2\sigma
\end{aligned}
\tag{1.39}
$$

$$
A(S) = \int d^2\sigma\sqrt{\det q} \tag{1.40}
$$

$$
q^S_{uv} = \frac{\partial x^\alpha}{\partial\sigma^u}\frac{\partial x^\beta}{\partial\sigma^v}q_{\alpha\beta} \tag{1.41}
$$

$$
n_a = \frac{1}{2}\epsilon^{uv}\epsilon_{\alpha\beta\gamma}\frac{\partial x^\beta}{\partial\sigma^u}\frac{\partial x^\gamma}{\partial\sigma^v} \tag{1.42}
$$

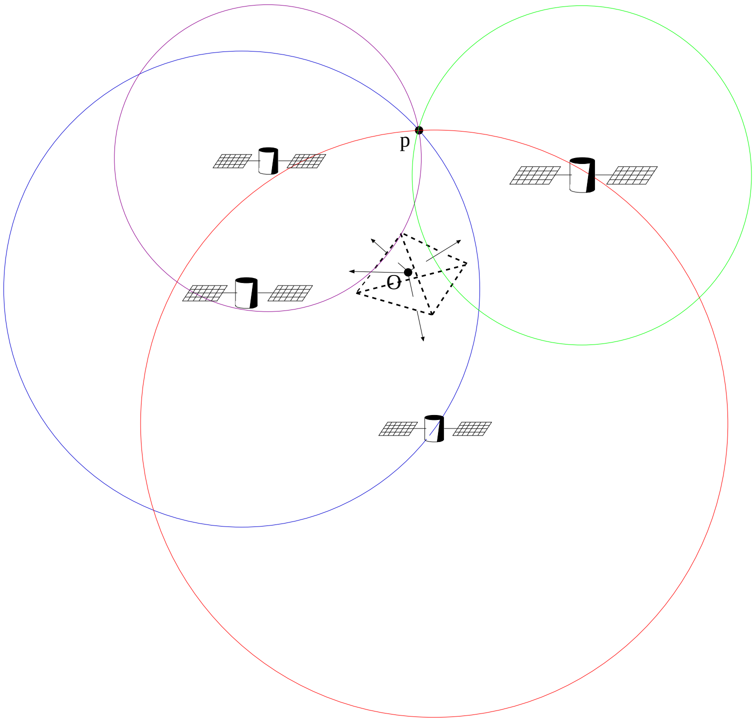


Figure 1.15: GPS3D A spacetime point in Minkoski spacetime can be expressed as a relation amogst 4? measurable variables. This definition of spacetime location retains meaning in the jump to GR.

$$\begin{aligned} A[S] &= \int_S d^2\sigma \sqrt{\det q} = \sqrt{\frac{1}{2}\epsilon^{uu'}\epsilon^{vv'} q^S_{uv} q^S_{u'v'}} \\ &= \int_S d^2\sigma \sqrt{n_\alpha n_\beta \tilde{E}^{\alpha i} \tilde{E}^{\beta j}} \end{aligned} \tag{1.43}$$

1.3.3 Description of a Measurement of Area

i. the gravitational field,

ii. two particles,

iii. a two dimensional surface (the "table").

The simultaneity surface Σ is the set of points in $\mathcal{M}$ whose light cone intersects X in two points at the same proper time distance along X from P.

We also assume other physical objects exist light pulses travelling along geodesics, apparatus that detect the arrival of light pulses, clocks that measure proper time along world

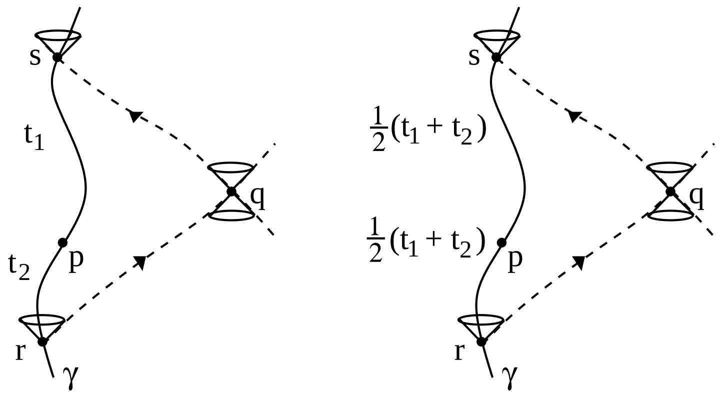


Figure 1.16: clock time.

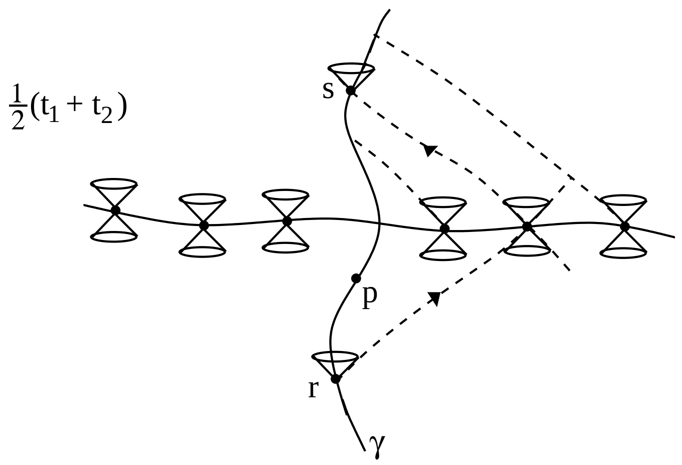


Figure 1.17:

lines, recording devises and so on. We do not consider these other physical objects as part of the dynamical system observed

it is invaraint under coordinate transformations that preserve the coordinate choice made??? what about acttive diffs??

The intersection between the surface of simultanity Σ of the observer and the table world history S is a two dimensional surface $S = \Sigma \cap T$.

A Coordinate choice

In our preffered coordinates :

(i) P is the origin,

(ii) The 3-surface T is defined by $-1 < x^1 < +1$, $-1 < x^1 < +1$ and $x^3 = 0$.

(iii) The world line X is defined by $x^1 = x^2 = x^3 = 0$

No physical quantities will not depend upon which coordinates we choose, just as in electrodynamics where the electric $E_i(x)$ and magnetic $B_i(x)$ fields dont depend on which particular gauge potential $A_\mu(x)$ we choose to calculate $E_i(x)$ and $B_i(x)$. Let us on this

1.3.4 Einstein's Field Equations

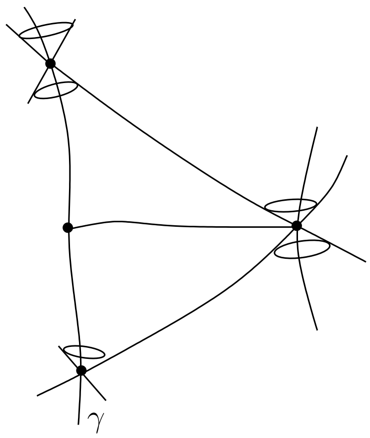


Figure 1.18: measLocation.

Measurement of Relative Velocities

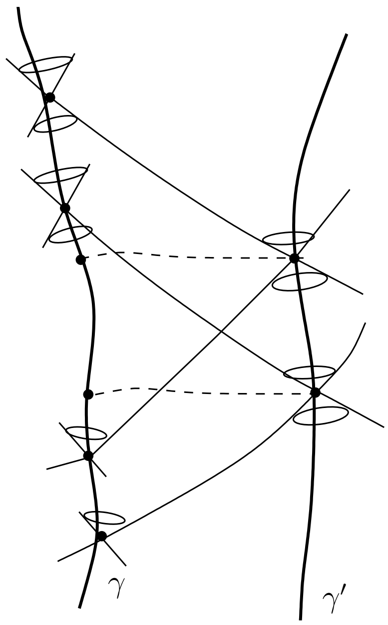


Figure 1.19: measVelocity.

Lorentz transformation laws from full GR

1.3.5 The velocity composition law

1.3.6 Dust as Matter Reference System

1.4 Some conceptual issues

There are suprising features of classical and quantum general relativity that take some getting used to. Here we present a some for the reader to ponder over. On a first reading the reader may want to skip this section and move straight to Connections verses Metrics. The talks are more focussed on techniqual issues - although of course the conceptual and techiqual interplay.

Since there is no background space-time metric, what does "time evolution" mean? the theory does not have a unique inert notion of time, fundemental ingredient in quantum mechanics. The predictons of classical general relativity do not depend explicitly on the coordinate time t. What the theory predicts are correlations between physical variables, not the way physical variables evolve with respect to a preffered time coordinate t.

This is a general property of generally covariant theories. $\mathcal{H} = 0$ - has to do with the fact that there is no physical meaning physical notion of time in GR. We illustrate ths with a simply example: we have a non-relativistic particle. We will put in a gauge symmetry by hand

$$\begin{aligned} S[q(t+\epsilon)] - S[q(t)] &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t} \epsilon + \frac{\partial \mathcal{L}}{\partial q^i} \dot{q}^i \epsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \frac{d}{dt} (\dot{q}^i \epsilon) \right] dt \\ &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t} \epsilon + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) \frac{d\epsilon}{dt} \right] dt \\ &= [\mathcal{L}\epsilon]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L} \right) \frac{d\epsilon}{dt} dt \end{aligned} \tag{1.44}$$

Parameterization-invariance means that the integral must zanish for arbitrary $d\epsilon/dt$, so that we have

$$\boxed{\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L} = 0.} \tag{1.45}$$

It is a general result that the Hamiltonian zanishes for invariant under parameterization-invariance.

There is an extensive literature on this subject. We will just settle by refering to C.Isham's paper "The problem of time in conanical quantum Gravity" and by quoting Rovelli, [36] gr-qc/0306059

"...In nonrelativistic physics, time and position are defined with respect to a system of reference bodies and clocks that are implicititly assumed to exist and not to intetract with the physical system studied. In gravatational physics, one discovers that no body or clock exists which does not interact with the gravitationalfield: the gravitaional field affects directly the motion and rate of any reference body or clock. Thereforeone cannot separate reference bodies and clocks from the dynamical variables of the system. General relativity - in fact, any general covariaant theory -is always a theory of interacting variables that necessary include the physial bodies and clocks used as references to characterize spacetime points. "

Our notion of time is meaningless in a generic situation. The features of time is not present at the fundemental level, rather it "emerges" as features of the semi-classical limit, (same sort of idea of how the notion of temperature is an emerges in the thermodynamical limit, but is meaningless at the fundemental level of molecules). How does this come about? Well, what the theory does is it describes relative location and relative evolution of dynamical objects. Pick observable degrees of freedom that act as a "clock" and find its correlations to some other set of measurable degrees of freedom. Then by changing the reading of the clock and observing the changes in the other measurable quantities, we can, *to some approximation*, recover our usual notion of mechanics defined by evolution in fixed background time. The notion is meanigless in a generic situation.

There is no longer a clear cut distinction between physical objects that form the reference system (laboratory walls) and physical objects whose dynamics we are describing.

An active spatial diffeomorphism $f : M \rightarrow M$ relates different objects in M in the same coordinate syatem. f is viewed as a map that associates one point in the manifold to another one. GR is the only theory of nature that is invariant under active difeomorphisms. It is this invariance that allows for the possibility of quantum gravity to be finite.

1.5 Relational Mechanics

1.5.1 Covariant Hamiltonian Formulation

dynamics fixes relations among vaiables, so that knowing some of them we can predict the others. The absence of any preffered time vairable. The relation aspect of evolution.

learn

rephrase it in the language of the presymplectic formulation of a conventional system, and from here, extend it to presympectic systems that do not correspond to a conventional system.

Parametrized Harmonic Oscillator

In its usual presentation, classical mechanics appears to give time a special role. However, mechanics can be formulated so as to treat the time variable on the same footing as the other variables in an extended phase space. Phase space variables being on the same footing.

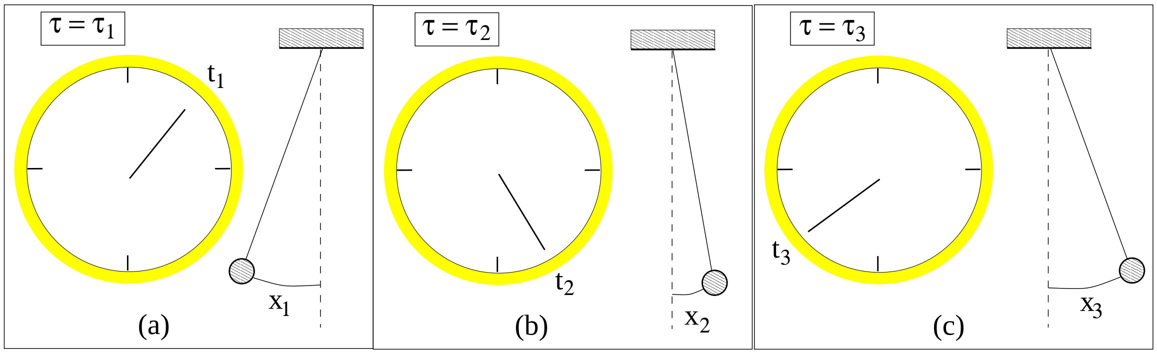


Figure 1.20: partialobs. τ is an unphysical parammeter labelling different possible correlations between the time reading t of the clock and the elongation x of the pendulm.

$$x(\tau), \qquad t(\tau) \tag{1.46}$$

The role of the coordinates (and in particular of the time coordinate) can be clarified by means of this analogy. The coordinates have the same physical status as the arbitrary parameter τ that we use in order to label and distinguish the set of relations between the reading on the clock and the elongation of the pendulm. The 'time coordinate' in GR has no physical meaning and serves only to label and distinguish points of space-time and is a mere matter of convenience.

[82]

$$S = \int d\tau \left[\frac{dx}{d\tau}p + \frac{dt}{d\tau}p_t - \lambda\left(p_t + \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\right)\right] \tag{1.47}$$

$$\begin{aligned} \frac{dp}{d\tau} &= -\lambda m\omega^2 x, & \frac{dx}{d\tau} &= \lambda\frac{p}{m}, \\ \frac{dp_t}{d\tau} &= 0, & \frac{dt}{d\tau} &= \lambda. \end{aligned} \tag{1.48}$$

variation with respect to λ gives the first class constraint. These constraints form a first-class system, which means that the Poisson bracket of two of them is again a linear combination (generally with phase-space dependent coeffcients) of the, any other constraints are called second class, see Apendix F for more on constrained Hamiltonian system.

$$\mathcal{C} = p_t + \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = 0 \tag{1.49}$$

$$\frac{dx}{dt} = \frac{dx}{d\tau} \Big/ \frac{dt}{d\tau} = \frac{\lambda p/m}{\lambda} = \frac{p}{m} \tag{1.50}$$

$$\frac{dp}{dt} = \frac{dp}{d\tau} \Big/ \frac{dt}{d\tau} = \frac{-\lambda m\omega^2 x}{\lambda} = -\omega^2 x \tag{1.51}$$

or

$$p = m\frac{dx}{dt}, \quad m\frac{d^2x}{dt^2} = -m\omega^2 x. \tag{1.52}$$

What we have done is promoted the time to a dynamical variable. This has conjugate momentum which we denote p_t. The original degrees of freedom and time are left as functions of some physically irrelevant parameter τ (physically meaningles as it does not correspond to a measurable quantity - just like the time coordinate in GR). Time t can be considered independent of the other degrees of freedom when a Lagrangian multiplier term is added to the action. Invariance of this extended Hamiltonian under variations of τ implies that it is constrained to vanish, (just as in the case of GR)).

have

$$\begin{aligned} x(\tau) &= A\cos(\omega\tau) + B\sin(\omega\tau), \\ t(\tau) &= \tau, \\ p(\tau) &= -m\omega A\sin(\omega\tau) + m\omega B\cos(\omega\tau), \\ p_t(\tau) &= -\frac{1}{2}m\omega^2(A^2 + B^2), \end{aligned} \tag{1.53}$$

as a solution, where (A,B) are the physical observables coordinatize the *physical* phase space, $\Re$

$$\begin{aligned} A &= \cos(\omega t)x - \frac{1}{m\omega}\sin(\omega t)p, \\ B &= \frac{1}{m\omega}\cos(\omega t)p + \sin(\omega t)x. \end{aligned} \tag{1.54}$$

Note $A = x(t=0) \equiv x_0$, and $B = \frac{p(t=0)}{m\omega} \equiv \frac{p_0}{m\omega}$, i.e. the position x_0 of the harmonic oscillator when the internal clock measures $t = 0$ physical observables.

The space-time arguments x and t are *not observables* exactly like the τ of the parametrized description of a pendulm and clock.

1.5.2 Depamerizable Mechanics: Identification of a "time" variable

If $q^a = (t, q^i)$ and

$$\mathcal{C} = p_t + \mathcal{C}_0(p^i, q^i) \tag{1.55}$$

then

$$S = \int d\tau \left(p_t \frac{dt}{d\tau} + p_i \frac{dq^i}{d\tau} + N(p_t + \mathcal{C}_0(p_i, q^i)) \right) \tag{1.56}$$

varying N:

$$p_t = -\mathcal{C}_0 \tag{1.57}$$

substituting this into the action

$$S = \int d\tau \left(p_i \frac{dq^i}{d\tau} - \mathcal{C}_0 \frac{dt}{d\tau} \right) \tag{1.58}$$

$$S = \int d\tau \frac{dt}{d\tau} \left(p_i \frac{dq^i}{dt} - \mathcal{C}_0 \right) \tag{1.59}$$

$$S = \int dt \left(p_i \frac{dq^i}{dt} - \mathcal{C}_0(p^i, q^i) \right) \tag{1.60}$$

it must be true that a canonical transformation can be made on the phse space such that the Hamiltonian constraintcan be written in the form (1.55).

GR cannot be *deparamitized*! This is because there is no unique notion of time

The absence of any preffered time vairable

1.5.3 Fully Constrained Hamiltonian Systems

true observables are composed of correspondences between partial observables, one of which is the reading of a clock.

One parameter families of that are given by the same partial observers at different clock feadings.

Presympletic phase space

The space of partial observables is the extended configuration space $\mathcal{C}$, and the dynamics is governed by a vanishing Hamiltonian $\mathcal{H}$ on the phase space associatedd with $\mathcal{C}$ (denoted $T^*\mathcal{C}$).

$$S[q^i, p_i, \lambda^m] = \int_{\tau 1}^{\tau 2} d\tau \left\{ \frac{dq^i}{d\tau} p_i - \lambda^m \mathcal{C}_m(q^i, p_i) \right\}, \tag{1.61}$$

which is invariant arbitrary reparametrizations of the parameter τ. The parameter is unphysical and unobservable, like the time coordinate in general relativity. The unreduced, or extended phase-space γ_{ex} is coordinatized by the canonical pairs (q^i, p_i); $i = 1, 2, \ldots, N$.

The variation of the action with respect to the canonical coordinates q^i, p_i gives the equations of motion

$$\frac{dq^i}{d\tau} = \lambda^m \frac{\partial \mathcal{C}_m(q^i, p_i)}{\partial p_i}, \tag{1.62}$$

$$\frac{dp_i}{d\tau} = \lambda^m \frac{\partial \mathcal{C}_m(q^i, p_i)}{\partial q^i}. \tag{1.63}$$

while variation with respect to the Lagrange multipliers λ^m (where the index m labels the M constraints) gives us the constraint equations

$$\mathcal{C}_m = \mathcal{C}_m(q^i, p_i) = 0, \quad m = 1, 2, \ldots, M \tag{1.64}$$

In the example above $q^1 = x$, $p_1 = p$, $\lambda^1 = \lambda$ and $\mathcal{C}_1 = \mathcal{C} = p_t + \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

The canonical 2-form on γ_{ex}

$$X_{d\mathcal{C}_m} = -\frac{\partial \mathcal{C}_m}{\partial p_i}\frac{\partial}{\partial q^i} + \frac{\partial \mathcal{C}_m}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{1.65}$$

The pair $(\Gamma_{ex}, \omega_{ex})$ forms a symplectic space.

The first class constraints satisfy, in general, a "non-Lie" algebra

$$\{\mathcal{C}_m, \mathcal{C}_n\} = C^l_{mn}(q^i, p_i)\mathcal{C}_l, \tag{1.66}$$

and the number of independent physical degrees of freedom of the theory is $D = N - M$. The constraint surface γ of γ_{ex} defined by the constraint equations (1.64) is a $(2D + M)-$dimensional manifold. The restriction ω of ω_{ex} to the consraint surface γ is of rank $2D$. The M null directions of ω are the infinitesimal transformations generated by the constraints. They define the gauge orbits on γ.

The space (γ, ω) is a presymplectic space, which contains the full dynamical information about the system. Hence dynamical systems in this form are also called "presymplectic systems". γ can be parameterized by the set of independent coordinates $(\tilde{q}^a, \tilde{p}_a, t^m)$, where $(\tilde{q}^a, \tilde{p}_a)$, $a = 1, 2, \ldots, D$ are canonical variables that coordinatize the physical phase space γ_{ph}, and t_m, $m = 1, 2, \ldots, M$ coordinatize the orbits. In general this coordinatization can hold only locally, and different charts may be needed to cover the entire space.

where t is the coordinate in $\mathbb{R}$, and corresponds to the external time variable. The difference between the conventional formulation and the presymplectic formulation is only in the fact that this time variable is treated on the same footing as the other variables. As a concrete example, we may imagine that H is the harmonic oscillator Hamiltonian describing the small oscillations of a pendulum, while t is the reading of a physical clock. Then the presymplectic system (??) describes how two equal-footing physical variables (the pendulum amplitude and the clock reading) evolve with respect to one another. In general covariant systems, such as any general relativistic system, this 'equal footing' status between all physical variables is an essential feature of the theory. It expresses the major physical discovery of general relativity: the complete relativity of spacetime localization.

Note that the canonical coordinates $\tilde{q}^a$, and $\tilde{p}_a$ are the physical observables of the system. They are gauge-invariant. They satisfy $\{\tilde{q}^a, \tilde{p}_b\} = \delta^a_b$ on the physical phase space. In these coordinates, the physical symplectic form on ω_{ph} is $\omega_{ph} = d\tilde{p}_a \wedge \tilde{q}^a$. The p q general solution of the equations of motion is simply given by the embedding equations of the orbits in γ_{ex}, that is

$$q^i = q^i(t^m; \tilde{q}^a, \tilde{p}_a), \tag{1.67}$$
$$p_i = p_i(t^m; \tilde{q}^a, \tilde{p}_a). \tag{1.68}$$

Each set $(\tilde{q}^a, \tilde{p}_a)$ determines a solution; along each solution, the quantities (q^i, p_i) depend on the M parameters t^m (instead than just on a single time variable) because of the gauge freedom in the evolution. The inverse relations of (??)-(??) give the dependence of the physical observables $\tilde{q}^a$, and $\tilde{p}_a$ from the original coordinates

$$\tilde{q}^a = \tilde{q}^a(q^i, p_i), \tag{1.69}$$

$$\tilde{p}_a = \tilde{p}_a(q^i, p_i), \tag{1.70}$$

as well as the orbit coordinates t^m

$$t^m = t^m(q^i, p_i). \tag{1.71}$$

The quantities (??) and (??) commute with all the constraints, and provide a complete set (in the sense of Dirac) of gauge-invariant observables. Every other physical observable can be obtained from them.

In general, $2N - M$ of these equations are independent. For each physical state of the system, determined by the value of $(\tilde{q}^a, \tilde{p}_a)$, these equations define an M dimensional subspace in the phase space. Therefore each state determines a set of relations on the original phase space variables. These relations represent the dynamical information on the system; they provide the full solution of the dynamics in a gauge-invariant fashion.

$$S = \int d\tau \left(p_a \frac{dq^a}{d\tau} + N\mathcal{C}(p_a, q^a) \right) \tag{1.72}$$

$$\tilde{q}^a = \tilde{q}^a(q^i, p_i), \tag{1.73}$$

$$\tilde{p}_a = \tilde{p}_a(q^i, p_i), \tag{1.74}$$

1.6 Action Principle for General Relativity

Just as the action principle for electrodynamics sould be invariant under gauge transformations, an action principle for general relativty should be invariant under its gauge transformations which we have learnt are infintesimal active diffeomorphisms, not to be confused with coordinate transformations.

It turns out, however, that actions that are invaraint under coordinate transformations are automatically invariant under active diffeomorphisms! This we prove in the following subsection.

In the second subsection we demonstrate the first order Palatini action principle of GR. This is an action formulation in which the connection is considered as an independent variable.

1.6.1 Invariance of Integral Scalars Under Active Diffeomorphisms

With a coordinate transformation the volume element

$$d^4x = \frac{1}{4!}\epsilon_{\mu\nu\sigma\rho}dx^\mu dx^\nu dx^\sigma dx^\rho$$

transforms as

$$d^4x' = d^4xJ \tag{1.75}$$

where

$$J = \det\left(\frac{\partial x^{\mu'}}{\partial x^\alpha}\right).$$

Now

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x^{\mu'}}g_{\alpha\beta}(x)\frac{\partial x^\beta}{\partial x^{\nu'}}$$

We look on the right-hand side as the product of three matrices, and take the determinant of both sides giving

$$g' = J^{-2}g.$$

Thus

$$\sqrt{-g} = J\sqrt{-g'}. \tag{1.76}$$

Therefore

$$\sqrt{-g}d^4x = \sqrt{-g'}d^4x'$$

and so the quantity

$$\sqrt{-g}d^4x$$

is invariant under coordinate transformations. Suppose F is a scalar field, $F = F'$. Then

$$\int F\sqrt{-g}d^4x = \int F\sqrt{-g'}Jd^4x = \int F'\sqrt{-g'}d^4x'$$

if the region of integration for x' corresponds to that for x. Therefore

$$\int F\sqrt{-g}d^4x$$

is invariant under coordinate transformations. We refer to as an integral scalar.

Now let us investigate how an integral scalar transforms under an active diffeomorphism. We start with seeing how $\sqrt{-g}d^4x$ transforms under active diffeomorphisms.

Consider the "small" volume depictured in fig (1.21). Under an active diffeomorphism the "corner" points of the infintesimal volume are changed but then evaluated at the original values of the coordinates, therefore

$$d^4x = d^4y.$$

and the quantity

$$d^4x$$

is invariant under active diffeomorphisms.

We see that in integrals over spacetime volume, the volume element d^4x does not change under an active diffeomorphism, while it does change under a coordinate transformation. However, the volume element $\sqrt{-g}\ d^4x$ is invariant under a coordinate transformation but not under an active diffeomorphism.

We will now show that any scalar integral is invariant under an active diffeomorphism that vanishes at the end points of integration. Let us find the effect of an active diffeomoprhism generated by the vector field ξ_ν on the metric $g_{\mu\nu}$.

$$y^\mu = x^\mu + \epsilon\xi^\mu(x)$$

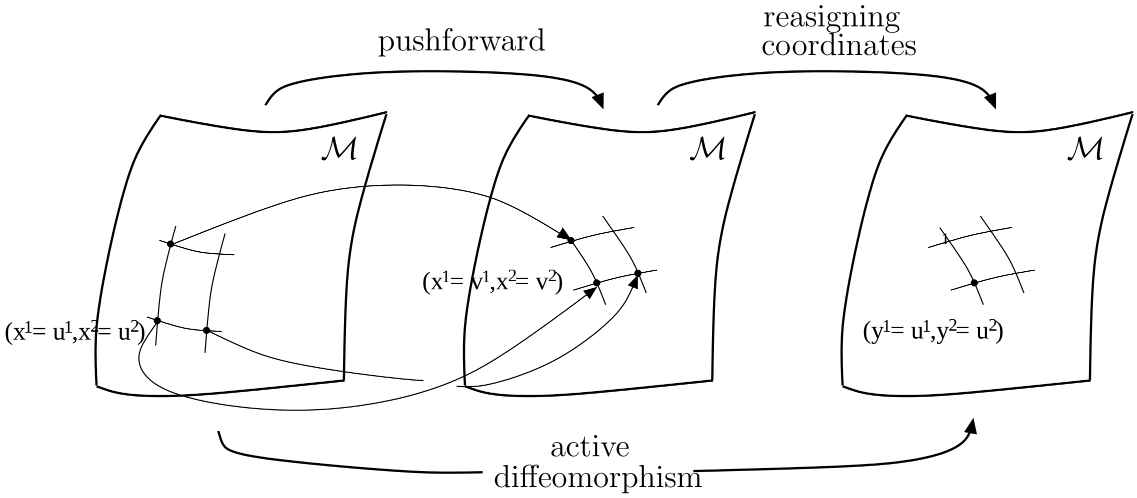


Figure 1.21: We first do a pushforward mapping the "corners" of the infintesimal volume to new points. We then do a coordinate transformation to assign the new points the original coordinate values.

Differentiating, we get

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu + \epsilon \partial_\nu \xi^\mu. \tag{1.77}$$

$$\begin{aligned} g^{\mu\nu}(y) &= \frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma} g^{\rho\sigma}(x) \\ &= (\delta^\mu_\rho + \epsilon\partial_\rho \xi^\mu)(\delta^\nu_\sigma + \epsilon \partial_\sigma \xi^\nu) g^{\rho\sigma}(x) \\ &= g^{\mu\nu}(x) + \epsilon g^{\rho\nu}(x)\partial_\rho \xi^\mu + \epsilon g^{\mu\rho}(x)\partial_\rho \xi^\nu \end{aligned} \tag{1.78}$$

$$g^{\mu\nu}(y) = g^{\mu\nu}(x^\rho + \epsilon\xi^\rho) = g^{\mu\nu}(x) + \epsilon \xi^\rho \partial_\rho g^{\mu\nu}(x) \tag{1.79}$$

$$\frac{\delta g^{\mu\nu}}{\epsilon} = \frac{g^{\mu\nu}(y) - g'^{\mu\nu}(y)}{\epsilon} \tag{1.80}$$

$$\begin{aligned} \delta g^{\mu\nu}/\epsilon &= \xi^\rho \partial_\rho g^{\mu\nu} - g^{\mu\rho}\partial_\rho \xi^\nu - g^{\rho\nu}\partial_\rho \xi^\mu \\ &= \xi^\rho \partial_\rho g^{\mu\nu} - g^{\mu\rho}(\nabla_\rho \xi^\nu - \Gamma^\nu_{\sigma\rho}\xi^\sigma) - g^{\rho\nu}(\nabla_\rho \xi^\mu - \Gamma^\mu_{\sigma\rho}\xi^\sigma) \\ &= \xi^\rho(\partial_\rho g^{\mu\nu} + g^{\mu\sigma}\Gamma^\nu_{\rho\sigma} + g^{\sigma\nu}\Gamma^\mu_{\rho\sigma}) - g^{\mu\rho}(\nabla_\rho \xi^\nu) - g^{\rho\nu}(\nabla_\rho \xi^\mu) \\ &= \xi^\rho \nabla_\rho g^{\mu\nu} - \nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu \\ &= -\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu \end{aligned} \tag{1.81}$$

We have under a diffeomorphism

$$\delta g^{\mu\nu} = -\epsilon \nabla^\mu \xi^\nu - \epsilon \nabla^\nu \xi^\mu .$$

Then from

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

we have

$$\delta \sqrt{-g} = \epsilon (\nabla_\alpha \xi^\alpha) \sqrt{-g}. \tag{1.82}$$

Putting this together and as the Lie derivative of a scalar is the directional derivative

$$\begin{aligned}
\delta S_{\epsilon\xi} &= \int \mathcal{L}_{\epsilon\xi} (F \sqrt{-g}) d^4 x \\
&= \epsilon \int (\xi^\mu \nabla_\mu F + F \nabla_\mu \xi^\mu) \sqrt{-g} d^4 x \\
&= \epsilon \int \nabla_\mu (F \xi^\mu) \sqrt{-g} d^4 x \\
&= \epsilon \int F \xi^\mu d^3 \Sigma_\mu
\end{aligned} \tag{1.83}$$

where we have used Gauuss' law. For variations with $\xi^\mu = 0$ at the bounadries, $\delta S_{\epsilon\xi} = 0$.

What is the physical implication of this? Note if the metric was a nondynamical object it could not be varied without changing the original action principle. Since we are assuming the metric is not a nondynamical object that would otherwise break the background independence of the theory - diffeomorphism invariance is formally equivalent to general covariance, namely the invariance of the field equations under arbitrary changes of the spacetime coordinates $\vec{x}$ and t. Therefore if the action principle is based on an integral scalar **it will have active diffeomorphisms as a gauge transformation**.

1.6.2 Palatini's first order formalism of Einstein's equation.

This is a special connection built out of the metric and its derivatives is the so-called metric connection uniquely determined by the relation

$$\nabla_\alpha g_{\mu\nu} = 0$$

which follows from the requirement that a vector parallely transported around an infintesimal loop retains the same length.

If the action is to be a scalar, the Lagrangian for the spacetime metric cannot depend on the first derivatives $\partial_\alpha g_{\mu\nu}$, because $\nabla_\alpha g_{\mu\nu} = 0$ and the first derivatives can all be transformed to zero at a point. Thus for the action to be an integral scalar, one is forced to include second derivatives of the metric. The Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ is the simplest scalar that can be formed from second derivatives of the metric.

$$\mathcal{L}_G = (-g)^{1/2} R \tag{1.84}$$

This as a functional of g_{ab} and its first and second derivatives, namely,

$$\mathcal{L}_G = \mathcal{L}_G(g_{ab}, g_{ab,c}, g_{ab,cd}), \tag{1.85}$$

The Euler-Lagrange equations include an extra term because of the dependence on $g_{ab,cd}$,

$$\frac{\partial \mathcal{L}}{\partial g_{ij}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{g}_{ij}}\right) = 0 \tag{1.86}$$

The direct calculation is horrendous and there is a more efficient method which we present in appendix D. But here we detail a particular approach which uses a choice of basic variables considered more suitable to for quantization, i.e. in which connections are considered as independent variables.

We wish to consider the connection as a dynamical degree of freedom in its own right, with no dependency on the metric. How can we relinquish the connection's relation to the metric but still have a theory with the same physical content?

Inspiration comes from single particle mechanics. The variables $x(t)$ and $p(t)$ become two independently adjustable functions in a new variational principle [?],

$$I = \int [p\dot{x} - H(p, x, t)] dt \tag{1.87}$$

$$\delta I = p\delta x|_{x',t'}^{x'',t''} + \int_{x',t'}^{x'',t''} \left[\left(\dot{x} - \frac{\partial H}{\partial p}\right)\delta p + \left(-\dot{p} - \frac{\partial H}{\partial x}\right)\delta x\right] dt. \tag{1.88}$$

Out of the extrematization with respect to the $p(t)$, one recovers the standard formula for the momentum in terms of the velocity,

$$\dot{x} = \frac{\partial H(p, x, t)}{\partial p}. \tag{1.89}$$

Extremisation with respect to the $x(t)$ give the equation of motion

$$\dot{p} = -\frac{\partial H(p, x, t)}{\partial x}. \tag{1.90}$$

Palatini's first order formalism of Einstein's equation.

$$\mathcal{L}_G = \mathcal{L}_G(g^{ab}, \Gamma^a_{bc}, \Gamma^a_{bc,d}) \tag{1.91}$$

$$S = \int_\Sigma (-g)^{1/2} g^{ab} R_{ab}(\Gamma^a_{bc}, \Gamma^a_{bc,d}) d^3x. \tag{1.92}$$

However, we consider the

$$\mathcal{L}_G = \mathcal{L}_G(\tilde{g}^{ab}, \Gamma^a_{bc}, \Gamma^a_{bc,d}) \tag{1.93}$$

$$S = \int_\Sigma \tilde{g}^{ab} R_{ab}(\Gamma^a_{bc}, \Gamma^a_{bc,d}) d^3x. \tag{1.94}$$

with respect to $\tilde{g}^{\mu\nu}$ only, and the principle of stationary action gives immediately the vacuum field equations

$$R_{ab}(\Gamma^a_{bc}) = 0. \tag{1.95}$$

The more difficult part is showing that independent variation of the connection Γ^a_{bc} in (1.94) implies the connection is the metric connection: $\Gamma^c_{ab} = g^{cd}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab})/2$, (or rather the condition $\nabla_c g_{ab} = 0$, which is equivalent). We do this now. In a freely falling frame the connection vanishes

$$\begin{aligned} \delta R^\alpha{}_{\beta\mu\nu} &\overset{*}{=} \partial_\mu(\delta\Gamma^\alpha_{\beta\nu}) - \partial_\nu(\delta\Gamma^\alpha_{\beta\mu}) \\ &\overset{*}{=} \nabla_\mu(\delta\Gamma^\alpha_{\beta\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\beta\mu}) \end{aligned} \tag{1.96}$$

If a tensor equation holds in one coordinate system it holds in all. The Palatini equation

$$\delta R^{\alpha}{}_{\beta\mu\nu} = \nabla_{\mu}(\delta\Gamma^{\alpha}_{\beta\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\beta\mu}) \tag{1.97}$$

Contracting on α and μ we get the useful formula

$$\delta R_{\beta\nu} = \nabla_{\alpha}(\delta\Gamma^{\alpha}_{\beta\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\beta\alpha}) \tag{1.98}$$

$$\begin{aligned} \delta S &= \int_V (\sqrt{-g}g^{\mu\nu})\delta R_{\mu\nu} d^4x \\ &= \int_V (\sqrt{-g}g^{\mu\nu})[\nabla_{\alpha}(\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\mu\alpha})] d^4x \end{aligned} \tag{1.99}$$

Integrating by parts

$$\begin{aligned} \delta S &= \int_V [\nabla_{\nu}(\sqrt{-g}g^{\mu\nu})\delta\Gamma^{\alpha}_{\mu\alpha} - \nabla_{\alpha}(\sqrt{-g}g^{\mu\nu})\delta\Gamma^{\alpha}_{\mu\nu}] d^4x \\ &= \int_V [(\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu}))\delta\Gamma^{\rho}_{\mu\nu}] d^4x \end{aligned} \tag{1.100}$$

The integrand vanishes:

$$\sum_{\mu,\nu}[\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu})]\delta\Gamma^{\rho}_{\mu\nu} = 0 \tag{1.101}$$

where we have reinstated the summation for clarity. The variations in $\Gamma^{\rho}_{\mu\nu}$ are aribtrary, but symmetric in μ and ν, so the above is equivalent to

$$(2\sum_{\mu<\nu} + \sum_{\mu=\nu})\,[\frac{1}{2}\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) + \frac{1}{2}\delta^{\mu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\nu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu})]\delta\Gamma^{\rho}_{\mu\nu} = 0$$

where the expression in the brakets of (1.101) has been replaced by its symmetric part. In this summation all the $\delta\Gamma^{\rho}_{\mu\nu}$ are independent of each other and so we have

$$\frac{1}{2}\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) + \frac{1}{2}\delta^{\mu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\nu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu}) = 0 \tag{1.102}$$

We now show in turn that the covariant derivatives of $(\sqrt{-g}g^{\mu\nu})$, $\sqrt{-g}$, $g^{\mu\nu}$, and $g_{\mu\nu}$ vanish. Setting $\nu = \rho$ in (1.102) and summing over ν implies

$$\nabla_\sigma(\sqrt{-g}g^{\mu\sigma}) = 0$$

Using this in (1.102) we get

$$\nabla_\rho(\sqrt{-g}g^{\mu\sigma}) = 0 \tag{1.103}$$

Taking the determinant of

$$g^{\mu\sigma}\nabla_\rho\sqrt{-g} + \sqrt{-g}\nabla_\rho g^{\mu\sigma} = 0$$

considered as a matrix with indices μ, σ implies

$$\nabla_\rho\sqrt{-g} = 0. \tag{1.104}$$

Which then implies

$$\nabla_\rho g^{\mu\nu} = 0. \tag{1.105}$$

Which we use in the following

$$\begin{aligned} 0 = g_{\mu\alpha}\nabla_\rho(\delta^\alpha_\nu) &= g_{\mu\alpha}\nabla_\rho(g^{\alpha\sigma}g_{\sigma\nu}) \\ &= g_{\mu\alpha}g^{\alpha\sigma}\nabla_\rho g_{\sigma\nu} \\ &= \delta^\sigma_\mu\nabla_\rho g_{\sigma\nu} \\ &= \nabla_\rho g_{\mu\nu}. \end{aligned} \tag{1.106}$$

This completes the proof.

1.7 Hamiltonian Formulation

1.7.1 Gauge Transformations in Phase Space

A dynamic system is said to be constrained if its physical phase space is a submanifold $\overline{\Gamma}$ of the original phase space Γ, called the constraint surface. The constraint surface is defined by the vanishing of a set of functions $C_k(q,p)$ called the constraints:

$$\overline{\Gamma} := \{p \in \Gamma \mid C_k = 0, \quad k = 1, \ldots, K\}. \tag{1.107}$$

Consistency resquires that the constraints are preserved under time evolution, this may imply further constraints, which in turn have to be checked for consistency, leading to the possibly of other constraints that go by the names of secondary, tertiary e.t.c. Hopefully at some point this iteration (the Dirac-Bergman algorithm) terminates and one arrives at a finite, total number of constraints

$$C_k = 0, \quad k = 1, \ldots, K, \tag{1.108}$$

that are consistent with evolution.

To understand the phase space of a constrained system, one needs to distinguish between first class and second class constraints. First class constraints are defined by the property that their Poisson brackets with all other constraints vanish on the constraint surface. All other constraints, i.e. those that are not first class, are called second class.

First class constraints play two roles: as well as specifying the constraint surface, along with the other constraints, they also generate flows in the constraint surface.

A constrained system is said to be of first class if for all covectors n_α normal to $\overline{\Gamma}$, $\Omega^{\alpha\beta} n_\alpha$ is tangent to $\overline{\Gamma}$.

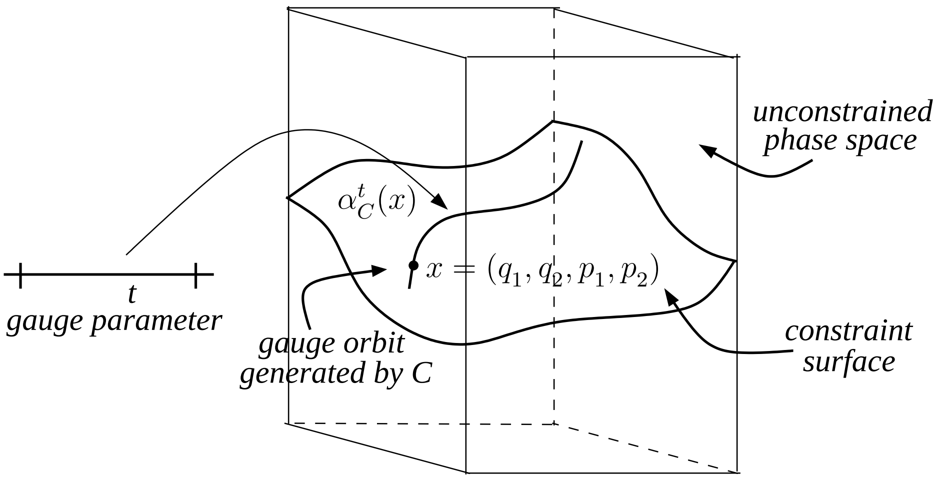


Figure 1.22: partComptDitt1.

The Hamiltonian constraint of the theory is a first class constraint and should therefore be viewed as a gauge transformation. However, since the constraint is responsible for generating time evolution is the unphysical unfolding of a gauge transformation!

The definition of an observable (invariant quanitity under gauge transformations) in a constrained system is a variable that weakly (on the constraint surface) commutes with all the first class constraints . However, since one of these constraints is the generator of time evolution (the Hamiltonian constraint), the observables are constants of motion.

These functions are called *structure functions.* If they are constants, they are called structure constants and the constraint functions C_k are generators of a Lie algebra of the set of functions on Γ.

Due to non-trivial structure *functions* in the commutators involving the hamiltonian constriant of GR makes the situation outside the range of standard techniques. A way out of this is to use an alternative set of constraints whose commutators only involve structure constants - for example, the so-called Master constriant.

1.7.2 Second Class Constraints and Gauge Fixing

E.g. The Lorentz gauge constraint

$$\partial^\mu A_\mu(x) = 0. \tag{1.109}$$

These appear in the constrained BF theory equivalent to the first order tetrad formulation of GR. How to promote these second class constraints is of cruical importance in formulating improved version spin foams (quantum spacetime) capable of reproducing the correct semiclassical limit (see chapter 4).

1.8 Partial, Complete and Dirac Observables

A partial observable is

A complete observable is

Dirac Observables

Different choices of clocks can be interpreted as different setups for a physical measurent.

$$\mathbf{x}(t) = \mathbf{x} - \frac{\mathbf{p}}{p_0}(t - x^0) \tag{1.110}$$

a true observable formed by two partial observables.

Has vanishing Poisson brackets with the constraint $\mathbf{p}^2 - m^2$:

$$\{x_1(t), p_1^2 + p_2^2 + p_3^2 - m^2\} = \{x_1(t), p_1^2\} \tag{1.111}$$

For systems with one constraint the idea works in the following way: Assume that the system is totally constrained so that the constraint generates the time evolution (which is then considered as a gauge transformation). Use a phase space function T, which is not a Dirac observable, as a clock which "measures" the time flow, denoted $T(\tau)$ i.e. the gauge transformation. Consider another phase space function f and calculate the value of f "at the time" at which T assumes the value τ, that is, the combined quantity $f(T)$ when $T = \tau$. Since the value of f at a fixed time τ does not change with time, the result will be time independent, i.e. a Dirac observable. Moreover varying τ gives a one-parameter family of Dirac observables. Following Rovelli, T and f partial observables and the one-parameter family of Dirac observables complete observables.

Dittrich generalizes these ideas to an arbitary number of gauge degrees of freedom [301].

Briefly discuss the notion.

In a familiar setting say we want the position or momentum some time later, what would do this for you is

$$q(t + \delta t) = q(t) + \delta t \, \frac{\partial q}{\partial t} = q(t) + \delta t \, \{\mathcal{H}, q(t)\}.$$

Now if we want to know what the position is at $t_0 + 2\delta t$ we have

$$\begin{aligned} q(t + 2\delta t) &= q(t + \delta t) + \delta t \, \{\mathcal{H}, q(t + \delta t)\} \\ &= q(t) + 2\delta t \, \{\mathcal{H}, q(t)\} + \delta t^2 \, \{\mathcal{H}\{\mathcal{H}, q(t)\}\}. \end{aligned} \tag{1.112}$$

Similarly

$$\begin{aligned} q(t + 3\delta t) &= q(t + \delta t) + \delta t\{\mathcal{H}, q(t + \delta t)\} \\ &= q(t) + 3\delta t\{\mathcal{H}, q(t)\} + 3 \cdot 2\delta t^2\{\mathcal{H}\{\mathcal{H}, q(t)\}\} + \delta t^3 \, \{\mathcal{H}\{\mathcal{H}\{\mathcal{H}, q(t)\}\}\}. \end{aligned} \tag{1.113}$$

Let us introduce the shorthand notation

$$\{\mathcal{H}, q(t_0)\}_{(2)} \equiv \{\mathcal{H}\{\mathcal{H}, q(t_0)\}\}, \quad \{\mathcal{H}, q(t_0)\}_{(3)} \equiv \{\mathcal{H}\{\mathcal{H}\{\mathcal{H}, q(t_0)\}\}\}, \quad \ldots$$

It is not too difficult find

$$
\begin{aligned}
q(t_0+t) = q(t_0+N\delta t) &= q(t) + t\,\{\mathcal{H}, q(t_0)\} + \frac{t^2}{2!}\{\mathcal{H}, q(t_0)\}_{(2)} + \cdots + \\
&\quad + \ldots \frac{t^n}{n!}\{\mathcal{H}, q(t_0)\}_{(n)} + \ldots \\
&=: \exp t\{\mathcal{H}, \}q(t_0).
\end{aligned} \tag{1.114}
$$

$$\alpha^t_{\mathcal{H}}(q)(x) := \exp t\{\mathcal{H}, \}q(t).$$

$$\alpha^t_C(x)$$

$$\alpha^t_C(f)(x) := f(\alpha^t_C(x)) \tag{1.115}$$

If $\alpha^s_C(x)$ is the flow generated by the constraint C starting from the point x, then the value of the function $\alpha^s_C(f)$ at the point x is given by

$$\alpha^s_C(f)(x) := f(\alpha^s_C(x)). \tag{1.116}$$

It can be calculated with the series

$$\alpha^s_C(f(x)) = \sum_{k=0}^{\infty} \frac{s^k}{k!}\{C, f(x)\}_k. \tag{1.117}$$

Partial observables

"a physical quantity to which we can associate a (measuring) procedure leading to a number", [51]. If we assume that one can associate to an arbitrary phase space function such a measuring procedure, then any phase space function is a partial observable. Partial observables need not be a Dirac observable.

T is a kind of clock, whose values .

$$\alpha^s_C(T)(x) = \tau \tag{1.118}$$

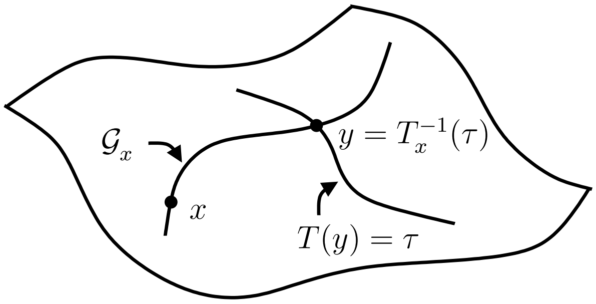


Figure 1.23: comptasGIE.

Complete observables

A complete observable is "a quantity whose value can be predicted by the theory in classical theory)".

Now given another phase space function f and a phase space point x one can calculate the value the function takes at the point of the gauge orbit through x at which T assumes the value τ. The complete observable predicts the value of f for the "time" τ. We denote this predicted value

$$F_{[f;T]}(\tau, x). \tag{1.119}$$

More formally, the condition defining $F_{[f;T]}(\tau, x)$ is

$$F_{[f,T]}(\tau, x) := \alpha_C^s(f(x))_{|\alpha_C^s(T(x))=\tau}. \tag{1.120}$$

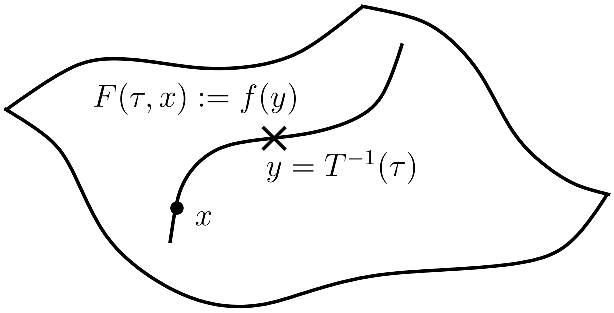


Figure 1.24: partComptDitt3a. $F_{[f,T]}(\tau, x)$ is defined as the value of f evaluated at the point y on the gauge orbit where $T(y) = \tau$. Solving for y we can write $F_{[f,T]}(\tau, x) = f(y) = f(T^{-1}(\tau))$. Note that $F_{[f,T]}(\tau, x)$ only depends on τ and not on the location of x along the gauge orbit.

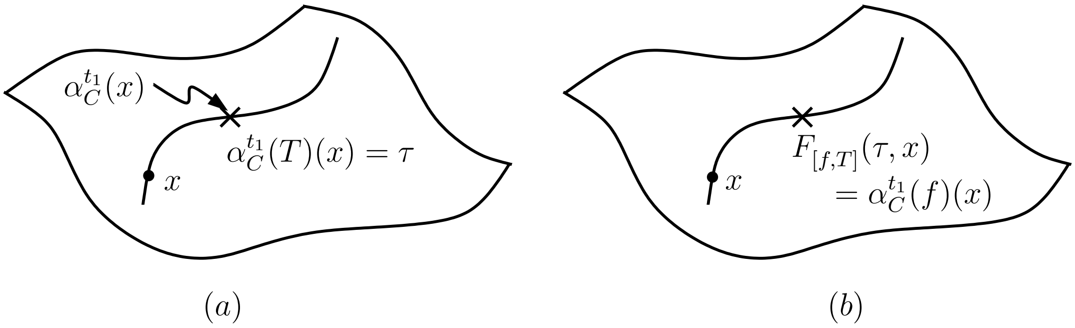


Figure 1.25: partComptDitt3. (a) $t = t_1$ when the clock function $T(\alpha_C^t(x))$ assumes the value τ. (b) The function $F_{[f,T]}(\tau, x)$ gives the value that the function $f(\alpha_C^t(x))$ assumes if the function $T(\alpha_C^t(x))$ assumes the value τ. $F_{[f,T]}(\tau, x)$ is a complete observable generated from the partial observables $T(x)$ and $f(x)$.

Clock variables

The gauge orbits are one-dimensional. Hence, we can parametrize a gauge orbit with the values of a phase space function T, if this phase space function changes strictly monotonously along the orbit, there should be only one point on each gauge orbit where the function T assumes the value τ.

1.8.1 A Complete Observable is a Dirac Observable

Since the definition of $F_{[f;T]}(\tau, x)$ depends on the phase space point, we have actually constructed a phase space function

$$F_{[f;T]}(\tau, \cdot).$$

However, by definition, $F_{[f,T]}(\tau, x)$ only depends on the gauge orbit through x and not on where x is located on the orbit. Hence, if the phase space points x and y both lie on the same gauge orbit we have

$$F_{[f,T]}(\tau, x) = F_{[f,T]}(\tau, y), \tag{1.121}$$

i.e. $F_{[f;T]}(\tau, \cdot)$ is constant along gauge orbits (see fig.(1.26)).

No matter which point on a particular gauge orbit we start at, f always assumes the same value where T assumes the value τ. Hence the complete observable $F_{[f;T]}(\tau, \cdot)$ is a Dirac observable.

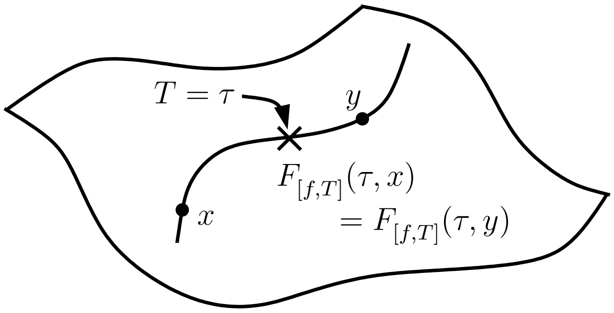


Figure 1.26: partComptDitt5. If we started at another phase space point y on the same orbit, we would still evaluate f at the same point on that gauge orbit as before, namely at that point, at which T assumes the value τ.

To every evolution orbit it assigns one value, namely the prediction of f for that moment at which T is equal to τ. But this value will change if we change τ, i.e. predict f for another value of the clock varaible T. In this sence we can 'evolve' $F_{[f;T]}(\tau, \cdot)$ through all values of the clock variable T.

We now consider some examples.

Examples

(i) Parameterized Harmonic oscillator

phase space point: $x = (q, p, t, p_t)$

$$C = p_t + \frac{p^2}{2m_1} + \frac{1}{2}m\omega^2 q^2 \tag{1.122}$$

where p_t is the conjugated momentum to the time variable t and p are conjugated to the position variables q. A natural choice for a clock variable is t and we can ask for the position of the first particle at that moment at which t assumes the value τ. We will denote this observable by $F_{[q;t]}(\tau)$ and it can be easily calculated to be

$$F_{[q;t]}(\tau) = q \cos \omega(\tau - t) + \frac{p}{m\omega} \sin \omega(\tau - t). \tag{1.123}$$

$$\begin{aligned} \{C, q(x)\} &= \{p_t + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, q\} \\ &= -\frac{p}{m} \end{aligned}$$

$$
\begin{aligned}
\{C,\{C,q(x)\}\} &= -\frac{1}{m}\{C,p(x)\} \\
&= -\frac{1}{m}\{p_t+\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2,p\} \\
&= -\omega^2 q
\end{aligned}
$$

$$
\begin{aligned}
\alpha_C^s(q) &= q(\alpha^s(x)) \\
&= \exp s\{C,\}q(x) \\
&= q(1-\frac{s^2}{2!}\omega^2+\frac{s^4}{4!}\omega^4-\dots)-\frac{p}{m\omega}(s\omega-\frac{s^3\omega^3}{3!}+\dots) \\
&= q\cos\omega s-\frac{p}{m\omega}\sin\omega s
\end{aligned}
$$

$$
\begin{aligned}
\{C,t\} &= \{p_t+\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2,t\} \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
\alpha_C^s(t) &= t(\alpha^s(x)) \\
&= \exp s\{C,\}t(x) \\
&= t-s
\end{aligned}
$$

$$
\begin{aligned}
F_{[q;t]}(\tau)(x=\{q,p,t,p_t\}) &= q(\alpha_C^s)_{|t(\alpha_C^s)=\tau} \\
&= q\cos\omega(\tau-t)+\frac{p}{m\omega}\sin\omega(\tau-t) \qquad (1.124)
\end{aligned}
$$

(ii)

phase space point: $x=(q_1,p_1,q_2,p_2,t,p_t)$

$$
C=p_t+\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2} \qquad (1.125)
$$

where p_t is the conjugated momentum to the time variable t and p_1, p_2 are conjugated to the two position variables q_1, q_2. A natural choice for a clock variable is t and we can ask for the position of the first particle at that moment at which t assumes the value τ. We will denote this observable by $F_{[q_1;t]}(\tau)$ and it can be easily calculated to be

$$F_{[q_1;t]}(\tau) = q_1 + \frac{p_1}{m_1}(\tau - t). \tag{1.126}$$

$$\begin{aligned}
\{C, q_1(x)\} &= \{p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}, q_1\} \\
&= -\frac{p_1}{m_1}
\end{aligned}$$

$$\begin{aligned}
\{C, \{C, q_1(x)\}\} &= \frac{1}{m_1}\{C, p_1(x)\} \\
&= -\frac{1}{m_1}\{p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}, p_1\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\alpha_C^s(q_1) &= q_1(\alpha^s(x)) \\
&= \exp s\{C, \}q_1(x) \\
&= q_1 - \frac{p_1}{m_1}s
\end{aligned} \tag{1.127}$$

$$\begin{aligned}
\{C, t(x)\} &= \{p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}, t\} \\
&= 1
\end{aligned} \tag{1.128}$$

$$\begin{aligned}
\alpha_C^s(t) &= t(\alpha^s(x)) \\
&= \exp s\{C, \}t(x) \\
&= t + s
\end{aligned} \tag{1.129}$$

$$\begin{aligned}
F_{[q_1;t]}(\tau)(x = \{q_1, p_1, t, p_t\}) &= q_1(\alpha_C^s)_{|t(\alpha_C^s)=\tau} \\
&= q_1 + \frac{p_1}{m_1}(\tau - t)
\end{aligned} \tag{1.130}$$

It Poisson commutes with the constraint and is therefore a Dirac observable.

$$\begin{aligned}
\{F_{[q_1;t]}, C\} &= \{q_1 + \frac{p_1}{m}(\tau - t)\ ,\ p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\} \\
&= -p_1\{p_t, t\}\frac{1}{m_1} + \{q_1, p_1^2\}\frac{1}{2m_1} \\
&= 0 \qquad (1.131)
\end{aligned}$$

The Poisson bracket of two observables $F[q_1;t](\tau_1)$ and $F[q_1;t](\tau_2)$ at two different clock values τ_1 and τ_2 is phase space independent

$$\begin{aligned}
\{F[q_1;t](\tau_1), F[q_1;t](\tau_2)\} &= \{q_1 + \frac{p_1}{m_1}(\tau_1 - t)\ ,\ q_1 + \frac{p_1}{m_1}(\tau_2 - t)\} \\
&= \frac{1}{m_1}(\tau_2 - \tau_1) \qquad (1.132)
\end{aligned}$$

Now one could also choose the position of the second particle as a clock variable and ask for the position of the first particle at that moment at which the second particle has position τ_2. The corresponding observable is

$$F_{[q_1;q_2]}(\tau') = q_1 + \frac{p_1}{m_1}\frac{m_2}{p_2}(\tau' - q_2). \qquad (1.133)$$

$$\alpha_C^s(q_1) = q_1 + \frac{p_1}{m_1}s, \qquad \alpha_C^s(q_2) = q_2 + \frac{p_2}{m_2}s$$

we rewrite the second equation as

$$s = \frac{m_2}{p_2}(\alpha_C^s(q_2) - q_2)$$

to use it to replace s

$$\begin{aligned}
F_{[q_1;q_2]}(\tau')(x = \{q_1, p_1, t, p_t\}) &= q_1(\alpha_C^s)_{|q_2(\alpha^s)=\tau} \\
&= \left(q_1 + \frac{p_1}{m_1}s\right)_{|q_2(\alpha_C^s)=\tau'} \\
&= q_1 + \frac{p_1}{m_1}\frac{m_2}{p_2}(\tau' - q_2) \qquad (1.134)
\end{aligned}$$

If one ignores that τ and τ' refer to different clocks, (1.133) looks of course quite different from (1.126). However if one takes into account that the value τ is reached at that moment at which

$$\tau' = F_{[q_2;t]} = q_2 + \frac{p_2}{m_2}(\tau - t) \tag{1.135}$$

and uses this to replace τ' in (1.133) one will get back to (1.126). In so far both choices of clock variables give us the same time evolution if one takes into account the "translation" (1.126) between the clock readings τ' and τ'.

(iii) FRW-cosmology with massless scalar field

phase space point: $x = (a, P_a, \phi, P_\phi)$

with constraint:

$$C = \frac{1}{2}(-\frac{P_a^2}{a} + \frac{P_\phi^2}{a^3})$$

Partial observables: clock

$$T = \phi \tag{1.136}$$

$$f = a \tag{1.137}$$

Clock observable:

$$F^\tau_{[f;T]}(x) = f(x') \tag{1.138}$$

where

$x' \sim x$ and $T(x') = \tau$

The complete observable for this example $\tau = 0$ is

$$F^{\tau=0}_{[a;\phi]}(x) = a \exp(-\mathrm{sgn}(P_a P_\phi)(0 - \phi)) \tag{1.139}$$

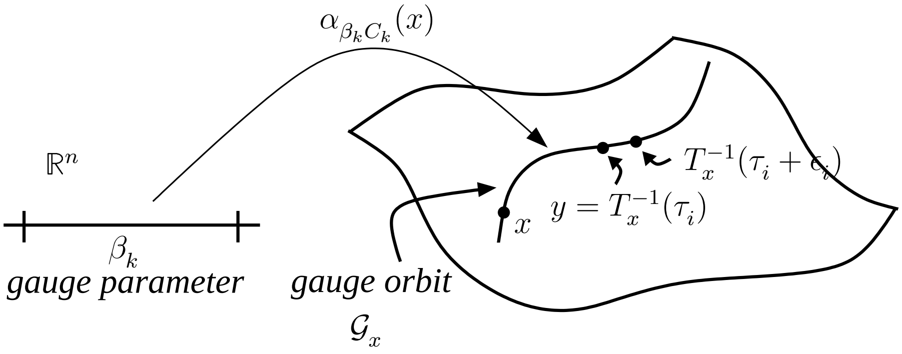


Figure 1.27: partComptDitt4.

1.8.2 Systems with Several Constraints

$$F_{[f;T^K]}(\tau) = \sum_{r=0}^{\infty} \frac{1}{r!}\{\cdots\{f, \tilde{C}_{K_1}, \cdots\}, \tilde{C}_{K_r}\}\ (\tau^{K_1} - T^{K_1}) \cdots (\tau^{K_r} - T^{K_r}). \tag{1.140}$$

1.8.3 Infinitely Many Constraints

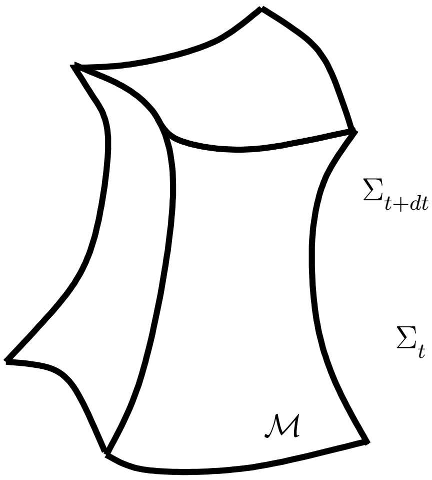


Figure 1.28: FolofSpacetime. Foliation of spacetime $\mathcal{M}$ by a family of spacelike hpersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$.

1.8.4 Observables for Canonical General Relativity

Discuss this in more detail in Appendix E and also in the chapter on the master constraint.

True observables of general relativity. To explicitly find them would this would amount to solving Einstein's equations. This is a problem of the classical theory, unrelated to the problem of its quantization.

For the case of gravity in four space time dimensions and for asymptotically flat boundary conditions there are 10 gauge invariant phase space functions known. These are the ADM charges [306] given by the generators of the Poincare transformations at spatial infinty. For gravity coupled to matter, in some cases gauge invariant functions describing matter are known but in general no phase space functions which describe the gravitational degrees of freedom (with the exception of the ADM charges). Yet there are infinitely many gauge invariant degrees of freedom.

Longstanding open problem in classical GR. Partial, complete observables provide a method for calculating Dirac observables, [303].

defines complete observables for systems with several, including infinite, number of constraints.

can construct explicit expression for complete observerable in canonical general relativity which is pameratized by one parameter insted of infinite - could develope approximation schemes.

1.8.5 Approximate Observables for Canonical General Relativity

• Approximate Dirac observables with a dynamical interpretation can be calculated explicitly to an arbitrary order

• Precise inderstanding of linearized theory and (quantum) field theory on a fixed background as approximations to full general relativity

• Formulism can be used to address construction and interpretation of Dirac observables

• Can be generalized to expansion around symmetry reduced/cosmological sectors $\Rightarrow$ use knowledge on symmetry reduced sectors to construct approximate Dirac observables for full theory

• Need a better understanding of the (quantum) interpretation of complete observables, in particular role of clock variables.

1.9 Recovering Time

The hypothesis that the fundamental theory of nature can be formulated in a timeless language [288], and that temporal phenomena could be emergent [289], [290], [291].

see week 41 beaz

Rovelli wants to use thermodynamics to **define** what we call time as we usualy mean. does this as follows. Given a classical statistical state ρ, find some Hamiltonian H such that ρ is the Gibbs state $\exp(-H/kT)$. In lots of cases this isn't hard; it basically amounts to

$$H_0 = -kT \ln \rho_0 \tag{1.141}$$

Therefore, in a statistical context we have in principle an operational procedure for determining which one is the time variable: Measure ρ_0; compute H_0 from (1.141); compute the hamiltonian flow $s(t)$ of H_0 on Σ: the time variable t is the parameter of this flow. The multiplicative constant in front of H_0 just sets the unit in which time is measured.

Of course, H will depend on T, but this is really is just saying that fixing your temperature fixes your units of time!

1.10 Why Quantise Gravity

Combine general relativity and quantum theory into a single theory that can claim to be the complete theory of nature.

General relativity has the problem with infiniteis as inside a black hole the density of matter and the strength of the gravitational field quickly become infinite. That also the case in the very early in the history of the universe.

Infinities in qunatum mechanics occur whenever you attempt to apply it to fields, such as the electromagnetic field. The problem is that the electric and magnetic fields have values at every point in space so an infinite number of variables. In quantum theory, there are uncontrollable fluctuations in the values of every quantum variable. An infinite number of variables, fluctuating uncontrollably, can lead to equations that get out of hand and predict infinite numbers.

It has been a long held hope that when gravity is taken into account, the fluctuations will be tamed and will be finite, and that the infinities of classical general relativity will be brought under control by quantum theory.

1.11 The Problem of Quantising Gravity

Say you didn't know Schrodinger's equation and wanted to formulate it for a general potential. It goes without saying that it is sensible for this equation to not depend on some particular classical tragectory! Of course the equation you arrive at will have solutions that approximate some particular trajectory, but this is a different matter. This here is what you need to appreicate in order to understand Ashtekar's statement!

Now say you want to formulate the equations of quantum gravity. As with Shcrodinger's equation for an electron in a general potential, the general equation for QG should not depend on some particular classical tragectory, that is, it should not depend on some particular classical spacetime!! As above, the equation's of QG have solutions that approximate some classical spacetime but this is something else.

Physics in the absence of spacetime. How can you do that? For example, if you want to (and this is what is mostly done in LQG) you can introduce a classical spacetime as a mathematical device on which to formulate the theory, but you do it in a way that the end result doesn't depend on the fidicual spacetime you used. What you end up with is anything, quite the opposite. Turns out that LQG is essentially unique (at least at the kinematic level).

If someone tells you that you're crazy, you'll believe anything! you can ask should we take Schrodinger's equation for a general system, that appears in all these physics texts books, and replace it with something that depends on some particular classical tragectory?

Obviously, there are interpretative issues that such a candidate for a theory of QG need to be addressed to be taken seriously which are just as important as the tequiniqual side! Significant progress in formulating an interpretative framework for all such theories - For example Rovelli has some very nice simple, intersting and natural ideas.

1.11.1 The Problem of Time

absolute time: Information brought in from outside.

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L} = 0. \tag{1.142}$$

The Schrödinger equation $\frac{i\hbar^2}{2m} \frac{\partial}{\partial t} |\psi >= \hat{\mathcal{H}} |\psi >$

$$\hat{\mathcal{H}} |\psi >= 0 \tag{1.143}$$

There are prblems with the presence of this constraint.

That the Shrödinger equation, the varaibles "x" and "t" play very different roles. In contrast t is assumed to be a continous external parameter. expected to have a clock that behaves perfectly classically and is completely external to the system under study. As we have seen, for a generally corvariant theory, any reference system must be part of the dynamical system under study; in light of GR there is no such thing as a perfect clock.

How is one to do quantum mechanics in this new conceptual setting? As the previous discussion of the classical situation would suggest the answer: "relationally". One could envision promoting *all* variables of a system to quantum operators, and choosing one of them to play the role of a "clock". Say we call such a variable t (it could for instance be, the angular position of a real clock, or maybe the elongation of a pendulumn). One could compute conditional probabilities for the other variables to take certain values x_0 when the "clock" variable takes the value t_0. If the "clock" we use does correspond to a variable that is behaving classicaly as a clock, then the conditional probabilities will approximate well the probabilities computed in ordinary Schrödinger theory.

the conditional probabilities are still well defined, but they do not approximate Schrödinger theory. If there is no variable that can be considered as a good classical clock, Shrödinger's quantum mechanics does not make any sense and the relational quantum mechanics is therefore a generalization of Shrödinger's quantum mechanics.

"... Conventional mechanics describes evolution in "background time": the independent time variable t is assumed to to represent a mearsurable but non-dynamical physical quantitiy. This is true in the classical as well as in the quantum theory, where evolution in background time is given by unitary transformations. With general relativity, however, we have learned that there is no non-dynamical measurable time in nature: there is no background time in particular. Therefore, at the fundemantal level physics cannot describe evolution in time. It can only describe relations, or correlatins, between mearsurable quantities. It is therefore neccessary to extend the formulation of classical and quantum mechanics to such background independen context. [[37]] ... "

Rovelli has the clearest head with conceptual issues. What is observable in classical and quantum gravity [279].

"...Precisely as classical mechanics, the conventional formulation of QM describes evolution of states and observables in time. Precicsely as classical mechanics, this is not sufficient to deal with general relativistic systems, because these systems do not describe evolution in time; they describe correlations between observables. Therefore a formulation of /qm slightly more general than the conventional one - or quantum version of the relativistic mechanics... ...The possibility of such a formulation is discussed in the first part of this chapter [chapter 5 [?]]. In the second part, I discuss the physical interpretation of QM.

relational QM. No general notion of time at the fundemental level. No good clocks at the Plank scale.

The gravitational interactions become huge so we need a quantum theory of gravity that preserves the non-perturbative and background independence of the classical theory. The big bang - creation of the universe The notion of time is meanigless in a generic situation. Asking what happens before the big bang in quantum gravity is unlikely to make any sense because the classical notion of realtional time breeaks down near to the singularity.

There are schools of thought in relation to the "problem of time" in generally covariant theories. One trys to single out the "correct" time variable out of all the variables out of the covariant theory. Another approach takes general covariance more seriously, and keeps all variables on an equal footing.

May not need a radical upheavle of the present formulation of quantum mechanics. various approaches:

Important

The intemaitly related problem of quantum cosmology.

1.12 The Problem of Quantum Cosmology

Quantum cosmology is the application of quantum theory to the universe as a whole. In quantum cosmology there is no external clock or observer to the system of interest.there is no a priori classical world. The Copenhagen interpretation assumes from the outset the existence of a classical world with respect to which the results of a quantum measurements are interpreted.

the quantum constraint equations define quantum states that are supposed to describe the whole universe.

we need an interpretaion of the states of a quantum theory that allows the observer to be part of the quantum system.

Many worlds: This interpretation disposes of the notion of collapse of the wave function altogether. Instead asserts that everything that can happen *does* happen. Not only is there a world in which Schödinger's cat lives, there is also a disconnected world in which Schödinger's cat dies! The many-worlds approach is an option to use when we describing the universe as a whole and there is no external large system...

Laws of physics give rise to many different solutions.

Chapter 2

The early beginnings of LQG (1984-1992)

- **Motivation for Non-Perturbative (Background Independent) Quantum General Relativity (Not String Theory!)**
- **Canonical Quatization Of GR.**
- **Ashtekar's New Variables.**
- **Loops and the Loop Representation.**

2.1 Introduction

Loop quantum gravity a non-perturbative, canonical approach is mathematical rigorous, and it demands no new hypotheses each of which is especially important in quantum gravity because the lack of experimental guides.

It rests completely on the classical theory of general relativity and the core tenants of quantum mechanics.

Both General relativity and Quantum Mechanics has been successful in every test that it has encountered. overwhelmingly tested regimes of quantum theory and general relativity.

The idea of quantizing gravity.

We recall what we learned in the previous chapter: one introduces fields, electromagnetic, gravitational, etc fields over the spacetime manifold. Physical theories are still defined over spacetime, but which are invariant under active diffeomorphisms $\phi : \mathcal{M} \rightarrow \mathcal{M}$ of the spacetime manifold $\mathcal{M}$ into itself.

The Hamiltonian approach, which is very much like the approach people usually learn in a first course on quantum mechanics: if you know the wavefunction of your system at $t = 0$, the Hamiltonian tells you how it evolves in time from then on.

The canonical, background indepenent, theory was one of the first attempts based on a Hamiltonian formulation of the field equations of general relativity. Has to do with how we can reparametrize the spacial slice.

An additional degree of freedom that arises because we formulate the theory with triads instead of the metric. We can rotate the triad with a element of $SO(3)$ without changing the metric they encode.

2.2 Motivation for Non-Perturbative Quantum General Relativity (as Apposed to String Theory!)

It involves splitting the spacetime metric g_{ab} into two parts, an inert flat background η_{ab}, and another h_{ab} that measures the deviation from the background. One sets $g_{ab} = \eta_{ab} + G h_{ab}$, where G is Newton's constant. So the metric g_{ab} is the fixed metric plus small perturbations, one quantizes the latter as an interacting theory of spin-2 fields living in the flat space. The resulting theory diverges badly and is non-renormalizable, as can be shown by simple power counting arguments.

In quantum electrodynamics one is able to overcome the difficulties with divergencies by absorbing them in a redefination of finite many physical parameters. However, this procedure was not successful with quantum gravity. Even though gravitation is such a weak interaction it is not possible to treat it perturbativly. The reason for this is that transition amplitudes of n-th order in the gravitational constant diverges like a momentum integral of the general form

$$\int p^{2n-1} dp, \tag{2.1}$$

leaving us with an infinite number of ultraviolet divergent Feynman diagrams that cannot be removed by redefining finitely many physical parameters.

These arguments apply to *effective* conventional field theories, and not to quantum gravity proper!

"Man shall not separate what Einstein put together"

Renormalization was one of the guiding principles in the construction of the Standard Model, our current understanding of particle physics phenomenology. Particle physicists draw an analogy to the theory of weak interactions, where non-renormalizability of the initial "Fermi theory" forced to replace it by the renormalizable Glashow-Weinberg-Salam theory. However this analogy overlooks a crucial fact that, in the case of GR, there is a qualitatively new element. Perturbative treatments pre-supose that the space-time can be assumed to be continuous even below the scale.

take recourse to the same perturbative scheme as was done for the electromweak and strong forces. Physically interesting theories exist in their own right and perturbation methods serve as approximation techniques to extract answer to "physically interesting questions.

Perturbation treatments pre-suppose that the space-time can be assumed to be continuous even below the Plank length. This is because of a general feature of perturbation theory that to finite order in perturbation theory does not change the qualative nature of the starting point. We can illustrate this point with a particle in a SHO potential: when we calculate the energy eigenvalues we find that there is a discrete spectrum. If we wanted to treat the same problem using perturbation theory we would start with a free particle and incorporate the effects of the potential term by term. To finite order in perturbative expansion the energy spectrum is still continuous.

In string theory gravity is the exchange by gravitons with the same spacetime behaviour as photons and gluons (except for the spin). General relativity is the discovery that spacetime and the gravitational field are the same entity. Somehow the whole idea of the gravitational interaction as a result of graviton exchange on a background metric contradicts Einstein's original and fundamental idea that gravity is geometry and not a force in the usual sense. One can not begin with Minkowskian spacetime and build, say, the Scharwzchild spacetime using linear gravitational radiation. Therefore such a perturbative description of the theory is very unnatural from the outset and can have at most a semi-classical meaning when the metric fluctuations are very tiny.

Maximally breaks general covariance. Dynamical spacetime is formally equivalent to background independence; by fixing a background metric one wipes out the main point of general relativity. The machinery itself should under go suitable modifications so as to be applicable to the problem at hand. Little attention paid to this background independent approach, probably because it had yet to be backed up by concrete calculations.

" if we remove life from Einstein's beautiful theory by steam-rolling it first to flatness and linearity, then we shall learn nothing from attempting to wave the magic wand of quantum theory over resulting corpse."

The failure of standard treatments may simply be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical microstructure of geometry may well be free of these inconsistencies.

In particular, it should be noted that what fails is the idea of perturbatively quantizing general relativity, and more precisely a given perturbative scheme fails.

It is not true that if a theory is not quantizable perturbatively it cannot be quantized. DeWitt has studied several non-linear Sigma models that do not exist pereturbatively but have been satisfactory quantum theories (constructed, for instance on a lattice)

Ashtekar

> *"The theory is not renormalizable; the perturbative quantum field theory has infinitely many undetermined parameters, rendering the theory pretty much useless as far as making physical predictions are concerned. In terms of the modern view, suggested by the theory of critical the phenomena, there is* ***new physics*** *at short distances* [here he is eluding to non-perturbative, background independent effects] *which is not captured by perturbative techniques and ot which the theory is highly sensitive. ... Renormalizabililty is a criterion of simplicity; such theories are "short-distance insensitive" so that, even in the absence of the detailed picture of the microscopic physics, one can make predictions using just a finite number of* ***effective*** *parameters which are relevant to the scale of observation. It is* ***not*** *a criterion to decide if a theory is consistent quantum mechanically.*

Einstein's theory can be stated as a variational problem: one takes a manifold (the manifold is simply a blank background upon which we place the metric to put the familiar features of space and time into it. We try all possible assignments of metrics to the manifold to find those that minnamise the Einstein-Hilbert action - these are the solutions to Einstein's equations. Nowhere in the action does there appear a fixed background metric, or any fixed geometric structure at all.

Background Independence: Experimentalists devising measurements after a while you come to realise that spacetime coordinates are a convenient mathematical device but has no physical relevance because they can always be made to fall out of the equations.

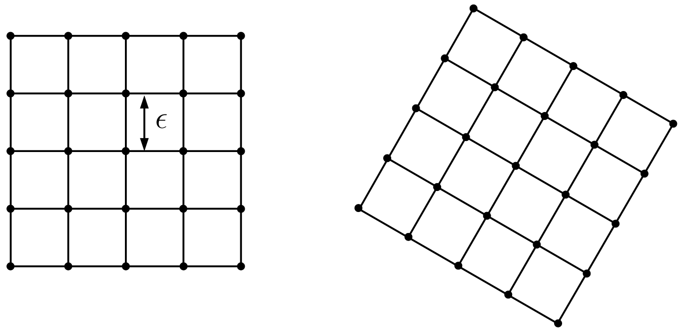
Gr is not a theory of configurations of fields over a spacetime manifold! So as you might imagine such background independent quantum field theories are very different to ordinary quantum field theories living on a fixed spacetime.

Background Independent Quantum Field Theory is Finite!

In the context of perturbation theory of quantum field theory, **regularization** is manipulating divergent integrals is not well defined, so we need to cut off the integration over d^4p. This renders each Feynmann diagram finite. Lattice regularization: Here, we assume that spacetime is actually a set of discrete points arranged. The lattice spacing then serves as the cuttoff for the spacetime integrals.

This Yang-Mills theory exists without reference to a background metric. Such a theory makes no distinction between small and large distances, as described by a background metric; take the same coordinate system but introduce two distinct metrics. According to one metric the proper distance between two points might be small and the other the proper distance between these two points might be large. The Yang-Mills theory is blind to either metric and as a result, one can argue that, such a *background independent* quantum theory will not suffer from UV divergences.

It should be noted that we are not considering I'm not talking about *passive diffeomorphisms*, which is when you view the same physical system in a different coordinate system; the invariance under passive transformations is a property of any theory of nature. GR is the only theory that is invariant under *active dffeomorphism transformations*.



(a) (b)

Figure 2.1: The regularized operator depends on the orientation of the lattice.

The lattice spacing serves as the cutoff for the spacetime integrals. a lattice spacing. The regularized operator depends on the choice of orientation of the lattice.

The techniques have to regularize a field theory uses the background metric, be it Minkowski or curved spacetime metric.

because they all depend on the presence of a background metric. new regularization procedures which may be applied to field theories constructed without a background metric. Additional background structure is introduced in the definition of the regularized operator. We introduce a fidicual metric. For given numerical value of the regularization parameter ϵ, if we change the fidicual metric we will get a different regularized operator. Its dependency on the metric is related to its dependency on the regularization parameter ϵ.

This argument can be made more precise more techniqual:

smolin

" *A background independent operator must always be finite. This is because the regulator scale and the background metric are always introduced together in the regularization procedure. This is necessary, because the scale that the regularization parameter refers to must be described in terms of a background metric or coordinate chart introduced in the construction of the regulated operator. Because of this the dependence of the regulated operator on the cuttoff, or regulator parameter, is related to its dependence on the background metric. When one takes the limit of the regulator parameter going to zero one isolates the non-vanishing terms. If these have any dependence on the regulator parameter (which would be the case if the term is blowing up) then it must also have dependence on the background metric. Conversely, if the terms that are nonvanishing in the limit the regulator is removed have no dependence on the background metric, it must be finite.* "

One must face this problem non-perturbatively. The canonical approach is well suited for the task. In the sixties a non-perturbative quantization programme was formulated within this approach. The idea was to first represent quantum states as functionals of 3-metrics and then select physical states by demanding that they be annihilated by the quantum constraint operators. Unfortunately, the task turned out to be too difficult in the quantum theory and not a single solution could be obtained. In fact, it was not even possible to make sense of the quantum constraint equations.

The approach aims at unifying quantum mechanics and general relativity by developing new non-perturbative techniques from the outset and by staying as close as possible to conventional quantum theory and experimentally tested GR. Superstring theory, based on the simple idea of flat space-time, has always been able to exploit the tools of standard quantum theory. In contrast, the connection approach has had to start from scratch.

short scale structure of Loop quantum gravity introduces a physical cutoff. very high momentum integrations that originate the ultraviolet divergences.

introduced by quantum gravity cure the ultraviolet difficulties of conventional quantum field theory.

2.3 Connections verses Metrics

On its own, the manifold is simply a blank background; the whole point of the metric is to put the familiar features of space and time into it. The metric is all about distances between points. To avoid making assumptions about the space-time metric it must abandon it.

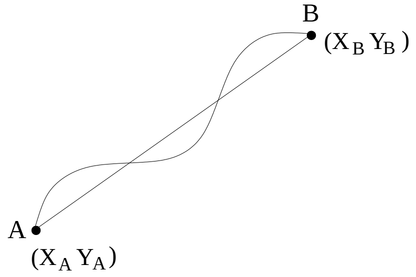


Figure 2.2: With the metric approach, the shortest distance from A to B is defined by coordinates, and a Phythagors-like metric equation for S.

We want to use variables that are in accordance with our wish to construct a background independent quantum theory?

While the metric is all about distances between points, a connection captures the notion of parallelism along curves. Connections are powerful, and allow physicists to do detailed calculations but without having to make any assumptions about the nature of space-time they are dealing with. Many mathematical concepts can also be freed from their reliance on specific assumptions of space and time by switching from metrics to connections.

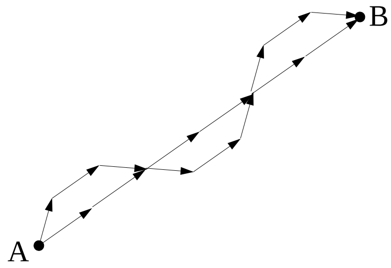


Figure 2.3: With connections, the shortest distance is defined as the route along which every tangent vector is parallel to its neighbour so there is no need for coordinates X and Y.

A key difference is that now there are no background fields whatsoever. Most of the techniques used in standard quantum field theory are deeply rooted in the availability of a flat background metric.

2.4 Canonical Quatization Of GR.

Rovelli [280]

"In nonrelativistic physics, time and spacial position are defined with respect to a system of reference bodies and clocks that are always implicitly assumed to exist and not to interact with the system studied. In gravitational physics, one learns that no body or clock exists which does not interact with the gravitational field: the gravitational field affects directly the motion and rate of any reference body or clock. ..."

In passing from the Lagranegian to Hamiltonian formalism. It is this stage by the use of a particular "time variable" plays such an important role in

the physical states describe only those aspects that are independent of any choice of coordinate

Saying spacetime is 'dynamic' does not mean that it is 'changes' with respect to any given external time. Time is within, not external to spacetime. Accordingly, solutions to Einstein's equations, which are whole spacetimes, do not describe anything evolving. In order to take such an evolutionary form, which is, for example, necessary to formulate an intial value problem, we have to re-introduce a notion of 'time' with reference to which we may speak of 'evolution'. This is done by introducing a structure the somehow allows us to spilt spacetime into space and time.

2.4.1 Sketch of canonical quantization:

1) Pick a Poisson algebra of classical quantities.
2) Represent these quantities as quantum operators acting on a space of quantum states.
3) Implement any constraint you may have as a quantum operator equation and solve to find physical states.
4) Construct an inner product on physical states.
5) Develop a semiclassical approximation and compute expectation values of physical quantities.

Now, the general scheme outlined above has been arund ever since the work of DeWitt

Quantum theory of gravity, I-III by Bryce S. DeWitt, Phys. Rev. 160 (1967), 1113-1148, 162 (1967) 1195-1239, 1239-1256.

However, the problem has always been making the scheme mathematically rigorous, or else to do some kind of calculations that shed some light on the meaning of it all.

see appendix ...

So the first thing you need is a canonical formulation of your theory, in this case general relativity. This was first worked out by Dirac and Bergmann in the late 60's.

This split is necessary to got to phase space and the Hamiltonian formalism, which we need to do in order to formulate a quantum theory.

What is the "configuration space" which is analogous to the set of particle positions in classical mechanics.

One starts by considering the Hilbert actions $S = \int Ldt = \int d^4x\sqrt{-g}R$

And considers a foiliation of space-time into space and time,

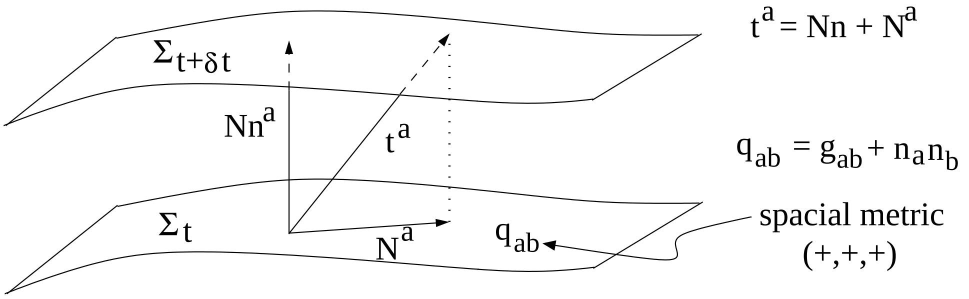
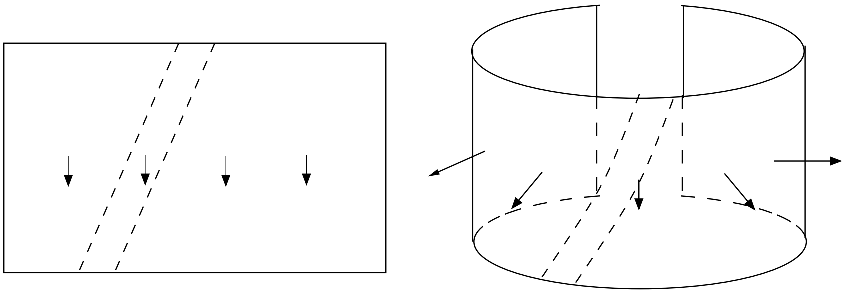


Figure 2.4: A spacetime diagram illustrating the definition of the lapse N and shift vector N^a

$$ds^2 = N^2dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \tag{2.2}$$

The quantity $N(t, x^a)$ is called the *lapse* function - it measures the difference between the coordinate time t, and the proper time, τ, on curves normal to the hyper-surfaces Σ_t, the normal being $n_a = (-N, 0, 0, 0)$ in the above coordinates. The quantity $N_a(t, x^a)$ is called the *shift vector* - it measures the difference between a spacial coordinates are comoving if $N_a(t, x^a)$

One can also define an *extrinsic curvature* which describes how a spacial hypersurfaces Σ_t curve with respect to the 4-dimensional spacetime manifold.



(a) (b)

Figure 2.5: Both have the same intrinsic geometry, i.e. both have a induced metric that is flat. However, the plane does not curve with respect to the 3-dimensional space manifold and the spilt-cylinder does. The spilt-cylinder is said to have non-zero extrinsic curvature.

The canonically conjugate variables to q_{ab} is called π^{ab}. We could have chosen a different foliation, and used the corresponding induced metric and its conjugate momenta as our dynamical variables. The foliation used is specified by the shift and lapse functions; these are freely specifiable and do not appear as dynamical variables in the Hamiltonian, instead they appears Lagrangian multipliers imposing constraints...

Canonical momenta are densities. I will denote this from here with a tilde (this turns out to have some importance later).

If one examines the Hilbert action, one notices that there are no time derivatives of the lapse N or shift N^a. This is understood since N merely represents a particular paramatrisation of time-like curves and so is not a dynamical degree of freedom. The shift N^a

As their canonically conjugate momenta vanish. These are conditions that have to be satisfied on any spatial surface. They are called constraints. There are four of them, three associated with the shift are called "momentum" or "diffeomorphism" constraints. They are either denoted as a co-vector (density) C_a or some times, for convenience they are integrated against a fixed test vector and denoted $C(N) = \int d^3x N^a(x) C_a(x)$

The one associated with the lapse is called the "Hamiltonian" constraint and is a (doubly densitized) scalar, which sometimes it is presented integrated against an arbitrary density of weight -1,

$C(N) = \int d^3xN(x)C(x)$

With the variables we are using, the explicit form of the constraints is,

It is a general result that the Hamiltonian vanishes for invariant under parameterization-invariance. We illustrate with a simple example,

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L} = 0. \tag{2.3}$$

Where/what is physical "time" in Quantum Gravity? Note: x0 is not time . Theory is reparametrization invariant. H does NOT generate time translation

$$\exp\left(\frac{-ix_0H}{\hbar}\right)\Psi[\mathcal{G}] = \Psi[\mathcal{G}] \tag{2.4}$$

The vector or diffeomorphism constraint has a simple geometric interpretation. If one considers its Poisson bracket with a function of the canonical variables, one finds,

That is, the constraint is in the canonical language, the infinitesimal generator of diffeomorphisms on the spacial surface.

I will use q^{ab} to represent the spacial metric.

In particular, the constraints themselves are covariant,

The diffeomorphism algebra is a "Lie" algebra, but the algebra involving the Hamiltonian constraint has non-constant structure functions.

If one performs a Legendre transform, one finds that the theory has a "Hamiltonian" that is a combination of the constraints.

$$H = N^aC_a + NC \tag{2.5}$$

If one considers the Poisson bracket of the "Hamiltonian" with a function of the canonical variables, one gets the "time" evolution along the vector t^a we introduced, interpreted as a "shift" given by the Hamiltonian constraint.

Notice that we are dealing with an unusual theory in the sense that the Hamiltonian vanishes. This is in line with the fact that we introduced a fiducial notion of the time by picking a foliation. The choice of a preferred Lorentz frame specifies a preferred time variable t. However, the choice of inertial frame should have no bearing on physical predictions. The choice of time slicing of space time specifies a preferred coordinate time variable t. The choice of time coordinate t should not explicitly appear.

How would one go about quantizing this theory? One could start by picking as the classical Poisson algebra just the three-metric and its canonically conjugate momenta. One would then consider functions, for instance, of the three metric and represent,

$$\hat{q}_{ab}(q) = q_{ab}(q); \qquad \hat{e}^{ab} = \frac{\delta}{\delta q_{ab}}(q) \tag{2.6}$$

One would then require that the wavefunctions be annihilated by the constraints, written as quantum operator equations. Geometrically this means that the wavefunctions be invariant under the symmetries of the theory.

Here one runs into trouble. The Hamiltonian constraint is a complicated, non-polynomial expression that needs to be regularized. Usual regularization procedure do not preserve the covariance of the theory or require external background structures.

Moreover, we have almost no experience on handling these kinds of wavefunctions. We know very little about the functional space we are dealing with and in particular do not have a measure of integration that would yield a physically valid inner product.

These difficulties stalled the progress on canonical quantization since the 60's.

At the beginning we claimed that the observables of the theory must be finite - that was the easy bit - we must then face the issue of what to compute with a quantum theory like this. One should only consider of physical interest quantities that have vanishing Poisson brackets with all the constraints. No such quantities are known for general relativity (in a compact manifold), We will see in chapter 6, that if the Master constraint programme works out, that they will be able to construct such quantities.

So we must think of this as a functional of $g^{\alpha\beta}$ and its first and second derivatives, namely,

$$\mathcal{L}_G = \mathcal{L}_G(g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\delta}) \tag{2.7}$$

$$\begin{aligned}
\left[\int_\Sigma N\mathcal{H}, \int_\Sigma M\mathcal{H}\right] &= i\frac{16\pi}{M_p^2}\int_\Sigma (N\partial_i M - M\partial_i N)h^{ij}\mathcal{H}_j, &(2.8)\\
\left[\int_\Sigma N^i\mathcal{H}_i, \int_\Sigma M^j\mathcal{H}_j\right] &= i\frac{16\pi}{M_p^2}\int_\Sigma [\vec{N},\vec{M}]^k\mathcal{H}_k, &(2.9)\\
\left[\int_\Sigma N\mathcal{H}, \int_\Sigma M^j\mathcal{H}_j\right] &= -i\frac{16\pi}{M_p^2}\int_\Sigma \mathcal{L}_{\vec{M}}N\mathcal{H}, &(2.10)
\end{aligned}$$

where $\mathcal{L}_{\vec{M}}$ denotes the Lie derivative along the vector field M^j and $[\vec{N},\vec{M}]$ denotes the commutator of the two vector fields. The operators $\int_\Sigma N^i\mathcal{H}_i$ generate diffeomorphisms of

Σ and, on classical solutions, the operators $\int_\Sigma N\mathcal{H}$ generate displacements of the hypersurface Σ along the vector field Nn^a, where n^a is the future-pointing spacetime normal to Σ.

These constraints form a first-class system, meaning that the Poisson bracket of them is again a linear combination (generally with phase-space dependent coefficients) of the constraints.

(2.9) says that the vector constraints form a Lie subalgebra which, however, according to (2.10), is not an ideal. This means that the flows generated by the scalar constraints are not tangential to the constraint-hypersurface that is determined by the vanishing of the vector constraints, except for the points where the constraint hypersurfaces for the scalar and vector constraints intersect. This means that generally one cannot reduce the constraints, simply because the scalar constraints do not act on the solution space of the vector constraints. This difficulty persists in the implementation of the constraint equations of operator constraints in canonical quantum gravity and is in fact the source of much difficulty in understanding the dynamics and semiclassical linit.

2.5 Ashtekar's New Variables.

As we have already discussed, a major obstacle to progress in the canonical approach had been the complicated nature of the field equations in the traditional variables, (q_{ab}, p^{ab}). This obstacle was removed in 1984 with the proposal by Ashtekar of a new set of variables for studying canonical quantum gravity. In terms of these, all the constraint become polynomial, in fact, at worst quartic.

with the introduction of Ashtekar's new canonical variables.

I The first step consists of using triads to encode the metric information. A triad is a set of three vector fields E^i_a, $i = 1, 2, 3$ that are orthogonal, that is,

$$\delta_{ij} = q_{ab} E^a_i E^b_j. \tag{2.11}$$

There are now two different types of "indices", "space" indices a, b, c tha behave like regular indices in curved space, and "internal" indices i, j, k which behave like indices in flat-space (the corresponding "metric" which raises and lowers internal indices is simply δ_{ij}). Defin the dual triad E^i_a as

$$E^i_a = q_{ab} E^b_i. \tag{2.12}$$

We then have the orthogonality conditions

$$\delta_{ij} = q^{ab} E^i_a E^j_b, \tag{2.13}$$

where q^{ab} is the inverse matrix the metric, and

$$E^i_a E^i_b = \delta^a_b. \tag{2.14}$$

The first orthogonality condition (2.13) comes from substituiting (2.12) in $q^{ab} E^i_a E^j_b$,

$$\begin{aligned} q^{ab} E^i_a E^j_b &= q^{ab}(q_{ac} E^c_i)(q_{bd} E^d_j) \\ &= \delta^b_c q_{bd} E^c_i E^d_j \\ &= q_{cd} E^b_i E^d_j = \delta_{ij}. \end{aligned} \tag{2.15}$$

The second orthogonality condition (2.14) comes from contraction $\delta_{ij} = q_{ab} E^a_i E^b_j$ with E^i_c,

$$\begin{aligned} 0 &= (\delta_{ij} - q_{ab} E^a_i E^b_j) E^i_c \\ &= E^j_c - (q_{ab} E^b_j)(E^a_i E^i_c) \\ &= E^j_c - E^j_a (E^a_i E^i_c) \\ &= E^j_a (\delta^a_c - E^a_i E^i_c) \end{aligned} \tag{2.16}$$

As the E^j_a are linearly independent this implies $\delta^a_c - E^a_i E^i_c = 0$. It is easy to verify from the first orthogonality condition (2.13), using (2.14), that

$$q^{\alpha\beta}(x) = E^\alpha_i E^\beta_i = \delta^{ij} E^\alpha_i E^\beta_j \tag{2.17}$$

We see that the triad can be thought of as the "square-root" of the (inverse) metric. Physically the triads can be thought of as coordinate transformations to locally flat space. Actually what is really considered is

$$(\det q(x)) q^{\alpha\beta}(x) = \tilde{E}^\alpha_i \tilde{E}^\beta_i \tag{2.18}$$

$\tilde{E}^\alpha_i(x)$ are the dentitised triads (densitised as $\tilde{E}^\alpha_i = \sqrt{det(q)} E^\alpha_i$).One recovers from $\tilde{E}^\alpha_i$ the metric times a factor gien by its determinant. It is clear that $\tilde{E}^\alpha_i$ and E^α_i contain the same information, just rearanged. It turns out that the use of such framefields brings out a different point of view on the connection and curvature, one in which GR has a strong resemblance to particle physics field theories.

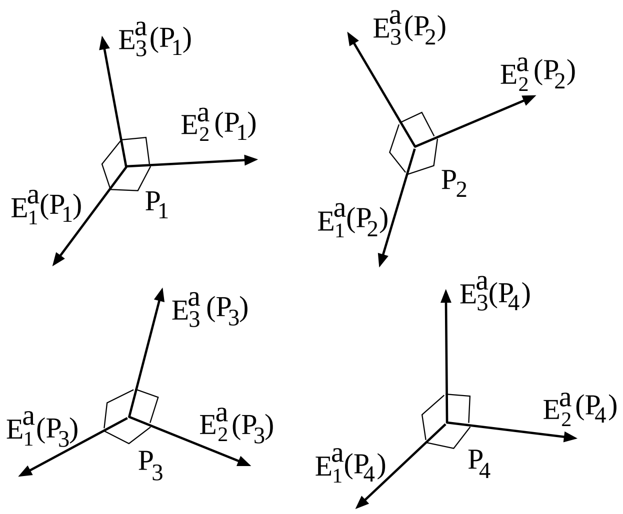


Figure 2.6: Triad field.

The basis system $\tilde{E}^{\alpha}_{i}(x)$ fulfilling (2.18) can be chosen somewhat arbitrary. We have the freedom to choose a different basis i.e. $\tilde{E}'^{\alpha}_{i}(x) = M^{j}_{i}(x)\tilde{E}^{\alpha}_{i}(x)$. The metric $q_{ab}(x)$ is left unchanged by local $SO(3)$ transformations such that

$$E^{a}_{i}(x^{a}) \rightarrow E'^{a}_{\ i}(x^{a}) = O^{j}_{\ i}(x^{a})E^{a}_{j}(x^{a}), \tag{2.19}$$

where $O^{j}_{\ i}(x^{a})$ is a matrix in $SO(3)$ which depends on position in space. When "Physical quantities" are left invariant, such transformations are known as *gauge transformations*, and theories invariant under them are called *gauge theories.*

Now we have introduced these frame fields we now need to know how to compare vectors in frames at different points.

$\nabla_{a}V_{b} = \partial_{a}V_{b} + \Gamma^{c}_{ab}V_{c}$. The same remedy is applied to $\partial_{a}V^{i}(x^{a})$ and we introduce a connection

$$\omega_{a\ j}^{\ i}(x) \tag{2.20}$$

with two tetrad indicies and one spacetime index.

$$\mathcal{D}_{a}V^{i}(x) = \partial_{a}V^{i}(x) + \omega_{a\ j}^{\ i}V^{j}(x) \tag{2.21}$$

The same information contained in $\omega_{a\ j}^{i}(x)$ is also contained in the field

$$\Gamma^{i}_{a} = \epsilon^{i}_{\ jk}\omega_{a}^{\ jk}, \tag{2.22}$$

The canonically conjugate variable is related to the extrinsic curvature, $K_a^i = K_{ab}E^{bi}$

The constraints become,

$$\epsilon_{ijk}K_a^j E^{ak} = 0 \tag{2.23}$$

Where $\zeta = -1$ for Lorentzian and $\zeta = 1$ for Euclidean space-time

Formulating general relativity with frame fields instead of metrics was not new, it had been tried before. Most noteably by Palatini who rewrote the standard Einstein-Hilbert (a functional of the spacetime metric) action in such a way that the spacetime metric, well actually tetrads (four-dimensional lorentzian version of triads), and an arbitrary Lorentz connection are the basic variables,

$$\mathcal{S}[e^I, \omega^{IJ}] = \frac{1}{4\kappa}\int_{\mathcal{M}} d^4x \epsilon^{abcd}\epsilon_{IJKL}e_a^I e_b^J \left(R_{cd}^{KL} - \frac{\Lambda}{6}e_c^K e_d^L\right) \tag{2.24}$$

However, as shown in appendix E of volume I, the $3+1$ Palantini theory collapses back to the standard geometric descrption of general relativity when one writes it in Hamiltonian form. And the problems are similar to those of using the metric formulation when one tries to build a quantum theory.

This presentation follows Barbero gr-qc/9410014

Ashtekars's new insight was to introduce a new variable canonically conjugate to the triad, via the (complex) canonical transformation,

$$A_a^i = \Gamma_a^i + iK_a^i \tag{2.25}$$

and it is in this field that is used as a basic variable in the canonical theory. The first constraint becomes a Gauss law, stating that the theory is invariant under frame rotations (Euclidean group in 3 dimensions, that is, SO(3)).

$$D_a E_i^a = 0 \tag{2.26}$$

The second constraint, which we know correspond to diffeomorphism invariance, are written simply as the vanishing of the Poynting vector.

$$F_{ab}^i E_i^b \equiv \vec{\mathbf{B}}_\mathbf{i} \times \vec{\mathbf{E}}^\mathbf{i} = 0 \tag{2.27}$$

where $\vec{\mathbf{B}}_\mathbf{i}$ is the "magnetic" field constructed from A_{ai}.

The complicated Hamiltonian constraint can be made a simple expression quadratic in the momenta,

$$\epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{abk} \equiv \epsilon^{ijk}\vec{\mathbf{B}}_{\mathbf{i}} \cdot \vec{\mathbf{E}}_{\mathbf{j}} \times \vec{\mathbf{E}}_{\mathbf{k}} = 0 \tag{2.28}$$

Another interesting aspect of this reformulation is that we can think of general relativity as an unusual type of SO(3) Yang-Mills theory. In particular its (unconstrained) phase-spaces are the same. General relativity can be viewed as Yang-Mills theory with some extra constraints (and a different Hamiltonian that vanishes - this implies the dynamics will be very different).

We therefore can think of attempting the quantization of general relativity as we would do for a Yang-Mills theory. In electromagnitism, A is like the "position" variable and E is like the "momentum", because the equation

$$\frac{dA}{dt} = E$$

(in the temporal gauge) is like

$$\frac{dq}{dt} = p.$$

Similarly, in Ashtekar's approach A is the configuration variable and E is the canonically conjugate "momentum". Therefore, it is natural to consider wavefunctions of the connection $\Psi[A]$. Notice that this significantly different from what one would have done in the metric representation. Just as we promote the canonical variables to operators in ordinary quantum mechanins according to

$$\hat{x} = x, \quad -i\hbar\frac{d}{dx} \tag{2.29}$$

we promote the canonical variables of the connection representation to operators according to

$$\hat{A}_i^a = A_i^a, \quad \hat{E}_a^i = -i\hbar\frac{\partial}{\partial A_i^a} \tag{2.30}$$

acting on the functional $\Psi[A]$, such that

$$\hat{C}_i\Psi = 0, \quad \hat{C}_a\Psi = 0, \quad \hat{C}\Psi = 0, \tag{2.31}$$

the **connection representation**.

However, let us instead consider the more general canonical transformation,

$$A_a^i = \Gamma_a^i + \beta K_a^i \tag{2.32}$$

where β is an arbitrary complex number called the Immirzi parameter. Then the constraints become,

$$\begin{aligned}
\tilde{G}_i &\equiv \nabla_a \tilde{E}_i^a = 0 &(2.33)\\
\tilde{V}_a &\equiv F_{ab}^i \tilde{E}_i^b = 0 &(2.34)\\
\tilde{\tilde{S}} &\equiv -\zeta \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{2(\beta^2\zeta - 1)}{\beta^2} \tilde{E}_{[i}^a \tilde{E}_{j]}^b (A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) = 0 &(2.35)
\end{aligned}$$

So we see that if we choose the Immirzi parameter to be the imaginary unit (in the Lorentzian case) or one (in the Euclidean case), the constraints become polynomial functions of the fundamental variables.

2.6 Loops and the Loop Representation.

In the of quantization we choose certain functions of the basic variables to be fundamental operators, in terms of which all other operators are constructed. In the case of ordinary text-book quantum mechanics the standard choice we take the basic variables position and momentum themselves to be the fundamental operators. Then using these we construct the Hamiltonian and other operators representing physical observables as made up . A seemingly sensible choice would be to choose the connection representation, (2.30), as it parallels Schödinger representation. We contemplate a different choice - the loop representation.

There is an alternative to thinking about geometry in terms of the curvature fields at each point in space is to instead think about the holonomy around loops in space.

The idea is to take any path that starts at one point and comes back to the same point (this is called a loop), and consider a vector which is carried along the path always keeping the new vector parallel to the previous vector. In curved space the intial and final vectors will be related by a rotation transformation. This rotation transformation is called the holonony of the loop. It can be calculated for any loop, so the holonomy of a curved space is an assignment of rotations to all loops in the space.

In the loop representation the fundamental operator is the **Wilson loop** rather than the connection field operator $\hat{A}_i^a$.

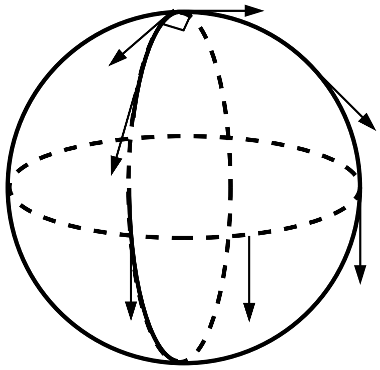


Figure 2.7: SphcurvEx. This is a manifestation of the curvature of the sphere.

In such a representation, the first constraint just requires that the wavefunctions be invariant under SO(3) gauge invariant, just like wavefunctions of Yang-Mills theory.

An example of wavefunction that is invariant under SO(3) gauge transformation is the trace of the holonomy of the connection along a loop, also called "Wilson loop",

$$h_\gamma[A] = \exp\left(\oint_\gamma A_a(x)\dot{\gamma}^a(x)ds\right) \tag{2.36}$$

where $\dot{\gamma}^a(x) = dx^a/ds$. Let us call $\Delta\phi = \int_\gamma A_a(x)\dot{\gamma}^a(x)ds$. Recalling Stokes's law

$$\oint_\gamma \mathbf{V} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{V} \cdot d\mathbf{S} \tag{2.37}$$

$$\begin{aligned}\Delta\phi = \frac{e}{\hbar c}\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} \\ = \frac{e\Phi}{\hbar c}\end{aligned} \tag{2.38}$$

$\Phi_B = \int \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux, (a gauge invariant quantity), through the surface. Hence, the holonomy is a gauge invariant quantity and so solves the Gauss constraint for Electro-magnetism.

Tells us how we are to compare basis at different points; we are to think of them as related to each other by a rotation.

A short digression into the use of two-component spinors as a mathematical object, first introduced as a mathematical tool to study aspects of calssical GR by Penrose.

Vectors and tensors are defined by their transformation rules, spinors are too..

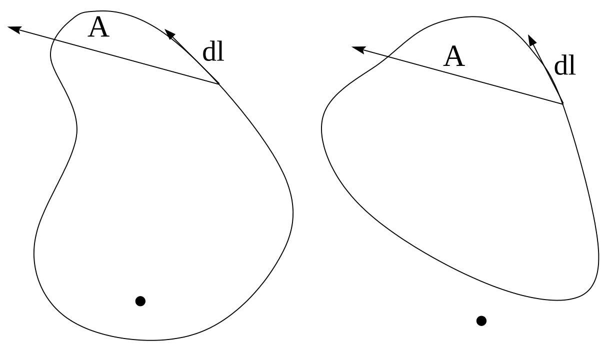


Figure 2.8: loop.

Going to introduce spinors, but only as a mathematical object - they anything quantum mechanical. We are familiar with the rotation matrix of a vector, from quantum mechanics courses, we know the rotation matrix of a spinor. When one parallel transports a vector around a closed loop, the final direction is related to the initial direction by a rotation matrix. We can read off the angles and then plug them into the rotation matrix of the spinor - this relates the initial direction of the spinor to the final one, (mod some subtleties 1-2 instead of 1-1 - remember when you rotate a spinor through angle 2π this changes the sign of the spinor, one has to rotate through 4π to return to the original value of the spinor). So we have two equivalent descriptions - we can, instead of using vectors, we can encode information about the curvature of space in terms of the rotation of a spinor under parallel propagation around a closed loops. It is more conveinent to use the $su(2)$ valued connection.

so if we transport a spinor along the entire closed curve γ, it will return multiplied by a group element.

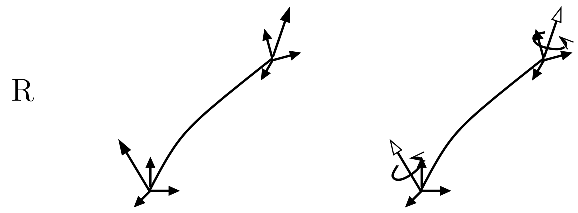


Figure 2.9: introduce spinors.

$$\hat{R} = \exp\left(i\mathbf{a}\cdot\mathbf{J}\right) \tag{2.39}$$

$$\hat{U} = \exp\left(\frac{i}{2}\mathbf{a}\cdot\sigma\right) \tag{2.40}$$

we specify a path and perform a series of infintesimal parallel transports along that path.

If the points $x_0, x_1, x_2, \ldots, x_n$ form a series of infintesimal separated points along a path γ, parallel transport is

$$
\begin{aligned}
U(x_0, x_n) &= (1 + A_a^i(x_0)(x_1 - x_0)^a \tau_i)(1 + A_a^i(x_1)(x_2 - x_1)^a \tau_i) \ldots \\
&\quad \cdots \times \; (1 + A_a^i(x_{n-1})(x_n - x_{n-1})^a \tau_i)
\end{aligned}
\tag{2.41}
$$

Let s be a parameter of the path γ, running from 0 at x_0 to 1 at x_n. Then define the Wilson line as the power-series expansion of the exponential, with matrices in each term ordered so that higher values of s stand to the left. This prescription is called *path-ordering* and is denoted by the symbol $\mathcal{P}$.

$$
\mathbf{H}_\gamma[A] = \mathcal{P} \exp \left(\int_\gamma A_a^i(x) \tau_i \frac{dx^a}{ds} ds \right) \tag{2.42}
$$

This transforms as

$$
\mathbf{g}(x_0) \mathbf{H}_\gamma[A] \mathbf{g}^{-1}(x_n) \tag{2.43}
$$

Consider when γ is a closed path. This transforms as

$$
\mathbf{g}(x_0) \mathbf{H}_\gamma[A] \mathbf{g}^{-1}(x_0) \tag{2.44}
$$

$$
W_\gamma[A] = \mathrm{Tr} W_\gamma[A] \tag{2.45}
$$

$A_a^i(x) = A_a^{i\,\prime}(x) + \tau_i \partial_a \phi(x)$

$$
W_\gamma[A] = \mathrm{Tr} \mathbf{H}_\gamma[A] = \mathrm{Tr}(g(x_0) \mathbf{H}_\gamma[A'] g^{-1}(x_0)) = W_\gamma[A'] \tag{2.46}
$$

Wilson loops are gauge invariant:

$$
W_\gamma[A] = W_\gamma[A'] \tag{2.47}
$$

will use Wilson loops[1] are basis variables instead of the connection. We are replacing 3-d fields with 1-d objects - aren't they going to be rather singular? Yes in QCD. Gravity has

[1]holonomies are defined without the use of a background metric and so are in accordance with with our wish to construct a background independent quantum theory

a cure for this problem: factorize out this dependencey by averaging over the position of the loop, thus in affect smearing the Wilson loop over the whole of space.

Supplementary: Going from vectors to spinors.

the use of two-component spinors as a mathematical object, first introduced into GR by Penrose.

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.48}$$

$$\begin{aligned} & a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \mathbf{a} \cdot \sigma \\ & = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \mathbf{A} \end{aligned} \tag{2.49}$$

The determinate of $\mathbf{A}$ is obviously $-(a_1^2 + a_2^2 + a_3^2) \equiv -\mathbf{a} \cdot \mathbf{a}$. Consider the effect of a similarity transformation on $\mathbf{A}$,

$$\mathbf{A'} := \mathbf{U}\mathbf{A}\mathbf{U^{-1}} \tag{2.50}$$

$\det \mathbf{A'} = \det \mathbf{U} \det \mathbf{A} \det \mathbf{U^{-1}} = \det \mathbf{A}$, which means

$$\mathbf{a}' \cdot \mathbf{a}' = \mathbf{a} \cdot \mathbf{a} \tag{2.51}$$

so the similarity transformation induces a transformation on a_1, a_2, a_3 - that preserves the length of the vector $\mathbf{a}$ i.e. a rotation. Let us write the matrix $\mathbf{A}$ as

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \times \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \equiv \begin{pmatrix} \eta_1\xi_1 & \eta_1\xi_2 \\ \eta_2\xi_1 & \eta_1\xi_2 \end{pmatrix} \equiv \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \tag{2.52}$$

then ξ transforms as

$$\xi' = \mathbf{U}\xi \tag{2.53}$$

If one checks one finds that the transformation is precisely that of the rotation transformation on a spinor!

$$\hat{U} = \exp\left(\frac{i}{2}\mathbf{a} \cdot \sigma\right) \tag{2.54}$$

$$\Psi \to \Psi' = \hat{U}\Psi(x) = \exp\left(i\mathbf{a}(x) \cdot \hat{\mathbf{T}}\right) \Psi(x) \tag{2.55}$$

$$\tau \cdot a \to \tau \cdot a' = \exp\left(\frac{i}{2}\mathbf{a}\cdot\sigma\right)(\tau \cdot a)\exp\left(\frac{i}{2}\mathbf{a}\cdot\sigma\right) \tag{2.56}$$

effects a rotation of the vector field r around the axis $\mathbf{n} = \mathbf{r}/a$ by an angle $|a|$.

There are many combinations of loops one can consider. In Figure (2.6) is an example of interconnected loops. We can also envisage Wilson loops that intersect themselves or other Wilson loops, see Figure (2.6), (these are usually called networks rather than loops).

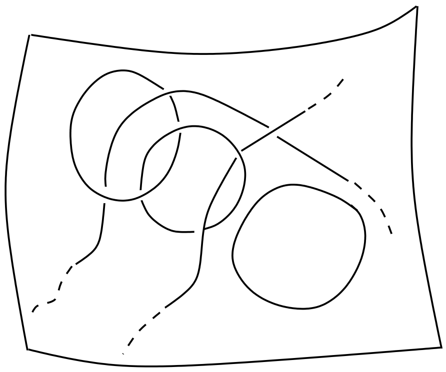


Figure 2.10: There are many combinations of loops one can consider.

Wilson loops are known to be annihilated by the Gauss law, but not by the diffeomorphism constraint.

Wilson loops are known to be an overcomplete (more later) basis for all gauge invariant functions (*Giles PRD24, 2160, (1981))*

With the Wilson loops we solve the Gauss law constraint. One is left with the diffeomorphism Hamiltonian constraint. It is clear that trying to solve the difeomorphism constraint in terms of the connection field, $A_i^a(x)$, will not be too effective.

Basing ourselves on the fact that Wilson loops are a basis we can formally expand any gauge invariant function of a connection as,

$$\Psi[A] = \text{“}\sum_{\gamma} \Psi[\gamma] W_{\gamma}[A]\text{”} \tag{2.57}$$

At the moment this is only a formal expression, we will see later that we can get better control of it. It is analogous to what one does when one goes to the momentum representation in quantum mechanics

$$\Psi[x] = \int dk \Psi[k] \exp(ikx) \tag{2.58}$$

The coefficients of the expansion are functions of loops. They are the "wavefunctions in the loop representation" given by the "inverse loop transform"

$$\Psi[\gamma] = \int dA \Psi[A] W_\gamma[A] \tag{2.59}$$

Position - Momentum representations of point particle QMs.	Connection - Loop representations of quantum gravity.
$< x\|\psi > \leftrightarrow \psi(x)$	$< A\|\psi > \leftrightarrow \psi(A)$
$< p\|\psi > \leftrightarrow \psi(p)$	$< \alpha\|\psi > \leftrightarrow \psi(\alpha)$
$< x\|p > \leftrightarrow e^{ipx}$	$< \alpha\|A > \leftrightarrow -\mathrm{Tr}\mathcal{P}e^{\oint_\alpha A\cdot dl}$
$\psi(p) = \int dx e^{ipx}\psi(x)$	$\psi(\alpha) = -\int d\mu[A]\mathrm{Tr}\mathcal{P}e^{\oint_\alpha A\cdot dl}\psi(A)$

The loop transform defines the loop representation. Given an operator $\hat{O}$ in the connection representation,

$$\Phi[A] = \hat{O}\Psi[A], \tag{2.60}$$

We define the corresponding operator, $\hat{O}'$, in the loop representatin by

$$\Phi[\gamma] = \hat{O}'\Psi[\gamma] \tag{2.61}$$

where $\Phi[\gamma]$ is defined by

$$\Phi[\gamma] = \int dA \Phi[A] W_\gamma[A]. \tag{2.62}$$

Equating the right hand sides of (2.61) and (2.62) and using (2.60) we obtain,

$$\hat{O}'\Psi[\gamma] = \int [dA] W_\gamma[A] \hat{O}\Psi[A], \tag{2.63}$$

or

$$\hat{O}'\Psi[\gamma] = \int [dA] (\hat{O}^\dagger W_\gamma[A]) \Psi[A], \tag{2.64}$$

where by $\hat{O}^\dagger$ we mean the operator $\hat{O}$ but with the opposite factor ordering. We evaluate the action of this operator on the Wilson loop as a calculation in the connection representation and rearranging the the result as a manipulation purely in terms of loops (one should remember that when considering the action on the Wilson loop one should chose the operator one wishes to transform with the opposite factor ordering to the one chosen for its action on wavefunctions $\Psi[A]$).This gives the phsyical meaning of the operator $\hat{O}'$. For example if $\hat{O}^\dagger$ were a spatial diffeomorphism, then this can be thought of as keeping the connection field A of $W_\gamma[A]$ where it is while performing a spatial diffeomorphism on γ, the argument of $\Psi[\gamma]$.

Why would we want to go to the loop representation? Because the constraint has a simple geometric action, it shifts the loop. This is illustrated in Figure (2.6).

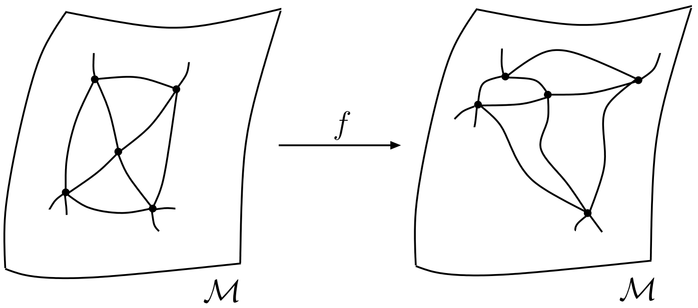


Figure 2.11: Discrete objects are ideal for dealing with the requirement of spatial difeomorphism invariance. Physically relevant information is represented by abstract combinatorics.

In the loop representation it is immediate to write solutions to the diffeomorphism constraint. One simply has to consider functions of loops $\Psi[\gamma]$ that are invariant when one applies a diffeomorphism to the loop. That is, they have to be what mathematicians call **knot invariants**.

This opens an unexpected connection between knot theory and quantum gravity, first noted by Rovelli and Smolin (1988). It also illustrates why using these new variables opens new perspectives on the problem not available with the traditional variables.

2.6.1 Projector Technique (Group Averaging)

We can illustrate the basic idea with rotational invariant wavefucntions in two dimensions: take any square integrable function in polar coordinates $\varphi(r,\theta)$

$$\tilde{\varphi}(r,\theta) = \int_0^{2\pi} d\alpha \varphi(r,\theta+\alpha) \tag{2.65}$$

$$\frac{d}{d\theta}\tilde{\varphi}(r,\theta) = 0 \qquad \text{i.e. } \tilde{\varphi}(r,\theta) = \tilde{\varphi}(r) \tag{2.66}$$

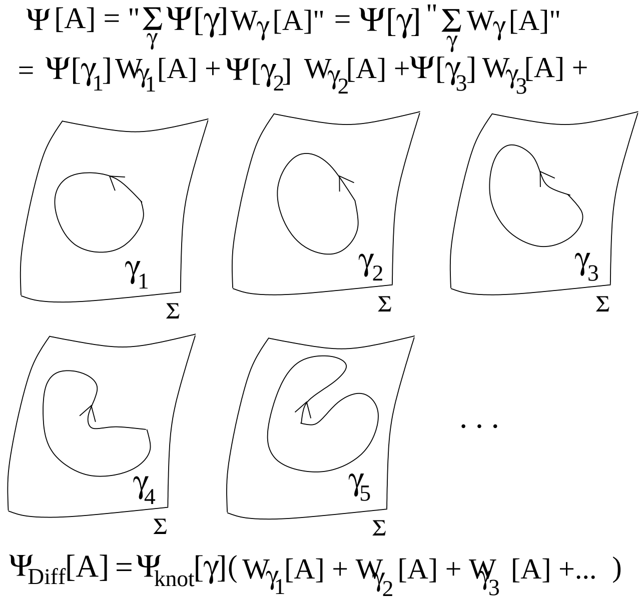


Figure 2.12: We can generate spatially diff invariant wavefunctions $\Psi_{Diff}[A]$ by averaging of the associated loop γ $\Psi[A]$. Each of the loops γ_i for $i = 1, 2, \ldots$ are topologically equivalent but geometrically inequivalent.

Note, (2.65) can be written as the formal equation

$$\tilde{\varphi}(r,\theta) = \int_0^{2\pi} d\alpha \exp\left(i\alpha \hat{p}_\theta\right) \varphi(r,\theta), \quad \text{where } \hat{p}_\theta = -i\frac{d}{d\theta}, \tag{2.67}$$

where $\exp\left(i\alpha\hat{p}_\theta\right)\varphi(r,\theta)$ has the meaning

$$\exp\left(i\alpha\hat{p}_\theta\right)\varphi(r,\theta) = \exp\left(\alpha\frac{d}{d\theta}\right)\varphi(r,\theta) = \left(1 + \frac{d}{d\theta} + \frac{1}{2}\frac{d^2}{d\theta^2} + \ldots\right)\varphi(r,\theta) = \varphi(r,\theta+\alpha) \tag{2.68}$$

which is evidently the Taylor expansion.

see Appendix A4.1. With (2.65) we were able to generate rotationally invariant functions, $\tilde{\varphi}_{rot}(r,\theta) = \tilde{\varphi}(r)$, from any given function $\varphi(r,\theta)$. This suggests the formal solution to the spatial diffeomorphism constraint $\hat{C}_a\Psi[A] = 0$,

$$\Psi_{Diff}[A] = \int \mathcal{D}[N^a] \exp\left(\int_\Sigma d^3x i N^a(x)\hat{C}_a(x)\right)\Psi[A] \tag{2.69}$$

where $\hat{C}^a$ generates infintesimal spatial transformations and where $N^a(x)$ plays the role of the parameter α in (2.67).

Alternatively, can be viewed as a "delta functional"

$$P \sim \prod_x \delta(C_a(x)) \sim \int \mathcal{D}[N^a] \exp\left(i \int d^3x N^a C_a(x)\right). \tag{2.70}$$

analogous with the representation of the delta function as the integral of an exponential

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ipx}. \tag{2.71}$$

In chapter 3 we will atempt to apply this same technique to solving the Hamiltonian constraint. Although serious difficulties arise in this procedure, it will bring us in contact to the idea of spinfoams - the quantum geometry of spacetime.

2.7 The Hamiltonian Constraint

What about the Hamiltonian constraint on a Wilson loop? We define a factor ordering for the constraint on a Wilson loop by

$$\hat{\tilde{H}}^\dagger W_\gamma[A] = \epsilon^{ijk} F^k_{ab}[A] \hat{\tilde{E}}^b_j \hat{\tilde{E}}^a_i W_\gamma[A]. \tag{2.72}$$

The Hamiltonian involves products of operators that need to be regularized. Let us be simple minded and consider a point-splitting regularization with a fiducial background metric. Let us also consider a factor ordering in which the electric fields are to the right. We then have

$$F^k_{ab}\ \tilde{E}^a_i \tilde{E}^b_j\ W_\alpha[A] = F^k_{ab}\ \frac{\delta^2 W_\alpha[A]}{\delta A^i_a \delta A^j_b} \sim F^k_{ab}\ \dot{\gamma}^a \dot{\gamma}^b = 0 \tag{2.73}$$

The Hamiltonian constraint in the loop representation is given by

$$\hat{\tilde{H}}' \Psi[\gamma] = \int [dA] W_\gamma[A] \hat{\tilde{H}} \Psi[A]. \tag{2.74}$$

We consider wavefunctions $\Psi[\gamma]$ that vanish if the loop has discontinuities and that are knot invariants. Such functions solve the Gauss law, the spatial diffeomorphism constraint and (formally) the Hamiltonian constraint.

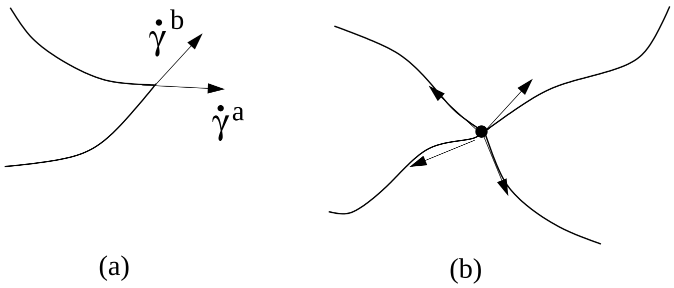


Figure 2.13: If there are kinks as in (a) or intersections as in (b) the product $\gamma^a(x_P)\gamma^b(x_P)$ is not always a symmetric quantity, (we are ignoring the very important regularization issue!). Hence, such loops don't solve the Hamiltonian constraint, see (2.73).

[45]

"The central result obtained in the loop representaion is that one can find a large class of solutions of the *full* set of quantum constraint equations. More precisely, we find the general solution of the diffeomorphism constraint and an infinite-dimensional space of solutions of all the constraints."

2.8 Need for Intersecting Loops

That means that not any wavefunction of the gravitational field. In particular, it implies that a function that is non-vanishing on smooth loops only" is not a good candidate.

As we see, just using functions of loops is a problem.

The other point we need to consider the expression of the (doubly densitized) metric in terms of the new variables

$$det(q)q^{ab} = \tilde{E}^a_i \tilde{E}^b_i \tag{2.75}$$

And think of the corresponding quantum operator, we see that the metric is, as a matrix, degenerate everywhere on loop states except at intersections; the metric has only one non-vanishing component, along the loop.

We include the cosmological constant term in the Hamiltonian constraint

$$H_\Lambda = H + \Lambda \det q \tag{2.76}$$

where Λ is the cosmological constant and $\det q$ the determinant of the three metric. This term is replaced by tetrad variables

$$H = \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{ab}^k + \frac{\Lambda}{6} \epsilon^{ijk} \epsilon_{abc} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c. \tag{2.77}$$

The cosmological term acting on a Wilson loop

$$\frac{\Lambda}{6} \epsilon^{ijk} \epsilon_{abc} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c W_\gamma[A] \sim \epsilon^{ijk} \epsilon_{abc} \gamma^a \gamma^b \gamma^c \tag{2.78}$$

which is obviously zero for smooth Wilson loops. General relativity with and without a cosmological constant are very different, this suggests these solutions don't have physical relevance.

The volume operator

$$V = \int \sqrt{q} = \int \sqrt{\frac{1}{3!} \epsilon^{ijk} \epsilon_{abc} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c} \tag{2.79}$$

is zero everywhere except at intersections.

For all these reasons we therefore must consider such loops.

2.8.1 Intersecting Loops

The Hamiltonian constraint has a simple action in the same representation in which the diffeomorphism constraint can be solved.

Acting on states the actin of the Hamiltonian constraint is concentrated at the intersection of loops. As a result of this one can find an infinite dimensional space of exact solutions to the Hamiltonian constraint.[64] [65].

These include an infinite space of solutions which consists of all the states which have support on loop that have no intersections. JacobSmolin????

There are, in addition, a large number of states with support on intersecting loops [66]

Supplementary: Solutions for all the constraints that have intersections.

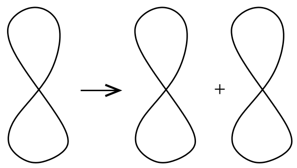


Figure 2.14: Action of the Hamiltonian on a four-valent spin network.

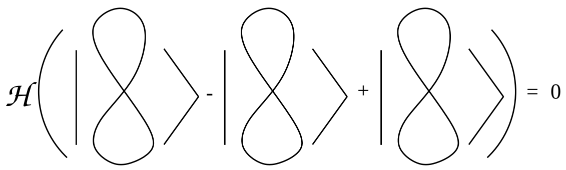


Figure 2.15: Solution of the Hamiltonian of four-valent spin network.

2.9 Difficulties

Many suggestive facts about the possibility of making breakthroughs in solving the quantum constraint equations of canonical quantized general relativity were discovered. But most of the results were formal and loaded with difficulties.

2.9.1 Over-Completeness of Wilson Loops

In the Loop representation quantum states are represented by functions of sets of loops, of the form $\Psi[\alpha, \beta, \gamma]$. We wish to use the space of functions of single loops - in doing so, however, we find difficulty we glossed over in the presentation. We said that the Wilson loops were an "overcomplete" basis of gauge invariant states.

Before I explain what that means, I first introduce some notation for the Wilson loops that we will be needing below. If γ_i represent segments of curves then $\gamma_i \circ \gamma_j$ says that the beginning of γ_j is joined to the end of γ_i. The segment γ is necessarily an open curve. Also the inverse γ^{-1} is the segment with opposite orientation. With this, the Wilson loop satisfies the following properties:

$$\begin{aligned} W[0] &= 1 \\ W[\gamma] &= W[\gamma^{-1}] \\ W[\gamma] &= W[\gamma \circ \lambda \circ \lambda^{-1}] \end{aligned} \tag{2.80}$$

In the fundamental representation of $SU(2)$ (Pauli) matrices are not generic 2×2 matrices, but they actually satisfy certain relationships, like for instance,

$$\mathrm{T}r\mathbb{A}\ \mathrm{T}r\mathbb{B} = \mathrm{T}r(\mathbb{A}\mathbb{B}) + \mathrm{T}r(\mathbb{A}\mathbb{B}^{-1}). \tag{2.81}$$

(which we will prove in a moment) or in terms of Wilson loops,

$$W_\alpha[A]W_\beta[A] = W_{\alpha\circ\beta}[A] + W_{\alpha\circ\beta^{-1}}[A] \tag{2.82}$$

It means wavefunctions in the loop representation are not free quantities. We are talking about the coefficients in the basis of the overcomplete vectors. It would be good to define a set of vectors that are free of such linear identities. In the above simple network it is easy to do (Talk 2). However, for more complicated networks the number of such identities sky-rockets.

We now prove the identity (2.81). Consider 2×2 marices $\mathbb{A}, \mathbb{B} \in SL(2, \mathbb{C})$ which have components written as A_{AB} where $A, B = 1, 2$. Then (2.81) follows from contracting the identity $\epsilon_{AB}\epsilon_{CD} = \delta_{AB}\delta_{CD} - \delta_{AD}\delta_{BC}$ with $B_{AD}A_{CB}$ and using that for any 2×2 matrix with unit determinant that $(B^{-1})_{BC} = \epsilon_{AB}\epsilon_{DC}B_{AD}$.

2.9.2 Complex General Relativity and Reality Conditions

Finally, we saw that the canonical transformation defining the Ashtekar connection had the form

$$A_a^i = \Gamma_a^i + iK_a^i \tag{2.83}$$

And the "Immirzi" parameter had to be the imaginary unit for the Hamiltonian to have the simple form we have been considering. This implies that one is dealing with a theory of complex variables. To recover real Lorentzian GR we must impose *reality conditions* at the end of any calculation. However, this did not appear to be easy.

2.9.3 Hamiltonian Constraint and Spacial Diffromorphism

A less immediately obvious but more serious problem is that the Hamiltonian constraint, used by Ashtekar, is a scalar density of weight two, (i.e. it transforms as $\tilde{\tilde{S}} \rightarrow \tilde{\tilde{S}}' = J^2\tilde{\tilde{S}}$ where J is the Jabcobian of the coordinate transformation); such objects cannot be promoted to quantum operators without breaking spatial diffeomorphism invariance.

2.9.4 Formulism Based on Surfaces of Simultaneity

absolute time provided a natural foliation of spacetime into spacial cross sections.

In special relativity because of relativity of simultaneity they do not agree on a unique time slicing

General covariance tells us that observables of quantum gravity do not depend on the time coordinate. So should we really formulate the quantum theory from a Hamiltonian mechanics based on the space of fields at fixed "time". We discuss this further in the last chapter.

The notion of a special spacelike surface over which initial data are fixed conflicts with diffeomorphism invariance. A generally covariant notion of observable at a given time, make very little physical sense. A consistent definition of state and observable in a generally covariant context cannot explicitly involve time.

2.9.5 Observables

We may have an infinite number of solutions but have no idea of the physical significance - need interpretation of operators acting on physical states. We need operators corresponding to classical observables. The solutions are useless until a way to interpret them is found.

2.9.6 Locality

Locality is a tricky issue in background independent quantum theories of gravity because there is no background metric with which to measure distances or intervals. It is non-trivial to construct diffeomoerphism invariant observables that that measure local properties of fields.

2.10 Summary

In loop quantum gravity, the gravitational field is described in terms of its effect on the parallel transport of spin-1/2 test particles, leading to a formalism in which the quantum geometry of space is described using Wilson loops. This theory makes specific predictions concerning the discreteness of geometrical observables such as area and volume.

Amazingly, the Hamiltonian constraint has a simple action in the same representation in which the diffeomorphism constraint can be solved. Acting on states the action of the

Hamiltonian constraint is concentrated at the intersection of loops. As a result of this one can find an infinite dimensional space of exact solutions to the Hamiltonian constraint.

Many suggestive facts about the possibility of making breakthroughs in solving the quantum constraint equations of canonical quantized general relativity were discovered. But most of the results were formal and loaded with difficulties.

Chapter 3

Formal Developments

- **Spin networks: how to generate independent Wilson loops**
- **Well defined operators: areas and volumes.**
- **Functional integration and functional calculus.**
- **Uniqueness of kinematic representation.**
- **Spatial Invariant Hilbert Space.**

In the previous talk we reviewed how a new canonical formulation of general relativity appeared to offer attractive possibilities, in particular

The phase space was identical to that of an SO(3) Yang-Mills theory.

One could solve the Gauss law using Wilson loops.

One can introduce a representation (the loop representation) where the diffeomorphism constraint can be naturally handled through knot invariants.

Promising results appeared when analyzing formal versions of the quantum Hamiltonian constraint.

We however found several aspects that need sharpening:

The calculations involving the Hamiltonian were formal, unregulated ones. We need more experience regulating operators in this formalism.

The Wilson loops were an over-complete basis of functions and that meant that wave-functions in the loop representations were bound up by complicated identities.

The variables that made the Hamiltonian constraint simple were complex variables requiring us to enforce additional reality conditions to make sure we were obtaining real general relativity.

We will see how developments that happened early in the 90s helped significantly with these issues.

3.1 Spin networks: how to generate independent Wilson loops.

Spin networks Rovelli and Smolin 1994 gr-qc/9411005

But also Penrose in the 60's Witten 1991, Kauffman Lins 90's

linear combinations of products of Wilson loops. All Wilson loops will be gauge invariant. We wish to do away with the identities that bound the wavefunctions in the loop representation. If we recall, we had

$$W_\alpha[A]W_\beta[A] = W_{\alpha\circ\beta}[A] + W_{\alpha\circ\beta^{-1}}[A] \tag{3.1}$$

Or, graphically,

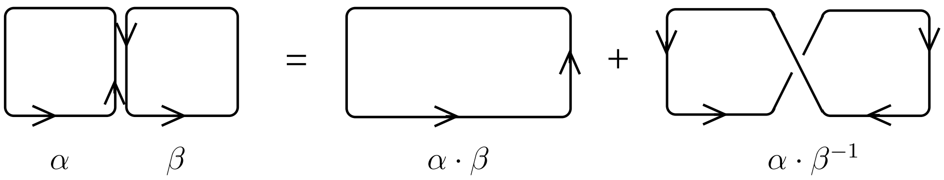


Figure 3.1: Graphical representation of the Mandelstam identity (3.1) relating different Wilson loops.

Which implies,

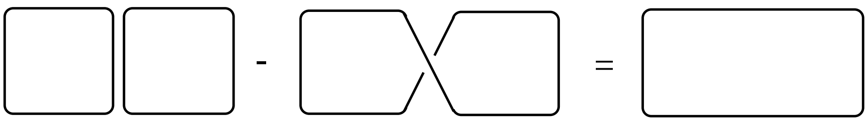


Figure 3.2: .

Let us put this a different way: suppose one has a lattice, and on this lattice we ask "which Wilson loops can I set up which are independent?".

Ignoring multiple windings, the possibilities are

Figure 3.3: .

And are not independent. From the previous diagram we learnt that one could choose and the symmetrized combination of the other two.

The moral is: in the center link if we take "no loop" and "symmetrized" loops, we exhaust all independent possibilities.

The result you'll have to half-believe me is that this construction is general. that is, given any graph, you can construct independent wilson loops by choosing these two possibilities for each line.

How could this be? The idea is that in considering Wilson loops we had unnecessarily straightjacketed into considering the fundamental representation of SO(3). In general one can construct a "generalized holonomy". To do this first consider a graph embedded in 3d with intersections of any order.

Now, along each line we consider a holonomy of an SO(3) connection in the j-th representation. We can generate a gauge invariant object by contracting the holonomies at the vertices using invariant tensors for the group.

The resulting object is a generalization of the Wilson loop.

Considering higher order representation is tantamount to the "symmetrization" of the lines we discussed in the simple example.

Spin networks are linear combinations of Wilson loops. They can simply be seen as an efficient graphical device for keeping track of which combinations are independent. They are also very natural to work with.

Draw a graph in space and associate a half-integer (or irreducible representation of $SU(2)$) with each link. We denote the graph by α. For a given a connection $A_a(x)$, we parallel propagate a spinor along the link - this gives you an element of $SU(2)$. The corresponding representation gives me a matrix. We tie these matrix indices by interwiners that make the function gauge invariant. These spin networks provide a basis of states.

As put by John Baez

"...a state of quantum geometry assigns an amplitude to any system of spinning test particles tracing out paths in space, merging and splitting. These are described by spin networks: graphs with edges labelled by spins together with 'interwining operators' at vertices saying how the spins are routed. These are described using the mathematics of the group $SU(2)$."

The total Hilbert space can be written as a direct sum of finite dimensional ordinary spin-system Hilbert spaces - we all learned about in 1st year quantum mechanics courses.

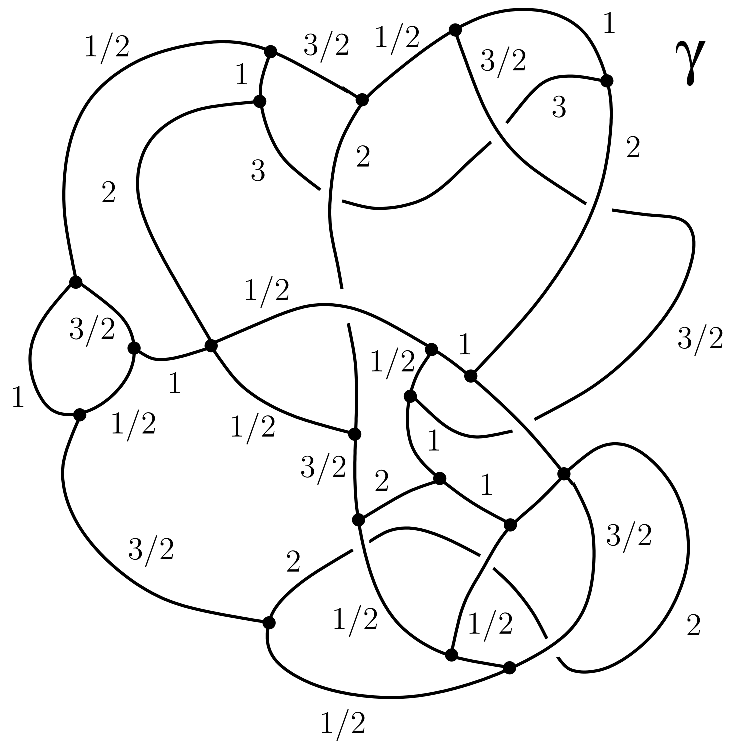


Figure 3.4: A state of quantum geometry $\Psi[\gamma]$ assigns an amplitude to any system of spinning test particles tracing out paths in space, merging and splitting. These are described by spin networks γ: graphs with edges labelled by spins together with 'interwining operators' at vertices saying how the spins are routed.

A quantum state of geometry of space assigns an amplitude to any spin network. operators corresponding to geometric quantities: lengths, areas and volumes.

The space around us is described by a huge linear combination of enormous spin networks that approximates the seemingly smooth geometry we see at distances much large than the plank length ($\sim 10^{-35}$ meters).

The internal labels i, j, k are positive integers determined by the external labels a, b, c: obviously $a = i + j$, $b = j + k$ and $c = i + k$ so that

$$i = (a + c - b)/2, \qquad j = (b + c - a)/2, \qquad k = (a + b - c)/2, \tag{3.2}$$

as in quantum mechanics of adding angular momentum the labels must satisfy the triangle inequalities

$$a + b \geq c, \qquad b + c \geq a, \qquad a + c \geq b \tag{3.3}$$

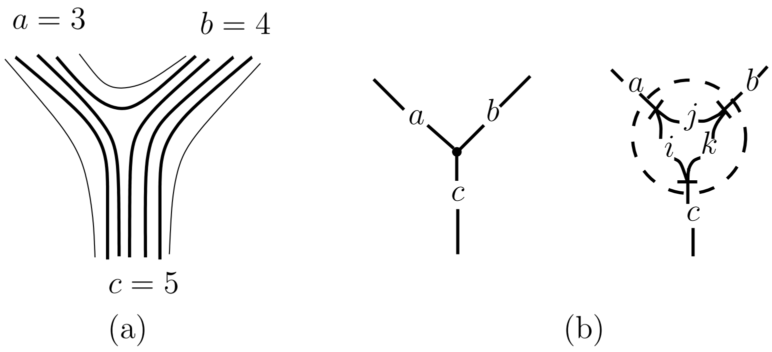


Figure 3.5: Trivalent nodes have to obey Clebsch-Gordon conditions.

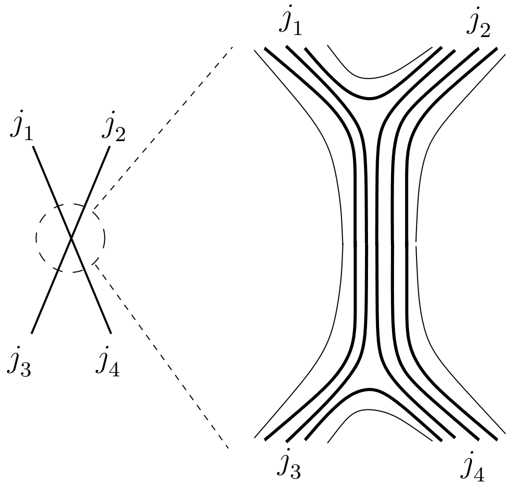


Figure 3.6: A node of valence 4 can be reduced to two trivalent nodes. The different ways of combining 4 spins

3.1.1 Some Maths of Spin Networks

mathematical - can skip this section more details given in appendix [].

$$h_e(A) \otimes h_e(A) \tag{3.4}$$

$$\underline{\underline{a}} \otimes \underline{\underline{b}} := \begin{pmatrix} a_{11}\,\underline{\underline{b}} & a_{12}\,\underline{\underline{b}} \\ a_{21}\,\underline{\underline{b}} & a_{22}\,\underline{\underline{b}} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \tag{3.5}$$

$i, j, k, l = 1, 2,\ A, B = 1, 2, 3, 4$ The row and column labels of C are ***composite*** labels $a_{ij} b_{kl} = c_{AB}$

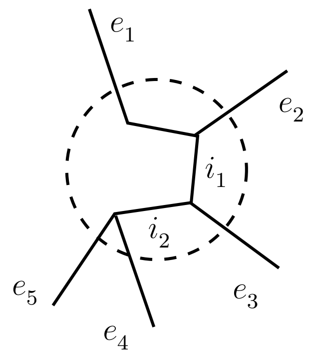


Figure 3.7: valdecomp. The decomposition of a higher valent intersection valence 4...

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{3.6}$$

$\underline{\underline{M}}$ a unitary representation of $SU(2)$, i.e., it satisfies

$$[\underline{\underline{M}}_i, \underline{\underline{M}}_j] = \epsilon_{ijk}\, \underline{\underline{M}}_k, \quad \text{for } i, j, k = 1, 2, 3. \tag{3.7}$$

block diagonalize $\underline{\underline{M}}$, that is put it into the form:

$$\underline{\underline{M}}_i = \begin{pmatrix} \underline{\underline{A}}_i & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{B}}_i & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{C}}_i \end{pmatrix}. \tag{3.8}$$

It is obvious that the satisfy the $SU(2)$ algebra individually, $[\underline{\underline{A}}_i, \underline{\underline{A}}_j] = \epsilon_{ijk}\, \underline{\underline{A}}_k$, ect.

$$\underline{\underline{M}}_i = \underline{\underline{A}}_i \oplus \underline{\underline{B}}_i \oplus \underline{\underline{C}}_i \quad \text{for } i = 1, 2, 3. \tag{3.9}$$

$$\underline{\underline{a}} \otimes \underline{\underline{b}} = \underline{\underline{A}} \oplus \underline{\underline{B}} \oplus \underline{\underline{C}} \tag{3.10}$$

The irreducible representations of $SU(2)$ have dimensions $j(j+1)$.

$A \otimes B$

$$a_{ij} b_{kl} = c_{ik;jl}. \tag{3.11}$$

the row and column labels of the matrix elements of C are *composite* labels: the row label ik, is obtained from the row labels of the matrix elements of A and B and the column label, jl is obtained from the corresponding column labels.

Class function

$$f(x, y, \ldots, z) = f(g^{-1}xg, g^{-1}yg, g^{-1} \ldots zg) \tag{3.12}$$

The character of an irreducible representation $\chi_i(x) := \sum_\alpha M_{\alpha\alpha}(x)$. The obvious property that $\chi_i(x) = \mathrm{Tr}(\mathbf{U}^{-1}\mathbf{M}\mathbf{U}) = \mathrm{Tr}\mathbf{M}$. Class functions can be expanded (**Peter-Weyl theorem**)

$$f(x) = \sum_i a_i \, \chi_i(x) \tag{3.13}$$

where summation is over the irreducible representations. This is easily generalized to case of more than one arguement

$$f(x, y, \ldots, z) = \sum_{i,j,k} a_{ij\ldots k} \, \chi_i(x)\chi_j(y) \ldots \chi_k(z) \tag{3.14}$$

The *Peter-Weyl Theorem* applied to $U(1)$ gives the Fourier series theory:

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \tag{3.15}$$

where $f(\theta) \in L^2(U(1))$.

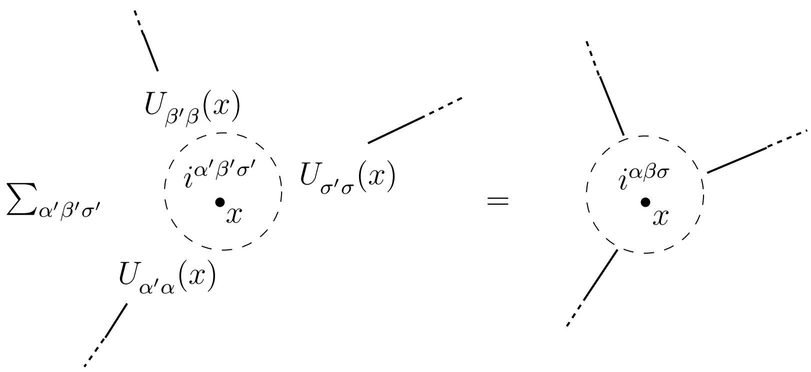


Figure 3.8: $i^{\alpha\beta\sigma}$. no free ends if diagrams closed will be Gauss gauge invariant.

$$h_e(A) \mapsto U_{\alpha\alpha'}(x) h_e(A) U_{\alpha\alpha'}(y) \tag{3.16}$$

where x is the starting point of the link and y the end point.

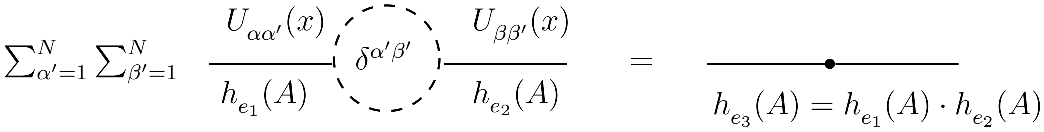


Figure 3.9: A node of valence 2 has $i^{\alpha\beta} = \delta^{\alpha\beta}$ as the trival interwiner.

the interwiners are constants $i^{\alpha\beta\sigma}$ $(\alpha, \beta, \sigma = 1, 2)$ such that

$$i^{\alpha'\beta'\sigma'} = i^{\alpha\beta\sigma}\, U_{\alpha\alpha'} U_{\beta\beta'} U_{\sigma\sigma'}. \tag{3.17}$$

In the case where the interwiners are unabiguisly defined, but for greater than 3 there are different choices. If we draw analogy with composition of angular momentum. This reflects the fact that there is more than one way to add 3 angular momentun.

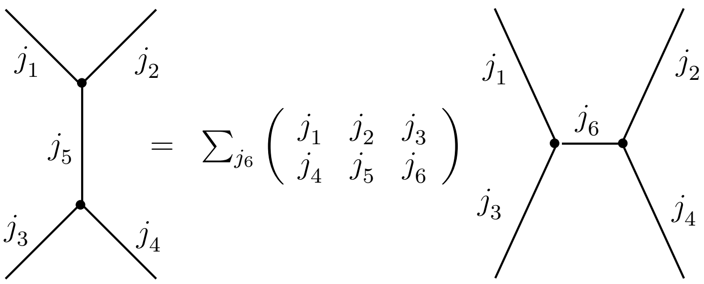


Figure 3.10: .

$$[\rho_{j_1}(H_{e_1}(A))]^{\alpha_1}_{\beta_1} \cdots [\rho_{j_n}(H_{e_1}(A))]^{\alpha_n}_{\beta_n}\, v^{\beta_1 \dots \beta_n} = v^{\alpha_1 \dots \alpha_n} \tag{3.18}$$

Case (a) ψ_{AB}:

$$\epsilon_{AB}\epsilon^{CD} = \delta^C_A \delta^D_B - \delta^D_A \delta^C_B. \tag{3.19}$$

This identity can be incorpriated

$$\begin{aligned}
\psi_{AB} &= \psi_{[AB]} + \psi_{(AB)} \\
&= \psi_{(AB)} + \frac{1}{2}\psi_{[CD]}(\delta^C_A \delta^D_B - \delta^D_A \delta^C_B) \\
&= \psi_{(AB)} + \left(\frac{1}{2}\psi_{[CD]}\epsilon^{CD}\right)\epsilon_{AB} \\
&\equiv \psi_{1AB} + \psi_0 \epsilon_{AB}
\end{aligned} \tag{3.20}$$

where

$$\psi_0 = \frac{1}{2}\psi_{CD}\epsilon^{CD} \tag{3.21}$$

Both $\psi_0\epsilon_{AB}$ and ψ_{1AB} are separately idependent of the binor identity. Any spinor can be decomposed in this way incorpriating the binor identitiy. Let us consider a three component spinor ψ_{ABC}.

Case (b) ψ_{ABC}:

$$\begin{aligned} 3\psi_{(ABC)} &= \psi_{A(BC)} + \psi_{B(AC)} + \psi_{C(AB)} \\ &= 3\psi_{A(BC)} - (\psi_{A(BC)} - \psi_{B(AC)}) - (\psi_{A(BC)} - \psi_{C(AB)}) \\ &= 3\psi_{A(BC)} - \epsilon_{AB}\sigma_C - \epsilon_{AC}\sigma_B \end{aligned} \tag{3.22}$$

where $\sigma_C = \epsilon^{AB}(\psi_{A(BC)} - \psi_{B(AC)})$. This rearanged gives

$$\psi_{A(BC)} = \psi_{(ABC)} + \frac{1}{3}\epsilon_{AB}\sigma_C + \frac{1}{3}\epsilon_{AC}\sigma_B. \tag{3.23}$$

$$\psi_{A(BC)} = \psi_{ABC} - \frac{1}{2}\epsilon_{BC}\psi_{AD}{}^{D} \tag{3.24}$$

Using (3.23) in (3.24) we arrive at for ψ_{ABC}

$$\psi_{ABC} = \psi_{(ABC)} + \frac{1}{2}\epsilon_{BC}\psi_{AD}{}^{D} + \frac{1}{3}\epsilon_{AB}\ \sigma_C + \frac{1}{3}\epsilon_{AC}\ \sigma_B. \tag{3.25}$$

obtaining the desired expansion in which all terms are separately independent of the binor identity.

Case (c) $\psi_{A...F}$: Proof is by induction. More generally (3.23) implies

$$\psi_{A(BC...F)} = \psi_{(ABC...F)} + \frac{1}{n}\epsilon_{AB}\rho_{(C...F)} + \cdots + \frac{1}{n}\epsilon_{AF}\rho_{(B...F)} \tag{3.26}$$

where

$$\rho_{(C...F)} = \epsilon^{AB}(\psi_{A(BC...F)} - \psi_{B(AC...F)}) \tag{3.27}$$

Spinor decomposition

Any decomposition involves

$$\psi_A, \psi_{(AB)}, \psi_{(ABC)}, \dots \text{ and products of } \epsilon_{AB} \tag{3.28}$$

$$\hat{\sigma}^2 = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k \hat{\sigma}^k}{4} \right) \otimes \cdots \otimes 1 \tag{3.29}$$

$$\hat{\sigma}^2 \psi_{(AB \dots F)} = n(n+1) \psi_{(AB \dots F)} \tag{3.30}$$

A representation of a group G in a vector space V over k is defined by a homomorphism

$$\pi : G \to GL(V).$$

The degree of the representation is the dimension of the vector space.

Direct Products and Clebsch-Gordan Coefficients

$$\tau^{\mathbf{i}}_{(\mathbf{j})} = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2} \right) \otimes \cdots \otimes 1 \tag{3.31}$$

We wish to calculate $\tau^{\mathbf{i}}_{(\mathbf{j})} \tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})} \tau^{\mathbf{i}}_{(\mathbf{j})}$. The terms below for $k \neq k'$ wont contribute to the commutator as the order of multiplication for $k = k'$ is irrelevant,

$$\sum_{k \neq k'} \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2} \right) \otimes \cdots \otimes 1 \right) \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^{k'}}{2} \right) \otimes \cdots \otimes 1 \right) - (k \leftrightarrow k') = 0 \tag{3.32}$$

$$\begin{aligned}
\tau^{\mathbf{i}}_{(\mathbf{j})} \tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})} \tau^{\mathbf{i}}_{(\mathbf{j})} &= \\
&= \sum_{k=1}^{2s+1} \left[1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i}{2} \right) \left(\frac{\hat{\sigma}^j}{2} \right) \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^j}{2} \right) \left(\frac{\hat{\sigma}^i}{2} \right) \otimes \cdots \otimes 1 \right] \\
&= \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i \hat{\sigma}^j}{4} - \frac{\hat{\sigma}^j \hat{\sigma}^i}{4} \right) \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk} \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \frac{\hat{\sigma}^k}{2} \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk} \tau^{\mathbf{k}}_{(\mathbf{j})}
\end{aligned} \tag{3.33}$$

We say the vector space V carries a representation π of $SU(2)$.

$$\begin{aligned}
\frac{1}{2}\hbar\hat{\sigma}^3_{AA'BB'CC'} &:= \frac{1}{2}\hbar\hat{\sigma}^3_{AA'}\delta_{BB'}\delta_{CC'} \\
&\vdots \\
&\quad \frac{1}{2}\hbar\hat{\sigma}^3_{CC'}\delta_{AA'}\delta_{BB'}\delta_{EE'} \\
&\quad (AA'; BB'; CC' = 0, 1).
\end{aligned} \tag{3.34}$$

Summarising

A spin network function is labelled by a graph γ, a set of non-trivial representations $\{\pi_e\}$ one for each edge of γ, and a set of contraction matrices $\{M_v\}$, one for each vertex of γ, which contract indices of the tensor product $\otimes_{e\in E(\gamma)}\pi_e(h_e)$ in such a way that the resulting function is gauge invariant.

One can show that these functions are linearly independent.

$\rightarrow$ The interwiner operators associated a vertex of a spin network can be understood as specifying how we could connect the ropes of the edges meet at the same vertex. In the language of representaion of representation theory of groups, this corresponds to the fact that tensor products of irreducible represenations can be completely decomposed into the direct sum of irreducible representations. Hence they are invariant tensors of irreducible representations of groups and given by the stanadrd Clebsch-Gordon theory. In the case of $SU(2)$ spin networks, when the vertex is tri-valent, the decomposition of the product is unique. For $n > 3$, an n-valent vertex can be divided into tri-valanetvertices by making use of interwiner operators. At the same time, the colors asssociated with edges which meet at a vertx must satify consistent conditions. $\leftarrow$

3.1.2 A Note on Spatial Diffeomorphism Invariance

The next step in the construction of the theory is to factor away diffeomorphism invariance. Diffeomorphism invariance identifies two spin-networks that can be deformed into each other as gauge equivalent; in the same way that two solutions of the Einstein equationsthat are related by a coordinate transformation. An s-knot is an equivalence classe of spin networks related by diffeomorphisms.

This is a key step for two reasons. First of all, $\mathcal{H}$ is a "huge" non-separable space (). Prevents Wilson loops are distributional . However, gravity has a cure for this; to factor out spatial diffeomorphism heuristically, we are averaging the loop over the position of the loop and so in some sense the loop is smeared over the whole manifold. The only remaining information contained in the loop is its knotting which are indexed by a countable index set (it isn't quite as straightforward as this; for networks with nodes of valence 5 and greater in factoring away smoothe diffeomorphisms are still labelled by continuous parameters, (see Appendix M)).

One can show that the space of square integrable functionals, i.e.$\int |\Psi[A]|^2 \leq \infty$, has a basis of "spin networks". In the nexted section we will look at rigorously quantized geometrically interesting obseravables area of surfaces and volumes of space, which have been obtained operators on $L^2(\mathcal{A})$. The matrix elements of these operators can be explicitly computed in the spin network basis

3.2 Constructing Well Defined Operators: Areas and Volumes

3.2.1 Area.

[94], [97]

Rovelli and Smolin 94;Ashtekar, Lewandowski et al 95

Given a surface, we want to compute its area quantum mechanically.

$$A = \int_S d^2\sigma \sqrt{E^{ai}E^{bi}n_a n_b} \tag{3.35}$$

Conceptual consideration What are the true observables as these are subject to Heisenberg's indeterminacy principle. The details of this are presented in appendix L - Physical geometery and geometric operators. Although the expression for area is invariant under a change of coordinates it fails to be invariant under a diff transformation as a consquence of the fact that the area of an abstract surface defined in terms of coordinates is not invariant under active diff transformations. In fact, physically mearsurable areas in general relativity corresponed to surfaces defined by physical degrees of freedom, for instance matter (the area of a table). However, it is reasonable to expect that the fully gauge invariant operator corresponding to a physically defined area (say defined with matter) has precisely the same mathematical form as the gauge invariant operator studied here. The reason is that one can use the matter degres of freedom to gauge fix the diffeomorphisms - so that a non-diff-invariant quantity in pure gravity correspondes to a diff-invariant quantity in a gravity+matter theory.)

To promote this quantity to an operator, we need to handle the product of triads and also the square root. To do this, we start by partitioning the surface in small elements of area and notice that since the triads are functional derivatives quantum mechanically, one only gets contributions from the small elements of area pierced by a line of the spin net.

So we have $\hat{A} = \lim n \to \infty \sum_I \sqrt{A^2}_I$,

We need the action of the triad on a spin network state, which is very similar to on a loop state,

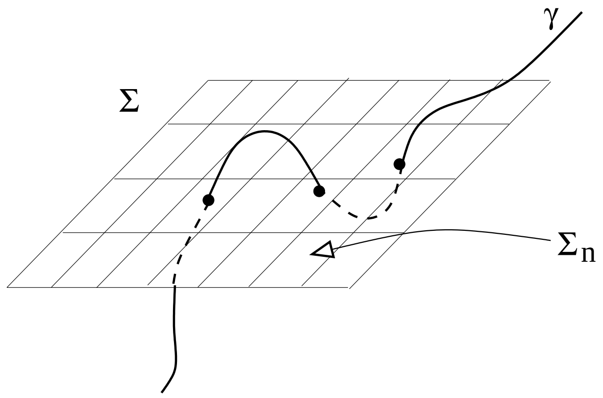


Figure 3.11: A partition of Σ.

Where X_i^j is a generator of SO(3) in the J level representation and s_X is a spin network that is opened at the point x and the generator X is inserted in that place. Since these quantities are distributional, we need to regularize. We will regularize by: a) smearing the E's along the small surface and point-splitting the product. The result is

$$A \tag{3.36}$$

Notice that we have six one dimensional Dirac deltas. All these "cancel each other" and we are left with a simple expression givenby the square of the SO(3) generator in the J-th spin representation. From angular momentum theory, we know that the value of such square is $j(j+1)$, so the end result for the area operator is,

$$A = 8\pi\hbar G\gamma \sum_i \sqrt{j_i(j_i+1)} \tag{3.37}$$

So we see that the area has a well defined action, and although we used background structures to regularize, the final result is topological and background independent. The spectrum of the operator is discrete, and admits a simple interpretation in which the spin of the lines of a spin network can be viewed as "quanta of area".

Ashtekar and Lewandowski have done a complete analysis that includes the possibility of lines being parallel to the surface and produced the complete spectrum of the operator. gr-qc/9602046,9711031

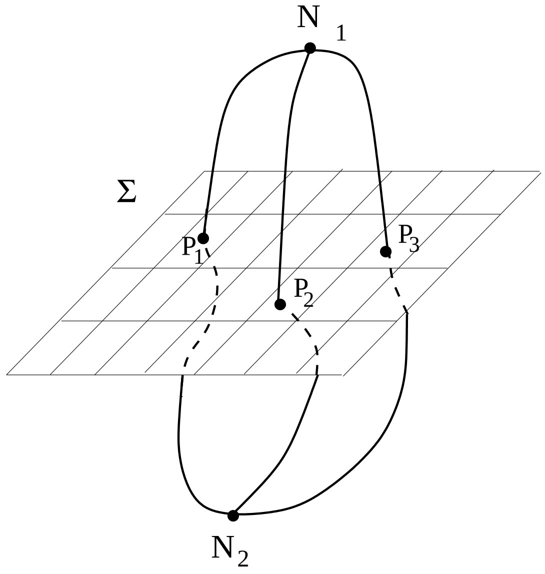


Figure 3.12: A simple spin network S intersecting the surface Σ.

While the quantum operator is well-defined onj states which are cylindrical with respect to any one graph, at the end we have to ensure that the resulting family of operators is consistent. This is a non-trivial issue.

The prediction of the spectrum is very specific and has immediate consequences. One naively would have assumed that the spectrum of the area would go as,

However, it doesn't. With the prediction of

The spacing of the eigenvalues diminishes rapidly for large values of the area. We will se that this has consequences at the time of computing the entropy of black holes.

[97] - The difference ΔA between are eigenvalues as $A \to \infty$ is

$$\Delta A \leq 4\pi\beta l_P^2 \frac{\sqrt{8\pi\beta}}{\sqrt{A}} + \mathcal{O}\left(\frac{l_P^2}{A}\right) l_P^2. \tag{3.38}$$

the area spectrum is used in the statisticalmechanical caculation of black-hole entropy.

A last note. Some objections have ben raised obout the last point... Some objections are based on the intuition that the position of the matter defining the surface could be subjected to quantum fluctuations, preventing the possibility of defining a sharp surface. This objection is incorect. Neither the position of the matter, nor the area of the surface, have physical independent reality. It is only the gravitaional field in the location determined by matter, or, the other way round, the location of the matter in the gravitaional field, that have physical reality. The two do not form independent sets of degrees of freedom subjected to independent quantum fluctuataions. See ...".

3.2.2 Volume operator:

[94], [98]

The volume operator

$$V = \int \sqrt{q} = \int \sqrt{\frac{1}{3!}\epsilon^{ijk}\epsilon_{abc}\tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c} \tag{3.39}$$

I will omit the details since the calculation goes very much as for the area operator: one first breaks the integral into a sum over little cubic regions. In each of these regions. In each of regions. In each of these regions one smears the E operators with two dimensional integrals. Because of the epsilons the quantity is only non-vanishing at a place where there is a vetex of the spin network. One is left with three two dimensional integrals, three one dimensional integrals and three-dimensional Dirac deltas so the result is finite.

The group factor is a bit more difficult to compute than before, but corresponds to three traced generators contracted with epsilon. It can be computed. The end result is that the volume is finite, has a discrete spectrum, and the non-vanishing contributions come from **four-valent** intersections or higher. The eigenvalues depends on the value of the valences that enter the intersections.

[94], [95]

From [27] Smolin says, *The area and volume operators can be promoted to genuine physical observables, by gauge fixing the time gauge so that at least locally time is measured by a physical field* [The papers smolin references are here [91], [92]]. *The discrete spectra remain for such physical observables, hence the spectra of area and volume constitute genuine physical predictions of hte quantum theory of gravity.*

3.2.3 Quantum Geometry

So a picture of quantum geometry emerges in which the lines of flux are associated with "quanta of area" and the intersections of the lines quanta of volume. A spin-network is not in space it is space. To ask where is the spin-network is like asking where is a solution to Einsein's equation.

In classical geometry the volumes of regions and areas of the surfaces depend on the values of the gravitaional fields. They are coded in the metric tensor. On the other hand in the quantum picture the geometry is is coded in the choice of spin network. These spin networks correspond to the classical description, one can find a spin network which describes, to some level of approximation, the same geometry.

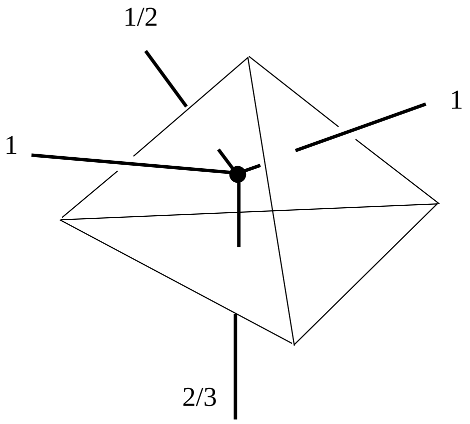


Figure 3.13: A node of a spin network (in bold) and its dual 3-cell (here a tetrahedron). The colouring of the node determines the quantized volume of the tetrahedron. The colouring of the edges determines the quantized area of faces via equation ().

It is extremely convenient that these spin-network states are eigenvectors of geometric operators. An example of this which we will come to is the microscopic source of the entropy of a black hole.

What are the knotting and linking of loops and graphs to do with?

It has been shown that the diffeomorphism invariant states were characterized by the knotting and linking of loops and graphs. What features of geometry do the knotting and linking measure? Observables sufficient to label the degrees of freedom of quantum geometry were identified in the area and volume operators, which measure combinatorial and labeling information, but which are insensitive to the topology of the embedding.

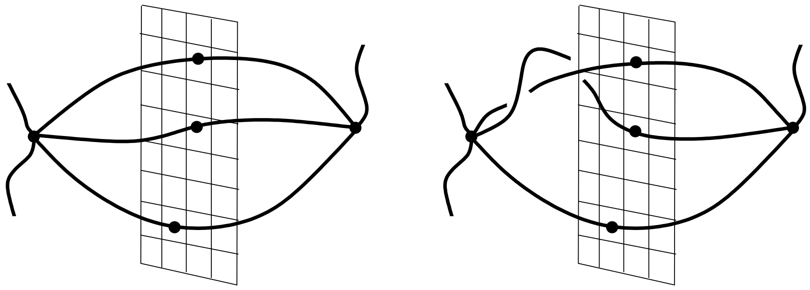


Figure 3.14: knotAresVolF. Area and volume operators are insensitive to the topology of the embedding.

Recently, results we describe in [276] show that some of the information in the embedding may have nothing to do with geometry, but instead describes emergent particle states. We will say more about this at the end of chapter 7.

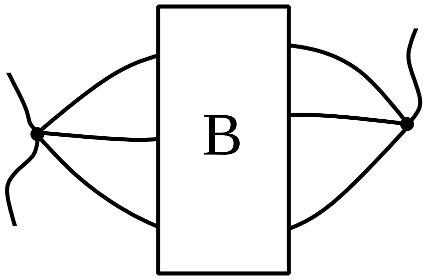


Figure 3.15: knotAresVolF2. Infinte degeneracy.

Other applications of spin network states

They are also employed in a different context; they employed in a technique for constructed a measure of integration called *clyndrical measure thoery*. In the next section They will appear as a natural candidates for *cylindrical functions* used in this construction.

Similarly in Ashtekar's approach A is like the "position" and E is like the canonically conjugate "momentum". Spin network states are eigenstates of E, so they are like "momentum" eigenstates.

out of place

3.3 Algebraic Quantization

The approach is conservative in the sence that one is following a non radical extension of the usual procedure of (algebraic) quantization.

Choose a subset, $\mathcal{S}$, of the set of all complex valued functions on phase space Γ such that

(1) $\mathcal{S}$ is large enough to allow any sufficiently regular function on Γ to be expressed as a sun of their products of its elements;

(2) $\mathcal{S}$ is closed under Poisson brackets and also closed under complex conjugation;

(3) elememnts of $\mathcal{S}$ are to have unambuguous quantum analogs.

In short, the elements of $\mathcal{S}$ should play the role played by the q's and p's when the phase space is $\mathbb{R}^{2n}$.

3.3.1 The GNS Construction

• A state F on $\mathcal{A}$ is a positive linear functional ('expectation-value'of the operators in $\mathcal{A}$): For any $a \in \mathcal{A}$, $F(a)$ is a complex number such that:

$$F(a + \lambda b) = F(a) + \lambda F(b), \qquad \lambda \in \mathbb{C}; \quad F(I) = 1; \quad F(a^* a) \geq 0.$$

• Given any F, the GNS construction provides a Hilbert space $\mathcal{H}$ and a representation of $\mathcal{A}$ by operators on $\mathcal{H}$ such that

i) the repressentation is cyclic; i.e. there exists a vector Ψ_F in $\mathcal{H}$ such that $\{\mathcal{A} \cdot \Psi_F\}$ is dense in $\mathcal{H}$; and

ii) $F(a) = (\Psi_F, a\Psi_F)$ for all $a \in \mathcal{A}$.

• Very general procedure. e.g., Every ireducible representation of $\mathcal{A}$ is cyclic.

If θ is an automorphism on $\mathcal{A}$ (i.e. a structure preserving map from $\mathcal{A}$ to otself), and if $F[\theta(a)] = F[a]$ then θ is unitarily implemented on $\mathcal{H}$; There exists a unitary operator U_θ on $\mathcal{H}$ such that

$$(\theta(a))\Psi = (U_\theta^{-1} a U_\theta)\Psi,$$

for all $\Psi \in \mathcal{H}$ and $U_\theta \Psi_F = \Psi_F$.

A powerful and economic way to ensure that (gauge-)symmetries are unitarily implemented. In Minkowski field theories, $F(a) =< 0|a|0 >$ is Poincare invariant.

3.3.2 C^*-algebras

Representations

Stuff discussed here will be proven in detail in appendix O.

Definition of a C^*-algebra.

first intro

An algebraic structure called a C^*-algebra. A concrete C^*-algebra is a linear space $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, that is, a bunch of operators closed under addition, multiplication, scalar multiplication, and taking adjoints which is also complete with respect to the operator norm.

A C^*-algebra can be defined abstractly without any reference to linear operators acting on a Hilbert space. An abstract C^*-algebra is given by a set on which addition, multiplication, adjoint conjugation, and a norm are defined, satisfying the same algebraic relations as their concrete counterparts.

second intro

The elements of $\mathcal{O}$ are abstract mathematical entities, completely unrestricted except for the above conditions. A representation of the C^*-algebra $\mathcal{O}$ is a set of particular objects that satisfy the conditions of a C^*-algebra.

A representation of a C^*-algebra $\mathcal{O}$ consists of $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space and π is a morphism of $\mathcal{O}$ to the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. Clearly, the conditions of the definition of a C^*-algebra $\mathcal{O}$ are satisfied.

Cyclic representations and states

The Hydrogen atom. We generate a complete set of eigenstates by acting on the ground state by ladder operators.

A cyclic representation of $\mathcal{O}$ is a triple $(\mathcal{H}, \pi, \Omega)$, where $\Omega \in \mathcal{H}$ such that $\|\Omega\| = 1$ and $\pi(\mathcal{O})$ is dense in $\mathcal{H}$.

If $(\mathcal{H}, \pi, \Omega)$ is a cyclic representation of a $*-$algebra, then

$$A \to \omega(A) :=< \Omega|\pi(A)\Omega > \tag{3.40}$$

defines a linear functional on $\mathcal{O}$. Remember a functional is something that acts on a vector and gives you back a complex number. The linear functional on $\mathcal{O}$ is usualy called, slightly confusingly at first, a *state*: A state is an assignment of an expectation value to each member of a collection of 'observables' (the elements of a C^*-algebra), if we know the expectation value of a complete set of commuting "observables" we know the state $|\Psi>$ of the system. For example, if we have the expectation value of the enery, z-component and the total angular momentum of an electron in a Hydrogen atom then we know the quntum state $\psi_{E_n,J^z_m,J}(x)$ of the electron.

Now the converse of the above statement is also true, and is known as the GNS construction: Given one has a state ω one can construct a cyclic rerpresentation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$.

However there is a proviso, a property you want the inner product to have is that it be positive definite, i.e. $(A, A) = 0$ implies that A is the null vector. A representation π is said to be faithful if, for $A \in \mathcal{O}$, $\pi(A) = 0$ implies $A = 0$. If the representation is not faithul the C^*-algebra would contain elements of $\mathcal{O}$ for which $\omega(A^*A) = 0$ does not imply $A = 0$. A representation can be forced to be faithful as is discussed below.

Given one has a faithul representation and a state ω on a $C\star-$algebra, one can construct a cyclic rerpresentation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$.

3.3.3 Null vectors

If an inner product $(A, A) = 0$ does not imply that A is null it is said to be only positive semidefinite.

The presence of a null ideal $\mathcal{N}$ requires to construct the Hilbert space by transferring our attention from the $C\star-$algebra $\mathcal{O}$ ... to consideration of equivalence classes ... and the induced multiplication, addition and scalar multiplification between and of classes. This forces the positive semidefinite inner product to become positive definite.

Must establish if induced multiplication, addition between classes, scalar multiplication and inner product is independent of the particular representative of the equivalence class.

Let

$$\mathcal{N} = \{A \in \mathcal{O} \ : \ \omega(A^*A) = 0\}.$$

Using the Cauchy-Schwarz inequality

$$|\omega(A^*B)|^2 \leq \omega(B^*B)\omega(A^*A),$$

we find that

$$\omega(N^*A) = \omega(A^*N) = \omega(N^*N) = 0 \tag{3.41}$$

whenever $N \in \mathcal{N}$. We consider each operation of a C^*-algebra in turn and then its inner product.

Conjugation $[A^*] = [A]^*$?

$[A^*]$ is comprised of those elements $A^* + N$ where N runs through the elements belonging to $\mathcal{N}$. Let N be any element in $\mathcal{N}$, we easily have

$$\omega((N^*)^*N^*) = \omega(NN^*) = \omega(N^*N) = 0$$

hence $N^* \in \mathcal{N}$. It follows from

$$(A + N)^* = A^* + N^*$$

that $A^* + N^*$ is a representative of $[A]^*$.

Multiplication $[A][B] = [AB]$?

Let N, N' be any two elements in $\mathcal{N}$,

$$(A + N)(B + N') = AB + (AN' + NB + NN') \equiv AB + N'' \tag{3.42}$$

Eq (3.41) implies $N'' \in \mathcal{N}$ so $AB + N''$ is a representative of $[AB]$.

Addition $[A] + [B] = [A + B]$?

Let N, N' be any two elements in $\mathcal{N}$, then obviously

$$\omega((N + N')^*(N + N')) = 0$$

so that $N + N' \in \mathcal{N}$. It then follows from

$$A + N + B + N' = A + B + (N + N')$$

that $A + B + (N + N')$ is a representative of $[A + B]$.

Scalar multiplication $\alpha[A] = [\alpha A]$?

As ω is linear

$$\omega((\alpha N)^* \alpha N) = \alpha^* \alpha \, \omega(N^* N) = 0$$

hence $\alpha N \in \mathcal{N}$. It then follows from

$$\alpha(A + N) = \alpha A + (\alpha N),$$

that $\alpha A + \alpha N$ is a representative of $[\alpha A]$.

Inner product $\omega([A]^*[B]) = \omega(A^* B)$?

Now we prove $(A + N, B + N')_\omega$ is independent of the representatives.

$$(A, B)_\omega = \omega(A^* B) \tag{3.43}$$

For any two elements A and B, $A \to \overline{A} = A + N$ and $B \to \overline{B} = B + N'$, where N and N' are elements of $\mathcal{N}$, we have

$$\begin{aligned}\omega(\overline{A}^*\overline{B}) &= \omega((A^*+N^*)(B+N')) \\ &= \omega(A^*B)+\omega(N^*B)+\omega(A^*N')+\omega(N^*N') \\ &= \omega(A^*B) \qquad (3.44)\end{aligned}$$

$\mathcal{N}$ is what is called a *two sided ideal* in $\mathcal{O}$.

Define the scalar product on $\mathcal{O}/\mathcal{N}$ by

$$([A],[B])_\omega = \omega(A^*B) \qquad (3.45)$$

Completion of $\mathcal{O}/\mathcal{N}$ with respect to this norm is the Hilbert space $\mathcal{H}_\omega$.

3.3.4 GNS Constrution

Given a state ω over an abstract C^*-algebra $\mathcal{A}$, the Gelfand-Naimark-Segal construction provides us with a Hilbert space $\mathcal{H}_\omega$ with a preferred state Ω_ω, and a representation π_ω of $\mathcal{A}$ as a concrete algebra of bounded operators on $\mathcal{H}_\omega$, such that

$$\omega(A) = <\Omega_\omega|\pi_\omega(A)\Omega_\omega> . \qquad (3.46)$$

Now one invokes the Gelfand-Naimark theorem which asserts that:

> Every C^*-algebra with identity is isomorphic to the C^*-algebra of all continous, bounded, complex functions on a compact, Hausdorff space.

Compact means finite in size. A Hausdorff space is a space with a certain separibility condition: for two distinct points there exists a neighbourhood for each point which are disjoint.

3.4 The Holonomy-Flux Algebra

3.4.1 Differentiability Classes of Manifolds and Loops

A function is called real analytic at a point if it possesses derivatives of all orders and given by a convergent power series locally. For example, a function on the real line $\mathbb{R}$

is analytic at the point p if there exists an interval (a, b) containing p such that in this interval the function can be expanded as a convergent series

$$f(x) = a_0 + a_1(x-p) + a_2(x-p)^2 + a_3(x-p)^3 + \ldots, \tag{3.47}$$

where

$$a_0 = f(p),\ a_1 = f'(p),\ a_2 = \frac{f''(p)}{2!},\ a_3 = \frac{f'''(p)}{3!}, \ldots \tag{3.48}$$

A function is analytic if it is analytic at each point in its whole domain. The set of all analytic functions is contained in the set of smooth functions. Analytic functions are also referred to as C^ω-smooth functions.

When we write parametric equations for a curve in the Euclidean space, say, $(x^1 = x^1(t), \ldots, x^n = x^n(t))$, for (a, b) we expect the curve to be continuous (as a function from (a, b) to $\mathbb{R}^2$) provided that each of the 'coordinate functions' $x^1(t), \ldots, x^{n-1}(t)$ and $x^n(t)$ are continuous (as a function from (a, b) to $\mathbb{R}$). A curve in Euclidean space $\mathbb{R}^n$ is analytic if it can be expanded as a Taylor series locally.

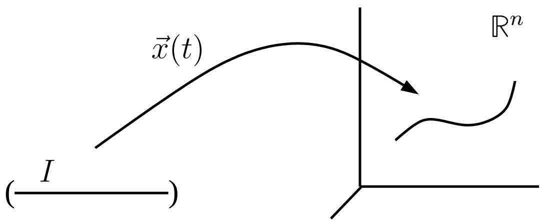


Figure 3.16: DiffCurveFig. The map $\lambda(t)$ from the open interval $I = (a, b)$ of the real line to the coordinates on $\mathbb{R}^n$ characterizes the differentiability class of the curve.

A curve in a manifold $\mathcal{M}$ is analytic if and only if its image under a chart is an analytic curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval $I = (a, b)$ to $\mathbb{R}^n$ in Fig.(C.7.1) is an analytic map. A curve is piecewise analytic if it is made up of a finite number of pieces, each of which is analytic.

Analytic and smooth diffeomorphims from the manifold to itself.

This is possible [??] but then the technical discussion becomes much more complicated because, e.g., two smooth curves can intersect one another at an infinite number of points.

3.4.2 Integration on a Manifold

One can not sum over vectors and tensors because the result will be ambiguous - at different points the vectors and tensors transform differently. Unlike tensors in general,

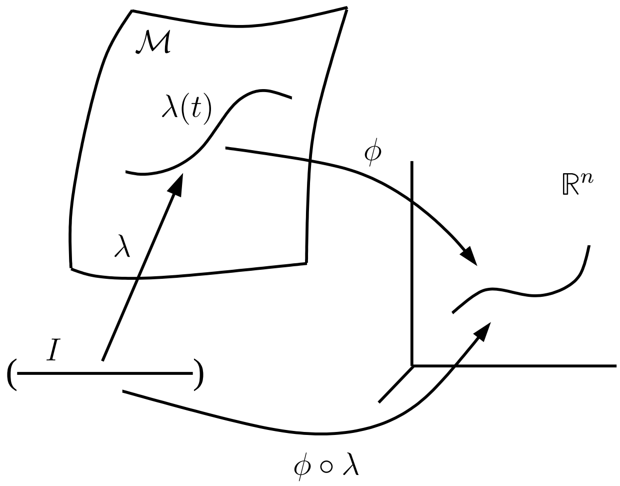


Figure 3.17: tangvectorForm. The map $\phi \circ \lambda$ from the open interval $I = (a, b)$ of the real line to the coordinates characterizes the differentiability class of the curve $\lambda(t)$.

we can add a scalar field field evaluated at two different points, x_1 and x_2 say, and the resulting quantity is still a scalr, since under a coordinate transformation, the sum transforms as

$$\phi'(x'_1) + \phi'(x'_2) = \phi(x_1) + \phi(x_2)$$

Summation must be over over scalars! As the volume element $d\Omega$ is a scalar density of weight -1 it follows that we can only integrate scalar density Φ of weight $+1$ over a region Ω, $\int_\Omega \Phi d\Omega$ since at each point $\Phi d\Omega$ is a scalar and can be added together. There are similar statements about which can be made about about integrations over curves and surfaces. It is natural to integrate a one-form X^a on a curve γ

$$\int_\gamma X_a dx^a \tag{3.49}$$

and a two form Y_{ab} over a surface S

$$\int_S Y_{ab} dS^{ab} \tag{3.50}$$

An *embeding* of one manifold into another.

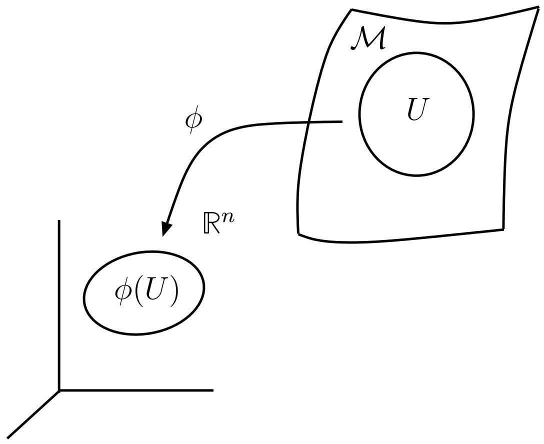


Figure 3.18: DiffClass0. A chart on $\mathcal{M}$ comprise an open set U of $\mathcal{M}$, called a coordinate patch, and a map $\phi : U \to \mathbb{R}^n$.

3.4.3 The Holonomy-Flux Algebra

There is no metric to raise and lower indices so we need to be honest about the type of tensor the variables of the theory are. There is the connection one-form

$$A_a^i(x)$$

the Electric field

$$E_i^a(x)$$

we also have the totally-antisymmteric tensor density

$$\epsilon_{abc}$$

From this tensor we can form the two-vector density

$$\tilde{E}_{ab}(x) := \epsilon_{abc} E_i^c(x)$$

Wilson loop functions are the obvious candidates for configuration variables. These will be associated with piecewise analytic loops on Σ, i.e., with piecewise analytic maps $\alpha : S^1 \to \Sigma$. (Thus, the loops do not have a preferred parameterization, although in the intermediate stages of calculations, it is often convenient to choose one.) The Wilson loop variables $T\alpha(A)$ are given by:

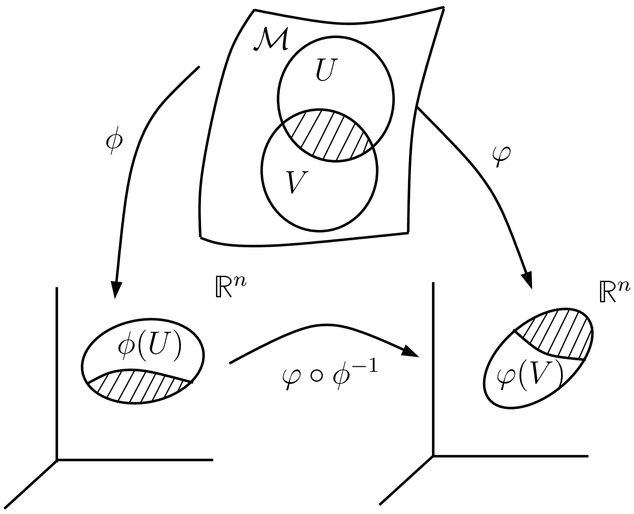


Figure 3.19: DiffClass2. The neighbourhoods U and V in $\mathcal{M}$ overlap. Their respective maps to $\mathbb{R}^n$, ϕ and φ, give two different coordinate systems to the overlap region. The relation between these coordinates characterizes the differentiability class of the manifold.

$$T_\alpha(A) := Trh_\alpha[A] \equiv Tr\mathcal{P}\exp\int_\alpha Ads \tag{3.51}$$

As defined, these are functions on the space of connections. However, being gauge invariant, they project down naturally to $\mathcal{A}/\mathcal{G}$. The momentum observables, T_S are associated with piecewise analytic strips S, i.e., ribbons which are foliated by a 1-parameter family of loops. For technical reasons, it is convenient to begin with piecewise analytic embeddings $S : (1,1)S17$ and use

$$T_S(A) := \int_S dS^{ab}\eta_{abc}T^c_{\alpha_\tau}(\sigma,\tau) \tag{3.52}$$

where

$$T^c_{\alpha_\tau}(A) := Tr\big(h_{\alpha_\tau}(\sigma,\tau)[A]\tilde{E}^c(\sigma,\tau)\big) \tag{3.53}$$

σ, τ are coordinates on S (with τ labeling the loops within S and σ running along each loop α_τ), η_{abc} denotes the Levi-Civita tensor density on Σ, and, as before h_{α_τ} denotes the holonomy along the loop α_τ. Again, the functions T_S are gauge invariant and hence well-defined on the phase space (cotangent bundle) over $\mathcal{A}/\mathcal{G}$. They are called "momentum variables" because they are linear in $\tilde{E}^a_i$.

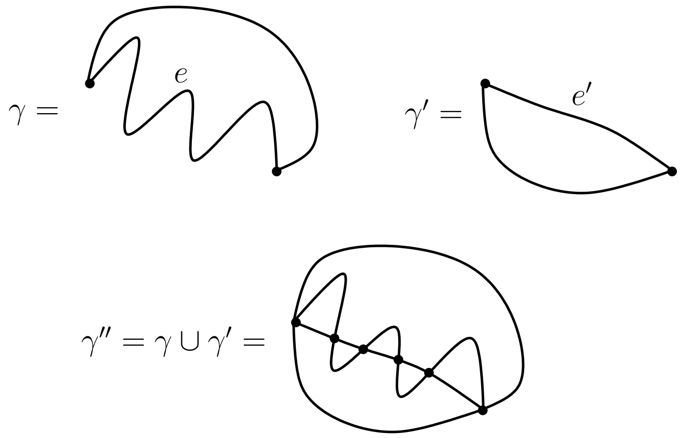


Figure 3.20: The union of the two graphs γ and γ' has a finite number of edges. If the edge e were allowed to “oscillate” arbitarily rapidly the union of the two graphs would have an inifinte number of edges.

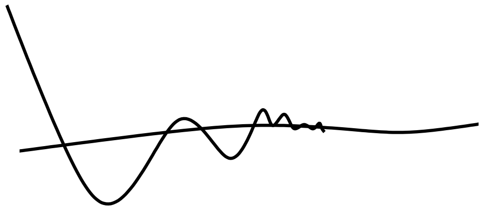


Figure 3.21: smoothIntersecF. Piecwise analytic curves which interesect at least a countable number of times *must* coincide everwise, i.e. be the same edge. However, smooth edges can intersect one another at an infinite number of points without coinciding everywhere and so the union has an infinite number of independent edges.

3.4.4 Flux Operators

The left invariant vector field in the $i-$th internal direction on the copy of G corresponds to the e-th edge

$$L_e^i \cdot \psi(h_{e_1}, \ldots, h_{e_N}) = (h_e \tau^i)^A_B \frac{\partial \psi}{\partial (h_e)^A_B} = \left(\frac{d}{dt}\right)_{t=0} \psi(h_{e_1}, \ldots, h_e e^{t\tau_i}, \ldots, h_{e_N})$$

$$R_e^i \cdot \psi(h_{e_1}, \ldots, h_{e_N}) = (\tau^i h_e)^A_B \frac{\partial \psi}{\partial (h_e)^A_B} = \left(\frac{d}{dt}\right)_{t=0} \psi(h_{e_1}, \ldots, e^{t\tau_i} h_e, \ldots, h_{e_N})$$

Let us remark on edges which are tangential to S. In this case, although $\dot{e}^3_I$ vanishes, we also have a singular term $\delta(0,0)$ (in the x^3 direction). Hence, to recover an unambiguous answer, for these edges, we need to smear also in the thir direction using an additional

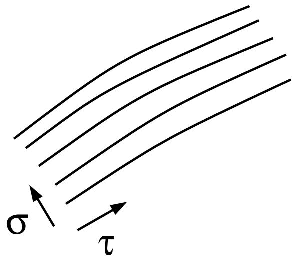


Figure 3.22: CongHolVarbF.

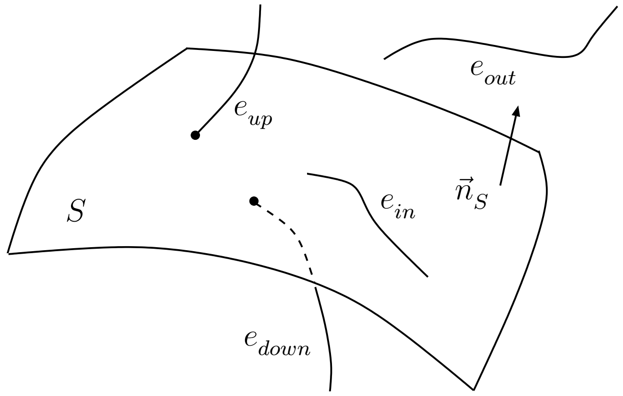


Figure 3.23: Types of edges with respect to a face.

regulator, say $g_{\epsilon'}(x^3, y^3)$, At the end, one finds that the contribution of the tangential eges do not contribute.

$$X_{S,n}[f] := \frac{1}{2} \sum_{p \in S \cap \gamma} \sum_{e_p} \sigma(e_p, S) n_i(p) X^i_{e_p}[f],$$

where the second sum is over the edges of γ adjacent to p,

$$\sigma(e_p, S) = \begin{cases} 1 & \text{if } e_p \text{ lies above } S \\ 0 & \text{if } e_p \cap S = \emptyset \text{ or } e_p \cap S = e_p \\ -1 & \text{if } e_p \text{ lies below } S \end{cases}$$

and $X^i_{e_p}$ is the *ith* left-invariant (right-invariant) vector field on $SU(2)$ acting on the argument of f corresponding to the holonomy h_{e_p} if e_p is pointing away from (towards) S.

3.5 Implimentation of Quantization

3.5.1 GNS Constrution for the Holonomy-Flux Representation

we concentrate on the commutative sub-algebra of the W_γ's. These functions serve to separate the points of $\mathcal{A}/\mathcal{G}$ i.e. if two potentials are inequivalent then there exists at least one loop for which the corresponding W_γ's are different. This sub-algebra is the holonomy algebra and is denoted $\mathcal{HA}$. If γ is a trivial loop, H_γ is $\mathbf{1}$ and is the identity element of the sub-algebra. We define a norm:

$$\|H_\gamma\| := \sup_{[A]\in\mathcal{A}/\mathcal{G}} |H_\gamma[A]| \tag{3.54}$$

Completion with respect to this norm

Now one invokes the Gelfand-Naimark theorem which asserts that every C^*-algebra with identity is isomorphic to the C^*-algebra of all continous, bounded, complex functions on a compact, Hausdorff space.

We now have a completed holonmy algebra. We still have construct $\mathcal{H}_{aux}$.

The C^*-algebra allows construction of its representations on Hilbert spaces. For every cyclic representation of $\overline{\mathcal{HA}}$ there is a Borel measuer μ on $\mathcal{A}/\mathcal{G}$ using which we get a Hilbert space:

$$\mathcal{H}_{aux} := L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu). \tag{3.55}$$

Thus $\mathcal{H}_{aux}$ consits of square integrable functions on the "quantum configuration space" $\overline{\mathcal{A}/\mathcal{G}}$. The H_γ's act multiplicatively (just as position operators does in ordinary quantum mechanics) and are bounded operators on $\mathcal{H}_{aux}$.

What about the momentum conjugates?

This characterization of $\mathcal{A}/\mathcal{G}$ as a limit of finite dimensional spaces allows the introduction of integral calculus on $\mathcal{A}/\mathcal{G}$ using integration theory on finite dimensional spaces.

3.6 A Measure for Integration:

A key ingredient for discussing quantum physics is to have at hand an inner product to compute expectation values.

a loop transform,

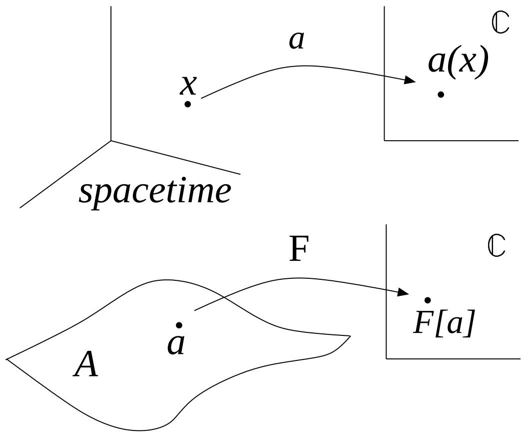


Figure 3.24: funcspace. (a) a is a function on spacetime. It maps points in spacetime to real or complex numbers. (b) a is apoint in the function space $\mathcal{A}$. The functional $F[a]$ maps points in the function space $\mathcal{A}$ to real or complex numbers. That is, the functional $F[a]$ turns functions into numbers.

$$\Psi[s] = \int \mathcal{D}A \Psi[A] W_A[s], \tag{3.56}$$

3.6.1 Primer on Functional Integration

Consider, for definiteness, a scalar field theory. The key step then is that of giving meaning to the Euclidean functional integrals by defining a rigorous version $d\mu$ of the heuristic measure

$$\text{“}[\exp(S(\phi))] \prod_x d\phi(x)\text{”} \tag{3.57}$$

on the space of histories of the scalar field, where $S(\phi)$ denotes the action governing the dynamics of the model.

an approximation scheme to solve the formal equations of the theory and then use the approximation method to define what is meant by one's equations in the first place.

$$\int \mathcal{D}A \delta[A - \xi] = 1. \tag{3.58}$$

We think of $\mathcal{D}A$ as the infinite product,

$$\mathcal{D}A = \prod_{\vec{x}} dA(\vec{x}). \tag{3.59}$$

$$\delta[a-\xi] = \prod_{\vec{x}} \delta(A(\vec{x}) - \xi(\vec{x})), \tag{3.60}$$

may find it helpful to think of $A(x)$ as infinite dimensional column vectors where $\mathbf{x}$ plays the role of an index. A functional integral is an infinite-dimensional limit of ordinary finite-dimensional integrals.

Consider

$$\int \mathcal{D}[A] e^{-\int dx A^2(x)} \tag{3.61}$$

$$\begin{aligned}
\int \mathcal{D}[A] &\rightarrow \int \prod dA(x) e^{-\sum_x A^2(x)} \\
&= \int \prod_x dA(x) \prod_x e^{-A^2(x)} \\
&= \prod_x \int dA(x) e^{-A^2(x)} \\
&= \prod_x \sqrt{\pi},
\end{aligned} \tag{3.62}$$

$$\int \mathcal{D}[A] e^{-\int \int dx\, dy A(x) M(x,y) A(y)} = \frac{(\sqrt{\pi})^\infty}{\sqrt{\det M}} \tag{3.63}$$

Motivation for Constructing Measures From of Finite Subspace Measuress?

constructing measures.

whenever a self-consistent sequence $\{\mu_n\}$ is given, does there exist a $\sigma-$additive[1] measure μ which satisfies the condition

$$\mu_n(E) = \mu(\pi_n^{-1}(E)), \quad \text{for all } E \subset X^n. \tag{3.64}$$

[1] $\sigma-$additive means $\mu(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$

$$d\mu := d^d x e^{-\alpha x^2} \tag{3.65}$$

$$\chi(k) = \int \exp(ik \cdot x)(d^d x e^{-\alpha x^2}) = \int \exp(ik \cdot x) d\mu \tag{3.66}$$

Contains all the information on $d\mu$.

3.6.2 Cylidrical Intergration Theory

$$F_e(\phi) = \int_{\mathbb{R}^{d+1}} \phi(x) e(x) \; d^{d+1}x \tag{3.67}$$

depends on ϕ only through their "n-components" $F_{e_1}(\phi), \ldots, F_{e_n}(\phi)$

$$f(\phi) = \tilde{f} \; (F_{e_1}(\phi), \ldots, F_{e_n}(\phi)). \tag{3.68}$$

where $\tilde{f}$ is a well behaved function on $\mathbb{R}^d$.

A cylindrical measure is a measure that allows to integrate cylindrical functions. Any measure would allow us to integrate cylindrical functions, but the tricky part is that there has to be consistency of these measures as there will be nesting and overlapping between these finite dimensional subspaces.

$$\int_C d\mu(\phi) f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{\langle e_1, \phi \rangle, \ldots, \langle e_n, \phi \rangle} \tag{3.69}$$

$$f(\phi) = \tilde{f}_1 \; (F_e(\phi)) := \exp[i\lambda \int_{\mathbb{R}^{d+1}} e(x)\phi(x) \; d^{d+1}x] \tag{3.70}$$

a function of $F_e(\phi)$ and $F_{\hat{e}}$ that just so happens to not depend on $F_{\hat{e}}$:

$$f(\phi) = \tilde{f}_2 \; (F_e(\phi), F_{\hat{e}}(\phi)) := \exp[i\lambda \int_{\mathbb{R}^{d+1}} e(x)\phi(x) \; d^{d+1}x] \tag{3.71}$$

$$\int_{\mathbb{R}} e^{i\lambda\eta} d\mu_e(\eta) = \int_{\mathbb{R}^2} e^{i\lambda\eta} d\mu_{e,\hat{e}}(\eta, \hat{\eta}) \tag{3.72}$$

And therefore one has to have that,

$$d\mu_e(\eta) = \int_{\mathbb{R}} d\mu_{e,\hat{e}}(\eta, \hat{\eta}). \tag{3.73}$$

Any set of measures one finite dimensional spaces satisfying these conditions for any cylindrical function F, defines a cylindrical measure via,

$$\int_C d\mu(\phi) f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n) \tag{3.74}$$

And conversely, a cylindrical measure defines consistent sets of measures in finite dimensional settings.

The situation is strikingly similar to ordinary quantum mechanics, where the Hilbert space of physical states is obtained by suitable completions of square integrable functions on the configuration space. In field theory the situation is more involved. Not every physical state is a function on just the configuration space, but distributions on the time=constant hypersurface are also generically involved.

$$\chi(e) = \text{“}\frac{1}{Z}\int \exp\left(\int_{\mathbb{R}^{d+1}} \Big(ie(x)\phi(x) - \mathcal{L}[\phi(x), \partial\phi(x)] \Big)\, d^{d+1}x\right)\text{”} \tag{3.75}$$

3.6.3 Integrating Gauge Field Theories

Simple example

consider the integral

$$Z = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; e^{(x-y)^2} \tag{3.76}$$

$$\begin{aligned} x &\to x + a \\ y &\to y + a \end{aligned} \tag{3.77}$$

the gauge orbits are lines of constant $x - y$. The “action”, $(x - y)^2$ is a gauge invariant.

$$\overset{-1}{\det}\left(\frac{\partial f}{\partial x}\right)\bigg|_{f=a} = \int dx \delta(f(x) - a) \tag{3.78}$$

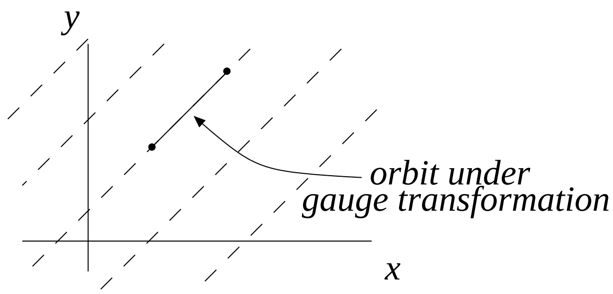


Figure 3.25: The motion of a "configuration," (x, y), in the configuration space under the "gauge" transformation defined in (D.-1). The path is called a gauge orbit. In the simple example here, the gauge orbits are lines of constant $x - y$.

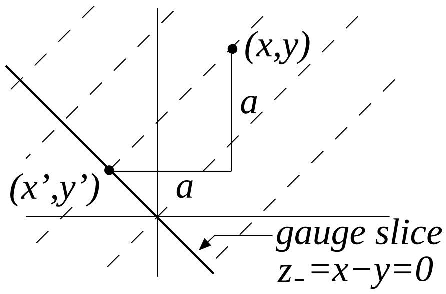


Figure 3.26: The gauge choice $x + y = 0$ defines a "gauge slice" through the configuration space. (x', y') is a configuration on the slice, that is, it satifies the gauge condition. (x, y) is a gauge equivalent configuration, since both (x, y) and (x', y') reside on the same gauge orbit. a is the gauge transformation that takes us from the slice (x, y).

The degeneracy is caused by the fact that we integrate over a redundent set of integration variables which results in an infinite volume factor. This situation occurs because of the way we formulate the theory as based on the principle of a local gauge invariance. The complete physical content is contained by one contribuation out of each equivalance class. One selects to one such member by imposing a condition called a *gauge fixing condition.*

Infinite Dimensional Non-Linear spaces

It is not easy to develop functional measures in infinite dimensional non-linear spaces like the space of connections modulo gauge transformations. Measures have been introduced for connections in the cases like 1+1 Yang-Mills with finite dimensional spaces.

We wish to have a rigorous defintion for the "loop transformation"

$$\Psi[\gamma] = \int \mathcal{D}[A] \Psi[A] W_\gamma[A] \tag{3.79}$$

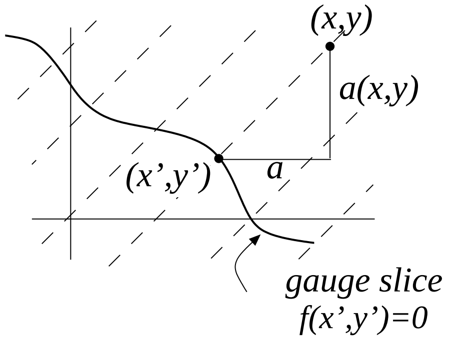


Figure 3.27: Illustration of a general choice of gauge, $f(x, y) = 0$. The desired change of coordinates is from (x, y) to (s, y), s being a variable that runs along the slice,and a deing the gauge transformation that runs from the slice (x, y).

which is how the loop representation was originally introduced by Rovelli and Smolin in 1988.

As part of the development of the techniques for dealing with quantum gravity, mathematically rigorous measure were introduced in these kinds of spaces, in some cases for the first time ever.

The Rovelli

" The space of physical states must have the structure of a Hilbert space, namely a scalar product, in order to be able to compute expectation values. This Hilbert structure is determined by the requirement that real physical observables correspond to self-adjoint operators. In order to define a Hilbert space of physical states, it is convenient to define first a Hilbert space of unconstrained states. This is because we have a much better knowledge of the unconstrained observables than the physical ones. If we choose a scalar product on the unconstrained state space which is gauge invaiant then there exist standard techniques to "bring it down" to the space of the physical states. Thus, we need a gauge and diffeomorphism invariant scalar product, with respect to which real observable are self-adjoint operators. "

The space of connections is an infite-dimension space is a well defined limit -a projective limit - of a family of finite dimensional spaces. This ***projective*** structure reduces the task of dealing with an infinite dimensional space to a problem of finite dimensional spaces with certain consistency conditions. The idea is to work with graphs embedded in space, and for each graph to define a Hilbert space of wave functions depending only on the holonomies of the connection along the edges of the graph. This work is due to Ashtekar, Lewandowski, Marolf, Mourao, Thiemann. Due to its heavy mathematical nature, we will only give a brief sketch here to highlight the main concepts.

One wishes to compute:

$$(\psi_1, \psi_2) = \int_{\mathcal{A}/\mathcal{G}} d\mu([A])\psi_1([A])\psi_2([A]) \tag{3.80}$$

3.7 How does one apply cylindrical measure theory to non-Abelian connections?

The physical degrees of freedom, the gauge orbits, is the quotient of the space of connections $\mathcal{A}$ by the gauge transformations $\mathcal{G}$, which is neatly written

$$\mathcal{A}/\mathcal{G}. \tag{3.81}$$

The idea is to work with graphs embedded in space, and for each graph to define a Hilbert space of wave functions depending only on the holonomies of the connection along the edges of the graph.If the graph γ has n edges, the holonomies along its edges are summarized by a point in $\mathcal{A}_\gamma \cong SU(2)$, and the Hilbert space we get is $L^2(\mathcal{A}_\gamma)$, is defined using a measure on $SU(2)$.

Any cylindrical function, f', based on a graph Σ' can be written as a cylindrical function, f, based on a larger graph containing Σ. We simply choose the function f to agreee with f' on the links Σ shares with Σ' and to be independent of the links in Σ but not in Σ'.

If the graph γ is contained in a larger graph γ' then $\mathcal{A}_\gamma$ is contained in $\mathcal{A}_{\gamma'}$ and one has $L^2(\mathcal{A}_\gamma) \subseteq L^2(\mathcal{A}_{\gamma'})$. We can thus form the union of all these Hilbert spaces and complete it to obtain the desired Hilbert space.

Pick a graph defined above. For each of the n links γ_i of Σ_n consider the holonomy $U_i(A) \equiv U[A, \gamma_i]$ of the connection A along γ_i. Every (smooth) connection assigns a $SU(2)$ matrix to each link γ_i of Σ_n via the holonomy $g_i \equiv U_i(A) = m\mathcal{P} \exp \int_{\gamma_i} A$. Thus an element of $[SU(2)]^n$ is assigned to the graph Σ_n. The next step is to consider complex-valued functions $f_n(g_1, \dots, g_n)$ on $[SU(2)]^n$,

$$f_n : [SU(2)]^n \to \mathbb{C} \tag{3.82}$$

These functions are finite with respect to the Haar measure of $[SU(2)]^n$.

Given a graph and a function f_n, we define

$$\Phi_{\Sigma_n, f_n}(A) = f_n(U_1, \dots, U_n). \tag{3.83}$$

These are fake infinite functions as they depend on the connection only via the graphs finite number of holonomies. They are called cylindrical functions. They form a dense subset of states in $\mathcal{L}$, the space of continuous smooth functions on $\mathcal{A}$. This justifies the exclusive use of this special class of functions for the construction of the Hilbert space.

duality between connection and holonomies of graphs is non-liner :

$$h_\gamma(A_1 + A_2) \neq h_\gamma(A_1) + h_\gamma(A_1) \tag{3.84}$$

as apposed to Eq.(3.67).

We introduce the notion of "hoops" (holonomic loops), that is, loops that yield the same holonomy for any connection. Such quantities form a group (Gambini 1980's) under composition at a given base point x_0.

Let us consider a set of independent hoops $(\beta_1, \dots, \beta_n)$ (hint: use spin networks). If we consider the holonomy along each of these loops for a given connection, I get a map from the space of connections modulo gauge transformations to n copies of the gauge group modulo the adjoint action.

$$\pi_{\beta_1,\dots,\beta_n}([A]) \quad \mathcal{A}/\mathcal{G} \to G^n/Ad \tag{3.85}$$

$$\pi_{\beta_1,\dots,\beta_n}([A]) \;\; = \;\; [H(\beta_1, A), \dots, H(\beta_n, A)], \tag{3.86}$$

We can now define a cylindrical function very much as we did before, in this case considering a function on G^n/Ad,

$$\int_{\mathcal{A}/\mathcal{G}} f([A]) d\mu([A]) = \int_{G^n/Ad} F([g_1, \dots, g_n]) d\mu_{\beta_1,\dots,\beta_n}([g_1, \dots, g_n]) \tag{3.87}$$

A particularly simple choice of measure is to consider the Haar measure on each G/Ad. This choice turns out to be consistent (hard to prove with loops, easier with spin nets).

A scalar product is defined on the space of these functions as follows. Given two cylindrical functions defined by the same graph Γ, we define

$$(\Psi_{\Gamma,f}|\Psi_{\Gamma,g}) := \int_{G^n} dU_1 \overline{f(U_1, \dots, U_L} \; g(U_1, \dots, U_L) \tag{3.88}$$

Since the measure was defined without reference to any background structure it is naturally diffeomorphism invariant!

The construction looks intimidating but the end result is amazingly simple, especially if one casts it in terms of spin nets. It simply states that

$$\langle s_1 s_2 \rangle = \int_{\mathcal{A}/\mathcal{G}} D[A] W_{s_1}[A] W_{s_2}[A] = i_{s_1;s_2} \tag{3.89}$$

Which means that the inner product of two spin networks states vanishes if the two spin network states are different. More precisely, if no representative of the diffeo-equivalence class of spin networks s_1 is present in the class s_2.

That is, not only have we made sense precisely of the infinite dimensional integral present in the inner product, but the result is remarkably simple at the time of doing calculations.

the inner product was obtained on this set of states by requiring that the classical reality conditions be implemented as adjointness conditions on the corresponding quantum operators.

The Hamiltonian constraint takes diff invariant wave functions and maps them onto non-diff invariant wave functions. Master constraint is diff invariant and so can make full use of...

3.8 Weyl Rather than Heisenberg

The only way to construct diffeomorphism invariant theories is to start with exponentiated objects, like holonomies: $h_e(A) = \mathcal{P}\exp(\oint A)$. The quantum theory is discontinous. This means that for a system with $\hat{p}$ and $\hat{q}$ as fundermental coordinates, one of them becomes ill-defined.

Say if the Hamiltonian is of the form $H = p^2 + V(q)$, we can not define it on the kinematic Hilbert space $\mathcal{H}$.

We approximate the non-existing operator by a different (finite) operator that does exist.

$- - - - - - - - - -$

$$[P, Q] = i\hbar \tag{3.90}$$

The algebraic relations between Q and P expressed in are replaced by

$$U(a)V(b) = e^{i2\pi ab/\hbar} V(b)U(a). \tag{3.91}$$

and the product is

$$U(a)U(b) = U(a+b), \qquad V(a)V(b) = V(a+b) \tag{3.92}$$

This is the Weyl form of the CCR for one degree of freedom. We can then ask *formally* what the algebra for "generated",

$$\begin{aligned} U(a)QU^{-1}(a) &= \exp(iaP)Q\exp(-iaP) \\ &= Q + ia[P,Q] + \frac{(ia)^2}{2!}[P,[P,Q]] + \dots \\ &= Q + a\hbar \quad (\text{as } [P,Q] = i\hbar \text{ a scalar}). \end{aligned} \tag{3.93}$$

$$(U(a)\Psi)(x) = \Psi(xa) \ \text{ and } \ (V(b)\Psi)(x) = e^{-i2\pi bx/\hbar}\Psi(x), \tag{3.94}$$

The formula involving bounded operators will typically imply the one for unbounded operators but not vice versa.

$$[e^{i\mu\hat{x}}, e^{i\nu\hat{p}}] = \tag{3.95}$$

The Weyl algebra $\mathcal{W}$ is generated by taking finite linear combinations of the generators $U(a)$ and $W(b)$. Quantization means finding a unitary representation of the Weyl algebra $\mathcal{W}$ on a Hilbert space.

(exponetiated) diffeomorphisms only. It is given by

$$\begin{aligned} U(\varphi)U(\varphi')U(\varphi)^{-1} &= U(\varphi \circ \varphi' \circ \varphi^{-1}) \\ U(\varphi)\hat{H}(N)U(\varphi)^{-1} &= \hat{H}(N \circ \varphi^{-1}) \end{aligned} \tag{3.96}$$

3.9 Projective Limits

Summarize the basic ideas which used to construct the measure on $\overline{\mathcal{A}/\mathcal{G}}$ from the measure on cylindrical functions.

3.9.1 Introduction to Measure Theory and Topology

The Riemann integral we split the x coordinate into small intervals and approximating $f(x)$ in every interval by its maximum and minimum. The problem with this approach is that the difference between the maximum and minimum will only tend to zero, as the interval gets smaller, if $f(x)$ is sufficiently well behaved.

To circumvent these difficulties, the idea is to portion the range instead of the domain. Fig(J.14).

We considering instead of an interval, the set of x for which $f(x)$ lies between two numbers $a < b$. Now we need the size of the set of these x, that is, the size of the preimage $f^{-1}((a,b))$.

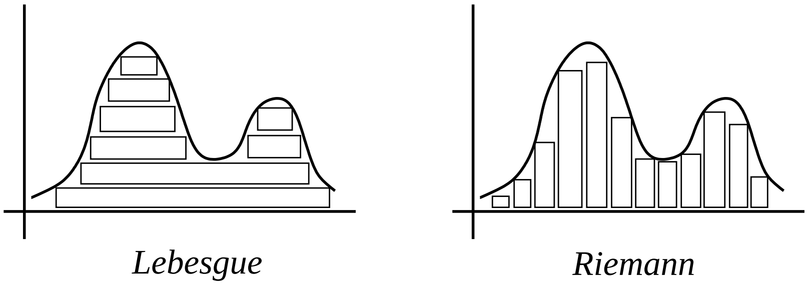


Figure 3.28: LebRien.

What are reasonable to take for subsets? There is no reasonable way of adding up an uncountable set of numbers each of which is zero. However, there is for subsets that satisfy the conditions that the union of any two subsets is also in the family, the intersection of any two subsets is also in the family, the completment of any subset is also in the family.

$\sigma-$algebra

Let X be a set. Then a collection of subsets $\mathcal{U}$ of X is called a $\sigma-$algebra provided that

1) $X \in \mathcal{U}$

2) $U \in \mathcal{U}$ implies $X - U \in \mathcal{U}$ and

3) $\mathcal{U}$ is closed under countable unions, that is, $U_n \in \mathcal{U},\ n = 1, 2, \ldots$ then also is $\cup_{n=1}^{\infty} U_n \in \mathcal{U}$.

Definition of topological spaces

Topology starts with the observation that the basic idea of continuity can be derived without reference to epsilons and deltas, from a small set of axioms.

Let M be a metric space with metric d. The open and closed ball of radius r around x_0 are the subsets

$$B_r(x_0) := \{x \in X : d(x_0, x) < r\} \text{ and } \overline{B}_r(x_0) := \{x \in X : d(x_0, x) \leq r\}$$

respectively. Suppose $U \subseteq M$ is a subset of M. U is called open or an open set, if, given $x \in U$, there is $\delta > 0$ so that

$$B_\delta \subseteq U \subseteq M$$

where $B_\delta(x) = \{y : y \in M, d(x, y) < \delta\}$.

Next let U and V be open and $x \in U \cap V$. Since U is open we find $\delta > 0$ such that $U_\delta(x) \subset U$, and since V is also open we can choose $\epsilon > 0$ with $U_\epsilon(x) \subset V$. Replacing both δ and ϵ by the smaller of the two we may assume δ and obtain $U_\delta(x) \subset U \cap V$. Thus $U \cap V$ is open. See fig (3.9.1)

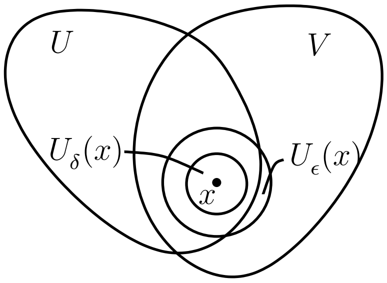


Figure 3.29: The intersection of two open sets contains an open ball which is contained in both U and V, and hence is open itself.

Induction proves that finite intersections of open sets are also open. Examples prove this does not extend to infinite intersections. Finally consider a family $\{U_I\}_\mathcal{I}$ of open subsets $U_I \subset X$, and let x be in their union. We pick an arbitrary $I \in \mathcal{I}$ and a $\delta > 0$ with $U_\delta(x) \subset U_I$. Since

$$U_\delta(x) \subset U_I \subset \bigcup_{I \in \mathcal{I}} U_I$$

we have that $\cup_{I \in \mathcal{I}}$ is an open subset of X.

A topology on X is a family of open sets, containing $\emptyset$ and X, which is closed under the uncountible union and finite intersection.

Any metric space is a topological space, with open sets defined above. Our three conditions are satisfied.

Let X be a set. Then a collection of subsets $\mathcal{U}$ of X is called a topology provided that

1) $X \in \mathcal{U}$

2) $U \in \mathcal{U}$ and $V \in \mathcal{V}$ implies $U \cap V \in \mathcal{U}$ and

3) $\mathcal{U}$ is closed under arbitrary, possibly uncountable, unions, that is, if $U_I \in \mathcal{U}$ then also is $\cup_{I \in \mathcal{I}} U_I \in \mathcal{U}$.

In general, a given set admits many different topologies, that is different families of open sets which satify the definition.

In rough terms, one topology is weaker than another if it has fewer open sets, and stronger than another if it has more open sets. topology on X which makes all the f_i's are continuous mappings. This is clearly a topology on X which makes all the f_i's continuous, and it is weaker than any topology which has this property.

Notice the similarity between a collection of sets $\mathcal{U}$ that qualify as a $\sigma-$algebra and a topology; in both cases $X, \emptyset$ belong to $\mathcal{U}$ but while open sets are closed under arbitrary unions and finite intersections, measurable sets are closed under countable unions and intersections.

Separation and compactness of topological spaces

The topologies of particular interest to us have addition structure to do with separation and compactness properties.

Compactness is a mathematical notion of finiteness.

A space is called **Hausdorff** if and only if for any two points $x \neq y$ there exist opne neighbourhoods U, V of x, y respectively that are disjoint.

A topological space is called **compact** if every open cover of X contains a finite subcovering.

3.9.2 Tychonov' Theorem and Topology on Infinte Dimensional Manifolds

We must consider the cartesian product of sets. It carries its own natural topology.

Let $(X,\mathcal{U})$ be a topological space. A subset $\mathcal{B} \subset \mathcal{U}$ is a **basis** of $\mathcal{U}$ if each open set $U \in \mathcal{U}$ is a union of elements of $\mathcal{B}$.

Let X and Y be topological spaces. Then

$$\mathcal{B} := \{U \times V : U \subset X \text{ and } V \subset Y \text{ open}\}$$

is a basis of a topology on the cartesian product $X \times Y$. The resulting topology is called the product space of X and Y.

The compactness of each Hausdorff space will imply that every X_α is compact and Haudorff. Now on a direct product space (independent of the cardinality of the index set) in which each factor space is compact and Haudorff one can naturally define a topology, the Tychonov topology, such that X_∞ is itself compact. If $\overline{X}$ is closed in X_∞ then $\overline{X}$ will be compact and Hausdorff as well in the subspace toplogy. However, for compact Hausdorff spaces powerful measure theoretic theorems hold which enable one to equip to relavent infinite dimensional spaces associated to background independent gauge theories with the structure of the so-called $\sigma-$algebra and to develop measure theory thereon.

3.9.3 Measure Theory on Infinite Dimensional Manifolds

Borel sets

Let X be a topological space. The smallest $\sigma-$algebra on X that contains all open set of X is called the Borel $\sigma-$algebra of X.

" $\mathcal{A}$ lies topologically dense, but measure theoretically thin in $\overline{\mathcal{A}}$ this section we will see that $\mathcal{A}$ (similar results apply to $\mathcal{A}/\mathcal{G}$ with respect to $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$) with respect to the uniform measure μ_0. More precisely, there is a dense embedding (injective inclusion) $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ but $\mathcal{A}$ is embedded into a measurable subset of $\mathcal{A}$ of measure zero. The latter result demonstrates that the measure is concentrated on non-smooth (distributional) connections so that $\overline{\mathcal{A}}$ is indeed much larger than $\mathcal{A}$. "

Riesz representation theorem

Let X be a locally compact Hausdorff space and let $\Lambda : C_0(X) \rightarrow \mathbb{C}$ be a positive linear functional on the space of continuous, complex-valued functions of compact support in X. Then there exists a $\sigma-$algebra $\mathcal{U}$ on X which contains the Borel $\sigma-$algebra and a unique positive meausre μ on $\mathcal{U}$ such that Λ is represented by μ, that is,

$$\Lambda(f) = \int_X d\mu(x) f(x) \tag{3.97}$$

for all $f \in C_0(X)$.

3.10 The C^* Algebraic Viewpoint

3.10.1 Basic Definitions

Definition: Linear space.

Let V be be a non-empty set, and assume there is an operation called addition, denoted $+$, such that is $x, y \in V$ then $z = x + y$ is in V. Assume also that this operation of adittion satisfies the following conditions:

(i) $x + y = y + x$;
(ii) $x + (y + z) = (x + y) + z$;
(iii) there exists in V an element, denoted by 0 and called the zero element such that $x + 0 = x$ for every x ;
(iv) to each element x in V there corresponds a unique element in V, denoted by $-x$ and called the negative of x, such that $x + (-x) = 0$.

We now assume that each scalar α and each element $x \in V$ can be combined by a process called scalar multiplication to yield an element $y \in V$ denoted by $y = \alpha x$ in such a way that

(v) $\alpha(x + y) = \alpha x + \alpha y$;
(vi) $(\alpha + \beta)x = \alpha x + \beta y$;
(vii) $(\alpha\beta)x = \alpha(\beta x)$
(viii) $1 \cdot x = x$.

The algebraic system V, defined by the above properties, is called a linear space.

Definition: Normed linear space.

A *normed linear space* is a linear space on which there is defined a norm, i.e. a function which assigns to each element x in the space a real number $\|x\|$ in such a manner that

(i) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|x + y\| \leq \|x\| + \|y\|$;
(iii) $\|\alpha x\| = |\alpha| \, \|x\|$.

From (iii) and the fact that $-x = (-1)x$, we obtain $\| - x\| = \|x\|$. A normed linear space is a metric space with respect to the induced metric defined by

$$d(x, y) = \|x - y\|.$$

Check it is a metric space:

(a) $\|x - y\| \geq 0$, and $\|x - y\| = 0$ if and only $x = y$ by property (i) ;
(b) $\|x - y\| = \|y - x\|$ (symmetry) ;
(c) $\|x - y\| = \|x - z - y + z\| \leq \|x - z\| + \|y - z\|$ by property (ii) (triangle inequality).

Definition: A Banach space.

A *Banach space* is a normed linear space which is complete as a metric space.

Definition: A Banach algebra.

A *Banach algebra* is a complex Banach space which is also an algebra with identity 1, and in which the multiplication structure is related to the norm by the following requirements:

(i) $\|xy\| \leq \|x\| \, \|y\|$;
(ii) $\|1 \,| = 1$.

Definition: A Banach $*$-algebra.

A Banach algebra A is called a *Banach $*$-algebra* if it has an involution, that is, if there exists a mapping $x \to x^*$ of A into itself with the following properties

(i) $(x+y)^* = x^* + y^*$;
(ii) $(\alpha x)^* = \overline{\alpha} x^*$;
(iii) $(xy)^* = y^* x^*$;
(iv) $x^{**} = x$.

It is an easy consequence of (iv) that the involution $x \to x^*$ is actually a one-to-one mapping of A onto itself.

Definition: A C^*-algebra.

A *C^*-algebra*, A, is a Banach over the field of complex numbers, together with a map * : $A \to A$. The map * has the properties

(i) $(x+y)^* = x^* + y^*$;
(ii) $(\alpha x)^* = \overline{\alpha} x^*$;
(iii) $(xy)^* = y^* x^*$;
(iv) $x^{**} = x$;
(v) $\|xx^*\| = \|x\| \, \|x^*\|$.

The last identity is called the *C^*-identity* and is equivalent to

$$\|xx^*\| = \|x\|^2.$$

3.10.2 Projective Limits and the Associated C^*-Algebra of Cylindrical Functions

Let L be a partially ordered, directed set; i.e. a set equipped with a relation '$\geq$' such that, for all S, S' and S'' in L we have:

$$\begin{aligned} S &\geq S; \\ S \geq S' \quad &\text{and} \quad S' \geq S \Rightarrow S = S'; \\ S \geq S' \quad &\text{and} \quad S' \geq S'' \Rightarrow S \geq S''; \end{aligned} \tag{3.96}$$

and, given any $S', S'' \in L$, there exists $S \in L$ such that

$$S \geq S' \text{ and } S \geq S''. \tag{3.96}$$

L will serve as the label set. A projective family $(\chi_S, p_{SS'})_{S,S' \in L}$ consists of sets χ_S indexed by elements of L, together with a family of onto projections,

$$p_{SS'} \; \chi_{S'} \rightarrow \chi_S \tag{3.96}$$

and satisfies the consistency condition

$$p_{SS'} \circ p_{S'S''} = p_{SS''} \tag{3.96}$$

for $S'' \geq S' \geq S$.

The continuum theory will be recovered in the limit as one considers lattices of increasing number of loops of arbitrary complexicity.

$$\begin{aligned} \overline{\chi} \;\; := \;\; & \{\text{elements}, x_S, \text{ of the cartesian porduct, } \times_{S \in L} \chi_S, \\ & \qquad\qquad \text{such that } : S' \geq S \Rightarrow p_{SS'} x_{S'} = x_S\}. \end{aligned} \tag{3.96}$$

This limit gives us the gives us the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$.

the minimal graph the function is defined on.

$$\bigcup_{S \in L} C^0(\chi_S) \tag{3.96}$$

Let us define the following equivalence relation. Given $f_{S_i} \in C^0(\chi_{S_i})$, $i = 1, 2$, we say that

$$f_{S_1} \;\sim\; f_{S_2} \quad \text{if} \quad p^*_{S_1 S_3} f_{S_1} = p^*_{S_2 S_3} f_{S_2} \tag{3.96}$$

for every $S_3 \geq S_1, S_2$, where $p^*_{S_1 S_3}$ denotes the pull-back map from the space of functions on χ_{S_1} to the space of functions on χ_{S_3}. Note that to be equivalent, it is sufficient if the equality (3.10.2) holds for just one $S_3 \geq S_1, S_2$. To see this, suppose (3.10.2) holds for S_1, S_2 and S_3 and let $S_4 \geq S_1, S_2$. Since L is a directed set we find $S_5 \geq S_1, S_2, S_3, S_4$ and due to the consistency condition among projections we have

$$\begin{array}{lr} p_{S_1 S_4} \circ p_{S_4 S_5} = p_{S_1 S_5} = p_{S_1 S_3} \circ p_{S_3 S_5} & (i) \\ p_{S_2 S_4} \circ p_{S_4 S_5} = p_{S_2 S_5} = p_{S_2 S_3} \circ p_{S_3 S_5} & (ii) \end{array} \tag{3.96}$$

whence

$$\begin{array}{rcl} p^*_{S_4 S_5} \circ p^*_{S_1 S_4} f_{S_1} & =_{(i)} & p^*_{S_3 S_5} p^*_{S_1 S_3} f_{S_1} \\ & = & p^*_{S_3 S_5} p^*_{S_2 S_3} f_{S_2} \\ & =_{(ii)} & p^*_{S_4 S_5} p^*_{S_2 S_4} f_{S_2} \end{array} \tag{3.95}$$

where in the second inequality we have used (3.10.2). We conclude that

$$p^*_{S_4 S_5}[p^*_{S_1 S_4} f_{S_1} - p^*_{S_2 S_4} f_{S_2}] = 0.$$

Now for any $f_{S_4} \in C^0(\chi_{S_4})$ the condition $f_{S_4}(p_{S_4 S_5}(x_{S_5})) = 0$ for all $x_{S_5} \in \chi_{S_5}$ means that $f_{S_4} = 0$ because the $p_{S_4 S_5} : \chi_{S_5} \to \chi_{S_4}$ is onto.

Using the equivalence relation we can now introduce the set of cylindrical functions associated with the projective family $(\chi_S p_{SS'})_{S,S' \in L}$,

$$\text{Cyl}(\overline{\chi}) := (\bigcup_{S \in L} C^0(\chi_S)\ )\ /\ \sim . \tag{3.95}$$

The quotient just gets rid of a reundancy.

Henseforth we denote the element of $\text{Cyl}(\overline{\chi})$ defined by $f_S \in C^0(\chi_S)$ by $[f_S]_\sim$. We have:

Lemma A : Given any $f, f' \in \text{Cyl}(\overline{\chi})$, there exists $S \in L$ and $f_S, f'_S \in C^0(\chi_S)$ such that

$$f = [f_S]_\sim, \quad g = [g_S]_\sim \tag{3.95}$$

Proof: Choose any two representatives $f_{S_1} \in C^0(\chi_{S_1})$ and $f_{S_2} \in C^0(\chi_{S_2})$ such that $f = [f_{S_1}]_\sim$, $f' = [f_{S_2}]_\sim$. Choose any $S \geq S_1, S_2$ then $f_S := p^*_{S_1 S} f_{S_1} \sim f_{S_1}$ (write (3.10.2) as $p^*_{S_1 S} f_{S_1} = p^*_{S_2 S} f_{S_2}$, choose $S_2 = S$ and use $p_{SS} = \text{id}_{\chi_S}$) and $f'_S := p^*_{S_2 S} f_{S_2} \sim f_{S_2}$. Thus $f = [f_S]_\sim$ and $f' = [f'_S]_\sim$.

□

Lemma B :

(i) Let $f, f' \in \mathrm{Cyl}(\overline{\chi})$ then the following operations are well-defined (independent of the representations)

$$\begin{aligned} f+f' &:= [f_S + f'_S]_\sim, \\ ff' &:= [f_S f'_S]_\sim, \\ zf &:= [zf_S]_\sim \\ \overline{f} &:= [\overline{f}_S]_\sim \end{aligned} \tag{3.93}$$

(ii) $\mathrm{Cyl}(\overline{\chi})$ conxtains the constant functions.

(iii) The sup-norm for $f = [f_S]_\sim$

$$\|f\| := \sup_{x_S \in \chi_S} |f_S(x_S)| \tag{3.93}$$

is well-defined.

Proof:

(i) We consider only pointwise multiplication, the other cases are similar. Let S, f_S, f'_S and $S', f_{S'}, f'_{S'}$ be as in Lemma A. We find $S'' \geq S, S'$ and have $p^*_{SS''} f_S = p^*_{S'S''} f_{S'}$ and $p^*_{SS''} f'_S = p^*_{S'S''} f'_{S'}$. Thus

$$\begin{aligned} p^*_{SS''}(f_S f'_S) &= p^*_{SS''}(f_S) p^*_{SS''}(f'_S) \\ &= p^*_{S'S''}(f_{S'}) p^*_{S'S''}(f'_{S'}) \\ &= p^*_{S'S''}(f_{S'} f'_{S'}) \end{aligned} \tag{3.92}$$

so $f_S f'_S \sim f_{S'} f'_{S'}$.

(ii) The function $f^z_S : \chi_S \to \mathbb{C}$ for any $z \in \mathbb{C}$ is certainly an element of $C^0(\chi_S)$ and for any $S'' \geq S, S'$ we have

$$z = (p^*_{SS''} f^z_S)(x_{S''}) = (p^*_{S'S''} f^z_{S'})(x_{S''})$$

for all $x_{S''} \in \chi_{S''}$ so $f^z = [f^z_S]_\sim$ is well-defined.

(iii) If $f = [f_S]_\sim = [f_{S'}]_\sim$ is given, choose any $S'' \geq S, S'$ so that we know that $p^*_{SS''} f_S = p^*_{S'S''} f_{S'}$. Then from the surjectivity of $p_{SS''}$, $p_{S'S''}$ we have

$$
\begin{aligned}
\sup_{x_S\in\chi_S}|f_S(x_S)| &= \sup_{x_{S''}\in\chi_{S''}}|(p^*_{SS''}f_S)(x_{S''})| \\
&= \sup_{x_{S''}\in\chi_{S''}}|(p^*_{S'S''}f_{S'})(x_{S''})| \\
&= \sup_{x_{S'}\in\chi_{S'}}|f_{S'}(x_{S'})|. \qquad (3.91)
\end{aligned}
$$

□

The norm is well defined according to part (iii) of Lemma B. Consider the Banach algebra $\overline{\mathrm{Cyl}}(\overline{\chi})$ obtained by the Cauchy completion of $\mathrm{Cyl}(\overline{\chi})$. The norm is easily seen to satify the C^**-identity*,

$$
\begin{aligned}
\|f\overline{f}\| &= \|[f_S]_\sim[\overline{f}_S]_\sim\| \\
&= \sup_{x_S\in\chi_S}|f_S(x_S)\overline{f_S}(x_S)| \\
&= \sup_{x_S\in\chi_S}(|f_S(x_S)|\ |\overline{f_S}(x_S)|) \\
&= \sup_{x_S\in\chi_S}|f_S(x_S)| \times \sup_{x_S\in\chi_S}|\overline{f_S}(x_S)| \\
&= \|[f_S]_\sim\|\ \|[\overline{f}_S]_\sim\| \\
&= \|f\|\ \|\overline{f}\|. \qquad (3.87)
\end{aligned}
$$

Therefore $\overline{\mathrm{Cyl}}(\overline{\chi})$ is a C^*-algebra. By construction, it is Abelian and has an identity element. We refer to it as the C^*-algebra associated with the projective family $(\chi_S, p_{SS'})_{SS'\in\mathcal{G}}$.

3.10.3 The Gelfand Spectrum of $\overline{\mathrm{Cyl}}(\overline{\chi})$

the minimal graph the function is defined on.

$$\bigcup_{S\in L} C^0(\chi_S) \qquad (3.87)$$

$$\mathrm{Cyl}(\overline{\chi}) := (\ \bigcup_{S\in L} C^0(\chi_S)\ )\ /\ \sim . \qquad (3.87)$$

quantum configuration space. [2]

[2]Typical measures used in quantum field theories are. The classical configuration spaces are contained in sets of measure zero. As a result, the action S inside the functional integral is ill-defined

In the Hilbert space language, this is the origin of field theoretic infinites, e.g., the reson why we can not naively multiply field operators.

3.10.4 Functional Calculus

3.11 Generalized Eigenstates as Solutions to Constraint Equations

This may seem it may sound outlandish but... outlandish: looking or sounding very strange or foreign, bizarre.

Consider a finite Hilberts pace upon which we want to impose a constraint.

$$\hat{C}|\psi_n >= 0 \tag{3.87}$$

The subset $(\psi(), \psi(), \ldots, \psi())$ of the unconstrained Hilbert space that have zero eigenvalue with respect to the constraint. For calculating expectation values between these we simply use the inner-product of the unconstrianed Hilbert space, restricted to states.

However, we run into trouble even in simple quantum mechanical systems constraint equation. Say we have quauntum mechanics of a particle a two dimensional plane. The inner-product is

$$||\psi(x,y)|| := \int dx \int dy \overline{\psi(x,y)} \psi(x,y) \tag{3.87}$$

Consider the constraint

$$\hat{P}_y \psi(x,y) = -i\hbar \frac{\partial}{\partial y} \psi(x,y) = 0 \tag{3.87}$$

we easily see that the solutions are

$$\psi(x,y) = \phi(x) \tag{3.87}$$

These solutions have are non-normalizable in the natural inner-product:

The presnce of an infinite number of degres of freedom causes only one major modificatoin: the classical configuration space $\mathcal{C}$ of smooth fields is enlarged to the quantum configuration space $\mathcal{S}'$ of (tempered) distributions. Quantum theoretical difficulties associated with defining products of operators can be directly traced back to this enlargement [34].

This enlaregment from $\mathcal{A}$ to $\overline{\mathcal{A}}$ which occurs in the passage to the quantum theory is ver similar to the enlargement from $\mathcal{C}$ to $\mathcal{S}'$ in the case of scalar fields. This enlargement plays a key role in the quantum theory (especially in the discussion of surface states of the quantum horizon).

$$||\psi(x,y)|| = \int dx\overline{\phi(x)}\phi(x) \times \int dy = \int dx\overline{\phi(x)}\phi(x) \times \infty = \infty. \tag{3.87}$$

Therefore there are no solutions to the constraint eq(3.11) that lie in this Hilbert space (the Hilbert space, by definition, being the space of square integrable wavefunctions). The constraint is imposing the symmetry that nothing changes in the $y-$direction. The translation group in the $y-$direction and this group volume diverges: $\int_{-\infty}^{\infty} dy = \infty$. This circumstance often occurs with constrained systems for the same simply mathematical reason: by satisfying the constraints, the physical wavefunctions must be constant on some degrees of freedom on which unconstrained wavefunctions can depend. If the "volume" obtained by integrating over these degrees of freedom diverges, the wavefunction will be non-normalizable.

The physical states that solve the "Wheeler-DeWitt" equation are infinite-norm states in the natural Hilbert space structure of the unconstained states space.

In the continuous part of the spectrum of an observable does not charaterize the state of the system because the corresponding state is non-normalizable, and hence does not belong to the Hilbert space. However, there is always at least one state in which the distribution of values around the mean value is as sharp as we want.

We shall use the Dirac notation for vectors and vector duals in $\mathcal{H}$: a vector will be written as a 'ket' $|\psi>$, and a linear functional (something that maps vectors to a complex number) on $\mathcal{H}$ is written as a 'bra' $<\psi|$.

Solving the Hamiltonian constraint using a projection operator. The idea if the Hilbert space were finite: say we the equation

$$\hat{H}|v>= 0 \tag{3.87}$$

$$\hat{H}|v_\lambda>= \lambda|v_\lambda> \tag{3.87}$$

$$\hat{H}(\hat{P}|v>) = \hat{H}(\exp \hat{H}|v>) = \tag{3.87}$$

Since these physical states do not belong to H_{kin}, the scalar product between them is well-defined - they are too distributional to be normalizable states in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$.

this can be expressed as

$$<\Psi_{phys}|\Phi_{phys}>=<P\Psi|P\Phi>=<P\Psi|\Phi>= \int \tag{3.87}$$

position eigenstates

We call ϕ a distribution or dual state

We think about think of this states in the momentum representation in which we find it is a delta function in p.

$$\hat{p}|\psi >= p|\psi > \tag{3.87}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{ipx} \tilde{\psi}(p), \quad \text{where } \mathcal{L}_2(R). \tag{3.87}$$

$$\begin{aligned}
\left(-i\frac{d}{dx}\right)\psi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \left(-i\frac{d}{dx}\right) e^{ipx} \tilde{\psi}(p) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\ p\ e^{ipx} \tilde{\psi}(p)
\end{aligned} \tag{3.87}$$

for the function $\tilde{\psi}(p)$we, for example, use $\psi(p) = \frac{1}{2\pi d} e^{-(p-p_0)^2 d}$ $d >> 1$ approximate the delta fucntion $\delta(p - p_0)$ centered at p_0,

$$\begin{aligned}
\left(-i\frac{d}{dx}\right)\psi(x) &\approx p_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{ipx} \tilde{\psi}(p) \\
&= \approx p_0 \psi(x).
\end{aligned} \tag{3.87}$$

a smeared momentum eigenstate

$$< \omega| \iff e^{ipx}[.] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{ipx}(.) \tag{3.87}$$

We think about this state as a dual state. For this construction to be well defined we must use a smaller subset $\mathcal{D} \subset \mathcal{H}$ for which

$$e^{ipx}[\psi] < \infty \ \text{ for every } \psi \in \mathcal{D}. \tag{3.87}$$

The space of functionals is called the dual of $\mathcal{D}$ and is denoted $\mathcal{D}^*$. The dual space contains all the functions in $\mathcal{H}$ i.e. $\mathcal{H} \subset \mathcal{D}^*$. Altogother we have that

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^* \tag{3.87}$$

This construction is called a Gel'fand triple or a rigged Hilbert space. It turns out that there is always an analogous construction for any self-adjoint operator. In fact, it can be shown that there is a sence in which all self-adjoint operators have a complete set of generalized eigenststes, (this is refered to as the **Nuclear spectral theorem**).

This isn't a state in the Hilbert space itself but has a finite inner product with some large class of states in the Hilbert space. An example as the Schwartz space - which are functions that fall off rapidly and are smooth (we can differentiate them as many times as we want) and correspondingly have smooth Fourier transforms and the inner product of our position independent state with one of these nice rapidly falling off smooth states should be well defined.

3.11.1 Induced Hilbert Space and the New Inner Product

$$-i\frac{d}{dx}\psi(x,y)=0 \tag{3.87}$$

are functions $\psi(x,y)$ constants in the y and are non-normalizable in $\mathcal{K}$. However, the decomposition

$$\mathcal{K}=\int_{\mathbb{R}} dyH_y. \tag{3.87}$$

where $H(y)=L^2[\mathbb{R},dx]$.

$$(\psi,\phi)_{\mathcal{K}}=\int_{\mathbb{R}} d^2x\overline{\psi(x,y)}\phi(x,y)=\int_{\mathbb{R}}(\psi_y,\phi_y)_{H_y}, \tag{3.87}$$

where $\psi_y(x)=\psi(x,y)$ and

$$(\psi_y,\phi_y)_{H_y}=\int_{\mathbb{R}} dx\overline{\psi_y(x)}\phi_y(x). \tag{3.87}$$

The space of solutions of (M.5) is $H(0)$ and has the natural Hilbert structure $H(0)=L^2[\mathbb{R},dx]$. Introduce some notation,

3.12 Spatial Diffeomorphism Inner-Product Structure

3.12.1 Solving the Diffeomorphism Constraint

Unlike the strategy in solving Gaussian constraint, one cannot define an operator for quantum diffeomorphism constraint as the infinitesimal generator of finite diffeomorphism transformations (unitary operators since the measure is diffeomorphism invariant) represented on $\mathcal{H}_{kin}$. The representation of finite diffeomorphisms is a family of unitary operators $\hat{U}_\varphi$ acting on cylindrical functions ψ_α by

$$\hat{U}_\varphi \psi_\alpha := \psi_{\varphi\circ\alpha}, \tag{3.87}$$

for any spatial diffeomorphism φ on Σ.

There is a subset of Diff(Σ) which leaves the curve γ invariant, and only reparamertizes it. These elements of this subset are (finite) diffeomorphisms generated by the vector fields on Σ that are tangent to γ.

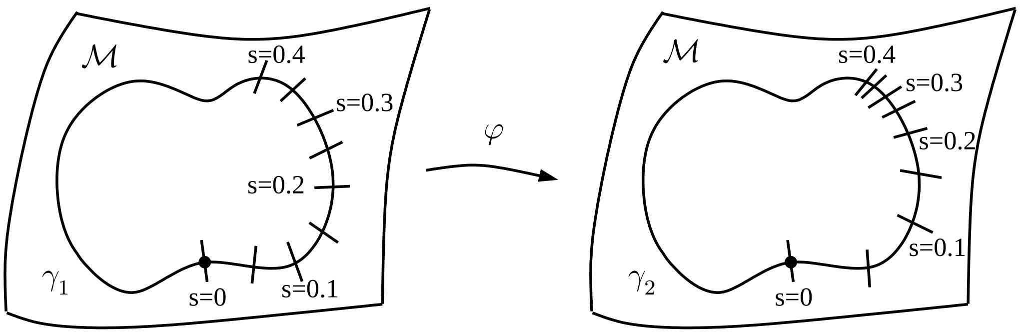


Figure 3.30: GraSymmFig0. The diffeomorphism φ maps a loop to itself resulting in a reparametrization of the loop γ_1.

Notice that a diffeomorphism may change more than just the graph of a spin network, that is, the equation $\hat{U}_\varphi|\alpha, j_l, i_n\rangle = |\varphi\circ\alpha, j_l, i_n\rangle$ is not always correct. A diffeomorphism that leaves the graph α invariant may still affect a spin network state $|\alpha, j_l, i_n\rangle$. This is because, for each graph, the definition of an orientation (an assignment of a direction for each link) and ordering (enumeration of the links) and these get changed by a diffeomorphism.

Averaging over the group of graph symmetries

Also let $GS(\gamma)$ be the group of graph symmetries of γ, that is, the group $Iso(\gamma)/TA(\gamma)$, where the isotropy group $Iso(\gamma)$ is the group of diffeomorphisms mapping γ to itself, and

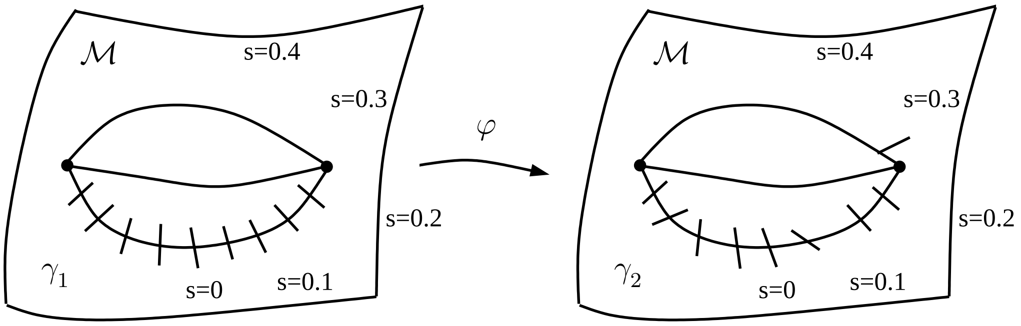


Figure 3.31: GraSymmFig0a. The diffeomorphism φ maps a loop to itself resulting in a reparametrization of the loop γ_1.

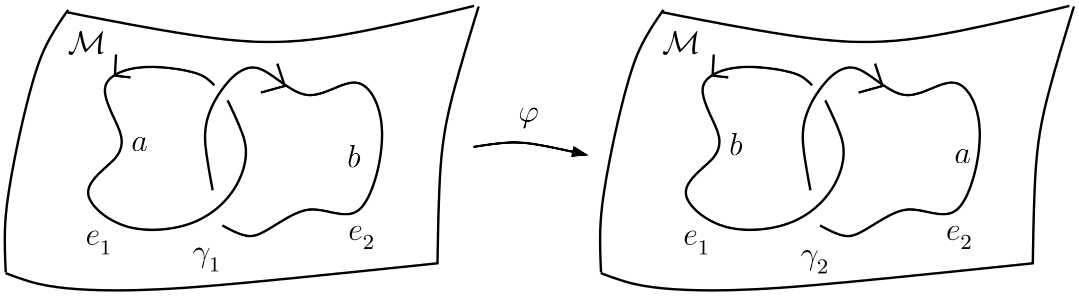


Figure 3.32: GraSymmFig. The diffeomorphism φ maps the edge e_1 to e_2 and at the same time the edge e_2 to the edge e_1. This has the effect of swaping around the labels on the two edges.

the trivial action group $TA(\gamma)$ is the subgroup fixing each edge of γ . We may define an element

If we to average over these two graphs

$$\frac{1}{2}(\Psi_{\gamma_1} + \Psi_{\gamma_2}) \tag{3.87}$$

will be a state that remains unchanged by the diffeomorphism. Let us denote the effect induced on the state Ψ_α by the diffeomorphism φ as $\varphi \star \Psi_\alpha$, this is known as the pull-back of Ψ_α under φ. In our example

$$\mathbf{1} \star \Psi_{\gamma_1} = \Psi_{\gamma_1} \quad \text{and} \quad \varphi \star \Psi_{\gamma_1} = \Psi_{\gamma_2}$$

where $\mathbf{1}$ is a diffeomorphism which maps each edge to themselves.

This obviously has the structure of a group, this is called the *group of graph symmetries*. We can collect all the information on group operations in a group table.

	I	a	b
I	I	b	b
a	a	b	b
b	b	b	b

For each row (and in each column) of a group table every group element appears exactly once. The statement that b occors in row a and column x is the equation

$$ax = b,$$

this is solved by $x = a^{-1}b$. Say there were another element $y \in G$ such that $ay = b$, then $ax = ay$. Multiplying both sides of this from the left by a^{-1} gives $x = y$. So there is a unique solution for x, i.e. given by an element b, there is precisely one column x in which it occurs, i.e. each element occurs exactly once in each row. Now, this implies that if we multiply the sum of all group elements by any one group element the result will be the same summation, but now in a different order. This observation allows us to factorize out the group of graph symmetries.

First, given any $\Psi_\alpha \in \mathcal{H}'_\alpha$, we average it using *only the group of graph symmetries* and obtain a projection map $\hat{P}_{Diff,\alpha}$ from $\mathcal{H}'_\alpha$ to its subspace which is invariant under $\hat{GS}_\alpha$:

$$\hat{P}_{Diff,\alpha}\Psi_\alpha := \frac{1}{N_\alpha} \sum_{\varphi \in GS_\alpha} (\varphi \star \Psi_\alpha) \tag{3.87}$$

where N_α is the number of the elements of GS_α (the volume of the obit of GS_α).

For the simply example above this is

$$\hat{P}_{Diff,\alpha}\Psi_\alpha = \frac{1}{2}(1 \star \Psi_{\gamma_1} + \varphi \star \Psi_{\gamma_1}) = \frac{1}{2}(\Psi_{\gamma_1} + \Psi_{\gamma_2}) \tag{3.87}$$

Mathematical excursion: Diagonalizing the quadratic form and obtaining orthogonal states

Diagonalizing the quadratic form ("overlap matrix") and obtaining orthogonal states

These are general results that applies to any finite dimensional inner-product space.

Say you have a set of states $\{|\mathbf{v}_i\rangle\}_{i=1,\ldots,N}$. The you can form the quadratic form ("overlap matrix"):

$$O_{ij} = \langle \mathbf{v}_i | \mathbf{v}_j \rangle$$

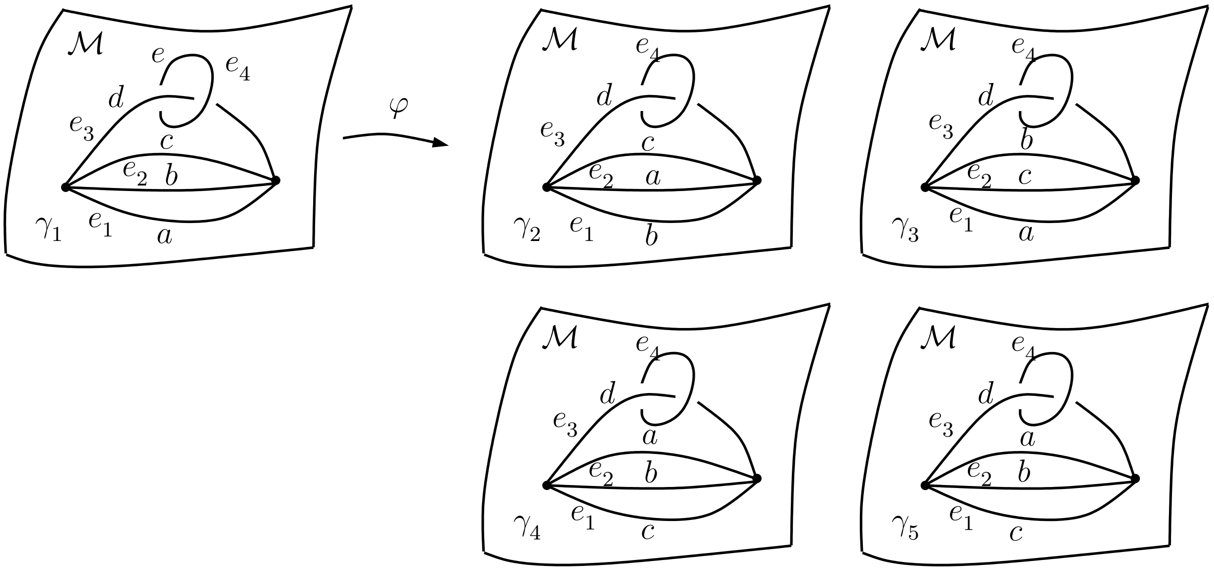


Figure 3.33: GraSymmFig2. The diffeomorphism φ maps swaping around the labels on the two edges.

Define

$$|\mathbf{v}\rangle = \sum_{i=1}^{N} \lambda_i |\mathbf{v}_i\rangle, \qquad |\mathbf{v}'\rangle = \sum_{i'=1}^{N} \lambda'_{i'} |\mathbf{v}_{i'}\rangle$$

Then

$$\begin{aligned}
\langle \mathbf{v}|\mathbf{v}'\rangle &= \sum_{i=1}^{N} \sum_{i'=1}^{N} \langle \lambda_i \mathbf{v}_i | \lambda'_{i'} \mathbf{v}_{i'}\rangle \\
&= \sum_{i=1}^{N} \sum_{i'=1}^{N} \lambda_i^* \lambda'_{i'} \langle \mathbf{v}_i | \mathbf{v}_{i'}\rangle \\
&= \sum_{i=1}^{N} \sum_{i'=1}^{N} \lambda_i^* O_{ii'} \lambda'_{i'}.
\end{aligned}$$

If we are dealing with an orthogonal basis, then $O_{ij} = \langle \mathbf{v}_i | \mathbf{v}_j \rangle = 0$ if $i \neq j$ and O_{ij} is diagonal. Then

$$\langle \mathbf{v}|\mathbf{v}'\rangle = \sum_{i=1}^{N} \lambda_i^* \lambda'_i O_{ii}.$$

If we are dealing with an orthonormal basis, then additionally $O_{ii} = \langle \mathbf{v}_i | \mathbf{v}_i \rangle = 1$ and so:

$$\langle \mathbf{v} | \mathbf{v}' \rangle = \sum_{i=1}^{N} \lambda_i^* \lambda_i'.$$

The RHS is the "usual" inner product for $\mathbb{C}^N$.

A method for producing an orthogonal collection of vectors is the Gram-Schmidt orthogonalisation. A second method is to diagonalise the overlap matrix. A new orthogonal basis can be constructed from the eigenvectors corresponding to the non-vanishing eigenvalues.

Given $O_{ij} = \langle \mathbf{v}_i | \mathbf{v}_j \rangle$ we have the eigenvalue equations:

$$\sum_{j=1}^{N} O_{ij} \lambda_j^{(m)} = o^{(m)} \lambda_i^{(m)}$$

Define

$$|\mathbf{n}^{(m)}\rangle = \sum_{i=1}^{N} \lambda_i^{(m)} |\mathbf{v}_i\rangle, \qquad o^{(m)} \neq 0$$

then the $|\mathbf{n}^{(m)}\rangle$ are orthogonal.

Proof:

We have

$$\begin{aligned}
\langle \mathbf{n}^{(m)} | \mathbf{n}^{(m')} \rangle &= \sum_{i=1}^{N} \sum_{i'=1}^{N} \langle \lambda_i^{(m)} \mathbf{v}_i | \lambda_j^{(m')} \mathbf{v}_j \rangle \\
&= \sum_{i=1}^{N} \sum_{i'=1}^{N} \lambda_i^{(m)*} \lambda_j^{(m')} \langle \mathbf{v}_i | \mathbf{v}_j \rangle \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i^{(m)*} O_{ij} \lambda_j^{(m')} \\
&= \sum_{i=1}^{N} \lambda_i^{(m)*} o^{(m')} \lambda_i^{(m')} \\
&= o^{(m)} \delta_{mm'} \sum_{i=1}^{N} \lambda_i^{(m)*} \lambda_i^{(m)}
\end{aligned}$$

where we have used that for a Hermitian matrix eigenvectors are orthogonal, and $\langle \mathbf{v}_i | \mathbf{v}_j \rangle = \langle \mathbf{v}_j | \mathbf{v}_i \rangle^*$ is an axiom for the inner product.

□

Orthonormal states can be defined as

$$|\tilde{\mathbf{n}}^{(m)}\rangle = \frac{1}{\sqrt{o^{(m)} \sum_{i=1}^{N} \lambda_i^{(m)*} \lambda_i^{(m)}}} \sum_{i=1}^{N} \lambda_i^{(m)} |\mathbf{v}_i\rangle.$$

The following result proves that if the original states $\{|\mathbf{v}_i\rangle\}_{i=1,\dots,n}$ are linearly independent then the eigenvalues are all non-zero, and hence $\{|\mathbf{n}^{(m)}\rangle\}_{m=1,\dots,n}$ form an orthogonal basis and $\{|\tilde{\mathbf{n}}^{(m)}\rangle\}_{m=1,\dots,n}$ form an orthonormal basis. This follows from the fact that the overlap matrix is invertible if and only if the eigenvalues are all non-zero.

Theorem: A list of vectors is linearly independent if and only if its overlap matrix is invertible.

Proof:

First we prove that invertibility of the Overlap matrix implies linear independence. Suppose that $\sum_{j=1}^{N} c_j |\mathbf{v}_j\rangle = 0$. Take the inner product with $\langle \mathbf{v}_i|$. This gives

$$\sum_{j=1}^{N} O_{ij} c_j = 0$$

for each i. Since the matrix is invertible, it follows that the coefficients c_j are all zero. This proves linear independence.

Then we prove that linear independence implies that the overlap matrix is invertible. Consider a coordinate vector of c_j that is in the null space of the matrix O, that is, such that

$$\sum_{j=1}^{N} O_{ij} c_j = 0$$

for each i. It follows that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_i^* O_{ij} c_j = 0$$

or

$$\langle\sum_{i=1}^{N} c_i \mathbf{v}_i | \sum_{j=1}^{N} c_j \mathbf{v}_j\rangle = 0$$

or

$$\| \sum_{i=1}^{N} c_i |\mathbf{v}_i\rangle \|^2 = 0$$

or

$$\sum_{i=1}^{N} c_i |\mathbf{v}_i\rangle = 0.$$

By linear dependence the coefficients c_j are all zero. This proves that the null space of O is trivial. Therefore O is invertible.

$\square$

Averaging over the remaining diffeormorphisms

test function space - $\Phi^\times$ is the topological dual of Φ - it corresponds to the complete space of continuous (bounded) linear functionals. $\Phi^\times$ denotes the space of distributions. Because the elements of Φ are so well-behaved $\Phi^\times$ is very large and contains solutions to the operators through their adjoint action.

$$(\mathbf{\Phi}^*_{Kin})_{Diff} \tag{3.87}$$

$$\mathcal{H}_{Diff} \tag{3.87}$$

Construct a space of diffeomorphism invariant states $H_{Diff} \subset H^*_{kin}$, which are invariant under the action of $\hat{U}(\phi)$. These are the diffeomorphism invariant states and they live inside the dual of the kinematical Hilbert space.

A continous index set for diff-invariant states

Finally we mention that $\mathcal{H}_{Diff}$ just like $\mathcal{H}$ is still not separable because the set of singular knot classes $[\gamma]$ has uncountably infinite cardinality [??].

This is easy to understand. By the definition of tangent space T_p loops define directions in T_p and diffiomorphisms act linearly on it. Thus the equivalence loops under diffeomorphisms implies the corresponding equivalence of T_p under liner transformtions induced by the diffeomorphism. So from the fact that the group of semianalytic diffeomorphism reduces to $GL(3, \mathbb{R})$ at each vertex. This transformation depends on nine parameters (per point).

Hence, for vertices of valence higher than nine one cannot arbitrarily change, in a coordinate chart, all the angles between the tangents of the adjacent edges. Any two vertices with the same valence which is above nine are not related through a semianalytic diffeomorphism and so belong to inequivalence classes. This leads to the emergence of equaivalence classes labelled by continuous parameters. It turns out that valence five is already sufficient, that is, there are diffeomorphism invariant "angles", called moduli θ in all vertices of valence five or higher.

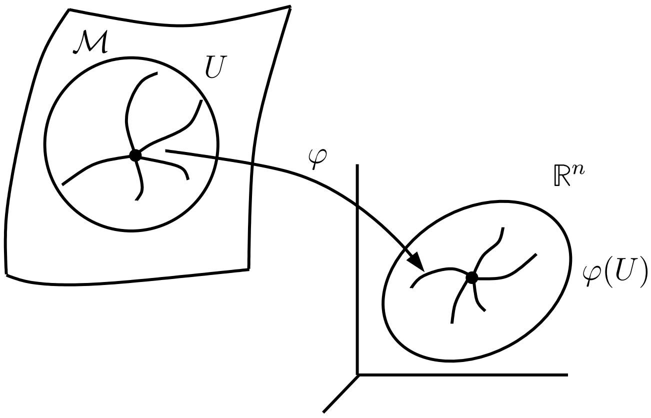


Figure 3.34: tangnodeF. The map .

There are several proposals for an enlargement of the group of diffeomorphisms [??], however, these groups do not interact well with certain crucial operators in the the theory such as the volume operator which depend on at least $C^{(1)}(\Sigma)$ structures while those extensions basically replace diffeomorphisms by homeomorphisms or even mor general bijective maps on Σ. We will see, however, that the non separability of $\mathcal{H}_{Diff}$ is immaterial when we pass to the physical Hilbert space $\mathcal{H}_{Phys}$.

[102]:

"We investigate the action of diffeomorphisms in the context of Hamiltonian Gravity. By considering how the diffeomorphism-invariant Hilbert space of Loop Quantum Gravity should be constructed, we formulate a physical principle by demanding, that the gauge-invariant Hilbert space is a completion of gauge- (i.e. diffeomorphism-)orbits of the classical (configuration) variables, explaining which extensions of the group of di eomorphisms must be implemented in the quantum theory. It turns out, that these are at least a subgroup of the stratified analytic diffeomorphisms. Factoring these stratified diffeomorphisms out, we obtain that the orbits of graphs under this group are just labelled by their knot classes, which in turn form a countable set. Thus, using a physical argument, we construct a separable Hilbert-space for di eomorphism invariant Loop Quantum Gravity, that has a spin-knot basis, which is labelled by a countable set consisting of the combination of knot-classes and spin quantum numbers. It is important to notice, that this set of di eomorphism leaves the set of piecewise analytic edges invariant, which ensures, that one can construct flux-operators and the associated Weyl-operators."

3.12.2 Non Weak Continuity of Diffeomorphisms in LQG

We consider an aspect of the action of diffeomorphisms on the quantum theorey. The space of quantum configurations $\mathcal{A}$, i.e. the space of (distributional) connections on Σ carries a natural action of the diffeomorphism group Diff Σ. An element $\phi \in$ Diff Σ simply acts by $A \to \phi^* A$ on a (distributional) connection A. With this, one can simply define the action of Diff Σ on $\mathcal{H}_{kin}$ by

$$\alpha_\phi f(A) := f(\phi^* A)$$

where $\phi^* A$ is the pullback of the connection A under the diffeomorphism ϕ. By this definition, under a diffeomorphism ϕ the kinematic Hilbert space transforms as

$$\alpha_\phi \mathcal{H}_\gamma \to \mathcal{H}_{\phi(\gamma)}. \tag{3.87}$$

where $\phi(\gamma)$ is the image of γ under ϕ. Note the action (3.12.2) is not weakly continuous in ϕ, since two graphs can be arbitrary "close" to each other, but still be not intersecting, which means that their corresponding Hilbert spaces are mutually orthogonal subspaces of $\mathcal{H}_{kin}$. This is natural in the LQG picture, since the notion of "being close to each other" has no meaning as the metric can be transformed into another one which imposes a differernt geometry by an active diffeomorphism.

We say that the representation is not weakly continuous with respect to the holonomies, by which we mean that matrix elements of cylindrical functions are not continuous under continuous deformations of edges. While the Weyl representation is not weakly continuous with respect to the holonomies it is with respect to the flux.

3.12.3 Ergodicity of Spacial Diffeomorphisms

3.13 Uniqueness Theorem for the Holonomy-Flux Representation

Under reasonable physical assumptions, there is a unique representation of the Holonomy-Flux algebra. This means that, once the Holonomy-Flux algebra has been chosen, we can be confident to use that unique, kinematic representation as a basis for the constraint quantization programme. Uniquness will lead to make definiative predictions that may be confronted with experiments.

In section ?? we introduced the Ashtekar-Isham-Lewendowski state ω_0 - existence of a representation is established.

One will be interested in those representation which are distinguished by physical selection criteria. One such criterion is a unitary representation of the diffeomorphism group (rather than an infintesimal version). It turns out that such a representaion is actually unique. We need to introduce some defintions and technical results.

In general any representation of a $*-$algebra is a direct sum of cyclic representations (the proof is analogous to the proof of the existence of a orthonormal basis for a Hilbert space). But every cyclic represenation comes from a state ω (positive linear functional) on the algebra $\mathcal{O}$ via the GNS construction.

If the state is invariant the associated representation is automatically unitary. Hence, it sufficse to look for invariant states and it has now been proven [][??] that the only such state is the Ashtekar-Isham-Lewendowski state ω_0.

The uniqueness theorem not only requires that the expenentiated fluxes are represenated weakly continuously but actually smoothly. So as compared to the Stone-von Neumann theorem of ordinary quantum mechanics, one considers representations in which continuity is slightly relaxed in one direction and slightly tightened in the other.

Theorem 3.13.1 *There is a unique, semi-weakly smooth, diffeomorphism-invariant state on (equivalently, cyclic representation of) $\mathfrak{A}$. Moreover, the corresponding cyclic GNS representation is irreducible.*

3.13.1 Assumptions of the Uniqueness Theorem

Quantum Geometry: Representation on $L_2(\overline{\mathcal{A}}/\overline{\mathcal{G}}, \mu_0)$ is unique if

1. diffeomorphism invariant;

2. semianalytic.

Diffeomorphism invariance

A natural idea is to first look at irreducible or at least cyclic representations as the simple building blocks, out of which more complicated representations could eventually be built.

If $(\mathcal{H}, \pi, \Omega)$ is a cyclic representation of a $*-$algebra, and $\pi(G)$ is the representation of a group symmetry then

$$< \Omega|\pi(g)^{-1}\pi(A)\pi(g)\Omega > = < \Omega|\pi(A)\Omega >$$

As $\pi(g)$ is unitary, this condition is equivalent to

$$\omega(gA) = \omega(A)$$

for all $g \in G$ and $A \in \mathcal{O}$.

Now we consider the converse of the above statement. A morphism is a map from an algebra to itself that, that is $g : \mathcal{O} \to \mathcal{O}$. A morphism that has an inverse g^{-1} is called an automorphism. We say a state ω is invariant under a group of automorphisms G if

$$\omega(gA) = \omega(A)$$

for all $g \in G$ and $A \in \mathcal{O}$. If the state is invariant the associated representation is automatically unitary.

Let G be a group of automorphisms of the $C*-$algebra $\mathcal{O}$ and ω a corresponding $G-$invariant state on $\mathcal{O}$. Then there is a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$

$$\pi_\omega(gA) = U_\omega(g)\pi_\omega(A)U_\omega(g)^{-1}, \quad U_\omega(g)\Omega_\omega = \Omega_\omega, \tag{3.87}$$

for all $g \in G$ and $A \in \mathcal{O}$.

A simple formulation of these properties can be given by asking for a state (i.e. a positive, normalized, linear functional) on $\mathcal{U}$ that it is invariant under the classical symmetry automorphisms of $\mathcal{U}$. Given a state ω on $\mathcal{U}$ one can define a representation via the GNS construction. This representation will be cyclic by construction, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$,. If the state is invariant under some automorphism of $\mathcal{U}$, its action is automatically unitarily implemented in the representation.

Semianalyticity

Now we move to condition 3. A necessary condition for the Poisson bracket between the variables to be finite is that every edge intersects every face in at most a finite number of isolated intersection points plus a finite number of connected segments (i.e. the edges themselves). It is vital for the uniqueness proof that the intersection between paths with surfaces contain finitely many isolated points. A simple condition that ensures this property uses a real analytic structure on Σ, analytic paths and analytic surfaces. We could consider the class , analytic diffeomorphisms. However, analytic functions are uniquely determined by the value of it and all its derivatives at one point and, thus, in particular, is uniquely determined by its value in an arbitrarily small neighbourhood of a point. This implies that analytic diffeomorphisms, in a sense, have no local degrees of freedom. To alleviate this restriction while retaining the finite intersection property of analytic structures, one considers the larger group of diffeomorphisms - semianalytic diffeomorphisms.

Briefly, 'semianalytic' means 'piecewise analytic'. For example, a semianalytic sub-manifold would be analytic except for on some lower dimensional sub-manifolds, which in turn have to be piecewise analytic. We have already met the idea of semianalycity, see fig (L.9) (a). To convey the general idea, fig (L.9) (b) depicts a semi-analytic surface in $\mathbb{R}^3$.

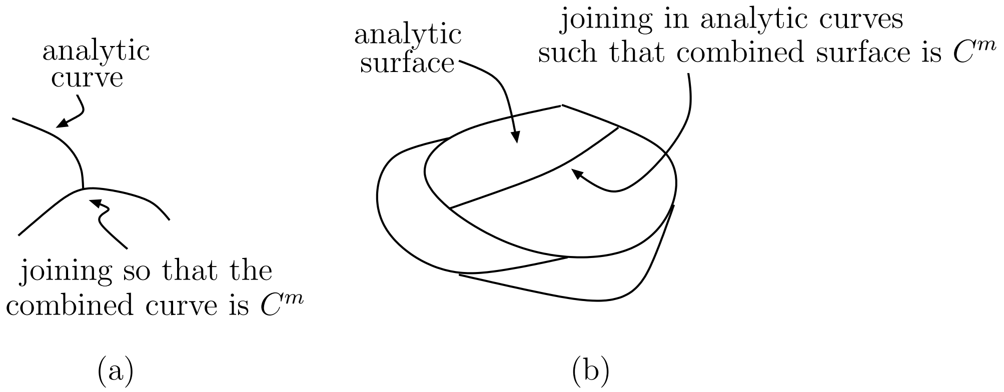


Figure 3.35: (a) A semianalytic curve in $\mathbb{R}^3$. (b) A semianalytic surface in $\mathbb{R}^3$.

analytic paths are determined everywhere once they are kniwn on an open set, thus making them non-local. If we make it semianalytic then these data only determine the path up to the next point where analyticity is reduced to C^n, $n > 0$. This is important because we need to make sure that certain local constructions do not have an impact on rdegions far away from the region of interest.

Semianalytic diffeomorphisms do have local degrees of freedom, in the sense that for every point $x \in \mathcal{M}$ and its neighbourhood $\mathcal{U}'$, there is a semianalytic diffeomorphism of $\mathbb{R}^n$ which moves x, but restricted to $\mathbb{R}^n/\mathcal{U}'$ is the identity map. With the semianalytic case, one is not requiring that the diffeomorphisms we consider be analytic everywhere but, roughly speaking, analytic only up to submanifolds of lower dimension. Some care has to be taken in the precise definition of this notion (see section J.19), mainly to insure that they form a group and that application of these diffeomorphisms produce surfaces and edges that still have finitely many isolated intersection points. The analyticity of the entire analytic patches will ensure the finite intersection property with piecewise analytic edges discussed before. The important point is that this larger symmetry group now contains local diffeomorphisms, and this is instrumental for proving the uniqueness result.

3.13.2 Outline of Proof

We have already seen that a diffeomorphism invariant state exists, the Ashtekar-Lewandowski state ω_0. Given a $^*-$algebra and a symmetry group, assuming the existence of a diffeomorphism invariant state is a strong condition and it turns out that ω_0 is the only one - this is the result of the uniqueness theorem.

One defines a $*-$algebra $\mathfrak{A}$ called the Sahlmann holonomy-flux $*-$algebra whose elements correspond to cylindrical functions and fluxes, and then studies $*-$representations of the algebra on a Hilbert space.

Consider the space of cylindrical functions on $\mathcal{A}$, that is, functions of the form

$$\Phi[A] \equiv \phi(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_n}[A])$$

where ϕ is a complex-valued continuous function. From the Poisson brackets of A and E one can compute the Poisson brackets for the Φ, $E_{S,f}$.

We then consider the Hamiltonian vector fields $X_{S,f}$ of the fluxes $E_f(S)$ defined by

$$X_{S,f} \cdot A(p) = \{E_f(S), A(p)\}.$$

Then the association of the classical functional E with elements of $\mathfrak{A}$ is given by

$$E_{S,f} \quad \mapsto \quad \hat{X}_{S,f}. \tag{3.88}$$

We let $\mathfrak{A}$ be generated by

$$\begin{aligned} \Phi \quad &\mapsto \quad \hat{\Psi}\Phi := \Psi\Phi, &(3.89)\\ \Phi \quad &\mapsto \quad \hat{X}_{S,f}\Phi, &(3.90)\end{aligned}$$

where Φ, Ψ are C^∞ cylindrical functions. We introduce the $*$ operation on $\mathfrak{A}$ by specifying its action on the generating set:

$$\hat{\Psi}^* := \hat{\overline{\Psi}}, \quad \hat{X}^*_{S,f} := \hat{X}_{S,f}.$$

Every element of the algebra $\mathfrak{A}$ is a finite linear combination of elements of the form

$$\hat{\Psi},\ \hat{\Psi}_1\hat{X}_{S_{11},f_{11}},\ \hat{\Psi}_2\hat{X}_{S_{21},f_{21}}\hat{X}_{S_{22},f_{22}},\ \dots,\ \hat{\Psi}_k\hat{X}_{S_{k1},f_{k1}}\dots\hat{X}_{S_{kk},f_{kk}},\ \dots\ , \tag{3.90}$$

$$\begin{aligned} a &= \hat{X}_{S,f}\hat{\Psi}\hat{X}_{S',f'} \\ &= -i\widehat{\{\hat{X}_{S,f},\hat{\Psi}\}}\hat{X}_{S',f'} + \hat{\Psi}\hat{X}_{S,f}\hat{X}_{S',f'} \\ &= -i\widehat{\hat{X}_{S,f}(\Psi)}\hat{X}_{S',f'} + \hat{\Psi}\hat{X}_{S,f}\hat{X}_{S',f'}. \end{aligned} \tag{3.89}$$

Note that since $\mathfrak{A}$ is generated by the elements of Cyl and X, a represention π of $\mathfrak{A}$ is completely determined once the representors π (Cyl) and $\pi(X)$ are known.

Let us use the GNS notation

$$[a] = \pi_\omega(a)\Omega_\omega,$$

then

$$[\hat{\Psi}],\ [\hat{\Psi}_1\hat{X}_{S_{11},f_{11}}],\ [\hat{\Psi}_2\hat{X}_{S_{21},f_{21}}\hat{X}_{S_{22},f_{22}}],\ \dots,\ [\hat{\Psi}_k\hat{X}_{S_{k1},f_{k1}}\dots\hat{X}_{S_{kk},f_{kk}}],\ \dots\ , \tag{3.89}$$

where $[a] := \{a + b : b \in \mathfrak{A} \text{ such that } \omega(b^*b) = 0\}$ is the equivalence class of $a \in \mathfrak{A}$ with respect to the Gel'fand ideal of null vectors, discussed above. For the $*$ operation to be well defined we must have that b^* in the equivalence class of a (we go into the details of this in appendix P). This algebra is a unital one with a unit given by a constant cylindrical function of the value equal to 1.

The product provides a norm $\|a\|_\omega = \sqrt{< [a], [a] >}$ in $[\mathfrak{A}]$, and the completion

$$\mathcal{H}_\omega := \overline{[\mathfrak{A}]} \tag{3.89}$$

together with the inner product $< \cdot , \cdot >$ is a Hilbert space.

To every element a of $\mathfrak{A}$ we asign a linear but in general unbounded operator $\pi_\omega(a)$ acting in $[\mathfrak{A}]$

$$\pi_\omega(ab)\Omega_\omega = \pi_\omega(a)\pi_\omega(b)\Omega_\omega, \quad \text{for every} \quad b \in \mathfrak{A}$$

or

$$\pi_\omega(a)[b] := [ab], \quad \text{for every} \quad b \in \mathfrak{A} \tag{3.89}$$

The action of π_ω preserves the suspace $[\mathfrak{A}]$, hence a dense set of vectors in $\mathcal{H}_\omega$ is given by the linear span of all the vectors of the form

$$\begin{aligned}[\hat{\Psi}],\ \pi_\omega(\hat{\Psi}_1)[\hat{X}_{S_{11},f_{11}}],\ \pi_\omega(\hat{\Psi}_2\hat{X}_{S_{21},f_{21}})[\hat{X}_{S_{22},f_{22}}],\dots \\ \dots,\pi_\omega(\hat{\Psi}_k\hat{X}_{S_{k1},f_{k1}}\dots)[\hat{X}_{S_{kk},f_{kk}}]\end{aligned} \tag{3.89}$$

The space of cylindrical functions used to construct $\mathfrak{A}$ leads naturally to the space of generalized connections $\overline{\mathcal{A}}$.

In the Ashtekar-Lewandowski representation the fluxes vanish. It turns out that the only representation with this property is indeed the Ashtekar-Lewandowski representation. By vanishing of fluxes we mean that for any face S and any smearing vector field f in any GNS-representation coming from the invariant state

$$[X_{S,f}] = 0. \tag{3.89}$$

Once this rather technical result is established the rest of the proof of uniqueness is fairly straightforward.

This is proved by first making a certain decomposition and proving that each individual term satisfies this condition.

The proof of this result relies crucially on the local character of semianalycity. By local character is meant there is a partition of unity subordinate to a local finite covering $\{\mathcal{U}_I\}$, i.e., there exists a family of differentiable functions $\phi_I(x)$ such that

(i) $0 \leq \phi_I(x) \leq 1$

(ii) $\phi_I(x) = 0$ if $x \neq U_I$

(iii) $\sum_I^N \phi_1(x) = 1$ for any point $x \in \mathcal{M}$.

Even though a partition of unity does not exist in the analytic case, it does for the semianalytic one, just as in the smooth case.

Analytic diffeomorphisms - an analyic function is already determined by its values in an arbiary small neighbourhood of any point. The local aspect of semianalytic diffeomorphisms is required in the above result.

Because of this result, only the terms of the form $[\hat{\Psi}]$ in (3.89) are non-zero. It follows that all the information on the state ω is determined by its restriction to smooth cylindrical functions Cyl^{∞}. As Cyl is a unital Abelian $C^{*}-$algebra we can now use some powerful results of representation theory. Due to the theorem of Gelfand, since Cyl is Abelian, it is isomophic to the algebra of continuous functions on the spectrum $\overline{\mathcal{A}}$, on a compact Hausdorff space, of Cyl. From this and the representation theorem of Riesz and Markow, there exists a measure μ on $\overline{\mathcal{A}}$ such that

$$\omega(\hat{\Psi}) = \int_{\overline{\mathcal{A}}} \Psi d\mu.$$

Note that (3.13.2) implies in what follows

$$\begin{aligned}
\int_{\overline{\mathcal{A}}} \overline{\Psi} X_{S,f}(\Psi') d\mu &= \big\langle [\overline{\Psi}], [\widehat{X_{S,f}(\Psi')}] \big\rangle_{\mathcal{H}_\omega} \\
&= i\big\langle [\hat{\Psi}], [X_{S,f}\Psi'] - [\Psi' X_{S,f}] \big\rangle_{\mathcal{H}_\omega} \\
&= i\big\langle [\hat{\Psi}], [X_{S,f}\Psi'] \big\rangle_{\mathcal{H}_\omega} \\
&= i\big\langle [X_{S,f}\hat{\Psi}], [\Psi'] \big\rangle_{\mathcal{H}_\omega} \\
&= -\int_{\overline{\mathcal{A}}} \overline{X_{S,f}(\Psi)} \Psi' d\mu.
\end{aligned}$$

Setting $\Psi = I$ (i.e. the constant function on $\overline{\mathcal{A}}$ on the value 1) it follows that for any face S and any smearing vector field f and for any function $\Psi' \in \mathrm{Cyl}$

$$\int_{\overline{\mathcal{A}}} X_{S,f}(\Psi') d\mu = 0.$$

As was shown in [88] the only measure satifying the above condition coincides with the measure defined on $\overline{\mathcal{A}}$ by the Ashtekar-Lewandowski state ω_0.

3.13.3 Irreducibility

If the theory has non-trivial closed invariant subspaces signals the possibility that the representation space $\mathcal{H}_0$ is too large because interesting physics can already be captured by one of its invariant subspaces [114]. It has been proven that this is not the case for π_0, further strenghing physical assumption undrlying LQG.

Irreduciblity: In this situation, it is worthwhile to note that there is a strong analogy between the AIL representation of $\mathcal{U}$ and the Schrödinger representation of the Heisenberg algebra in quantum mechanics. In both representations, the representation spaces are roughly speaking L^2 spaces over the configuration space, the configuration variables act by multiplication and the momenta by derivations. The Heisenberg algebra is again an algebra of unbounded operators, which makes the definition of irreducibility diffcult. Moreover it is dubious that its Schrödinger representation can be irreducible in any sense, since for example the sub-spaces generated by functions which vanish on fixed open sets are invariant under action with multiplication operators and differentiation (if defined). To see this consider the space of C^∞ functions non-vanishing in the interval $I = [-a, a]$ of the real line, which is non-empty by example of

$$f(x) = \begin{cases} \exp(-1/(x-a)^2) + \exp(-1/(x+a)^2) & |x| \leq a \\ 0 & |x| > a \end{cases} .$$

Any function in C_I^∞ acted on by $\hat{x}, \hat{p}$ an arbitary number of times results in another function belonging to C_I^∞. Obviously, this space of functions is orthogonal to any function belonging to C_J^∞ where $J \subset \mathbb{R}$ with $I \cap J = \emptyset$. Hence there are many invariant subspaces under the operators $\hat{x}, \hat{p}$.

However,

$$(V(b)f)(x) = f(x+b)$$

changes the support and hence there is a chance that the Weyl algebra is represented in an irreducible fashion.

However, the Schrödinger representation of the Heisenberg algebra can be obtained from the Schrödinger representation of the corresponding Weyl algebra, and it is this representation that is irreducible. That is, if $\rho(a, b)$ $(a, b \in \mathbb{R})$ is the Schrödinger representation of the Heisenberg algebra, then the only nonzero closed invariant subspace of $L^2(\mathbb{R}^n)$ is $L^2(\mathbb{R}^n)$ itself.

3.13.4 Outline of Proof

Introduce a flux vector field satisfying

$$Y_\gamma(t_\gamma) f_\gamma = \sum_{e \in E(\gamma)} t_e^j R_J^e f_\gamma$$

$$\begin{aligned} M_{\psi,\psi'}(t_\gamma, I_\gamma) &:= \; < \psi, T_{\gamma, I_\gamma} \; W_\gamma(t_\gamma) \psi' >_{\mathcal{H}_0} \\ &= \int_{\overline{A}} d\mu_0(A) \; T_{\gamma, I_\gamma}(A) W_\gamma(t_\gamma)(A) \end{aligned} \tag{3.84}$$

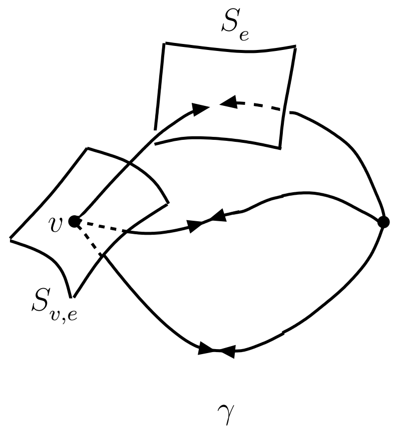


Figure 3.36: Action of $Y_\gamma(t_\gamma)$ on T_s

The inner product of the type

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma := \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \overline{M_{\psi_1,\psi_1'}(t_\gamma, I_\gamma)} M_{\psi_2,\psi_2'}(t_\gamma, I_\gamma)$$

is a crucial ingredient in an elementary irreducibility proof of the Schrödinger representation of ordinary quantum mechanics.

i) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_0$ we have

$$|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \leq \|\psi_1\| \, \|\psi_1'\| \, \|\psi_2\| \, \|\psi_2'\| \tag{3.84}$$

ii) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_{0,\gamma}$ we have

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma = <\psi_2, \psi_1>_{\mathcal{H}_0} <\psi_1', \psi_2'>_{\mathcal{H}_0} \tag{3.84}$$

where $\mathcal{H}_{0,\gamma}$ denotes the closure of the cylindrical functions over γ.

In the evaluation of $M_{\psi,\psi'}(t_\gamma, I_\gamma)$ this the cylindrical delta function

$$\delta_\gamma(A, A') = \prod_{e \in E(\gamma)} \delta_{\mu_H}(h_e[A], h_e[A'])$$

arises from the Plancherel formula

$$\delta_{\mu_H}(g, g') = \sum_{\pi,m,n} \overline{T_{\pi,m,n}(g)} T_{\pi,m,n}(g')$$

(recall the formula $\delta(x,y)=\sum_n e_n^*(x)e_n(y)$ for a complete basis $e_n(x)$).

We subdivide the degrees of freedom of $A\in\overline{\mathcal{A}}$ into the set $\overline{\mathcal{A}}_\gamma$ and the complement. We need to be more presice about this definition which we do in appendix P but here we will be slightly heuristic. For any function $f\in\mathcal{H}_0$

$$f(A)=F(A_{|\overline{\gamma}},A_{|\gamma}).$$

We have

$$\begin{aligned}
&\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}})\int_{\overline{\mathcal{A}}_{\gamma}} d\mu_{0\gamma}(A_{|\gamma})\int_{\overline{\mathcal{A}}_{\gamma}} d\mu_{0\gamma}(A'_{|\gamma})\;\delta_\gamma(A,A')F(A_{|\overline{\gamma}},A'_{|\overline{\gamma}},A_{|\gamma},A'_{|\gamma})\\
&=\int_{\overline{\mathcal{A}}_{\gamma}} d\mu_{0\gamma}(A_{|\gamma})\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}})F(A_{|\overline{\gamma}},A'_{|\overline{\gamma}},A_{|\gamma},A_{|\gamma})
\end{aligned}$$

First simply estimate we can make is

$$\begin{aligned}
|(M_{\psi_1,\psi_1'},M_{\psi_2,\psi_2'})_\gamma| \;&\leq\; \int_{D_\gamma} d\mu(t_\gamma)\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\\
&\times\;\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\;\big|\Psi_1(A_{|\overline{\gamma}},A_\gamma)\big|\;\big|[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}},A_\gamma)\big|\\
&\times\;\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}})\;\big|\Psi_2'(A'_{|\overline{\gamma}},A_\gamma)\big|\;\big|[W_\gamma(t_\gamma)\Psi_2](A'_{|\overline{\gamma}},A_\gamma)\big|
\end{aligned}\tag{3.80}$$

Then by the Cauchy-Schwartz inequality

$$\begin{aligned}
&\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\;\big|\Psi_1(A_{|\overline{\gamma}},A_\gamma)\big|\;\big|[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}},A_\gamma)\big|\\
&\leq\Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\big|\Psi_1(A_{|\overline{\gamma}},A_\gamma)\big|^2\Big)^{1/2}\Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\big|[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}},A_\gamma)\big|^2\Big)^{1/2}
\end{aligned}$$

We expand Ψ_1' in terms of spin-network functions

$$\Psi_1'(A_{|\overline{\gamma}},A_\gamma)=\sum_{n=1}^{\infty}T_{s_n}(A_{|\overline{\gamma}},A_\gamma).$$

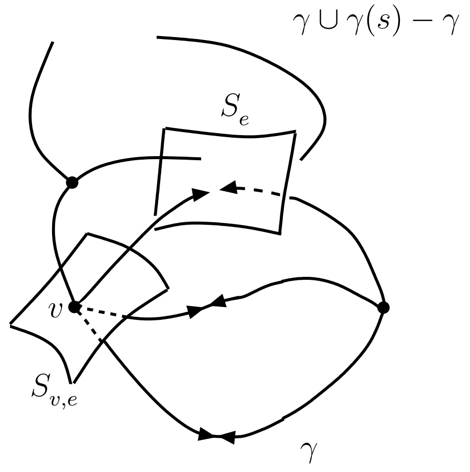


Figure 3.37: Action of $Y_\gamma(t_\gamma)$ on T_s. T_s is composed of degrees of freedom $A_{|\gamma}$ of A related to the edges $E(\gamma)$ and degrees of freedom $A_{|\overline{\gamma}}$, the complement of $A_{|\gamma}$, related to the edges $E(\gamma \cup \gamma(s) - \gamma)$.

This is where our choice of flux vector field $Y_\gamma(t_\gamma)$ simplifies the rest of the calculation.

Our conctrete vector field $Y_\gamma(t_\gamma)$ involves a finite collection of surfaces to which edges $e \in E(\gamma)$ are already adapted to. From fig (3.37) it is easy to see that the action of $Y_\gamma(t_\gamma)$ on T_s is given by

$$Y_\gamma(t_\gamma) T_s = [\sum_{e' \in E(\gamma \cup \gamma(s)) - E(\gamma)} t_j^{e'}(t_\gamma) R^j_{e'} + \sum_{e \in E(\gamma)} t^e_j R^j_e] F.$$

where $T_j^{e'}(t_\gamma)$ is a certain linear combination of the t^e_j depending on e' and the concrete sutfaces $S_e, S_{v,e}$ used in the construction of $Y_\gamma(t_\gamma)$.

The right action R^j_e on T_s is easily computed

$$\begin{aligned}
R^j_e T_s &= [\tau^j \pi(h_e)]_{mn} \frac{\partial T_s}{\partial [\pi(h_e)]_{mn}} \\
&= \sqrt{d_{\pi_{e_1}}} [\pi_{e_1}(h_{e_1})]_{m_{e_1} n_{e_1}} \ldots [\tau_j]_{m_e l_e} \sqrt{d_{\pi_e}} [\pi_e(h_e)]_{l_e n_e} \ldots \sqrt{d_{\pi_{e_N}}} [\pi_{e_N}(h_{e_N})]_{m_{e_N} n_{e_N}}
\end{aligned}$$

$$\begin{aligned}
W_\gamma(t_\gamma) \Psi'_1(A_{|\overline{\gamma}}, A_\gamma) &= \sum_{n=1}^{\infty} W_\gamma(t_\gamma) T_{s_n}(A_{|\overline{\gamma}}, A_\gamma) \\
&= \sum_{n=1}^{\infty} F_{s_n}(\{e^{t_j^{e'}(t_\gamma)\tau_j} h_{e'}\}_{e' \in E(\gamma) \cup E(\gamma(s)) - E(\gamma)}, \{e^{t^e_j \tau_j} h_e\}_{e \in E(\gamma)})
\end{aligned}$$

The integral

$$\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\big|[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}}, A_\gamma)\big|^2$$

can be written as a countable linear combination of integrals over spin-network functions T_s and then the prescription is to integrate over the degrees of freedom $A(e'), e' \in E(\gamma(s) \cup \gamma) - E(\gamma)$ for each individual integral with the corresponding product Haar measure. By the left invariance of the Haar measure

$$\begin{aligned}
& \int_G d\mu_H(h_{e'})\overline{T_{s_m}(\{e^{t_j^{e'}(t_\gamma)\tau_j}h_{e'}\},\{e^{t_j^e\tau_j}h_e\})}T_{s_n}(\{e^{t_j^{e'}(t_\gamma)\tau_j}h_{e'}\},\{e^{t_j^e\tau_j}h_e\}) \\
= & \int_G d\mu_H(h_{e'})\overline{T_{s_m}(\{h_{e'}\},\{e^{t_j^e\tau_j}h_e\})}T_{s_n}\{(h_{e'}\},\{e^{t_j^e\tau_j}h_e\}).
\end{aligned}$$

This allows us to make the replacement

$$[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}}, A_\gamma) \to \Psi_1'(A_{|\overline{\gamma}}, \{e^{t_j^e\tau_j}h_e\}_{e\in E(\gamma)})$$

in the integral.

Introduce the notation

$$\Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\big|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)\big|^2\Big)^{1/2} = \|\Psi_1(A_\gamma)\|_{\overline{\gamma}}.$$

In (3.80) we introduce new integration variables $h_e' := g(t_e)h_e$ where $g(t_e) = \exp(t_j^e\tau_j)$. Since by defintion

$$d\mu(t_\gamma) = \prod_{e\in E(\gamma)} d\mu(t_e) = \prod_{e\in E(\gamma)} d\mu_H(g(t_e))$$

$$\begin{aligned}
|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \;\le\; & \int_{D_\gamma} d\mu(t_\gamma)\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\;\|\Psi_1(A_\gamma)\|_{\overline{\gamma}}\;\|\Psi_1'(\{g_eA(e)\}_{e\in E(\gamma)})\|_{\overline{\gamma}} \\
& \times\|\Psi_2(A_\gamma)\|_{\overline{\gamma}}\;\|\Psi_2'(\{g_eA(e)\}_{e\in E(\gamma)})\|_{\overline{\gamma}} \\
= \;& [\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\;\|\Psi_1(A_\gamma)\|_{\overline{\gamma}}\;\|\Psi_2(A_\gamma)\|_{\overline{\gamma}}] \\
& \times[\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}')\;\;\|\Psi_1'(A_\gamma')\|_{\overline{\gamma}}\;\;\|\Psi_2'(A_\gamma')\|_{\overline{\gamma}}] \\
\le \;& \|\;\|\Psi_1\|_{\overline{\gamma}}\;\|_\gamma\;\;\|\;\|\Psi_1'\|_{\overline{\gamma}}\;\|_\gamma\;\;\|\;\|\Psi_2\|_{\overline{\gamma}}\;\|_\gamma\;\;\|\;\|\Psi_2'\|_{\overline{\gamma}}\;\|_\gamma
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality again. Noting

$$\begin{aligned}
\| \, \|\Psi_1\|_{\overline{\gamma}} \, \|_\gamma &= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \, \left| \, \Psi_1(A_{|\gamma}) \, \right|^2 \\
&= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) |\Psi_1(A_{|\overline{\gamma}}, A_{|\gamma})|^2 = \int_{\overline{\mathcal{A}}} d\mu_0(A) |\psi_1(A)|^2 \\
&= \|\psi_1\|^2_{\mathcal{H}_0}
\end{aligned} \tag{3.65}$$

completes the proof of the inequality i). The proof of ii) is simply as the integrals over $\overline{A}_{|\overline{\gamma}}$ are trivial.

With these results established the rest of the proof is straightforward. Suppose the representation π_0 is not irreducible, that is, not every vector is cyclic. Thus we can find non-zero vectors $\psi, \psi' \in \mathcal{H}_0$ such that

$$< \psi, a\psi' > = 0 \tag{3.65}$$

for all $a \in \mathfrak{M}$.

Since the cylindrical functions are dense in $\mathcal{H}_0$, for any $\epsilon > 0$ we can find a graph γ and functions f, f' cylindrical over γ such that

$$\|\psi - f\| < \epsilon, \quad \|\psi' - f'\| < \epsilon. \tag{3.65}$$

$$\begin{aligned}
(M_{\psi-f,\psi'}, M_{\psi,\psi'}) &= (< \psi, \psi >_{\mathcal{H}_0} - < \psi, f >_{\mathcal{H}_0}) < \psi', \psi' >_{\mathcal{H}_0} \\
(M_{\psi,\psi'-f'}, M_{\psi,\psi'}) &= < \psi, f >_{\mathcal{H}_0} (< \psi', \psi' >_{\mathcal{H}_0} - < f', \psi' >_{\mathcal{H}_0}) \\
(M_{f,f'}, M_{\psi-f,\psi'}) &= (< \psi, f >_{\mathcal{H}_0} - < f, f >_{\mathcal{H}_0}) < f', \psi' >_{\mathcal{H}_0} \\
(M_{f,f'}, M_{f,\psi'-f'}) &= < f, f >_{\mathcal{H}_0} (< f', \psi' >_{\mathcal{H}_0} - < f', f' >_{\mathcal{H}_0}) \\
(M_{f,f'}, M_{f,f'}) &= < f, f >_{\mathcal{H}_0} < f', f' >_{\mathcal{H}_0}
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
0 &= (M_{\psi,\psi'}, M_{\psi,\psi'})_\gamma \qquad (3.63) \\
&= (M_{\psi-f,\psi'}, M_{\psi,\psi'})_\gamma + (M_{f,\psi'-f'}, M_{\psi,\psi'})_\gamma + (M_{f,f'}, M_{\psi-f,\psi'})_\gamma + (M_{f,f'}, M_{\psi,\psi'-f'})_\gamma \\
&\quad + \|f\|^2 \, \|f'\|^2
\end{aligned}$$

Obviously

$$\|f\|^2 \, \|f'\|^2 \leq |(M_{\psi-f,\psi'}, M_{\psi,\psi'})| + \cdots + |(M_{f,f'}, M_{f,\psi'-f'})|$$

Recall for any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_0$ we have

$$|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})| \leq \|\psi_1\| \, \|\psi_1'\| \, \|\psi_2\| \, \|\psi_2'\|.$$

hence

$$\begin{aligned} \|f\|^2 \, \|f'\|^2 \leq & \|\psi - f\| \, \|\psi'\| \, \|\psi\| \, \|\psi'\| + \|f\| \, \|\psi' - f'\| \, \|\psi\| \, \|\psi'\| \\ & + \|f\| \, \|f'\| \|\psi - f\| \, \|\psi'\| + \|f\| \, \|f'\| \, \|f\| \, \|\psi' - f'\| \end{aligned}$$

therefore

$$\begin{aligned} (\|\psi\| - \epsilon)^2 (\|\psi'\| - \epsilon)^2 \leq & \epsilon \big\{ \|\psi'\|^2 \, \|\psi\| + (\|\psi\| + \epsilon) \|\psi\| \, \|\psi'\| \\ & + (\|\psi\| + \epsilon) \, (\|\psi'\| + \epsilon) \, \|\psi'\| + (\|\psi\| + \epsilon)^2 \, (\|\psi'\| + \epsilon) \big\} \end{aligned}$$

Since this inequality holds for all ϵ we can take $\epsilon \to 0$ and find

$$\|\psi'\|^2 \, \|\psi\|^2 = 0 \tag{3.57}$$

that is, either $\psi = 0$ or $\psi' = 0$ in contradiction to our assumption. Hence π_0 is irreducible.

3.13.5 Alternative Uniqueness Theorem of Fleischhack

Regular: We wish to require that the Weyl algebra is represented weakly continuously. States whose GNS representation have this property are said to be regular.

Stone-von Neuman theorem says that if a representation is regular and irreducible then the representation is unique.

Quantum geometry:

1. diffeomorphism invariant;

2. regular;

3. irreducible;

4. semianalytic - stratified diffeomorphisms.

3.14 Summary:

· Significant problem in majour technical issues that plagued the formalism (and related ones) from the beginning.

· Spin networks provide an elegant and powerful calculational tool.

· Discreteness of areas and volumes.

· Well-defined functional intergation via the cyclindrical measure theory.

· Well-defined and well understood Kinematic Hilbert space.

· Solving the spacial diffeomorphism constraint by the group averaging technique.

· Uniqueness Theorem for the Ashtekar-Isham-Lewendowski representation of the holonomy-flux algebra.

Chapter 4

Dynamics: The Hamiltonian constraint and Spin foams

· **Real Formalism and the Hamiltonian constraint.**

· **Quantization of the Hamiltonian constraint.**

· **Spin foams - the quantum geometry of spacetime.**

· **Unsettled concerns.**

4.1 Introduction

The task before us is:

Construct a sequence of regularized operators, $\mathcal{H}^\epsilon(x)$ to represent the Hamiltonian constraints, in H_{kin}. Prove that the limit as $\epsilon \to 0$ takes diffeomorphism invariant states to diffeomorphism invariant states, and thus defines a finite operator in H_{diff}. Prove that the limit has a kernel in H_{Diff} that is infinite dimensional. This kernel $H_{Physical} \subset H_{Diff}$ is the physical Hilbert space.

$$\hat{S}_L \equiv \hat{W}\hat{S}_E\hat{W} \tag{4.1}$$
$$\mathcal{H}^{Mink}_{phys} \equiv \hat{W}\mathcal{H}^{Eucl}_{phys} \tag{4.2}$$

With the introduction of a generalized Wick transform to map the constraint equations of Riemannian general relativity to those of the Lorentzian theory. This opens up the possibility within "connection-dynamics" where one can work, throughout, only with real variables. The resulting quantum theory would then be free of complicated reality conditions.

At the quantum mechanical level, we need to impose extra conditions to ensure that at the end of the day we are dealing with real general relativity. One possibility that was suggested was to request that observables (perennials) of the theory be real. At a quantum mechanical level this means they should be self-adjoint operators with respect to the inner product one selects. In particular, this is used as a selection criterion for the inner product. We will also see that it is harder to handle theories with non-compact groups as complex $SO(3)$

We discussed in the first lecture how one could write the Hamiltonian constraint and suprisingly find some solutions. The results were unregulated and formal. Part of the problem stemmed from the fact that we were looking at the Hamiltonian constraint in its double-densitized form,

$$\tilde{\tilde{H}} = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} \tag{4.2}$$

Suppose one wants to promote this to an operator in the loop representation. We have at hand a maifold and loops. How could one build a density of weight two? The answer is: one can't, without the aid of external structures. Most regularizations of the "early years" did exactly that.

Couldn't we consider the single-densitized Hamiltonian? Then one could represent it as a Dirac delta, which is defined intrinsically.

The reason that stopped people from trying this is the complicated, non-polynomial form of the single-densitized constraint in terms of Ashtekar's new variables,

$$H = \frac{\epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j}{\sqrt{\epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k}} F^k_{ab} \tag{4.2}$$

4.2 Dynamics: The Hamiltonian constraint

In geometrodynamics la Wheeler and DeWitt, the basic canonically conjugate variables were the 3-metric and extrinsic curvature . The idea was to quantize these, making them into operators acting on wavefunctions on the space of 3-metrics, and then to quantize the Hamiltonian and diffeomorphism constraints and seek wavefunctions annihilated bythese quantized constraints. However, this program soon became regarded as dauntingly difficult for various reasons, one being the non-polynomial nature of the Hamiltonian constraint:

$$\mathcal{H} = \sqrt{\det q}(K_{ab}K^{ab} - (K_a^a)^2 - {}^3R), \tag{4.2}$$

where ${}^3R$ is the scalar curvature of the 3-metric. It is often difficult to quantize non-polynomial expressions in the canonically conjugate variables and their derivatives. The factor of $(\det q)^{1/2}$ is not even an entire function of the 3-metric!

In the 1980's Ashtekar found a new formulation of general relativity in which the canonically conjugate variables are a densitized complex triad field $\tilde{E}_i^a$ and a chiral spin connection A_a^i. When all the constraints are satisfied, these are related to the original geometrodynamical variables by

$$(\det q)q^{ab} = \delta^{ij}\tilde{E}_i^a\tilde{E}_j^b, \quad A_a^i = \Gamma_a^i - iK_a^i, \tag{4.2}$$

where Γ_a^i is built from the Levi-Civita connection of the 3-metric and K_a^i is built from the extrinsic curvature. In terms of these new variables the Hamiltonian constraint appears polynomial in form, reviving hopes for canonical quantum gravity.

Actually, in this formulation one works with the densitized Hamiltonian constraint, given by

$$\tilde{\mathcal{H}} = \epsilon^{ijk}F_{abi}E_j^aE_k^b = \mathrm{tr}(F_{ab}[E^a, E^b]) \tag{4.2}$$

where F_{abi} is the curvature of A_a^i, and the trace and commutator are interpreted by thinking of as indices. Clearly $\tilde{\mathcal{H}}$ is a polynomial in A_a^i, E_i^a, and their derivatives. However, it is related to the original Hamiltonian constraint by $\tilde{\tilde{\mathcal{H}}} = (\det q)^{1/2}\tilde{\mathcal{H}}$ so in a sense the original problem has been displaced rather than addressed. It took a while, but it was eventually seen that many of the problems with quantizing $\tilde{\tilde{\mathcal{H}}}$ can be traced to this fact (or technically speaking, the fact that it has density weight 2).

A more immediately evident problem was that because E_i^a is complex-valued, the corresponding 3-metric is alsocomplex-valued unless one imposes extra 'reality conditions'. The reality conditions are easy to deal with in the Riemannian theory, where the signature of spacetime is taken to be . There one can handle them by working with a real densitized triad field and an connection given not by the above formula but by

In the physically important Lorentzian theory, however, no such easy remedy is available.

Despite these problems, the enthusiasm generated by the new variables led to a burst of work on canonical quantum gravity.Many new ideas were developed, most prominently the loop representation. In the Riemannian theory, this allows one to rigorously construct a Hilbert space of wavefunctions on the space of connections on space. The idea is to work with graphs embedded in space, and for each such graph to define a Hilbert space of wavefunctions depending only on the holonomies of the connection along the edges of the graph. Concretely, if the graph has edges, the holonomies along its are summarized by a point in , and the Hilbert space we get is , defined using Haar measure on. If the graph is contained in a larger graph then is contained in and one has . We can thus form the union of all these Hilbert spaces and complete it to obtain the desired Hilbert space.

One can show that has a basis of 'spin networks', given by graphs with edges labeled by representations of –i.e., spins – and vertices labeled by vectors in the tensor product of the representations labeling the incident edges. One can also rigorously quantize geometrically interesting observables such as the total volume of space, obtaining operators on . The matrix elements of these operators can be explicitly computed in the spin network basis.

Thiemann's approach applies this machinery developed for the Riemannian theory to Lorentzian gravity by exploiting theinterplay between the Riemannian and Lorentzian theories. He takes as his canonically conjugate variables an connection and a real densitized triad field , and takes as his Hilbert space as defined above. This automatically deals with the reality conditions, as in the Riemannian case. Then he writes the Lorentzian Hamiltonian constraint in terms of these variables, and quantizes it to obtain a densely defined operator on – modulo some subtleties we discuss below. Interestingly, it is crucial to his approach that he quantizes the Hamiltonian constraint rather than the densitized Hamiltonian constraint . This avoids the regularization problems that plagued attempts to quantize .

He writes the Lorentzian Hamiltonian constraint in terms of and in a clever way, as follows. First he notes that

$$\mathcal{H} = -\mathcal{H}_R + \frac{2}{\sqrt{\det q}} \mathrm{Tr}([K_a, K_b][E^a, E^b]) \tag{4.2}$$

where the commutators and trace are taken in $su(2)$, and is the Riemannian Hamiltonian constraint, given by

Then he notes that

where

is the total volume of space (which is assumed compact). This observation lets him get rid of the terrifying factors of . Similarly, he notes that

where

Thus he obtains

where

Finally, he eliminates from the formula for using the formula

If we use the standard trick of replacing Poisson brackets by commutators, these formulas reduce the problem of quantizing to the problem of quantizing , , and . As noted, the volume has already been successfully quantized, and the resulting 'volume operator' is known quite explicitly. This leaves the connection and curvature.

Now, a fundamental fact about the loop representation – at least as currently formulated – is that the connection and curvature do not correspond to well-defined operators on , even if one smears them with test functions in the usual way. Instead, one has operators corresponding to the holonomy along paths in space. The holonomy along an open path can be used to define a kind of substitute for , and the holonomy around an open loop to define a substitute for . One cannot, however, take the limit as the path or loop shrinks to zero length. Thus the best one can do when quantizing a polynomial in and is to choose some paths or loops and use the substitutes built from holonomies. This eliminates problems associated with multiplying operator-valued distributions, but it introduces another kind of ambiguity: dependence on the arbitrary choice of path or loop.

So, ironically, while the factors of in the Hamiltonian constraint are essential in Thiemann's approach, the polynomial expressions in and introduce problematic ambiguities! Accepting but carefully minimizing this ambiguity, Thiemann obtains for any lapse function a large family of different versions of the smeared Hamiltonian constraint operator. The ambiguity is such that two different versions acting on a spin network give spin networks differing only by a diffeomorphism of space. Mathematically speaking we may describe this as follows. Let be the space of finite linear combinations of spin networks, and let be the space of finite linear combinations of spin networks modulo diffeomorphisms. Then Thiemann obtains, for any choice of lapse function, a Hamiltonian constraint operator

independent of the arbitrary choices he needed in his construction.

Since these operators do not map a space to itself we cannot ask whether they satisfy the naively expected commutation relations, the 'Dirac algebra'. However, this should come as no surprise, since the Dirac algebra also involves other operator-valued distributions that are ill-defined in the loop representation, such as . Thiemann does check as far as possible that the consequences one would expect from the Dirac algebra really do hold. Thus if one is troubled by how arbitrary choices of paths and loops prevent one from achieving a representation of the Dirac algebra, one is really troubled by the assumption, built into the loop representation, that , , and are not well-defined operator-valued distributions. Ultimately, the validity of this assumption can only be known through its implications for physics. The great virtue of Thiemann's work is that it brings us closer to figuring out these implications.

Remarkably, Thiemann (1996) discovered the identity,

$$\frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} = 2\epsilon^{abc}\{A^k_c, V\} \tag{4.2}$$

Which alows us to write the single densitized Hamiltonian as,

$$\tilde{H} = -2Tr(F_{ab}\{A_c, V\})\epsilon^{abc} \tag{4.2}$$

This is not only remarkably simple, but Thiemann found something else as well...

Recall that when Ashtekar introduced the new variables, there was a free parameter in the canonical transformation (the Immirzi β parameter) and if one did not choose it equal to the imaginary unit the Hamiltonian constraint looked like,

$$\begin{aligned} \tilde{\tilde{S}} &\equiv -\zeta\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j F_{abk} + \frac{2(\beta^2\zeta - 1)}{\beta^2}\tilde{E}^a_{[i}\tilde{E}^b_{j]}(A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b) \\ &= \mathcal{H}_E(N) - 2(1+\gamma^2)\mathcal{T}(N) = 0 \end{aligned} \tag{4.2}$$

Thiemann found that the ugly looking second piece (divided by det(q)) can be written as,

$$\mathcal{T}(N) = 4\epsilon^{abc}Tr(\{A_a, K\}\{A_b, K\}\{A_c, V\}) \tag{4.2}$$

where,

$$K = -\{\frac{V}{\kappa}, \int_\Sigma d^3x H^E(x)\} \tag{4.2}$$

That is, not only is the "Euclidean" piece of the Hamiltonian constraint (single densitized) easy. The full Hamiltonian constraint without using complex variables is reasonably simple too!

4.3 Regularization of the Hamiltonian Constraint

$$[\hat{C}(N)'\Psi](f) := \Psi(\hat{C}^\dagger(N)f) \tag{4.2}$$

does not preserve $(\Phi^*_{Kin})_{Diff}$

Simple Regularization

$$\mathcal{C}^{Eucd}(N) = -\frac{2N(v_\Box)}{k^2\gamma^{3/2}}\sum_i Tr\Big((\overline{A}(\beta_i) - \overline{A}(\beta_i^{-1}))\overline{A}(s_i)^{-1}\{\overline{A}(s_i), V\}\Big) \tag{4.2}$$

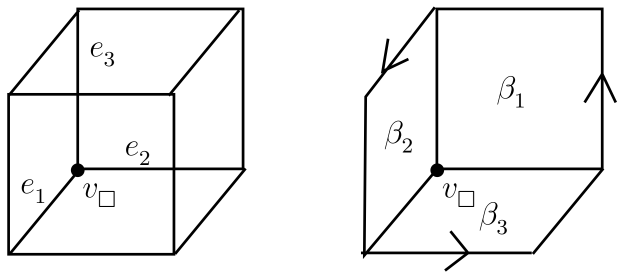


Figure 4.1: regfigHam1

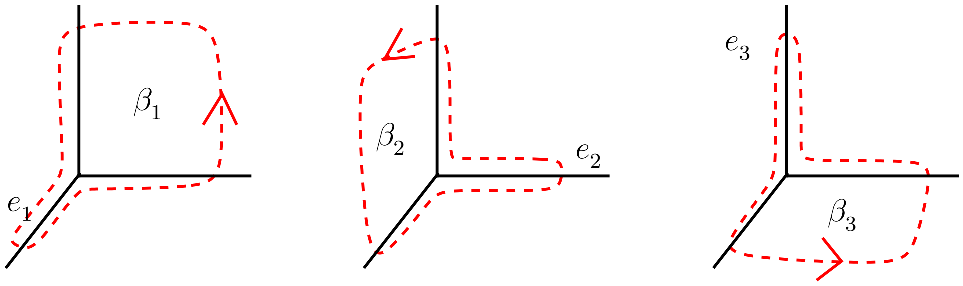


Figure 4.2: regfigHam2

State-Dependant Triangularization

Thiemann proceeds to quantize this expression. He starts by discretizing it on a lattice, obtained by triangulating the spatial manifold in terms of tetrahedra,

$$\tilde{H} = -2Tr(F_{ab}\{A_c, V\})\epsilon^{abc} \tag{4.2}$$

$$H^E_\Delta[N] = -\frac{2}{3}N_v\epsilon^{ijk}tr(h_{\alpha_{ij}}h_{s_k}(\Delta)\{h^{-1}_{s_k}(\Delta), V\}) \tag{4.2}$$

When $\Delta \to 0$, $h_{e_{ij}} = 1 + F_{ij}s^i s^j \tilde{E}^2$; $\quad h_{s_k} = 1 + A_k s^k \tilde{E}$.

So the expression tends (pointwise) to the above one (times Δ^3) when the triangulation is shrunk. Moreover it is manifestly gauge invariant since loops are all closed in the end. Therefore,

$$H^E_T[N] = \sum_{\Delta T} H^E_\Delta[N] \tag{4.2}$$

Is a good approximation to the "smeared" classical Hamiltonian.

Since both the holonomy and the volume operator are well defined quantities in the space of cylindrial functions of spin networks, it is immediate to promote H to a quantum operator.

via a state-dependent triangulation T on Σ??

$$\hat{H}^E_\Delta[N] = -2\frac{N(v(\Delta))}{3il_p^2}\epsilon^{ijk}tr(h_{\alpha_{ij}}h_{s_k}(\Delta)[h^{-1}_{s_k}(\Delta),\hat{V}]) := N_v\hat{H}^E_\Delta \tag{4.2}$$

clearer?:

$$\hat{H}_\tau[N] = \frac{1}{\kappa}\sum_{\Delta\in\tau}N(p_\Delta)\epsilon^{IJK}Tr((A(\alpha_{IJ}(\triangle))A_{s_K}(\Delta)[A_{s_K}(\Delta)^{-1},\hat{V}(R_{p(\Delta)}]) \tag{4.2}$$

Here $T(\gamma, v)$ is the number of ordered triples of edges incident at v (taken with outgoing orientation), see fig. (**??**). This is

$$T(\gamma, v) = \frac{n_v(n_v-1)(n_v-2)}{6}$$

where n_v is the valence of vertex v. Proof. First assume that the three vectors e_1, e_2, e_3 and the 'empty edges' don't come in any particular order - there are $n_v!$ ways of arranging these. Now, we are only interseted in only one particular ordering of the vectors e_1, e_2, e_3 so we must divide by 3!. Similarly, we must divide by $(n_v - 3)!$ because the 'empty edges' have only one ordering themselves.

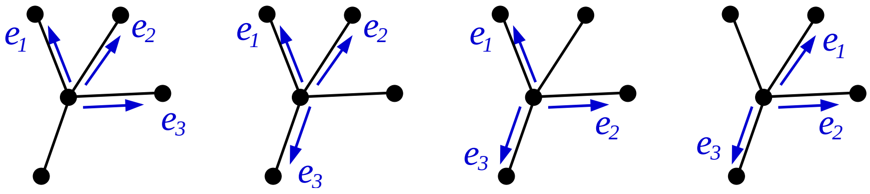


Figure 4.3: NumTripF. .

It is at this point where we must regularise (??). We consider a triangulation τ of Σ by tetrahedra $\triangle$. For each $\triangle$, let us single out a corner$p(\triangle)$ and denote the edges of $\triangle$ outgoing from $p(\triangle)$ by $s_I(\triangle)$, $I = 1, 2, 3$.

$$\int_\Sigma = [\int_{\Sigma - W^\epsilon_\gamma}] + \sum\frac{1}{T(\gamma, v)}\sum_{e_1\cap e_2\cap e_3 = v}\{[\int_{W^\epsilon_\gamma - W^\epsilon_\gamma(e_1,e_2,e_3)}] + [\int_{W^\epsilon_\gamma(e_1,e_2,e_3)}]\} \tag{4.2}$$

tetSaturate1ChD

4.4 Details of Thiemann's Hamiltonian Constraint

The first task here is to rewrite Thiemann's Hamiltonian (Eq.) in terms of the Poisson braket the connection and the volume and with the function K (see Eq.). This task will be broken down into managible parts.

$$\tilde{H} := \frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}}F_{abk} + \frac{2}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}}\tilde{E}^a_{[i}\tilde{E}^b_{j]}(A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b) \tag{4.2}$$

$$\tilde{H} = -2Tr(F_{ab}\{A^i_c(x), V\})\epsilon^{abc} + 4\epsilon^{abc}Tr(\{A^i_a(x), K\}\{A^j_b(x), K\}\{A^k_c(x), V\}) \tag{4.2}$$

First we need the important identity,

Detailed calculation (I1.1)——————————————————————————

$$2\epsilon^{abc}\{A^k_c, V\} = \frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} \tag{4.2}$$

where

$$V[\sigma]_\epsilon = \int_\sigma \sqrt{g} = \int \sqrt{\epsilon_{abc}\epsilon_{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k} \tag{4.2}$$

Proof:

$$\{A^b_j(x), V\} = \frac{\delta A^b_j(y)}{\delta A^a_i(x)}\frac{\delta V}{\delta\tilde{E}^a_i(x)} + 0 \tag{4.2}$$

the second part of the Poisson bracket is zero because $\delta A^b_j/\delta\tilde{E}^b_j = 0$ as they are independent variables.

$$\frac{\delta V}{\delta\tilde{E}^b_j} = \frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} \tag{4.2}$$

$$\{A^k_c, V\} = \frac{\epsilon_{abc}\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} \tag{4.2}$$

We now use

$$\epsilon_{abe}\epsilon^{cde} = \delta_a^c \delta_b^d - \delta_b^c \delta_a^d \tag{4.2}$$

to get,

$$2\epsilon^{abc}\{A_c^k, V\} = \frac{\epsilon^{ijk}\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{\epsilon^{ijk}\tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c}} \tag{4.2}$$

$$\begin{aligned} e_a^j(x) &= -\frac{2}{\kappa\beta}\{A_a^j, V(R_x)\} \\ K_a^j(x) &= -\frac{1}{\kappa\beta}\{A_a^j, \{C_E(1), V\}\} \\ V(R_x) &= \int_\sigma \chi_{R_x}(y) d^3y \sqrt{det(q(x))}(y) \\ C_E(1) &= C_E(N)_{|N=1} \end{aligned} \tag{4.3}$$

Thiemann proceeds to quantize this expression. He starts by discretizing it on a lattice, obtained by triangulating the spatial manifold in terms of tetrahedra,

$$\tilde{H} = -2Tr(F_{ab}\{A_c, V\})\epsilon^{abc} \tag{4.0}$$

$$H^E[N] = \frac{2}{\kappa}\int_\Sigma d^3x N(x)\epsilon^{abc} Tr(F_{ab}\{A_c, V\}) \tag{4.0}$$

$$H_\Delta^E[N] = -\frac{2}{3}N_v \epsilon^{ijk} tr(h_{\alpha_{ij}} h_{s_k}(\Delta)\{h_{s_k}^{-1}(\Delta), V\}) \tag{4.0}$$

When $\Delta \to 0$, $h_{e_{ij}} = 1 + F_{ij}s^i s^j \tilde{E}^2$; $\quad h_{s_k} = 1 + A_k s^k \tilde{E}$.

So the expression tends (pointwise) to the above one (times Δ^3) when the triangulation is shrunk. Moreover it is manifestly gauge invariant since loops are all closed in the end. Therefore,

$$H_T^E[N] = \sum_{\Delta \in \mathcal{T}(\epsilon)} H_\Delta^E[N] \tag{4.0}$$

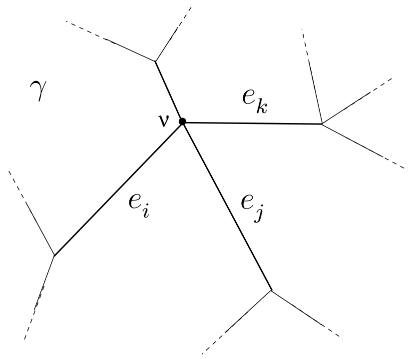


Figure 4.4: graphofnet. Given a graph γ we will construct one triangularization of the triple of edges meeting at the given vertex ν.

An elamentary tetrahedron tetrahedra shrink to their base points the expression (Eq. 4.4) converges (pointwise) to the expression (Eq. 4.4)

Is a good approximation to the "smeared" classical Hamiltonian.

In order to prepare the classsical expression for quantizaton, we triagulate space-like hypersurface Σ in terms of an elementary tetradra $\triangle$. The triangulation has the following properties: we select a finite set of distict points of Σ, denoted as $\{v\}$. At each of these points we choose three independent directions $(\hat{u}_1), (\hat{u}_2), (\hat{u}_3)$ and construct the eight tetrhedra with vertex v and edges $(\pm u_1), (\pm u_2), (\pm u_3)$, with $u_i = \epsilon \hat{u}_i$. The eight terahedra saturating v define a closed region of scale length ϵ.

The rest of the region $\Sigma-$is triangulated by arbitrary tetrahedra Σ'. The motivation for this particular discretizaton is the when we promote the classical expression to a quantum operator, we will adapt the triangularizaton to the spin network of the state in question by choosing the points v to coincide with the vertices of the spin network.

We define the regulated operators on different $\mathcal{H}'_\alpha$ separately, then remove the regulator ϵ so that the limit operator is defined on $\mathcal{H}_{kin}$ cylindrically consistenly.

then we average with respect to the triples of edges meeting at the given vertex.

For each vertex v of γ and each three distinct edges (e_I, e_J, e_K) incident to v, the number $C_{n(v)}$ of such distinct triples of edges is easily seen to be

$$C_{n(v)} = \frac{n(v)}{3!}(n(v)-1)(n(v)-2) \tag{4.0}$$

where $n(v)$ is the valence of the vertex.

• The graph γ in $T(\epsilon)$ for all , so that every vertex v of γ coincides with a vertex $v(\triangle)$ in $T(\epsilon)$.

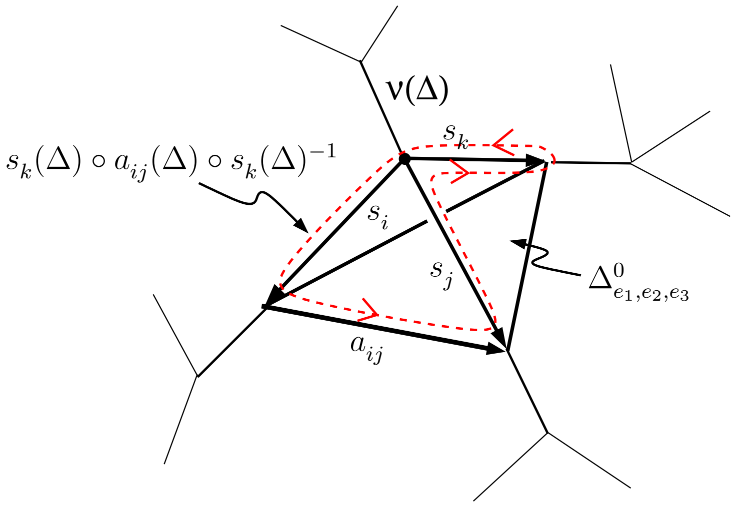


Figure 4.5: tetrahedron. An elementary tetrahedron and the choice of edges. Where $s_i(\Delta) \subset e_i$ for all $i = 1, 2, 3$. $a_{ij}(\Delta) := s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta)^{-1}$ is a "loop with a nose". It is clear that the regulated formular for $S(N)$, based on such paths, is invariant under internal gauge transformations.

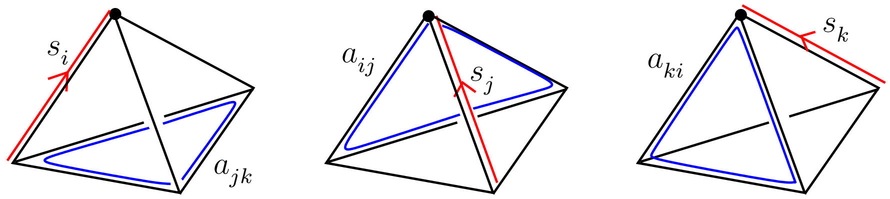


Figure 4.6: ThiemHam2. An elamentary tetrahedron and the edges.

- The triangulation must be fine enough so that the neighbourhoods $U(\nu) := \cup_{e_1,e_2,e_3} U_{e_1,e_2,e_3}(\nu)$ are disjoint for different vertices ν and ν' of γ. Thus for any open neigbourhood U_γ of the graph γ, there exists a triangulation $T(\epsilon)$ such that $\cup_{\nu \in V(\gamma)} U(\nu) \subseteq U_\gamma$.

- The distance between a vertex $\nu(\Delta)$ and the corresponding arcs α_{ij} is described by the parameter ϵ. For any two different ϵ and ϵ', the arcs $\alpha_{ij}(\Delta^\epsilon)$ and $\alpha_{ij}(\Delta^{\epsilon'})$ with respect to one vertex $\nu(\Delta)$ are semi-analytically diffeomorpic with each other .

- With the triangulations $T(\epsilon)$, the integral over Σ is replaced by the Riemanian sum:

$$\int_{U(v)} = \frac{1}{C^3_{n(v)}} \sum_{e_1,e_2,e_3} [\int_{U_{e_1,e_2,e_3}(v)} + \int_{U(v) - U_{e_1,e_2,e_3}(v)}] \tag{4.1}$$

where $n(v)$ is the valence of the vertex $v = s(e_1) = s(e_2) = s(e_3)$. Observe that

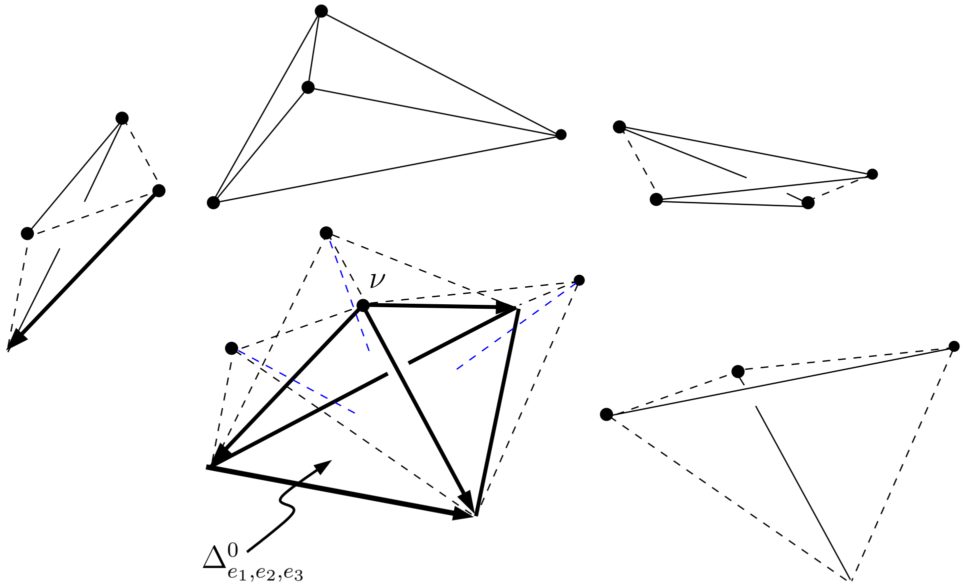


Figure 4.7: tetSaturate1. Visualizing the 7 additional tetrahedra $\Delta^{\mathfrak{p}}_{e_1,e_2,e_3}$, $\mathfrak{p}= 1, \ldots, 7$, which together with $\Delta^0_{e_1,e_2,e_3}$, form a neighbourhood, $U_{e_1,e_2,e_3} := \cup^7_{\mathfrak{p}=0}\Delta^{\mathfrak{p}}_{e_1,e_2,e_3}$, of ν.

$$\int_{U_{e_1,e_2,e_3}(v)} = 8 \int_{\Delta^0_{e_1,e_2,e_3}(v)} \tag{4.1}$$

in the limit $\epsilon \to 0$, that is when the tetrahedra shrink to there base points. As we will see because of this the dependency on the additional tetrahedra will drop out.

- Triangulation in the regions:

$$\begin{aligned} &U(v) - U_{e_1,e_2,e_3}(v) \\ &U_\gamma - \cup_{v\in V(\gamma)} U(v) \\ &\Sigma - U_\gamma \end{aligned} \tag{4.0}$$

are arbitrary. These regions do not contribute to the construction of the operator, since the commutator term $[A(s_i), V_{R_{v(\Delta)}}]\psi_\alpha$ vanishes for all tetrahedron Δ in the regions (4.0).

$$H(N) = \frac{2}{G} \lim \delta \to 0 \int d^3y \sum_{\in\{,\Delta\}} \tag{4.0}$$

$$\mathcal{V}\epsilon^{abc} = \frac{\alpha}{6}\epsilon^{ijk}u^a_i u^b_j u^c_k, \tag{4.0}$$

where$\alpha = 8$ if *square* $=$, and $\alpha = 1$ if *square* $= \Delta$. In this last case, the u_i' represent the edges of the tetrahedra Δ' adjacent to $v_{\Delta'}$. We also have adapted the regularizaton to the Dirac delta funcion to the tetrahedral decomposition by defining,

$$f_d(y) = \frac{\Omega_d(y)}{\mathcal{V}_d} \tag{4.0}$$

where $\Omega_d(y)$ is one if $y \in D$ and zero otherwise.

$$\hat{H}^E_{T(\gamma)}[N]f := \hat{H}^E_\gamma[N]f = \sum_{v\in V(\gamma)} N_v \frac{8}{C_{n(v)}} \sum_{v(\Delta)=v} \hat{H}^E_\Delta f =: \sum_{v\in V(\gamma)} N_v \hat{H}^E_v f \tag{4.0}$$

there is no state in $\mathcal{H}_{Kin}$ that could be interpreted as "$\lim_{\epsilon\to 0} \hat{H}[N,\epsilon]\Psi$"

We assume we are acting on a state given by a loop transform,

$$\Psi[s] = \int \mathcal{D}A\Psi[A]W_A[s], \tag{4.0}$$

$$H(N) = \lim D \to 0 \int d^3y \sum \frac{\alpha}{3G}\epsilon_{ijk}u^a_i u^b_j N(y) Tr[F_{ab}() h(u_k)\{h^{-1}(u_k), V\}] f_D(y), \tag{4.0}$$

and we realize the Hamiltonian operator over spin network wavefunctions by promoting the classiacal expression Eq.(4.4) as an operator acting on the Wilson net appearing in the loop transform.

$$\mathcal{H}^\Delta_T[N] := \frac{2}{3GC(m)} N(v(\Delta))\epsilon^{ijk}\mathrm{tr}\left[h_{\alpha_{ij}} h_{s_k}\{h^{-1}_{s_k(\Delta)}, V\}\right], \tag{4.0}$$

4.4.1 Quantization of the Regulated Constraint

In order to quantize this expression one now replaces all apearing quantities by operators and Poisson brackets by commutator divided by $i\hbar$.

$$\begin{aligned} &A(e) \mapsto \hat{A}(e), \quad V_R \mapsto \hat{V}_R, \quad \{\ ,\ \} \mapsto \frac{[\ ,\ ]}{i\hbar} \\ &K \mapsto \hat{K}^\epsilon = \frac{\gamma^{-2}}{i\hbar}[S^\epsilon_1(1), \hat{V}_\Sigma]. \end{aligned} \tag{4.0}$$

to arrive at an unambiguous result one had to make the triagulation *state dependent*. That is, the regulated operator is defined on a certain (so called spin-network??) basis elements T_s of the Hilbert space in terms of an adapted triangularization τ_s and extented by linearity. This is justified because the Riemann sum that enters the definition of $C_\tau(N)$ converges to $C(N)$ no matter how we refine the triangulation.

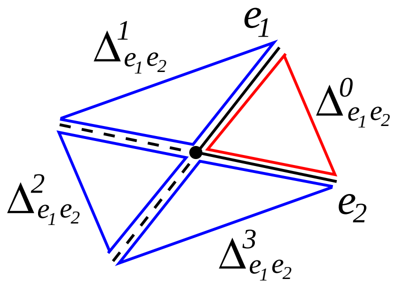


Figure 4.8: TriangHam. so that $U_{e_1,e_2}; = \cup_{\mathcal{P}=0}^{3} \triangle^{\mathcal{P}}_{e_1,e_2}$ is a neighborhood of v.

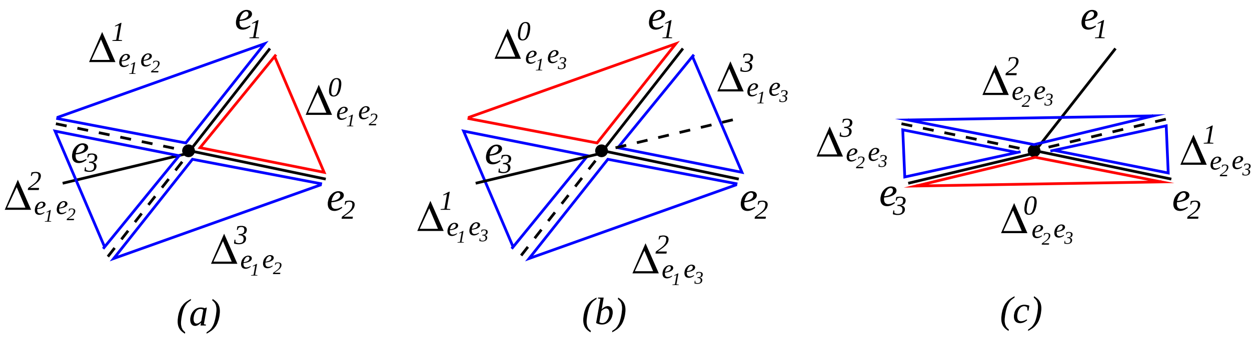


Figure 4.9: TriangHam2. trivalent node we average over the three triangles based on using only one such quadrupel of triangles.

4.4.2 Removal of the Regulator

The *weak operator topology* on $\mathcal{B}(H)$ is the weak topology generated by all functions of the form $T \to (Tx, y)$; that is, it is the weakest topology with respect to which all these functions are continuous. It is easy to see from the inequality $|(Tx, y) - (T_0 x, y)| \leq \|T - T_0\| \, \|x\| \, \|y\|$ that this topology is weaker than the usual norm topology, so that its closed sets are also closed in the usual sense.

(from intro topology and modern analysis)

The action of the Hamiltonian consraint operator on ψ_γ adds an arc with a $1/2$-representation with respect to each vertex $\nu(\Delta)$ of γ. - the action of $\hat{S}^\epsilon(N)$ on cylindrical functions is graph changing. Hence the operator does not converge with respect to the weak operator topology in $\mathcal{H}_{kin}$ when $\epsilon \to 0$, since different $\mathcal{H}'_{\alpha(\gamma)}$ with different graphs γ are mutually orthogonal. Thus on has to define a weaker operator topology to make the operator limit meaningful.

To take the infinite refinement limit (or continuum) $\tau \to \sigma$ of the resulting regulated operator $\hat{C}^\dagger_\tau(N)$ is non-trival because the holonomy operators are not even weakly continuous represented on the Hilbert space, hence the limit cannot exist in the weak oerator topology

The weak operator topology on $\mathcal{L}(X,Y)$ is the weakest topology such that the maps

$$E_{x,\ell} : \mathcal{L}(X,Y) \to \mathbf{C} \tag{4.0}$$

given by $E_{x,\ell}(T) = \ell(Tx)$ are all continuous for all $x \in X$, $\ell \in Y^*$. A basis at the origin is given by sets of the form

$$\{S|S \in \mathcal{L}(X,Y),\ |\ell_i(Tx_j)| < \epsilon,\ \ i = 1,\dots,n,\ \ j = 1,\dots,m\} \tag{4.0}$$

where $\{x_i\}_{i=1}^n$ and $\{x_j\}_{j=1}^m$ are finite families of elements of X and Y^* respectively.

It turns out that it exists in the, what one would cal the *weak Diff* topology* [32], [33]:

Let $\mathbf{\Phi}_{Kin}$ be a dense invariant domain for the closable operator $\hat{C}^\dagger_\tau(N)$ on the Hilbert space $\mathcal{H}_{Kin}$ and let $(\mathbf{\Phi}^*_{Kin})_{Diff}$ be the set of all spacially diffeomorphism invariant distributions over Φ_{Kin} (equipped with the topology of pointwise convergence - so algebraic dual of Φ_{Kin}?). Then $\lim_{\tau\sigma} \hat{C}_\tau(N) = \hat{C}_\tau(N)$ if and only if for each $\epsilon > 0$, T_s, $\ell \in (\Phi^*_{Kin})_{Diff}$ there exists $\tau_s(\epsilon)$ *independent of* ℓ such that

$$|\ell([\hat{C}_{\tau_s}(N) - \hat{C}(N)]T_s)| < \epsilon \ \text{ for all } \ \tau_s(\epsilon) \subset \tau_s \tag{4.0}$$

That the limit is uniform in ℓ is crucial because it excludes the existence of the limit on spaces larger than $(\Phi^*_{Kin})_{Diff}$ [] which would be unphysical because the space of physical solutions to all the constraints must obviously be a subspace of $(\Phi^*_{Kin})_{Diff}$. Notice that the limit is required refinements of adapted triangulations only.

Action of the regularised Hamiltonian

Given a spin network S, the operator (4.3) in two ways:

The graph is modified by the two operators $h_{\alpha_{ij}}$ and h_{s_k}. The first superposes a path of lenth ϵ to links of Γ. The second superposes a triangle two sides length ϵ alond links of Γ, and a third side that is not on Γ.

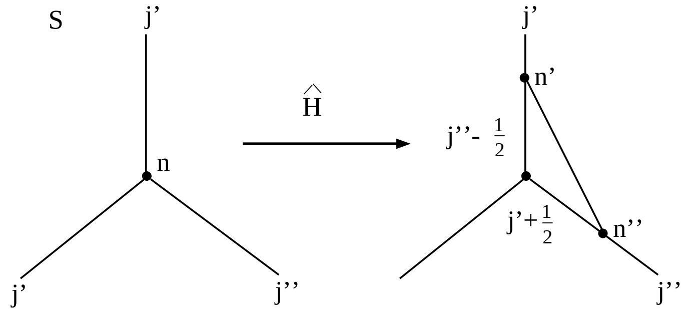


Figure 4.10: The action of the Hamiltonian constraint on a trivalent node.

The action of the Hamiltonian constraint on spin network states is the creation of a new edge of the underlying graph and rerouting of the quanta of angular momentum.

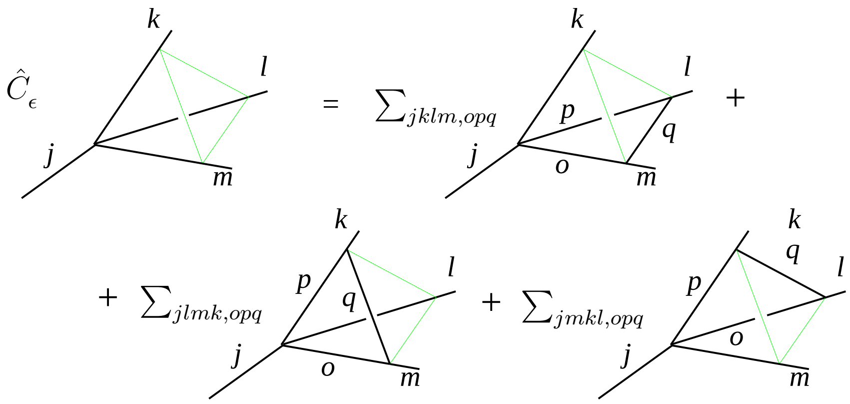


Figure 4.11: actionham2. The action of the Hamiltonian constraint on a node of valence 4.

$$(\phi|\ \hat{C}(N)\ |\psi >= \lim_{\epsilon \to 0} (\phi|\ \hat{C}_\epsilon(N)\ |\psi > \tag{4.0}$$

The operator is finite. It is well defined if we are acting on diffeomorphism invariant states. Otherwise we would have to specify where to add the “crossings” line of the loop α.

4.5 Concerns About the Hamiltonian Constraint

Quantum anomalies

There are complications. The classical condition for first class constraints,

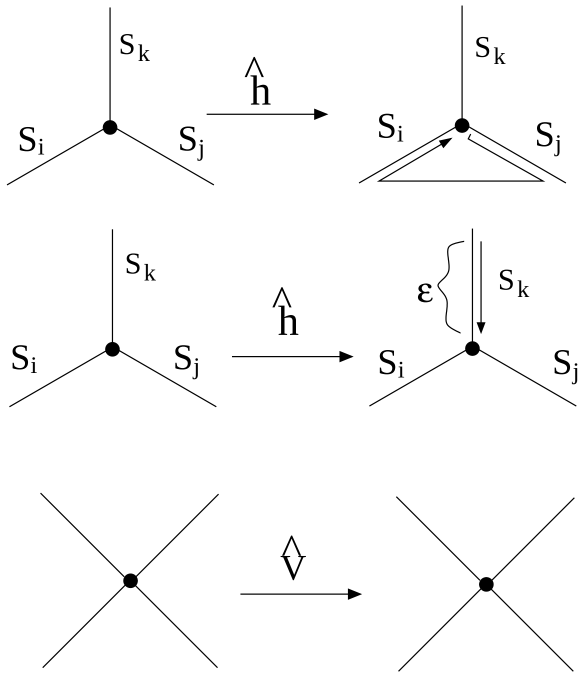


Figure 4.12: actionofham. The action of the Hamiltonian constraint on a trivalent node.

$$\{C_a, C_b\} = C^c_{ab} C_c, \tag{4.0}$$

could be spoiled by quantum corrections of order $\hbar^2$,

$$[\hat{C}_a, \hat{C}_b] = C^c_{ab} \hat{C}_c + \hbar^2 \hat{D}_{ab}. \tag{4.0}$$

In this case () would imply

$$\hat{D}_{ab} |\psi >= 0. \tag{4.0}$$

This could restrict the physical subspace.

Also, the classical relation

$$\{\mathcal{H}, C_a\} = V_a{}^b C_b \tag{4.0}$$

$$[\bar{\mathcal{H}}, \hat{C}_a] = V_a{}^b \hat{C}_b + \hbar^2 \hat{C}_a. \tag{4.0}$$

To apply the Dirac quantization method, one has to asume $\hat{D}_{ab} = \hat{C}_c = 0$

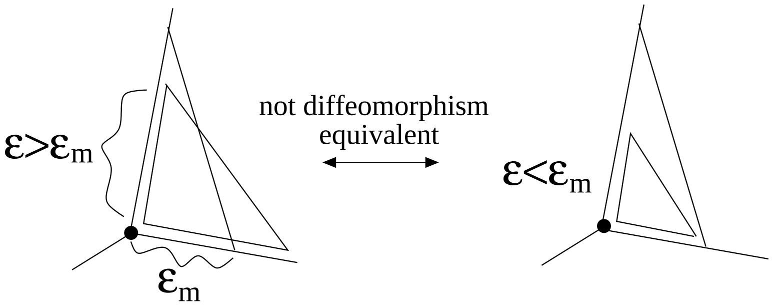


Figure 4.13: ϵ_{max}.

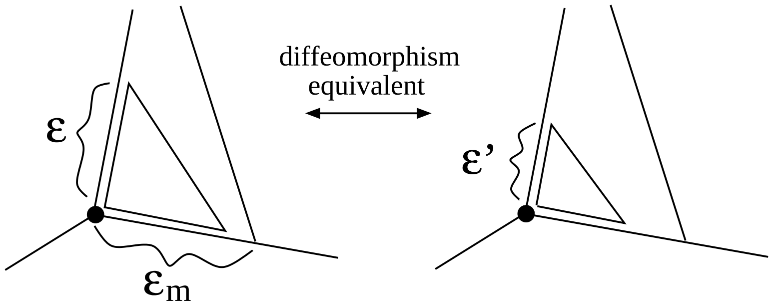


Figure 4.14: rescale. ϵ_{max}.

Due to this property, the operator is also anomaly-free. That is, the commutator of two Hamiltonians, vanishes, which agrees with the classical Poisson algebra on diffeomorphism invariant functions.

$$[\hat{H}(N), \hat{H}(M)] = 0 \quad \{\hat{H}(N), \hat{H}(M)\} = \int d^3x (N\partial_a M - M\partial_a N) g^{ab} C_b \tag{4.0}$$

If states have to solve additional constraints then the quantum theory would not have as many physical degrees of freedom as the classical theory. To ensure that this is so with the Hamiltonian constraint, we have to restrict the way it acts upon spinnetwork states: The way an operator acts on a state depends on the triangularisation prescription one is adhering to, the requirement for there to be no anomalies places a restriction on this triangularisatoin prescription.

Notice the crucial role that diffeomorphism invariance played in the construction. For ϵ and ϵ', sufficently small, $\hat{H}_\epsilon|\Psi>$ and $\hat{H}_{\epsilon'}|\Psi>$ are equivalent under diffeomorphisms. If the functions were not diffeomorphism invariant, the added line would have to be shrunk to the vertex and possible divergences could appear.

The same construction can be applied to the Hamiltonian of general relativity coupled to matter: scalar fields, Yang-Mills fields, fermions. In all cases the theory is finite, anomaly free and well defined. Gravity indeed appears to be acting as a fundemental regulator" of theories of matter.

A similar construction, applies in 2+1 dimensions, yeilds a theory of quantum gravity that correctly contains the physical states found by Witten through his quantization.

This technique can be applied, in particular, to the standard model coupled to gravity [110], [?]. In particular, it works for *Lorentzian* gravity while all other earlier proposals could at best work in the Euclidean context only (see [25] and references therein). The algebra of important operators of the resulting field theories was shown to be consistent [?]. Notice that as far as these operators are concerned, is stronger than the believed but unproven finiteness of scattering amplitudes order by order in perturbation theory of the five critical string theories, in a sense we claim the perturbation series converges. (Scatering amplitudes for background independent theories are conceptually very difficult and it is only recently that substantial progress has been made. How to define, interpret and calculate background independent scattering amplitudes are currently being developed [?].) The absence of the divergences that usually plague interacting field theories in a Minkowskian background spacetime can be understood intuitively from the diffeomorphism invariance of the theory "short and long distances are gauge equivalent".

What we therefore have a well defined theory of quantum gravity. Is this "THE" theory?

Von-Neumann's uniques theorem tells us that there is only one representation of the canonical commutation relations In the context in the field theories, on the other hand, the operator algebra admit infinely many inequivalent representations, that is, there may be For example, a ferromagnetic material is described by different spaces, $\mathcal{H}_E$, according to whether it is magnetized or not. The algebra is the same. These diferent representations have different "vacuum" states and their low energy behaviour is different. One does not know a priori which of them will provide the physical situation under consideration. There is concern that they are working in an unphysical sector.

Although this could only be settled in detail when the semiclasssical limit is worked out, there are certain worries.

4.5.1 A "Failure to Propagate"?

The first one is the sort of action the Hamiltonian has. It only acts at vertices and it acts by "dressing up" the vertex with lines. It does not interconnect vertices nor change the valences of the lines (outside the "dressing"). This imediately suggests superselection rules and quantities that are anomalously conserved, that is, relations, amongst the quantum mechanical degrees of freedom, that have no classical counterpart. For instance, one can consider surfaces that enclose a vertex (diffeomorphically invariantly defined). The area of such surfaces would commute with the Hamiltonian.

This hints at the theory "failing to propagate" (we will clarify what to "propagate" in the next section).

However, this is inconclusive because the constraint acts everywhere.

Infact, it is actually technically incorrect that the actions of the Hamiltonian constraints $\hat{H}_v$, $\hat{H}_{v'}$ at different vertices v, v' do not influence each other: In fact, these two operators do not commute, for instance if v, v' are next neighbour, because for any choice function $\gamma \mapsto \epsilon_\gamma$ what is required is that the loop attachments at v, v' do not intersect which requires that the action at v' after the action at v attaches the loop at v' closer to v' than it would before the action at v and vice versa.

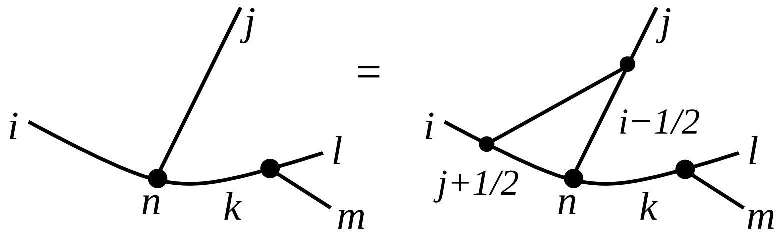


Figure 4.15: HamNonComfig0. .

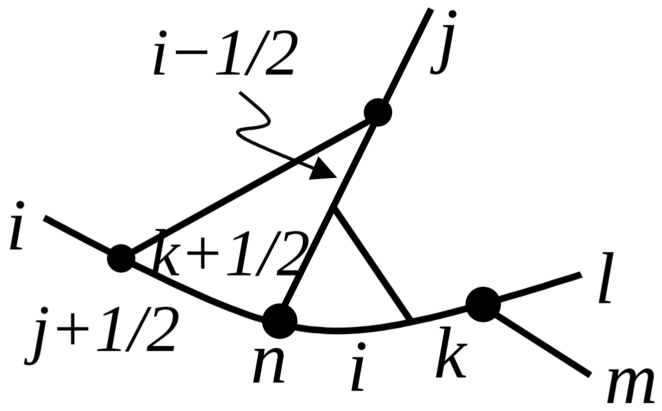


Figure 4.16: HamNonComfig0a. .

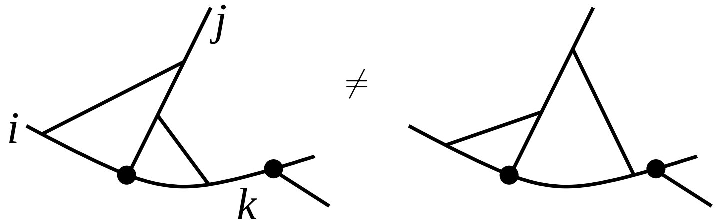


Figure 4.17: HamNonComfig. .

4.5.2 Anomaly-freeness

• First, we consider the subalgebra of the quantum diffeomorphism constraint. Recall

$$\hat{U}_\varphi \psi_\gamma := \psi_{\varphi \circ \gamma},$$

for any spatial diffeomorphism φ on Σ. Therefore, this subalgebra is free of anomaly by construction:

$$\hat{U}_\varphi \hat{U}_{\varphi'} \hat{U}_\varphi^{-1} \hat{U}_{\varphi'}^{-1} = \hat{U}_{\varphi \circ \varphi' \circ \varphi^{-1} \circ \varphi'^{-1}} \tag{4.0}$$

which coincides with the exponentiated version of the Poisson bracket between two diffeomorphism constraints generating the transformations $\varphi, \varphi' \in Diff(\Sigma)$.

• The quantum constraint algebra between the dual Hamiltonian constraint $S'(N)$ and the finite diffeomorphism transforamtion $\hat{U}_\varphi$ on diffeomorphism-invariant states coincides with the classical Poisson algebra between $\mathcal{V}(\vec{N})$ and $S(M)$.

Given a cylindrical fucntion ϕ_γ associated with a graph γ and triangulations $T(\epsilon)$ adapted to the graph α??, triangulations $T(\varphi \circ \epsilon) \equiv \varphi \circ T(\epsilon)$ are compatible with the graph $\varphi \circ \gamma$.

$$\begin{aligned}
\Big(-([\hat{S}(N), \hat{U}_\varphi])'\Psi_{Diff}\Big)[\phi_\gamma] &= ([\hat{S}'(N), \hat{U}'_\varphi]\Psi_{Diff})[\phi_\gamma] \\
&= \Psi_{Diff}[\hat{S}^\epsilon(N)\phi_\gamma - \hat{S}^\epsilon(N)\phi_{\varphi\circ\gamma}] \\
&= \sum_{\nu \in V(\gamma)} \big(N(\nu)\Psi_{Diff}[\hat{S}^\epsilon_\nu \phi_\gamma] - N(\varphi \circ \nu)\Psi_{Diff}[\hat{S}^{\varphi\circ\epsilon}_{\varphi\circ\nu}\phi_{\varphi\circ\gamma}]\big) \\
&= \sum_{\nu \in V(\gamma)} [N(\nu) - N(\varphi \circ \nu)]\Psi_{Diff}[\hat{S}^\epsilon_\nu \phi_\gamma] \\
&= \big(\hat{S}'(N - \varphi^* N)\Psi_{Diff}\big)[\phi_\gamma]
\end{aligned} \tag{4.-3}$$

Thus there is no anomoly.

• The commutator between two Hamiltonian constraint operators

$$\{H(M), H(N)\} = \int_\Sigma d^3x (M_{,a} N - M N_{,a}) q^{ab} V_b \tag{4.-3}$$

where V_b is the vector constraint.

$$
\begin{aligned}
[\hat{H}(N), \hat{H}(M)]\phi_\gamma &= \sum_{v,v' \in V(\gamma), v \neq v'} [M(v)N(v') - N(v)M(v')] \hat{H}^{\epsilon'}_{v'} \hat{H}^{\epsilon}_{v} \, \phi_\gamma \\
&= \frac{1}{2} \sum_{v,v' \in V(\gamma), v \neq v'} [M(v)N(v') - N(v)M(v')] [\hat{H}^{\epsilon'}_{v'} \hat{H}^{\epsilon}_{v} - \hat{H}^{\epsilon}_{v} \hat{H}^{\epsilon'}_{v'}] \, \phi_\gamma \\
&= \frac{1}{2} \sum_{v,v' \in V(\gamma), v \neq v'} [M(v)N(v') - N(v)M(v')] [(U_\varphi - U_\varphi) \hat{H}^{\epsilon}_{v}] \, \phi_\gamma
\end{aligned}
\tag{4.-5}
$$

It is mathematically consistent with the classical expression of two Hamiltonian constraint operators commute on diffeomorphism states.

Obviously, we have in the uniform Rovelli-Smolin topology

$$
([\hat{S}(N), \hat{S}(M)])' \Psi_{Diff} = 0 \tag{4.-5}
$$

for all $\Psi_{Diff} \in Cly^*_{Diff}$.

4.5.3 Recovering Poisson bracket between two Hamiltonian constraints

While the commutator of two Hamiltonian constraints then is anomaly free in the sense explained, in addition one would like to check that the classical limit of the commutator between quantum Hamiltonian constraints is precisely the corresponding Poisson bracket between the classical constraints. Failure to do so could raises concerns that the theory does not have the correct semiclassical limit. At present this can't be checked.

Summary?

The Hamiltonian constraint remains the major unsolved problem in LQG. These unsettled questions that concern the algebra of commutators among smeared Hamiltonian constraints which must be faced in order to make progress. We will see in chapter on the Master constrain program aimed at doing just that.

4.5.4 Solutions and Physical Inner Product

Solutions to all constraints can be constructed algorithmically []. The Hamiltonian constraint acts on spin network nodes by adding *exceptional* edges. These exceptional edges are added in the vicinity of nodes and are anihilated by subsequent applications of the Hamiltonian constraint. It follows that solutions of the Hamiltonian constraint can be built by appropriate superpositions of states corresponding to graphs with 'spiderweb' nodes as illustrated in fig. 4.18.

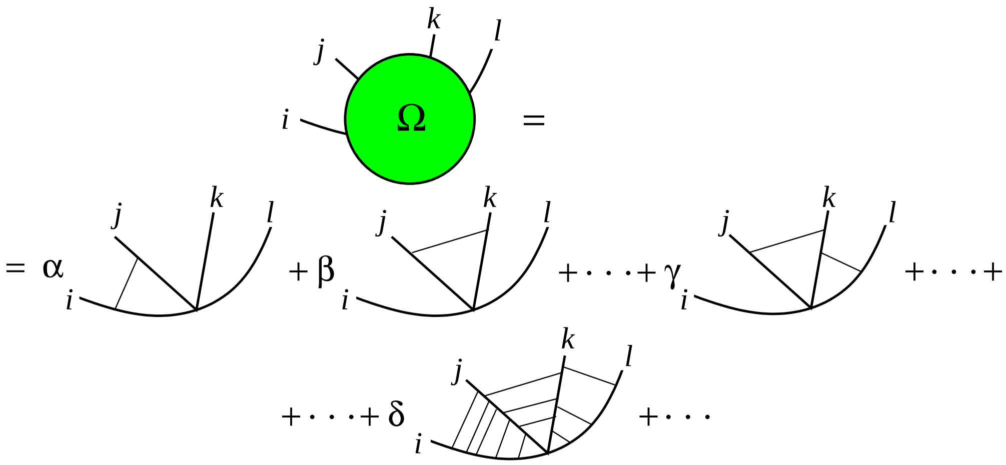


Figure 4.18: spinderfig. (spiderweb) around each vertex.

where we have left out the spin labels for clarity and the coefficients $\alpha, \beta, \ldots, \delta$ depend on the defition of the Hamiltonian constraint and the spin labelling of the corresponding spin networks.

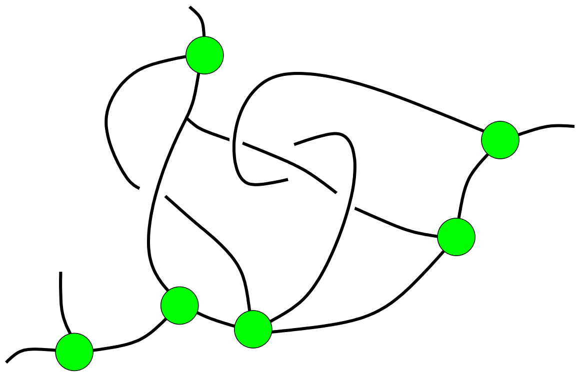


Figure 4.19: solutionfig. Solutions to all the constraints of canonical quantum gravity as dressed (Diff-invariant) spin networks.

Finding solutions to all the constraints of LQG reduces to solving algebraic relations between coefficients. They are the first rigorous solutions ever constructed in canonical quantum gravity, have non zero volume and are labelled by fractal knot classes because the iterated action of the Hamiltonian constraint creates a self-similar structure (spiderweb) around each vertex.

These are the full LQG analogs of the LQC solutions of the difference equation that results from the single Hamiltonian constraint of LQC which will be discussed in chapter 5. However, as in LQC these solutions are not systematically derived from a rigging map which is why a physical inner product is currently missing for these solutions.

4.5.5 Semi-Classical Limit

The second is that the existing semiclassical tools are only appropriate for non-graph-changing operators such as the volume operator. Namely, as we will see in chapter 7, in order to be normalisable, coherent states are (superpositions of) states defined on specific graphs. The Hamiltonian constraint operator, however, is graph changing. This means that it creates new modes on which the coherent state does not depend and whose fluctuations are therefore not suppressed. Therefore the existing semiclassical tools are insufficient for graph changing operators such as the Hamiltonian constraint. The development of improved tools is extremely diffcult and currently out of reach.

4.6 Spin Foams

Spin foam models [152] are an attempt at a path integral definition of LQG. They were heuristically defined in the seminal work [151][92] which attempted at the construction of the physical inner product via the formal exponentiation of the Hamiltonian constraints of [85].

After the construction of Hamiltonian constraint operator, formal, Euclidean functional integral was constructed in [M. Reisenberger, C. Rovelli, Phys.**D56** (1997) 3490-3508] and gave rise to the so-called spinfoam models (a spin foam is a history of a graph with faces as the history of edges).

4.6.1 Obtaing Physical Solutions with the "Projection" Operator

One is naturally lead to consider a functional evolution formalism to describe the dynamics of the quantum theory.??

$$W(x,t;x',t') \sim \int_{x(t')=x'}^{x(t)=x} \mathcal{D}[x(t)] e^{iS[x]}, \tag{4.-5}$$

The amplitude to go from one 3 geometry to another is given by summing over a phase factor (as usual calculated from the action of classical general relativity) over all 4-spacetimes interpolating the two 3-spaces.

This amplitude does not depend upon any times.

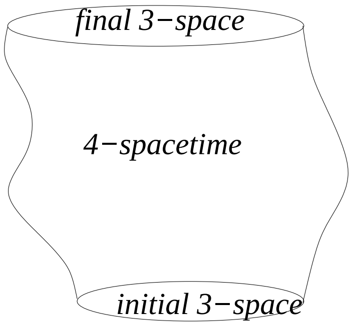


Figure 4.20: histories. “Sum over histories” formalism in quantum gravity.

It can be viewed as a mathematical well defined and possibly divergence free version of Hawking's “sum over geometries” formulation of quantum gravity.

$$W(s, s') =< s|P|s' > \tag{4.-5}$$

$W(s, s')$ giving the probability amplitude of measuring the quantized three geometry described by the spin network s is the three geometry described by the spin nework s' has been measured.

$$P \int [\mathcal{D}N] e^{i\hat{\mathcal{H}}[N]} = \int [\mathcal{D}N] e^{iN\hat{\mathcal{H}}}. \tag{4.-5}$$

In the spin network basis, the matrix elements of P are

$$< s|P|s' >=< s| \int [\mathcal{D}N] e^{iN\hat{\mathcal{H}}} |s' > \tag{4.-5}$$

It can be shown that a diffeomorphism invariant notion of integration exists for this functional integral.

$$< s|P|s' >\sim< s|s' > + \int [\mathcal{D}N] \left(N < s|\hat{\mathcal{H}}|s' > + NN < s|\hat{\mathcal{H}}\hat{\mathcal{H}}|s' > + \dots \right) \tag{4.-5}$$

$$W(s, s') =< s|P|s' > \tag{4.-5}$$

we can now . Inserting resolutions of identity $I = \sum_s |s >< s|$, we obtain an expansion of the form

$$
\begin{aligned}
W(s,s') &= \lim_{N\to\infty}\sum_{s_1,\ldots,s_N} < s|e^{-\int d^3x\mathcal{H}(x)dt}|s_N >< s_N|e^{-\int d^3x\mathcal{H}(x)dt}|s_{N-1} > \\
&\quad \cdots < s_1|e^{-\int d^3x\mathcal{H}(x)dt}|s' > \qquad (4.\text{-}5)
\end{aligned}
$$

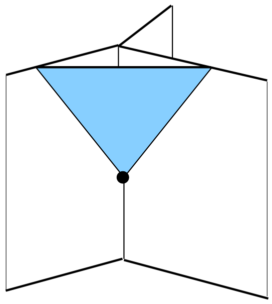


Figure 4.21: A vertex of a spinfoam.

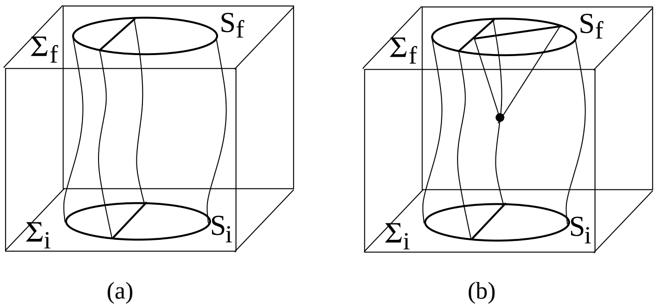


Figure 4.22: (a) Surface corresponding to a term of order zero. (b) Surface corresponding to first order term.

This is "unfreezing the frozen formalism" - in the classical theory this would be like reintroducing the unphysical time coordinate. So we can think of evolution of spin networks.

4.6.2 Difficulties with this Spin Foams Model

Both canonical and spin foam programme try to construct $\mathcal{H}_{Phys}$. Heuristic "projector" onto physical states,

$$
\begin{aligned}
&P\cdot|s>:=\delta[C']|s>:=[\prod_{x\in\sigma}\delta(C'(x))]|s>:=\int_{\mathcal{N}}[dN]e^{iC'(N)}|s> \\
&< P\cdot, P\cdot l' >_{Phys}:=< l, P\cdot l' >_{Diff} \qquad (4.\text{-}5)
\end{aligned}
$$

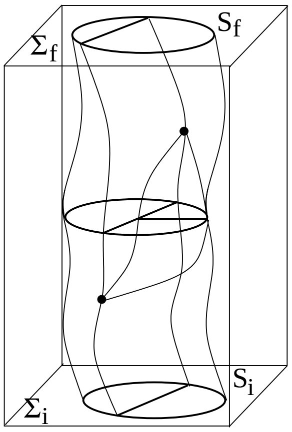


Figure 4.23: A term of second order.

Open issues:

1. $C'(N)$ does not preserve $\mathcal{H}_{Diff}$, hence $< l, P \cdot l' >_{Diff}$ is ill-defined.

2. Possibly final expression well-defined after intergrating over N, however, convergence issues.

The reason that this approach was formal is that the Hamiltonian constraints do not form a Lie algebra and they are not even self-adjoint. Thus, there are mathematical (exponentiation of non normal operators) and physical (non Lie group structure of the constraints prohibiting the possibility that functional integration over N of $\exp(i(N))$ leads to a (generalised) projector) issues with this proposal.

It tries to define a generalized projector of the form

$$\prod_{x \in \sigma} \delta(\hat{C}(x)) \tag{4.-5}$$

at least formally where $\hat{C}(x)$ is the Hamiltonian constraint of [2, 3, 4??]. However, this is quite diffcult to turn into a technically clean procedure for several reasons:

1. First of all the $\hat{C}(x)$ are not self-adjoint whence the exponential is defined at most on analytic vectors of $\mathcal{H}_{Kin}$.

$C'(N)$ not self-adjoint, hence $\exp(iC'(N))$ cannot be defined via spectral theorem. Only the formal power expansion of the exponetial can be defined.

Defintion analytic vectors: Let A be an (unbounded) linear operator on a Banach space B. An *analytic vector* for A is an element $u \in B$ such that $A^n u$ is defined for all n and

$$\sum_{n=0}^{\infty} \frac{\|A^n u\|}{n!} t^n < \infty \tag{4.-5}$$

for some $t > 0$, that is, the power series expansion of $e^{tA}u$ is defined and has positive radius of absolute convergence.

2. Secondly there is an infinite number of constraints and thus the generalized projector must involve a path integral over a suitable Lagrange multiplier N and one is never sure which measure to choose for such an integral without introducing anomalies.

Possibly final expression well-defined after intergrating over N, however, convergence issues.

3. Thirdly and most seriously, the $\hat{C}(x)$ are not mutually commuting and since products of projections define a new projection if and only if the individual projections commute, the formal object $\prod_{x \in \sigma} \delta(\hat{C}(x))$ is not even a (generalized) projection.

Proof: Projection operators are meant to be Hermitian and equal to their square (this second condition needs to be modified for generalised projection operators). Note that since P_1 and P_2 are projection operators they are Hermitian, $P_1 = P_1^\dagger$ and $P_2 = P_2^\dagger$. Therefore

$$(P_1 P_2)^\dagger = P_2^\dagger P_1^\dagger = P_2 P_1$$

So that a necessary condition for the product of two projection operators to be a projection operator is that

$$P_1 P_2 = P_2 P_1.$$

So we have $[P_1, P_2] = 0$. Now consider the square

$$(P_1 P_2)^2 = (P_1 P_2)(P_1 P_2)$$

If P_1 and P_2 commute, we have

$$(P_1 P_2)^2 = (P_1 P_2)(P_1 P_2) = P_1^2 P_2^2 = P_1 P_2.$$

So we see that the necessary and sufficient condition for the product of two projection operators to be itself a projection operator is that the two operators commute.

$\square$

If one defines it somehow on diffeomorphism invariant states (which might be possible because, while the individual $\hat{C}(x)$ are not diffeomorphism invariant, the product might be up to an (infinite) factor) then that problem could disappear because the commutator of two Hamiltonian constraints annihilates diffeomorphism invariant states [2, 3, 4??], however, this would be very hard to prove rigorously.

Fourthly there is a somewhat subtle problem: $\hat{C}(x)$, while defined on $\mathcal{H}_{Kin}$ are not explicitly known (they are known up to a diffeomorphism; they exist by the axiom of choice).

It is probably due to these diffculties and the non-manifest spacetime covariance of the amplititudes computed in [20??] for the Euclidean Hamiltonian constraint that the spin foam approach has chosen an alternative route that, however, has no clear connection with Hamiltonian formalism so far. Such an approach is the Barret-Crane model which we come to in the next section.

Another approach has been proposed which could not only remove the above four problems but also has the potential to combine the canonical and spin foam programme rigorously. This is the Master constraint programme which we will come to in chapter 6.

How about using $C(N) \rightarrow \frac{1}{2}(C(N) + C^{\dagger}(N))$.

5. Power expansion $\approx$ state sum model (sum over representation labels of SNW), motivaes PI with respect to Palantinin action (=spin foam models). Connection to canonical theory?

4.7 Canonical Reduced Phase Space Quantization of LQG

4.7.1 Introduction

The attractive feature of this reduced phase space approach is that we no longer need to deal with the constraints: No anomalies can arise, no master constraint needs to be constructed (this is an alternative way of formulating the dynamics, to be covered in the next chapter), no physical Hilbert space needs to be derived by complicated group averaging techniques. We map a conceptually complicated gauge system to the conceptually safe realm of an ordinary dynamical Hamiltonian system. The kinematical results of LQG such as discreteness of spectra of geometric operators now become physical predictions.

There is, however, a difference between the constraint quantization programme and the reduced phase space quantization programme which leads to physically different predictions: As we shall see in the reduced phase space programme the clock variables are replaced by real numbers and their conjugate momenta by the functions. Thus the clock variables are not quantized. On the other hand, in the Dirac quantization programme also the clock variables are quantized as well as the conjugate momenta which are not replaced, via the constraints, in terms of q^a, p_a, T_j . Hence, the representations of the Dirac observables that come from constraint quantization know about the quantum fluctuations of the clock variables while those of the reduced phase space quantization do not.

4.7.2 Reduced Phase Space Quantization of Some Toy Systems

For systems with one constraint

$$F^{\tau}_{f,T} := \sum_{k=0} \frac{(\tau - T)^k}{k!} (X)^k \cdot f \tag{4.-5}$$

where $X \cdot f$ is defined by

$$X \cdot f = \{A^{-1}C, f\}, \qquad A := \{C, T\}.$$

$$\alpha^{\tau}_{C}(f(x)) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \{C, f(x)\}_k. \tag{4.-5}$$

Example 1 A point in phase space $x = (q^1, p_1, q^2, p_2)$. We have constraint equation

$$\mathcal{C} = p_2 + \frac{p_1^2}{2m} + \frac{1}{2} m\omega^2 q_1^2$$

There are canonical coordinates (q, p) and (T, P_T) on the unreduced phase space,

$$\{q, p\} = 1, \qquad \{T, P_T\} = 1$$

Here $q^1 = q$, $p_1 = p$, $q^2 = t$, $p_2 = p_t$, and the constraint equation is then

$$\mathcal{C} = p_t + \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

We define with $F_T(\cdot) := F^{\tau=0}_{\cdot,T}$ the weak Dirac observables at multi-fingered time $\tau = 0$ (or any other fixed allowed value of τ)

$$Q := F_T(q) = q \cos \omega t - \frac{p}{m\omega} \sin \omega t \tag{4.-5}$$

$$P := F_T(p) = q\cos\omega t - \frac{p}{m\omega}\sin\omega t \tag{4.-5}$$

F_T is a homomorphism with respect to pointwise operations:

$$F_T(f+f') = F_T(f) + F_T(f'), \qquad F_T(ff') = F_T(f)F_T(f') \tag{4.-5}$$

where f, f' are arbirary phase space functions, we have

$$F_T(P_T) \approx E(F_T(q), F_T(p), F_T(T)) \approx E(Q, P, \tau) \tag{4.-5}$$

and thus also does not give rise to a Dirac observable which we could not already construct from Q, P, namely

$$\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 =$$

Due to the homomorphism property (4.7.2)

$$\{P, Q\} \approx F^0_{\{p,q\}^*,T} = F^0_{1,T} = 1, \qquad \{Q, Q\} \approx \{P, P\} \approx 0. \tag{4.-5}$$

In other words, even though the functions P, Q are complicated expressions in terms of q, p, T they nevertheless have canonical brackets at least on the constraint surface.

Now the reduced phase space quantization consists in quantizing the subalgebra of $\mathcal{D}$, spanned by our preferred Dirac observables Q, P evaluated on the constraint surface. As we have just seen, the algebra $\mathcal{D}$ itself is given by the Poisson algebra of the functions of the Q, P evaluated on the constraint surface. Hence all the weak equalities now become exact. We are therefore looking for a representation

$$\pi : \mathcal{D} \to \mathcal{L}(\mathcal{H})$$

of that subalgebra of $\mathcal{D}$ as self-adjoint, linear operators on a Hilbert space such that

$$[\pi(P), \pi(Q)] = i\hbar.$$

The constraint has disappeared, it has been solved and reduced. Instead of a constraint on the gauge variant phase space coordinatised by gauge variant canonical pairs one has a gauge invariant phase space coordinatised by (4.7.2) and (4.7.2). At this point it looks as if we have completely trivialized the reduced phase space quantization problem of our constrained Hamiltonian system because there is no Hamiltonian constraint to be considered and so it seems that we can just choose the standard kinematical representations for quantizing the phase space coordinatized by the q, p (representations unitarily equivalent to the free particle Schrodinger representation) and simply use it for Q, P because the respective Poisson algebras are (weakly) isomorphic. However, this is not the case. In addition to satisfying the canonical commutation relations we want that the parameter group of automorphisms α_τ on $\mathcal{D}$ be represented unitarily on $\mathcal{H}$. In other words, we want that the parameter group of automorphisms be represented unitarily on $\mathcal{H}$, i.e., that there exist a one parameter group of unitary operators $U(\tau)$ on $\mathcal{H}$ such that

$$\pi(\alpha^\tau(Q)) = U(\tau)\pi(Q)U(\tau)^{-1} \qquad \text{and} \qquad \pi(\alpha^\tau(P)) = U(\tau)\pi(P)U(\tau)^{-1}.$$

Notice that due to the relation (which is exact on the constraint surface)

$$\alpha_\tau(Q) = F_{\alpha'_\tau(q),T} = \sum_{k=0} \frac{\tau^k}{k!} F_{X^k \cdot q,T} \tag{4.-5}$$

and where on the right hand side we may replace any occurence of P_T, T by functions of Q, P according to the above rules. Hence the automorphism α^τ preserves the algebra of functions of the Q, P, although it is a very complicated map in general and in quantum theory will suffer from ordering ambiguities. On the other hand, for short time periods (4.7.2) gives rise to a quickly converging perturbative expansion.

One way to implement the time evolution unitarily is by quantizing the Hamiltonian that generates the Hamiltonian flow. The constraint is of the form

$$C = P_T + E(q, p, T)$$

We now set the

$$H(Q, P) := F^0_{E,T} \approx E(F^0_{q,T}, F^0_{p,T}, F^0_{T,T}) \approx E(Q, P, 0)$$

$$\begin{aligned}
\{H, F^0_{f,T}\} &\approx F^0_{\{E,T\}^*,T} = F^0_{\{E,T\},T} = F^0_{\{C,T\},T} \\
&= \sum_k \frac{(\tau - T)^k}{k!} X^k \cdot \tilde{X} \cdot f \qquad \text{as } \{T, E\} = \{T, f\} = 0 \\
&\approx \tilde{X} \cdot F^0_{f,T} - \sum_k (\tilde{X} \cdot \frac{(\tau - T)^k}{k!} X^k \cdot f)
\end{aligned} \tag{4.-6}$$

The problem of implementing the flow unitarily can be reduced to finding a self adjoint quantization of the function H.

$$H(Q,P) \approx E(Q,P,0) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$$

$$[\pi(P),\pi(Q)] = i\hbar, \qquad U(\tau) = \exp i\left(\frac{\pi(P)^2}{2m} + \frac{1}{2}m\omega^2\pi(Q)^2\right)\tau \tag{4.-6}$$

$$\pi(Q) = Q, \qquad \pi(P) = -i\hbar\frac{\partial}{\partial Q}$$

As $U(\tau)$ is unitarily implemented on $\mathcal{H}$ for an orthonormal basis $\psi_n \in \mathcal{H}$

We no longer have a gauge symmetry but rather a symmetry group of the physical Hamiltonian H_{ph}. We require a covariant representation of the group symmetry.

$$U(\tau)\psi(Q,t) = \psi(Q,t-\tau).$$

The function $\psi(t)$ can be expanded uniquely into a Fourier integral

$$\psi(Q,t) = \int_{-\infty}^{\infty} \tilde{\psi}(Q,E)e^{iEt}dE$$

The functions $e_E(Q,t) = \psi(Q)e^{iEt}$ thus form a basis. Moreover,

$$(U(\tau)e_E)(t) = e^{iE(t-\tau)} = e^{-iE\tau}e_E(t),$$

Example 2

The double harmonic oscillator was first studied by Rovelli [278] as a toy model to help understand the "problem of time".

The Hamiltonian for the double harmonic oscillator

$$C = \lambda\left(\frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) - E\right) \tag{4.-6}$$

One of the oscillators can be thought of as a "clock" for the other oscillator.

Here we choose $q^1 = q$, $p_1 = p$, $q^2 = t$, $p_2 = p_t$. These are canonical coordinates

$$(q, p) \qquad \text{and} \qquad (T, P_T) \tag{4.-6}$$

on the unreduced phase space,

$$\{q, p\} = 1, \qquad \{T, P_T\} = 1$$

and the constraint equation is then

$$\mathcal{C} = \lambda \left(\frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}(p_t^2 + \omega^2 T^2) - E \right).$$

As before we define the weak Dirac observables at multi-fingered time $\tau = 0$ (or any other fixed allowed value of τ)

$$Q := F_T(q) = \tag{4.-6}$$

$$P := F_T(p) = \tag{4.-6}$$

Since at least locally we can solve the constraint C for the momentum P_T, that is

$$p_t = -\sqrt{\left(\frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}\omega^2 T^2) - E \right)}$$

and F_T is a homomorphism property (4.7.2)

$$\begin{aligned} F_T(P_T) &\approx E(F_T(q), F_T(p), F_T(T)) \\ &\approx E(Q, P, \tau) \\ &= \sqrt{\left(\frac{1}{2}(P^2 + \omega^2 Q^2) + \frac{1}{2}\omega^2 \tau^2) - E \right)} \end{aligned} \tag{4.-7}$$

and thus also does not give rise to a Dirac observable which we could not already construct from Q, P.

Again by the homomorphism property

$$\{P, Q\} \approx F^0_{\{p,q\}^*,T} = F^0_{1,T} = 1, \qquad \{Q, Q\} \approx \{P, P\} \approx 0,$$

we have produced gauge invariant canonical pairs from gauge variant canonical pairs.

This can be done as follows: The original constraint C can be solved for the momenta P_T conjugate to T and we get an equivalent constraint

$$\tilde{C} \approx p_t + \sqrt{\left(\frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}\omega^2 T^2) - E\right)}$$

We now set the

$$H(Q,P) := F^0_{E,T} \approx E(F^0_{q,T}, F^0_{p,T}, F^0_{T,T}) \approx E(Q,P,0) \tag{4.-7}$$

$$\{H, F^0_{f,T}\} \approx (\frac{\partial}{\partial \tau})_{\tau=0}\, \alpha^\tau(F_T(f))$$

This means that the strongly Abelian group of Poisson bracket automorphisms α^τ is generated by the "Hamiltonians" H_j . Thus, if we interpret the T_j as clocks then we have a multi-fingered time evolution with Hamiltonians H_j.

The problem of implementing the flow unitarily can be reduced to finding a self adjoint quantization of the function H.

The constraint has disappeared, they have been solved and reduced. Instead of a constraint on the gauge variant phase space coordinatised by (4.7.2) which generates gauge transformations, there is a physical Hamiltonian (4.7.2) which generates physical time evolution on the gauge invariant phase space coordinatised by (4.7.2) and (4.7.2).

4.7.3 The General Scheme

except in asymptoticy flat spacetimes where all metrics under consideration approach a fixed metric because here all active diffeomorphisms universally reduce to active Poincare transformations. we have at infinity we have the symmetry group the Poincare group the Poincare charges at spacial infinity

The Hamiltonian constraints are not spacially diffeomorphism independent and this has meant solving the Hamiltonian constraint has been slow. One cannot first solve the spatial diffeomorphism constraints and then solve the Hamiltonian constraint because the latter do not preserve the space of solutions to the spacially diffeomorphism constraint.

f and T_j functions on phase space. Weak Dirac observable there are n T_j's

$$F^\tau_{f,T} := \sum_{k_1,\ldots,k_n=0} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} (X_1)^{k_1} \ldots (X_n)^{k_n} \cdot f. \tag{4.-7}$$

where $X_r \cdot f$ is defined as

$$X_j \cdot f := \{(A^{-1})_{jk} C_k, f\}, \quad A_{jk} := \{C_j, T_k\}. \tag{4.-7}$$

Poisson algebra

$$\{F^\tau_{f,T}, F^\tau_{f',T}\} = F^\tau_{\{f,f'\}^*,T} \tag{4.-7}$$

defining the automorphism on ?? generated by the Hamiltonian vector field of $\sum_j \tau^j C'_j$

$$\alpha'_\tau(f) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_j \tau^j X_j \right)^n \cdot f \tag{4.-7}$$

$$\{\alpha^\tau(F^\tau_{f,T}), \alpha^\tau(F^\tau_{f',T})\} \approx \alpha^\tau(\{F^\tau_{f,T}, F^\tau_{f',T}\}) \tag{4.-7}$$

In other words, α^τ is a weak, abelian, multi-parameter group of automorphisms on the each $F^{\tau_0}_{f,T}$.

In [224] they perform a canonical, reduced phase space quatization of General Relativity by Loop Quantum Gravity methods.

It is assumed that it is possible to to choose the functions T_j as canonical coordinates. In other words, we choose a canonical coordinate system consisting of canonical pairs (q^a, p_a) and (T_j, P^j) where the first system of coordinates has vanishing Poisson brackets with the second so that the only non vanishing brackets are

$$\{p_a, q^b\} = \delta^b_a, \qquad \{P^j, T_k\} = \delta^j_k.$$

The virtue of this assumption is that the Dirac bracket reduces to the ordinary Poisson bracket on functions which depend only on q^a, p_a. We will shortly see why this is important.

Usually the Dirac brackets make the Poisson structure so complicated that one cannot find representations thereof. However, as we see, if the system deparametrises, if one uses as clocks T the configuration varaibles conjugate to the momenta P in $C = P + H$ and if one considers functions f which dont depend on T, P then F_T becomes a Poisson bracket isomorphism. This is no loss of generality because P can be eliminated in terms of the other degrees of freedom via the constraints and T is pure gauge.

We define with $F_T(\cdot) := F^0_{\cdot,T}$ the weak Dirac observables at multi-fingered time $\tau = 0$ (or any other fixed allowed value of τ)

$$Q^a := F_T(q^a), \qquad P_a := F_T(p_a) \tag{4.-7}$$

Notice that

$$F_{T_j,T} \approx \tau_j,$$

so the Dirac observable corresponding to T_j is just a constant and thus not very interesting (but evolves precisely as a clock). Likewise

$$F^\tau_{Cj,T} \approx 0$$

is not very interesting. Since at least locally we can solve the constraints C_j for the momenta P^j, that is

$$P^j \approx E_j(q^a, p_a, T_k)$$

and F_T is a homomorphism with respect to pointwise operations:

$$F_T(f + f') = F_T(f) + F_T(f'), \qquad F_T(ff') = F_T(f)F_T(f') \tag{4.-7}$$

where f, f' are arbirary phase space functions, we have

$$F_T(P_j) \approx E_j(F_T(q^a), F_T(p_a), F_T(T_k)) \approx E_j(Q^a, P_a, \tau_k) \tag{4.-7}$$

and thus also does not give rise to a Dirac observable which we could not already construct from Q^a, P_a. The importance of our assumption is now that due to the homomorphism property

$$\{P_a, Q^b\} \approx F^0_{\{p_a,q^b\}^*,T} = F^0_{\delta^b_a,T} = \delta^b_a, \qquad \{Q^a, Q^b\} \approx \{P_a, P_b\} \approx 0. \tag{4.-7}$$

In other words, even though the functions P_a, Q^a are very complicated expressions in terms of q^a, p_a, T_j they nevertheless have canonical brackets at least on the constraint surface. If we would have had to use the Dirac bracket then this would not be the case and the algebra among the Q^a, P_a would be too complicated and no hope would exist towards its quantization. However, under our assumption there is now a chance.

Now the reduced phase space quantization consists in quantizing the subalgebra of $\mathcal{D}$, spanned by our preferred Dirac observables Q_a, P_a evaluated on the constraint surface. As we have just seen, the algebra $\mathcal{D}$ itself is given by the Poisson algebra of the functions of the Q_a, P_a evaluated on the constraint surface. Hence all the weak equalities now become exact. We are therefore looking for a representation

$$\pi : \mathcal{D} \to \mathcal{L}(\mathcal{H})$$

of that subalgebra of $\mathcal{D}$ as self-adjoint, linear operators on a Hilbert space such that

$$[\pi(P_a), \pi(Q^b)] = i\hbar\delta_a^b.$$

At this point it looks as if we have completely trivialized the reduced phase space quantization problem of our constrained Hamiltonian system because there is no Hamiltonian to be considered and so it seems that we can just choose any of the standard kinematical representations for quantizing the phase space coordinatized by the q^a, p_a and simply use it for Q^a, P_a because the respective Poisson algebras are (weakly) isomorphic. However, this is not the case. In addition to satisfying the canonical commutation relations we want that the multi parameter group of automorphisms α_τ on $\mathcal{D}$ be represented unitarily on $\mathcal{H}$ (or at least a suitable, preferred one parameter group thereof). In other words, we want that there exists a multi parameter group of unitary operators $U(\tau)$ on $\mathcal{H}$ such that

$$\pi(\alpha_\tau(Q^a)) = U(\tau)\pi(Q^a)U(\tau)^{-1}$$

and similarly for P_a.

Notice that due to the relation (which is exact on the constraint surface)

$$\alpha_\tau(Q^a) = F_{\alpha'_\tau(q^a),T} = \sum_{\{k\}} \prod_j \frac{T_j^{k_j}}{k_j!} F_{\prod_j X_j^{k_j} \cdot q^a, T} \tag{4.-7}$$

and where on the righthand side we may replace any occurence of P_j, T_j by functions of Q^a, P_a according to the above rules.

Hence the automorphism α^τ preserves the algebra of functions of the Q^a, P_a, although it is a very complicated map in general and the quantum theory will suffer from ordering ambiguities. On the other hand, for short time periods (4.7.3) gives rise to a quickly converging perturbative expansion. Hence we see that the representation problem of $\mathcal{D}$ will be severly constrained by our additional requirement to implement the multi time evolution unitarily, if at all possible. Whether or not this is feasible will strongly depend on the choice of the T_j.

A possible way to implement the multi-fingered time evolution unitarily is by quantizing the Hamiltonians $\mathcal{H}_j$ that generate the Hamiltonian flows $\tau_j \mapsto \alpha^\tau$ where $\tau_k = \delta_{jk}\tau_j$. This can be done as follows: The original constraints C_j can be solved for the momenta P^j conjugate to T_j and we get equivalent constraints

$$C_j = P^j + E_j(q^a, p_a, T_k).$$

These constraints have a strongly Abelean constraint algebra. Proof: We must have $\{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}l\tilde{C}_l$ for some new structure functions $\tilde{f}$ by the first class property. The left hand side is independent of the functions P^j, thus must be the right hand side, which may therefore be evaluated at any value of P^j. Set $P^j = -E_j$.

We may write

$$C'_j = K_{jk}\tilde{C}_k$$

for some regular matrix K. Since

$$\{C'_j, T_k\} \approx \delta_{jk} = \{\tilde{C}_j, T_k\}$$

it follows that $K_{jk} \approx \delta_{jk}$. In other words $C'_j = \tilde{C}_j + \mathcal{O}(C^2)$ where the notation $\mathcal{O}(C^2)$ means that the two constraints set differ by terms quadratic in the constraints. It follows that the Hamiltonian vector fields $X_j, \tilde{X}_j$ of $C'_j, \tilde{C}_j$ are weakly commuting. We now set

$$H_j(Q^a, P_a) := F^0_{E_j,T} \approx E_j(F^0_{q^a,T}, F^0_{p_a,T}, F^0_{T_k,T}) \approx E_j(Q^a, P_a, 0)$$

$$\{H_j(\tau), F^0_{f,T}\} = (\frac{\partial}{\partial\tau_j})_{\tau=0}\, \alpha^\tau(F_T(f)) \tag{4.-6}$$

where in the derivation one uses that $\{T_j, E_k\} = \{T_j, f\} = 0$, $\{P_j, f\} = 0$, that the $X_j, \tilde{X}_k$ are weakly commuting, that $F_{f,T}$ is a weak observable, and the definition of the flow. We conclude that the Dirac observables H_j generate the multifingered flow on the space of functions of the Q^a, P_a when restricted to the constraint surface. The algebra of the H_j is weakly Abelean because the flow α^τ is a weakly Abelean group of automorphisms.

Thus, the problem of implementing the flow unitarily can be reduced to finding a self adjoint quantization of the functions H_j. Preferred one parameter subgroups will be those for which the corresponding Hamiltonian generator is bounded from below. Notice, however, that in () we have computed the infinitesimal flow at $\tau = 0$ only. For an arbitrary value of τ the infintesimal generator $H_j(Q^a, P_a, \tau)$ defined by

$$\{H_j(\tau), F^\tau_{f,T}\} := \frac{\partial}{\partial \tau_j} \alpha^\tau(F_T(f)) \tag{4.-6}$$

4.7.4 Outline of a Reduced Phase Space Quantization of LQG

Since the resulting algebra of observables is very simple, one can quantise it using the methods of LQG. Basically, the kinematical Hilbert space of non reduced LQG now becomes a physical Hilbert space and the kinematical results of LQG such as discreteness of spectra of geometrical operators now have physical meaning. The constraints have disappeared, however, the dynamics of the observables is driven by a physical Hamiltonian which is related to the Hamiltonian of the standard model (without dust) and which we quantise.

In the quantum theory we are looking for representations of the Poisson *-algebra generated by the which supports a quantised version of the physical Hamiltonian H_{ph}.

One wants a unitary representation of the spacial diffeomorphism group of the spacial coordinate manifold which is a *gauge group*. Since all our observables are gauge invariant, we have no diffeomorphism gauge group any longer, hence that physical selection criterion is absent. However, it is replaced by another: it turns out that the physical Hamiltonian has the difeomorphism group of the dust label space as a *symmetry group*. These difeomophisms change our observables, they map between physically distinguishable dust label space. This allows us to apply the same selection criterion.

In our case we do not have a diffeomorphism gauge group but rather a diffeomorphism symmetry group $Diff(X)$ of the physical Hamiltonian H_{ph}. This is physical input enough to also insist on cyclic $Diff(S)$ covariant representations and correspondingly we can copy the uniqueness result.

Thus we simply choose the background independent and active diffeomorphism covariant Hilbert space representation of LQG used extensively in [??] and we ask whether that representation supports a quantum operator corresponding to H_{ph}.

4.7.5 Physical Hilbert Space

Like in unreduced LQG we consider the holonomy-flux algebra.

In unreduced LQG representation by covariance under Diff(σ).

In reduced LQG the representation is fixed by covariance under DiffΣ.

Uniqueness result applies: $\mathcal{H}_{phys}$ = usual LQG Hilbert space but here is physical.

In particular: usual kinematic coherent states now physical coherent states.

4.8 Consistent Discrete Classical and Quantum General Relativity

One of the alternatives, to be presented in chapter 6, is thhe "master constraint" project of Thiemann and collaborators. Others considered covariant "spin foam" spproach as an alternative, sinse one may bypass the construction of the canonical algebra entirely, at least in some settings. In this section we introduce another approach.

However, when one discretizes the equations, the resulting system of algebraic equations is in general incompatable.

We are explicitly working at the *physical* level.

4.8.1 Semi-Discrete Approach

Significant departure from what everyone else was doing in LQG. They could not make use of of kinematic tools of LQG, like spin-network states, Astekar-Lewendowski diff-invariant measure, ... Thiemann paper *"One way out could be to look at constraint quantization from an entirely new point of view which proves useful also in discrete formulations of classical GR, that is, numerical GR. While being a fascinating possibility, such a procedure would be a rather drastic step in the sense that it would render most results of LQG obtained so far obsolete."*, [77].

discrete time but keep space continuous in the classical action. But would be not be a discord with GR in which space and time on same footing?

Get coupled non-linear PDEs.

Same kinematics but deal with spacial constraint in the usual way.

$$\begin{aligned} S = \int & dtd^3x \left[Tr \left(\tilde{P}^a (A_a(x) - V(x) A_{n+1,a}(x) V^{-1}(x) + \partial_a (V(x)) V^{-1} \right) \right. \\ - & \left. N^a C_a - NC + \mu Tr(V(x) V^\dagger(x) - 1) \right] \end{aligned} \tag{4.-6}$$

4.9 Summary:

· Real variables.

· Construction of a anomoly free, mathematically rigorous finite Hamiltonian constraint.

· But is it a physically viable theory of quantum GR?

4.10 Biblioliographical notes

In this chapter I have relied on the following references:

[].

Chapter 5

Physical Applications of LQG: Black Hole Entropy and Loop Quantum Cosmology

- **Thermodynamics of Black holes and Hawking radiation.**
- **Microscopic sourse of black hole entropy. Entropy of black holes.**
- **Loop Quantum Cosmology.**
- **Inflation From Loop Quantum Cosmology.**
- **Consistent histories interpretaion.**
- **Relational quantum cosmology.**

5.1 Introduction

5.2 Thermodynamics of Black holes and Hawking radiation

Second law of black hole mechanics

$$\delta\mathcal{A} \geq 0 \tag{5.0}$$

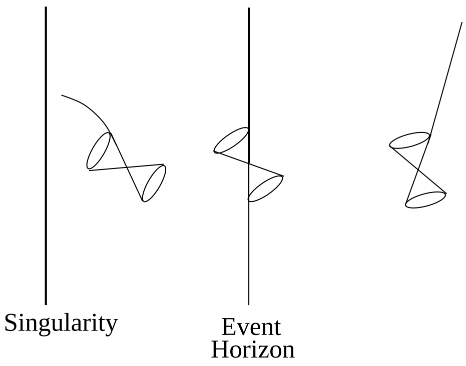


Figure 5.1: .

Second law of thermodynamics

$$\delta S \geq 0 \tag{5.0}$$

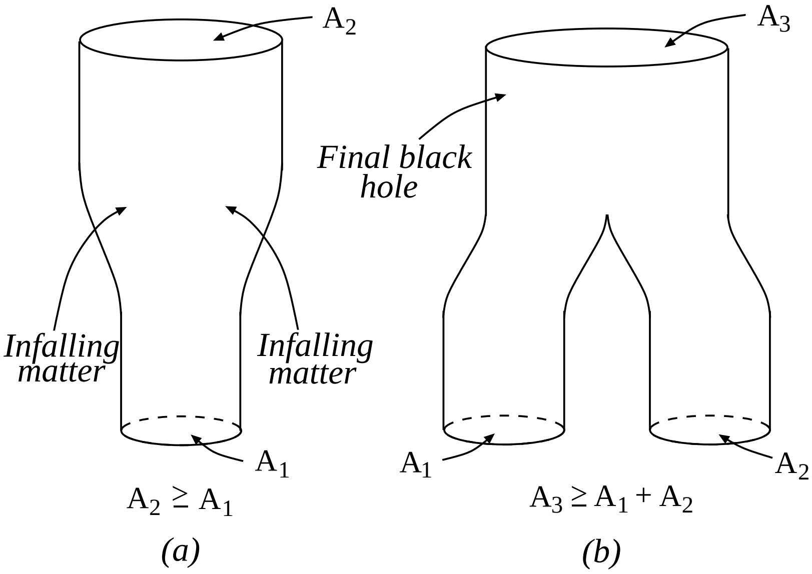


Figure 5.2: areaincrease.

First law of black hole mechanics:

$$\delta E = \frac{\kappa}{8\pi}\delta\mathcal{A} + \Omega\delta J + \Omega\delta\mathcal{Q} \tag{5.0}$$

It was immediately noticed by Bardeen, Carter and Hawking (1973) that there was a close similarity between these laws of black hole mechanics and the usual laws of thermodynamics, with κ proportional to the temperature and A proportional to the entropy.

First law of thermodynamics:

$$\delta E = T\delta + P\delta V \tag{5.0}$$

Zeroth law of black hole mechanics

κ is the same everywhere on the horizon of a time-independent black hole.

Zeroth law of thermodynamics

T is the same everywhere for a system in thermal equilibrium.

When the expansion goes negative, that is when the geodesics are converging the attractive nature of gravitation means that they will intersect within a finite affine parameter time.

However, it was originally thought that this could only be an analogy, since if a black hole really had a nonzero temperature, it would have to radiate and everyone knew that nothing could escape from a black hole. This view changed completely when Hawking (1975) showed that if matter is treated quantum mechanically, black holes do radiate. This showed that black holes are indeed thermodynamic objects with a temperature and entropy given by

$$T_{Bh} = \frac{\hbar\kappa}{2\pi}, \qquad S_{Bh} = \frac{A}{4\hbar} \tag{5.0}$$

5.3 Hawking Radiation

See [130] for note containing a step-by-step presentation of Hawking's calculation.

Rough idea:

See [130] for note containing a step-by-step presentation of Hawking's calculationd. In 1974, Steven Hawking showed that black holes do radiate quantum mechanically, thereby shinking in area. Only matter was treated quantum mechanically; there were no quanta of geometry.

$$\left(i\frac{\partial}{\partial x^\mu} - eA_\mu \right)^2 \psi(x) = m^2 \psi(x). \tag{5.0}$$

$$i\frac{\partial \varphi}{\partial \lambda} = -\frac{1}{2} \left(i\frac{\partial}{\partial x^\mu} - eA_\mu \right)^2 \varphi \tag{5.0}$$

Because the functions A_μ are independent of λ, (5.3) has solutions of the form $\varphi(x,\lambda) = \exp(im^2/2)\psi(x)$ with satisfying (5.3).

$$\mathcal{L} = \frac{1}{2}\left(\frac{dx^\mu}{d\lambda}\right)^2 + e\frac{dx^\mu}{d\lambda}A_\mu \tag{5.0}$$

$$G(x_A, x_B) = \int_0^\infty e^{-im^2\lambda/2} G(x_A, x_B; \lambda)\, d\lambda \tag{5.0}$$

with $G(x_A, x_B; \lambda)$ given by the path integral in the Schwarzschild metric

$$G(x_A, x_B; \lambda) = \int_{x(0)=x_A}^{x(\lambda)=x_B} \mathcal{D}[x] \exp\left(i\int_0^\lambda \frac{1}{2} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\, d\lambda\right). \tag{5.0}$$

$$\mathcal{H}_s \otimes \mathcal{H}_B \tag{5.0}$$

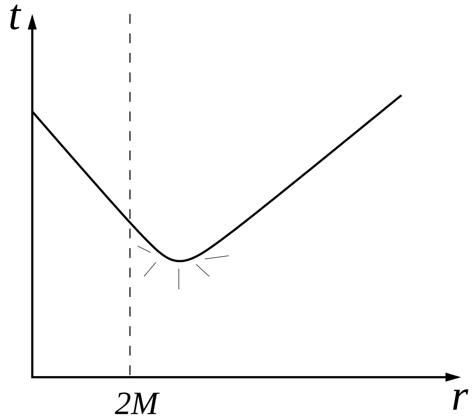


Figure 5.3: Reflects forward in time by the strong gravitaional field outside the event horizon.

Feynmann told us that we can interpret particles that going backwards in time as anti-particles (see details on page 1828). So anti-particles get swallowed by the black hole leaving a particle which has a chance of escaping off to infinity. That is, the black hole evapourates particles reducing its mass.

5.4 Laws of Thermodynamics from Statistical Mechanics

There is an area of theoretical physics that gives us indirect information on quantum gravity: black hole thermodynamics. The great power of thermodynamics to put con-

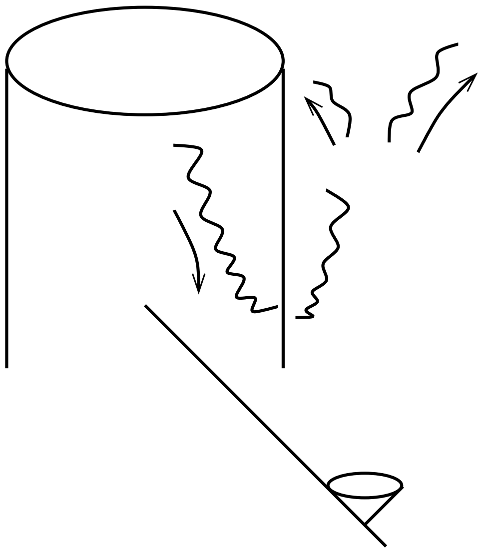


Figure 5.4: timereflection2. The antiparticle mode falling into the black hole can be interpreted as a particle travelling backwards in time, form the singularity down to the horizon.

straints on theoretical constructions, and even provide precise quantitative indications on microscopic theories is well known: quantum mechanics itself was born to a large extent in order to satisfy thermodynamical consistency requirements (Planck's spectrum, solid state...).

The first law of thermodynamics is an expression of conservation of energy including internal energy, heat and work.

We can just follow the usual argument that under some ergodicity hypothesis tells us what are the equilibrium states, if they exist.

In statistical mechanics, the relation $dS = dE/T$ is derived as a relation between nearby equilibrium states, not as a description of a dynamical process of approach to equilibrium.

$$\sum_i n_i = N; \tag{5.1}$$

$$\sum_j m_j = M. \tag{5.2}$$

and

$$\sum_i n_i \epsilon_i + \sum_j m_j \eta_j = E. \tag{5.2}$$

We write down the number of stationary states of the two ensembles combined; this number is given by

$$P = \frac{N!}{n_1!n_2!\ldots n_i!} \times \frac{M!}{m_1!m_2!\ldots m_j!} \tag{5.2}$$

This is just the product of the number of stationary states in ensemble A with the number of stationary states of ensemble B. We are treating the two systems as if any stationary state of the composite system is produced by the combination of any stationary state of one system with any stationary state of the other system, that is, as if they were isolated (apart from the condition that the total energy is conserved).

We want the values of the n_i's and of the m_j's which make P a maximum subject to the constraints.

$$\delta \ln P = -\delta \ln \sum_i n_i! - \delta \ln \sum_j m_j! = 0 \tag{5.2}$$

subject to the condition that

5.4.1 First Law

What do we mean by deriving the first law from a microscopic description? In a typical setting of a system in thermodynamic equilibrium, the statistical description of the system is given by a density matrix ρ. Let's fix on the canonical ensemble, so $\rho = \exp(-\beta\mathcal{H})/Z$, where β is the inverse temperature, $\mathcal{H}$ is the Hamiltonian, and Z is the trace of $\exp(-\beta\mathcal{H})$. Suppose some energy is added to the system, but in such a way that $\delta\rho \ll \rho$. Then varying the entropy $S = -Tr(\rho \ln \rho)$ and the mean energy $< E >= Tr(\rho\mathcal{H})$, one finds the relation

$$\delta S = \beta\delta < E >, \tag{5.2}$$

hence the thermodynamic relation $dE = TdS$ the first law. Thus once we have a system whose density matrix is a canonical ensemble, we have a derivation of the first law.

Let the number of system plus exterior states characterized by E_s and E_e be N. By ergodicity,

$$N = N_e(E_e) \times N_s(E_s),$$

where $N_e(E_e)$ is the number of states of the exterior with Energy E_e and similarly for $N_s(E_s)$. Consider a different macroscopic equilibrium state obtained by an adiabatic process that decreases E_e by dE and increases E_s by a corresponding amount. The corresponding number of microstates will be

$$N(dE) = N_e(E_e - dE) \times N_s(E_s + dE).$$

If our original state is the most probable, N must be the d maximum of $N(dE)$, obtained for $dE = 0$. Hence

$$\frac{d}{dE} \ln N(E) = 0.$$

Namely

$$\frac{dS_e(E_e)}{dE_e} = \frac{dS_s(E_s)}{dE_s} \equiv \frac{1}{T}.$$

That is, an equilibrium situation is characterized by a macroscopic parameter T which must be the same for the exterior and the surface and such that $dE = TdS$.

5.5 Black Hole Statistical Mechanics

Now, black hole thermodynamics derives a surprising set of simple laws just from classical general relativity and quantum field theory in curved spacetime. These laws have not been experimentally tested, but are very well motivated. However, they are thermodynamical "phenomenological" laws, and their derivation from first principles requires a quantum theory of gravity.

Phase space of a system is a space each of whose points represents an entire physical state. A single phase-space point provides all the microscopic coordinates of all individual constitutes of the physical system. We group together the all the states which look alike from the macroscopic properties.

Thus, for two completely independent physical systems, the total entropy of the two systems combined will be the sum of the entropies of each system separately.

The statistical ensemble is the region of phase space over which the system could wander if it were isolated, namely if it did not exchange energy with its surroundings.

from theomodynamics we see the temperature and entropy arises from underlying statistical mechanics. What microstaes are responsible for black hole thermodynamics?

a classical, stationary black hole is determined completely by its mass, charge, and angular momentum, with no room for additional microscopic states to account for thermal behaviour.

If black hole thermodynamics has a statistical mechanical origin then the relavent states must therefore be non-classical.

In the classical theory a realistic black hole with vanishing cahrge and angular momnetum evolves very rapidly towards the Schwarzschild solution, by rapidly radiating away all excess energy. In the qunatum theory, the Heisneberg principle prevents the hole from converging exactly to a Schwarzschild metric.

A microstate is not given by the Schwarzschild metric, but by some complicated time-dependent non-symmetric metric.

such time-dependent non-symmetric microstates of the geometry into account is essential for a statistical understanding of the thermal behavior of black holes

The interior degrees of freedom of the black hole are indistguishable to an exterior observer - classically because there is a causal barrier at horizon stops the interior effecting the exterior[1], hence these degrees of freedon do not contribute to the entropy and so don't effect the energy exchange between the black hole and the exterior.

5.5.1 First Law

There is no energy conservation that allows us to follow the usual statistical mechanical line of thinking. In fact, a strict attachment to the nonrelativistic notion of energy is not useful in the general relativistic context, where energy is a more delicate notion. Now, what is the role of energy conservation in the microcanonical setting? It is to reduce the region of phase space where the system is free to wander (ergodicity then tells us the systems are all over the allowed region). Now, in general relativity there is no analogous energy conservation. Thus, we have to look for a dynamical input from the Einstein equation that can play the same role that energy conservation plays in the nonrelativistic context. We have precisely what we need: the theorem stating that

> "the area of the horizon does not decrease".

The limit in which we disregard quantum effects, the area is compatible with the Einstein dynamics, as a parameter characterizing an equilibrium state. This theorem captures the relevant information from the general relativistic dynamics needed for understanding the statistical mechanics of the horizon. They play the same role as the microscopic energy conservation in the usual nonrelativistic thermodynamic context.

[1] assuming that the horizon is a strict causal barrier even at the quantum level.

The area is the only quantity that (classically) cannot decrease, and therefore the ensemble is formed by all the microstates with a certain area. The Einstein equation tells us also that if we throw a certain amount of energy dE across the horizon, then the area will increase by a certain amount. Hence a change in area is connected to a change in external energy. If we throw an energy dE inside the hole, then its horizon will fluctuate over a different ensemble, characterized by a larger area. There is no need to think that there is an actual "Energy" that remains on the surface degrees of freedom without continuing inside the hole. The Einstein dynamics allows us to apply the usual statistical mechanics logic, with just what is needed: a condition on the set of microstates in which the system can be. This is an answer to the question about the fact that energy enters the hole and therefore cannot thermalize on the surface degrees of freedom.

5.6 Entropy of Black Holes (Isolated Horizons)

Early idas: Smolin (95) Krasnov (96), Rovelli (96) More refined treatment: Ashtekar, Baez, Krasnov, Corichi, Lewandowski, Beetle. Fairhurst, Dreyer, Krishnan(99-00).

(Easy to read presentation: Ashtekar gr-qc/9910101)

They used the full theory but probes consequences of quatum geometry which are not sensitive to the full quantum dynamics.

Standard treatments isolated black holes are represented by stationary solutions of the field equations. This simple idealization it seems over restrictive.

Isolated horizons are generalizations of the event horizon of stationary black holes to physically more realistic situations. The generaliztion is in two directons. First, while one needs the entire space-time history to locate an event horizon, isolated horizons are defined using properties of space-time at the horizon. Second, although the horizon itself is stationary, the outside space-time can contain non-stationary fields and admit radiaton.

the event horizons can only be identified after knowing the complete evolution of the space-time. Consider fig (a) marginally trapped surfaces and would form part of the event horizon as nothing else hapens. Suppose instead (fig(b)) that after a long time, that a thin shell of mass δM collapses. Then Δ_1 would no longer be part of the event horizon which would lie slightly outside Δ_1.

Physically, it should be sufficient to impose boundary conditions at the horizon the horizon that ensure only the back hole itself is isolated. That a spacetime possess such a boundary defines a certain subset of the phase space of full general relativity.

Yet, the structure available on isolated horizons is sufficiently rich to allow a natural extension of the standard laws of black hole mechanics. Finally, cosmological horizons to which thermodynamical considerations also apply are special cases of isolated horizons. One can The precise notion of an isolated horizon can be arrived at by extracting

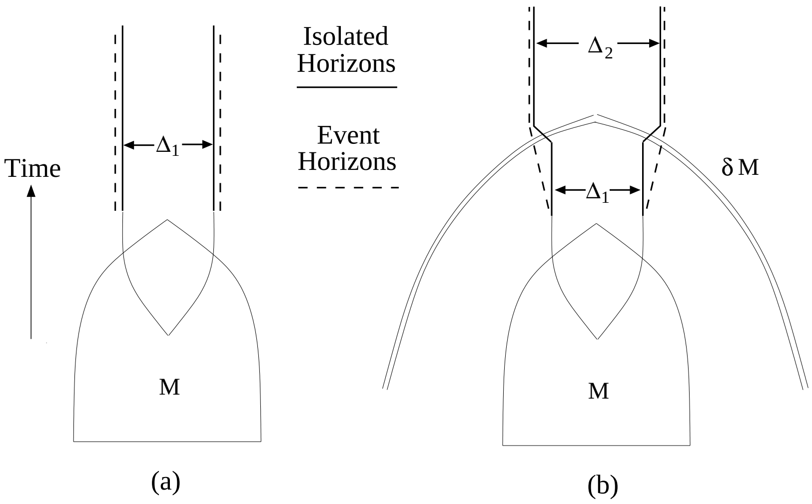


Figure 5.5: a) A spherical star of mass M undergoes collapse. b) Later, a spherical shell of mass δM falls into the resulting black hole. With Δ_1 and Δ_2 are both isolated horizons, only Δ_2 is part of the event horizon.

from the definition of a Killing horizon the minimum conditions necessary for black hole thermodynamics.

The first task consisits on choosing a surface that is reasonably close in its definition to a horizon, but that is an actual observable of the theory. This was accomplished by Ashtekar and collaborators through the notion of “isolated horizon”.

Since spherical symmetry is assumed only at the horizon, the concept embodies the natural idea that even in the dynamical spacetimes with radiation, even arbitrarly close to the horizon.

Δ is a null 3-surface $R \times S^2$. With zero shear and expansion.

Field equations hold on Δ.

(1) H is null and $\approx R \times S^2$

(2) S is outer marginally trapped

(3) no gravitational radiation or matter falls in H

(4) the area A of S is time-independent.

If one studies in detail the Einstein action with the set of boundary conditions above, one finds that for the action to be differentiable one needs to add boundary terms. The boundary terms have the form of the integral of a Chern-Simons form built with the Ashtekar connection.

One can then construct a quantum theory with Hilbert space,

$$H = H_{bulk} \otimes H_{surface} \tag{5.2}$$

These two spaces are not entirely disconnected, it turns out that the "level" k of the Chern-simons theory is determined by the bulk.

One then wishes to consider a microcanonical ensemble, in terms of the area, $(a_0 - \delta a, a_0 + \delta a)$. The quantum boundary conditions dictate that for a given state in the bulk that punctures P times the surface, the Chern-Simons state has its curvature concentrated at the punctures, one therefore has,

$$H^{phys}_{surface} = \oplus_P H^P_{surface} \tag{5.2}$$

Each puncture adds an element of area $8\pi\beta$ and introduces a deficit angle of value $2\pi m_i/k$ Where m is in the interval of $[-j_i, +j_i]$ and k is the "level" of the Chern-simons theory.

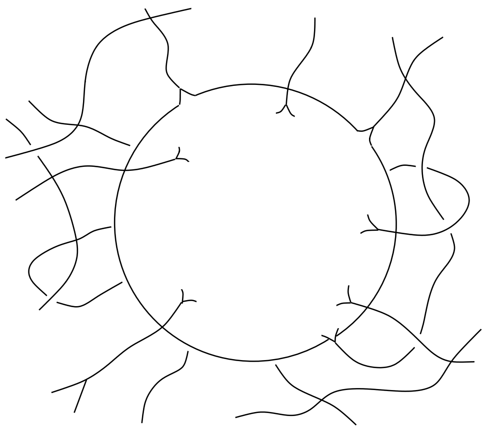


Figure 5.6: Quantum Horizon. Polymer exicitations in the bulk puncture the horizon, endowing it with quantized area. Intrinsically, the horizon is flat except at punctures where it aquires a quantized deficit angle. These angles add up to 4π.

The picture of the quantum geometry of the horizon that appears is that it is flat except at punctures where the lines of gravitational flux "pull" the surface up and introduce curvature. Let us state more precisely what we mean by this quantity: this curvature is defined in an intricsic way - imagine starting at a point p and moving a geodesic distance ϵ in all directions. What you would form is the closests thing to a "circle" on this surface. If a space is flat, the circumference C of the circle is $C = 2\pi\epsilon$. However, on a curved surface the circumference would be slightly smaller or greater that this depending on whether the surface has positive or negative curvature at the point p. Gauss suggested the following definition for curvature

$$K_{Gauss} = \lim_{\epsilon} \frac{6}{\epsilon^2}\left(1 - \frac{C}{2\pi\epsilon}\right). \tag{5.2}$$

The curvature of the isolated horizon is rather sigular: consider the diagram fig (5.7). The circle is flat. We glue together the edges to form a cone. The surface is flat yet by (5.6) there is curvature - the curvature must all be concentrated at the tip of the cone, whats known as a *canonical singularity* (vortices in a two-dinensional fluid with angular momentum however the vorticity is zero everywhere except at the centre).

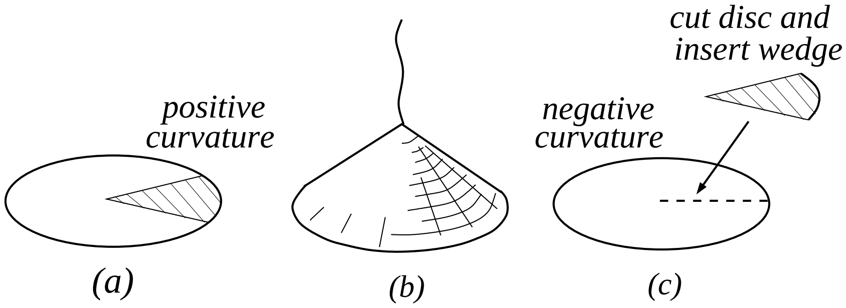


Figure 5.7: Deficit angle in horizon 2-geometry. (a) surface is flat (b) A 3-dimensional perspective: a bulk polymer excitation “exerts a tug” on the horizon exciting a quantum of curvature. (c) A cut is made in the disc and a wedge is incerted corresponds to negative curvature by (5.6).

At each puncture there is a quantized deficit angle. These deficit angles add up to endow the horizon with a 2-sphere topolgy. For a solar mass black hole, a typical horizon state would have 10^{77} punctures, each contributing a tiny deficit angle. So, although the quantum geometry is distributional, it can be well approximated by a smooth metric. Suppose one has a black hole. The area of the horizon will have an eigenvalue S. There are many rearrangements of the spin networks that yield the same eigenvalue. Counting the number of these quantum states gives a measure of the entropy of the black hole.

In the quantum theory neither the intrinsic geometry nor the horizon geometry are frozen; neither is a classical field. Each is allowed to undergo quatum fluctuations but because of the operator equation relating them, they have to flutuate in tandem.

They do not have complete control over the quatum Hamiltonian constraint. To proceed they make the rather weak assumptionthat for each microstate of the horizon geometry there corresponds at least one spin network state in the bulk that solves all the constraints of quantum GR

For large black holes the microsatates wich assign to each puncture the smallest quantum of area ($j = 1/2$) dominate the counting as they maximise the number of punctures required for a given horizon area. In this case, each contribution to the area of the horizon is $4\pi\sqrt{3}\gamma$. For each puncture m takes two possible values $\pm 1/2$. For n punctures, we have $AS = 4\pi\sqrt{3}\gamma n$ and entropy

$$\text{Entropy} \;\; S \approx \ln(2^n) = \frac{\ln(2)}{4\pi\sqrt{3}\gamma} A \tag{5.2}$$

We have agreement with Hawkings formula $S = A/4$ if we take $\gamma = \frac{\ln 2}{\pi\sqrt{3}}$. The smallest quantum of area is then $4\ln(2)$.

We have made use of the assumption that for each set of punctures on the horizon that there is at least one solution to the hamiltonian constraint. This allowed us to write

S $= \ln N + n\ln 2 \approx n\ln 2$

where N is the number of bulk solutions, with $N > 0$. Since thermodynamic entropy is defined only up to an additive constant, we may argue that the bulk states do not play any role in black hole thermodynamics.

It is good that the result (log) is proportional to the area (without the tight boundary conditions one gets results proportional to square root of area, for instance). The results depends on a free parameter, the Immirzi parameter γ.

Considering black holes with elecromagnetic or dilatonic charge one finds that one gets the correct result for the same value of the Immirzi parameter. The result has been extended to rotating black holes apparently! In this asnalysis, all black holes and cosmological horizons are treated in a universal fashion; there is no restriction , e.g. to near-extremal black holes.

To determine the Immirzi parameter γ we called on Hawkikng's result that seemed like cheating. However, it has been argued that by looking at the quasinormal damped modes of a classical black hole one is able to derive the quanta of area in a different way. They conclude that $\Delta A = 4\ln(3)$. But this would correspond not to a minimum value $j_m in = 1/2$ but to $j_m in = 1$. There is the sugestion that one should think of a conserved fermion number being assigned to each spin-1/2 edge so that

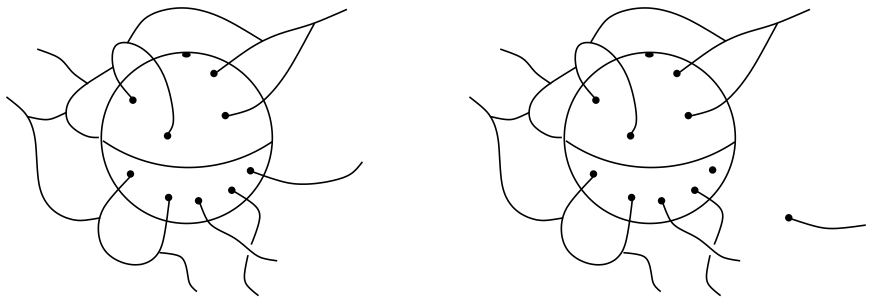


Figure 5.8: An example of the emission in which one of the flux lines breaks, with one of the ends falling into the black hole and the other escaping to infinity.

In 1974, Steven Hawking showed that black holes do radiate quantum mechanically, thereby shinking in area. This is a strong hint that the geometrical area of a black hole horizon can be converted into matter. Only matter was treated quantum mechanically; there were no quanta of area. With quantum geometry, they could re-examine the situation. Now, it is literally true that the black hole horizon aquires its area through its interactions with polymer geometry. In the Hawking process, quanta of area are converted to quanta of matter.

Details: Illustrative example: Maxwell Field in Minkowskian space-time outside $r = r_0$.

$$S = \int_{\mathcal{M}} d^4x \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} = -\frac{1}{4} \int d^4x \; F_{\mu\nu}(x) F^{\mu\nu}(x) \tag{5.2}$$

Boundary conditions:

1. At ∞: Standard ones.
2. At Δ: $F^{(\Delta)}_{\mu\nu} = \lambda \, {}^*F^{(\Delta)}_{\mu\nu}$

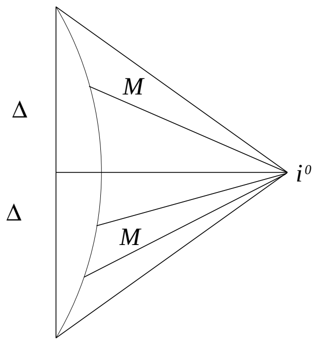


Figure 5.9: Maxwell Field in Minkowskian space-time outside $r = r_0$.

Then to make the action differentiable we have to add a surface term.

$$S = \int_{\mathcal{M}} d^4x \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \underbrace{\frac{\lambda}{2} \oint_{\Delta} \epsilon^{\mu\nu\sigma} F_{\mu\nu} A_{\sigma}}_{U(1) \; Chern-Simons} , \tag{5.2}$$

or in differerntial geometry language

$$S = \int_{\mathcal{M}} F \wedge {}^*F + \frac{\lambda}{2} \underbrace{\oint_{\Delta} F \wedge A}_{U(1) \text{ Chern−Simons}} \tag{5.2}$$

Chern-Simons for Maxwell field

5.6.1 A new Hint: Quasinormal Modes of Black Holes

[155]

Corrections to counting

In [138], [139] it is shown that the initial calculation of the blackhole entropy was actually incorrect: there was a miscounting of the microscopic states contributing to the entropy as states labeled with higher spins also contribute to the entropy calculation.

one has that [139]

$$\frac{\ln 2}{\pi} \leq \gamma_1 \leq \frac{\ln 3}{\pi}. \tag{5.2}$$

Further issues:

• The Barbero-Immirzi pararmeter remains somewhat puzzling. quasinormal modes of black holes, still ...

• It is largley kinematic and doesn't address dynamical issues to do with black hole formation or black hole evapouration

• It does not address "what happens at the singularity.

See zero mass light shells [].

These last two issues are also being studed using the techniques which have already been applied to the subject of the next section.

5.6.2 Entropy of Rotating and Axially-Symmetric Distorted Blackholes (Type II Isolated Horizons)

the geometry is assumed to be axisymmetric. These are more representative of equilibrium states of *generic* horizons in realistic astrophysical and comological situations.

5.7 Black Hole Information Paradox

$$-i\hbar \frac{\partial}{\partial t} = \mathcal{H} \tag{5.2}$$

conservation of probabilities requires evolution to be *unitrary.*

In GR there is no time evolution

$$0 = \mathcal{H} \tag{5.2}$$

Probabilities preserved with respect to what? there is no time evolution - obseverables don't depend on the coordinate time and so there is no reason why the Hamiltonian needs to be unitrary.

We do have relational evolution. Since there is no such thing as a idel clock - the variable representing the clock too is subject to quantum fluctuations. Hence, there is a *natural decoherence*. We must answer wether the decoherence comming from the evapouration of the black hole is greater than the natural decohernce.

The system has an unitary evolution in n. At t cannot br perfectly correlated with n, even in the semi-classical regime of the clock, the evolution in t is not perfectly unitary. In fact one can show that the density matrix evolves according to

$$\frac{\partial}{\partial t}\rho_2 = -i[\mathcal{H}_2, \rho_2(t)] - \sigma[\mathcal{H}_2, [\mathcal{H}_2, \rho_2(t)]]. \tag{5.2}$$

This equation was first proposed by Milburn based on phenomenological arguments, and is a particular tpye of non-unitary evolutions considered by Lindblad.

Their derivation allows to estimate σ that is of order of the Planck time.

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t} \tag{5.2}$$

This equation doesnot violate the conservation of energy like Hawking prosal for information loss. One could exect to confirm this type of equation by studying some mesoscopic quatum systems.

Information loss problem in Black Holes

It provides a new and very effective mechanism for treating the information loss problem in Black Holes.

They have shown that for any Black hole bigger than 600 Plank masses the information loss induced by our equation is enough to dissipate all the black hole information before to its evaporation.

For very small black holes, Hawking's semi-classical analysisis not valid.

5.7.1 Singularity Avoidance

Quantum Blackholes

Using ideas first developed in the context of quantum cosmology, effects of the quantum nature of geometry on blackhole singularities have been recently analyzed [156] (Ashtekar and Bojowald in preparation). It is found that the black hole singularity is resolved but the classical spacetime dissolves in the Plank regime.

Singularity avoidance by collapsing shells in quantum gravity

Quantum Theory of Gravitation Collaspe - Zero Mass Light Shells

To investigate singularity we have to find a different approximation to WKB.

Isolated Horizons and Dynamical Horizons and Their Application [137]

5.8 Black Holes in Full Quantum Gravity

We would like to study black holes in the full non-perturbative quantum theory, without recurring to semiclassical considerations. Here a proposal is given of a definition of a quantum black hole as the collection of the quantum degrees of freedom that do not influence observables at infinity. From this definition, it follows that for an observer at infinity a black hole is described

So far, however, the description of black holes has relied on some mixture of quantum theory and classical analysis of black hole geometry: for instance, one can characterize a black hole classically, and then quantize the part of the classical theory phase space that contains the black hole. Is it possible, instead, to describe black holes entirely within the non perturbative quantum theory of spacetime?

In this section a suggestion is given for a direction for answering this question. We propose a simple definition of a quantum black hole within the full quantum loop theory, as a region of a spin network which is not "visible" from infinity. This is in the same spirit of the global analysis that is possible in classical general relativity, where properties of horizons and black holes can be obtained by studying their implicit definition, even without being able to solve the equations of motion and writing the metric explicitly

From the point of view of the outside observer, a black hole behaves as a possibly (gigantic) single interwiner, which interwines all the links puncturing its horizon.

The notion of a horizon used here is based on the traditional one (the boundary of the past of future null infinity), and has the same limitations. It would be interesting to find an extension of the above construction that could capture also the notion of an isolated horizon. In particular we could extend the result presented here also to the scenario where information is recovered during, or at the end, of the Hawking evaporation, and where, according to the traditional definition, there is no horizon.

Another suggestion

To fully understand the final evolution of black holes one needs a description...

[145] do not allow for the analysis of black hole evaporation.

Formulate quantum versions of the horizon boundary conditions and impose them at the quantum level so that they capture the intuiative idea of that classical black holes are defined by the presence of surfaces from which light cannot escape outward. They can used to identify quantum staes that describe quantum black hole. They allow for a defintion of

various dynamical processes of quantum black holes such as formation, mergers and evaporation.

5.8.1 Quantum geometry and the Schwarzschild singularity

[162]

5.9 Strings and Black-Hole Entropy

Andrew Strominger and Cumrum Vafa

Extremal brane systems turn out to share many properties with extremal black holes. In particular, the thermodynamical properties of the two systems are identical.

The original calculation pertained to 5-dimensional spacetime. Later results do apply to 4-dimensional spacetime, but the initial overblown proclamations were elicited by the original 5-dimensional calculation.

All "these string theory string-theoretical results referred only to the extremely special limitimg case of an 'extremal hole' (or to pertubations away from this) for which the Hawking temperature is zero - and where the hole involves additional supersymmetric Yang-Mills-type fields which have no clear justification from known physics".

It has not been settled whether the results correspond to a relationship between the two systems that is just an accidental consequence of the fact that both have a lot of extra symmetry and do not lead to general insights about black holes, or whether on the contrary, that all black holes can be understood using the same ideas, and that the extra symmetries present in special cases simply allow us to calculate more precisely.

Moreover, the calculations were performed in flat space, where there is no actual event horizon. In the calculation, you begin by turning off the gravitational force (you do this by putting the string coupling constant to zero). It is conjectured that they would become black-holes if the gravitational force is slowly turned on. But the problem is that it has been very difficult to define consistent string theories (or any compellingly proof that such a string theory exists) where the graviational field changes in time.

Do the black hole results of string theory concern actual black holes or do they just concern ensembles of states in free string theory in flat spacetime, with gravitational coupling turned off?

5.10 Quantum Cosmology

We make no attempt to make sense of the quantum theory of a single universe in which the notion of an external observer has no place, and usual probabilistic interpretation of the wavefunction is questionable.

Take semi-classical solution an evolve backwards using the Wheeler-De Wit equation you still end up with a singularity [].

Restrict techniques of loop quantum gravity to cosmologies with compact (isotropic) and homogeneous spacelike silces. Take Thiemann's ideas on the Hamiltonian constraint and apply them to mini super space quantization.

5.10.1 Classical theory

$$d\tau^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{5.2}$$

Where a is the scale factor, and in solving Einstein's equations the task is to determine a as a function of t.

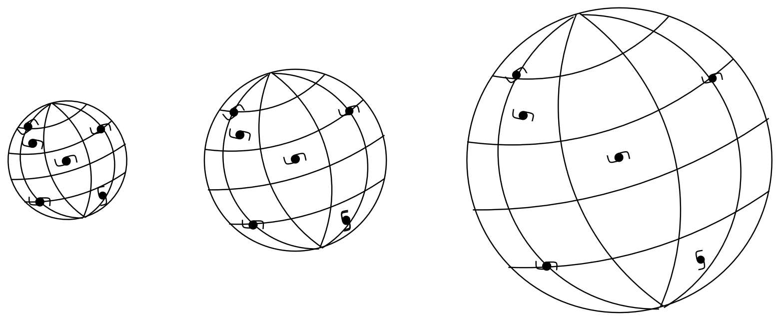


Figure 5.10: Expansion of closed universe.

The space itself is brought into being at the big bang. It is only for visualization purposes that the sphere sits in the 3-dimensional Euclidean space; we imagine we are a 2D creature living on the sphere and that there is no "ambient space" (it is not contained in a further space - here the Euclidean space), in which the sphere is embedded. a hypershere - a 3-dimensional version of a spherical surface, we have the ability to circumnavigate the univeerse by travelling always in the same direction until one returns to one's starting point from the other way. there is no boundary. We can't envisage a hypersphere mentally, but we can .

The universe is *expanding* - the further away we look, the more rapidly the distant galaxies homogeneous means "the same at every point" and isotropic about a point p means that all directions at p are equivalent. The same recessiion of galaxies is seen wherever one is located in the universe. The expansion is uniform on a large scale,

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\gamma} g_{\beta\delta} - R_{\beta\gamma} g_{\alpha\delta} + R_{\beta\delta} g_{\alpha\gamma} - R_{\alpha\delta} g_{\beta\delta} - \frac{1}{2} R (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}). \tag{5.2}$$

Consider R^α_β which is a symmetric matrix and hence always has a complete set of orthogonal eigenvectors.

$$R^\alpha_\beta l^\beta = \lambda l^\alpha \tag{5.2}$$

Unless $R^\alpha_\beta = \delta^\alpha_\beta$ the eigenvectors would pick out a preffered direction. Hence, $R_{\alpha\beta} = K g_{\alpha\beta}$ for some K. Taking the trace we have

$$R_{\alpha\beta} = \frac{1}{3} R g_{\alpha\beta}, \tag{5.2}$$

substitutaing this into (N.-19) we obtain

$$R_{\mu\nu\gamma\delta} = \frac{1}{6} R (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}). \tag{5.2}$$

The clocks move with space not through it.

Accelaration Equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho \left(1 + \frac{3w}{c^2} + \frac{\Lambda}{3} \right) \tag{5.2}$$

Distant objects can *appear* to move away faster than the speed of light. However, it is space itself which is expanding - the expansion of space is just like that of the balloon, and pulls the galaxies along with it. Special relativity refers to the relative speeds of objects passing each other, and can not be used to compare the relative speeds of distant objects.

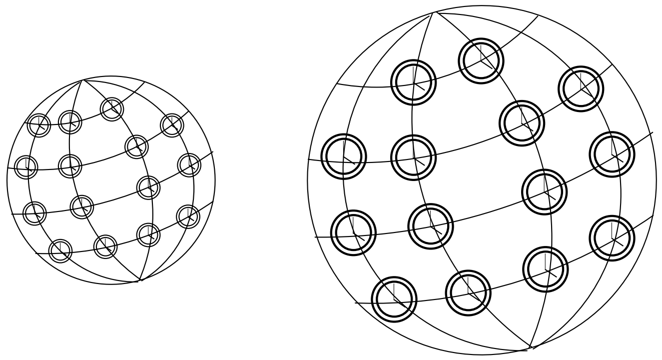


Figure 5.11: Cosmic time.

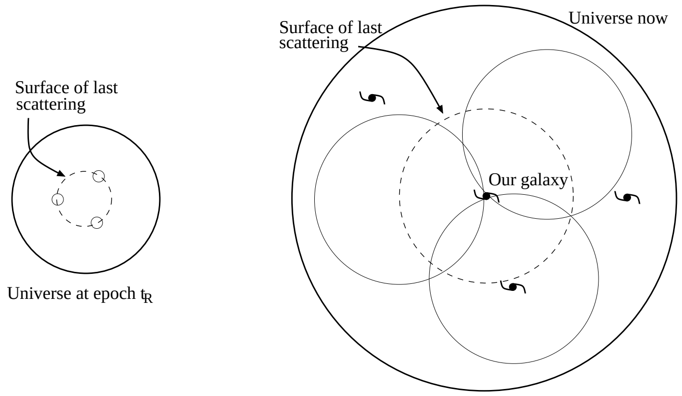


Figure 5.12: (a) Photon. (b) At the last scattering surface, the photons had a much higher frequency, which has been redshifted as the photons travel toward us.

The universe has expanded by a huge factor sinse the time of the Big bang, this initial fireball has dispersed by an absolutely enormous factor.

Standard Model of Cosmology

Models with a Cosmological Constant

5.10.2 Problems in Classical Cosmology

Dark Matter

There are two different basic ways of accessing the actual average density of matter in the universe.

There is missing matter that we can't see it referred to as dark matter.

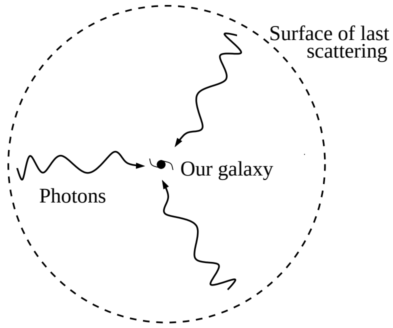


Figure 5.13: At the last scattering surface, the photons had a much higher frequency, which has been redshifted as the photons travel toward us.

Cosmological Constant

When positive it causes the universe to expand and to expand at an accelerating rate.

Ordinary matter causes the universe to contract because of mutual gravitational attraction of all the matter it contains.

Quantum theory appears to require a huge cosmological constant. If something is at rest it has definite position and momentum but this contradicts the uncertainty principle. As a consequence and degree of freedom cannot have zero energy. But a field

Quantum theory predicts a huge vacuum energy and by general relativity a huge cosmological constant. We know this is wrong.

However, we must keep in mind that the above considerations were made in the context of the semiclassical approximation of quantum fields on a fixed background. Let us go into more details. In this approximation the fixed background is given by the equation

$$G_{\mu\nu} + \Lambda^0 g_{\mu\nu} = 8\pi G \ < \Psi|\hat{T}_{\mu\nu}|\Psi > \tag{5.2}$$

where Λ^0 is a fundamental (or bare) cosmological constant and $|\Psi>$ is a "vacuum" state. Here the expectation of $\hat{T}_{\mu\nu}$ contributes to the observed cosmological constant and leads to a quantum state dependent effective cosmological constant Λ given by

$$\Lambda = \Lambda^0 - 2\pi G \ < \Psi|\hat{T}_{\mu\nu}|\Psi > \tag{5.2}$$

If the background is Minkowski or highly symmetric spacetime, the matter field vacuum is more or less unambiguous. In such cases

$$< \Psi|\hat{T}_{\mu\nu}|\Psi > = \rho g_{\mu\nu} \tag{5.2}$$

holds at least to leading order in $\hbar$, and can say that ρ in this equation contributes to the cosmological constant.

The observed value of the cosmological constant using the cold dark matter model is of the order 10^{-120} in Plank units, whereas the theoretical contribution to it from ρ alone, using a suitable cutoff, is far from this number. Indeed, to have a match between theory and experiment requires the finely tuned cacellation of Λ^0 and ρ to 120 decimal places. This is the usual statement of the cosmological constant problem.

Does the problem really exist at a more fundamental level in a non-perturbative context, outside the semiclassical approximation where there is no fixed background spacetime?

We should really be seeking a vacuum (or ground state) of a full non-perturbative- matter-geometry Hamiltonian derived from general relativity coupled to matter fields

Having determined a non-vanishing Hamiltonian, the task is then to find its ground state(s) $|q, \phi >_0$ and compute the ground state (or vacuum) energy. It is at this stage that there may be an emergent "vacuum energy problem" if the energy of the relevant state of $\hat{H}$ does not match the observed one.

From simply models it is demonstrated that at the non-perturbative level there is a relationship between the cosmological constant, time and the vacuum energy which is rather complex, and fundamentally different from what one would conclude from naive use of the semiclassical equation.

Dark Energy

The expansion of the universe appears to be accelerating, whereas, given the observed matter plus the calculated amount of dark matter, it should be decelerating. One proposed explaination is that there is this strange new energy, postulated to fit the data, thhe so called dark energy.

We do not know what the dark energy is. We only know about it because we can measure its effect on the expansion of the universe. It manifests itself as a source of gravitational attraction spread uniformaly through space. Its only effect it can have is on the average speed at which the galaxies move away from each other. In 1998 observations of supernovas in distant galaxies appeared to show that the expansion of the universe was accelerating in a way best explained by the existtence of dark energy.

The Flatness Problem

$$\Omega \approx 1 \text{ to an accuracey of } 10^2. \tag{5.2}$$

The Horizon Problem

The CMB looks more or less the same no matter which direction we look at in the sky. The CMB was emitted when the universe was about 3000,000 years old. Any area in the universe that is that has size gerater than 1 degree

Causal communication between two regions A and A' can eist only if they are within a distance $2R_H(t) = 4t$. Two regions separated by a proper distance greater than $2R_H(t)$ at epoch t could never have influenced each other. Hence, there is non a priori reason to expect points A and A' to have smilar physical environments.

5.10.3 Inflationary Universe

There is a big difference between this scenario and the old idea that the while universe was created at the same moment of time (Big Bang), in a nearly uniform state with indefinitely large temperature.

Inflationary cosmology has been regarded as an attractive scenario. Loop quantum cosmology is the first direct link to a propsed fundamental theory.

Power-law Inflation

[hep-ph/0505249]

$$a(t) = \text{const. } t^p \tag{5.2}$$

Slow-roll Approximation

$$H^2 = \frac{8\pi}{3M_p^2} V(\varphi) \tag{5.2}$$

5.10.4 Matter Driven Inflation

There are two equations that describe the evolution of a homogoneous scalar field in the model, the field equation

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi, \tag{5.2}$$

and Einstein's equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{5.2}$$

The first equation is similar to the equation of motion of a damped harmonic oscillator, where instead of $x(t)$ we have $\phi(t)$. The thrid term is analogous to the term describing friction in the equation for a harmonic oscillator.

Inflation was a period of the early universe when $\dot{a}$ increses

$$\ddot{a} > 0 \tag{5.2}$$

The standard is a material with the unusual property of having negative pressure. However, such a particle plays a key role in particle physics theory of the electrweak force. for now we just The pressure and energy of the inflation field are given by

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{5.3}$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{5.4}$$

where $V(\phi)$ is the potential energy. During the inflationary phase the inflation field . This is

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{5.4}$$

By substituting the scalar field pressure and energy into the fluid equation (**??**), we obtain the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \tag{5.4}$$

where $'$ denotes differentiation with respect to ϕ. will sustain inflation while

$$\dot{\phi}^2 < V \tag{5.4}$$

i.e. while the kinetic energy is small. This is called the *slow-roll approximation.* Friedmann equation

$$H^2 \approx \frac{8\pi G}{3}V \tag{5.4}$$

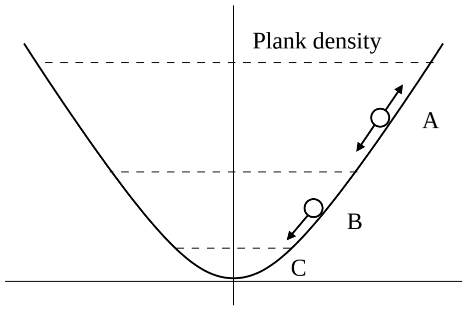


Figure 5.14: Motion of the scalar field in the theory with $\frac{m^2}{2}\phi^2$..

and the corresponding Klein-Gordon equation is

$$\dot{\phi}^2 \approx -\frac{V'}{3H}. \tag{5.4}$$

$|V'|$ is very small so H is nearly constant implying that $a(t)$ grows nearly exponentionally

$$\frac{\dot{a}}{a} \approx const. \tag{5.4}$$

The amount of inflation can be measured in terms of e-folding number N given by

$$N(t) = \ln\left(\frac{a(t_{after})}{a(t_{initial})}\right) \tag{5.4}$$

we need at least 50 e-folding to solve the hot Big Bang problems.

Reheating

During inflation the universe is supercooled by very rapid expansion. The radiation produced by this procces starts to *reheat* the universe, whereby the period of inflationary expansion gives way to the standard hot Big Bag phase.

The key point is that at the time of reheating, the entropy of the universe increases by a large factor, and this drives the universe towards spaciatial flatness, as can be seen from the FRW equations (N.-19).

5.10.5 Canonical Quantization

$$p_i = -i\frac{\partial}{\partial q^i} \tag{5.4}$$

Since the Hamiltoian is quadratic in p_i one obtains a second order differential equation called the Wheeler-DeWitt equation.

WKB solutions

the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\psi(x) = E\psi(x) \tag{5.4}$$

with the WKB solution of the form

$$\psi(x) = e^{\frac{i}{\hbar}S(x)}, \tag{5.4}$$

may lead to the Hamilton-Jacobi equation in the limit $\hbar \to 0$,

$$\frac{1}{2m}(\nabla S)^2 + V(x) = E. \tag{5.4}$$

With the identification $p = \nabla S$ and $H = E$,

The same idea may be applied to quantum cosmology.

$$\Psi(\Omega, \phi) \sim \exp\left(\pm\frac{1}{3V(\phi)}(1 - e^{2\Omega}V(\phi))^{3/1})\right), \tag{5.4}$$

when the scale factor is large, the WKB solutions form

$$\Psi(\Omega, \phi) \sim \exp\left(\pm\frac{1}{3V(\phi)}(e^{2\Omega}V(\phi) - 1)^{3/1})\right). \tag{5.4}$$

5.10.6 Why does the Wheeler-DeWitt Equation fail?

$$-\frac{1}{6}l_P^4 a^{-1}\frac{d}{da}\left(a^{-1}\frac{d}{da}(a\psi(a,\phi))\right) = \kappa\hat{\mathcal{H}}_{matter}\psi(a,\phi) \tag{5.4}$$

quation still singular, possible rescue

boundary conditions,

e.g. $\psi(0) = 0$ motivaed by intuition (DeWitt)

no-boundary proposal, (Hartle-Hawking)

tunneling (Vileukia)

advantage: selects a unique state (up to norm) appropriate for a unique universe

but not enough to remove singularity:

a-spectrum continuous arbitrarity to $a = 0$ singulariy point of creation.

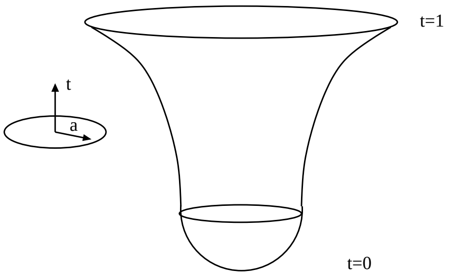


Figure 5.15: The universe emerging from nothing.

Wheeler-DeWitt Equation

Symmetry redudction at the classical level and then quantization. No direct relation to quantum gravity. Not derived from quantum gravity.

Loop quantum gravity is a well defined quantum theory at kinematic level (prior to imposing the Hamiltonian constraint) Symmetric reduction at this level after quantization.

5.11 Loop Quantum Cosmology

$$d\tau^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{5.4}$$

that we have because we formulate the theory with triads instead of the metric. We can perform a Gauss gauge transformation without changing the metric they encode.

$$A_a \mapsto g^{-1} A_a g + g^{-1} \frac{\partial g}{\partial x^a}, \qquad E^a \mapsto g^{-1} E^a g \tag{5.4}$$

where A_a and E^a are the matrices $A^i_a \tau_i$ and $E^a \tau_i$ respectively.

We can rotate them with an $SO(3)$ element without changing the metric they encode.

$$
\begin{aligned}
qq^{ab} &= E^a_i E^b_j \delta_{ij} \\
&= \mathrm{tr}(E^a E^b) \\
&= \mathrm{tr}(g^{-1} E^a g g^{-1} E^b g)
\end{aligned} \tag{5.3}
$$

the Poisson brackets of any two functions f and g on this phase space is given by:

$$
\{f, g\} = \frac{\kappa\gamma}{3}\left(\frac{\partial f}{\partial c}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial c}\frac{\partial f}{\partial p}\right) \tag{5.3}
$$

because of homogeneity and isotropy, we do not need all edges e and surfaces S. Symmetric connections A in $\mathcal{A}$ can be recovered knowing holonomies $h(e)$ along straight lines in $\mathcal{M}$. Similary,

Hamiltonian constraint uses volume operator, creation of loops - It is difficult to find solutions with interesting classical correspondence. There are quantum ambiguities. We can exploit simplified context of symmetric models.

The Hamiltonian constraint is a *difference equation*

$$
\hat{\mathcal{H}}|n >= f(n)(|n + 4 > -2|n > |n - 4 >) \tag{5.3}
$$

where $|n >$ iare eigenstates of the volume operator. Using the volume of the universe as "time", we take $\hat{\mathcal{H}}\psi = 0$ and interpret it as a discretized time evolution equation.

evolution equations don't breakdown even though the volume of the universe goes to zero!

$$
\mathcal{H} = -12\gamma^{-2}\kappa^{-1}(c(c - k) + (1 + \gamma^2)k^2/4)\sqrt{|p|} \tag{5.3}
$$

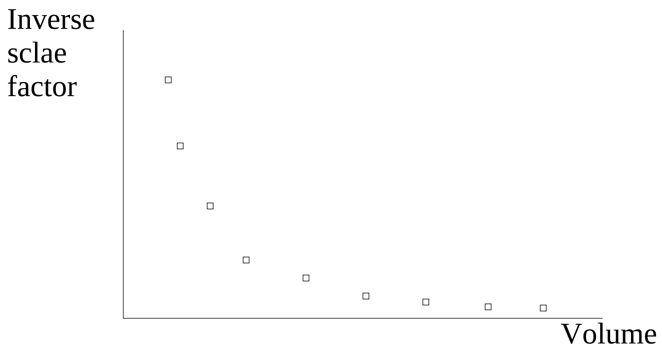


Figure 5.16: inversescale.

One has $|p| = a^2$ while $c = (k - \gamma\dot{a}$. If we insert this constraint equation $\mathcal{H} + \mathcal{H}_{\text{matter}}$

journal Advances in Theoretical and Mathematical Physics:

"The question of whether the universe had a beginning at a finite time is now 'transcended'. At first, the answer seems to be 'no' in the sense that the quantum evolution does not stop at the big-bang. However, since space-time geometry 'dissolves' near the big-bang, there is no longer a notion of time, or of 'before' or 'after' in the familiar sense. Therefore strictly, the question is no longer meaningful. The paradigm has changed and meaningful questions must now be phrased differently, without using notions tied to classical space-times."

• a^{-1} bounded (curvature cut-off)

• non-singular evolution (matter-independent, fixes ordering)

• dynamical intial conditions

• Wheeler-DeWitt quantum cosmology as continuum limit applies to realistic spacetimes

• Further insight from complicated models

• All except a finite number of degrees of freedom are "frozen". The shift function is zero and the lapse is homogeneous. N is still a function of τ, so that separation beween two succesive three-surfaces is still undetermined. Reparametrization invariance is what remains of general covariance of the full theory.

• Are there corresponding solutions in full theory?

• Only considering states that are symmetric at the microscopic level.

5.11.1 Details of the LQC Hilbert Space Representation

holonomies can be evaluated, starting from those associated to straight paths $\mu^0 e^i_a$, which gives

$$h(A)_a = \cos\frac{\mu c}{2} I + 2\sin\frac{\mu c}{2}\tau_a. \tag{5.3}$$

Hence, one may take almost periodic functions $N_\mu = e^{\frac{i\mu c}{2}}$ as a basis in the configuration space. The algebra generated by $\{N_\mu, p\}$ plays the same role as the holonomy-flux algebra. The analogous construction of full LQG turns out to be

$$\mathcal{H}_{kin} = L_2(\mathbb{R}_{Bohr}, d\mu_{Bohr}),$$

where $\mathbb{R}_{Bohr}$ is the so-called Bohr compactification of the real line. An orthonormal basis is given by the N_μ's, with measure

$$< N_{\mu'} | N_{\mu} > = \delta_{\mu'\mu}. \tag{5.3}$$

One has a countable set of basis vectors despite μ being conitinuous. The states are normalizable - this represents discreteness in a weaker sense.

The action of the operators is given by

$$N_{\mu}\psi(c) = e^{\frac{i\mu c}{2}}\psi(c), \qquad p\psi(c) = -i\frac{8\pi\gamma l_P^2}{3}\frac{d}{dc}\psi(c) \tag{5.3}$$

so that states $|\mu >$ can be defined so that

$$< c|\mu > = e^{\frac{i\mu c}{2}}, \qquad p|\mu >= \frac{8\pi\gamma l_P^2}{6}\mu|\mu > \tag{5.3}$$

We only consider the case $k = 0$.

The evaluation of the Hamiltonian constraint involves teh expression of the field strength F_{ij}^a, which can be obtained by the limiting procedure from holonomies.

The regualrization procedure involves the fixing of a minimum value $\mu = \overline{\mu}$, just as in the general case.

Since the Hamiltonian involves the volume operator V, it is advantageous to introduce the basis

$$|v > = \frac{2^{3/2}}{3^{7/4}} sign(\mu)|\mu|^{3/2}|\mu >,$$

such that

$$V|v > = \frac{2^{3/2}}{3^{7/4}}\left(\frac{8\pi\gamma}{6}\right)^{3/2} l_P^3|v||v > \tag{5.3}$$

Then the Hamiltonian H reads

$$H^{\overline{\mu}} = sign(p)\frac{4i}{8\pi\gamma^3\overline{\mu}^3 l_P^2}\sin^2\overline{\mu}c\left(\sin\frac{\overline{\mu}c}{2}V\cos\frac{\overline{\mu}c}{2} - \cos\frac{\overline{\mu}c}{2}V\sin\frac{\overline{\mu}c}{2}\right) \tag{5.3}$$

It is worth noting that the limit $\overline{\mu} \to 0$ does not exist. There are different ways of fixing $\overline{\mu}$; one choice is

$$\overline{\mu}|p| = 2\sqrt{3}\pi\gamma l_P^2,$$

such that the minimum area enclosed by a loop is given by the minimum area gap predicted by LQG.

The expression (5.11.1) can be made Hermitian by a certain factor ordering (though not necessary, improves solving the problem)

$$H^{\overline{\mu}} = sign(p)\frac{4i}{8\pi\gamma^3\overline{\mu}^3 l_P^2}\sin\overline{\mu}c\left(\sin\frac{\overline{\mu}c}{2}V\cos\frac{\overline{\mu}c}{2} - \cos\frac{\overline{\mu}c}{2}V\sin\frac{\overline{\mu}c}{2}\right)\sin\overline{\mu}c \tag{5.3}$$

which, when applied to $\psi(v)$, leads to the following difference equation

$$H^{\overline{\mu}}\psi(v) = f_+\psi(v+4) + f_0\psi(v) + f_-(v-4) \tag{5.3}$$

with

$$\begin{aligned} f_+(\mu) &= \frac{3^{5/4}}{2^{5/2}}\sqrt{\frac{8\pi}{6}}\frac{l_P}{\gamma^3/2}|v+2|\Big||v+1| - |v+3|\Big| \\ f_-(\mu) &= f_+(v-4), \qquad f_0(\mu) = -f_+(\mu) - f_-(\mu). \end{aligned} \tag{5.3}$$

Striking aspects of LQC is the replacement of the Wheeler-DeWitt differential equationn with a difference one. When a clock-like scalar field ϕ is introduced, the full Hamiltonian reads

$$H_{tot} = H + 8\pi G\frac{p_\phi^2}{|p|^{3/2}} \tag{5.3}$$

and canonical quantization leads to the follwoing difference equation for the dynamics

$$\frac{\partial^2}{\partial\phi^2}\Psi(v,\phi) = B^{-1}(v)H\Psi(v,\phi) = \Theta\Psi(v,\phi) \tag{5.3}$$

where the inverse volume operator's eigenvalues are

$$B(v) = \frac{1}{|p|^{3/2}} = \frac{3^{5/4}}{2^{3/2}}|v|\Big||v+1|^{1/3} - |v-1|^{1/3}\Big|^3. \tag{5.3}$$

The boundedness of the operator corresponding to $|p|^{3/2}$ was the first sign towards teh resolution of the cosmological singularity.

$$i\frac{\partial^2}{\partial\phi^2}\Psi(v,\phi) = -\sqrt{\Theta}\Psi(v,\phi) \tag{5.3}$$

The physical Hilbert space can be induced from $\mathcal{H}_{kin}$ by the group averaging technique.

If one starts from a semi-classical universe, described by a state sharply peaked around $\mu = \mu^* \gg \overline{\mu}$, and evolves it backwards in time, the state remains semi-classical during the evolution and a bounce occurs for $\mu \sim \overline{\mu}$, followed by an expansion phase.

5.11.2 On the Reliability of the Results of LQC

The scale operator, defined on the Kinematic Hilbert space, does not correspond to a Dirac observable as it does not commute with all the first class constraints generating the gauge transforamtions. Although the inverse scale factor operator commutes with the other constraints, it does not commute with the Hamiltonian constraint (except for very special choices of $\hat{C}_{matter}$). Thus the inverse scale factor operator is not a physical operator and its spectrum can not be considered as an indication of possible measurements of the spatial curvature. Consequently, the fact that this spectrum is bounded from above does not give a reliable indication that the kinematic singularity is avoided. In order for a legitiment conclusion to be drawn, a physical operator correspnding to an observable encoding spactial curvature is neccessary.

Although the inverse scale factor operator doesn't commute with the Hamiltonian constraint and does not qualify as a Dirac observable, its commutator with both the $SU(2)$ and the spatial diffeomorphism constraint vanishws. Thus, it constitutes an operator corresponding to a partially invariant partial observable [301]. As explained in chapter 1, one can turn such a partial observable into a Dirac obesrvable invariant under all the constraints. However, the spectrum of the operator corresponding to the Dirac observable will in general differ from the Kinematic spectrum, and sometimes drastically so. This means that the fact that the inverse scale factor operator is bounded from above cannot be used to produce reliable statements concerning the removal of the classical singularity in LQC [217].

5.11.3 Inflation from Loop Quantum Cosmology

effective Friedmann equation

A generalized Hamiltonian Constraint Operator in Loop Quantum Gravity and its simplest Euclidean Matrix Elements, Class.Quant.Grav. 18 (2001) 1593-1624, gr-qc/0011106

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{16\pi}{3}Ga^{-3}\left(\frac{1}{2}a^{-3}p(3a^2/jl_P^2)^6p_\phi^2 + a^3V(\phi)\right). \qquad (5.3)$$

Since the right hand side now depends on a for small a the classical behaviour, the dynamics is clearly modified.

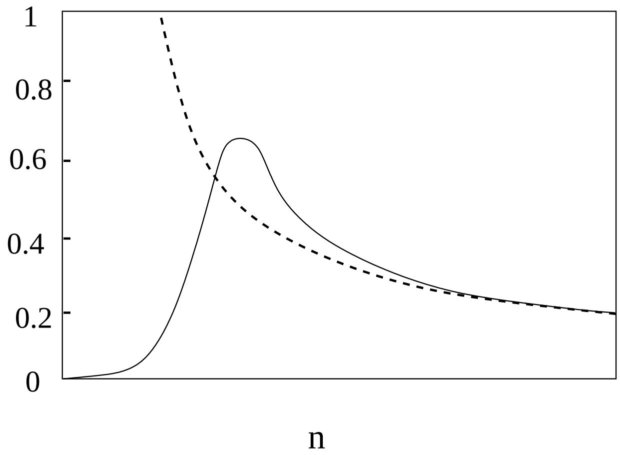


Figure 5.17: eigendensity Eigenvalues of the density operator for twp choices of the ambiguity parameter, compared to the classical expectation (thick, dashed).

The scalar field is quantum mechanical and will have characteristic quantum fluctuations. It is possible that these quantum fluctuatons to eventually manifest as classical density fluctuations.

5.12 Path Integral Formulation of Loop Quantum Cosmology: The ACH Expansion

A reminder: The kinematic Hilbert space is a tensor product $\mathcal{H} = \mathcal{H}_{kin}^{grav} \otimes \mathcal{H}_{kin}^{matt}$ of the gravitational and matter Hilbert spaces. Elements $\Psi(\nu)$ of $\mathcal{H}_{kin}^{grav}$ are functions of ν with support on a countable number of points and with a finite norm $\|\Psi\|^2 := \sum_\nu |\Psi(\nu)|^2$. The matter Hilbert space is the standard one: $\mathcal{H}_{kin}^{matt} = L^2(\mathbb{R}, d\phi)$. Thus the kinematic states of the model are functions $\Psi(\nu, \phi)$ with finite norm $\|\Psi\|^2 := \sum_\nu \int d\phi |\Psi(\nu, \phi)|^2$.

To obtain the physical Hilbert space, one first note that the quantum constraint can be written as

$$-C\Psi(\nu, \phi) \equiv \partial_\phi^2 \Psi(\nu, \phi) + \Theta\Psi(\nu, \phi) = 0 \qquad (5.3)$$

where Θ is a positive and self-adjoint operator on $\mathcal{H}_{kin}^{grav}$. More explicitly, Θ is a second order difference operator

$$\begin{aligned}(\Theta\Psi)(\nu) \quad = \quad & -\frac{3\pi G}{4\ell_0^2}\left[\sqrt{|\nu(\nu+4\ell_0)|}(\nu+2\ell_0)\Psi(\nu+4\ell_0)-2\nu^2\Psi(\nu)\right. \\ & \left. +\sqrt{|\nu(\nu-4\ell_0)|}(\nu-2\ell_0)\Psi(\nu-4\ell_0))\right] \end{aligned} \tag{5.3}$$

where ℓ_0 is related to the 'area gap' $\Delta = 4\sqrt{3}\pi\gamma\ell_{pl}^2$ via $\ell_0^2 = \Delta$.

Based on "Casting Loop Quantum Cosmology in the Spin Foam Paradigm" Abhay Ashtekar, Miguel Campiglia and Adam Henderson, [gr-qc] arXiv:1001.5147v2 [190]

5.12.1 Sum over Histories

The gravitational amplitude A_G

The idea of the Ashtekar, Campiglia and Henderson (ACH) expansionis is decribed here. They apply the standard Feynman procedure they regard $e^{-i\alpha\Theta}$ as an 'evolution operator' with 'Hamiltonian' $\alpha\Theta$ and a 'time 'interval' $\Delta\tau = 1$. It is unerstood that this 'evolution' is just a mathematical construct and does not correspond to the physical evolutionwith respect to the relational time variables ϕnormally used in LQC. Rather, since it is generated by the constraint C, physically it represents gauge transformations (or time reparameterisations).

As in the standard Feynman construction of the path integral, they write

$$e^{-i\alpha\Theta} = \underbrace{e^{-i\epsilon\alpha\Theta}\cdots e^{-i\epsilon\alpha\Theta}}_{N\text{ times}} \tag{5.3}$$

where N is an arbitrary positive integer and $\epsilon \equiv 1/N$ and write the gravitational amplitude $A(\nu_f, \nu_i; \alpha)$ as

$$\langle\nu_f|e^{-i\alpha\Theta}|\nu_i\rangle = \sum_{\overline{\nu}_{N-1},\ldots,\overline{\nu}_1}\langle\nu_f|e^{-i\epsilon\alpha\Theta}|\overline{\nu}_{N-1}\rangle\langle\overline{\nu}_{N-1}|e^{-i\epsilon\alpha\Theta}|\overline{\nu}_{N-2}\rangle\ldots\langle\overline{\nu}_1|e^{-i\epsilon\alpha\Theta}|\nu_i\rangle \tag{5.3}$$

where we have introduced a decomposition of identity at every intermediate 'time' $\tau = n\epsilon$, $n = 1, 2, \ldots, N-1$. We can write this as

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{\overline{\nu}_{N-1},\ldots,\overline{\nu}_1} U_{\overline{\nu}_N\overline{\nu}_{N-1}} U_{\overline{\nu}_{N-1}\overline{\nu}_{N-2}} \ldots U_{\overline{\nu}_1\overline{\nu}_0} \tag{5.3}$$

where we have denoted the matrix element $\langle \overline{\nu}_n | e^{-i\epsilon\alpha\Theta} | \overline{\nu}_{n-1} \rangle$ as $U_{\overline{\nu}_n\overline{\nu}_{n-1}}$ and where we have set $\nu_f = \overline{\nu}_N$ and $\nu_i = \overline{\nu}_0$.

They consider a path σ_N^M which involves M volume transitions:

$$\sigma_N^M = (\nu_M, \ldots, \nu_M; \ldots\ldots; \overbrace{\nu_1, \ldots, \nu_1; \underbrace{\nu_0, \ldots, \nu_0}_{N_0}}^{N_1}) \tag{5.3}$$

The second key idea is to perform the sum over all these amplitudes in three steps. First we fix the ordered set of volumes $(\nu_M, \ldots, \nu_0)$, but sum over all the transitions to occur over all permissible values of $N_0, N_1, \ldots, N_M$. A moment's reflection shows that this is given by

$$A_N(\nu_M, \ldots, \nu_0; \alpha) = \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \cdots \sum_{N_2=2}^{N_3-1} \sum_{N_1=1}^{N_2-1} A(\sigma_N^M) \tag{5.3}$$

the range of the sum being determined by $N < N_M < N_{M-1} < \cdots < N_2 < N_1$ where all the N_m are different from zero. Next we sum over all possible intermediate values of ν_m such that $\nu_m \neq \nu_{m-1}$, keeping $\nu_0 = \nu_i$ and $\nu_M = \nu_f$ to obtain the amplitude $A_N(M)$ associated with the set of paths in which there are precisely M volume transitions:

$$A_N(M; \alpha) = \sum_{\nu_{M-1},\ldots,\nu_1; \nu_m \neq \nu_{m+1}} A_N(\nu_M, \ldots, \nu_0; \alpha). \tag{5.3}$$

Finally, the total amplitude $A(\nu_f; \nu_i; \alpha)$ is obtained by summing over all volume transitions

$$A_G(\nu_f; \nu_i; \alpha) = \sum_{M=0}^{N} A_N(M; \alpha). \tag{5.3}$$

We take the continuuml limit of $A_N(\nu_M, \ldots, \nu_0; \alpha)$. This can be done by noticing that the $N \to \infty$ limit of the sums is nothing else than the definition of the Riemann integral. Thus for instance the first sum in N_M converges to an integral from zero to $\Delta\tau$, and so on. This gives

$$\sum_{N_M=M}^{N-1} \epsilon \sum_{N_{M-1}=M-1}^{N_M-1} \epsilon \cdots \sum_{N_2=2}^{N_3-1} \epsilon \sum_{N_1=1}^{N_2-1} \epsilon \to \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \tag{5.3}$$

where $\tau_m = \epsilon N_m$. For large enough N we can write

$$\langle \nu_n | e^{-i\epsilon\alpha\Theta} | \nu_n \rangle \sim e^{-i\epsilon\alpha\Theta_{\nu_n\nu_n}} \tag{5.3}$$

while if $\nu_{n'} \neq \nu_n$

$$\langle \nu_m | e^{-i\epsilon\alpha\Theta} | \nu_n \rangle \sim -i\epsilon\alpha\Theta_{\nu_m\nu_n} \tag{5.3}$$

(the details are given in appendix I.4). Bringing it all together, we have

$$\begin{aligned} A(\nu_M, \ldots, \nu_0; \alpha) &:= \lim_{N\to\infty} A_N(\nu_M, \ldots, \nu_0; \alpha) \\ &= \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 \, A(\nu_M, \ldots, \nu_0; \tau_M, \ldots, \tau_1; \alpha) \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} A(\nu_M, \ldots, \nu_0; \tau_M, \ldots, \tau_1; \alpha) &:= e^{-i(1-\tau_M)\alpha\Theta_{\nu_M\nu_M}} \left(-i\alpha\Theta_{\nu_M\nu_{M-1}}\right) \times \\ &\qquad \ldots e^{-i(\tau_2-\tau_1)\alpha\Theta_{\nu_1\nu_1}} \left(-i\alpha\Theta_{\nu_1\nu_0}\right) e^{-i\tau_1\alpha\Theta_{\nu_0\nu_0}} \end{aligned} \tag{5.2}$$

Note that the matrix elements $\Theta_{\nu_m\nu_n} = \langle \nu_m | \Theta | \nu_n \rangle$ of Θ in $\mathcal{H}^{grav}_{kin}$ can be calculated from (5.3) and vanish if $(\nu_m - \nu_n) \notin \{0, \pm 4\ell_0\}$. Therefore, explicit evaluation of the limit is rather straightforward.

The expression (5.2) still contains some integrals. These can be performed exactly (see appendix I.6). The result is

$$\begin{aligned} A(\nu_M, \ldots, \nu_0; \alpha) &= \Theta_{\nu_M\nu_M} \Theta_{\nu_{M-1}\nu_{M-1}} \ldots \Theta_{\nu_1\nu_1} \times \\ &\quad \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial\Theta_{w_kw_k}}\right)^{n_k-1} \sum_{m=1}^{p} \frac{e^{-i\alpha\Theta_{w_mw_m}}}{\prod_{j\neq m}^{p}(\Theta_{w_mw_m} - \Theta_{w_jw_j})} \end{aligned} \tag{5.2}$$

where, since the volumes can repeat along the discrete path, w_m label the p distinct values taken by the volume and n_m the number of times that each value occurs in the sequence.

5.12.2 Perturbative Series

We show that the expression (N.-19) for the transition amplitude can also be obtained using a particular perturbative expansion.

Consider the diagonal and off- diagonal parts D and K of the operator Θ in the basis $|\nu = 4n\ell_0\rangle$. Thus, the matrix elements of D and K are given by:

$$D_{\nu'\nu} = \Theta_{\nu\nu}\delta_{\nu'\nu}, \qquad K_{\nu'\nu} = \begin{cases} \Theta_{\nu'\nu} & \nu' \neq \nu \\ 0 & \nu' = \nu \end{cases} . \tag{5.2}$$

Clearly $C = p_\phi^2 - D - K$. The idea is to think of $p_\phi^2 - D$ as the 'main part' of C and K as a 'perturbation'. To implement it, we introduce a 1-parameter family of operators

$$C_\lambda = p_\phi^2 - \Theta_\lambda = p_\phi^2 - D - \lambda K \tag{5.2}$$

as an intermediate mathematical step. The parameter λ simply keeps track of powers of K in the paerturbative expansion, we have to set $\lambda = 1$ at the end of the calculation.

The λ dependence appears in the gravitational part:

$$A_G^{(\lambda)}(\nu_f, \nu_i; \alpha) = \langle \nu_f | e^{-i\alpha\Theta_\lambda} | \nu_i \rangle. \tag{5.2}$$

Let us consider a perturbative expansion of this amplitude. Again we think of $e^{-i\alpha\Theta_\lambda}$ as a mathematical 'evolution operator' defined by the 'Hamiltonian' $\alpha\Theta_\lambda$ and a 'time interval' $\Delta\tau = 1$. The 'unperturbed Hamiltonian' is αD and the 'perturbation' is $\lambda\alpha K$. the usual procedure (outlined in appendix I.7) let us define the 'interaction Hamiltonian' as

$$H_I(\tau) = e^{i\alpha D\tau}\alpha K e^{-i\alpha D\tau}. \tag{5.2}$$

Then the evolution in the interaction picture is dictated by the 1-parameter family of unitary operators on $\mathcal{H}_{kin}^{grav}$

$$\tilde{U}_\lambda(\tau) = e^{i\alpha D\tau}e^{-i\alpha\Theta_\lambda\tau}, \qquad \text{satisfying} \quad \frac{d\tilde{U}_\lambda(\tau)}{d\tau} = -i\lambda H_I(\tau)\tilde{U}_\lambda(\tau). \tag{5.2}$$

The solution of this given by a time-ordered exponential:

$$\begin{aligned} \tilde{U}_\lambda(\tau) &= \mathcal{T} e^{-i\int_0^\tau H_I(\tau)d\tau} \\ &= \sum_{M=0}^{\infty} \lambda^M \int_0^\tau d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 \, [-iH_I(\tau_M)] \ldots [-iH_I(\tau_1)]. \end{aligned} \tag{5.2}$$

Next we use the relation $e^{-i\alpha\Theta_\lambda} = e^{-i\alpha D}\tilde{U}_\lambda(1)$, with $\tilde{U}_\lambda$ given by (5.2), take the matrix element of $e^{-i\alpha\Theta_\lambda}$ between initial and final states, $|\nu_i \equiv \nu_0\rangle$ and $|\nu_f \equiv \nu_M\rangle$, and write out explicitly the product of the H_I's. The result is

$$
\begin{aligned}
& A_G^{(\lambda)}(\nu_f, \nu_i; \alpha) \\
= & \langle \nu_f | e^{-i\alpha D} \tilde{U}_\lambda(1) | \nu_i \rangle \\
= & \sum_{M=0}^{\infty} \lambda^M \int_0^\tau d\tau_M \ldots \int_0^{\tau_2} d\tau_1 \ \langle \nu_M | e^{-i\alpha D} [-i H_I(\tau_M)] \ldots [-i H_I(\tau_1)] | \nu_0 \rangle \\
= & \sum_{M=0}^{\infty} \lambda^M \int_0^\tau d\tau_M \ldots \int_0^{\tau_2} d\tau_1 \sum_{\nu_{M-1},\ldots,\nu_1} \langle \nu_M | e^{-i\alpha D} [-i H_I(\tau_M)] | \nu_{M-1} \rangle \ldots \\
& \quad \ldots \langle \nu_2 [-i H_I(\tau_2)] | \nu_1 \rangle \langle \nu_1 [-i H_I(\tau_1)] | \nu_0 \rangle \\
= & \sum_{M=0}^{\infty} \lambda^M \int_0^\tau d\tau_M \ldots \int_0^{\tau_2} d\tau_1 \sum_{\nu_{M-1},\ldots,\nu_1} e^{-i\alpha D_{\nu_M \nu_M}} \times \\
& \quad e^{i\alpha \tau_M D_{\nu_M \nu_M}} [-i\alpha K_{\nu_M \nu_{M-1}}] | e^{-i\alpha \tau_{M-1} D_{\nu_{M-1} \nu_{M-1}}} \ldots \\
& \quad \ldots e^{i \tau_2 D_{\nu_2 \nu_2}} [-i\alpha K_{\nu_2 \nu_1}] e^{-i\tau_2 \alpha D_{\nu_1 \nu_1}} e^{i\tau_1 \alpha D_{\nu_1 \nu_1}} [-i\alpha K_{\nu_1 \nu_0}] e^{-i\tau_1 \alpha D_{\nu_0 \nu_0}} \\
= & \sum_{M=0}^{\infty} \lambda^M \int_0^\tau d\tau_M \ldots \int_0^{\tau_2} d\tau_1 \sum_{\nu_{M-1},\ldots,\nu_1} [e^{-i(1-\tau_M)\alpha D_{\nu_M \nu_M}}] \times \\
& \quad (-i\alpha K_{\nu_M \nu_{M-1}}) \ldots [e^{-i(\tau_2 - \tau_1)\alpha D_{\nu_1 \nu_1}}] (-i\alpha K_{\nu_1 \nu_0}) [e^{-i\tau_1 \alpha D_{\nu_0 \nu_0}}].
\end{aligned}
\tag{5.-7}
$$

We can now replace D and K by their deinitions (5.12.2). Because K has no diagonal matrix elements, only terms with $\nu_m \neq \nu_{m+1}$ contribute and the sum reduces precisely to

$$
A_G^{(\lambda)}(\nu_f, \nu_i; \alpha) = \sum_{M=0}^{\infty} \lambda^M \Big[\sum_{\nu_{M-1},\ldots,\nu_1; \nu_m \neq \nu_{m+1}} A(\nu_M, \ldots, \nu_1) \Big], \tag{5.-7}
$$

where $A(\nu_M, \ldots, \nu_1)$ is given by () as in the sum over histories expansion of section 5.12.1.

5.13 Path Integral Formulation of Loop Quantum Cosmology

Recent developements have allowed for the construction of new spin foam models that are more closely related to the canonical theory. It is important to make connections to spin-foams by building path integrals from the canonical theory when possible.

5.13.1 Introduction

We construct a path integral for the exactly soluble Loop Quantum Cosmology starting with the canonical quantum theory.

The construction defines each component of the path integral. Each has non-trivial changes from the standard path integral.

We see the origin of singularity resolution in the path integral representation of LQC.

The structure of the path integral features similarities to spin foam models.

The path integral can give an argument for the surprising accuracy of the effective equations used in more complicated models.

5.13.2 LQC Canonical Theory as a Sum Over Histories

We examine the simply case of $k = 0$ FRW model with a scalar field

$$\nu = \epsilon \frac{a^3\ {}^oV}{2\pi \ell_p^2 \gamma}, \qquad b = -\epsilon \frac{4\pi\gamma G}{3\ {}^oV} \frac{p_a}{a^2} \tag{5.-7}$$

With Poisson bracket

$$\{b, v\} = 2/\hbar$$

Classically they have range $(-\infty, \infty)$.

To simplify the constraint the lapse is chosen to be $N = a^3 N'$.

The phase space action is then

$$S = \int dt[\dot{b}v\frac{\hbar}{2} + p_\phi \dot{\phi} - \frac{N'}{2}(p_\phi^2 - 3\pi\hbar^2 G b^2 v^2)] \tag{5.-7}$$

We will construct the path integral starting from the physical Hilbert space of Schrödinger LQC.

The physical states are solutions to the quantum Hamiltonian constraint.

$$\partial_\phi^2 \psi(\nu, \phi) = -\hat{\Theta}\psi(\nu, \phi) \tag{5.-7}$$

The physical Hilbert space can be obtained by group averaging procedure.

Find that physical states satisfy a Schrodinger like equation.

$$-i\partial_\phi \psi(\nu,\phi) = \widehat{\sqrt{\Theta}}\psi(\nu,\phi) \tag{5.-7}$$

The physical inner product is.

$$< \psi_1|\psi_2 > = \frac{\lambda}{\pi} \sum_{\nu=4n\lambda} \frac{1}{|\nu|} \overline{\psi}_1(\nu,\phi_0)\psi_2(\nu,\phi_0) \tag{5.-7}$$

We have Schrodinger LQC written in the form of a Schrodinger equation with ϕ as time, so we apply the construction above to obtain a path integral.

Similar to the construction in non-relativistic quantum mechanics we want a path integral representation of the propagator.

$$< \nu'|e^{i\sqrt{\Theta}\Delta\phi}|\nu > = < \nu',\phi'|\nu,\phi > . \tag{5.-7}$$

More generally we could construct the path integral from the definition of the physical inner product in terms of group averaging.

We have some knowledge of the exact propagator.

One can show that

$$< \nu',\phi'|\nu,\phi > = 0 \quad \text{if} \quad \nu' < 0 \quad \text{and} \quad \nu > 0 \tag{5.-7}$$

This allows us to simplify the calculations by restricting to positive or negative ν.

The propagator can be written as an integral

$$< \nu',\phi'|\nu,\phi > = \frac{\lambda}{2\pi\nu} \int_0^{\pi/\lambda} db \, e^{\frac{b\nu}{2} - \frac{1}{\lambda}\tan^{-1}(e^{\Delta\phi}\tan(\lambda b/2))\nu'} + (\Delta\phi \to -\Delta\phi) \tag{5.-7}$$

5.13.3 Vertex expansion: Reorganising the sum over histories

$$\begin{aligned} U_{\nu_M\nu_M} &= < \nu_M|e^{i\epsilon\mathcal{H}}|\nu_M > \\ &= 1 + i\epsilon < \nu_M|\mathcal{H}|\nu_M > + \mathcal{O}(\epsilon^2) \end{aligned} \tag{5.-7}$$

$$\begin{aligned} U_{\nu_M \nu_{M-1}} &= \; < \nu_M | e^{i\epsilon \mathcal{H}} | \nu_{M-1} > \\ &= \; < \nu_M | \nu_{M-1} > + i\epsilon < \nu_M | \mathcal{H} | \nu_{M-1} > + \mathcal{O}(\epsilon^2) \\ &= \; i\epsilon \mathcal{H}_{\nu_M \nu_{M-1}} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{5.-8}$$

5.13.4 Derivation of Path Integral from the Canonical Theory

Step 1 - Split Exponetial/Insert Complete Basis

We split the exponential into N copies and insert a complete basis of ν between each.

$$1 = \frac{\pi}{\lambda} \sum_{\nu = 4n\lambda} |\nu| \; |\nu >< \nu| \tag{5.-8}$$

Giving

$$< \nu', \phi' | \nu, \phi > = \prod_{n=1}^{N-1} \left[\frac{\pi}{\lambda} \sum_{\nu_n} |\nu_n| \right] \prod_{n=1}^{N} \left[< \nu_n | e^{i\epsilon \widehat{\sqrt{\Theta}}} | \nu_{n-1} > \right] \tag{5.-8}$$

Important difference: Instead of continuous integrals there are discrete sums over ν at each ϕ.

The next step is to compute each term of the product:

$$< \nu_n | e^{i\epsilon \widehat{\sqrt{\Theta}}} | \nu_{n-1} > \tag{5.-8}$$

Step 2 - Evaluate Each Term

We want to evaluate each term to first order in ϵ

$$< \nu_n | e^{i\epsilon \widehat{\sqrt{\Theta}}} | \nu_{n-1} >$$

Problem: Even computing to first order in ϵ requires knowing the spectrum of Θ.

The resolution is to rewrite each term as

$$< \nu_n|e^{i\epsilon\widehat{\sqrt{\Theta}}}|\nu_{n-1}> = \int_{-\infty}^{\infty} dp_{\phi_n}|p_{\phi_n}|\Theta(p_{\phi_n})\int_{-\infty}^{\infty} dN_n\frac{\epsilon}{2\pi\hbar}e^{ip_{\phi_n}\epsilon/\hbar} < \nu_n|e^{-i\epsilon\frac{N_n}{2\hbar}(p_{\phi_n}^2-\hbar^2\widehat{\Theta})}|\nu_{n-1}> \tag{5.-8}$$

We then only need to evaluate the the following to first order in ϵ.

$$< \nu_n|e^{i\epsilon\frac{\hbar N_n}{2}\widehat{\Theta}}|\nu_{n-1}> \tag{5.-8}$$

Evaluating this term is simple given the action of

$$\begin{aligned} < \nu_n|e^{i\epsilon\frac{\hbar N_n}{2}\widehat{\Theta}}|\nu_{n-1}> &= \frac{\lambda}{\pi\nu_{n-1}}\Big(\delta_{\nu_n,\nu_{n-1}} - i\epsilon\frac{\hbar N_n}{2}\frac{3\pi G}{4\lambda^2}\nu_n\frac{\nu_n+\nu_{n-1}}{2} \\ &\times[\delta_{\nu_n,\nu_{n-1}+4\lambda} + \delta_{\nu_n,\nu_{n-1}-4\lambda} - 2\delta_{\nu_n,\nu_{n-1}}] + \mathcal{O}(\epsilon^2)\Big) \end{aligned} \tag{5.-8}$$

Expressed as an integral by writing the delta functions as integrals over b.

$$\frac{1}{\nu_{n-1}}\frac{\lambda^2}{\pi^2}\int_0^{\pi/\lambda} db_n e^{-i(\nu_n-\nu_{n-1})b_n/2}[1 + i\epsilon\frac{\hbar N_n}{2}\frac{3\pi G}{\lambda^2}\nu_n\frac{\nu_n+\nu_{n-1}}{2}\sin^2(\lambda b_n)] \tag{5.-8}$$

Step 3 - Rexponentiate

Combining together the results from the previous slides and re-exponentiating the product we arrive at the path integral:

$$\begin{aligned} < \nu',\phi'|\nu,\phi> &= \lim_{N\to\infty}\frac{1}{\nu_0}\prod_{n=1}^{N-1}\left[\sum_{\nu_n}\right]\prod_{n=1}^{N}\left[\frac{\lambda}{\pi}\int_0^{\pi/\lambda} db_n\int_{-\infty}^{\infty} dp_{\phi_n}|p_{\phi_n}|\Theta(p_{\phi_n})\int_{-\infty}^{\infty} dN_n\right] \\ &\exp\frac{i}{\hbar}\sum_{n=1}^{N}\epsilon\Bigg[p_{\phi_n} - \frac{\hbar}{2}\frac{(\nu_n-\nu_{n-1})}{\epsilon}b_n \\ &\qquad -\frac{N_n}{2}(p_{\phi_n}^2 - \frac{3\pi G\hbar^2}{\lambda^2}\nu_{n-1}\frac{\nu_n+\nu_{n-1}}{2}\sin^2(\lambda b_n))\Bigg] \end{aligned} \tag{5.-9}$$

Where the discretized action is

$$S_N = \sum_{n=1}^{N} \epsilon \left[p_{\phi_n} - \frac{\hbar}{2} \frac{(\nu_n - \nu_{n-1})}{\epsilon} b_n - \frac{N_n}{2} (p_{\phi_n}^2 - \frac{3\pi G \hbar^2}{\lambda^2} \nu_{n-1} \frac{\nu_n + \nu_{n-1}}{2} \sin^2(\lambda b_n)) \right] \tag{5.-9}$$

There are non-trivial changes to the space of paths, measure, and action.

Allowed Paths in b, ν

Space of Paths: Defined by the range of integration at each time.

Paths $\nu(\phi)$ are discrete : $\nu(\phi)$ $(4\lambda, 8\lambda, 12\lambda, ...)$

Paths $b(\phi)$ are continuous, but bounded: $b(\phi) \in [0, \pi/\lambda]$

We are integrating only over discrete quantum geometries

Similar situation to that of spin foam models

Would try to define path integral over continuous fields ${}^4e$ and ${}^4A$

Instead integrate over discrete geometries.

The space of paths has been modified due to the kinematical structure of LQC.

The measure has changed - but is a natural measure on this space of paths.

Phase - Effective Action

The phase associated to each path is not the classical action.

$$S_N = \sum_{n=1}^{N} \epsilon \left[p_{\phi_n} - \frac{\hbar}{2} \frac{(\nu_n - \nu_{n-1})}{\epsilon} b_n - \frac{N_n}{2} (p_{\phi_n}^2 - \frac{3\pi G \hbar^2}{\lambda^2} \nu_{n-1} \frac{\nu_n + \nu_{n-1}}{2} \sin^2(\lambda b_n)) \right]$$

This is a discretized version of an effective action which contains non-perturbative quantum corrections

$$S = \int_{\phi}^{\phi'} d\phi \left[p_\phi - \frac{\hbar}{2} \dot{\nu} b - \frac{N}{2} \left(p_\phi^2 - \frac{3\pi G \hbar^2}{\lambda^2} \nu^2 \sin^2(\lambda b) \right) \right] \tag{5.-10}$$

This is the effective action that well approximates the quantum dynamics.

Simplify Path Integral

Possible to integrate out variables to obtain a simpler expression?

Want configuration space path integral.

We can integrate out N and p_ϕ to obtain a path integral over b and ν only.

$$\begin{aligned} < \nu', \phi' | \nu, \phi > \quad = \quad & \lim_{N\to\infty} \frac{1}{\nu_0} \prod_{n=1}^{N-1} \left[\sum_{\nu_n} \right] \prod_{n=1}^{N} \left[\frac{\lambda}{\pi} \int_0^{\pi/\lambda} db_n \right] \\ & \exp \frac{i}{\hbar} \sum_{n=1}^{N} \epsilon \left[+ \sqrt{\frac{3\pi G\hbar^2}{\lambda^2} \nu_{n-1} \frac{\nu_n + \nu_{n-1}}{2}} \sin(\lambda b_n) \right. \\ & \left. \qquad - \frac{\hbar}{2} \frac{(\nu_n - \nu_{n-1})}{\epsilon} b_n \right] \end{aligned} \tag{5.-11}$$

Equivalent to solving the constraint → Path integral on constraint surface.

5.13.5 Configuration Path Integral's Connection to Spin Foams

The expansion in λ resembles the vertex expansion in group field theory.

5.13.6 Singularity Resolution

The path integral is then dominated by the extrema of the effective action

$$S[v, b, p_\phi, N] = \int d\phi [p_\phi - \frac{\hbar}{2} b\dot{v} - \frac{N}{2} (p_\phi^2 - \frac{3\pi G\hbar^2}{\lambda^2} v^2 \sin^2(\lambda b))] \tag{5.-11}$$

The "classical solutions", x_{eff} to this action are the bouncing solutions.

The action can be computed along these bouncing solutions between v and v'.

$S_{eff}(x_{eff})$ is large in units of $\hbar \to 0$ loop corrections are negligible.

Provides an additional explanation for the accuracy of the effective equations.

5.14 Disordered Locality: A Possible Origin for Dark Energy and the Value of the Cosmological Constant

5.14.1 Introduction

the low energy limit may suffer from a disordered localilty characterised by identifications of far away points.

If macroscopic locality, as defined by the classical metric, is an approximation small departures could be indicators of an underlying quantum geometry. The consequences of spacetime being quantum mechanical can contribute to observable effects throughout the lifetime of the universe. Under some assumptions, it turns out that such phenomena at the macroscopis scale could lead to a contribution to the energy-momentum tensor that looks very much like a cosmological term, which has the potential of explaining the present value of the cosmological constant. However, there is currently no unique observational signiature that would distibguish this model from other dark energy pictures.

Any good model of quantumm gravity in which the classical metric is emergent (as with LQG) has to expalin why non-local connections are enough not to disrupt local physics. This has not yet been addressed, here it is just assumed that we have such a theory. Non-local connections on a lattice type structure can be both common cosmologically and very difficult to detect locally.

5.14.2 Notions and Hypothesis

Two kinds of locality:

Microlocality: connectivity of a single spin net graph causal structure of a single spin foam history.

Macrolocality: nearby in the classical metric that emerges

The issues: Semicassical states may involve superpositions of large numbers of graphs. In addition being semiclassical is a course grained, low energy property.

Could there not be mistmatches between micro and macrolocality?

It is not hard to see that they are generic.

For an emergent manifold scenario to work these mismatches must be suppressed.

Suppose they are, what scale measures how suppressed they are? What if it is the cosmological scale? Would there be observable consequences?

Hypothesis: the low energy limit of a theory with emergent manifold is characterised by a small worlds network

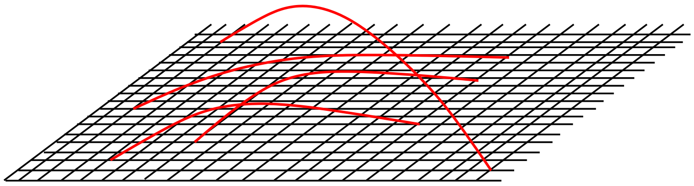


Figure 5.18: We begin with well-studied semiclassical states and contaminate them by adding a small proportion of links that are non-local in the classical metric q_{ab} without, however, weakening the correspondence between expectation values of coarse grained observables and the classical metric.

Suppose the ground state is contained by small proportion of non-local links (locality defects)??

If this room had a small proportion of non-local link, with no two nodes in the room connected, but instead connecting to nodes at cosmological distances, could we tell?

Studied the Ising model on a lattice contaminated by random non-local links.

Conclusion was a small amount of disordered-locality would be hard to observe by local measurements.

What if the scale of disordered locality were cosmological?

Could there be observable consequences?

Hypothesis: disordered locality contributions to dark energy.

5.14.3 Disordered FRW Cosmology

The flat FRW metric

$$ds^2 = -\mathcal{N}^2 dt^2 + a^2(t) q^0_{ij} dx^i dx^j$$

We assum there is an underlying spin network $\Gamma(t)$,

That there are mostly local links

There are non-local links between N_{NL} pairs of nodes.

Macroscopically, N_{NL} pairs of points (x_i, y_i) identified. $i = 1, \ldots, N_{NL}$.

Both ends within present Hubble volume, distributed randomly.

N_{NL} evolves slowly

$$N_{NL}(a) = N_0 \left(\frac{a}{a_0}\right)^p$$

5.14.4 Energetics of Non-Local Interactions

We model interactions by a simple nearest neighbour coupling σ_i are degrees of freedom on each node (dimensionless).

$$H^{nodes} = -\epsilon \frac{1}{l_P} \sum_{<mn>} \sigma_m \cdot \sigma_n \qquad \epsilon = \pm 1.$$

This splits into a local and non-local part

$$H^{matter} = H^{local} + H^{NL}.$$

The local part becomes an ordinary matter coupling:

$$H^{local} = -\epsilon \frac{1}{l_P} \sum_{<mn>}^{local} \sigma_m \cdot \sigma_n \qquad \epsilon = \pm 1.$$

Effective scalar field at point x_n closest node n.

$$\phi(x_n) = l_P^{-1} \sigma_n \qquad \partial_a \phi(x_n) = \frac{1}{l_P^2}(\sigma_{n+\hat{a}} - \sigma_n).$$

This implies a local matter coupling

$$H^{local} = \frac{\epsilon}{2} \int d^3x \sqrt{q(x)} [q^{ab} \partial_a \phi \partial_b \phi - \mu^2 \phi^2]$$

$$\mu^2 = \frac{\sqrt{2}}{l_P^2}, \qquad \sum_n l_P^3 \to \int d^3x \sqrt{q}$$

The energetics of the non-local coupling:

Consider two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ conected by N_{12} non-local links. The average field in a region is

$$< \sigma(x) >_{\mathcal{R}} = \frac{\int_{\mathcal{R}} \sqrt{q} \sigma}{\int_{\mathcal{R}} \sqrt{q}}.$$

We approximate the interaction energy between the two regions, averaged over possible ways to connect them with N_{12} non-local links

$$\begin{aligned} H^{\mathcal{R}_1 \mathcal{R}_2} &= -\frac{\epsilon}{l_P} \sum_{I}^{x_I \in \mathcal{R}_1 y_I \in \mathcal{R}_2} \sigma(x_I) \cdot \sigma(y_I) \\ &\approx -\frac{\epsilon}{l_P} N_{12} < \sigma(x) >_{\mathcal{R}_1} < \sigma(x) >_{\mathcal{R}_2} \end{aligned} \tag{5.-11}$$

This is called the annealing approximation in statistical mechanics.

Now consider the case that the two regions are the Hubble volume

$$H^{NL} = -\frac{\epsilon}{l_P} N^{NL}(t) \ < \sigma >_a \cdot < \sigma >_a$$

average field in the Hubble volume: $< \sigma >_a$.

Now put in the time dependence of $N_{NL}(t)$

$$H^{NL} = -\frac{\epsilon}{l_P} \frac{N_0}{a_0^p} \left(\int_a d^3x \sqrt{q} \right)^{p/3} < \sigma >_a \cdot < \sigma >_a$$

We choose $p = 3$ to find

$$H^{NL} = -\frac{\epsilon}{l_P} \frac{N_0}{a_0^3} \left(\int_a d^3x \sqrt{q} \right) < \sigma >_a \cdot < \sigma >_a$$

This gives an effective action

$$S^{NL} = \int dt \mathcal{N} H^{NL} = -\frac{\epsilon}{l_P} \frac{N_0}{a_0^3} \int_a d^4x \sqrt{-g} < \sigma >_a^2 .$$

This in turn gives a contribution to the energy momentum tensor

$$T_{NL}^{ab} = \frac{1}{\sqrt{-g}} \frac{\delta S^{NL}}{\delta g_{ab}} = -g^{ab} m^4 < \sigma >_a^2$$

defines a mass

$$m^4 = -\frac{\epsilon}{2l_P}\frac{N_0}{a_0^3}$$

We see that $p = 3$ and $\epsilon = -1$ are necessary to get adrk energy $w = -1$. That is, the number of non-local links within the comoving volume increases in time proportionately to the comoving volume.

To match the dark energy we need (recalling $< \sigma >$ is dimensionless)

$$m^4 = \frac{N_0}{2l_P a_0^3} = \frac{10^{-120}}{l_P^4}.$$

Let us evaluate this at present. It gives:

$$N_{NL}(now) = 10^{-120}\left(\frac{a_{now}}{l_P}\right)^3 \approx 10^{60}.$$

A typical distance between ends of non-local links is 100km.

5.14.5 The Evolution of $N_{NL}(a)$ in a Simply Model of LQG

The Evolution of $N_{NL}(a)$, the number of non-local connections.

Let us consider a simple model of LQG with trivalent nodes and the related dual Pachner moves.

i) There are microscopic processes by which non-local links split into two and processes in which pairs annihilate.

ii) These must be in balance with the increase in volume which comes from 1 to 3 moves.

Exchange moves can increase the non-local edges.

The two left and two right edges can now evolve away from each other, leading to two non-local edges.

The probability for this to happen on each local move is

$$\alpha N^{NL}.$$

Exchange moves that decrease the non-local edges.

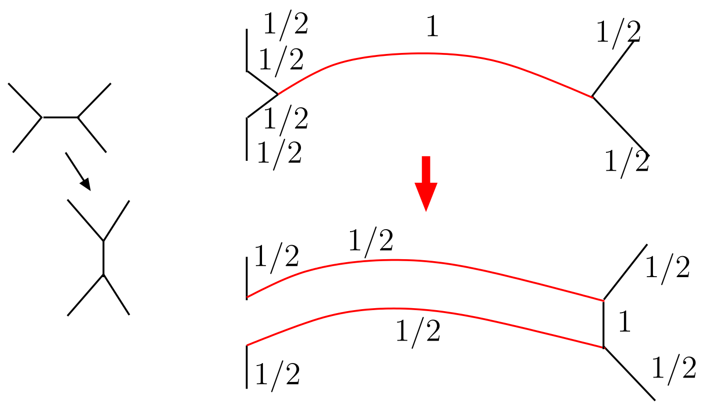


Figure 5.19: Exchange moves increasing the number of non-local edges.

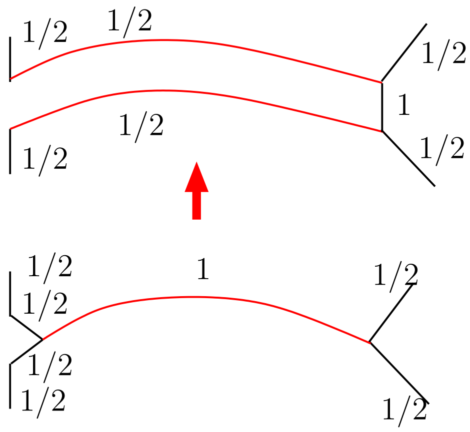


Figure 5.20: Exchange moves decrease the number of non-local edges.

This requires the inverse move on two non-local edges both of whose ends are coincident:

The probability is the probability that there are two non-local edges that coincide on each end times the probability that the move acts on one of them

$$-\beta(N^{NL})^3/V$$

The volume inceases by the net number of 1 to 3 moves over 3 to 1 moves.

$$\frac{dV}{dt} = VP_{1to3} = 3HV.$$

The change in the number of non-local links is dominated by

$$\frac{dN^{NL}}{dt} = P_{2to2}N^{NL}$$

Per node we have, on everage, $P_{2to2} = cP_{1to3}$ so:

$$\frac{dN^{NL}}{dt} = \frac{3H}{c} N^{NL}$$

$N^{NL} \sim$ volume implies that $c = 3$.

5.14.6 Conclusion

The conclusion is that disordered locality provides a possible explanation of dark energy.

5.15 Problems with Interpretations of Quantum Mechanics Applied to Quantum Cosmology

5.15.1 The Problem

Gravity governs the entire universe, so from a theory of quantum gravity we should formulate a quantum theory of cosmology.

Quantum gravity is to be applied to the universe as a whole. Issues are present.

$$\Psi_{univ} \tag{5.-11}$$

There is nothing outside the universe. Do not have a satisfacotry physical interpretation of the state of the whole universe $|\Psi_{univ}>$ because it would only be accessible to an observer outside the universe.

We only understand the quantum mechanical evolution of ordinary systems.

5.15.2 Many Worlds Interpretation

The many worlds interpretation succeeds in letting the observer be part of the universe, but the world we observe is only a small part of reality.

5.15.3 Consistent Histories Interpretation

The probabiltiy for taking the branch I_n at time t_n *after* having gone through the branches $I_1, \ldots, I_{n-1}$ at $t_1, \ldots, t_{n-1}$ is given by

$$Tr\left(\hat{P}^n_{I_n}\hat{U}(t_n - t_{n-1})\hat{\rho}_{(I_1,t_1)\ldots(I_{n-1},t_{n-1})}\hat{U}(t_n - t_{n-1})^{-1}\hat{P}^n_{I_n}\right) \tag{5.-11}$$

and when this branch is chosen, the density matrix for that branch is given by

$$\hat{\rho}_{(I_1,t_1)\ldots(I_n,t_n)} = \frac{\hat{P}^n_{I_n}\hat{U}(t_n - t_{n-1})\hat{\rho}_{(I_1,t_1)\ldots(I_{n-1},t_{n-1})}\hat{U}(t_n - t_{n-1})^{-1}\hat{P}^n_{I_n}}{Tr\left(\hat{P}^n_{I_n}\hat{U}(t_n - t_{n-1})\hat{\rho}_{(I_1,t_1)\ldots(I_{n-1},t_{n-1})}\hat{U}(t_n - t_{n-1})^{-1}\hat{P}^n_{I_n}\right)} \tag{5.-11}$$

5.15.4 Problems with the Consistent Histories Interpretation

The set of questionswe can ask are consrtained by having to be solutions of certain equations. Although we can solve the equations that determine the quantum states of the universe, it is much more difficult to determine the questions that can be asked of the theory and it is unlikely that this can ever be done.

5.16 Relational Quantum Cosmology

There are many quantum theories, corresponding to as many observers there are. They are interelated, because when two observers can ask the same question they should get the same answer.

Crane

Fontini

Smolin

Isham and Butterfield

Classical logic demands that every statement be either true or false. Topos is a category that has properties in common with the category of sets **Set**.

5.16.1 Relational Quantum Cosmology a la Smolin

[186]

[185] (Crane)

5.16.2 Rovelli's Relation Quantum Mechanics

quickly introduce rovelli's **relation quantum mechanics**.

In conventional quatum mechanics a measurement entails: the superpostion principle holds alternative, and eventually mutually exclusive possibilities, right until a measurement is made that suddenly selects one of them alone as the realized actuality on this occasion.

There are many different mathematical descriptions, each corresponding to what each different observer can see. Each is incomplete, because no observer can see the whole universe. Each observer

when they ask the same question, they must agree.

Take quantum mechanics to be a fundamental description, that is, applies equaly to both the microscopic and the macroscopic world. Also that there is no special observer which can collapse a wavefunction. The conceptual difficulties boils down to one issue, an apparent discord: a system has a determined outcome according to one observer, while at the very same time is still in a quantum superpostion according to a second observer, that is, until the second observer makes a measurement. Now, what exactly is our reason for regecting this as a description of reality? There is no experimental evidence that telling us that such a discord is impossible, infact the very formulation of quantum mechanics rules out the possibility for any such experiment to be devised! Quantum mechanics is internally consistent because the time at which the wavefunction collapsed does not affect the predictions of the final observations. What is stopping us from accepting this as a description of reality isn't any logical insonsistenceis in the fromulation of quantum mechanics or any empirical evidence at the microscopic or macroscopic level, it is a philosophical pregudece!

"... the common unease with taking quantum mechanics as a fundemental description of nature could derive from an incorrect notion - as the unease with the Lorentz transformations before Einstein derived from the notion of observer independent time."

"... We suggest that the incorrect notion that generates the unease with quantum mechanics is the notion of observer-independent state os a system.

different obsevers may give different descriptions of the same sequence of events.

Smolin incorporates R to address problem of quantum cosmology

The most complete information one can have of the universe is the collection of these incomplete, but mutually consistent quantum state descriptions. There is no way in principle of formimg one wavefunction for the whole universe.

advantages when it comes to metaphysical issues many worlds

Instead of ***many worlds*** there is one universe with ***many observers***.

5.16.3 Causal sets

The internal description of a causal set: What the universe looks like from the inside [367]

Quantum causal histories [368]

An insider's guide to quantum causal histories [369]

Planck-scale models of the Universe [370]

Evolution in Quantum Causal Histories [371]

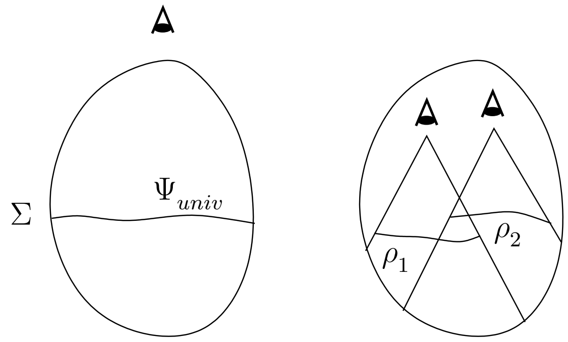


Figure 5.21: cosmologyeyes.

5.16.4 Discrete Quantum Gravity Applied to Cosmology

A very simple example

$$L = E\dot{A} + \pi\dot{\phi} - NE^2(-A^2 + (\Lambda + m^2\phi^2)|E|) \tag{5.-11}$$

The system has four hase space variables and one constraint. Therefore it has two independent observables ($\{\mathcal{O}, C\}$):

$$\mathcal{O}_1 = \phi, \quad \mathcal{O}_2 = \pi + \frac{2}{3}\frac{m^2\phi}{\Lambda + m^2\phi^2}AE. \tag{5.-11}$$

$$L(n, n+1) = E_n(A_{n+1} - A_n) + \pi_n(\phi_{n+1} - \phi_n) - N_nE_n^2(-A_n^2 + (\Lambda + m^2\phi_n^2)|E_n|) \tag{5.-11}$$

Notice that the discrete theory has four phase space degrees of freedom instead of the two of the continuum theory. The additional degrees of freedom charactrize the step of the discretization and encode remnants of the gauge invariance in the discrete theory.

Although the graphs suggest that the triad goes to zero at $n = 0$ and therefore one has a singularity this is not the case.

We here show the approach to the singularity in the discrete and continuum case. The discrete theory has a small but non-vanishing triad a $n = 0$. R. Gambini and J. Pullin gr-qc/0212033

The rate of contraction/expansion changes when going through the big crunch/bang. Question: is that a remnent of the reparametrization invariance or does it have physical consequences?

The answer to this queation is related with the existence of more degrees of freedom and therefore more constants of motion. In fact, the discrete canonical transformation is singuluar for $A = 0$. If one tries to introduce a generator of this evolution:

$$A_{n+1} = A_n + \{A_n, \mathcal{H}_n\} + \frac{1}{2!}\{\{A_n, \mathcal{H}_n\}, \mathcal{H}_n\} + \cdots \tag{5.-11}$$

$$\mathcal{H}_n = \frac{\mathcal{C}_n^2}{4\Theta A_n}\left[1 + \sum_{k=1}^{\infty} a_k \left(\frac{\mathcal{C}_n}{A_n^2}\right)^k\right] \tag{5.-11}$$

H_n diverges for $\left(\frac{\mathcal{C}_n}{A_n^2}\right)^k > 2$. This happens for $n = 0$ when the system goes through the singularity.

H_n, which is constant on each region characterizes the spacing of the discretization in an invariant way, and in that sense suggest a procedure for taking the continuum limit.

Tunneling through the singularity the lapse gets modified and therefore the “lattice spacing” before and after is different. in lattice gauge theories the sacing is related to the “dressed” values of the fundamental constants a mechanism for fundamental constants to change when tunneling through a singularity.

Have some bare charge then you have an effetive renormalization but the effective renormalization of the lattice.

It may provide a mechanism for changing the values of the fundamental constants: They argue that this result for the particular cosmological model, this feature of tunneling through a singularity should to exists in a variety of models. This can be extrapolated to the internal of black holes. Each black hole will have its singularity replaced by tunneling into a new universe, in which the dressed value of the fundamental constants will be different. This allows to construct a picture of the universe in which “evolution” takes place every time a black hole is formed, as was the original proposal of Smolin in “The life of the cosmos”.

5.17 Summary

• Black hole entropy: very detailed calculation. Microscopic degrees of freedom for black-hole entropy. Once this parameter is fixed the correct formula for any no extremum blackhole (except rotating ones maybe)

• Removle of cosmological singurlarity. Evolutional equations don't break down at the place where the classical singularity is.

• Initial conditions are derived rather than guessed at - the evolution equations supplies the boundary conditions. Perhaps not suprising as the constraint equations are admissible conditions on the initial data.

• First direct derivation of inflation from a candidate for quantum gravity. Due to a quatnum geometry effect in the early kinematic dominated universe.

• Some features in the minisuperspace are shared with the full theory. Allows proper investigation of the dynamics of minisuperspace that could shed some light on the dynamics of the full theory.

• Disordered locality as a possible contribution to dark energy with the possible consequence of a naturally small vacuum energy.

• Relational Quantum Cosmology

Chapter 6

The Master Constraint Programme and an Improved Understanding of the Dynamics

MUCH IS TAKEN DIRECTLY FROM PAPERS, TO BE REPLACED

6.1 Introduction

We left chapter 4 with several problems to do with the Hamitonian constraint:

(i) The dual Hamiltonian constraint operator does not preserve the Hilbert space $\mathcal{H}_{Diff}$. As a result the inner product structure of $\mathcal{H}_{Diff}$ cannot be employed in the construction of the physical inner product.

(ii) Classically the collection of Hamiltonians do not form a Lie algebra. We cannot use the group averaging strategy to solve the corresponding quantum constraint equation.

(iii) Their proposal takes a fixed graph in space and uses it to construct coherent states that approximate any given metric/extrinsic curvature pair in the classical theory. However, the Hamiltonian constraint is a graph changing operator - the new graph it generates has degrees of freedom upon which the coherent state does not depend and so their quantum fluctuations are not suppressed. In the Hamiltonian constraint programme the recovery of the low energy physics appears cumbersome.

6.1.1 The Master Constraint

In [77] the Master Constraint Progaremme was launched which proposes to replace $\mathcal{D}$ by a much simpler Master Constraint Algebra $\mathcal{M}$. Basically, the infinite number of

Hamiltonian constraints are replaced by a single constraint, namely the weighted integral of their squares such that the associated Master Constraint **M** is spatially diffeomorphism invariant. One can show that $\mathcal{D}$, $\mathcal{M}$ are classically equivalent.

The physical Hilbert is then readily available using standard spectral analysis techniques [77],[209] provided one manages to implement **M** as a self-adjoint operator $\hat{\mathbf{M}}$ on either $\mathcal{H}_{aux}$ or $\mathcal{H}_{Diff}$ (and provided that the Hilbert space is a direct sum of separable subspaces invariant for $\hat{\mathbf{M}}$).

To take the sum of squares of constraints rather than the consraints themselves has successfully been tested for various toy models including those with an infinite number of degrees of freedom and with structure functions [210], [211], [212], [213].

2. after having shown that the solution theory is consistent reduces to QTF on a background spacetime in the semiclassical limit of low geometry fluctuations.

This is precisely the purpose of the Master Constraint Programme to complete this task.

• **Possibility of having:**

(i) Control over Physical space of solutions,

(ii) Control over Quantum Dirac Observables of LQG,

(iii) An Answer to Whether LQG has the Correct Semi-classical limit.

In his paper he proposes a solution to this set of problems based on the so-called Master Constraint which combines the smeared Hamiltonian for all smearing functions into a single constraint.

If certain mathematical conditions hold, which still have to be proved (which now have been see [215]),

then not only the problems with the commutator algebra could disappear, also chances are good that one can control the solution space and the (quantum) Dirac observables of LQG. Even a decision on whether the theory has the correct classical limit and connection with the path integral (or spin foam) formulation could be within reach.

Summary:

Hence we see that the problem of investigating the classical limit of LQG and to verify the quantum algebra of constraints are very much interlinked:

1. Spatial diffeomorphism invariance enforces a weakly discontinuous representation of spatial diffeomorphisms.

2. Anomaly freeness in the presence of only finite diffeomorphisms enforces graph changing Hamiltonian constraints.

3. Graph changing Hamiltonians seem to prohibit appropriate semiclassical states.

· Problems with the commuter algebra disappear
· Could have control of the solution space
· Could have control of the (quantum) Dirac observables LQG.
· Even a decision on whether the theory has the correct classical limit.
· and the connection with the path (or spin foam) formulation could be within reach.

Can be traced back to simple facts about the constraint algebra:

1. *The (smeared) Hamiltonian constraint is not a spacially diffeomorphism invariant function.*
2. The algebra of (smeared) Hamiltonian constraints does not close, it is proportional to a spatial diffeomorphism constraint.
3. The coefficient of proportionality is not a constant, it is a non-trivial function on the phase space.

$$\begin{aligned}
\{\vec{C}(\vec{N}), \vec{C}(\vec{N}')\} &= \kappa \vec{C}(\mathcal{L}_{\vec{N}} \vec{N}') \quad (6.1)\\
\{\vec{C}(\vec{N}), C(N')\} &= \kappa C(\mathcal{L}_{\vec{N}} \vec{N}') \quad (6.2)\\
\{C(N), C(M)\} &= \int d^3x (N\partial_a M - M\partial_a N) g^{ab} C_b \quad (6.3)
\end{aligned}$$

where $C(N) = \int_\sigma d^3 N(x) C(x)$ is the smeared Hamiltonian constraint, C_b is the spacial diffeomorphism constraint, $\vec{C}(\vec{N}) = \int_\sigma d^3 N^a(x) C_a(x)$ is the smeared spatial diffeomorphism constraint, q^{ab} is the inverse spatial metric tensor, N, N, N^a, N'^a are smearing functions on the spacial three-manifold σ and κ is the gravitational constant.

The righthand side of the commutator does not obviously resemble the right hand side of the Poisson bracket of two Hamiltonian constraints.

All these problems would disappear if it would be possible to reformulate the Hamiltonian constraint in such a way that it is equivalent to the original formulation but such that it becomes spatially diffeomorphim invariant function with an honest Lie algebra. There is a natural candidate, namely

$$\mathbf{M} = \int_\sigma d^3x \frac{[C(x)]^2}{\sqrt{det(q(x))}} \quad (6.4)$$

This has been called the **Master Constraint** corresponding to the infinite number of constraints $C(x)$, $x \in \sigma$ because due to the positivity of the integrand, the Master Equation $\mathbf{M} = \mathbf{0}$ is equivalent with $C(x) = 0\ \forall x \in \sigma$ since $C(x)$ is real valued. The factor has been incorporated in order to make the integrand a scalar of density of weight one. This guarantied

1) that $\mathbf{M}$ is spatially diffeomorphism invariant quantity and
2) that $\mathbf{M}$ has a chance to survive quantization.

Invariance under active spacial diffeomorphisms follows from section 1.6.1.

Secondly, only scalar densisties of weight 1 can be promoted to spacially diffeomorphism invaraiant operators.

Why was it that anyone didn't think of such a quantity before? There was an a priori problem with which prevented Thiemann to from considering it seriously much earlier: consider the Poisson bracket

$$\begin{aligned} \{O, \mathbf{M}\} &= \{O, \int_\Sigma d^3x \frac{C(x)}{\sqrt{q(x)}} C(x)\} \\ &= \{O, \int_\Sigma d^3x \frac{C(x)}{\sqrt{q(x)}}\} \, C(x) + \int_\Sigma d^3x \frac{C(x)}{\sqrt{q(x)}} \, \{O, C(x)\} \end{aligned} \tag{6.4}$$

(where we used the product rule of the Poisson bracket). On the constraint surface $\mathbf{M} = 0$ ($C(x) = 0$) we obviously have $\{O, \mathbf{M}\} = 0$ for *any* differentiable function O on the phase space. This is a problem because (weak) Dirac observables for first class constraints such as $C(x) = 0$ are selected precisely by the condition $\{O, C(x)\} = 0$ for all $x \in \sigma$ on the constraint surface. Thus the Master Constraint seems to fail to detect Dirac observables with respect to the original set of Hamiltonian constraints $C(x) = 0$, $x \in \sigma$.

if it satisfies the **Master Equation**

$$\boxed{\{O, \{O, \mathbf{M}\}\}_{\mathbf{M}=0} = 0} \tag{6.5}$$

The price we have to pay is this Master Equation is that it is a non-linear condition on O. Now from the theory of differential equations one knows that non-linear differential equations (such as the Hamlitonian-Jacobi equation) are often easier to solve if one transforms them into a system of linear partial differential equations and one might think that in order to find solutions to the Master-Constraint one has to go back to the original infinite system of conditions. However, also this is not the case: As we will show, one can explicitly solve the Master Condition for a subset of *strong* Dirac observables by using **Ergodic Theory Methods**.

i) The Hamiltonian constraints $C(x)$ do not preserve $\mathcal{H}_{Diff}$.

$$\hat{C}^a(x)|\Phi >= 0 \tag{6.5}$$

$$[\hat{C}(x), \hat{C}^a(x')] \neq 0 \tag{6.5}$$

$$\hat{C}^a(x')\hat{C}(x)|\Phi > \neq \hat{C}(x)\hat{C}^a(x')|\Phi >= 0 \tag{6.5}$$

$$\hat{C}(x)|\Phi >\neq 0. \tag{6.5}$$

$C[N]$ is not diffeomorphism invariant, and therefore $C[N]|s>$ is not in H_{Diff}, because a diffeomorphism modifies N. More precisely it is not invariant under diffeomorphisms that move the positions of the nodes; the state $C[N]|s>$ has a factor $N(x_i)$ where x_i is the position of a node of the state $|s\rangle$. Under a diffeomorphism in which the position x_i is sent to the position x'_i the field $N(x_i)$ is replaced by $N(x'_i)$.

Thus the inner product structure of H_{Diff} cannot be employed, via the same powerful techniques used to construct inner product structure of H_{Diff} from the kinematic inner product structure ... , in the construction of the physical inner product.

ii) *Physical States and the Physical Inner Product*

The constraint operators are defined on a common dense domain, S, consisting of the space of infinitely differentiable wavefunctions, ψ, for which $\hat{x}^j\hat{p}^k\psi$ is normalizable for all positive integers j and k. This is small so it has a very large dual space, S', called the space of *tempered distributions*, to which the operators can be transposed.

$$\mathbf{\Phi}^*{}_{Aux} \tag{6.5}$$

We cannot define the Hamiltonian constraint on

$$(\mathbf{\Phi}^*{}_{Kin})_{Diff} \tag{6.5}$$

iii) *Strong Dirac Observables*

$$\boxed{[\hat{O}] := \lim_{T\to\infty}\int_{-T}^{T} dt\hat{U}(t)\hat{O}_{Diff}\hat{U}(t)^{-1} \qquad (6.6)}$$

Hence, anomaly freeness has been transformed into the issue of the size of the $\mathcal{H}_{Phys}$

Instead of solving all the (possibly infinitely many) equations $C_i(m) = 0$, one can also define the so-called master constraint []

$$M := \sum_i C_i K_{ij} C_i. \tag{6.6}$$

Here K_{ij} is a symmetric, positive definite matrix in the case of i being a discrete set. Otherwise, Kij has to be a positive definite operator kernel and the summation over i turns into an integration. It is straightforward to see that

$$M(m) = 0 \qquad \Leftrightarrow \qquad C_i(m) = 0 \text{ for all } i. \tag{6.6}$$

Furthermore, for any phase space function f weakly commuting with the constraints:

$$\{\{M, f\}, f\} \approx 0 \qquad \Leftrightarrow \qquad \{C_i, f\} \text{ for all } i. \tag{6.6}$$

So M enables us to derive the complete set of observables on the physical pase space. This means that the physical phase space itself can be constructed from the knowledge of M, so one does not lose any information if one goes over to M from the C_i. The final classical systems defined by both are in fact equivalent. This is in fact independent of the actual choice of K_{ij}, so there are a priori many possible master constraints. One can choose the one that is most useful, makes the sum (6.1.1) converge and is the most convenient to compute. This freedom is quite useful in the quantized theories [].

6.2 The Master Constraint Programme versus the Hamiltonian Constraint Programme for GR.

Dirac programme:

· given (first class) constraints C_i

· implement C_i on $\mathcal{H}_{kin}$

· look for solutions $\hat{C}_i\phi = 0$ (RAQ: in $\Phi^* \to \mathcal{H}_{kin} \to \Phi^*$

· construct inner product (rigging map) $\to \mathcal{H}_{phys}$

Master Constraint programme:

· given (first class) constraints $M = \sum K_{ij} C_i C_j$ (finite case)
$M(m) =< C(m), KC(m) >_{\mathcal{H}}$ (field theory)

· quantize M on $\mathcal{H}_{kin}$ (pos., s.a.)

· determine direct integral decomposition wrt $\hat{M}$
$\mathcal{H}_{kin} = \int_{R_+}^{\oplus} d\mu(\lambda) \mathcal{H}_{kin}^{\oplus}(\lambda)$

· $\mathcal{H}_{phys} = \mathcal{H}_{kin}^{\oplus}(\lambda_{min})$ with induced inner product

in order to quantize this expression one no replaced all appearing quantities by operators and the Poisson bracket by a commutator divided by $i\hbar$.In addition, in order to arrive at an unambiguous result one had to make the triangularization *state dependent*. That is, the regulated operator is defined on a certain spin network basis elements T_s of the

Hilbert space in terms of an adapted triangularization τ_s and extended by linearity. This is justified because the Riemann sum that enters the definition of $C_\tau(N)$ converges to $C(N)$ no matter how we refine the triangularization.

Quantum Dirac Algebra

We may compute the commmutator $[\hat{C}, \hat{C}(N')]$ on Ψ_{Kin} corresponding to the Poisson braacket $\{\hat{C}, \hat{C}(N')\}$ which is proportional

Classically the constraint algebra $\{C_I, C_J\} = f_{IJ}{}^K C_K$ for structure functions $f_{IJ}{}^K$ that are constants for all cases other than the Hamiltonian Poison bracket.

$$[\hat{H}_I, \hat{H}_J] = i\hbar \hat{H}_K \hat{f}_{IJ}{}^K = i\hbar\{[\hat{H}_K, \hat{f}_{IJ}{}^K] + \hat{f}_{IJ}{}^K \hat{H}_K\} \tag{6.6}$$

and it follows that any $l \in \mathcal{D}^*_{phys}$ also solves the equation $([\hat{H}_K, f_{IJ}{}^K])l = 0$ for all I, J. If that commutator is not itself a constraint again, then it follows that l solves more that only the equations $\hat{C}_I l = 0$. The quantum solutions are subject to more conditions than the classical solutions and thus the quantum theory has less physical degrees of freedom than the classical theory.

The Master Constraint approach improves on these issues:

1) Quantizaton of the Regulated Constraint

$$\hat{C}^a(x)|\Phi >= 0 \tag{6.6}$$

$$[\hat{M}(x), \hat{C}^a(x')] = 0 \tag{6.6}$$

$$\hat{C}^a(x')\hat{M}(x)|\Phi >= \hat{M}(x)\hat{C}^a(x')|\Phi >= 0 \tag{6.6}$$

therefore

$$\hat{C}^a(x')\hat{M}(x)|\Phi >= 0. \tag{6.6}$$

So we either have that:

$$\hat{M}(x)|\Phi >= 0, \quad \text{or} \quad \hat{M}(x)|\Phi > \in \mathcal{H}_{Diff}. \tag{6.6}$$

We say the master constraint preserves the space of Diff solutions.

We can quantize $\mathbf{M}$ directly on $(\mathbf{\Phi}^*{}_{Kin})_{Diff}$,

2) Removal of the Regulator

By the same methods of the last talk one can remove the triangulation dependence. Diffeomorphism invariance ensures that the limit does not depend on the representative index set $\mathcal{I}$.

3) Quantum Dirac algebra.

If states have to solve additional constraints then the quantum theory would not have as many physical degrees of freedom as the classical theory. To ensure that this is so with the Hamiltonian constraint, we have to restrict the way it acts upon spinnetwork states: The way an operator acts on a state depends on the triangularisation prescription one is adhering to, the requirement for there to be no anomalies places a restriction on this triangularisatoin prescription.

Hamiltonian constraint was to have an anomaly free constraint algebra among the smeared Hamiltonian constraints $\hat{C}(N)$. This motivation is void with respect to $\hat{\mathbf{M}}$ since there is only one $\hat{\mathbf{M}}$ so there cannot be any anomaly (at most in the sense that H_{Phys} is too small, that is, has an unsucient number of semiclassical states). So we have more freedom in the way the Master Constraint acts on spin network states.

However with the Master Constraint one does no. So we have more freedom in the way the Master Constraint acts on spin network states.

There is no constraint algebra anymore, the issue of mathematical consistency (anomoly freenes) is trivalized. However, the issue of physical consistency as not answered yet. The operator ordering choices will have influence on the sizze of the physical Hilbert space and thus on the number of semiclassical states, see below.

4) Classical Limit

The issue could improve on the level of $\mathcal{H}_{Diff}$ for two resons:

i) First of all, $\mathcal{H}_{Diff}$ in contrst to $\mathcal{H}_{Kin}$ does carry an inner product.

ii) The Hilbert space is separable hence coherent states are not distributional but honest elements of $\mathcal{H}_{Diff}$.

Finally, there is a less ambitious programme where $\hat{\mathbf{M}}$ exists as a diffeomorphism invariant operator on $\mathcal{H}_{Kin}$ and where one can indeed try where one could try to answer the queston about the corectness of classical limit using existing semiclassical tools.

Construct semiclassical, spacially diffeomorphism invariant states, maybe by applying the map η to the states consructed in [], and compute expectation values and fluctuations of the Master constraint operator. Show that these quantities coincide with the expected clasical values up to $\hbar$ corrections. This is the second most important step because the existence of suitable semiclassical states at the spatially diffeomorphism invariant level is not a priori granted. Once this step is established, we would have shown that the classical limit of $\hat{\mathbf{M}}$ is the correct one and therfore the quantization really qualifies as a quantum field theory of GR.

Solution of all the Quantum Constraints

The Hilbert space is not a priori separable because there are continous moduli associated with intersecting knot classes with vertices of valence higher than four. It turns out that there is a simple way to remove those moduli by performing additional averaging in the rigging map η mentioned above. This should not affect the classical limit

Thus, if $\mathcal{H}_{Diff}$ is separable, then we can construct the direct integral representation of $\mathcal{H}_{Diff}$ associated with the self-adjoint operator $\hat{\mathbf{M}}$, that is,

$$\mathcal{H}_{Diff} = \int_R^{\oplus} d\mu(\lambda)\mathcal{H}^{\oplus}_{Diff}(\lambda) \tag{6.6}$$

and since $\hat{\mathbf{M}}$ acts on the Hilbert space $\mathcal{H}^{\oplus}_{Diff}(\lambda)$ by multiplication with λ it follows that

$$\mathcal{H}_{Phys} = \mathcal{H}^{\oplus}_{Diff}(0) \tag{6.6}$$

is the physical Hilbert space and a crucial open quesion to be answered is whether it is large enough (has sufficient number of smiclassical solutions).

The physical inner product is given by

$$< s, s' >:= \lim_{T\to\infty} < s, \int_{-T}^{T} dt e^{it\,\hat{\mathbf{M}}} s' > \tag{6.6}$$

Simple example: “Master constraint” direct decompostion of the kinnematic Hilbert space.

We wish to find the solution space for the constraint

$$\hat{C}\psi(p_1, p_2) := \hat{p}_2\psi(p_1, p_2) = 0 \tag{6.6}$$

and an induced inner product. Let us first write,

$$\hat{C}\psi(p_1, p_2) = \lambda\psi(p_1, p_2) \tag{6.6}$$

the solutions to the constraint equation (N.-19) are the eigenstates with zero eigenvector, $\lambda = 0$.

$$\begin{aligned}\int dp_1 dp_2 \overline{\psi(p_1,p_2)}\phi(p_1,p_2) &= \int d\lambda \left[\int dp_1 dp_2 \delta(\lambda - p_2)\overline{\psi(p_1,p_2)}\phi(p_1,p_2)\right] \\ &= \int d\lambda \left[\int dp_1 \overline{\psi(p_1,\lambda)}\phi(p_1,\lambda)\right] \qquad (6.6)\end{aligned}$$

$$(\psi,\phi)_K = \int d\mu(\lambda)(\psi_\lambda,\phi_\lambda)_{\mathcal{H}}, \quad d\mu(\lambda) = d\lambda \qquad (6.6)$$

$$(\psi_\lambda,\phi_\lambda)_{\mathcal{H}} = \int dp \overline{\psi_\lambda(p)}\phi_\lambda(p) \text{ where } \phi_\lambda(p) := \phi(p,\lambda) \qquad (6.6)$$

and so the kinematic Hilbert space is the direct summation:

$$\mathcal{H}_{Kin} = \int d\mu(\lambda)\mathcal{H}^{\oplus}_{Kin}(\lambda) \qquad (6.6)$$

where $\mathcal{H}^{\oplus}_{Kin}(\lambda)$ is the subset of the kinematic Hilbert space on which $\hat{C}$ operates by multiplication, i.e. for every $\psi(p) \in \mathcal{H}^{\oplus}_{Kin}(\lambda)$, $\hat{C}\psi(p) = \lambda\psi(p)$.

Now the "Master constraint"

We wish to find the solution space for the constraint

$$\hat{\mathbf{M}}\psi(p_1,p_2) = \hat{p}_2^2\psi(p_1,p_2) = 0 \qquad (6.6)$$

and an induced inner product. Let us first write,

$$\hat{\mathbf{M}}\psi(p_1,p_2) = \lambda\psi(p_1,p_2). \qquad (6.6)$$

the solutions are the eigenstates with zero eigenvector, $\lambda = 0$.

$$\int dp_1 dp_2 \overline{\psi(p_1,p_2)}\phi(p_1,p_2) = \int d\lambda \left[dp_1 dp_2 \delta(\lambda - p_2^2)\overline{\psi(p_1,p_2)}\phi(p_1,p_2)\right] \qquad (6.6)$$

with a change of variables

$$u = p_2^2, \;\; dp_2 = \frac{du}{2\sqrt{u}} \qquad (6.6)$$

eq(N.-19) evaluates as

$$
\begin{aligned}
\int dp_1 dp_2 \overline{\psi(p_1,p_2)}\phi(p_1,p_2) &= \int d\lambda \left[\int dp_1 \int \frac{du}{2\sqrt{u}}\delta(\lambda-u)\overline{\psi(p_1,u)}\phi(p_1,u)\right] \\
&= \int d\lambda \frac{1}{2\sqrt{\lambda}}\left[\int dp_1 \overline{\psi(p_1,\sqrt{\lambda})}\phi(p_1,\sqrt{\lambda})\right] \\
&= \int d\mu(\lambda)\left[\int dp_1 \overline{\psi(p_1,\sqrt{\lambda})}\phi(p_1,\sqrt{\lambda})\right] \qquad (6.5)
\end{aligned}
$$

$$
(\psi,\phi)_K = \int d\mu(\lambda)(\psi_\lambda,\phi_\lambda)_{\mathcal{H}}, \quad \text{with } \; d\mu(\lambda) = d\lambda\frac{1}{2\sqrt{\lambda}} \qquad (6.5)
$$

$$
(\psi_\lambda,\phi_\lambda)_{\mathcal{H}} = \int dp\overline{\psi_\lambda(p)}\phi_\lambda(p) \; \text{ where } \; \phi_\lambda(p) := \phi(p,\sqrt{\lambda}) \qquad (6.5)
$$

and

$$
\mathcal{H}_{Kin} = \int d\mu(\lambda)\mathcal{H}^{\oplus}_{Kin}(\lambda). \qquad (6.5)
$$

$$
\mathcal{H}_{phys} = \mathcal{H}^{\oplus}_{Kin}(0) \qquad (6.5)
$$

with induced inner product $L_2[R,dp]$. Where $\mathcal{H}^{\oplus}_{Kin}(\lambda)$ is the subset of the kinematic Hilbert space on which $\hat{\mathbf{M}}$ operates by multiplication, i.e. for every $\psi(p) \in \mathcal{H}^{\oplus}_{Kin}(\lambda)$, $\hat{\mathbf{M}}\psi(p) = \lambda\psi(p)$.

Quantum Dirac Observables

They are much cleaner than the ones involved in the Hamiltonian Constraint Programme.

$$
[O]_T := \frac{1}{2T}\int_{-T}^{T} dt e^{t\mathcal{L}_{\chi_{\mathbf{M}}}} O \qquad (6.5)
$$

$$
\{[O]_T, \mathbf{M}\} = \frac{1}{2T}\int_{-T}^{T} dt \{e^{t\mathcal{L}_{\chi_M}} O, \mathbf{M}\} \qquad (6.5)
$$

$$
\frac{df}{dt} := \mathcal{L}_{\chi_{\mathbf{M}}} \equiv \{f, \mathbf{M}\}. \qquad (6.5)
$$

$$
\{[O]_T, \mathbf{M}\} = \frac{1}{2T}\int_{-T}^{T} dt \frac{d}{dt} e^{t\mathcal{L}_{\chi_M}} O = \frac{e^{T\mathcal{L}_{\chi_{\mathbf{M}}}} - e^{-T\mathcal{L}_{\chi_{\mathbf{M}}}}}{2T} O \qquad (6.5)
$$

Since O is bounded (in sup-norm) on by assumption so is $e^{\pm T \mathcal{L}_{\chi_{\mathbf{M}}}} O$, hence

$$\lim T \to \infty \{[O]_T, \mathbf{M}\} = 0 \tag{6.5}$$

Thus provided that we can interchange the limit $\lim T \to \infty$ with the Poisson bracket, we get $\{[O], \mathbf{M}\}$

Summary of Master Constraint

$$\mathbf{M} := \int_\sigma d^3x \frac{[C(x)]^2}{\sqrt{\det(q)(x)}} \tag{6.5}$$

Properties:

i) Positive: $\mathbf{M} = 0$ if and only if $C(x) = 0$ for all $x \in \sigma$.

ii) Weak Dirac observables: $\{O, \{O, \mathbf{M}\}\}_{\mathbf{M}=0} = 0$ if and only if $\{O, C(x)\}_{\mathbf{M}=0} = 0$ for all $x \in \sigma$.

iii) Spatially diffeomorphism invariant: $\{\mathbf{M}, C_a(x)\}$ for all $x \in \sigma$.

iv) Trivial commutator: $\{\mathbf{M}, \mathbf{M}\} = 0$.

4) Since $\mathbf{M}$ spatial diffeomorphic we can define $\mathbf{M}$ directly on $\mathcal{H}_{Diff}$! - we can solve one constraint onto the other.

6.3 Quantization of the Master Constraint Operator for GR.

Clasical preliminaries

$$\mathbf{M} := \int_\sigma d^3x \frac{C(x)^2}{\sqrt{\det(q)}}(x) = \int_\sigma d^3x (\frac{C}{\det(q)^{1/4}})(x) \int_\sigma d^3y \delta(x,y) (\frac{C}{\det(q)^{1/4}})(y) \tag{6.5}$$

A slight variant of

$$\begin{aligned} \frac{C(x)}{\det(q)^{1/4}} &= H^{(1)} + H^{(2)} \quad \text{where} \\ H^{(1)}_\epsilon &= -2Tr(F_{ab}\{A_c, V^{1/2}\})\epsilon^{abc} && (6.4) \\ H^{(2)}_\epsilon &= \frac{\gamma^2+1}{\gamma^2}\epsilon^{abc} Tr(\{A_a, K\}\{A_b, K\}\{A_c, V^{1/2}\}) && (6.5) \end{aligned}$$

A regularized expression for the **Master constraint**:

$$\mathbf{M} = \lim \epsilon \to 0 \int_\sigma d^3x Tr H_\epsilon(x) \int_\sigma d^3y \chi_\epsilon(x,y) Tr H_\epsilon(y) \tag{6.5}$$

the integrands will be *precisely those* of (N.-19) but with $\{A_c^k, V^{1/2}\}$ instead of $\{A_c^k, V\}$ and with $H_\epsilon = H_\epsilon^{(1)} + H_\epsilon^{(2)}$ where

Here $\chi_\epsilon(x,y)$ is any one parameter family of functions such that $\lim \epsilon \to 0\ \chi_\epsilon(x,y)/\epsilon^3 = \delta(x,y)$ and $\chi_\epsilon(x,x) = 1$.

$$V_\epsilon(x) := \int_\sigma d^3y \chi_\epsilon(x,y) \sqrt{det(q)}(y) \tag{6.5}$$

We recognize that the integrands of the two integrals in are *precisely* those of Eq.(), the only difference being that the last factor in the wedge product is given by $\{A_c, V_\epsilon^{1/2}\}$ rather than $\{A_c, V\}$ which comes from the additional factor of $(det(q))^{-1/4}$ in the point-split expression. Thus we proceed exactly as in the last talk and introduce a partition $\mathcal{P}$ of σ into cells ∇, splitting both integrals into sums $\int_\sigma = \sum_{\nabla \in \mathcal{P}}$

$$H(N) = \lim_{\epsilon \to 0} \sum_{\triangle \in \mathcal{T}(\epsilon)} N(v(\Delta))\ H(\Delta) \tag{6.5}$$

where $H(\Delta) = H(\chi_\Delta) = \int_\sigma d^3x\ \chi_\Delta(x)\ H(x)$

$$\mathbf{M} = \lim_{\epsilon \to 0} \sum_{\triangle \in \mathcal{T}(\epsilon)} \frac{H(\Delta)^2}{V(\Delta)} \tag{6.5}$$

$$C(\Delta) := \frac{H(\Delta)}{\sqrt{V(\Delta)}} = \int_\Delta d^3x\ \epsilon^{abc} Tr(F_{ab} \frac{\{A_c, V(\Delta)\}}{\sqrt{V(\Delta)}}) = 2 \int_\Delta d^3x\ \epsilon^{abc} Tr(F_{ab} \{A_c, \sqrt{V(\Delta)}\}) \tag{6.5}$$

where we have used $\{\ .\ , V(\Delta)\}/\sqrt{V(\Delta)} = 2\{\ .\ , \sqrt{V(\Delta)}\}$.

$$\mathbf{M} = \lim_{\epsilon \to 0} \sum_{\triangle \in \mathcal{T}(\epsilon)} C(\Delta)^2 \tag{6.5}$$

6.3.1 Spatially Diffeomorphism Invariant Operators on $\mathcal{H}_{Kin}$

Non graph changing operators only change the labels of a graph but not change the graph it self. Diffeomorphism invariant operators which are graph-changing cannot exist on $\mathcal{H}_{Kin}$. This is due to the "infinite volume of the diffeomorphism group". We sketch the proof here missing out some of the technical details. Recall that for each φ in the group of diffeomorphisms of Σ there is a unitary operator $\hat{U}(\varphi)$ acts on the kinematical Hilbert space and as it is a unitary operator it is in the collection of bounded operators on $\mathcal{H}_{Kin}$. We call

$$Q(f,f') := \left\langle f, \hat{O} f' \right\rangle_{Kin}$$

the quadratic form of $\hat{O}$. Let $Q_{s,s'} := Q(T_s, T_{s'})$

$$Q_{s,s'} = 0 \;\; \text{whenever} \;\; \gamma(s) \neq \gamma(s') \tag{6.5}$$

A spacially diffeomorphism invariant quadratic form is defined by

$$Q(\hat{U}(\varphi)f, \hat{U}(\varphi)f') = Q(f,f')$$

for all f, f' in the domain of Q and for all φ in the difeomorphism group. Note that if an operator $\hat{O}$ is graph changing then there will exist a spin network state T_s such that $\left\langle T_s, \hat{O} T_{s'} \right\rangle_{Kin} \neq 0$ for $T_s \neq T_{s'}$. Consider any $\gamma(s) \neq \gamma(s')$ where $\gamma(s)$ is the graph underlying spin network s.

Recall that at the kinematic level two spin network states are orthogonal if their underlying graphs are topologically distinct or positioned at different places in σ. It should be obvious that one can find a countable infinite number of diffeomorphisms φ_n, $n = 0,1,2,\dots$ such that $T_{s'}$ is invariant but $T_{s_n} = \hat{U}(\varphi_n) T_s$ are mutually orthogonal spin network states, that is, we can find φ_n, $n = 0,1,2,\dots$ such that

$$U(\varphi_n) T_{s'} = T_{s'} \quad \text{and} \quad \left\langle \hat{U}(\varphi_m) T_s, \hat{U}(\varphi_n) T_s \right\rangle_{Kin} = 0 \quad \text{for } m \neq n.$$

Suppose now that Q is the quadratic form of a spacially diffeomorphism invariant operator $\hat{O}$ on $\mathcal{H}_{Kin}$, that is, $\hat{U}(\varphi)\hat{O}\hat{U}(\varphi)^{-1} = \hat{O}$ for all φ in the diffeomorphism group.

Let us expand $\hat{O} T_s'$ as an uncountable formal summation

$$\hat{O} T_{s'} = \sum_t a_t T_t$$

The coefficents a_t are obtained from

$$
\begin{aligned}
Q_{s,s'} = \left\langle T_s, \hat{O} T_{s'} \right\rangle_{Kin} &= \sum_t a_t \left\langle T_s, T_t \right\rangle_{Kin} \\
&= \sum_t a_t \delta_{st} = a_s
\end{aligned}
$$

Now we consider the norm square of $\hat{O} T_{s'}$

$$
\begin{aligned}
\|\hat{O} T_{s'}\|^2 &= \| \sum_t Q_{t,s'} T_t \|^2 \\
&= \sum_{t,t'} Q^*_{t,s'} Q_{t',s'} \left\langle T_t, T_{t'} \right\rangle_{Kin} \\
&= \sum_s |Q_{s,s'}|^2 .
\end{aligned} \tag{6.2}
$$

As summation over all graph labels s includes summation over the countable collection $\{s_n\}$, $n = 0, 1, 2, \ldots$, we have

$$
\sum_s |Q_{s,s'}|^2 \geq \sum_{n=0}^{\infty} |Q_{s_n,s'}|^2
$$

Now we use the spacial diffeomorphism invariance on the RHS

$$
\begin{aligned}
\sum_{n=0}^{\infty} |Q_{s_n,s'}|^2 &= \sum_{n=0}^{\infty} | \left\langle \hat{U}(\varphi_n) T_s, T_{s'} \right\rangle |^2 \\
&= \sum_{n=0}^{\infty} | \left\langle T_s, \hat{U}(\varphi_n)^{-1} \hat{O} \hat{U}(\varphi_n) \; \hat{U}(\varphi_n)^{-1} T'_s \right\rangle |^2 \\
&= \sum_{n=0}^{\infty} | \left\langle T_s, \hat{O} T'_s \right\rangle |^2 = \sum_{n=0}^{\infty} |Q_{s,s'}|^2
\end{aligned} \tag{6.1}
$$

where we have used that $\hat{U}(\varphi_n)$ has an inverse which is also a diffeomorphism. Hence

$$
\|\hat{O} T'_s\|^2 \geq |Q_{s,s'}|^2 [\sum_{n=0}^{\infty} 1] \tag{6.1}
$$

diverges unless $Q_{s,s'} = 0$. We conclude that diffeomorphism invariant operators which are graph changing cannot exist on the kinematic Hilbert space $\mathcal{H}_{Kin}$.

Diffeomorphism invariant operators that are graph changing must be defined on $\mathcal{H}_{Diff}$ and not on $\mathcal{H}_{Kin}$. Roughly speaking, this works because all the terms in the infinite summation (6.3.1) are equivalent under diffeomorphisms, hence we only need one of them, whence the infinite sum becomes finite.

For an operator to be well defined on $\mathcal{H}_{Kin}$ it must be non graph changing, but of course, this isn't a sufficient condition.

6.3.2 Diffeomorphism Invariant Hilbert Space

6.3.3 Regularizaton and Quantization of the Master Constraint

Basic building blocks of (??) are the integrals

$$C_{\epsilon\mathcal{P}()} = Tr\left(\left[F_{ab} + \frac{\gamma^2+1}{\gamma^2}\{A_a, \{C_E(1), V\}\}\{A_b, \{C_E(1), V\}\}\right)\right]\{A_c, V_\epsilon^{1/2}\}\right)\epsilon^{abc} \quad (6.1)$$

It frequently happens in quantum mechanics that an operator is not closed but has an extension which is closed.

6.3.4 Non Graph Changing (Extended) Master Constraint

In order to have control on the semiclassical limit one must currently use a non graph changing operator and an operator which can be defined on $\mathcal{H}_{Kin}$. The advantage of having a non-graph changing Master Constraint Operator is that one can quantize it directly as a positive operator on $\mathcal{H}_{Kin}$ and check its semi-classical properties by testing it with the semi-classical tools developed in [], [], [].

In order to define such an operator we need the notion of a minimal loop: Given a vertex v of a graph γ and two edges e, e' outgoing from v, a loop $\alpha(\gamma, v, e, e')$ within γ based at v, outgoing along e and incoming along e_2 is said to be minimal if there is no other loop within γ with the same properties and fewer edges traversed (see fig. 6.3.4). Let $L(\gamma, e_1, e_2, e_3)$ be the set of minimal loops with the data indicated. Notice that this set is always non empty but may consist of more than one element. If $L(\gamma, e_1, e_2, e_3)$ has more than one element then the corresponding Master constraint operator averages over the finite number of elements of $L(\gamma, e_1, e_2, e_3)$. We now define

Here $T(\gamma, v)$ is the number of ordered triples of edges incident at v (taken with outgoing orientation)

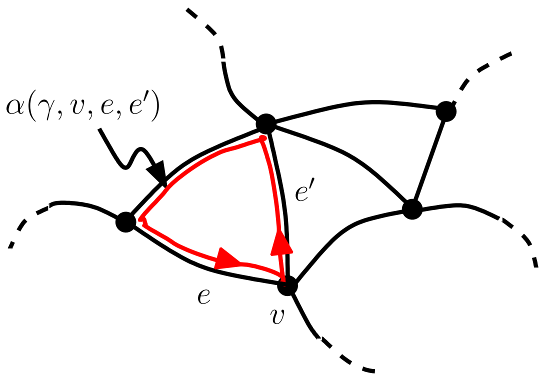


Figure 6.1: minimalloop1. minimal loop.

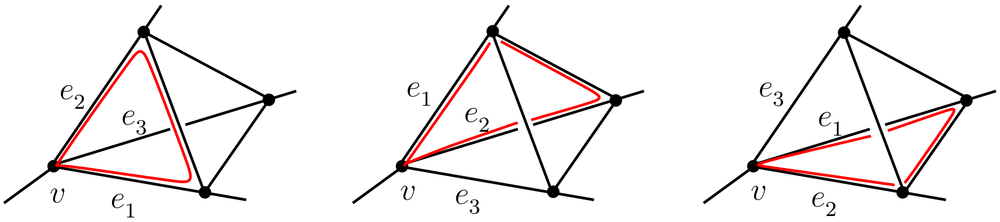


Figure 6.2: minimaloop2. minimal loop. $|L(\gamma, e_1, e_2, e_3)| = 3$

$$T(\gamma, v) = \frac{n(n-1)(n-2)}{6}$$

The expression below is a good approximation to the classical Master constraint

$$\begin{aligned}
\hat{M}_\gamma &:= \sum_{v \in V(\gamma)} \hat{\tilde{C}}_v^\dagger \hat{\tilde{C}}_v & (6.2)\\
\hat{\tilde{C}}_v &:= \frac{1}{|T(\gamma, v)|} \sum_{e_1, e_2, e_3 \in T(\gamma, v)} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(\gamma, e_1, e_2, e_3)|} \times \\
&\qquad \times \sum_{\alpha \in L(\gamma, e_1, e_2, e_3)} Tr\big([A(\alpha) - A(\alpha)^{-1}] A(e_3) [A(e_3)^{-1}, \sqrt{\hat{V}_v}]\big)
\end{aligned}$$

whose tangents are linearly independent and $\epsilon_v(e_1, e_2, e_3) = \text{sgn}(\det(\dot{e}_1(0), \dot{e}_2(0), \dot{e}_3(0))$. The volume operator is given explicitly by

$$\hat{V}_v = \sqrt{|\frac{i}{48} \sum_{e_1, e_2, e_3 \in T(\gamma, v)} \epsilon_v(e_1, e_2, e_3) \epsilon_{jkl} X_{e_1}^j X_{e_2}^k X_{e_3}^l|} \qquad (6.0)$$

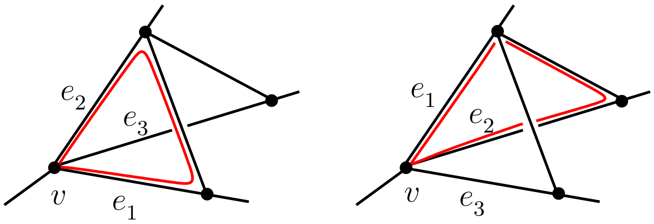


Figure 6.3: minimaloop3. minimal loop. $|L(\gamma, e_1, e_2, e_3)| = 2$

It is easy to see that the definition (6.2) is spatially diffeomorphism invariant. Moreover, the results of [221], [222], [223] imply that expectation values with respect to the coherent states constructed in [][86] defined on graphs which are suffieciently fine, the zeroth order in $\hbar$ of $\hat{M}_\gamma$ coincides with the classical expression. In other words, the correctness of the classical limit of $\hat{M}$ has been established recently.

The results of [221], [222], [223] also imply that the commutator between the $\sum_v N_v \hat{C}_v$ reproduces the third relation (6.3) in the sense of expectation values with respect to coherent states where $\hat{C}_v$ is the same as $\hat{\hat{C}}_v$ in (6.2) just that $\sqrt{\hat{V}_v}$ is replaced by $\hat{V}_v$. This removes a further criticism mentioned in section 4.5.3, namely we have off-shell closure of the Hamiltonian constraints to zeroth order in $\hbar$. Possible higher order corrections (anomalies) are no obstacle for the Master constraint programme as already said.

The disadvantage of a non-graph-changing operator is that it uses a prescription like the above minimal loop prescription as an ad hoc quantization step. While it is motivated by the more fundamental quantization procedure of the previous section and is actually not too drastic a modification thereof for sufficiently fine graphs, the procedure of the previous section should be considered as more fundamental. Maybe one could call the operator as formulated in this section an effective operator since it presumably reproduces all semiclassical properties.

6.3.5 Brief Note on the Volume Operator

In order to show this one has to calculate the matrix elements of (6.3.4) which is non trivial because the spectrum of that operator is not accessible exactly (The matrix elements of the square root are known in closed form [206]). However, one can perform an error controlled $\hbar$ expansion within coherent state matrix elements and compute the matrix elements of every term in that expansion analytically [221], [222], [223].

The idea is extremely simple: In applications we are interested in expressions of the form Q^r where Q is a positive operator, $0 < r \leq 1/4$ rational number and its relation to the volume operator is

$$V = Q^{1/4.}$$

The matrix elements of Q in coherent states can be computed in closed form. Now use the Taylor expansion of the function $f(x) = (1+x)^r$ up to some order N including the remainder with $x = Q/ < Q > 1$ where $< Q >$ is the expectation value of Q with respect to the coherent state of interest.

$$\begin{aligned} Q^{1/r} &= 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + R_{N+1}(x) \\ &= 1 + r\left(\frac{Q}{< Q >} - 1\right) + \frac{r(r-1)}{2!}\left(\frac{Q}{< Q >} - 1\right)^2 + \cdots + R_{N+1}\left(\frac{Q}{< Q >} - 1\right) \end{aligned}$$

The operators x^n in that expansion can be explicitly evaluated in the coherent state basis

$$\begin{aligned} < \psi|\left(\frac{Q}{< Q >} - 1\right)^n |\psi > &= \sum_{i_1, i_2, \ldots i_N} < \psi|x|\psi_{i_1} >< \psi_{i_1}|x|\psi_{i_2} >< \psi_{i_2}|x|\psi > \\ &= \sum_{i_1, i_2, \ldots i_n} x_{ii_1} x_{i_1 i_2} x_{ii_1} \ldots x_{i_n i} \end{aligned}$$

while the remainder $R_{N+1}(x)$ can be estimated from above and provides a higher $\hbar$ correction than any of the x^n, $0 \le n \le N$.

6.4 Spectral Decomposition

One of the basic results of linear algebra is the theorem of the existence of a complete system of eigenvectors for any self-adjoint linear operator A in an n-dimensional Eucldean space $\mathbb{R}^n$. This theorem states that if A is a self-adjoint operator in an n-dimensional Eucldean space $\mathbb{R}^n$, then an orthonormal basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ can be found, each vector of which is an eigenvector of the operator A:

$$Ae_k = \lambda_k e_k,$$

where λ_k is a real number. Expanding any vector f of the spaace $\mathbb{R}^n$ by means of the vectors $e_1, \ldots, e_n$:

$$f = a_1 e_1 + \cdots + a_n e_n,$$

where $a_k = (f, e_k)$, we can write the operator in the form:

$$Af = \sum_{k=1}^{n} \lambda_k (f, e_k) e_k. \qquad (6.\text{-}4)$$

The situtation beccomes complicated upon passing from the finite to the infinite dimensional case.

For a self-adjoint and unitary operators the eigenvalues are ordered in a natural way. For a self-adjoint operator this is so because the eigenvalues are real numbers and for a unitary operators the eigenvalues are represented by points on the unit circle of the complex plane.

Let us first consider self-adjoint operators. We assume that $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, and we use the P_i's to define new projections:

$$\begin{aligned}
E_{\lambda_0} &= 0 \\
E_{\lambda_1} &= P_1 \\
E_{\lambda_2} &= P_1 + P_2; \\
&\vdots \\
E_{\lambda_m} &= P_1 + P_2 + \cdots + P_m. \qquad (6.\text{-}7)
\end{aligned}$$

$$\begin{aligned}
A &= \lambda_1 P_1 + \lambda_2 P_2 + \cdots \lambda_m P_m \\
&= \lambda_1 (E_{\lambda_1} - E_{\lambda_0}) + \lambda_2 (E_{\lambda_2} - E_{\lambda_1}) + \cdots \lambda_m (E_{\lambda_m} - E_{\lambda_{m-1}}) \\
&= \sum_{i=1}^{m} \lambda_i (E_{\lambda_i} - E_{\lambda_{i-1}}). \qquad (6.\text{-}8)
\end{aligned}$$

$$A = \sum_{i=1}^{m} \lambda_i \Delta E_{\lambda_i} \qquad (6.\text{-}8)$$

Theorem For each self-adjoint operator there is a unique spectral family of projection operators P_x such that

$$(\phi, A\varphi) = \int_{\infty}^{\infty} x d(\phi, P_x \varphi) \qquad (6.\text{-}7)$$

for all vectors φ and ϕ.

The spectral decomposition of A

$$A = \int_{-\infty}^{\infty} dE_x \tag{6.-7}$$

$$\begin{aligned}
(\phi, \psi) &= (\phi, P_1\psi) + (\phi, P_2\psi) + \cdots (\phi, P_m\psi) \\
&= (\phi, (E_{\lambda_1} - E_{\lambda_0})\psi) + (\phi, (E_{\lambda_2} - E_{\lambda_1})\psi) + \cdots (\phi, (E_{\lambda_m} - E_{\lambda_{m-1}})\psi) \\
&= \sum_{i=1}^{m} (\phi, \Delta E_{\lambda_i}\psi). \qquad (6.\text{-}8)
\end{aligned}$$

$$\begin{aligned}
(\phi, A\psi) &= (\phi, \lambda_1 P_1\psi) + (\phi, \lambda_2 P_2\psi) + \cdots (\phi, \lambda_m P_m\psi) \\
&= (\phi, (E_{\lambda_1} - E_{\lambda_0})\psi) + (\phi, (E_{\lambda_2} - E_{\lambda_1})\psi) + \cdots (\phi, (E_{\lambda_m} - E_{\lambda_{m-1}})\psi) \\
&= \sum_{i=1}^{m} \lambda_i (\phi, \Delta E_{\lambda_i}\psi). \qquad (6.\text{-}9)
\end{aligned}$$

for any vectors φ and ϕ we have

$$(\phi, \psi) = \int_{-\infty}^{\infty} d(\phi, E_x\psi) \tag{6.-9}$$

and

$$(\phi, A\psi) = \int_{-\infty}^{\infty} xd(\phi, E_x\psi). \tag{6.-9}$$

Unitary operators can be treated in a similarly. For a unitary operator U let its eigenvalues be $u_i = e^{i\theta_i}$ labelled in the order

$$0 < \theta_1 < \theta_2 < \cdots < \theta_{m-1} < \theta_m \leq 2\pi \tag{6.-9}$$

For each real number x let

$$E_x = \sum_{\theta_i \leq x} P_i \tag{6.-9}$$

This is the progection operator onto the space spanned by all eigenvectors for eigenvalues $e^{i\theta_i}$ with $\theta_i \leq x$. If $x \leq 0$, then $E_x = 0$. If $x \geq 2\pi$, then $E_x = 1$. Evidently E_x increses by incriments P_i the same as for the Hermitian operator with eigenvalues θ_i. For

$$U = \sum_{i=1}^{m} u_i P_i = \sum_{i=1}^{m} e^{i\theta_i} P_i \tag{6.-9}$$

we can write

$$U = \int_0^{2\pi} e^{ix} dE_x. \tag{6.-9}$$

For any vectors ψ and ϕ we have

$$(\phi, U\psi) = \int_0^{2\pi} e^{ix} d(\phi, E_x \psi). \tag{6.-9}$$

6.5 Rigged Hilbert Space and Direct Integral Decomposition

The concept of a direct integral of a Hilbert space is a generalization of the concept of the orthogonal direct sum of a countable family of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \dots$.

6.6 Symmetric Operators and their Extensions

In the construction of operators a in physics one often starts from its matrix elements $Q_a(\psi, \psi')$, which should equal $< \psi, a\psi' >$. However, this is not enough to define an operator in inifite dimensions because given an ortho-normal basis (b_n) we must have

$$\|a\psi\| = \sum_n |Q_a(b_n, \psi)|^2 < \infty$$

in order that $\psi \in D(a)$. Hence it may happen that the quadratic form $Q_a(\psi, \psi')$ exists for ψ, ψ' in a dense subset of $\mathcal{H}$ but on the other hand it could be that $D(a) = \{0\}$.

For symmetric operators, there are always closed extensions. A smallest closed extension always exists (the double adjoint), but it is possible that none of these closed extensions is self-adjoint. On the other hand, semibounded forms need not have any closed extensions (defintions for quadratic forms to be given below), but when such extensions exsit and are semibounded, they are the quadratic forms associated with self-adjoint operators.

6.7 Weak Dirac Observables *a la* Dittrich

f and T_j functions on phase space. Weak Dirac observable there are n T_j's

$$F^{\tau}_{f,T} := \sum_{k_1,\ldots,k_n=0} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} (X_1)^{k_1} \ldots (X_n)^{k_n} \cdot f. \tag{6.-9}$$

where $X_r \cdot f$ is defined as

$$X_j \cdot f := \{(A^{-1})_{jk} C_k, f\}, \quad A_{jk} := \{C_j, T_k\}. \tag{6.-9}$$

Poisson algebra

$$\{F^{\tau}_{f,T}, F^{\tau}_{f',T}\} = F^{\tau}_{\{f,f'\}^*,T} \tag{6.-9}$$

defining the automorphism on ?? generated by the Hamiltonian vector field of $\sum_j \tau^j C'_j$

$$\alpha'_{\tau}(f) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_j \tau^j X_j \right)^n \cdot f \tag{6.-9}$$

$$\{\alpha^{\tau}(F^{\tau}_{f,T}), \alpha^{\tau}(F^{\tau}_{f',T})\} \approx \alpha^{\tau}(\{F^{\tau}_{f,T}, F^{\tau}_{f',T}\}) \tag{6.-9}$$

In other words, α^{τ} is a weak, abelian, multi-parameter group of automorphisms on the each $F^{\tau_0}_{f,T}$.

6.7.1 Equivalent Hamiltonians that are Spacially Diffeomorphism Invariant

An unfortunate feature of the Hamiltonian constraint was that it could not be implemented on the spatially diffeomorpism invariant Hilbert space because it would map spatially diffeomorpism invariant states onto non-diffeomorpism invariant states - i.e. it did not close on the space.

In [301] it was shown that if one is given a constraint algebra of the form

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, C_k\} = f_{Jk}{}^l C_l, \quad \{C_j, C_k\} = f_{jk}{}^L C_L \tag{6.-9}$$

$$A_{lj} := \{C_l, T_j\}. \tag{6.-9}$$

$$\tilde{C}_j := \{M, T_j\} \approx \sum_{k,l} Q_{kl} C_k A_{lj} \tag{6.-9}$$

the constraint algebra can be simplified to

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, \tilde{C}_k\} = 0, \quad \{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l. \tag{6.-9}$$

Using the Master constraint we can generate *we may generate new Hamiltonian constraints for GR which are spatially diffeomorphism invariant.* Employing these new Hamiltonian constraints it is possible to first solve the spatially diffeomorpism constraints and then implement the new Hamiltonian on the spatially diffeomorpism invariant Hilbert space where they close among themselves. In []

(Lie algebras an commutators???)

The powerful methods of [**?**]an then be used to induce the physical Hilbert space and its inner product.

6.8 Spin Foams From the Master Constraint

New proposal:

Instead of an infinite number of constraints there is only one, then there is an ordiary integral. One splits the T-parameter into discrete steps and writes

$$e^{it\,\hat{\mathbf{M}}} = \lim_{N\to\infty} [e^{it\,\hat{\mathbf{M}}/N}]^N = \lim_{N\to\infty} [1 + it\hat{\mathbf{M}}/N]^N. \tag{6.-9}$$

The action of $1 + it\hat{\mathbf{M}}/N$ on a spin network can be written as a linear combination of of new spin networks whose graphs whose labels have been modified by the creation of new nodes.

6.8.1 Removal of Difficulties of the Sum over Histories

Formally $\hat{\mathbf{M}}$ solved by bf rigging map with physical scalar product.

$$\eta \ : \ \Phi_{Diff} \to \Phi_{phys}; l \to \int_R \frac{dT}{2\pi} < e^{iT\hat{M}} l, . > \tag{6.-9}$$

$$< \eta(l), \eta(l') >_{Phys} := \eta(l')[l] = \int_R \frac{dT}{2\pi} < e^{iT\hat{M}} l', l > \tag{6.-9}$$

The integration variable T is nothing more than a variable of integration.

An approximate calculation:

$$\begin{aligned} & < \eta(l)| \ \exp[-it\frac{\hat{\mathbf{M}}}{\mathbf{N}}] \ |\eta'(l) >_{Diff} \\ = & < \eta(l)| \ [1 - it\frac{\hat{\mathbf{M}}}{\mathbf{N}}] \ |\eta'(l) >_{Diff} + \mathcal{O}(\frac{1}{N^2}) \\ = & \ \delta_{[s][s']} - \frac{it}{N} < \eta(l)| \ \hat{\mathbf{M}} \ |\eta'(l) >_{Diff} + \mathcal{O}(\frac{1}{N^2}) \end{aligned} \tag{6.-10}$$

$$< \eta(l)| \ \exp[-it\frac{\hat{\mathbf{M}}}{\mathbf{N}}] \ |\eta'(l) >_{Diff} = \ \delta_{[s][s']} - \frac{it}{N} Q_{\hat{\mathbf{M}}}(\eta(l), \eta'(l)) + \ \mathcal{O}(\frac{1}{N^2}) \tag{6.-10}$$

Observaions:

1. $\hat{\mathbf{M}}$ is positive

2. Strong existence theorems for PI's (Osterwalder-Schrader Reconstrusction).

3. Definition of Spin Foam model has precise connection to the Hamiltonian formulism and better convergence prop.($\hat{\mathbf{M}}$ = pos). Formally

4.

5.

6. If can be constructed this way lead to rigorous impleentation of Reisenberger-Rovelli idea, including **sum over all triangulations**.

Hamiltonian and vector constraints

$$M = \int_\Sigma d^3x \frac{C(x)^2 - q^{ab} V_a(x) V^b(x)}{\sqrt{\det q(x)}} \tag{6.-10}$$

Obviously, $\hat{\mathbf{M}}_E = 0$ if and only if $C(x) = C_a(x) = 0$ for all $a = 1, 2, 3; \ x \in \sigma$.

6.9 Summary:

· They are much cleaner than the ones involved in the Hamiltonian Constraint Programme.

6.9.1 Problems

(i) $\mathcal{H}_{phys} \not\subset \mathcal{H}_{Kin}$ in general.

(ii) Non-commuting constraints (spectral analysis not possible)

GR

(i) Cannot define $\hat{C}(N)$ directly on $\mathcal{H}_{Diff}$, but concrete implementation uses $\mathcal{H}_{Diff}$.

(iii) Makes semi-classical analysis difficult.

6.9.2 The Master Constraint

Simplifies constraint algebra!

Finite dimension system

$$M = \sum_{i,j} C_i K_{ij} C_j$$

where K_{ij} is strictly positive matrix.

Field theory has infinitely many constraints

$$M = \int d^n x C(x)(K \cdot C)(x)$$

$(K \cdot C)$ is a strictly positive operator.

Gravity

$$M = \int d^3 x \frac{C(x)C(x)}{\sqrt{\det q}}$$

3-dffeomorphism invariant.

$$M = 0 \iff \begin{cases} C_i = 0 & \text{for all } i \\ C = 0 & \text{for all } x \end{cases}$$

We are left with one constraint!

6.9.3 MCP: Physical Hilbert Space

$\mathcal{H}_{Kin}$ (separable), $\hat{M}$ (positive, self-adjoint) then:

Direct integral decomposition ('generalized eigenspace' decomposition)

$$\mathcal{H}_{Kin} \simeq \int_{spec(\hat{M})}^{\oplus} \mathcal{H}^{\oplus}_{Kin}(\lambda)\, d\mu(\lambda) \tag{6.-10}$$

$\mathcal{H}^{\oplus}_{Kin}$ carries an induced inner product and $d\mu(\lambda)$ the spectral measure

$$\mathcal{H}_{phys} := \mathcal{H}^{\oplus}_{Kin}(0) \tag{6.-10}$$

6.9.4 Recipe for $\mathcal{H}_{phys}$

(i) Find cyclic ON system $\{\Omega_j\}_{j\in\mathcal{J}}$

$$\mathcal{H}_{Kin} = \oplus_j \overline{\text{span}\{\hat{M}^k\Omega_j | k \in IN\}}$$

(ii) Calculate spectral measures.

$$\mu_j(\lambda) =< \Omega_j, \Theta(\lambda - \hat{M})\Omega_j >$$

(iii) Calculate spectral measure $\mu(\lambda)$

$$\mu(\lambda) = \sum_j a_j \mu_i(\lambda), \qquad \sum_j a_j = 1$$

(iv) Find the Radon-Nikodym derivatives

$$\rho_j(\lambda) = \frac{d\mu_j(\lambda)}{d\mu(\lambda)}$$

$\mathcal{H}^{\oplus}_{Kin}(\lambda)$ has an orthonormal basis $\{e_j(\lambda) \mid \rho_j(\lambda) > 0\}$

$\Rightarrow$ determines $\dim\big(\mathcal{H}^{\oplus}_{Kin}(\lambda)\big)$ = multiplicity of λ.

(v) Result does not depend on choice of $\{\Omega_j\}$ (in the following sense).

6.9.5 Uniqueness

$$\mathcal{H}_{Kin} \simeq \int^{\oplus} \mathcal{H}^{\oplus}_{Kin}(\lambda)\, d\mu(\lambda)$$

- measure theoretical formula

(given $\hat{M}$) uniqueness (of $\dim\mathcal{H}^{\oplus}_{Kin}(\lambda)$) μ a.e.

(i) for mixed spectrum points λ_i have **finite** measure

$\Rightarrow$ contributions to $\mathcal{H}^{\oplus}_{Kin}(\lambda_j)$ from ac-sectrum are suppressed

- decompose $\mathcal{H}_{Kin} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$

(ii) dependence of $\mathcal{H}_{phys}$ on $\hat{M}$ (that is K)?

6.9.6 Example: Abelian Constraints

$\mathcal{H}_{Kin} = \mathcal{L}^2(\mathbb{R}^2, d^2x)$, $\hat{C}_1 = \hat{p}_1$, $\hat{C}_2 = \hat{p}_2$

$$\hat{M} = \hat{p}_1^2 + \hat{p}_2^2 = -\hbar^2\nabla$$

soc: harmonic functions

'eigenfunctions' $|k_1, k_2 >= \exp(i\vec{k}\cdot\vec{x})$

(i) change to $\mathcal{L}^2(\mathbb{R}^2, d^2p)$

$$\Omega_j = N_j p^{|j|}\exp(ij\varphi)\exp(-\frac{p^2}{2}), j \in$$

p and φ spherical coordinates in $\mathbb{R}^2$

$$\rho_j(\lambda) = \frac{\lambda^{|j|}/|j|!}{2\exp(\frac{\lambda}{2}) - 1} \qquad \to \lim_{\lambda\to 0}\left\{\begin{array}{cc} 0 & |j| \geq 1 \\ 1 & j = 0 \end{array}\right.$$

$\Rightarrow \mathcal{H}^{\oplus}_{Kin}(0) = \mathcal{H}_{phys} \simeq \mathbb{C}$

$\mathcal{H}^{\oplus}_{Kin}(\lambda > 0) \simeq \mathcal{L}^2(S^1, dp)$

6.9.7 Observables

Classical: How to obtain them?

(i) framework available CBD. this chapter and chapter 1

(ii) gives in general **weak** Dirac observables.

Quantum: for 'strong' Dirac observables, i.e. $[\mathcal{O}, M] = 0$

$$(A|\psi>)_{Kin} \simeq \sum_j a_j(\lambda)(A(\lambda)|\psi>)_{\mathcal{H}(\lambda)}\ \ d\mu(\lambda)$$

can be calculated explicitly

(i) representation for 'weak' Dirac observables?

6.9.8 Examples

(i) Finite dimensional examples

(ii) Maxwell theory, linearized gravity:

squaring of constraints worsened

UV behaviour $\rightarrow$ choose k such that $\hat{M}$ is densely defined

k provides regularization of $\hat{M}$

(iii) Gauss constraints in Einstei - YM

$$M = \int d^3x\ \frac{G_z G^z}{\sqrt{\det(q)}}$$

background independent theories regulate themselves.

result: gauge invariant states

metric-degenerate states

6.9.9 Conclusions and Outlook

(i) method is widely applicable, in articular to open algebras.

(ii) Provides construction of physical inner product and representation of 'strong observables'

(iii) representation of 'weak' observables?

(iv) dependence on $\hat{M}$?

LQG:

(i) $\hat{M}$ can be defined on $\mathcal{H}_{Diff}$

(ii) need separable Hilbert spaces

(iii) semiclassical limit is easier now

(iv) 3-diffeomorphism invariant semiclassical states.

(v) MCP gives construction principle for $\mathcal{H}_{phys}$.

6.10 Biblioliographical notes

In this chapter I have relied on the following refferences:

Fundamental structure of Loop Quantum Gravity [215].

Introduction to Loop Quantum Gravity and Spin foams A. Perez [28].

Chapter 7

Semi-Classical Limit

- Introduction
- Relating Loop Representation to Fock-Space Descrition in the Low Energy Limit.
- Coherent States.
- Minkowskian Spacetime and Scattering Amplitudes.
- Noiseless Subsystems.
- Infinite tensor product

7.1 Introduction

how the background independed, non-peturbative quantum and low energy physics described by perturbation field theory in Minkowskian spacetime. In fully non-perturbative approaches to quantum gravity because there is no background spacetime to begin with. Both conceptually .

the relations between dynamical objects give the apperance of a background spacetime in the appropriate limits. We recover a Minkowski spacetime

In his model, he was able to show he cosmological constant paradox appears only if spacetime is regarded as fundemental rather than emergent, [267].

This cannot be done using Fock space states, as the inner product on Fock space depends on a background metric (most of the time Minkowski metric), whose presence breaks diffeomorphism invariance.

to do with the passage from the classical field to the quantum field

Algebraic Quantization

1. The set $\mathcal{S}$ should be a vector space large enough so that every function on Γ can be obtained

2. The set $\mathcal{S}$ should be small enough so that it is closed under Poisson brackets

$$[\hat{A}, \hat{B}] = i\hbar \widehat{\{A, B\}} \tag{7.0}$$

We must now find a vector space V and a representation of the elements of $\mathcal{A}$ as operators on V. Real observables must be represented by Hermition operators. One then completes V to get the Hilbert space $\mathcal{H}$ of the theory.

We give the Schrödinger picture where the Hilbert space is $\mathcal{H} = L_2(\mathbf{R}^3, d^3x)$ and the operators are represented by:

$$(\hat{\mathbf{1}}\Psi)(q) = \Psi(q), \quad (\hat{q}^i\Psi)(q) = q^i\Psi(q), \quad (\hat{p}_i\Psi)(q) = -i\hbar\frac{\partial}{\partial q^i}\Psi(q) \tag{7.0}$$

This is of course just the conventional Schrödinger representation of the CCR.

Ernfest's Theorem

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}_x\rangle}{m}, \tag{7.1}$$

$$\frac{d\langle\hat{p}_x\rangle}{dt} = -\left\langle\frac{\partial\hat{V}}{\partial x}\right\rangle. \tag{7.2}$$

When we Take the Semiclassical Limit Do we Always Get Back the Classical Theory we Quantized??

Perhaps surprisingly the answer is *no*, at least when we have an infinite number of degrees of freedom. When we quantize a classical theory, however, when there is an infinite number of degrees of freedom things get more complicated and not so clear-cut.

no analog of the Stone- von Neuman uniqueness theorem for quantum mechanics on a vector space. Hence there are infinitely many inequivalent representations of the Poisson bracket algebra. The Fock space representation is singled out by the additional requirement of Poincare invariance. This choice is equivalent to the requirement that the Fock vacuum be a zero eigenstate of the Fock space annihilation operators.

The operator algebra is associated with the system. However, the system can be in any number of different phases - with each phase there corresponds a different, mutually exclusive Hilbert space. For example: a ferreomagnetic system can be in a magnetic phase or a non-magnetic phase - each is described by a different Hilbert space, but the operator algebra is the same in either. Now, different phases involve different physics and so different low energy behaviour. Some representations may reduce to the classical theory in its low energy regime, however, others wont!

Analogies with between quantum field theory and statistical field theory.

$$“\sum_x e^{itS[\Phi(x)]} \;\; t \to -i\tau \;\; \sum e^{-\tau S[\Phi]}” \tag{7.2}$$

Aspects of one have counterparts in the other. In statistical field theory different phases - different states - different physics . In quantum field theory different **sectors** - different states - different physics!

The total Hilbert space is non-countable. The action of operators of the algebra, when apllied to a particular state, is countable. Hence any irreducible representation of the operator alegbra is a subset of the total Hilbert space - in fact there are an uncountable number of irreducible representations.

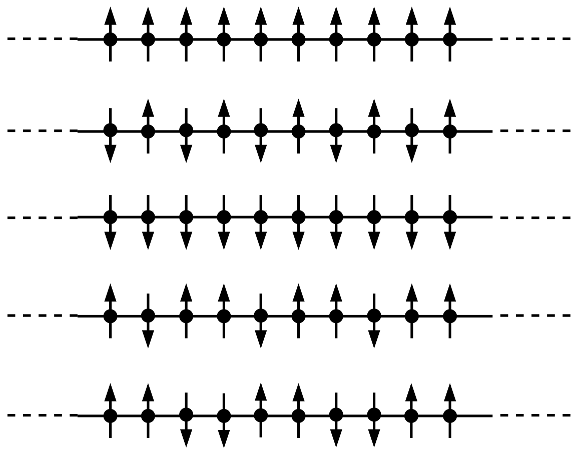


Figure 7.1: NonSepExF. No amount of algebraic action can one of these states into another. The different states describe states ‘infinitly’ different from each other. In statistical mechanics, different representations of the operator algebra describe different phases.

7.1.1 Properties of Semi-Classical States

the quantum representation

a property one need is that the semi-classical states be labelled by points of a classical phase space, i.e. $|\omega>, \omega \in \Sigma$. second property needed is "peaking". given any quantum observable $\hat{A}$, its expectation value $a(\omega) \equiv <\omega|\hat{A}|\omega>$. Quantum fluctuations should be small in a suitable sense. specifying $a_i(\omega) = C_i$ will enable one to obtain a unique point $\omega(C_i)$ in the phase space.

The requirement that the theory admit such semi-classical states is non-trivial as an arbitrarly constructed Hilbert space may or may not admit $|\omega>$.

7.1.2 The problem with establishing the semiclassical limit of LQG has to do with the quantum dynamics:

For graph changing operators such as the Hamiltonian constraints it turns out to be extremely diffcult to define coherent (or semiclassical) states. That is, states labelled by points in the classical phase space with respect to which the operator assumes an expection value which reproduces the value of the corresponding classical function at that point in phase space and with respect to which the (relative) fluctuations are small. The reason for why this happens is that the existing coherent states for LQG [] are defined over a finite collection of finite graphs and these suppress very e ectively the fluctuations of those degrees of freedom that are labelled by the given graph. However, the Hamiltonian constraints add degrees of freedom to the state on which they act and the fluctuations of those are therefore no longer suppressed. Indeed, the semiclassical behaviour of the Hamiltonian constraints with respect to these coherent states is rather bad.

bf cite: "T. Thiemann. Quantum Spin Dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of gauge field theories. Class. Quant. Grav. 18 (2001) 3293-3338. [hep-th/0005232] T. Thiemann. Gauge Field Theory Coherent States (GCS): I. General properties. Class. Quant. Grav. 18 (2001), 2025-2064. [hep-th/0005233]"

Hence we see that the problem of investigating the classical limit of LQG and to verify the quantum algebra of constraints are very much interlinked:

1. Spatial diffeomorphism invariance enforces a weakly discontinuous representation of spatial diffeomorphisms.

2. Anomaly freeness in the presence of only finite diffeomorphisms enforces graph changing Hamiltonian constraints.

3. Graph changing Hamiltonians seem to prohibit appropriate semiclassical states.

7.2 Some of the Schemes

(i) scattering amplitudes in a background independent field theory:

Rovelli *et al* are devising the a formulation for calculate scattering amplitudes in a background independent field theory of gravity plus matter theory. They already have a tentative expression representing the quantum state corresponding to Minkowskian spacetime in terms of a spin foam model. If this scheme works out they will explicitly demonstrate that LQG has the correct semi-classical limit.

(ii) Percolation model from microcausal spin foam models:

It would appear that, as is no notion of background time in GR there can be no background causal structure at the fundamental level. However, one might take the view that the notion of causality is actually more primitive than that of time, more fundamental, being already present in the notion of ordering between events. Smolin and Markopoulou have introduced microcausality into spin foam models - it has resulted into a *directed percolation model*, (however involving amplitude rather than probabilities as have been considered in condensed matter applications). If the "percolation" leads to critical behaviour, there are long established results guarantee that the leading order in the effective action is the Einstein-Hilbert action of classical GR.

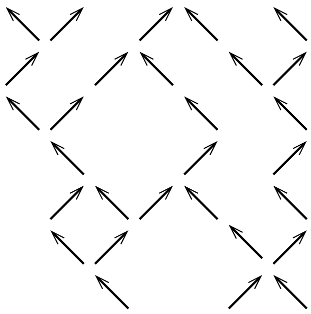


Figure 7.2: "perculation" leads to critical behaviour.

[258]

(iii) Lorentzian three-dimensional dynamical triangulations:

The model of Lorentzian three-dimensional dynamical triangulations provides a non-perturbative definition of three-dimensional quantum gravity. The theory has two phases:

In this approach, (diff-equivalent classes of) smooth metric configurations are approximated by spacetime rtiangulations where 1-simplexes have the same fixed proper length ℓ. The smaller the length scale the better the approximation; therefore, the proper length represents the regulator in the theory to be removed in a certain limit.

(iv) Causal Dynamical Triangulations:

Causal Dynamical Triangulations in four dimensions provide a background-independent definition of the sum over geometries in nonperturbative quantum gravity, with a positive cosmological constant. We present evidence that a macroscopic four-dimensional world emerges from this theory dynamically. [266]

(v) non-graph-changing Master constraint:

A spatially diff invariant operator can not be defined on $\mathcal{H}_{Diff}$ if it is graph changing. can give an ad hoc *non-graph-changing* Master constraint. Can check its semiclassical properties with tools already developed due to recent progress [206]

(vi) Lattice Quantum Gravity and Supercomputers:

non-graph-changing Master constraint arbitrary but fixed graph and study how the theory changes under coarsening of the graph with background independent renormalization schemes, [123], [124].

(vii) quantum geometry in quantum information terms:

Loop Quantum Gravity defines the quantum states of space geometry as spin networks and describes their evolution in time. We reformulate spin networks in terms of harmonic oscillators and show how the holographic degrees of freedom of the theory are described as matrix models. This allow us to make a link with non-commutative geometry and to look at the issue of the semi-classical limit of LQG from a new angle. This work is thought as part of a bigger project of describing quantum geometry in quantum information terms. [256], [257].

(vii) Evolution in Quantum Causal Histories:

[371] "This does mean that the structure of a QCH encompasses a reasonable notion of a quantum field theory, and hence is capable of describing matter degrees of freedom. It also indicates how quantum fields on curved spacetime might be obtained as a limit of some quantum gravity model based on QCH's".

(viii) Noiseless subsystems: Noiseless subsystems are useful for describing the long-term behaviour of the system because they are conserved.

If we divide the quantum gravitational field into subsystems, those properties that are **conserved** under interactions between subsystems will characterize the low energy-limit of spacetime geometry. If we have ... behaving as modes propagating through spacetime then we have spacetime.

(vii) Reduced Phase Space Quantization of LQG: In [224] they perform a canonical, reduced phase space quatization of General Relativity by Loop Quantum Gravity methods.

The kinematic Hilbert space becomes a physical Hilbert space.

We no longer have to deal with the constraints and so no anomalies cannot arise which cast doubt on the semiclassical limit. The physical Hilbert space doesn't need to be derived by complicated methods such as group averaging techniques.

7.3 Semi-Classical Limit

Weaves states we then average over the position of the loops same yield a spacially invariant state. The idea is that in the mean field limit the state approximates the described by the metric used in the construction of the weave state in the firstplace. Such states are by construction, eigenstates of the "momentum operator" $\tilde{E}_i^a(x)$.

In ordinary quantum mechanics

$$\psi(x) = [2\pi(\Delta x)^2]^{-1/4} \exp\left[-\frac{(x- < x >)^2}{4(\Delta x)^2} + \frac{i < p > x}{\hbar}\right] \tag{7.2}$$

7.4 Emergence of Spacetime from Dynamics of Background Independent Quantum Theories

a weak-coupling phase with quantum fluctuations around a "semiclassical" background geometry which is generated dynamically despite the fact that the formulation is explicitly background-independent, and a strong-coupling phase where "classical" space disintegrates into a foam of baby universes. [265]

A Simple Model

In a course-grained low energy approximation

$$\mathcal{H} = J_\perp \sum_{i=1}^{N} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+). \tag{7.2}$$

Is an abstraction of the relationships that exist between physical objects that make up the world. We specify how the spins are in relation with one another without having to talk about these spins sitting in physical space. In this sense it is a background independent theory. The relations are a connectivity between these spins. However, this connectivity does not necessarily have to be represented by points in a pre-existing physical space.

$$\{f_i^\dagger, f_j\} = \delta_{ij}. \tag{7.2}$$

$$\mathcal{H} = \sum_{i=1}^{N} \epsilon(k) f_k^\dagger f_k \tag{7.2}$$

where the energy is given by

$$\epsilon(k) = 4\pi J_\perp \cos\frac{2\pi}{N}k. \tag{7.2}$$

linear dispersion relation

$$\Delta\epsilon = 4\pi J_\perp \frac{2\pi}{N}\Delta k \equiv v_F \Delta k. \tag{7.2}$$

7.5 Semi-Classical Limit of Completely Constrained Systems

Generalized Uncertainty Relation

$$\Delta q\ \Delta p = \hbar/2. \tag{7.2}$$

The generalized uncertainty relation for two arbitrary self adjoint operators $\hat{a}$ and $\hat{b}$:

$$\Delta a \Delta b \geq \frac{\omega([\hat{a},\hat{b}])}{2i} \tag{7.2}$$

where by definition

$$(\Delta a)^2 :=\ \omega((\hat{a}-\omega(\hat{a}))^2), \quad (\Delta b)^2 :=\ \omega((\hat{b}-\omega(\hat{b}))^2). \tag{7.2}$$

Proof: Consider the zero mean operators $A := \hat{a}-\omega(\hat{a})$, $B := \hat{b}-\omega(\hat{b})$ then

$$(\Delta a)^2 = \omega(A^2), \qquad (\Delta b)^2 = \omega(B^2). \tag{7.2}$$

$(\Delta a)^2(\Delta b)^2 = \omega(A^2)\omega(B^2)$. From the Schwartz inequality we have:

$$\omega(A^2)\omega(B^2)\ \geq |\omega(AB)|^2 =\ \omega(AB)\omega(BA). \tag{7.2}$$

We have

$$\begin{aligned}\omega(AB)\omega(AB)^* &= Re(\omega(AB))^2 + Im(\omega(AB))^2 \\ &\geq Im(\omega(AB))^2 \\ &= ((\omega(AB) +\ \overline{\omega(AB)})/2i)^2.\end{aligned} \tag{7.1}$$

and also

$$\begin{aligned}\omega(AB) &= \omega\big((\hat{a}-\omega(\hat{a})(\hat{b}-\omega(\hat{b})\big) \\ &= \omega(\hat{a}\hat{b})-\omega(\hat{a})\omega(\hat{b}). \end{aligned} \tag{7.1}$$

Swapping $\hat{a}$ and $\hat{b}$ around gives $\omega(BA) = \omega\big((\hat{b}-\omega(\hat{b})(\hat{a}-\omega(\hat{a})\big) = \omega(\hat{b}\hat{a})-\omega(\hat{a})\omega(\hat{b})$. Putting all this together:

$$\begin{aligned}(\Delta a)^2(\Delta b)^2 \geq &= ((\omega(AB)-\omega(BA))/2i)^2 \\ &= ((\omega(\hat{a}\hat{b})-\omega(\hat{b}\hat{a}))/2i)^2 \\ &= \left(\frac{\omega([\hat{a},\hat{b}])}{2i}\right)^2. \end{aligned} \tag{7.0}$$

Coherent states for the subalgebra of observables $\mathcal{S}$

$$\Delta a\ \Delta b = \hbar\omega([\hat{a},\hat{b}])/2i. \tag{7.0}$$

there exists a certain non-self adjoint operator

$$\hat{z} = \hat{A}+i\lambda\hat{B} \tag{7.0}$$

with $\hat{A},\hat{B}\in\mathcal{S}$ and a certain 'squeezing' paramter λ.

$$\omega(z^*z)\geq 0, \qquad \hat{z}=\hat{A}+i\lambda\hat{B}, \tag{7.0}$$

for all $\lambda>0$, that is

$$\begin{aligned}\omega\big((\hat{A}+i\lambda\hat{B})^*(\hat{A}+i\lambda\hat{B})\big) &= \omega(\hat{A}^2)+\omega(\lambda^2\hat{B}^2)+\omega(i\lambda\hat{A}\hat{B}-i\lambda\hat{B}\hat{A}) \\ &= (\Delta a)^2+\lambda^2(\Delta b)^2+\lambda\omega(i[\hat{a},\hat{b}]) \\ &\geq 0 \end{aligned} \tag{7.-1}$$

or

$$(\Delta a)^2-\lambda\hbar\omega(i[\hat{a},\hat{b}])+\lambda^2(\Delta b)^2\geq 0. \tag{7.-1}$$

rewritten

$$\left(\lambda - \frac{\hbar\omega(i[\hat{a},\hat{b}])}{2(\Delta b)^2}\right)^2 + \frac{(\Delta a)^2}{(\Delta b)^2} - \frac{\hbar^2\omega(i[\hat{a},\hat{b}])^2}{4(\Delta b)^4} \geq 0. \tag{7.-1}$$

implying

$$(\Delta a)^2(\Delta b)^2 \geq \hbar^2(\omega([\hat{a},\hat{b}])/2i)^2 \tag{7.-1}$$

so for this ineqlality to hold for all λ implies the Heisenberg uncertainty relation (7.5), also that equality holds only if for some positive λ

$$\left(\frac{\hat{a}+i\lambda\hat{b}}{\sqrt{2\lambda\hbar}}\right)|\psi_0>=0. \tag{7.-1}$$

For $\{q,p\} = i\hbar$

$$[q,p]/2i = \hbar/2. \tag{7.-1}$$

the state

$$(\hat{q}+i\lambda\hat{p})|\psi_0>=0. \tag{7.-1}$$

for $\lambda = 1$ $|\psi>_0$ is the ground state (for other values the state is what's called a squeezed state).

This implies that the minimal uncertainty relation is saturated for the pair of elements $(\hat{a},\hat{b})$, i.e.,

$$\Psi_m([\hat{a}-\Psi_m(\hat{a})]^2) = \Psi_m([\hat{b}-\Psi_m(\hat{b})]^2) = \frac{1}{2}|\Psi_m([\hat{a},\hat{b}])|. \tag{7.-1}$$

□

An important issue is whether the constraint operators have the correct semi-classical limit. This has to be done by using the kinematic semiclassical states in $\mathcal{H}_{Kin}$.

The physical Hilbert space must contain enough semiclassical states to guaratee that the quantum theory one obtains returns the correct classical theory when $\hbar \to 0$.

The semiclassical states in a Hilbert space should have the following properties

Given a class of observables $\mathcal{S}$ which comprises a subalgerba in the space $\mathcal{L}(\mathcal{H})$ of linear operators on the Hilbert space, a family of (pure) states $\{\omega_m\}_{m\in\mathcal{M}}$ are said to be semiclassical with respect to $\mathcal{S}$ if and only if

(1) The observables in $\mathcal{S}$ should have the correct semiclassical limit on semiclassical states and the fluctuations should be small, i.e.,

$$\begin{aligned}\lim_{\hbar\to 0}\left|\frac{\omega_m(\hat{a})-a(m)}{a(m)}\right| &= 0,\\ \lim_{\hbar\to 0}\left|\frac{\omega_m(\hat{a}^2)-\omega_m(\hat{a})^2}{\omega_m(\hat{a})^2}\right| &= 0,\end{aligned}$$

for all $\hat{a}\in\mathcal{S}$.

• Coherent states for QGR, based on the general complexifier method, with built-in semiclassical properties.

• Polymer-like states for Maxwell theory and linearized gravity constructed by Varadarajan.

By making explicit use of the Minkowski background metric, he was able to construct an image of the usual Fock states on a distributional extension of the type of background independent Hilbert space $\mathcal{H}_0$ on which quantum general relativity currently is based. Varadarajan's states are complexifier coherent states.

7.6 Coherent States

7.6.1 Reminder of Coherent States

It is well known that coherent states provide a useful bridge between a classical theory and the corresponding quantum theory. Consider quantum mechanics of a particle on the real line, without specifying the potential. The basic observables are configuration and momentum, $\hat{X}$, $\hat{P}$, with

$$[\hat{X},\hat{P}] = i\hbar I$$

From these, we can build an annihilation operator

$$\hat{a} = \sqrt{\frac{\omega}{2\hbar}}\hat{X} + i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}$$

whose classical counterpart we denote by z:

$$z = \sqrt{\frac{\omega}{2\hbar}} X_0 + i \frac{1}{\sqrt{2\hbar\omega}} P_0.$$

Here (X_0, P_0) is a point in the classical phase space.

Let $(\hat{a}^\dagger)$ be the creation operator. If we set $N := a^\dagger a$ (the number operator), then

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -1. \tag{7.-3}$$

Let H be a Fock space generated by a and $a^\dagger$. The actions of a and $a^\dagger$ on H are given by

$$\begin{array}{rcl} a|n> & = & \sqrt{n}|n-1> \\ a^\dagger|n> & = & \sqrt{n+1}|n+1> \\ N|n> & = & n|n>. \end{array} \tag{7.-4}$$

$$|n> = \frac{(a^\dagger)^n}{\sqrt{n!}}|0>. \tag{7.-4}$$

where the state $|0>$ is defined by $\hat{a}|0> = 0$. These states satisfy the orthogonality and completeness conditions

$$< m|n> = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n>< n| = I. \tag{7.-4}$$

the following three conditions are equivalent

$$(i) \quad a|z> = z|z> \text{ and } < z|z> = 1 \tag{7.-3}$$

$$(ii) \quad |z> = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n> = e^{-|z|^2/2} e^{za^\dagger}|0> \tag{7.-2}$$

$$(iii) \quad |z> = e^{za^\dagger - \bar{z}a}|0>. \tag{7.-1}$$

The equivalence of (ii) and (iii) follows from the *Baker-Campbell-Hausdorff formula*

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B \tag{7.-1}$$

which holds whenever $[A,[A,B]] = [B,[A,B]] = 0$.

We prove that (ii) implies (i). We first consider

$$\hat{a}(z) = e^{-z\hat{a}^\dagger}\hat{a}e^{z\hat{a}^\dagger} \tag{7.-1}$$

which obeys

$$\begin{aligned} \frac{\partial}{\partial z}\hat{a}(z) &= e^{-z\hat{a}^\dagger}(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})e^{z\hat{a}^\dagger} \\ &= e^{-z\hat{a}^\dagger}e^{z\hat{a}^\dagger} = 1. \end{aligned} \tag{7.-1}$$

Hence

$$\hat{a}(z) = \hat{a}(0) + z = \hat{a} + z. \tag{7.-1}$$

Now from (ii) we have

$$|z >= e^{-|z|^2/2}e^{z\hat{a}^\dagger}|0>$$

then

$$\begin{aligned} \hat{a}|z> &= e^{z\hat{a}^\dagger}e^{-z\hat{a}^\dagger}\hat{a}\left(e^{-|z|^2/2}e^{z\hat{a}^\dagger}|0>\right) \\ &= e^{-|z|^2/2}e^{z\hat{a}^\dagger}\left(e^{-z\hat{a}^\dagger}\hat{a}e^{z\hat{a}^\dagger}\right)|0> \\ &= e^{-|z|^2/2}e^{z\hat{a}^\dagger}(\hat{a}+z)|0> \\ &= z|z>. \end{aligned} \tag{7.-3}$$

We give a proof that (i) implies (ii). Assuminmg the "number" states $|n>$ form a complete set we can write

$$|z >= \sum_{n=0}^{\infty} c_n|n>. \tag{7.-3}$$

From $\hat{a}|z >= z|z>$ we have

$$\sum_{n=0}^{\infty} c_n\hat{a}|n >= z\sum_{n=0}^{\infty} c_n|n>$$

or

$$\sum_{n=0}^{\infty} c_n \sqrt{n} |n-1> = z \sum_{n=0}^{\infty} c_n |n>$$

which can then be rewritten

$$\sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n> = z \sum_{n=0}^{\infty} c_n |n> .$$

So that

$$c_{n+1} = \frac{c_n}{\sqrt{n+1}} z.$$

If we set $c_0 = \mathcal{N}$ then

$$\begin{aligned}
c_1 &= c_0 z = \mathcal{N} z \\
c_2 &= \frac{1}{\sqrt{2}} c_1 z = \mathcal{N} \frac{1}{\sqrt{2}} z^2 \\
c_3 &= \frac{1}{\sqrt{3}} c_2 z = \mathcal{N} \frac{1}{\sqrt{3!}} z^3
\end{aligned}$$

etc. Then (7.6.1) becomes

$$|z> = \mathcal{N} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n> .$$

The condition $< z|z >= 1$ implies

$$\begin{aligned}
1 &= < z|z > \\
&= |\mathcal{N}|^2 \sum_{n,n'=0}^{\infty} \frac{z^{n+n'}}{\sqrt{n!}\sqrt{n'!}} < n|n' > \\
&= |\mathcal{N}|^2 \sum_{n,n'=0}^{\infty} \frac{z^{n+n'}}{\sqrt{n!}\sqrt{n'!}} \delta_{nn'} \\
&= |\mathcal{N}|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} = |\mathcal{N}|^2 e^{|z|^2}
\end{aligned}$$

so we have

$$|z>=^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n> .$$

See also worked exercises on how (i) implies (ii).

Overcompleteness

It can be shown that (see worked exercises)

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z><z^*| &= \sum_n \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0><0|\frac{\hat{a}^n}{\sqrt{n!}}\\ &= \sum_n |n><n|. \qquad (7.\text{-}10)\end{aligned}$$

Therefore the coherent states form a complete set.

Let us compute the inner-product of two coherent states:

$$\begin{aligned}<z_1|z_2> &= e^{-\frac{|z_1|^2+|z_2|^2}{2}}\sum_{n,n'=0}^{\infty}\frac{\overline{z}_1^n z_2^{n'}}{\sqrt{n!n'!}}<n|n'>\\ &= e^{-\frac{|z_1|^2+|z_2|^2}{2}}\sum_{n=0}^{\infty}\frac{(\overline{z}_1 z_2)^n}{n!}\\ &= e^{-\frac{|z_1|^2+|z_2|^2}{2}+\overline{z}_1 z_2} \qquad (7.\text{-}11)\end{aligned}$$

and so

$$\begin{aligned}|<z_1|z_2>|^2 &= e^{-\frac{|z_1|^2+|z_2|^2}{2}+\overline{z}_1 z_2}\times e^{-\frac{|z_1|^2+|z_2|^2}{2}+z_1\overline{z}_2}\\ &= e^{-\left(|z_1|^2+|z_2|^2-\overline{z}_1 z_2-z_1\overline{z}_2\right)}\\ &= e^{-(\overline{z}_1-\overline{z}_2)(z_1-z_2)}\\ &= e^{-|z_1-z_2|^2}. \qquad (7.\text{-}13)\end{aligned}$$

So coherent states are not orthogonal. Making the coherent states an overcomplete basis.

Configuration space and momentum space representation:

In configuration space and momentum space representation:

$$\begin{aligned}
\psi_z(x) &= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\omega}{2\hbar}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, && (7.\text{-}12) \\
\psi_z(p) &= \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\hbar}{2\omega} \frac{(p - P_0)^2}{\hbar^2} + \frac{i}{\hbar} p X_0 \right\}. && (7.\text{-}11)
\end{aligned}$$

substituting $t = \hbar/\omega$ this is rewritten

$$\begin{aligned}
\psi_z(x) &= \frac{1}{(\pi t)^{\frac{1}{4}}} \exp - \left\{ \frac{1}{2t}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, && (7.\text{-}10) \\
\psi_z(p) &= \left(\frac{t}{\pi}\right)^{\frac{1}{4}} \exp - \left\{ \frac{t}{2}(p - P_0)^2 + \frac{i}{\hbar} p X_0 \right\}. && (7.\text{-}9)
\end{aligned}$$

From these formulae we can read off the most important properties of coherent states: In both configuration and momentum representation the wavefunctions are Gaussian distributions, centered at

$$< \hat{X} >_{\Psi_z} = X_0, \quad \text{and} \quad < \hat{P} >_{\Psi_z} = P_0, \qquad (7.\text{-}9)$$

respectively. Furthermore we can see that the width of the distribution in the configuration representtion is inversely proportional to that on the momentum representation.

$$\begin{aligned}
(\Delta \hat{q}_i)^2 &\equiv \; < \Psi_\alpha | \hat{q}_i^2 | \Psi_\alpha > - [< \Psi_\alpha | \hat{q}_i | \Psi_\alpha >]^2 = \frac{1}{2}\ell_i^2, \\
(\Delta \hat{p}_i)^2 &\equiv \; < \Psi_\alpha | \hat{p}_i^2 | \Psi_\alpha > - [< \Psi_\alpha | \hat{p}_i | \Psi_\alpha >]^2 = \frac{1}{2}\hbar^2/\ell_i^2 && (7.\text{-}9)
\end{aligned}$$

$$< \Psi_\beta | : F(\hat{a}_i^\dagger, \hat{a}_i) : | \Psi_\alpha >= F(\overline{\beta}_i, \alpha_j) < \Psi_\beta | \Psi_\alpha > \qquad (7.\text{-}9)$$

We come to an important property. However, we must first derive the result

$$e^{t\Delta/2} \delta_y(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{1}{2t}(x-y)^2}. \qquad (7.\text{-}9)$$

To do this we use the Dirac delta-function representation

$$\delta_y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)}.$$

Then note

$$\begin{aligned}
e^{t\Delta/2}\delta_y(x) &= e^{t\Delta/2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip\cdot y}\exp\left\{\frac{t}{2}\frac{d^2}{dx^2}\right\}e^{ip\cdot x} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip\cdot y}\left[1-\frac{t}{2}p^2+\frac{1}{2!}\left(\frac{t}{2}\right)^2 p^4-\ldots\right]e^{ip\cdot x} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)}e^{-tp^2/2} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\frac{t}{2}\left\{p^2-\frac{2ip}{t}(x-y)\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\frac{t}{2}\left\{\left(p-\frac{i}{t}(x-y)\right)^2+\frac{1}{t^2}(x-y)^2\right\} \\
&= e^{-\frac{1}{2t}(x-y)^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tp^2/2} \\
&= \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(x-y)^2}. \qquad (7.\text{-}15)
\end{aligned}$$

Now the coherent state can be obtained as an *analytic continuation* of the heat kernel:

$$\begin{aligned}
\left[\frac{1}{\sqrt{2\pi t}}e^{\frac{1}{2t}(x-y)^2}\right]_{y\to z}(x) &\propto \exp\left\{-\frac{1}{2t}\left(x-X_0-\frac{it}{\hbar}P_0\right)^2\right\} \\
&= \exp\left\{-\frac{1}{2t}\left[(x-X_0)^2-2\frac{it}{\hbar}(x-X_0)P_0-\frac{t^2}{\hbar^2}P_0^2\right]\right\} \\
&\propto \exp\left\{-\left[\frac{1}{2t}(x-X_0)^2-\frac{i}{\hbar}xP_0\right]\right\}
\end{aligned}$$

that is,

$$\psi_z(x) \sim \left[e^{-t\Delta/2}\delta_y(x)\right]_{y\to z}(x). \qquad (7.\text{-}18)$$

These eigenstates have many interesting properties.

Overcompleteness

There is a resolution of unity

$$1_{\mathcal{H}} = \int_{\mathcal{M}} d\nu(m)\psi_m \ < \psi, \cdot > \tag{7.-18}$$

for some measure ν on $\mathcal{M}$.

Saturation of the Heisenberg uncertainty relation

For self-adjoint operators $\hat{x} := (\hat{g}+\hat{g}^\dagger)/2$, $\hat{y} := (\hat{g}-\hat{g}^\dagger)/2i$ uncertaintly relation is saturated

$$< (\hat{x}- < \hat{x} >_m)^2 >_m = < (\hat{y}- < \hat{y} >_m)^2 >_m = \frac{1}{2}| < [\hat{x}, \hat{y}] >_m | \tag{7.-18}$$

Annihilation operator property

Peakedness in phase space

Ehrenfest theorems

7.6.2 Quantum Gravity Coherent states

One introduces coherent states for the harmonic oscillator as eigenvalues of the annihilation operator in terms of superposition of energy eigenstates. Here one had a preferred Hamiltonian (unlike quantum gravity) and the problem is straightforward because of the linearity of the system.

The construction of coherent states for full nonlinear, non-Abelean Quantum General Relativity with all the desired properties like overcompleteness, saturation of the Heisenberg uncertainty relation, peakedness in phase space (thus both connection and electric flux are well approximated), construction of annihilation and creation operators and corresponding Ehrenfest theorems.

They are non-normalizable.

Cut-off Coherent states

The coherent state ψ_m with respect to a finite graph as a graph dependent coherent state in $\mathcal{H}_{Kin}$. These are then normalizable, graph dependent states produced by the complexifier method as well.

For a graph γ containg edges $e_1, \ldots, e_N$ the coherent state is constructed by taking the tensor product for all edges,

$$\psi^t_{g_1,\ldots,g_N}(h_1, \ldots, h_N) = \prod_{n=1}^{N} \psi^t_{g_n}(h_n). \tag{7.-18}$$

However, as cut-off coherent states are defined for each graph separately, for graph changing operators, the new graph generated has degrees of freedom upon which the coherent state does not depend and so whose fluctuations are not supressed. This makes it difficult to investigate the semmiclassical behaviour of graph changing operators like the Hamiltonian constraint.

Coherent states for the subalgebra of observables $\mathcal{S}$

Given a Hilbert space $\mathcal{H}$ for a dynamical system with constrints and a subalgebra of observables $\mathcal{S}$ in the space $\mathcal{L}(\mathcal{H})$ of linear operators on $\mathcal{H}$, the semiclassical states with respect to $\mathcal{S}$ are defined in ....??

7.6.3 The Complexifier Approach

Given a phase space $\mathcal{M} = T^*\mathcal{Q}$ for some dynamical system with configuration coordinates q and momentum coordinates p, a complexifier $C_q(p) = C(q,p)$, is a positive smooth function on $\mathcal{M}$, such that

(1) $C/\hbar$ is dimensionless;

(2) $\lim_{\|p\|\to\infty} \frac{|C(m)|}{\|p\|} = \infty$ for some suitable norm on the space of momentum;

(3) Certain complex coordinates $(z(m), \overline{z}(m))$ of $\mathcal{M}$ (given $z(m)$ and $\overline{z}(m)$ we can invert them to find real coordinates m for $\mathcal{M}$) can be constructed from C.

As a simple example, in the case of one-dimensional harmonic ascillator with Hamiltonian $H = \frac{1}{2}\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\right)$, with complexifier $C = p^2/(2m\omega)$, we illustrate the construction and obtain the usual harmonic ascillator coherent state up to phase factor.

Given a well-defined complexifier C on phase space Γ, we construct coherent states associated with C:

(i) Complex polarization

$$z(m) := \sum_{n=0}^{\infty} \frac{i^n}{n!} \{q, C\}_{(n)}(m). \tag{7.-18}$$

For the simple harmonic oscialltor this is simply

$$z(m) = q + \frac{i}{m\omega} p \tag{7.-18}$$

which together with the complex conjugate $\overline{z}(m) = q - ip/m\omega$ form complex coordinates $(z(m), \overline{z}(m))$ for the phase space $\Gamma = \mathbb{R}^2$.

(ii) Defining the annihilation operator

Given a suitable Hilbert space $\mathcal{H} = L^2(\mathcal{Q}, d\mu)$, we assume C can be defined as a positive self-adjoint operator $\hat{C}$ on $\mathcal{H}$. Then a corresponding operator $\hat{z}$ can be defined by replacing the Poisson brackets in (7.6.3) by commutators,

$$\hat{z} := \sum_{n=0}^{\infty} \frac{i^n}{n!} \frac{1}{(i\hbar)^n} [\hat{q}, \hat{C}]_{(n)}(m), \tag{7.-18}$$

This can also be written as $e^{-\hat{C}/\hbar} \hat{q} e^{\hat{C}/\hbar}$ as can be proven by employing the quantity

$$e^{-\alpha \hat{C}/\hbar} \hat{q} e^{\alpha \hat{C}/\hbar} = \sum_{n=0}^{\infty} \hat{X}_{(n)} \frac{\alpha^n}{n!}. \tag{7.-18}$$

It is obvious that $\hat{X}_{(0)} = \hat{q}$, as can be seen from setting $\alpha = 0$. In order to determine the other terms in the expansion, we will first prove by induction that

$$\frac{d^k}{d\alpha^k} (e^{-\alpha \hat{C}/\hbar} \hat{q} e^{\alpha \hat{C}/\hbar}) = \frac{1}{\hbar^k} e^{-\alpha \hat{C}/\hbar} [\hat{q}, \hat{C}]_{(k)} e^{\alpha \hat{C}/\hbar}. \tag{7.-18}$$

holds for all $k \geq 1$. First assume (7.6.3) for fixed k and then differentiate both side with respect to α,

$$
\begin{aligned}
\frac{d^{k+1}}{d\alpha^{k+1}}(e^{-\alpha\hat{C}/\hbar}\hat{q}e^{\alpha\hat{C}/\hbar}) &= \frac{d}{d\alpha}\left(\frac{1}{\hbar^k}e^{-\alpha\hat{C}/\hbar}[\hat{q},\hat{C}]_{(k)}e^{\alpha\hat{C}/\hbar}\right) \\
&= -\frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}\hat{C}[\hat{q},\hat{C}]_{(k)}e^{\alpha\hat{C}/\hbar} + \frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}\hat{C}[\hat{q},\hat{C}]_{(k)}\hat{C}e^{\alpha\hat{C}/\hbar} \\
&= \frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}([\hat{q},\hat{C}]_{(k)}\hat{C} - \hat{C}[\hat{q},\hat{C}]_{(k)})e^{\alpha\hat{C}/\hbar} \\
&= \frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}[\hat{q},\hat{C}]_{(k+1)}e^{\alpha\hat{C}/\hbar}. \qquad (7.\text{-}20)
\end{aligned}
$$

This shows that if the relation holds for k then it also holds for $k+1$. Obviously we have

$$
\frac{d}{d\alpha}(e^{-\alpha\hat{C}/\hbar}\hat{q}e^{\alpha\hat{C}/\hbar}) = \frac{1}{\hbar}e^{-\alpha\hat{C}/\hbar}[\hat{q},\hat{C}]e^{\alpha\hat{C}/\hbar}
$$

so the relation holds for $k=1$, which completes the proof that (7.6.3) holds for all $k \geq 1$. Now we differentiate both sides of (7.6.3) k times with respect to α and set $\alpha = 0$, and arrive at

$$
\hat{X}_{(k)} = \frac{1}{\hbar^k}[\hat{q},\hat{C}]_{(k)},
$$

Substituting this into (7.6.3) and setting $\alpha = 1$ gives

$$
\hat{z} = \sum_{n=0}^{\infty} \frac{i^n}{n!}\frac{1}{(i\hbar)^n}[\hat{q},\hat{C}]_{(n)}(m) = e^{-\hat{C}/\hbar}\hat{q}e^{\hat{C}/\hbar}. \qquad (7.\text{-}20)
$$

For the simple harmonic oscialltor this is just

$$
\hat{z} = \hat{q} + \frac{1}{m\omega\hbar}\hat{p}. \qquad (7.\text{-}20)
$$

which is the annihilation operator. $\hat{z}$ in (7.6.3) will be called the annihilation operator.

(iii) Constructing coherent states

Now, let $\delta_{q'}(q)$ be the delta function (distribution) on C with respect to the measure $d\mu$, i.e. $\int \delta_{q'}(q)f(q)d\mu = f(q')$.

One may analytically extend the variable q' in $e^{-\hat{C}/\hbar}\delta_{q'}(q)$ to complex values $z(m)$

$$
\psi'_m(q) := [e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)}, \qquad (7.\text{-}20)
$$

such that one has

$$
\begin{aligned}
\hat{z}\,\psi'_m(q) &:= e^{-\hat{C}/\hbar}\hat{q}e^{\hat{C}/\hbar}\,[e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \\
&= [e^{-\hat{C}/\hbar}\hat{q}\delta_{q'}(q)]_{q'\to z(m)} \\
&= [q'e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \\
&= z(m)\,[e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \\
&= z(m)\psi'_m(q).
\end{aligned} \tag{7.-23}
$$

We define our coherent states $\psi_m(q)$ as the normalization of $\psi'_m(q)$.

We illustrate how to form an eigenstate of the annihilation operator of the simple harmonic oscillator. Let $\delta_{q'}(q)$ be the delta function (distribution) on $\Gamma = \mathbb{R}^2$. Consider

$$
\psi'_m(q) := [e^{-\hat{p}^2/2m\omega\hbar}\delta_{q'}(q)]_{q'\to\hat{q}+i\hat{p}} \tag{7.-23}
$$

$$
\begin{aligned}
\psi'_m(q) &:= [e^{-\hat{p}^2/2m\omega\hbar}\sum_n e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= [e^{-\hat{H}(\hat{q})/\hbar\omega}e^{\hat{q}^2}\sum_n e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= [e^{q^2}\sum_n e^{-E_n/\hbar\omega}e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= [e^{q^2}\sum_n e^{-(n+\frac{1}{2})}e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= e^{q^2}\sum_n e^{-(n+\frac{1}{2})}e_n(q+i\hat{p})e^*_n(q)
\end{aligned} \tag{7.-26}
$$

Constructing kinematic coherent states for LQG

The complexifier appraoch can be used to construct kinematic coherent states in loop quantum gravity. Given a suitable complexifier C, for each path $e \subset \Sigma$ one can define

$$
A^C(e) := \sum_{n=0}^{\infty}\frac{i^n}{n!}\{A(e), C\}_{(n)}(m). \tag{7.-26}
$$

where $A(e) \in SU(2)$ is assigned to e.

the $\delta-$distribution on the quantum configuration space $\overline{\mathcal{A}}$ can be formally expressed as

$$\delta_{A'}(A) = \sum_s T_s(A')\overline{T_s(A)}$$

Thus by () we obtain the coherent states

$$\psi'_{A^C}(A) = \sum_s e^{-\tau\lambda_s} T_s(A^C)\overline{T_s(A)}. \tag{7.-26}$$

Since this is an uncountable sum of an infinite number of terms, the norm of $\psi'_{A^C}(A)$ will be divergent. So $\psi'_{A^C}(A)$ does not belong to $\mathcal{H}_{Kin}$ but rather to the dual of a dense subset of $\mathcal{H}_{Kin}$.

A candidate complexifier C

A candidate complexifier C is considered in [??] such that the corresponding operator acts on on cylindrical functions f_γ by

$$(\frac{\hat{C}}{\hbar})f_\gamma = \frac{1}{2}(\sum_{e\in E(\gamma)} l(e)\hat{J}_e^2)f_\gamma, \tag{7.-26}$$

where $\hat{J}_e^2$ is the Casimir operator

Complexification of a Lie Group

The complexification of a Lie group $\mathbf{g}$ is denoted $\mathbf{g}_{\mathbb{C}}$. Recall the defining properties of a Lie algebra: antisymmetric $[X,Y] = -[Y,X]$, bilinearity $[\alpha X, Y] = \alpha[X,Y] = [X, \alpha Y]$ for any real number α, and satisfies the Jacobi identity $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] \equiv 0$. $X_1 + iX_2$ where $X_1, X_2 \in \mathbf{g}$

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1,Y_1] - [X_2,Y_2]) + i([X_1,Y_2] + [X_2,Y_1]). \tag{7.-26}$$

It is clear that (B) is *real* bilnear and skew-symmetric. If we prove that it is *complex* linear in the first factor, it will be complex linear in the second because of the skew-symmetry. As we already know it is real linear in the first factor, it suffices to show that it is *imaginary* linear. This is not difficult to prove, all we need to do is verify that

$$[i(X_1 + iX_2), Y_1 + iY_2] = i[X_1 + iX_2, Y_1 + iY_2] \tag{7.-26}$$

is true. This is easily done by expanding each side and seeing they are indeed equal.

It remains to check the Jacobi identity. First consider Y and Z to be in $\mathbf{g}$ but take X to be in $\mathbf{g}_{\mathbb{C}}$. Now, $X = X_1 + iX_2$ is linear in the Jacobi identity and the Jacobi identity holds separately for X_1 and X_2,

$$[[X_1, Y], Z] + [[Z, X_1], Y] + [[Y, Z], X_1] + i([[X_2, Y], Z] + [[Z, X_2], Y] + [[Y, Z], X_2]) \equiv 0, \tag{7.-26}$$

and so the Jacobi identity holds for $X \in \mathbf{g}_{\mathbb{C}}$ and $Y, Z \in \mathbf{g}$. Similarly for Y and Z. Therefore we have shown that the elements of the complexification $\mathbf{g}_{\mathbb{C}}$ satisfy the Jacobi identity.

The cut-off state of the corresponding coherent state,

$$\psi_{A^C,\gamma}(A) = \psi'_{A^C,\gamma}(A) / \|\psi'_{A^C,\gamma}(A)\|, \tag{7.-26}$$

where

$$\psi'_{A^C,\gamma}(A) := \sum_{s,\gamma(s)=\gamma} e^{-\frac{1}{2}\sum_{e\in E(\gamma(s))} l(e) j_e(j_e+1)} T_s(A^C)\overline{T_s(A)}. \tag{7.-26}$$

There are various choices for the complexier coherent states on a compact Lie group G are given by the different possible complexifier $\hat{C}$. We have seen one possible complexiefier: the negative Laplacian on G.

$$\psi^t_g(h) = \sum_\pi e^{l_\pi t/2} d_\pi tr\pi(gh^{-1}) \tag{7.-26}$$

where we are summing over all irreducible finite-diemnsional representations π of G.

7.6.4 $U(1)$ Coherent states

Since all irreducible representations of $U(1)$ are know to be e^{-in} with $n \in \mathbb{Z}$, $d_\pi = 1$ and $l_\pi = n^2$ (7.6.3) becomes (for $n \to -n$).

$$\psi^t_z(\phi) = \sum_{n\in\mathbb{Z}} e^{-n^2 t/2} e^{-in(z-\phi)} \tag{7.-26}$$

where $g = e^{iz}$ and $h = e^{i\phi}$. By the Poisson resummation theorem (see worked exercises), for a function

$$\psi(y+nt)=\sum_{n=-\infty}^{\infty} f(y+nt)$$

we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f(y+nt) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{iny}\int_{-\infty}^{\infty} f(\tau)e^{-in\tau} \\
&= \frac{2\pi}{t}\sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{t}\right) \cdot \exp(2\pi i y n/t), \qquad (7.\text{-}26)
\end{aligned}$$

where $\tilde{f}(k)=\int_{\mathbb{R}}(dx/2\pi)e^{-ikx}f(x)$. Now

$$\begin{aligned}
\exp(n^2t/2+yn) &= \exp(\frac{1}{t}(nt+y)^2/2)\exp(-y^2/2t) \\
&= \exp(-y^2/2t)f(nt+y) \qquad (7.\text{-}26)
\end{aligned}$$

where $f(x)=\exp(x^2/2t)$. In our case $y=-i(z-\phi)$.

$$\begin{aligned}
\sum_{-\infty}^{\infty}\exp(ny)\exp(n^2t/2) &= \exp(-y^2/2t)\sum_{-\infty}^{\infty}\exp((y+nt)^2/2t) \\
&= \exp(-y^2/2t)\,\frac{2\pi}{t}\sum_{n=-\infty}^{\infty}\tilde{f}(\frac{2\pi n}{t}) \cdot \exp(2\pi i y n/t) \; (7.\text{-}26)
\end{aligned}$$

As

$$\tilde{f}(z)=\frac{1}{2\pi}\int f(x)e^{izx}dx=\frac{1}{2\pi}\int e^{-x^2/2t}e^{izx}dx=\sqrt{\frac{t}{2\pi}}e^{-z^2t/2}$$

$$\tilde{f}(\frac{2\pi n}{t})=\sqrt{\frac{t}{2\pi}}e^{-(2\pi n)^2t/2} \qquad (7.\text{-}26)$$

The Poisson summation formula (7.-26) converts a term in the exponention of t to an exponation where $1/t$ appears instead of t. This allows us to investigate the classical limit, $t\rightarrow 0$.

$$\begin{aligned}
\psi_z^t(\phi) &= e^{-(z-\phi)^2/2t}\,\frac{2\pi}{t}\sum_{n=-\infty}^{\infty}\sqrt{\frac{t}{2\pi}}e^{-(2\pi n)^2/2t}\cdot e^{-4\pi n(z-\phi)/2t} \\
&= \sqrt{\frac{2\pi}{t}}\sum_{n=-\infty}^{\infty} e^{-\frac{(z-\phi-2\pi n)^2}{2t}}. \qquad (7.\text{-}26)
\end{aligned}$$

inner product

$$\begin{aligned}
<\psi_g^t|\psi_{g'}^t> &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{-m^2t/2}e^{-n^2t/2}\,e^{im\overline{z}-inz'}\int\frac{d\phi}{2\pi}e^{i(-m+n)\phi} \\
&= \sum_{n=-\infty}^{\infty} e^{-n^2t}e^{in(\overline{z}-z')} \\
&= \sqrt{\frac{\pi}{t}}\sum_{n=-\infty}^{\infty} e^{-(\overline{z}-z'-2\pi n)^2/t}. \qquad (7.\text{-}27)
\end{aligned}$$

where again we have used the Poisson summation formula.

7.6.5 $SU(2)$ Coherent states

Coherent states are cylindrical states, obtained as a product of cylindrical over edges of the graph. In contrast to weave states, the functions of the edges are not eigenstates of the geometry, but are chosen to have good classical behaviour for both configuration and momentum degrees of freedom. Cylindrical functions on the edges are chosen as coherent states on $SU(2)$.

the coherent states of the simple harmonic oscillator coherent states can be obtained as analytic continuation of the heat kernel on $\mathbb{R}^n$:

$$\psi_z^t(x) = e^{-t\Delta}\delta_{x'}(x)\Big|_{x'\to z} \quad x\in, z\in\mathbf{C}, \qquad (7.\text{-}27)$$

the Laplacian Δ playing the role of a *complexifier*.

It was shown by Hall [243] that coherent states on a connected compact Lie group G can analogously be defined as an analytic continuation of the heat kernel

$$\psi_g^t(x) = e^{-t\Delta_G}\delta_{h'}^{(G)}(h)\Big|_{h'\to u}, \qquad (7.\text{-}27)$$

to an element u of the complexification $G^{\mathbb{C}}$ of G.

the complexification of $SU(2)$ is $SL(2,\mathbb{C})$.

$$u = \exp[i\tau_j^j/2] \tag{7.-27}$$

ψ_u^t is exponentially (Gaussin) peaked with respect to multiplication operator $\hat{h}$ on the group at the point h. The width of the peak is approximately given by $\sqrt{t}$,

ψ_u^t is Gaussian peaked with respect to the invariant vector fields at a point p/t in the associated momentum representation. The width of the peak is approximately given by $1/\sqrt{t}$.

Feature of $\mathcal{H}_{Diff}$ is separable as compared to $\mathcal{H}_\omega$: (can be made) separable! $\Longrightarrow$ Can construct *normalized* coherent states in $\mathcal{H}_{Diff}$ with support on all knot classes.
$\Longrightarrow$ Chance to take the classical limit on knot class changing operators just like in Varadarajans's polymer states do for the graph changing operators in his polymer version of Maxwell's theory.

We have to better understand mathematics of $\mathcal{H}_{Diff}$ (**generalized knot theory**)

Coherent states of a compact manifold

$$\begin{aligned}
h_e^{\mathbf{C}} &:= g_e = \sum_{n=0}^{\infty} \frac{i^n}{n!}\{h_e, C\}_{(n)} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}(-{}^e_j\frac{\tau_j}{2})^n \, h_e \\
&= e^{-i\tau_j p_j^e/2} h_e
\end{aligned} \tag{7.-28}$$

polar decomposition.

$$\psi_g^t(h) = \sum_{j\in\frac{1}{2}\mathbb{N}} e^{-j(j+1)t/2}(2+1)tr_j(g^{-1}h) \tag{7.-28}$$

with $g \in SU(2)^{\mathbb{C}} = SL(2,\mathbb{C})$. It has been shown in [244], [245], [**?**], that these states are sharply peaked around their labels $g \in SL(2,\mathbb{C})$ i.e. the overlap

$$\frac{|<\psi_{g_1}^t|\psi_{g_2}^t>|^2}{\|\psi_{g_1}^t\|^2\|\psi_{g_2}^t\|^2} \tag{7.-28}$$

equals 1 for $g_1 = g_2$, and tends to zero faster that any power of t as $t \to 0$.

7.6.6 Gauge-invariant Coherent states for LQG

from [254], [255].

This is a first step to construct *physical* coherent states.

project corresponding complexifier coherent states to the gauge-invariant Hilbert space.

form (7.6.2) it is straightforward to see that it is not invariant under Gause-gauge transformations. One can write the projector onto the gauge-invariant Hilbert space

$$P f(h_{e_1}, \ldots, h_{e_N}) := \int_{G^V} d\mu_H(U_1, \ldots, U_V)\, f(U_{b(e_1)} h_{e_1} U^{-1}_{f(e_1)}, \ldots, U_{b(e_N)} h_{e_N} U^{-1}_{f(e_N)}) \tag{7.-28}$$

7.7 Relating Loop Representation to Fock-Space Description in the Low Energy Limit.

We now turn to the second approach. As we have seen, loop quantum gravity is based on quantum geometry, where the fundamental exctitations are one-dimensional polymer-like. On the other hand, low energy physics based on quantum field theories which are constructed in flat Minkowski spacetime.

The configuration variables are holonomies along curves in the spacial slices of the spacestime, the basic momentum variables are integrals of the triad filed over surfaces in the spacial slices of spacetime. This is in contrast to ordinary background dependent quantum field theories, where both the basic configuration and momentum variables are three-dimension fields in the spacial slices of the spacetime.

We must understand how to recover background-dependent ordinary QFT in order to make contact with low energy physics. How can the techniques of $\overline{\mathcal{A}}, \mu_0$ can be used to describe the Fock states of Maxwell theory and linearized gravity on a Minkowski background spacetime?

7.7.1 Polymer representation of Maxwell's Field

The space $\overline{\mathcal{A}}$ carries a diffeomorphism and gauge invariant measure μ_0 induced by the Haar measure on $U(1)$, which gives rise to the Hilbert space $\mathcal{H}_0 := L^2(\overline{\mathcal{A}}, \mu_0)$, of polymer states.

the set of cylindrical functions

$$\mathcal{N}_{\alpha,\vec{n}}(A) = [h(e_1)]^{n_1}[h(e_2)]^{n_2}\cdots[h(e_N)]^{n_N} \tag{7.-28}$$

Fock space:

- 3-dimensional excitations,

- Basic operators are the connection $\hat{A}_a(x)$ and 'electric field" $\hat{E}$,

- Holonomies do have well defined operator in the Fock space.

Quantum geometry:

- One dimensional excitations,

- Basic operators are holonomies h_A and the "electric field" E^a,

- The connection operator is not well defined on this representation.

$\hat{A}(e)$ fails to be well defined. Introduce a test function using the Euclidean background metric on $\mathbb{R}^2$,

$$f_r(x,y) = \frac{1}{(2\pi)^{3/2}} \frac{e^{|x-y|^2/2r^2}}{r^3} \tag{7.-28}$$

approximates the Dirac delta function for small r. The Gaussian smeared form factor for an edge e is defined

$$X^a_{\gamma(r)}(\vec{x}) = \int_e ds f_r(\vec{e}(s) - \vec{x})\dot{e}^a. \tag{7.-28}$$

Then, the smeared holonomy is defined as

$$H_{\gamma(r)}(A) := \exp i \int_{\mathbb{R}^3} X^a_{\gamma(r)}(x) A_a(x) d^3x. \tag{7.-28}$$

where $A_a(\vec{x})$ is the $U(1)$ connection one-form of the Maxwell field on Σ. Similarly one can define Gaussian smeared electric fields by

$$E_{\gamma(r)}(g) := \int_{\mathbb{R}^3} g_a(\vec{x}) d^3x \int_{\mathbb{R}^3} f_r(\vec{y} - \vec{x}) E^a(\vec{y}) d^3y. \tag{7.-28}$$

Poisson bracket algebra:

We are given that

$$\{A_a(x), E^b(y)\} = \frac{1}{q_0}\delta_a^b\delta(x,y) \tag{7.-28}$$

$$\begin{aligned}
\{H_{\gamma(r)}, H_{\alpha(r)}\} &= \{E^a(x), E^b(y)\} = 0, \\
\{H_{\gamma(r)}, E^a(x)\} &= \frac{1}{q_0}X^a_{\gamma(r)}(x)H_{\gamma(r)}.
\end{aligned} \tag{7.-28}$$

we define the classical Gaussian smeared electric field $E^a_r(x)$ by

$$E^a_r(x) := \int d^3y f_r(y-x)E^a(y). \tag{7.-28}$$

The Poisson bracket algebra generated by the (unsmeared) holonomies and the Gaussian smeared electric field is

$$\begin{aligned}
\{H_\gamma, H_\alpha\} &= \{E^a_r(x), E^b_r(y)\} = 0, \\
\{H_{\gamma(r)}, E^a_r(x)\} &= \frac{1}{q_0}X^a_{\gamma(r)}(x)H_\gamma.
\end{aligned} \tag{7.-28}$$

the **r-Fock representation**

In the r-Fock representation the holonomies are *well defied operators* and so we can

The relation betwen the Fock and r-Fock representations. There is an isomorphism between them,

$$I_r : (A_{(r)}(e), E(g)) \mapsto (A(e), E_{(r)}(g)). \tag{7.-28}$$

For a measurement at a given length scale there always exists a sufficiently small r such that the prediction from r-Fock theory is experimently indistinguishable from that of usual Fock theory.

The image of the Fcok vacuum can be shown to be the following element of Cyl^*,

$$(V| = \sum_{\alpha,\vec{n}} \exp\big(-\frac{\hbar}{2}\sum_{IJ} G_{IJ}n_I n_J\big)(\mathcal{N}_{\alpha,\vec{n}}|, \tag{7.-28}$$

where $(\mathcal{N}_{\alpha,\vec{n}}| \in Cyl^*$ maps the flux network function $|\mathcal{N}_{\alpha,\vec{n}} >$ to one and every other flux network functions to zero.

$$G_{IJ} = \int_{e_I} dt\ \dot{e}^a_I(t) \int_{e_J} dt'\ \dot{e}^b_J(t') \int d^3x \delta_{ab}(\vec{x})[f_r(\vec{x} - \vec{e}_I(t))|\Delta|^{-1/2} f(\vec{x}, \vec{e}_J(t'))], \qquad (7.-28)$$

where δ_{ab} is th eflat Euclidean metric and Δ its Laplacian. Therefore, one can single out the Fock vacuum state directly in the polymer representation.

the action of the Fock vacuum $(V|$ on $(\mathcal{N}_{\alpha,\vec{n}}|$ reads

$$(V|\mathcal{N}_{\alpha,\vec{n}} > = \int_{\overline{A}_\alpha} d\mu^0_\alpha \overline{V}_\alpha \mathcal{N}_{\alpha,\vec{n}}, \qquad (7.-28)$$

where the state V_α is in the Hilbert space $\mathcal{H}_\alpha$ for the graph α and given by

$$V_\alpha(A) = \sum_{\vec{n}} \exp[-\frac{\hbar}{2} \sum_{IJ} G_{IJ} n_I N_J] \mathcal{N}_{\alpha,\vec{n}}(A). \qquad (7.-28)$$

Thus for any cylindrical functions φ_α associated with α,

$$(V|\varphi_\alpha > = < V_\alpha|\varphi_\alpha >, \qquad (7.-28)$$

where the inner product in the right hand side is taken in $\mathcal{H}_\alpha$. $V_\alpha(A)$ are refered to as "shadows" of $(V|$ on the graphs α.

7.8 Minkowskian Spacetime and Scattering Amplitudes

In the traditional in the traditional perturbative framework, the gravitational attration between two point masses aries from an exchange of virtual gravitons, described by the Feynman propagator.

Loop quantum gravity has no background metric. Therefore one cannot even begin to calculate begin to calculate the the propagator along these traditional lines.

the problem of deriving scattering amplitudes is still an open one. formulism available yet for deriving particle's scattering amplitudes from boundary amplitudes.

The aim of a semiclassical analysis would be to show that, for suitable choices of initial and final states, the transition amplitudes of LQG reduce to the transition amplitudes of an effective field theory on a background geometry. If this effective field theory contained gravity, this would be the prove that LQG is indeed a quantum theory of gravity.

7.8.1 Quantum states Representing Minkowskian Spacetime

dynamically generated quantum geometry that acts as a background geometry around which small quantum fluctuations take place.

corresponding object in the loop representation.

$$\Psi_{Mink}(A) = \sum_{\gamma} \Psi_{Mink}[\gamma]\, \overline{W}_{\gamma}[A] \tag{7.-28}$$

where $\Psi_{Mink}[\gamma]$ are invariants for the flat-space vacuum.

7.8.2 Scattering Amplitudes

In the standard formulism on associates A Hilbert space of states with each time-slice of a global foliation of space-time. An evolution takes place between two such time-slices and is represented by a unitary operator. Associated with states in the two such time-slices is a transition amplitude, whose modulus square determines the probability of finding the final state given that the initial one was prepared.

$$\mathcal{A}_{12} =< \psi_{int}|\hat{U}(t_1,t_2)|\psi_{fin} >, \quad \text{where } < \psi_{int}| \in \mathcal{H}^* \text{ and } |\psi_{fin} >\in \mathcal{H}. \tag{7.-28}$$

$$\mathcal{A}_{12} = \int d^3x \int d^3y\; \psi^*_{int}(x,t_{int}) W(x,y) \psi_{fin}(y,t_{fin}) \tag{7.-28}$$

Just as GR doesn't determine the distance between spacetime points, it doesn't determine this probability; the only way to preserve general covariance is if $W(x,y)$ is constant.

Transition amplitudes are associated with regions of spacetime and states are associated with their boundaries.

The boundary value of the gravitational field determine the geometry of the boundary surface Σ.

Scattering probabilities are determined internally. Scattering amplitude and the spacetime geometry both encoded in the state. Reflects the fact that there are no *external* reference bodies.

Introduce an S Cauchy data (q_{ab}, K^{ab}) induced by a flat Euclidean 4-metric. By 'evolving' this data via Einstein's equation one would recover an Euclidean 4-metric g_{ab} in the interior of S. However, this 4-metric, of course, will be unique up to active diffeomorphisms (which are identity at the boundary S). Here we have used the reference systems defining the region of the space, to do a partial gauge fixing. The propagator is then obtained by fixing points x, x' on the boundary S and summing over all configurations in the interior bulk which agree with the boundary data (q_{ab}, K^{ab}). In the quantum then one fixes a LQG boundary state $\Psi_{q,k}(s)$ peaked at some flat space initial data (q_{ab}, K^{ab}), where s is a (diffeomorphism equavalence class of) spin network(s) on the 3-sphere S.

Wave function on time-slice

A wave function at constant "time" has no physical meaning in a generally covariant setting.

$$\mathcal{H}^* \otimes \mathcal{H} \tag{7.-28}$$

in which no reference is made to infinitely extended spacial surfaces. it suggests a way to derive particles' scattering amplitudes from a spinfoam model.

$$W[\varphi, \Sigma] = W[\varphi] \tag{7.-28}$$

$$\Psi_M[s] =< s|0_M >= \lim_{T\to\infty} \int \mathcal{D}\Phi f_{s\# s_T}[\Phi] e^{-S[\Phi]}. \tag{7.-28}$$

$$W[s] = \int \mathcal{D}\Phi f_s[\Phi] e^{-S[\Phi]}, \tag{7.-28}$$

The spin foam polynomial is defined as

$$f_s[\Phi] = n \int dg_{n_1} \dots dg_{n_4} D_\alpha \beta_{n_1} \dots C^{i_n}_{\beta\dots\beta} l \delta^{l_1 l_2} \tag{7.-28}$$

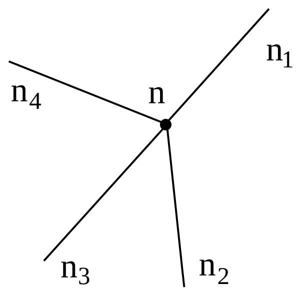


Figure 7.3: node.

7.9 Semiclassicallity from Spin Foams

large spin limit

The vertex is dominated in the large spin limit by semiclassical states satisfying the closure condition for each tetrahedron and the relation adjacent tetrahedra in the same 4-simplex.

7.10 The Infinite Tensor Product Extension

Quantum field theory on curved spacetimes is best understood if the spacetime is flat Minkowski space on the manifold $\mathcal{M} = \mathbb{R}^4$. Thus, when one wants to compute the low-energy limit of canonical Quantum General Relativity to show that one gets the standard model (pluss corrections) on a background metric one should do this first for the Minkowski background metric. In order that we can define a semiclassical limit for all the initial value data slice σ we must necessarily work with at least countably infinite embeded graphs.

However, the Hilbert spaces used in LQG have as dense subspace the space of cylindrical functions labelled by graphs with a finite number of edges.

When the number of edges of graphs are infinite, it turns out, that a much larger Hibert space is required. Actually the construction ofthe appropriate structure was already developed by von Neumann more that 60 years ago, know as the Infinite Tensor Product (ITP).

7.10.1 Von Neumann's Infinite Tensor Product (ITP)

We first consider the tensor product of a finite number of Hilbrt spaces. Say $f_k, g_k \in \mathcal{H}_k$ with inner product $(\cdot,\cdot)_k$ on $\mathcal{H}_k$. If an element of $\otimes_k \mathcal{H}_k$ is f, the inner product of the tensor product is defined as

$$(f, g) = \prod_{k=1}^{n} (f_k, g_k)_k$$

and the norm

$$\|f\| = \sqrt{\prod_{k=1}^{n} (f_k, f_k)_k} = \sqrt{\prod_{k=1}^{n} \|f_k\|_k^2} = \prod_{k=1}^{n} \|f_k\|_k.$$

We are lead to the consideration of the mathematics of products of arbitrary complex numbers.

Now when one forms the infinite tensor product of a collection of Hilbert spaces, a physical requirement is that this product must not depened on the order of the individual Hilbert spaces (whether the collection is countably or uncountably infinite).

Hence we are interested in the convergence properties of a countable or uncountable product of complex numbers which are independent of the ordering of the product. This was developed in the paper by von Neumann (availble at http://www.numdam.org/item?id=1939 0).

As we will see, convergence of products is related to convergence of corresponding summations. Now, it is a remarkable fact that whether an infinite series converges or not can depend on the ordering of the terms of that series. From which it follows that whether or not the coresponding product of complex numbers converges depends on the ordering of the terms of that product.

Absolutely and conditionally convergent series have completely different behaviours under rearrangement

Theorem 7.10.1 *(***Riemann's Rearrangement Theorem***) Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. For each real number s, there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges and has sum s.*

Proof:

The nonnegative series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty}(-q_n)$ diverge. In fact, if both were to converge, it would follow that $\sum_{n=1}^{\infty} |a_n|$ converges, that is, $\sum_{n=1}^{\infty} a_n$ would be absolutely convergent. On the other hand, if one of these series converged and the other diverged, it would follow that the partial sums of $\sum_{n=1}^{\infty} a_n$ diverge to either $+\infty$ or to $-\infty$. The convergence of $\sum_{n=1}^{\infty} a_n$ itself implies that both $\{p_n\}$ and $\{q_n\}$ have limit zero.

Now we construct the rearrangement. Choose terms $p_1, p_2, \ldots$ up to the first index k_1 such that

$$p_1 + p_2 + \cdots + p_{k_1} > s.$$

This will occur , because $\sum_{n=1}^{\infty} p_n = \infty$. Note, as $\sum_{n=1}^{k_1-1} p_n \leq s$,

$$|p_1 + p_2 + \cdots + p_{k_1} - s| \leq p_{k_1}.$$

Next, we choose $q_1, q_2, \ldots$ up to the first index l_1 such that

$$(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) < s. \tag{7.-28}$$

Note, as $\sum_{n=1}^{k_1} p_n + \sum_{n=1}^{l_1-1} q_n \geq s$,

$$|(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) - s| \leq \max\{p_{k_1}, |q_{l_1}|\}.$$

We then add just enough new p's to (7.10.1) to make the left hand side greater than s, followed by just enough q's to make it less than s, and continue. At each phase of the $2n-$th step, the difference between s and the partial sum of the new series has absolute value less than or equal to $\max\{p_{k_n}, |q_{l_n}|\}$, and at each phase of the $2n+1-$th step, the difference between s and the partial sum of the new series has absolute value less than or equla to $\max\{p_{k_{n+1}}, |q_{l_n}|\}$. As these have the limit 0, the rearranged series has sum s.

□

Corollary 7.10.2 *Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. It has a rearrangement that diverges to $+\infty$.*

Proof:

Start by adding consecuative positive terms p_n until you get a sum > 1. Then add q_1 which reduces the sum a bit. Then add consecuative p_n's from the remaining terms until the sum > 2, and then add in q_2. By proceeding in this manner we create a reagrrangement of the original series $\sum_n a_n$, a subsequence whose partial sums is unbounded. Thus the rearrangement diverges.

□

Theorem 7.10.3 *Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement. Then $\sum_{n=1}^{\infty} b_n$ converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.*

Proof: Let $\{s_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$, i.e,

$$s_n = \sum_{k=1}^{n} a_k,$$

and let s be the limit. Let $\{t_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} b_k$. Given $\epsilon > 0$, choose M so large that

$$\sum_{k=M+1}^{\infty} |a_k| < \frac{\epsilon}{2}. \tag{7.-28}$$

It follows from this that $|s - s_M| < \frac{\epsilon}{2}$,

$$|s - s_M| = |\sum_{k=M+1}^{\infty} a_k| \leq \sum_{k=M+1}^{\infty} |a_k| < \frac{\epsilon}{2}..$$

Choose N so large that every one of the first M terms of $\{a_k\}$ occurs among the first N terms of $\{b_k\}$. So that for any $n \geq N$

$$|t_n - s_M| = |t_n - \sum_{k=1}^{M} a_k| \leq \sum_{k=M+1}^{\infty} |a_k|.$$

Therefore

$$|t_n - s| \leq |t_n - s_M| + |s_M - s| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

The product of positive real numbers

$$\prod_{n=1}^{\infty} a_n$$

converges to a nonzero real number if and only if the sum

$$\sum_{n=1}^{\infty} \ln(a_n)$$

converges. This allows the translation of convergence criteria for infinite sums into convergence criteria for infinite products.

For products of reals in which each $a_n \geq 1$, written as, for instance, $a_n = 1 + p_n$, where ≥ 0, the bounds

$$1 + \sum_{n=1}^{N} p_n \leq \prod_{n=1}^{N} (1 + p_n) \leq \exp\left(\sum_{n=1}^{N} p_n\right)$$

show that the infinite product converges if the infinite sum of the p_n converges (the second bound in the above inequality follows from $1 + x \leq e^x$ for $x \geq 0$). This relies on the Monotone convergence theorem. We can show the converse by observing that, if $p_n \to 0$, then

$$\lim_{n\to\infty} \frac{\ln(1+p_n)}{p_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1,$$

and by the limit comparison test it follows that the two series

$$\sum_{n=1}^{\infty} \ln(1+p_n) \quad \text{and} \quad \sum_{n=1}^{\infty} p_n,$$

are equivalent meaning that either they both converge or they both diverge.

The same proof also shows that if $a_n = 1 - q_n$ for some $0 \leq q_n < 1$, then $\prod_{n=1}^{\infty}(1-q_n)$ converges to a non-zero number if and only if $\sum_{n=1}^{\infty} q_n$ converges. To prove this we will need $1 - \sum_{n=1}^{N} q_n \leq \prod_{n=1}^{N}(1-q_n)$, which is easy to prove via induction. The base case is $1 - q_1 \leq 1 - q_1$. The inductive step reads

$$\begin{aligned}
\prod_{n=1}^{N+1}(1-q_n) &= (1-q_{N+1})\prod_{n=1}^{N}(1-q_n) \\
&\geq (1-q_{N+1})(1-\sum_{n=1}^{N} q_n) \\
&= 1 - \sum_{n=1}^{N+1} q_n + q_{N+1}\sum_{n=1}^{N} q_n \\
&\geq 1 - \sum_{n=1}^{N+1} q_n. \qquad (7.\text{-}30)
\end{aligned}$$

Also, we will need $\prod_{n=1}^{N}(1-q_n) \leq \exp(-\sum_{n=1}^{N} q_n)$ which can be obtained from $\int_{1-x}^{1} \frac{dt}{t} \geq \int_{1-x}^{1} dt$ for $0 \leq x < 1$.

If the series $\sum_{n=1}^{\infty} \ln(a_n)$ diverges to $-\infty$, then the sequence of partial products of the a_n converges to zero. The infinite product is said to diverge to zero.

□

Definition: If the product $\prod_{n=1}^{\infty}(1+|p_n|)$ converges, the product $\sum_{n=1}^{\infty}(1+p_n)$ is said to be absoloutely convergent.

Theroem: If $\sum_{n=1}^{\infty} |p_n|$ is convergent (i.e. if $\sum_{n=1}^{\infty} p_n$ converges absolutely), then the product $\prod_{n=1}^{\infty}(1+|p_n|)$ converges, and thus $\prod_{n=1}^{\infty}(1+p_n)$ converges absolutely. The converse holds.

Proof is obvious.

$\square$

Theorem: If the infinite product $\prod_{n=1}^{\infty}(1+p_n)$ is absolutely convergent, its values is independent of the order of its factors.

Proof:

Consider

$$\prod_{n=1}^{\infty}(1+p_n)$$

which ius absolutely convergent. Its absolute convergence implies that $\sum_{n=1}^{\infty} p_n$ converges. Let $\sum_{n=1}^{\infty} r_n$ be a series obtained by rearranging the terms of $\sum_{n=1}^{\infty} p_n$ in any order. Write $a_n = |p_n|$ and $b_n = |r_n|$, then

$$\begin{aligned} P_n &= (1+p_1)\cdots(1+p_n), \qquad & R_n &= (1+r_1)\cdots(1+r_n) \\ A_n &= (1+a_1)\cdots(1+a_n), \qquad & B_n &= (1+b_1)\cdots(1+b_n). \end{aligned}$$

Then choose $p > n$ so that P_p contains the whole of R_n and consequently A_p contains the whole of B_n. It is evident that $\dfrac{A_p}{B_n} - 1$ contains every term in $\dfrac{P_p}{R_n} - 1$, but with all the signs made positive. Hence

$$\left|\frac{P_p}{R_n} - 1\right| \leq \frac{A_p}{B_n} - 1$$

and

$$R_n \leq B_n$$

so that

$$|P_p - R_n| \leq A_p - B_n. \tag{7.-32}$$

Choose p' so that the first n' factors of $A_{n'}$ are contained in $B_{p'}$. Then

$$A \geq B_r \geq A_{n'} \qquad \text{if } r \geq p'.$$

Now n' can be taken large enough to bring $A_{n'}$ as close to A as desired. Consequently,

$$\lim_{r'\to\infty} B_r = \lim_{n'\to\infty} A_{n'} = A.$$

Consequently an n_0 can be found so that

$$A > A_p > B_n > A - \frac{1}{2}\epsilon, \qquad \text{if } p > n > n_0.$$

Hence

$$A_p - B_n < \frac{1}{2}\epsilon, \qquad \text{if } n > n_0,$$

and using this in (7.10.1) we have

$$|P_p - R_n| < \frac{\epsilon}{2}, \qquad \text{if } n > n_0.$$

But $\lim_{p\to\infty} P_p = P$, and therefore an n_1 can be found so that if $p > n > n_1$, then $|P - P_p| < \frac{\epsilon}{2}$. Thus if $n > n_0$ and $n > n_1$,

$$|P - R_n| \leq |P - P_p| + |P_p - R_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

that is, $\lim_{n\to\infty} R_n = P$, and the theorem is proved.

□

As in the case of infinite series, a non-absolutely convergent infinite product may be made to converge to any values, or diverge, by altering the order of the factors.

Definition:

The inifinite product

$$\prod_e z_e$$

of complex numbers $z_e = |z_e|e^{i\varphi_e}$ is defined by

$$\prod_e z_e := \left[\prod_e |z_e|\right] e^{i\sum_e \varphi_e}$$

$\varphi_e \in [-\pi, \pi)$ provided that both

i) $\sum_e \big||z_e| - 1\big|$

ii) $\sum_e |\varphi_e|$

converge, in which case we also say that $\prod_e z_e$ is convergent. One would naively expect the product to converge if $\sum_e \varphi_e$ converged - this is a consequence of the defintion of convergence which is indepndent of ordering. too vehement oscillation of the φ_e. We say that $\prod_e z_e$ is quasi-convergent if $\prod_e |z_e|$ converges, and asign to $\prod_e z_e$ the value zero.

Two vectors $\otimes_f$, $\otimes_{f'}$ are said to be strongly equivalent if and only if

$$\sum_e |(f_e, f'_e)_{\mathcal{H}_e} - 1|$$

converges. We denote by $[f]$ the strong equivalence class of f. It follows that

$$(\otimes_f, \otimes_{f'}) = 0$$

if either $[f] \neq [f']$ or $[f] = [f']$ and $(f_e, f'_e)_{\mathcal{H}_e}$ for at least one e.

If we set

$$(z \cdot f)_e := z_e f_e$$

then the product formula

$$\otimes_{z \cdot f} = (\prod_e z_e) \otimes_f$$

fails to hold if $\prod_e z_e$ is (quasi-convergent but) not convergent. We say that f, f' are weakly equivalent provided that there exists z such that

$$[z \cdot f] = [f'].$$

This is equivalent to the convergence of

$$\sum_e \big||(f_e, f'_e)_{\mathcal{H}_e}| - 1\big|$$

We denote by (f) the weak equivalemce class of f. Obviously, strong equivalence implies weak equivalence. One can show that the closure of the linear span of all vectors in the same strong equivalence class $[f]$, denoted by $\mathcal{H}^{\otimes}_{[f]}$ is separable, consisting of the completion of the finite linear span of the vectors of the form $\otimes_{f'}$ where $f'_e = f_e$ for all but finitely many e. The ITP Hilbert space $\mathcal{H}^{\otimes}$ is the direct sum of the $\mathcal{H}^{\otimes}_{[f]}$.

Let also $\mathcal{H}^{\otimes}_{(f)}$ be the closure of the finite linear span of the $\otimes_{f'}$ with $(f') = (f)$. Then the strong equivalence subspaces of $\mathcal{H}^{\otimes}_{(f)}$ are unitarily equivalent, the corresponding unitary operators being of the form $U_z \otimes_f := \otimes_{z \cdot f}$ with $\prod_e z_e$ quasi-convergent.

A state f in the infinite tensor product Hilbert space which is a direct product of normalized states, one for each edge of the graph, generates so-called strong and weak equivalence classes of so-called C_0-sequences. It turns out that the corresponding C_0-vector plays the role of a cyclic vector (vacuum state) for a Fock-like tiny closed subspace of the complete ITP Hilbert space.

Those Fock-like spaces that correspond to the same weak class but different strong classes are unitarily equivalent while those that correspond to different strong and weak classes are unitarily inequivalent. This way the ITP gives rise to an uncountably infinite number of mutually unitarily inequivalent representations of the canonical commutation relations.

Infinite spin chain.

Now suppose our system becomes infinitely large. The dimension of the system will be infinite, of course, but it will be a larger infinity than those to which we are used - specifically it will be $2^{\aleph_0}$, the cardinality of the continuum, which is strictly larger than the cardinality $\aleph_0$ of the integers.

It follows that systems with infinitely many components have a Hilbert space which is non-separable (i.e., has uncountable dimension). To see the consequences of this, consider the operator algebra of our set of two-state systems. It consists of the set of linear combinations of spin operators, and hence has countably many linearly independent elements. The action of this algebra on any given state will generate only countably many linearly independent states, hence the action of the operator algebra on the total, non-separable space is highly reducible.

Our label set will be the integers $I = \mathbb{Z}$ and for each $n \in Z$ we have the Hilbert space $\mathcal{H}_n = \mathbb{C}^2$ with standard inner product

$$< f_n, f'_n >_n = \overline{f}^+_n f^{+'}_n + \overline{f}^-_n f^{-'}_n .$$

In each Hilbert space we have the standard orthogonal basis of vectors $e^{\pm}_n$ and spin operators $\sigma_n = \sigma_3$ (Pauli matrix) so that $\sigma_n e^{\pm}_n = \pm e^{\pm}_n$ corresponds to spin up/down. We also have ladder operators $\sigma^{\pm}_n = \frac{1}{2}[\sigma_1 \pm \sigma_2]$ so that $\sigma^{\pm}_n e^{\pm}_n = 0$ and $\sigma^{\pm}_n e^{\mp}_n = e^{\pm}_n$. Consider the positive semi-definite, self-adjoint Hamiltonian

$$\mathcal{H} := \frac{1}{2}\sum_n [1+\sigma_n] = \sum (\sigma_n^-)^\dagger \sigma_n^-$$

on the ITP Hilbert space

$$\mathcal{H}^\otimes = \otimes_n \mathcal{H}_n$$

which is non-separabe even though each $\mathcal{H}_n$ has finite dimension two.

We will first consider a C_0-vector $\otimes_f$ with $\|f_n\|_n = 1$ and a second one $\otimes_{f'}$ with $f_n = -f_n$. Are the corresponding C_0-sequences in the same strong (weak) eqivalence class? Since $(f_n, f_n)_n = -1$ we see that

$$\sum_n |(f_n, f_n)_n - 1| = \sum_n 2 = \infty$$

but

$$\sum_n \big||(f_n, f_n)_n| - 1\big| = \sum_n 0 = 0,$$

thus they are in different strong classes within the same weak class. In fact, the unitary operator $\hat{U}$ on $\mathcal{H}^\otimes$ defined densely on arbitrary C_0-vectors by $\hat{U}\otimes_g = \otimes_{g'}$ with $g_n = -g_n$ maps the two unit C_0-vectors into each other and thus the strong equivalence class Hilbert spaces built from them will be unitarily equivalent subspaces of the whole ITP. Notice that indeed $(\otimes_f, \otimes_{f'}) = \prod_n (-1)^n = 0$ since the product of numbers $z_n = -1$ is only quasi-convergent.

To see what these sectors are, suppose we start with all components having spin up. Then the action of any element of the algebra can, at most, cause finitely many components to have spin down. So no amount of algebraic action can transform such a state into one in which, say, every second component has spin up. This state, in turn, can be transformed into other states differing from it in finitely many places, but not into a state in which all components are spin down... or every third component is spin down... or where half the states are spin up but the spin-up states are grouped in pairs...

7.11 Emergent Coherent Excitations as Noiseless Subsystems

A few of preliminaries first

Open quantum system

We have a complete quantum experiment, which we want to divide into the system S and environment $\mathcal{E}$. We want to understand what quantum properties of the system may survive stably in spite of continual and uncontrollable interactions with the environment. The joint Hilbert space decomposes into the product of system and environment,

$$\mathcal{H}^{total} = \mathcal{H}^{\mathcal{S}} \otimes \mathcal{H}^{\mathcal{E}} \tag{7.-32}$$

while the Hamiltonian decomposes into the sum

$$H = H^{\mathcal{S}} + H^{\mathcal{E}} + H^{int} \tag{7.-32}$$

where $H^{\mathcal{S}}$ acts only on the system, $H^{\mathcal{E}}$ acts only on the environment and all the interactions between them are contained in H^{int}.

We are interested in the state of the system alone, and want to disregard the state of the environment. If total is the state of the whole system + environment, then the state of the system alone is

$$\begin{aligned} \rho_{sys} &= Tr_{env}\Big[\rho_{total}\Big] \\ &= \sum_{env} < env|\rho_{total}|env > . \end{aligned} \tag{7.-32}$$

Noiseless sub-systems

When an environment has a symmetry to it, there exists a subspace of the Hilbert space which is protected against decoherence effect of the environment - this is called a noiseless subsystem.

• Noiseless Subsystems are useful for describing long-term behaviour of the system because they are conserved.

• If we divide the quantum gravitational field into subsystems, those properties that are **conserved** under interactions between subsystems will characterize the low energy-limit of spacetime geometry.

• The commutant $A^{int\prime}$ should include the symmetries that characterize classical spacetime (e.g. Poincare).

Particles in quantum geometry

It is argued that a particle could be some kind emergent excitation, some subsystem, of microscopic quantum dynamics. The quantum spacetime, being dynamical, is constantly changing. In order for the emergent excitation to behave as if it were a particle moving through a fixed, non-dynamical background spacetime it must be a noiseless sub-system. So Markopoulou identified emergent particles as noiseless subsystems of quantum geometry corresponding to a Lorentz invariance symmetry at the appropriate scale.

7.11.1 The Standard Model - a Reminder

Matter

The Standard Model fermion are the leptons and quarks

(i) Leptons:

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}, \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} \tag{7.-32}$$

The electron, muon and tau electron all have $-1e$, their anti particles have charge $+1e$, but the neutrinos are all uncharged.

They form doublets under $SU(2)$

(ii) Quarks:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix} \tag{7.-32}$$

Left handed fermions in the Standard Model

Fermion left-handed	symbol	spin	electric charge	colour charge	mass
electron	e	1/2	-1	1	0.510999 Mev
electron neutrino	ν_e	1/2	0	1	~ 0 ev $< 2ev$ Mev
positron	e^c	1/2	$+1$	1	0.510999 Mev
electron antineutrino	μ_e^c	1/2	0	1	~ 0 ev < 2 ev
up quark	u	1/2	$+2/3$	3	~ 3 Mev
down quark	d	1/2	$-1/3$	3	~ 6 Mev
anti-up quark	u^c	1/2	$-2/3$	$\overline{3}$	~ 3 Mev
anti-down quark	d^c	1/2	$+1/3$	$\overline{3}$	~ 6 Mev

Forces

(i) Forces mediated by bosons (spin $0, 1, 2, \dots$)

(ii) The electromagnetic force is mediated by the photon (γ)

(iii) The weak force is mediated by the $W^{\pm}$ (posite, negative charge) and Z^0 (uncharged) bosons

(iv) The strong force is mediated by gluons (g)

Discrete symmetries

$$\begin{aligned} \mathcal{P} &: \mathbf{x} \to -\mathbf{x} \\ \mathcal{C} &: e \to -e \\ T &: t \to -t \end{aligned} \tag{7.-33}$$

Parity transfomation:

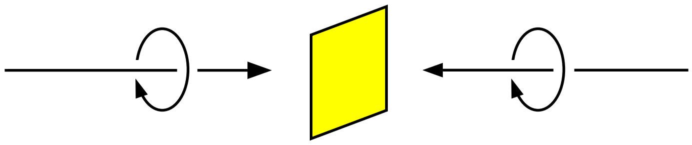


Figure 7.4: parityPartF. The operation $\mathcal{P}$ reverses the momentum of the particle whithout flipping its spin.

Charge conjugation $\mathcal{C}$: is the particle-antiparticle symmetry. Charge conjugation is defined to take a fermion with a given spin orientaion into an antifermion with the same spin orientaion.

• $\mathcal{P}$ converts a left-handed electron into a right-handed electron,

and

• $\mathcal{C}$ converts a left-handed electron into a left-handed positron.

the combination of these two operations interchanges left-handed particles with right-handed antiparticles.

zero-mass fermions

A wave function of negative energy and momentum $-\mathbf{p}$ corresponds to an anti-fermion with positive energy and momentum $\mathbf{p}$. Zero mass fermions and anti-zero mass ferminons with positive chirality both also have helicity (they are "right-handed"). Similarly, fermions and anti-fermions with negative chirality both barry negative helicity (they are "left-handed").

Although strong and electromagnetic forces make no distinction between right-handed or left-handed particles (particle invariance), particles subject to weak forces do make this distinction. (A right-handed particle is a particle spinning in the direction the right-hand fingers curl when the particle is traveling in the direction pointed-to by the right thumb). Thus, left-handed neutrinos are matter, whereas right-handed neutrinos are antimatter.

Gauge symmetry group of fermions

$SU(3) \times SU(2) \times U(1)$

Electroweak

iso-spin doublets

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix} = \tag{7.-33}$$

Strong

What is conserved in the Standard Model

(i) Energy, momentum, but not mass

(ii) Angular momentum

(iii) Charge and colour

(iv) Lepton flavour number (and therefore lepton number)

(v) Quark number (but not quark flavour number)

(vi) Charge conjugation ($\mathcal{C}$), parity ($\mathcal{P}$), and time reversal (T) are conserved by the strong and electromagnetic interactions, but not by the weak interaction ($\mathcal{CP}T$ is always conserved).

7.11.2 Conventional Preon Models and their Problems

Unlike hadrons, the light quarks and leptons are much lighter than the inverse of the largest experimentally allowed binding scale for their proposed subcomponents. This made it challenging to bind such hypothetical subcomponents by means of ordinary gauge interactions.

The binding mechanism proposed here operates at Planck scales, below the scales at which effective field theory would be a good description. The states are bound here, not by fields, but by quantum topology, because the configurations that we interpret as quarks and leptons are conserved under the dynamics of the quantum geometry.

7.12 The Standard Model from Loop Qunatum Gravity?

7.12.1 Introduction

These theories already contain elementary particles which can be identified with particles of the standard model so that the symmetries of the standard model are symmetries of the dynamics of the quantum geometry.

When they are continually in interaction with the quantum fluctuations of the microscopic theory - they are protected by symmetries in the dynamics.

7.12.2 The Standard Model from Loop Qunatum Gravity?

Preons:

"Quarks, leptons and heavy vector bosons are suggested to be composed of a stable spin-1/2 preons, existing in three flavours, combined according to simple rules."

Now onto the emergence of (part) of the Standard Model:

Sundance O. Bilson-Thompson (who was working on preons, not quantum gravity) noticed that a simple braiding of ribons that captured precisely the structure of the preon models of particle physics.

From this insight of Bilson-Thompsons', Markopoulou that the different ways of braid and knot the edges of graphs in a quantum spacetime must be different kinds of elementary particles!!! - So loop quantum gravity is not just about quantum spacetime - it already has elemenmtary particle physics in it.

From abstract of "Quantum Gravity and the Standard Model" (hep-th/0603022):

"We show that a class of background independent models of quantum spacetime have local excitations that can be mapped to the first generation fermions of the standard model of particle physics. These states propagate coherently as they can be shown to be noiseless subsystems of the microscopic quantum dynamics[2]. These are identified in terms of certain patterns of braiding of graphs, thus giving a quantum gravitational foundation for the topological preon model proposed in [1]."

7.12.3 Substitution Moves

Ribbon graph theories

Quantum $SU(2)$ group.

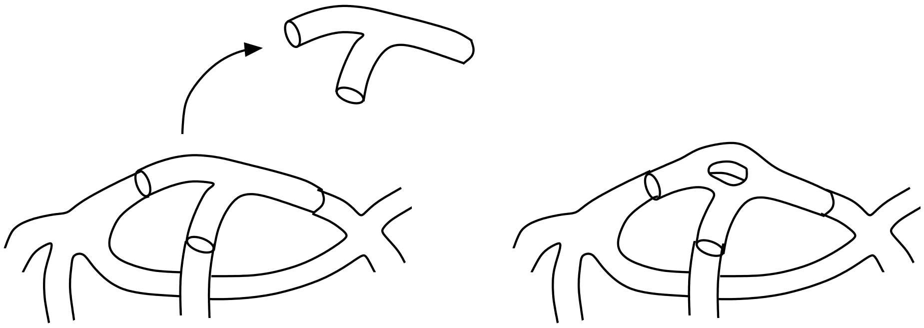


Figure 7.5: submoveF. An example of a substitution move.

The result is a sequence of states, $|0>, |1>, \ldots, |N>$, esch following from the previous one by a substitution move.

7.12.4 Discrete Symetries of Braids

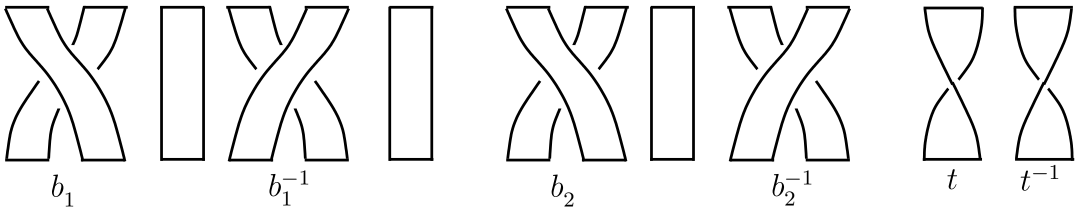


Figure 7.6: basucMovesF. The basic braiding moves on three strands, and the basic twisting moves.

Parity. A parity transformation is defined as a reflection of a braid, while not effecting the handedness of the twists on the strands.

Charge conjugation.

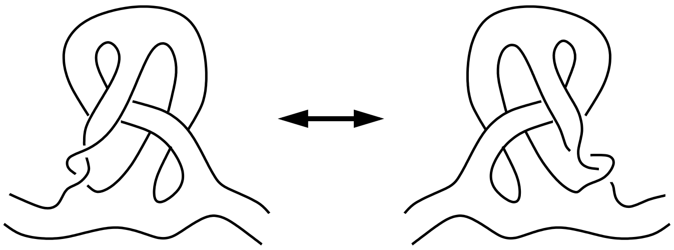


Figure 7.7: paritytranF. The effect of a parity transformation on a braid.

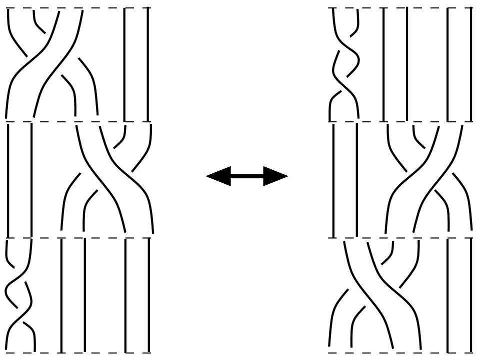


Figure 7.8: chargeDefF. The definition of a charge conjugation transformation on a braid.

7.12.5 Identification of the First Generation Fermions of the Standard model

Assumptions

As we will see, in order to arrive at the standard model fermions we have to make two assumptions:

• The lightest states are the simplest non-trivial braids made of ribbons with no twists (no charge) or one full twist (positive or negative charge).

• Quantum numbers are assigned only to braids with no positive and negative charge mixing. Such a rule is necessary in preon models as discussed [276].

These should be justified in a fundamental theory. At the level that an effective dynamics in spacetime emerges a notion of mass of emergent excitations will arise. This requires that there be in the low energy limit an emergent translation invariance in space and time. This will imply the conservation of energy and momentum for small excitations around the ground state. When the effective Hamiltonian H is evaluated on the states described here at zero momentum it will give us a mass matrix.

(1) Given two braids as just described, which have the same number of strands and

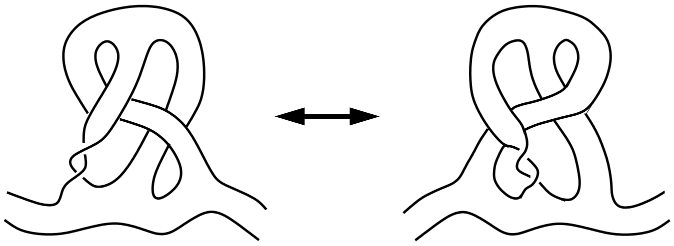


Figure 7.9: chargetranF. The effect of a charge conjugation transformation on a braid.

twistings, but differ by the number of crossings, the mass will increase with the number of crossings.

(2) Non-trivially braided states with both positive and negative twisted strands incident on the same vertex should have a heavy mass M. All other states are light relative to the scale M.

Identification of fermions

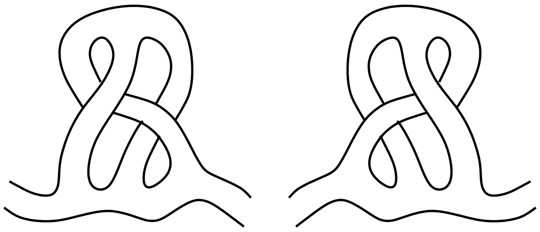


Figure 7.10: elecneutrinoF. The electron neutrino and anti-neutrino - two unchaged states.

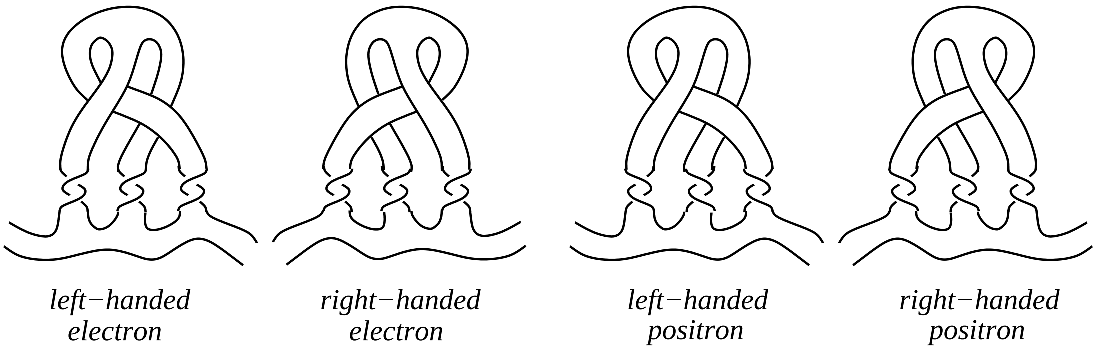


Figure 7.11: elecpositronF. The electron and positron - four maximally charged states.

These are partially charged states, with one and two +'s, and the rest zeroes. Theses are shown in figures 7.12 and 7.13. These are the quarks, with total charges $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$. In each of these there is an "odd strand out" - one that is different from the other two. The non-trivial nature of the braid means that there are three distinct positions in the braid which the "odd strand out" can occupy. Hence each of the partially charged states comes in three versions. We will equate these permutations with colour.

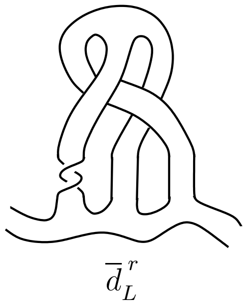


Figure 7.12: dquarkleftF. The left-handed down states - showing tripling of states for fractional charge.

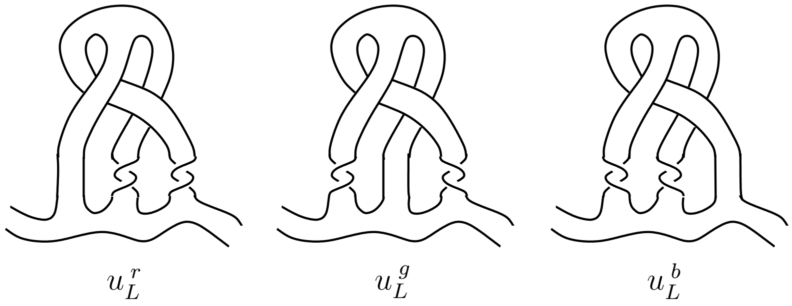


Figure 7.13: uquarkleftF. The three left-handed up states.

The correspondence between braids and particles suggests that more properties are waiting to be derived from the theory. The most substantial achievement would be to calculate the masses of the elementary particles from first principles.

It is natural to hypothesize then that the second generation standard model fermions come from the next most complicated states, which have three crossings. There are two kinds of three crossing states. There are three stranded braids, such as shown in Figure N.-19??. It is straightforward to see that by adding twists to this state one gets a repeat of the pattern for the first generation. However, this appears to imply no upper bound on the number of allowed generations.

generation = crossings - 1

one new $SU(2) + SU(3)$ singlet appears for all higher generations

Weak vector bosons come from unbraided triplets of lines.

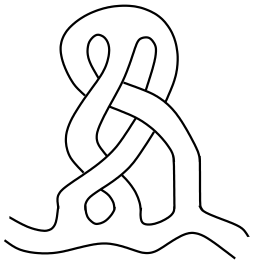


Figure 7.14: muonneutrF. A possible identification of the muon neutrino as a three crossing state.

With one more rule to suppress quark-lepton mixing weak interactoins are a consequence of loacl moves.

From all allowed twists we get a copyh of the 1st generation.

These give additional states which are $SU(2) + SU(3)$ singlets but come in left and right versions. Could these be the right handed neutrinos.

7.13 Semi-Classical Limit in Constitent Discrete Quantum Gravity

7.14 Worked Exercises

Prove that the states

$$|n>= \frac{(a^\dagger)^n}{\sqrt{n!}}|0>$$

are orthogonal:

$$< m|n >= \delta_{mn} \tag{7.-33}$$

Proof:

Using $[a, a^\dagger] = 1$ we obtain

$$
\begin{aligned}
< m|n > &= \frac{1}{\sqrt{m!n!}} < 0|(a)^m (a^\dagger)^n|0 > \\
&= \frac{1}{\sqrt{m!n!}} < 0|(a)^{m-1}(a^\dagger a + 1)(a^\dagger)^{n-1}|0 > \\
&= \frac{1}{\sqrt{m!n!}} < 0|(a)^{m-1}a^\dagger a(a^\dagger)^{n-1}|0 > + \frac{1}{\sqrt{mn}} < 0|(a)^{m-1}(a^\dagger)^{n-1}|0 > \\
&= \frac{1}{\sqrt{m!n!}} < 0|(a)^{m-1}a^\dagger(a^\dagger a + 1)(a^\dagger)^{n-2}|0 > + \frac{1}{\sqrt{mn}} < 0|(a)^{m-1}(a^\dagger)^{n-1}|0 > \\
&= \frac{1}{\sqrt{m!n!}} < 0|(a)^{m-1}(a^\dagger)^2 a(a^\dagger)^{n-2}|0 > +2\frac{1}{\sqrt{mn}} < 0|(a)^{m-1}(a^\dagger)^{n-1}|0 > \\
&\qquad \vdots \\
&= \frac{1}{\sqrt{m!n!}} < 0|(a)^{m-1}(a^\dagger)^n a|0 > +n\frac{1}{\sqrt{mn}} < 0|(a)^{m-1}(a^\dagger)^{n-1}|0 > \\
&= n\frac{1}{\sqrt{m!n!}} < 0|(a)^{m-1}(a^\dagger)^{n-1}|0 >
\end{aligned}
$$

Continuing in this manner we find the end result will bo zero if $m \neq n$. However, if $m = n$ then the end result will be

$$
\begin{aligned}
< m|n > &= n(n-1) \times \cdots \times 2\frac{1}{n!} < 0|aa^\dagger|0 > \\
&= n(n-1) \times \cdots \times 2\frac{1}{n!} < 0|(a^\dagger a + 1)|0 > \\
&= n!\frac{1}{n!} \\
&= 1.
\end{aligned}
\tag{7.-43}
$$

□

Using the assumption that

$$f(za^\dagger)|0 >= |z >$$

where f can be Taylor expanded, and that

$$a|z >= z|z > \quad \text{and} \quad < z|z >= 1$$

prove that

$$|z> = e^{-|z|^2/2} e^{za^\dagger}|0> .$$

Proof:

Note

$$af(za^\dagger)|0> = a|z> = z|z> = [a, f(za^\dagger)]|0> .$$

Write

$$f(za^\dagger) = N\left(1 + c_1 za^\dagger + c_2 z^2 (a^\dagger)^2 + \ldots\right)$$

We will need the result,

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$$

which is easily proved by induction: First we prove it explicitly for $n = 2$:

$$\begin{aligned}
[a, (a^\dagger)^2] &= a(a^\dagger)^2 - (a^\dagger)^2 a \\
&= (aa^\dagger - a^\dagger a)a^\dagger + a^\dagger(aa^\dagger - a^\dagger a) \\
&= [a, a^\dagger]a^\dagger + a^\dagger[a, a^\dagger] \\
&= 2a^\dagger
\end{aligned}$$

Now assume $[a, (a^\dagger)^m] = m(a^\dagger)^{m-1}$ and then consider

$$\begin{aligned}
[a, (a^\dagger)^{m+1}] &= a(a^\dagger)^{m+1} - (a^\dagger)^{m+1} a \\
&= [a, a^\dagger](a^\dagger)^m + a^\dagger[a, (a^\dagger)^m] \\
&= (a^\dagger)^m + m(a^\dagger)^m \\
&= (m + 1)(a^\dagger)^m.
\end{aligned}$$

Now we can calculate the commutator,

$$\begin{aligned}
[a, f(za^\dagger)] &= N[a, 1 + c_1 za^\dagger + c_2 z^2(a^\dagger)^2 + c_3 z^3 (a^\dagger)^3 + c_4 z^4 (a^\dagger)^4 + \ldots] \\
&= N\left(c_1 z[a, a^\dagger] + c_2 z^2 [a, a^\dagger)^2] + c_3[a, (a^\dagger)^3] + \ldots\right) \\
&= Nz\left(c_1 + c_2 za^\dagger + 3c_3 z^2 (a^\dagger)^2 + 4c_4 z^3 (a^\dagger)^3 \ldots\right).
\end{aligned}$$

We now use $[a, f(za^\dagger)]|0>= zf(za^\dagger)|0>$:

$$Nz\left(c_1 + 2c_2 za^\dagger + 3c_3 z^2 (a^\dagger)^2 + 4c_4 z^3 (a^\dagger)^3 \ldots\right)|0>= \\ = zN\left(1 + c_1 za^\dagger + c_2 z^2 (a^\dagger)^2 + c_3 z^3 (a^\dagger)^3 + \ldots\right)|0>$$

and read off

$$c_1 = 1, \quad 2c_2 = c_1, \quad 3c_3 = c_2, \quad 4c_4 = c_3, \ldots$$

or

$$c_1 = 1, \quad c_2 = \frac{1}{2!}, \quad c_3 = \frac{1}{3!}, \quad c_4 = \frac{1}{4!}, \ldots$$

So we have $|z>= f(za^\dagger)|0>= Ne^{za^\dagger}|0>$. We now use $<z|z>= 1$ to find N,

$$\begin{aligned}
1 &= <z|z> \\
&= N^2 <0|e^{\overline{z}a} e^{za^\dagger}|0> \\
&= N^2 <0|\left(1 + a\overline{z} + \frac{1}{2!}a^2\overline{z}^2 + \ldots\right)\left(1 + za^\dagger + \frac{1}{2!}(a^\dagger)^2 + \ldots\right)|0> \\
&= N^2\left(<0| + \overline{z} <1| + \overline{z}^2\frac{1}{\sqrt{2!}} <2| + \ldots\right)\left(|0> + z|1> + z^2\frac{1}{\sqrt{2!}}|2> + \ldots\right) \\
&= N^2\left(1 + |z|^2 + \frac{|z|^4}{2!} + \ldots\right) \\
&= N^2 e^{|z|^2}
\end{aligned}$$

where we have used (7.14). We have $N = e^{-|z|^2/2}$.

□

Coherent states in position representation

Use

$$\hat{a}|z>= |z>$$

to find the wavefunction in the position represenation.

Proof:

Let $|X>$ and $|P>$ be the eigenstates of $\hat{X}$ and $\hat{P}$. We write

$$\hat{a}|z>=\left(\sqrt{\frac{\omega}{2\hbar}}\hat{X}+i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}\right)|z>=z|z>$$

where

$$z=\sqrt{\frac{\omega}{2\hbar}}X_0+i\frac{1}{\sqrt{2\hbar\omega}}P_0 \tag{7.-62}$$

and act on the right with a state $<X|$ and obtain

$$<X|\left(\sqrt{\frac{\omega}{2\hbar}}\hat{X}+i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}\right)|z>=z<X|z>$$

Since $\hat{X}|X>=X|X>$ and $\hat{P}=-i\hbar\frac{\partial}{\partial X}$, we have $(\hat{X}|X>)^\dagger=<X|\hat{X}^\dagger=<X|(\hat{X}=(X|X>)^\dagger=<X|X$ and $(i\hat{P}|X>)^\dagger=-i<X|\hat{P}^\dagger=-i<X|\hat{P}=(\hbar\frac{\partial}{\partial X}|X>)^\dagger=-\hbar\frac{\partial}{\partial X}<X|$. That is,

$$<X|\hat{X}=<X|X \qquad \text{and} \qquad <X|i\hat{P}=\hbar\frac{\partial}{\partial X}<X|,$$

from which we obtain a differential equation

$$\sqrt{\frac{\omega}{2\hbar}}<X|z>X+\frac{1}{\sqrt{2\hbar\omega}}\hbar\frac{\partial}{\partial X}<X|z>=z<X|z>$$

or

$$\frac{\partial}{\partial X}\psi_z(X)=-\frac{\omega}{\hbar}\left(X-\sqrt{\frac{2\hbar}{\omega}}z\right)\psi_z(X)$$

This has the solution

$$\psi_z(X)=\mathcal{N}\exp-\left\{\frac{\omega}{2\hbar}\left(X-\sqrt{\frac{2\hbar}{\omega}}z\right)^2\right\}$$

or upon subsituting (7.14),

$$\begin{aligned}
\psi_z(X) &= \mathcal{N}\exp -\left\{\frac{\omega}{2\hbar}\left(X - X_0 - i\frac{P_0}{\omega}\right)^2\right\} \\
&= \mathcal{N}\exp -\frac{\omega}{2\hbar}\left\{(X - X_0)^2 - 2i(X - X_0)\frac{P_0}{\omega} - \frac{P_0^2}{\omega^2}\right\} \\
&= \mathcal{N}\exp -\left\{\frac{\omega}{2\hbar}(X - X_0)^2 - i(X - X_0)\frac{P_0}{\hbar} - \frac{P_0^2}{2\hbar\omega}\right\} \\
&= \mathcal{N}e^{P_0^2/2\hbar\omega}e^{-i\frac{X_0P_0}{\hbar}}\exp -\left\{\frac{\omega}{2\hbar}(X - X_0)^2 - \frac{i}{\hbar}XP_0\right\}.
\end{aligned}$$

Normalisation requires

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}|\psi_z(X)|^2 dX \\
&= |\mathcal{N}|^2 e^{P_0^2/\hbar\omega}\int_{-\infty}^{\infty}\exp -\left\{\frac{\omega}{\hbar}(X - X_0)^2\right\}dX \\
&= |\mathcal{N}|^2 e^{P_0^2/\hbar\omega}\sqrt{\frac{\pi\hbar}{\omega}}.
\end{aligned}$$

We can choose $\mathcal{N}$ to be

$$\mathcal{N} = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-P_0^2/2\hbar\omega}e^{i\frac{X_0P_0}{\hbar}}$$

so that

$$\psi_z(X) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp -\left\{\frac{\omega}{2\hbar}(X - X_0)^2 - \frac{i}{\hbar}XP_0\right\}.$$

□

Prove coherent states saturate uncerntainty

Prove the states

$$|z>$$

which is the eigenvalue of $\hat{a}$ with

$$\hat{a}|z>= z|z>$$

saturate uncerntainty, i.e. we have

$$\Delta X \Delta P = \frac{\hbar}{2}.$$

For z we will have

$$z = \sqrt{\frac{\omega}{2\hbar}} X_0 + i \frac{1}{\sqrt{2\hbar\omega}} P_0$$

Proof:

Note, for example,

$$\begin{aligned} < z|\hat{X}|z> &= \int dX < z|X> X < X|z> \\ &= \int dX \psi_z^*(X) X \psi_z(X) \\ < X > . \end{aligned}$$

Therefore we can avoid integrals, instead just use $\hat{a}$ and $\hat{a}^\dagger$.

Recall that

$$\begin{aligned} (\Delta x)^2 &= <(x- < x >)^2 > = < x^2 > - < x >^2 \\ (\Delta p)^2 &= <(p- < p >)> = < p^2 > - < p >^2 . \end{aligned}$$

It is easier verified via the definitions

$$(\Delta X)^2 =< \hat{X}^2 > - < \hat{X} >^2, \qquad (\Delta P)^2 =< \hat{P}^2 > - < \hat{P} >^2$$

As

$$\hat{a} = \sqrt{\frac{\omega}{2\hbar}} \hat{X} + i \frac{1}{\sqrt{2\hbar\omega}} \hat{P}, \qquad \hat{a}^\dagger = \sqrt{\frac{\omega}{2\hbar}} \hat{X} - i \frac{1}{\sqrt{2\hbar\omega}} \hat{P}$$

we obtain

$$\begin{aligned}
\hat{X} &= \sqrt{\frac{\hbar}{2\omega}}(\hat{a}+\hat{a}^\dagger) \\
\hat{P} &= -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}-\hat{a}^\dagger).
\end{aligned}$$

Recall that

$$\hat{a}|z>= z|z>, \qquad <z|\hat{a}^\dagger = \overline{z} < z|.$$

First we find $< z|\hat{X}|z >$,

$$\begin{aligned}
< z|\hat{X}|z > &= \sqrt{\frac{\hbar}{2\omega}} < z|(\hat{a}+\hat{a}^\dagger)|z > \\
&= \sqrt{\frac{\hbar}{2\omega}}(z < z|z > +\overline{z} < z|z >) \\
&= \sqrt{\frac{\hbar}{2\omega}}(z+\overline{z}) \\
&= \sqrt{\frac{\hbar}{2\omega}}2\sqrt{\frac{\omega}{2\hbar}}X_0 = X_0.
\end{aligned}$$

Now consider $< z|\hat{P}|z >$,

$$\begin{aligned}
< z|\hat{P}|z > &= -i\sqrt{\frac{\hbar\omega}{2}} < z|(\hat{a}-\hat{a}^\dagger)|z > \\
&= -i\sqrt{\frac{\hbar\omega}{2}}(z < z|z > -\overline{z} < z|z >) \\
&= -i\sqrt{\frac{\hbar\omega}{2}}(z-\overline{z}) \\
&= -i\sqrt{\frac{\hbar\omega}{2}}2(i\frac{1}{\sqrt{2\hbar\omega}}P_0) \\
&= P_0.
\end{aligned}$$

Now let us compute $< \hat{X}^2 >$,

$$\begin{aligned}
< z|\hat{X}^2|z > &= \frac{\hbar}{2\omega} < z|(\hat{a}+\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega} < z|(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega} < z|(\hat{a}^2+2\hat{a}^\dagger\hat{a}+1+(\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega} \left(z^2+2\overline{z}z+1\overline{z}^2\right) \\
&= \frac{\hbar}{2\omega} \left((z+\overline{z})^2+1\right) \\
&= < z|\hat{X}|z >^2 + \frac{\hbar}{2\omega}
\end{aligned}$$

where we have used $[\hat{a}, \hat{a}^\dagger] = 1$. We can now write

$$\begin{aligned}
\Delta X &= \sqrt{< z|\hat{X}^2|z > - < z|\hat{X}|z >^2} \\
&= \sqrt{\frac{\hbar}{2\omega}}.
\end{aligned}$$

Now let us compute $< \hat{P}^2 >$,

$$\begin{aligned}
< z|\hat{P}^2|z > &= -\frac{\hbar\omega}{2} < z|(\hat{a}-\hat{a}^\dagger)^2|z > \\
&= -\frac{\hbar\omega}{2} < z|(\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+(\hat{a})^\dagger)^2|z > \\
&= -\frac{\hbar\omega}{2} < z|(\hat{a}^2-2\hat{a}^\dagger\hat{a}-1+(\hat{a}^\dagger)^2|z > \\
&= -\frac{\hbar\omega}{2} \left(z^2-2\overline{z}z+\overline{z}^2-1\right) \\
&= -\frac{\hbar\omega}{2} \left((z-\overline{z})^2-1\right) \\
&= < z|\hat{P}|z >^2 + \frac{\hbar\omega}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta P &= \sqrt{< z|\hat{P}^2|z > - < z|\hat{P}|z >^2} \\
&= \sqrt{\frac{\hbar\omega}{2}}.
\end{aligned}$$

For coherent states

$$\Delta X \Delta P = \sqrt{\frac{\hbar}{2\omega}}\sqrt{\frac{\hbar\omega}{2}} = \frac{\hbar}{2}.$$

therefore we have equality in the Heisenberg uncertainty relation.

□

Minimum uncertainty wavefunction

Directly derive the wavefunctions which minimises the Heisenberg uncertainty relations in configuration and momentum space respectively. They are given by

$$\psi_z(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\omega}{2\hbar}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, \quad (7.\text{-}100)$$

$$\psi_z(p) = \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp - \left\{ \frac{\hbar}{2\omega}\frac{(p - P_0)^2}{\hbar^2} + \frac{i}{\hbar} p X_0 \right\}. \quad (7.\text{-}99)$$

where

$$< \hat{X} >_{\Psi_z} = X_0, \quad \text{and} \quad < \hat{P} >_{\Psi_z} = P_0. \quad (7.\text{-}99)$$

Denote the fluctuation of an observable $\hat{O}$ by

$$\Delta_{\Psi_z}(\hat{O}) = \left(< \hat{O}^2 >_\Psi - < \hat{O} >^2_\Psi \right)^{\frac{1}{2}},$$

and find that

$$\Delta_{\Psi_z}(\hat{X})\Delta_{\Psi_z}(\hat{P}) = \frac{\hbar}{2}, \qquad \frac{\Delta_{\Psi_z}(\hat{P})}{\Delta_{\Psi_z}(\hat{X})} = \omega. \quad (7.\text{-}99)$$

Proof:

$$(\Delta x)^2 = < (x - < x >)^2 > = < x^2 > - < x >^2$$
$$(\Delta p)^2 = < (p - < p >) > = < p^2 > - < p >^2$$

Define

$$\alpha = x - < x >, \qquad \beta = -i\hbar\left(\frac{d}{dx} - \left\langle\frac{d}{dx}\right\rangle\right) \tag{7.-101}$$

then

$$\begin{aligned}(\Delta x)^2(\Delta p)^2 &= \int_{-\infty}^{\infty}\psi^*\alpha^2\psi dx\int_{-\infty}^{\infty}\psi^*\beta^2\psi dx\\ &= \int_{-\infty}^{\infty}(\alpha^\dagger\psi^*)(\alpha\psi)dx\int_{-\infty}^{\infty}(\beta^\dagger\psi^*)(\beta\psi)dx \qquad (7.\text{-}101)\end{aligned}$$

where for β we have performed integration by parts and used that the normalised wave-function vanishes at $x = \pm\infty$.

We have the iequality

$$\int\left|f - g\frac{\int fg^*dx}{\int |g|^2dx}\right|^2 dx \geq 0.$$

Equality holds only if $f = \gamma g$. Expanding gives

$$\begin{aligned}\int\left|f - g\frac{\int fg^*dx}{\int |g|^2dx}\right|^2 &= \int\left(f^* - g^*\frac{(\int fg^*dx)^*}{\int |g|^2dx}\right)\left(f - g\frac{\int fg^*dx}{\int |g|^2dx}\right)dx\\ &= \int |f|^2dx + \int |g|^2dx\frac{\left|\int fg^*dx\right|^2}{\left(\int |g|^2dx\right)^2} - 2\frac{\left|\int fg^*dx\right|^2}{\int |g|^2dx} \geq 0\end{aligned}$$

so

$$\int |f|^2dx\int |g|^2dx \geq |\int fg^*dx|^2$$

If we know replace f by $\alpha\psi$ and g by $\beta\psi$, (7.-101) becomes

$$(\Delta x)^2(\Delta p)^2 \geq |\int(\alpha^\dagger\psi^*)(\beta^\dagger\psi)dx|^2 = |\int\psi^*\alpha\beta\psi|^2. \tag{7.-103}$$

The last term can be wrtten

$$
\begin{aligned}
&\left|\int \psi^* \left[\frac{1}{2}(\alpha\beta - \beta\alpha) + \frac{1}{2}(\alpha\beta + \beta\alpha)\right] \psi dx\right|^2 \\
&= \frac{1}{4}\left|\int \psi^*(\alpha\beta - \beta\alpha)\psi dx\right|^2 + \frac{1}{4}\left|\int \psi^*(\alpha\beta + \beta\alpha)\psi dx\right|^2 \\
&\qquad + \frac{1}{2}Re\left(\int \psi^*(\alpha\beta - \beta\alpha)\psi dx \int \psi^*(\alpha\beta + \beta\alpha)\psi dx\right).
\end{aligned}
$$

The third term vanishes because $\int \psi^*(\alpha\beta - \beta\alpha)\psi dx \int \psi^*(\alpha\beta + \beta\alpha)\psi dx$ is purely imaginary. In deriving the Heisenberg uncertainty principle we note that the second term $\frac{1}{4}\left|\int \psi^*(\alpha\beta + \beta\alpha)\psi dx\right|^2$ is positive and write

$$
\begin{aligned}
(\Delta x)^2(\Delta p)^2 \geq \left|\int \psi^* \alpha\beta\psi dx\right|^2 &\geq \frac{1}{4}\left|\int \psi^*(\alpha\beta - \beta\alpha)\psi dx\right|^2 \\
&= \frac{1}{4}\left|\int \psi^*(i\hbar)\psi dx\right|^2 \\
&= \frac{\hbar^2}{4}.
\end{aligned}
$$

The minimum uncertainty product is obtained onlt when the two conditions are fulfilled:

$$
\begin{aligned}
\alpha\psi &= \gamma\beta\psi & (7.\text{-}108) \\
\int \psi^*(\alpha\beta + \beta\alpha)\psi dx &= 0. & (7.\text{-}107)
\end{aligned}
$$

Equations (7.-108) and (7.14) give the differential equation

$$
(x- < x >)\psi_0 = \gamma(-i\hbar\frac{d\psi_0}{dx} - < p > \psi_0)
$$

or

$$
\frac{d\psi_0}{dx} = \left[\frac{i}{\gamma\hbar}(x- < x >) + \frac{i < p >}{\hbar}\right]\psi_0
$$

which is readily integrated

$$
\int \frac{d\psi_0}{\psi_0} = \left[\frac{i}{\gamma\hbar}(x- < x >) + \frac{i < p >}{\hbar}\right] dx
$$

so that

$$\ln \psi_0 = \frac{i}{2\gamma\hbar}(x - < x >)^2 + \frac{i < p > x}{\hbar} - \ln \mathcal{N}$$

(where $\mathcal{N}$ is an arbitrary constant) or

$$\psi_0(x) = \mathcal{N} \exp\left[\frac{i}{2\gamma\hbar}(x - < x >)^2 + \frac{i < p > x}{\hbar}\right] \qquad (7.\text{-}107)$$

Equation (7.-107), using (7.-108), becomes

$$\begin{aligned}
\int \psi_0^*(\alpha\beta + \beta\alpha)\psi dx &= \int \left(\psi_0^* \frac{1}{\gamma}\alpha^2 + (\beta^\dagger \psi_0^*)(\alpha\psi_0)\right) dx \\
&= \int \left(\psi_0^* \frac{1}{\gamma}\alpha^2 + (\frac{1}{\gamma^*}\alpha^\dagger \psi_0^*)(\alpha\psi_0)\right) dx \\
&= \left(\frac{1}{\gamma} + \frac{1}{\gamma^*}\right) \int \psi_0^* \alpha^2 \psi_0 dx = 0 \qquad (7.\text{-}108)
\end{aligned}$$

which implies that γ is purely imaginary. Requiring the wavefunction (7.14) to be normalisable means γ is negative imaginary. We now obtain $\mathcal{N}$ from the mormalisation condition

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = 1,$$

that is

$$\begin{aligned}
1 &= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[\left(\frac{i}{2\gamma\hbar} - \frac{i}{2\gamma^*\hbar}\right)(x - < x >)^2\right] dx \\
&= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[-\frac{1}{|\gamma|\hbar}(x - < x >)^2\right] dx \\
&= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[-\frac{1}{|\gamma|\hbar}x^2\right] dx \\
&= \mathcal{N}^2 \sqrt{|\gamma|\hbar\pi}
\end{aligned}$$

so

$$\mathcal{N} = (|\gamma|\hbar\pi)^{-\frac{1}{4}}. \qquad (7.\text{-}112)$$

The value of γ can be found from

$$\int_{-\infty}^{\infty}(x- < x >)^2|\psi_0(x)|^2dx = (\Delta x)^2$$

that is

$$\begin{aligned}
(\Delta x)^2 &= \frac{1}{\sqrt{|\gamma|\hbar\pi}}\int_{-\infty}^{\infty}(x- < x >)^2\exp\left[-\frac{1}{|\gamma|\hbar}(x- < x >)^2\right]dx \\
&= \frac{1}{\sqrt{|\gamma|\hbar\pi}}\int_{-\infty}^{\infty}x^2\exp\left[-\frac{1}{|\gamma|\hbar}x^2\right]dx \\
&= \frac{1}{\sqrt{|\gamma|\hbar\pi}}\frac{1}{2}\sqrt{(|\gamma|\hbar)^3\pi} \\
&= \frac{1}{2}|\gamma|\hbar. \qquad (7.\text{-}114)
\end{aligned}$$

So the wavefunction is

$$\psi_0(x) = [2\pi(\Delta x)^2]^{-\frac{1}{4}}\exp\left[-\frac{(x- < x >)^2}{4(\Delta x)^2} + \frac{i < p > x}{\hbar}\right] \qquad (7.\text{-}114)$$

As the uncertainty priniple is saturated we have equality: $(\Delta x)(\Delta p) = \hbar/2$. As ω is defined by $\omega = \Delta p/\Delta x$,

$$\left(\Delta_{\Psi_z}(\hat{X})\right)^2 = \frac{\hbar}{2\omega}, \qquad \left(\Delta_{\Psi_z}(\hat{P})\right)^2 = \frac{\hbar\omega}{2}. \qquad (7.\text{-}114)$$

and the wavefunction can be written

$$\psi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{\omega}{2\hbar}(x- < x >)^2 + \frac{i < p > x}{\hbar}\right] \qquad (7.\text{-}114)$$

We now transform to the momentum representation

$$\begin{aligned}
\psi_0(p) &:= \frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty}e^{-ipx/\hbar}\psi_0(x)dx \\
&= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty}\exp\left[-\frac{\omega}{2\hbar}(x- < x >)^2 - \frac{i(p- < p >)x}{\hbar}\right]dx \\
&= \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}}\frac{1}{\hbar}\int_{-\infty}^{\infty}\exp\left[-\frac{\omega}{2\hbar}\left\{(x- < x >)^2 + \frac{i2(p- < p >)x}{\omega}\right\}\right]dx
\end{aligned}$$
$$(7.\text{-}116)$$

The contents of the curly brackets rearranged becomes

$$
\begin{aligned}
&(x- < x >)^2 + \frac{i2(p- < p >)x}{\omega} \\
&\quad = x^2 + 2\frac{- < x > \omega + i(p- < p >)}{\omega}x + < x >^2 \\
&\quad = \left(x + \frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 - \left(\frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 + < x >^2 \\
&\quad \mapsto x^2 - \left(\frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 + < x >^2 \\
&\quad = x^2 + \frac{(p- < p >)^2}{\omega^2} + 2\frac{i(p- < p >)}{\omega} < x > \qquad (7.\text{-}119)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \psi_0(p) \\
= \;& \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}\left\{x^2 + \frac{(p- < p >)^2}{\omega^2} + 2\frac{i(p- < p >)}{\omega} < x >\right\}\right] dx \\
= \;& e^{\frac{i<x><p>}{\hbar}} \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp\left[-\frac{\omega}{2\hbar}\frac{(p- < p >)^2}{\omega^2} - \frac{ip < x >}{\hbar}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}x^2\right] dx \\
= \;& e^{\frac{i<x><p>}{\hbar}} \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar}\sqrt{\left(\frac{2\pi\hbar}{\omega}\right)} \exp\left[-\frac{1}{2\hbar\omega}(p- < p >)^2 - \frac{ip < x >}{\hbar}\right] \\
= \;& e^{\frac{i<x><p>}{\hbar}} \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp\left[-\frac{\hbar}{2\omega}\frac{(p- < p >)^2}{\hbar^2} - \frac{ip < x >}{\hbar}\right] \qquad (7.\text{-}122)
\end{aligned}
$$

The norm in the momentum representation is defined by

$$
\int_{-\infty}^{\infty} |\psi_0(p)|^2 \frac{dp}{\hbar}
$$

Normalisation is guaranted by

$$
\begin{aligned}
1=\int_{-\infty}^{\infty}|\psi_0(x)|^2dx &= \int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty}e^{ipx/\hbar}\psi_0(p)dp\right|^2 dx\\
&= \frac{1}{2\pi\hbar^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{-ipx/\hbar}\psi_0^*(p)dp\right)\left(\int_{-\infty}^{\infty}e^{i\tilde{p}x/\hbar}\psi_0(\tilde{p})d\tilde{p}\right)dx\\
&= \frac{1}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\psi_0^*(p)\psi_0(\tilde{p})\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\tilde{p}-p)x/\hbar}\frac{dx}{\hbar}\right)\right]dpd\tilde{p}\\
&= \frac{1}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_0^*(p)\psi_0(\tilde{p})\delta(p-\tilde{p})dpd\tilde{p}\\
&= \int_{-\infty}^{\infty}|\psi_0(p)|^2\frac{dp}{\hbar}.
\end{aligned}
$$

$\square$

Overcompletenes

Let

$$
\hat{X}'=\sqrt{\frac{\omega}{2\hbar}}\hat{X} \qquad \text{and} \qquad \hat{P}'=\frac{1}{\sqrt{2\hbar\omega}}\hat{P}
$$

Prove

$$
z=\frac{1}{\sqrt{2}}(X'+iP'), \quad \overline{z}=\frac{1}{\sqrt{2}}(X'-iP')
$$

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z><z^*|=1.
$$

Proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z><z^*| &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}e^{-z\overline{z}}\sum_{n,n'}\frac{(z\hat{a}^\dagger)^n}{n!}|0><0|\frac{(\overline{z}\hat{a})^{n'}}{n'!}\\
&= \sum_{n,n'}(\hat{a}^\dagger)^n|0><0|(\hat{a})^{n'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}e^{-z\overline{z}}\frac{z^n\overline{z}^{n'}}{n!n'!}
\end{aligned}
$$

The integrals are easily performed by introducing “polar coordinates”

$$X' = \sqrt{2\rho}\cos\theta, \qquad P' = \sqrt{2\rho}\sin\theta$$

We use the volume element,

$$dX'dP' = \sqrt{2\rho}d\theta d\sqrt{2\rho} = d\theta d\rho$$

(see diagram) in the above integral. we have

$$z = \sqrt{\rho}e^{i\theta}, \qquad \overline{z} = \sqrt{\rho}e^{-i\theta}.$$

The integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}e^{-z\overline{z}}\frac{z^n\overline{z}^{n'}}{n!n'!}$$

becomes

$$\int_0^{\infty}\int_0^{2\pi}d\rho\frac{d\theta}{2\pi}e^{i(n-n')\theta}e^{-\rho}\frac{(\sqrt{\rho})^{n+n'}}{n!n'!}.$$

We have

$$\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i(n-n')\theta} = \delta_{nn'}$$

and

$$\begin{aligned}
\int_0^{\infty}d\rho\rho^n e^{-\rho} &= (-1)^n\frac{\partial^n}{\partial\alpha^n}\int_0^{\infty}d\rho e^{-\alpha\rho}\bigg|_{\alpha=1} \\
&= (-1)^n\frac{\partial^n}{\partial\alpha^n}\left(\frac{1}{\alpha}\right)\bigg|_{\alpha=1} \\
&= n!.
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z>< z^*| = \sum_n \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0><0|\frac{\hat{a}^n}{\sqrt{n!}}$$
$$= \sum_n |n><n|,$$

$\square$

Poisson summation theorem

Prove that if

$$\psi(y) = \sum_{\infty}^{\infty} f(y+ns)$$

then

$$\sum_{n=-\infty}^{\infty} f(y+ns) = \frac{2\pi}{s}\sum_{n=-\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{s}\right)\exp(2\pi i yn/s)$$

and

$$\sum_{\infty}^{\infty} f(ns) = \frac{2\pi}{s}\sum_{\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{s}\right)$$

where $\tilde{f}(k) := \int_{\mathbb{R}}\frac{dx}{2\pi}e^{-ik\cdot x}f(x)$.

Proof

For any integer m we have

$$\begin{aligned}\psi(y+ms) &= \sum_{n=-\infty}^{\infty} f(y+(n+m)s)\\ &= \sum_{n=\infty}^{\infty} f(y+ns)\\ &= \psi(y)\end{aligned}$$

so that $\psi(y)$ is periodic with period s. As such $\psi(y)$ can be written

$$\sum_{n=-\infty}^{\infty} f(y+ns) = \sum_{m=\infty}^{\infty} \alpha_m e^{im(2\pi/s)y}$$

where

$$\alpha_m = \frac{1}{s}\int_0^s \left(\sum_{n=-\infty}^{\infty} f(\tau+ns)\right) e^{-im(2\pi/s)\tau} d\tau$$

so that

$$\begin{aligned}
\sum_{n=\infty}^{\infty} f(y+ns) &= \sum_{m=\infty}^{\infty} \frac{1}{s}\int_0^s \left(\sum_{n=-\infty}^{\infty} f(\tau+ns)\right) e^{-im(2\pi/s)\tau} d\tau \; e^{im(2\pi/s)y} \\
&= \frac{1}{s}\sum_{m=-\infty}^{\infty} e^{im(2\pi/s)y} \sum_{n=-\infty}^{\infty} \int_0^s f(\tau+ns)e^{-im(2\pi/s)\tau} d\tau.
\end{aligned}$$

The sum over n may be rewritten as follows

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \int_0^s f(\tau+ns)e^{-im(2\pi/s)\tau} d\tau \\
= & \cdots + \int_0^s f(\tau-s)e^{-im(2\pi/s)\tau} d\tau + \int_0^s f(\tau)e^{-im(2\pi/s)\tau} d\tau + \int_0^s f(\tau+s)e^{-im(2\pi/s)\tau} d\tau + \ldots \\
= & \cdots + \int_{-s}^0 f(\tau)e^{-im(2\pi/s)(\tau+s)} d\tau + \int_0^s f(\tau)e^{-im(2\pi/s)\tau} d\tau + \int_s^{2s} f(\tau)e^{-im(2\pi/s)(\tau-s)} d\tau + \ldots \\
= & \sum_{n=-\infty}^{\infty} \int_{sn}^{s(n+1)} f(\tau)e^{-im(2\pi/s)(\tau-ns)} d\tau \\
= & \sum_{n=-\infty}^{\infty} \int_{sn}^{s(n+1)} f(\tau)e^{-im(2\pi/s)\tau} d\tau \\
= & \int_{-\infty}^{\infty} f(\tau)e^{-im(2\pi/s)\tau} d\tau.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f(y+ns) &= \frac{1}{s}\sum_{n=-\infty}^{\infty} e^{in(2\pi/s)y}\int_{-\infty}^{\infty} f(\tau)e^{-in(2\pi/s)\tau}d\tau \\
&= \frac{2\pi}{s}\sum_{n=-\infty}^{\infty} e^{in(2\pi/s)y}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-in(2\pi/s)\tau}d\tau \\
&= \frac{2\pi}{s}\sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{s}\right)\exp(2\pi iyn/s)
\end{aligned}$$

If, finally, we set $y = 0$, we obtain

$$\sum_{n=-\infty}^{\infty} f(ns) = \frac{2\pi}{s}\sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\tilde{f}\left(\frac{2\pi n}{s}\right).$$

□

Chapter 8

Covariant LQG and Spinfoams

8.1 Geometry

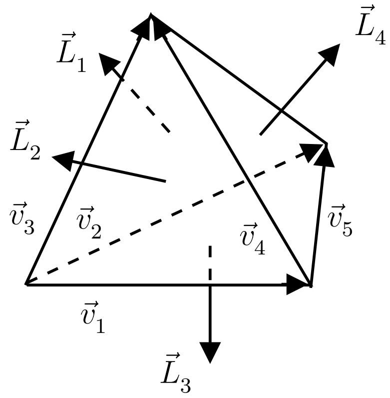


Figure 8.1: Four normals $\vec{L}_a$ to the faces.

The vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ define the tetrahedron. The vectors $\vec{v}_4$ and $\vec{v}_5$ can be given by

$$\begin{aligned} \vec{v}_4 &= \vec{v}_3 - \vec{v}_1 \\ \vec{v}_5 &= \vec{v}_2 - \vec{v}_1 \end{aligned} \tag{8.0}$$

$$\begin{aligned}
\vec{L}_1 &= -\frac{1}{2}\vec{v}_2 \times \vec{v}_3 \\
\vec{L}_2 &= -\frac{1}{2}\vec{v}_3 \times \vec{v}_1 \\
\vec{L}_3 &= -\frac{1}{2}\vec{v}_1 \times \vec{v}_2 \\
\vec{L}_4 &= -\frac{1}{2}\vec{v}_4 \times \vec{v}_5 \\
&= -\frac{1}{2}(\vec{v}_3 - \vec{v}_1) \times (\vec{v}_2 - \vec{v}_1) \qquad (8.\text{-}3)
\end{aligned}$$

(a) Closure.

$$\begin{aligned}
\vec{C} &:= \sum_{a=1}^{4} \vec{L}_a \\
&= -\frac{1}{2}\Big(\vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1 + \vec{v}_1 \times \vec{v}_2 \\
&\qquad + \vec{v}_3 \times \vec{v}_2 - \vec{v}_3 \times \vec{v}_1 - \vec{v}_1 \times \vec{v}_2\Big) \\
&= 0. \qquad (8.\text{-}5)
\end{aligned}$$

(b) (i) Areas.

The area of a triangle is

$$A = \frac{1}{2}|\vec{A} \times \vec{B}|$$

so the area of the $i - th$ face is

$$A_i = |\vec{L}_i|.$$

(ii) Volume.

In terms of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ the volume is

$$V = \frac{1}{6}\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \qquad (8.\text{-}5)$$

Consider the cross product

$$
\begin{aligned}
\vec{L}_1 \times \vec{L}_2 &= \frac{1}{4}(\vec{v}_2 \times \vec{v}_3) \times (\vec{v}_3 \times \vec{v}_1) \\
&= \frac{1}{4}[\vec{v}_3(\vec{v}_2 \times \vec{v}_3) \cdot \vec{v}_1 - \vec{v}_1(\vec{v}_2 \times \vec{v}_3) \cdot \vec{v}_3] \\
&= -\frac{3}{2}V\vec{v}_3 \qquad (8.\text{-}6)
\end{aligned}
$$

where we used the identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{C}(\vec{A} \cdot \vec{B}) - \vec{B}(\vec{A} \cdot \vec{C})$ and (8.1). Now

$$
\begin{aligned}
(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 &= \frac{3}{4}V\vec{v}_3 \cdot (\vec{v}_1 \times \vec{v}_2) \\
&= \frac{9}{2}V\frac{1}{6}\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \\
&= \frac{9}{2}V^2. \qquad (8.\text{-}7)
\end{aligned}
$$

So that

$$
V^2 = \frac{2}{9}(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 = \frac{2}{9}\epsilon_{ijk}L_1^i L_2^j L_3^k = \frac{2}{9}\det L. \qquad (8.\text{-}7)
$$

(iii) Angles between edges.

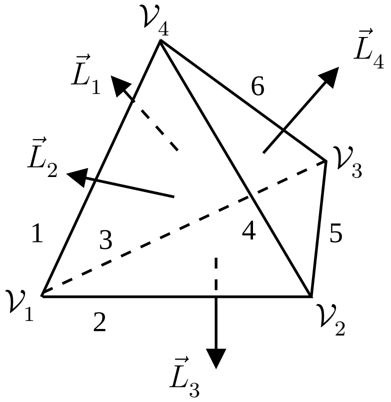


Figure 8.2: Edges.

As with (8.-6) we can derive

$$
\begin{aligned}
\vec{L}_1 \times \vec{L}_2 &= -\frac{3}{2} V \vec{v}_3 \\
\vec{L}_3 \times \vec{L}_1 &= -\frac{3}{2} V \vec{v}_2 \\
\vec{L}_2 \times \vec{L}_3 &= -\frac{3}{2} V \vec{v}_1
\end{aligned}
\tag{8.-8}
$$

Therefore we can write for the angles at vertex $\mathcal{V}_1$

$$
\begin{aligned}
\cos\theta_{12} &= \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{(\vec{L}_2 \times \vec{L}_3) \cdot (\vec{L}_3 \times \vec{L}_1)}{|\vec{L}_2 \times \vec{L}_3||\vec{L}_3 \times \vec{L}_1|} \\
\cos\theta_{23} &= \frac{\vec{v}_2 \cdot \vec{v}_3}{|\vec{v}_2||\vec{v}_3|} = \frac{(\vec{L}_3 \times \vec{L}_1) \cdot (\vec{L}_1 \times \vec{L}_2)}{|\vec{L}_3 \times \vec{L}_1||\vec{L}_1 \times \vec{L}_2|} \\
\cos\theta_{31} &= \frac{\vec{v}_3 \cdot \vec{v}_1}{|\vec{v}_3||\vec{v}_1|} = \frac{(\vec{L}_1 \times \vec{L}_2) \cdot (\vec{L}_2 \times \vec{L}_3)}{|\vec{L}_1 \times \vec{L}_2||\vec{L}_2 \times \vec{L}_3|}
\end{aligned}
\tag{8.-9}
$$

By symmetry at vertex $\mathcal{V}_2$ $(123 \to 254)$ we have

$$
\begin{aligned}
\cos\theta_{25} &= \frac{(\vec{L}_5 \times \vec{L}_4) \cdot (\vec{L}_4 \times \vec{L}_2)}{|\vec{L}_5 \times \vec{L}_4||\vec{L}_4 \times \vec{L}_2|} \\
\cos\theta_{54} &= \frac{(\vec{L}_4 \times \vec{L}_2) \cdot (\vec{L}_2 \times \vec{L}_5)}{|\vec{L}_4 \times \vec{L}_2||\vec{L}_2 \times \vec{L}_5|} \\
\cos\theta_{42} &= \frac{(\vec{L}_2 \times \vec{L}_5) \cdot (\vec{L}_5 \times \vec{L}_4)}{|\vec{L}_2 \times \vec{L}_5||\vec{L}_5 \times \vec{L}_4|}
\end{aligned}
\tag{8.-10}
$$

Similar results hold for vertices $\mathcal{V}_3$ and $\mathcal{V}_4$.

8.2 Three dimensional Quantum Gravity

8.3 Simplices in 2,3,4 Dimensions

We start with a point - this is a zero-simplex. A edge is a 1-simplex. A triangle is a 2-simplex. A tetrahedra is a 4-simplex.

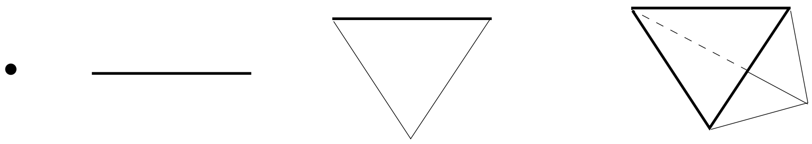


Figure 8.3: 0-simplex, 1-simplex, 2-simplex and 3-simplex.

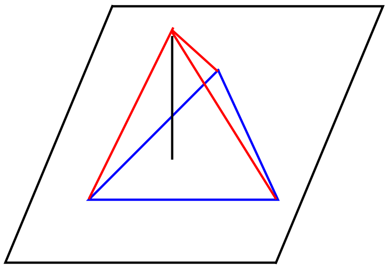


Figure 8.4: We cunstruct a 3-simplex by lifting the center of the 2-simplex.

8.3.1 A Crash Course in Simplexes and Complexes

Dimension of simplex	0	1	2	3	4
Number of vertices	1	2	3	4	5
Number of edges	0	1	3	6	10
Number of triangles	0	0	1	4	10
Number of tetrahedra	0	0	0	1	5

We construct a 3-simplex by lifting the center of the 2-simplex - see (8.4). This implies that the number of edges of the n-simplex, $\#e_n$, goes as $0+1+\cdots+(n-2)+(n-1)+n$ so is $(n+1)n/2$ or

$$\#e_n = \frac{(n+1)n}{2}. \tag{8.-10}$$

There are $n+1$ choices for the start point of an edge and n choices for the end point. Multiplying these numbers gives $(n+1)n$, but since this counts each edge twice, the total number of edges is $(n+1)n/2$, just as we found above.

Let us denote the number of triangles in an $n-$simplex as $\#t_n$. Then it is fairly easy to see that $\#t_n = \#t_{n-1}+$ [number of edges of the $(n-1)-$ simplex] or $\#t_n = \#t_{n-1} + \frac{n(n-1)}{2}$.So we need to solve:

$$\#t_n - \#t_{n-1} = \frac{n(n-1)}{2}.$$

We can use the trial solution $\#t_n = A(n+1)n(n-1)$,

$$\begin{aligned} A(n+1)n(n-1) - An(n-1)(n-2) &= An(n-1)[(n+1)-(n-2)] \\ &= 3An(n-1), \end{aligned}$$

so works if we choose $A = 1/6$. The number of triangles is

$$\#t_n = \frac{(n+1)n(n-1)}{6}, \tag{8.-12}$$

(note this obviously works for $n = 0$ and $n = 1$).

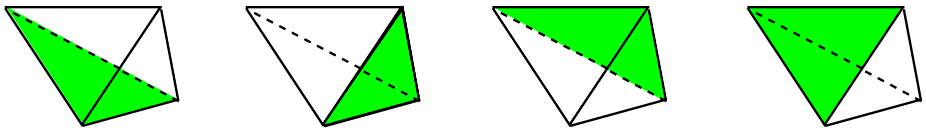


Figure 8.5: Structure of interaction vertex.

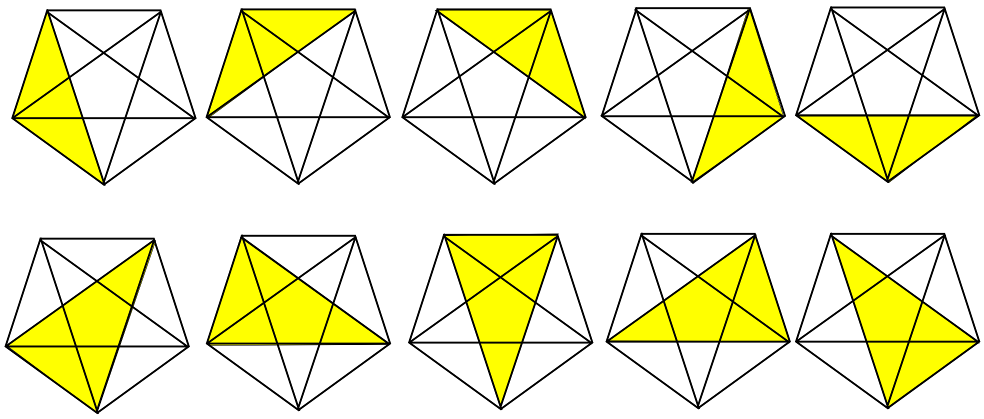


Figure 8.6: 4-simplex. All 10 faces of a 4-simplex.

Let us denote the number of tetrahedra in an $n-$simplex as $\#tet_n$. Then it is fairly easy to see that $\#tet_n = \#tet_{n-1}+$ [number of triangles of the $(n-1)-$ simplex] or $\#tet_n = \#tet_{n-1} + \frac{n(n-1)(n-2)}{6}$. So we need to solve:

$$\#tet_n - \#tet_{n-1} = \frac{n(n-1)(n-2)}{6}.$$

We can use the trial solution $\#t_n = A(n+1)n(n-1)(n-2)$,

$$
\begin{aligned}
A(n+1)n(n-1)(n-2) - An(n-1)(n-2)(n-3) \\
= An(n-1)(n-2)[(n+1)-(n-3)] \\
= 4An(n-1)(n-2),
\end{aligned}
$$

so works if we choose $A = 1/24$. The number of tetrahedra is

$$
\#tet_n = \frac{(n+1)n(n-1)(n-2)}{24}, \tag{8.-15}
$$

(note this obviously works for $n = 0$, $n = 1$ and $n = 1$).

In an n-dimensional simplex there are

$$
C(n,k) = \frac{(n+1)!}{(k+1)!(n-k)!} \tag{8.-15}
$$

$k-$simplicies. There is an easy derivation; an $n-$simplex is defined by a list of $n+1$ vertices. If we choose $k+1$ vertices out of the list of $n+1$ vertices they define a $k-$subsimplex. The number of ways of selecting $k+1$ vertices from a list of $n+1$ vertices is $\binom{n+1}{k+1}$.

8.3.2 Geometry of Simplicies

A 4-simplex in $\mathbb{R}^4$ is characterized by four vectors. These can be the vector pointing form one of the vertices. Let us denote these by e_a^I, where $I = 1, \ldots, 4$ is an index for a vector in $\mathbb{R}^4$, and $a = 1, \ldots, 4$ indecates a vertex at which the vector is directed. Thus, e_1^I is a vector pointing from the vertex (0) to the vertex (1).

Consider a tetrahedron. Its geometry is determined by the 3 displacement vectors from one vertex. Alternatively, it can be determined by a set of 4 bivectors E_i satifying the *closure constraint*

$$
E_0 + E_1 + E_2 + E_4 = 0 \tag{8.-15}
$$

These can be obtained for each triangle by taking the wedge product of the displacement vectors of the edges normal to then. The constraint says that the triangles close to form a tetrahedron.

Instead of vectors it is sometimes more convenuient to use the so-called bivectors E^{IJ} obtained by from two vectors $E_{ab}^{IJ} = e_a^{[I} e_b^{J]}$ where the brackets denote the antisymmetrization of the indices. These bivectors also characterize a 4-simplex and are obviously in one-to-one correspondance with faces of the 4-simplex h. For example, the bivector E^{IJ}_{12} corresponds to the face whose vertices are $(0),(1),(2)$. For the 4 vectors there correspond by 6 bivectors. The norm of each bivector is proportional to the squared area of the corresponding face

$$E^{IJ}_{ab} E_{abIJ} = 2A^2_{ab} \tag{8.-15}$$

where A_{ab} is the area of the face $(0),(a),(b)$ with no summation over a,b assumed. This is a norm with the geometric interpretation of the area of the face

$$\begin{aligned} |E_{ab}|^2 &= \frac{1}{2}(e^I_a e^J_b - e^J_a e^I_b)\frac{1}{2}(e_{aI}e_{bJ} - e_{aJ}e_{bI}) \\ &= \frac{1}{4}(e^I_a e_{aI}\; e^J_b e_{bJ} - e^I_a e_{bI}\; e^J_b e_{aJ} - e^J_a e_{bJ}\; e^I_b e_{aI} + e^J_a e_{aJ}\; e^I_b e_{bI}) \\ &= \frac{1}{2}(g_{aa}g_{bb} - g_{ab}g_{ab}) \end{aligned} \tag{8.-16}$$

$$|E_{ab}|^2 = g_{aa}g_{bb} - g_{ab}g_{ab} \equiv 2A^2_{ab} \tag{8.-16}$$

The volume of h can be obtained by wedging two bivectors that correspond to faces that do not share an edge. This is given by

$$V_h = \frac{1}{4!}\epsilon_{IJKL} E^{IJ}_{12} E^{KL}_{34}. \tag{8.-16}$$

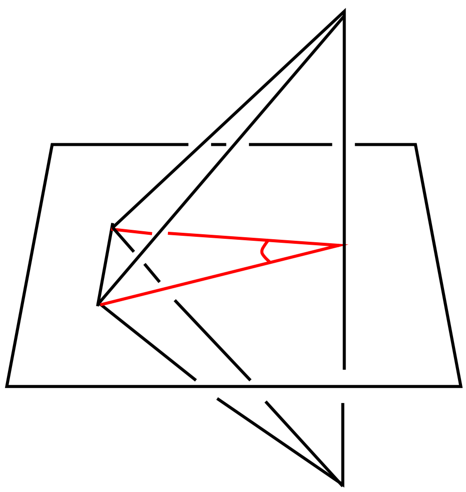


Figure 8.7: dihedralRov. The dihedral angle at the centre of one edge of a three-simplex.

3d Dihedral angles

$$E_{ab} \cdot E_{ac} = g_{aa} g_{bc} - g_{ab} g_{ac} = 2 J_{aabc} \tag{8.-16}$$

The angle between the infintesimal surface elements $dx^a dy^a$ and $dx^a dz^c$, if we take the scalar product of the normals of these two surface elements (in the 3-space they span),

$$A_{ab} A_{ac} \cos\theta_{aa\;bc} = g^{ef} (\epsilon_{egh} \delta^g_a \delta^h_b)(\epsilon_{fgh} \delta^g_a \delta^h_c) = g_{aa} g_{bc} - g_{ab} g_{ac} = 2 J_{aabc} \tag{8.-16}$$

8.3.3 The Dual Complex

In two dimensions the dual of a triangle is a point, and the dual of a edge is an edge.

In three dimension the dual of a tetrahedron is a point, the dual of a triangle is an edge, the dual of an edge is a triangle, and the dual of a point is a tetrahedron.

In four dimensions the dual of a 4-simplex is a point, the dual of a tetdrahedron is an edge, the dual of a face is a face, the dual of an edge is a tetdredron, and the dual of point is a 4-simplex.

Generally, in $d-$dimensions the dual of a $k-$simplex is a $(d-k)-$simplex.

The centres of the 4-simplices become the vertices of the dual complex. Connect the centres with an interval - these are the edges of the dual complex. The dual edges are in one-to-one correspondence with the tetrahedron t of the original simplicial decomposition.

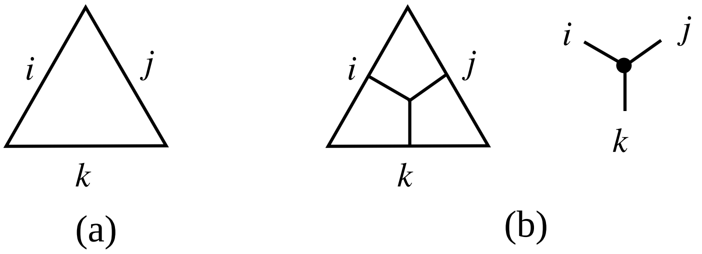


Figure 8.8: $1\triangle$

For a manifold $\mathcal{M}$ with an arbitrary cellular decomposition $\triangle$. There is a notion of the associated dual 2-complex of $\triangle$ denoted by $\triangle^*$. The dual 2-complex $\triangle^*$ is defined by a set of vertices $v \in \triangle^*$ (dual to 3-cells in $\triangle$) edges $e \in \triangle^*$ (dual to 2-cells in $\triangle$) and faces $f \in \triangle^*$ (dual to 1-cells in $\triangle$).

cellular decomposition $\triangle$ and associated dual 2-complex $\mathcal{F}_\triangle$.

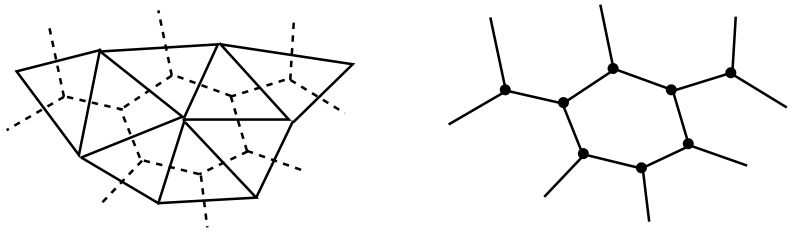


Figure 8.9: $2\triangle$

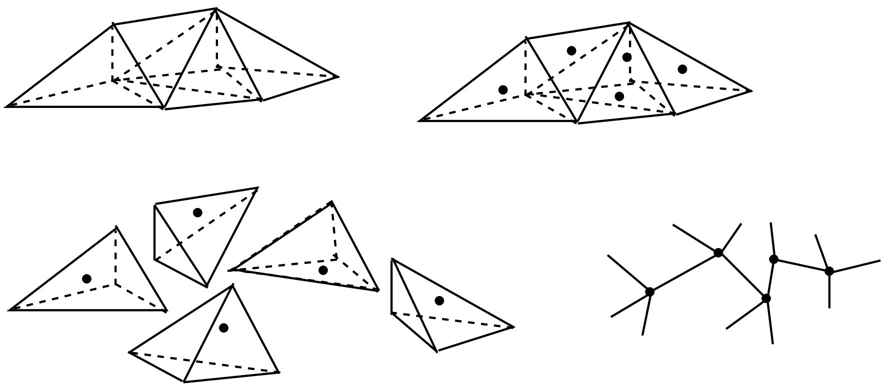


Figure 8.10: Part of a four-valent spin network corresponds to five tetrahedra of a cellular decomposition of 3-dimensional space.

vertices $v \in \mathcal{F}_\triangle$ (dual to 3-cells in $\triangle$)
edges $e \in \mathcal{F}_\triangle$ (dual to 2-cells in $\triangle$)
and faces $f \in \mathcal{F}_\triangle$ (dual to 1-cells in $\triangle$)

Each dual face f is goes inside, is surrounded by the triangles which all share the segment of $\triangle$ as a component of their boundary.

8.4 Barrett-Crane Model

Quantization is determined by the quantization of the topological field theory. As the constraints are non-derivative the gravitaional field has the same commutaion relations as the topological theory.

$$SO(4, \mathbb{C}) \tag{8.-16}$$

Topological field theory:

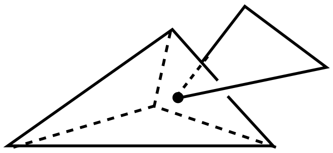


Figure 8.11: Structurex.

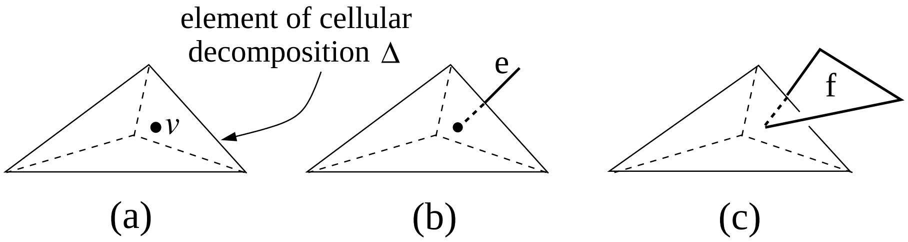


Figure 8.12: (a) $v \in \mathcal{F}_\triangle$ - dual to 3-cells in $\triangle$; (b) $e \in \mathcal{F}_\triangle$ - dual to 2-cells in $\triangle$; (c) faces $f \in \mathcal{F}_\triangle$ - dual to 1-cells in $\triangle$

$$S^{BF} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_i. \tag{8.-16}$$

No local degrees of freedom:

$$F^i = -\Lambda B^i, \qquad \mathcal{D} \wedge B^i = 0 \tag{8.-16}$$

We add a quadratic constraint

$$B^{(i} \wedge B^{j)} = \frac{1}{3} \delta^{ij} B^k \wedge B_k \tag{8.-16}$$

The result is general relativity:

$$S_{Plebanski} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_I - \frac{1}{2} \phi_{ij} B^i \wedge B^j \tag{8.-16}$$

One way to think of the affect of including the quadratic constraint is that it breaks the topological invariance.

Consider the action

$$S = \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2) \tag{8.-16}$$

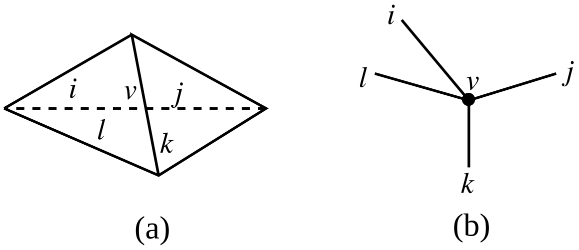


Figure 8.13: $\triangle$dual3

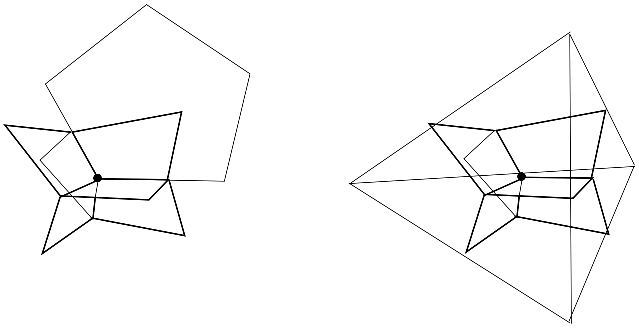


Figure 8.14: wedge.

which has two constrained degrees of freedom (q_1, q_2), which we assume to live on a circle, with conjugate momentum (p_1, p_2). This theory is completely constrained because both q_1 and q_2 must be zero. There are no degrees of freedom. Now let us impose another constraint with corresponding Lagrange multipier ξ. The action principle becomes

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2 + \xi(\lambda_1 - \lambda_2)) \tag{8.-16}$$

The two original Lagrange multipliers are constrained and we now have one degree of freedom: λ_1 has to be equal to λ_2 and thus only $q_1 + q_2$ has to be zero whereas the difference is free,

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1(q_1 + q_2)). \tag{8.-16}$$

This mimics the transition from BF-theory to gravity where additional constraints (the simplicity constraints) reduce the freedom the original Lagrange multipliers of the BF-theory and thereby introduce local degrees of freedom.

The $SO(4)$ (or $SO(3,1)$) BF theory

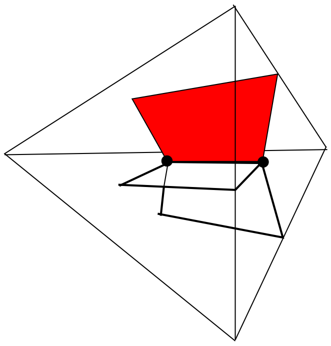


Figure 8.15: dualface3.

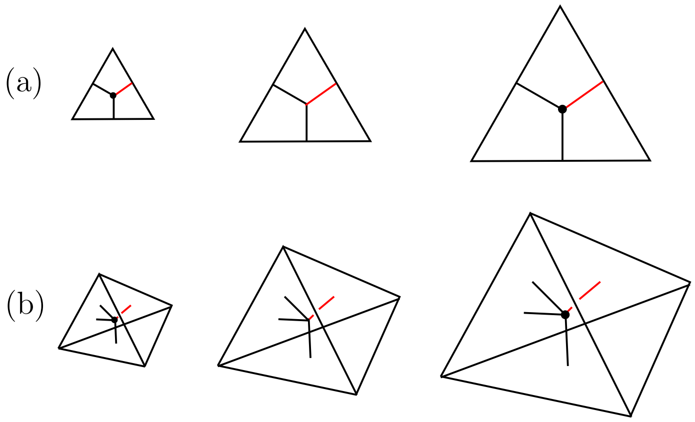


Figure 8.16: dualface3to4. (a) One dimension suppressed. Varing the normal coordinate sweeps out the 2-dimensional wedge in fig 8.15. (b) One dimension suppressed. Varing the normal coordinate sweeps out a 2-dimesional dual face corresponding to the dual edge from the centre of the 4-simplex to the centre of this boundary tetrahedron. There are 4 dual faces to the dual edge.

$$\int B^{IJ}_{\mu\nu} F(A)_{IJ\rho\sigma} \epsilon^{\mu\nu\rho\sigma} d^4x \tag{8.-16}$$

where $F(A)^{\mu\nu}_{IJ}$ is the curvature of an $SO(4)$ connection and $B^{IJ}_{\mu\nu}$ is a Lie-algebra two form field. The simplicity constraints

$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\nu\rho} \propto \epsilon_{\mu\nu\rho\sigma} \tag{8.-16}$$

The simplicty constraints ensures B comes from a tetrad field.

Sector I:

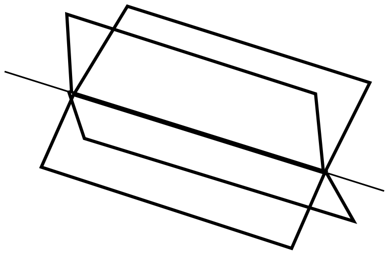


Figure 8.17: duale4duale. For a 4-simplex a dual edge e has four dual faces meeting at it.

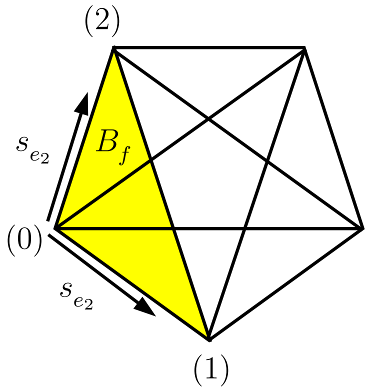


Figure 8.18: BCSim1. In the discretization of BF we assign a variable B_f to each triangle f of the triangulation. B_f can be taken to be the surface integral of B on f. We can discretize the gravitational field e, by assigning a variable e_s to each segment of the triangle. e_s can be taken to be the line integral of e along the segment s of the triangulation

$$B^{IJ}_{\mu\nu} = \pm e^I_\mu \wedge e^J_\nu \tag{8.-16}$$

or

Sector II:

$$B^{IJ}_{\mu\nu} = \pm \epsilon^{IJ}{}_{KL} e^K_\mu \wedge e^L_\nu . \tag{8.-16}$$

The case of sector II we obtain Einstein's theory and is referred to as the gravitational sector. In the case of sector I the corresponding thoery has no local degrees of freedom, this is the so-called topological sector.

8.4.1 The Partition Function

We approximate the B field with a distributional filed $B(t)$ with non-zero values on the triangles of the original triangulation. This gives an exact theory for a topological field theory like the BF one as there are no local degrees of freedom, but is only an approximation for gravity. However, this approximation would become better and better for more refined triangulations.

$$Z = \int \prod_t dg_t \prod_f \delta\left(\prod g_t\right), \tag{8.-16}$$

The $\delta-$functions impose the constraint that the holonomy around each dual face is trivial - flatness.

$$\delta\left(g\right) = \sum_{j \in irredreps} d_j tr[\rho_j(g)] \tag{8.-16}$$

where $tr[\rho_j(g)]$ is the character of g in the irreducible representation $\rho_j(g)$.

Simplicity constraints for 4-simplicies

Now we look into the details of the simplicity constraints for the 4-simplices which are the building elements of spinfoams.

(1) **Simplicity**. For each triangle Δ, the bivector B_Δ must satisfy

$$\epsilon_{IJKL} B^{IJ}_\Delta B^{KL}_\Delta = 0, \tag{8.-16}$$

i.e. it is simple.

(2) **Cross simplicity**. when the triangles Δ and Δ' belong to the same tetrahedron, that is, when they share a common edge, we have

$$\epsilon_{IJKL} B^{IJ}_\Delta B^{KL}_{\Delta'} = 0.$$

Which implies that the sum $B_\Delta + B_{\Delta'}$ of two bivectors is again simple

$$\epsilon_{IJKL}(B^{IJ}_\Delta + B^{IJ}_{\Delta'})(B^{KL}_\Delta + B^{KL}_{\Delta'}) = 0. \tag{8.-16}$$

(3)

Now we need to realize these conditions at the quantum level where the bivectors are represented by operators.

8.4.2 Hilbert Space of a Simplex

Quantizing a tetrahedron

$$\begin{aligned} \hat{E}_0^I &= J^I \otimes 1 \otimes 1 \otimes 1 \\ \hat{E}_1^I &= 1 \otimes J^I \otimes 1 \otimes 1 \\ \hat{E}_1^I &= 1 \otimes 1 \otimes J^I \otimes 1 \\ \hat{E}_1^I &= 1 \otimes 1 \otimes 1 \otimes J^I \end{aligned} \tag{8.-18}$$

with $I = 1, 2, 3$, and the closure constraint

$$\sum_i \hat{E}_i^I \psi = 0 \text{ for all } \psi \in \mathcal{H}^{\otimes 4}. \tag{8.-18}$$

This is a quantization of the condition that the bivectors associated to the faces of a tetrahedron must sum to zero.

Infact the closure constraint maintains the $SU(2)$ invariance of the state ψ: the tensor product the four spaces carries a reducible representation of $SU(2)$, that can be decomposed into its irreducible components. The spin zero component is the $SU(2)$ invariant part of the tensor product. The linear space of such states, satisfying the closure constraint, form the Hilbert space of a quantum tetrahedron denoted

$$\mathcal{H}_0 := Inv(j_0 \otimes j_1 \otimes j_2 \otimes j_3) \tag{8.-18}$$

This constraint ensures that the norms of these bivectors can be interpreted as areas of the faces of a geometric tetrahedron.

We can think of a spin network living in the dual complex of the triangulation. Each edge intersects each individual traingle of the tetrahedron labelled by an irreducible representation of $SU(2)$.

Thus a spin network completely characterizes a state of the quantum tetrahedron.

Quantizing a 4-simplex

In the quantum theory the variables B^{IJ}_Δ by $so(4)$ generators J^{IJ}_Δ.

The sum of four bivectors corresponding to the faces of each a tetrahedron of the 4-simplex is zero. This is imposed by summing over simple representations

$$\sum_{j\in irredreps} d_j tr[\rho_j(g)] \;\rightarrow \sum_{simple\ reps} d_j tr[\rho_j(g)] \tag{8.-18}$$

8.4.3 Simple Representations

A general element of the $so(4)$ algebra can be expressed by

$$\hat{X} = X^{IJ}\hat{J}_{IJ}, \tag{8.-18}$$

where $\hat{J}_{IJ}$ are generators of $so(4)$.

$$[J_{IJ}, J_{KL}] = i\delta_{IK}J_{JL} - i\delta_{JK}J_{IL} - i\delta_{IL}J_{JK} + i\delta_{JL}J_{IK}. \tag{8.-18}$$

There is a basis of generators, J_i^+, J_j^-, which form two commuting copies of $su(2)$,

$$[J_i^\pm, J_j^\pm] = i\epsilon_{ij}{}^k J_k^\pm, \quad [J_i^+, J_j^-] = 0. \tag{8.-18}$$

We pick a fixed 4-vector $n_0 = (1,0,0,0)$ and break X^{IJ} into "time-like", X^{0i}, and spacial parts, $X^i = (1/2)\epsilon^i{}_{jk}X^{ij}$. We can then define self dual and anti-self dual parts.

Given J_{IJ} we can form dual and self-dual parts

$$J_i^+ = \frac{1}{2}\Big(\frac{1}{2}\epsilon_i{}^{jk}J_{jk} + J_{0i}\Big)\ , \quad J_i^- = \frac{1}{2}\Big(\frac{1}{2}\epsilon_i{}^{jk}J_{jk} - J_{0i}\Big) \tag{8.-18}$$

First we find the commutators for J_i's and J_{0i}

$$\begin{aligned}
[J_i, J_{i'}] &= \frac{1}{4}\epsilon_i{}^{jk}\epsilon_{i'}{}^{j'k'}[J_{jk}, J_{j'k'}] \\
&= i\frac{1}{4}\epsilon_i{}^{jk}\epsilon_{i'}{}^{j'k'}(\delta_{jj'}J_{kk'} - \delta_{kj'}J_{jk'} - \delta_{jk'}J_{kj'} + \delta_{kk'}J_{jj'}) \\
&= i\frac{1}{4}\epsilon_i{}^{jk}\epsilon_{i'j}{}^{k'}J_{kk'} - \dots \\
&= iJ_{i'i}
\end{aligned} \tag{8.-20}$$

$$[J_i, J_{0i'}] = \frac{1}{2}\epsilon_i{}^{jk}[J_{jk}, J_{0i'}] = i\epsilon_{ii'}{}^k J_{k0}, \quad \text{and} \quad [J_{0i}, J_{0i'}] = iJ_{ii'} \tag{8.-19}$$

$$\begin{aligned}
[J_i^+, J_{i'}^+] &= \frac{1}{4}[J_i + J_{0i}, J_{i'} + J_{0i'}] \\
&= \frac{1}{4}([J_i, J_{i'}] + [J_i, J_{0i'}] + [J_{0i}, J_{i'}] + [J_{0i}, J_{0i'}]) \\
&= i\frac{1}{4}(J_{ii'} - J_{i'i} + 2i\epsilon_{ii'}{}^k J_{k0}) \\
&= i\frac{1}{2}\big(\frac{1}{2}\epsilon_{ii'}{}^k\epsilon_k{}^{lm} J_{lm} + i\epsilon_{ii'}{}^k J_{k0}\big) \\
&= i\frac{1}{2}\epsilon_{ii'}{}^k(J_k + J_{0k}) \\
&= \epsilon_{ii'}{}^k J_k^+
\end{aligned} \tag{8.-23}$$

Similarly we get for $J_i^- = (J_i - J_{0i})/2$

$$[J_i^-, J_j^-] = i\epsilon_{ij}{}^k J_k^-, \tag{8.-23}$$

Given a J_i^+ and J_i^- such that $[J_i^\pm, J_j^\pm] = i\epsilon_{ij}{}^k J_k^\pm$ and $[J_i^\pm, J_j^\mp] = 0$ then it can be verified that

$$J_{ij} = \epsilon_{ij}{}^k(J_k^+ + J_k^-), \qquad J_{0i} = J_i^+ - J_i^-. \tag{8.-23}$$

satisfies eq. (8.4.3). Thus there is a one to one relation between J_{IJ} and J_i^+, J_i^-. Thus the irreducible representations of $SO(4)$ have a one to one correspondance to two copies of $SU(2)$. (j^+, j^-). $so(4) \simeq su(2) \oplus su(2)$.

The number of generators of $so(4)$ is six as $J_{IJ} = -J_{JI}$. J_+ and J_- each span a three dimensional subalgebra of $so(4)$

$$\begin{aligned}
X^{IJ} J_{IJ} &= X^{k0} J_{k0} + X^{0k} J_{0k} + X^{ij} J_{ij} \\
&= (X^{k0} - X^{0k})(J_k^+ - J_k^-) + X^{ij}\epsilon_{ij}{}^k(J_k^+ + J_k^-) \\
&= (X^{k0} - X^{0k} + X^{ij}\epsilon_{ij}{}^k)J_k^+ + (X^{0k} - X^{k0} + X^{ij}\epsilon_{ij}{}^k)J_k^- \\
&= X^{+k}J_k^+ + X^{-k}J_k^-
\end{aligned} \tag{8.-25}$$

A group element U of $SO(4)$ can be written

$$
\begin{aligned}
U &= \exp(X^{IJ} J_{IJ}) \\
&= \exp(X^{+k} J_k^+ + X^{-k} J_k^-) \\
&= \exp(X^{+k} J_k^+) \exp(X^{-k} J_k^-) \\
&= U_+ U_-
\end{aligned} \tag{8.-27}
$$

$$
\begin{aligned}
UU' &= \exp(X^{IJ} J_{IJ}) \exp(Y^{IJ} J_{IJ}) \\
&= \exp(X^{+k} J_k^+ + X^{-k} J_k^-)\ \exp(X'^{+k} J_k^+ + X'^{-k} J_k^-) \\
&= [\exp(X^{+k} J_k^+) \exp(X'^{+k} J_k^+)]\ [\exp(X^{-k} J_k^-) \exp(X'^{-k} J_k^-)] \\
&= (U_+ U'_+)(U_- U'_-)
\end{aligned} \tag{8.-29}
$$

We see that $SO(4)$ can be put in the form of the direct product of two copies of $SU(2)$, $SO(4) \simeq SU(2) \otimes SU(2)$. In component form where indices in an $SO(4)$ representation are given by couples of indices in an $SU(2)$ representation

$$
U^{(aa')(bb')} = U_+^{ab}\, U_-^{a'b'} \tag{8.-29}
$$

are composite indices, with matrix multiplication

$$
U^{(aa')(bb')} U'^{(bb')(cc')} = (U_+^{ab} U_+'^{bc})(U_-^{a'b'} U_-'^{b'c'}) \tag{8.-29}
$$

that is

$$
UU' = U_+ U'_+ \otimes U_- U'_- \tag{8.-29}
$$

acting on a vector in components

$$
U^{(aa')(bb')} x_{(bb')} = (U_+^{ab} u_b)(U_-^{a'b'}\ v_{b'}) \tag{8.-29}
$$

and so

$$
U|x> = U(|u> \otimes |v>)\ = U_+|u> \otimes U_-\ |v> . \tag{8.-29}
$$

We introduce the notation

$$
U = \exp(X^{ij} J_{ij}) = \exp\left(X^{ij} \epsilon_{ij}{}^k (J_k^+ + J_k^-)\right) = (g_+, g_-) \tag{8.-29}
$$

$$(g_+, g_-)(g'_+, g'_-) = (g_+ g'_+, g_- g_-) \tag{8.-29}$$

For the case of $SO(4)$ we use the notation $\mathbf{g} = (g_+, g_-)$ for an $SO(4)$ group element, $\mathbf{j} = (j_+, j_-)$ for an $SO(4)$ irreducible representation, $d_{\mathbf{j}} = d_{j_+} d_{j_-}, d_j = 2j + 1$ for the corresponding dimension and $tr_{\mathbf{j}_f}(\mathbf{g}) = tr_{j_f^+}(g^+) tr_{j_f^-}(g^-)$ for the characters.

The simple representations

Casimir invariants:

The Lie group $SO(4)$ has two Casimir invariants: the scalar Casimir

$$C = J_{IJ} J^{IJ} = |J|^2 \tag{8.-29}$$

and the pseudo-scalar Casimir

$$\tilde{C} = \epsilon_{IJKL} J^{IJ} J^{KL}. \tag{8.-29}$$

Now say we impose the condition

$$\epsilon^{IJKL} J_{IJ} J_{KL} = 0 \tag{8.-29}$$

$$\begin{aligned}
\epsilon^{IJKL} J_{IJ} J_{KL} &= \epsilon^{0JKL} J_{0J} J_{KL} + \epsilon^{iJKL} J_{iJ} J_{KL} \\
&= \epsilon^{0jkl} J_{0j} J_{kl} + \epsilon^{iJ0L} J_{iJ} J_{0L} + \epsilon^{iJkL} J_{iJ} J_{kL} \\
&= \epsilon^{jkl} J_{0j} J_{kl} + \epsilon^{ij0k} J_{ij} J_{0k} + \epsilon^{iJjk0} J_{ij} J_{k0} + \epsilon^{i0kl} J_{i0} J_{kl}
\end{aligned} \tag{8.-30}$$

with $\epsilon^{ijk} := \epsilon^{0IJK}$

$$\begin{aligned}
\epsilon^{ijk} J_{0i} J_{jk} &= \epsilon^{ijk} (J_i^+ - J_i^-) \epsilon_{jk}{}^{l} (J_l^+ + J_l^-) \\
&= \delta^{il} (J_i^+ - J_i^-)(J_l^+ + J_l^-) \\
&= J^{+i} J_i^+ - J^{-i} J_i^- = 0
\end{aligned} \tag{8.-31}$$

in terms of the Casimir invariants of $SU(2)$ the condition (8.4.3) becomes

$$(J^+)^2 - (J^-)^2 = 0 \tag{8.-31}$$

This corresponds to a special representation of particular importance for which $j = j^+ = j^-$, this representation is called simple. Its algebra is that of the *diagonal* elements of $so(4)$ algebra: the ones of the form

$$X^i J_i^+ + X^i J_i^-. \tag{8.-31}$$

Exponentiating these we get the *diagonal* elements of the $SO(4)$ group for which we write (g, g). These digonal elements form an $SU(2)$ subgroup of $SO(4)$, which depends on n. It is the subgroup of $SO(4)$ that leaves the vector n invaraint.

Adjoint action of $SO(4)$ - inducing two $SU(2)$ transformations

$$\begin{aligned}
\mathbf{X}' &= X^{IJ}\, \mathbf{g} J_{IJ} \mathbf{g}^{-1} \\
&= \exp(iY^{KL} J_{KL})\; X^{IJ} J_{IJ}\; \exp(-iY^{IJ} J_{KL}) \\
&= \exp(iY^{+k} J_k^+ + iY^{-k} J_k^-)\; (X^i J_i^+ + X^i J_i^-)\; \exp(-iY^{+k} J_k^+ - iY^{-k} J_k^-) \\
&= X^i \exp(iY^{+k} J_k^+) J_i^+ \exp(-iY^{+k} J_k^+) \\
&\qquad + X^i \exp(iY^{-k} J_k^-) J_i^- \exp(-iY^{-k} J_k^-) \\
&= g^+ X^+ (g^+)^{-1} + g^- X^- (g^-)^{-1}
\end{aligned} \tag{8.-35}$$

So under the adjoint action of $SO(4)$ the self-dual and anti-self-dual elements transform as $SU(2)$ Lie algebra elements:

$$\mathbf{g}\mathbf{X}\mathbf{g}^{-1} = g^+ X^+ (g^+)^{-1} + g^- X^- (g^-)^{-1}. \tag{8.-35}$$

8.4.4 Quantizing a 4-Simplex

is of the form

$$\epsilon_{IJKL} J_\triangle^{IJ} J_\triangle^{KL} = (\vec{J}_\triangle^+)^2 - (\vec{J}_\triangle^-)^2 = 0. \tag{8.-35}$$

This means that the $so(4)$ representation (j^+, j^-) associated with each triangle $\triangle$ must carry the same spin on its self-dual and anti self-dual part,

$$j^+ = j^-.$$

The closure relation for each tetrahedron

see fig (8.19)

$$B_{AB} + B_{AC} + B_{AD} + B_{AE} = 0 \tag{8.-35}$$

for tetrahedron A, and so on. This is the discrete version of the Guass law ensuring the $SO(4)$ gauge invariance.

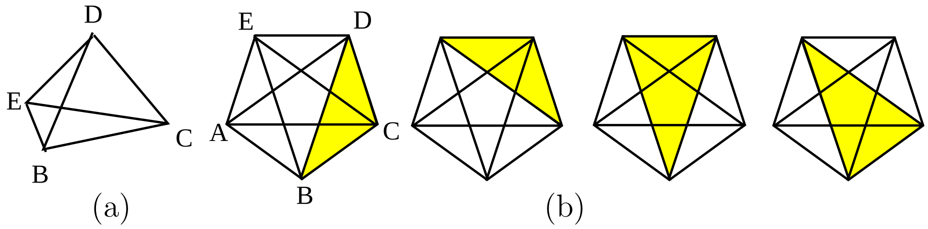


Figure 8.19: 4simclose. The closure condition for the tetrahedron A.

8.4.5 The Barrett-Crane Model

A tentative quantization of 4d Riemmanian GR on a fixed two-complex.

8.4.6 State Sum

How 6-j Symbols Appears:

Since every edge bounds three faces, each integral of the form

$$\int dg_{\sigma t}\, \rho^{j_1}(g_{\sigma t})^{\alpha}_{\alpha'}\, \rho^{j_2}(g_{\sigma t})^{\beta}_{\beta'}\, \rho^{j_3}(g_{\sigma t})^{\gamma}_{\gamma'} = v^{\alpha\beta\gamma\delta}\, v_{\alpha'\beta'\gamma'}. \tag{8.-35}$$

The closure condition $\sum_a J_a = 0$ means we are restricted to $so(4)$-invariant states in the tensor product $\mathcal{H} = \otimes_a \mathcal{H}_{(j_a,j_a)}, \ldots$

We use the standard recoupling basis of interwiners,

only simple representations will be including in the state sum

The interwiners are constrained by the simplicity constraints eq (??)-(??). Strongly imposing the cross-simplicity conditions $\epsilon_{IJKL} J_1 J_2 = 0$ forces the recoupled representation

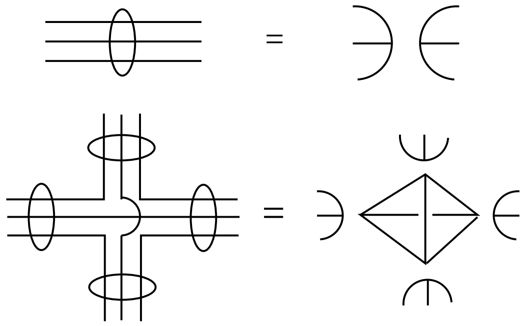


Figure 8.20: 6jappearsChD. How 6-j Symbols Appears.

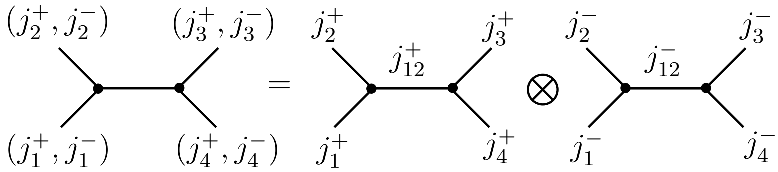


Figure 8.21: basisinter. The interwiner.

to be simple, $j_{12}^+ = j_{12}^-$. Imposing the othe cross-simplicity constraints $\epsilon_{IJKL} J_1 J_3 = 0$ and $\epsilon_{IJKL} J_1 J_4 = 0$ leads to a unqiue choice of interwiner, the Barrett-Crane interwiner.

$$i_{BC}^{(aa')(bb')(cc')(dd')} = \sum_j (2j+1)\ v^{abf} v^{fcd} v^{a'b'f'} v^{f'c'd'}, \tag{8.-35}$$

where indices in an $SO(4)$ representation are given by couples of indices in an $SU(2)$ representation, and the indices f and f' are in the representation j. This is the Barrett-Crane interwiner.

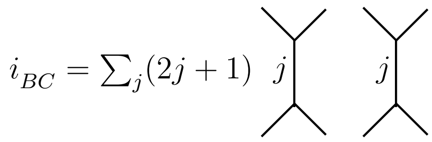


Figure 8.22: BCintdecom. The BC interwiner.

$$Z_{BC} = \sum_{simple\ j_f} \prod_f \dim(j_f) \prod_v \{10j\}. \tag{8.-35}$$

The set of two-complexes summed over is formed by a single two-complex. This is chosen to be the two-skeleton of the dual of a 4d triangulation.

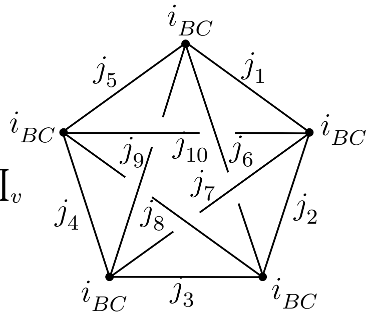
$Z_{BC} = \sum_{simple\, j_f} \prod_f \dim(j_f) \prod_v$ 

Figure 8.23: tenjsymbBC. Barrett-Crane partition function.

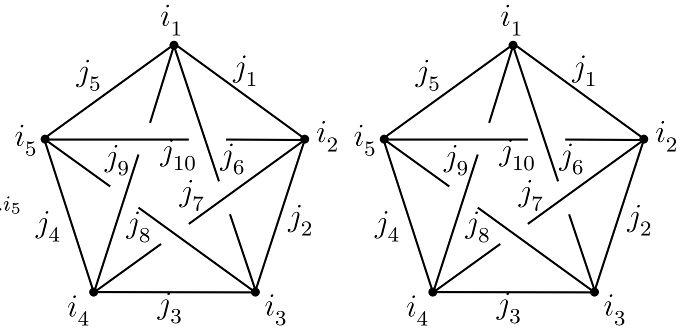
$A(j_1 \ldots j_{10}) = \sum_{i_1 \ldots i_5}$ 

Figure 8.24: tenjsymAmpl. This depends on ten spins and is called the $10j$ symbol. The vertex amplitude for the BC model.

$$1_{\mathcal{H}_0} = \sum |j_1, \ldots j_F, i_1 \ldots i_{F-3} >< j_1, \ldots j_F, i_1 \ldots i_{F-3}|. \tag{8.-35}$$

8.5 Lorentzian Barrett-Crane Model

opened up the way to the Lorentzian generalization [198] of the original Barrett-Crane model in Euclidean signature.

8.6 The Difficulties with Barrett-Crane Type Models

cast doubt on the physical correctness of the Barrett-Crane model.

8.6.1 Discretization Dependence

at odds with background independence

8.6.2 Boundary State Space

The Barrett-Crane model impliments the the constraints strongly at the quantum level identifying the boundary states. Although this bounadry state space is similar to, it does not exactly match, that of LQG. The Hilbert space of the boundary states arising from the Barrett-Crane model are too poor to allow the right tensorial structure for the graviton propagator in the semiclassical limit (see chapter 9).

leads to an over constrained Hilbert space, with not enough degrees of freedom to describe the 3-geometry. It is more natural to impose these constraints more weakly, using for instance coherent states.

8.6.3 Link to Canonical Approach

BC model imposes the constraints too, strongly and there is not a proper match with the states of the canonical theory (LQG) living on the boundary.

8.7 New Spinfoam Models for Quantum Gravity

The difficulties are related to the fact that in the BC model the interwiner quantum numbers are fully constrained. This follows from the fact that the simplicity constraints are imposed as strong operator equations,

$$C_n \psi = 0. \tag{8.-35}$$

However, these constraints are second class and the imposing of second class constraints strongly may lead to the incorrect elimination of physical degrees of freedom [Dirac]. As we will see the simplicity constraints can be imposed weakly,

$$< \phi C_n, \psi > = 0. \tag{8.-35}$$

i.e., the simplicity constraints hold at the level of expectation values. A familiar example is the Gupta-Bleuer quantization of Maxwell's field (see []??). If the Lorentz constraint $\partial_\mu A^\mu(x) = 0$ strongly as a quantum constraint

$$\partial_\mu \hat{A}^\mu | \psi > = 0$$

leads to the elimination of phsical degrees of freedom. However if only the positive frequency part is imposed strongly

$$\partial_\mu \hat{A}^{\mu+}|\psi>=0$$

We take this as the restriction on states which are allowed by the theory. This condition implies in its adjoint form

$$<\psi|\partial_\mu \hat{A}^{\mu-}=0$$

so that the Lorentz constraint is imposed weakly

$$<\phi|\partial_\mu(\hat{A}^{\mu+}+\hat{A}^{\mu+})|\psi>=<\phi|\partial_\mu \hat{A}^{\mu}|\psi>=0.$$

This ensures that the Lorenz condition and hence Maxwell's equations hold in the classical limit of the theory.

8.8 EPR

Acheivments of this approach

(i) the geometric interpretation for all the variables become fully transparent;

(ii) the boundary states fully capture the gravitational field variables;

and

(iii) correspond precisely to the spin network states of $SO(3)$ LQG. The boundary states of the model are precisely the eigenstates of the same quantities as the corresponding LQG states. It provides a novel independent derivation of LQG kinematics, in particular of the quantization of area and volume.

(iv) the vertex of this theory is similar to the BC vertex, but the corresponding dynamics may have a better low-energy behaviour and yield the correct low-enery (graviton) $n-$point functions (see chapter 9).

8.8.1 Outline of Derivation

We discretize GR via Regge and quantize.

Constraints on the Hilbert space associated to a bounadry of the triagulation.

Dynamics by giving the amplitude associated to each 4-simplex.

8.8.2 Area

$$A_f^2 = \frac{1}{2} B_f^{ij}\ B_{f\ ij} \tag{8.-35}$$

Writing this in terms of the momenta J gives

$$\begin{aligned}
A^2 &= \frac{1}{2} B_f^{ij}\ B_{f\ ij} \\
&= \frac{8\pi\hbar G\gamma}{\gamma^2 - 1}(J^{ij} - \frac{1}{\gamma}\ {}^*J^{ij})\ \frac{8\pi\hbar G\gamma}{\gamma^2 - 1}(J_{ij} - \frac{1}{\gamma}\ {}^*J_{ij}) \\
&= \left(\frac{8\pi\hbar G\gamma}{\gamma^2 - 1}\right)^2 \left(\frac{1}{2}C_1 - \left(1 - \frac{1}{\gamma^2}\right) L^2 - \frac{1}{2\gamma}C_2\right) \\
&= \left(\frac{8\pi\hbar G\gamma}{\gamma^2 - 1}\right)^2 \left(\frac{1}{2}\frac{\gamma}{2}\left(1 + \frac{1}{\gamma^2}\right) - \left(1 - \frac{1}{\gamma^2}\right) - \frac{1}{\gamma}\right) L^2 \\
&= (8\pi\hbar G\gamma)^2 \gamma^2 L^2
\end{aligned} \tag{8.-38}$$

That is

$$A = 8\pi\hbar G\gamma\sqrt{k(k+1)} \tag{8.-38}$$

8.9 Freidel, Krasnov

8.9.1 New Geometric Interpretation of Simplicity Constraints

Lemma 8.9.1 *A bivector X^{IJ} in $\mathbb{R}^4$ is an anti-symmetrized product of two vectors if and only if there exists a vector n^I such that $X^{IJ} n_J = 0$.*

Proof. First we prove that if a bivector is simple it defines a two-plane. To do this, pick a direction such that X^{IJ} in its orthogonal basis vectors components of X^{IJ} in are is X^{0i} then in this subspace the condition $\epsilon_{IJKL} X^{IJ} X^{KL} = 0$ becomes

$$X^{0j}(\epsilon_{jkl} X^{kl}) \ = \ X^{0j} \tilde{X}_j = 0, \tag{8.-37}$$

A rotation about the vector X^{0i} produces vectors spanning a two-plane.

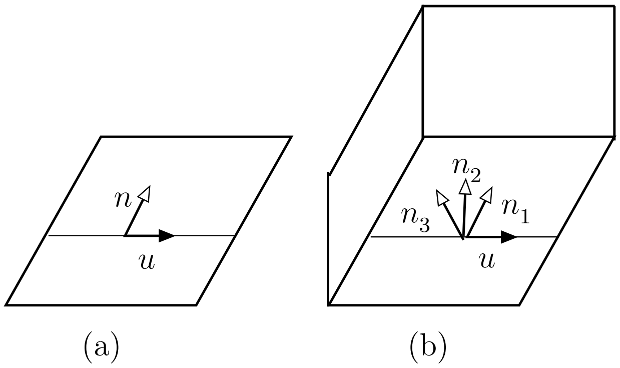


Figure 8.25: newgeom4. (a) In two dimensions there is only one independent norm $\vec{n}$ to the line with tangent $\vec{u}$. (b) In three dimensions there are many normals to $\vec{u}$ spaning a plane.

Now we move on to proving the lemma. A simple bivector defines a two-plane in $\mathbb{R}^4$. In $\mathbb{R}^4$ there is more than one normal vector - see fig.(8.25). Taking n^I to be any of the vectors orthogonal to this plane proves the only if case.

Now assume there exists a vector n^I such that $X^{IJ}n_J = 0$. Take the vector $n_0 = (1, 0, 0, 0)$ then $X^{0i} = 0$. The spacial part of the bivector, $X_i = \frac{1}{2}\epsilon_{ijk}X^{ij}$, is orthogonal to n^0. Now any vector in three dimensions can be written as the vector product of two 3-vectors $\vec{u}$ and $\vec{v}$, $X_i = \epsilon_{ijk}u_j v_k$. Contracting this with ϵ_{ijk} gives

$$X_{jk} = u_{[j}v_{k]}. \tag{8.-37}$$

Defining the 4-vectors $u = (0, \vec{u})$ and $v = (0, \vec{v})$, eq.(8.9.1) and $X^{i0} = 0$ can be written

$$X^{IJ} = u^{[I}v^{J]}. \tag{8.-37}$$

As this it is expressed as a 4-tensor equation it holds in general. This proves the if case.

An alternative proof that X^{IJ} is simple is based on the following. In three diemsions it is a straightforward fact that any bivector can always be written in the form $u^{[i}v^{j]}$ where u and v are vectors, visualized as a parallelogram. In dimensions greater than three the sum of two bivectors is not always expressible in the form $u^{[I}v^{J]}$. In general they are expressed as

$$u_1^{[I}v_1^{J]} + u_2^{[I}v_2^{J]}, \tag{8.-37}$$

which can be visualised a pair of parallelograms. Moreover, there always exists four vectors u_1, u_2, v_1, v_2 such that a bivector can be written in this form and u_1, u_2, v_1, v_2 span $\mathbb{R}^4$ forming a vector basis.

So assume that X^{IJ} is not simple. It can always be expressed as in (8.9.1) where u_1, u_2, v_1, v_2 can be taken as a basis. The vector n^I can be represented as a linear combination these basis vectors. It is then easy to see that the condition $X^{IJ} n_j = 0$ implies the four vectors u_1, u_2, v_1, v_2 are linearly dependent contradicting the assumption that X^{IJ} is not simple.

□

Lemma 8.9.2 *Two simple bivectors X_1^{IJ} and X_2^{IJ} span three-dimensional subspace of $\mathbb{R}^4$ if and only if there exists a vector n^I such that $X_1^{IJ} n_J = 0$ and $X_2^{IJ} n_J = 0$.*

Proof. If two bivectors are simple and the two-planes defined by them span a three-dimensional subspace, then n^I can be chosen to be a vector orthogonal to this subspace. This proves the only if case.

□

Lemma 8.9.3 *A bivector X_1^{IJ} in $\mathbb{R}^4$ is an antisymmetrized product of two vectors if and only if there exists a vector such that $\tilde{X}^{IJ} n_J = 0$.*

Proof. We need to show a bivector X^{IJ} is simple if and only if its dual bivector $\tilde{X}^{IJ}$ is simple. We know

$$4! \delta_{I_1}^{[J_1} \delta_{I_2}^{J_2} \delta_{I_3}^{J_3} \delta_{I_4}^{J_4]} = \epsilon^{J_1 J_2 J_3 J_4} \epsilon_{I_1 I_2 I_3 I_4}.$$

We use this to write

$$\begin{aligned} n_{[I_1} n_{I_2} n_{I_3} n_{I_4]} &= n_{[J_1} n_{J_2} n_{J_3} n_{J_4]} \delta_{I_1}^{[J_1} \cdots \delta_{I_4}^{J_4]} \\ &= \left(\frac{1}{4!} n_{[J_1} n_{J_2} n_{J_3} n_{J_4]} \epsilon^{J_1 J_2 J_3 J_4} \right) \epsilon_{I_1 I_2 I_3 I_4} \\ &= \alpha \epsilon_{I_1 I_2 I_3 I_4}. \end{aligned} \tag{8.-38}$$

Now say $X^{IJ} n_J = 0$ then

$$\tilde{X}_{IJ} n^J = \frac{1}{2} \epsilon_{IJKL} X^{KL} n^J = \frac{1}{2\alpha} n_{[I} n_J n_K n_{L]} X^{KL} n^J = 0.$$

As the dual of $\tilde{X}^{IJ}$ is X^{IJ},

$$
\begin{aligned}
\tilde{\tilde{X}}^{IJ} &= \frac{1}{2}\epsilon^{IJ}{}_{KL}\tilde{X}^{KL} \\
&= \frac{1}{2}\epsilon^{IJ}{}_{KL}(\frac{1}{2}\epsilon^{KL}{}_{MN}X^{MN}) \\
&= \frac{1}{2}(\delta^I_M\delta^J_N - \delta^I_N\delta^J_M)X^{MN} \\
&= X^{IJ},
\end{aligned}
\tag{8.-40}
$$

the converse is also true. Then this lemma follows from lemma 8.9.1.

□

Lemma 8.9.4 *Two simple bivectors X_1^{IJ} and X_2^{IJ} span a three-dimensional subspace of $\mathbb{R}^4$ if and only if there exists a vector such that $\tilde{X}_1^{IJ}n_J = 0$ and $\tilde{X}_2^{IJ}n_J = 0$.*

Proof. Two simple bivectors $X_{1,2}^{IJ}$ span a three-dimensional subspace if and only if their duals $\tilde{X}_{1,2}^{IJ}$ do, so this lemma follows from lemma.8.9.2.

□

Self- and anti-self-dual decomposition

$$
X^{\pm J}(X) = \frac{1}{2}\left(n_I\tilde{X}^{IJ} \pm \sqrt{\sigma}N_I X^{IJ}\right). \tag{8.-40}
$$

$$
\begin{aligned}
X^{IJ}n_J = 0 \;&\Leftrightarrow\; X^{+J} = X^{-J} = n_I\tilde{X}^{IJ} \\
\epsilon^{IJKL}n_J X^{KL} = 0 \;&\Leftrightarrow\; X^{+J} = -X^{-J} = n_I X^{IJ}.
\end{aligned}
\tag{8.-40}
$$

The graviataional sector the X_f^{IJ} is the dual of the area bivectors A_f^{IJ}

$$
X_f^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}A_f^{IJ}. \tag{8.-40}
$$

there exists a unit vector n_t such that $n_{t1}\tilde{X}_f^{IJ} = 0$ There exists an $SO(4)$ transformation that brings the vector n_t^I

$$
(X_f, -X_f) \to (g^+X_f(g^+)^{-1}, -g^-X_f(g^-)^{-1}). \tag{8.-40}
$$

the simplicity-intersection constraints on X_f for the gravitational sector can be expressed as

$$\mathbf{X}_f = (X_f, -n_t^{-1} X_f n_t). \tag{8.-40}$$

8.9.2 Coherent States in the Spin Foam Model

8.9.3 Coherent States

Coherent states for $SU(2)$

$$1_j = \sum_m |j, m >< j, m|, \tag{8.-40}$$

$$\delta_{mm'} = d_j \int_{SU(2)} dg \; t^j_{mj}(g) \overline{t^j_{m'j}(g)} \tag{8.-40}$$

$t^j_{mj}(g)$ and $t^j_{mj}(gh)$ differ only by a phase for any group element h from the $U(1)$ subgroup of $SU(2)$. The $U(1)$ subgroup being of the form

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}. \tag{8.-40}$$

$$1_j = d_j \int_{G/H} dn |j, n >< j, n|. \tag{8.-40}$$

$$< j, n|\hat{J}^i|j, n > \sigma_i = jn\sigma_3 n^{-1} \equiv jn^i \sigma_i \tag{8.-40}$$

or

$$< j, \hat{n}|\hat{J}^i|j, \hat{n} > = < j, j|\hat{J}^{i\prime}|j, j > \tag{8.-40}$$

where $J^{i\prime} = g(\hat{n})^{-1} J^i g(\hat{n})$ is the rotated generator.

Thus the state $|j, n >$ describes a vector in $\mathbb{R}^3$ of length j and of direction...

$$\Delta J^2 = j + j^2 - m^2 \tag{8.-40}$$

$|j,\hat{n}> = g(\hat{n})|j,j>$, where $\hat{n}$ is a unit vector defining a direction on the sphere S^2 and $g(\hat{n})$ an $SU(2)$ group element rotating the direction $\hat{z} \equiv (0,0,1)$ into the direction $\hat{n}$.

Just as $|j,j>$ has direction z with minimal uncertainty, $|j,\hat{n}>$ has direction $\hat{n}$ with minimal uncertainty.

Thus, the highest and lowest states $m = \pm j$ minimize the uncertainty relation correspond to cohernt states.

$$|j,j> \quad \text{and} \quad |j,-j>$$

Coherent states for $SO(4)$

$$|j,j> \otimes |j,j> \tag{8.-40}$$

for the gravitational sector

$$|j,g> \otimes \overline{|j,g>} \tag{8.-40}$$

8.9.4 Partition Function

$$Z = \sum_{j_f} \prod_f d_{j_f} \int \prod_{(t,\sigma)} dg_{t\sigma} \prod_f \left(\prod_\sigma g_{\sigma t} g_{t\sigma'} \right). \tag{8.-40}$$

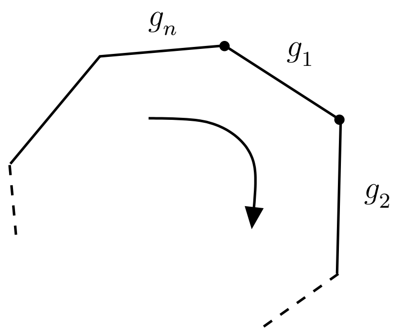


Figure 8.26: dualfacehol. .

We insert the decomposition of unity

$$1_j = d_j \int dn |j,n>< j,n|, \quad |j,n> \equiv |j_+, n_- > \otimes |j_-, n_- > . \tag{8.-40}$$

For example (see fig 8.30) one term is

$$\int dn_{tf} \int dn_{t'f} \ldots |j_f, n_{tf} >< j_f, n_{tf}| \, (g_{\sigma t})^{-1} g_{t\sigma'} \, |j_f, n_{t'f} >< j_f, n_{t'f}| \ldots \qquad (8.\text{-}40)$$

The partition function becomes

$$Z = \sum_{j_f} \prod_f d_{j_f} \int \prod_{(t,\sigma)} dg_{t\sigma} \prod_{t,f} d_{j_f} dn_{tf} \prod_{\sigma,f} < j_f, n_{tf}| \, (g_{\sigma t})^{-1} g_{\sigma t'} \, |j_f, n_{t'f}| > . \qquad (8.\text{-}40)$$

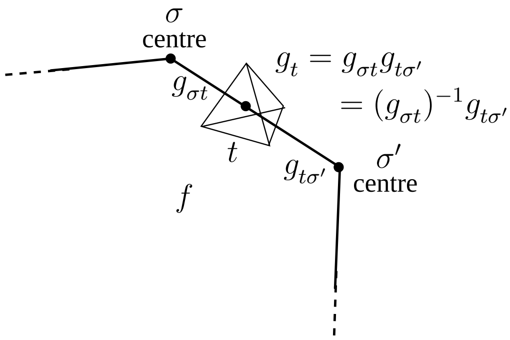


Figure 8.27: coherIncer. Schematic. .

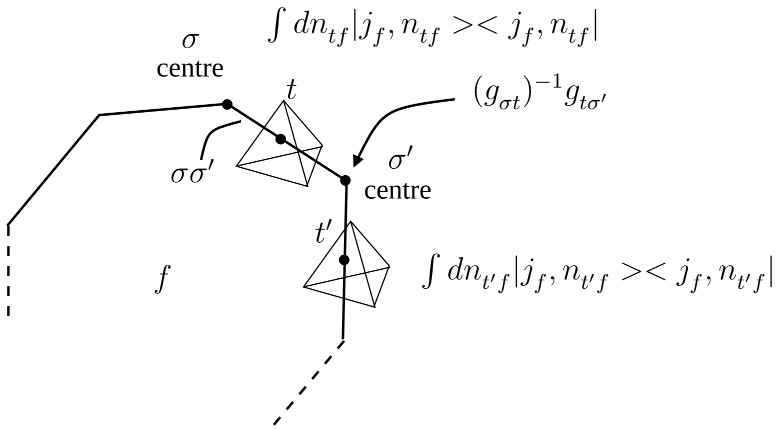


Figure 8.28: coherIncer2. Schematic. Have an edge $\sigma\sigma'$ in one-to-one correspondence with the common tetrahedra t. There is an insertion of a complete set of coherent states for each tetrahedron.

8.9.5 Imposing the Simplicty Constraints

it is possible to associate to each state $|\mathbf{j}_f, \mathbf{n}_{tf} >$ a bivector

$$\mathbf{X}_{(\mathbf{j}_f,\mathbf{n}_{tf})} = <\mathbf{j}_f,\mathbf{n}_{tf}|\hat{\mathbf{X}}|\mathbf{j}_f,\mathbf{n}_{tf}> \tag{8.-40}$$

where $\hat{\mathbf{X}}$ denotes an $SO(4)$ Lie algebra element.

demanding that these bivectors satisfying the geometrical simplicity and cross simplicity conditions is equivalent to the requirement that there exists an $SU(2)$ rgoup element u_t such that

$$\mathbf{X}_{(\mathbf{j}_f,\mathbf{n}_{tf})} = (X_f, -u_t X_f u_t^{-1}). \tag{8.-40}$$

rewrite the simplicity constraints in terms of the spins j_f and the $SU(2)$ elements n_f as:

$$j_f^+ = j_f^-, \qquad (n_{tf}^+, n_{tf}^-) = (n_{tf} h_{\phi tf}, u_t n_{tf} h_{\phi tf}^{-1} \epsilon) \tag{8.-40}$$

where $\phi_{tf} \in [0, 2\pi]$.

8.9.6 The Error in the way the Constraints were Imposed in the BC Model

For a triangulation of a 4-dimensional manifold there are exactly 4 dual faces that share each paricular dual edge (see fig 8.16), and thus each group element $g_{t\sigma}$ enters into 4 characters. Integrating over these holonomies one produces a product of $15j$ symbols one for each 4-simplex

$$\int dg_{\sigma t}\, \rho^{j_1}(g_{\sigma t})^{\alpha}_{\alpha'}\, \rho^{j_2}(g_{\sigma t})^{\beta}_{\beta'}\, \rho^{j_3}(g_{\sigma t})^{\gamma}_{\gamma'}\, \rho^{j_4}(g_{\sigma t})^{\sigma}_{\sigma'} = \sum_i \nu_i^{\alpha\beta\gamma\delta}\; \nu^i_{\alpha'\beta'\gamma'\delta'}. \tag{8.-40}$$

The edge amplitutdes require constraints are not imposed at each 4-simplex individually since a tetrahedron is shared by two 4-simplices.

If one ignores this requirement and impose the constraints at the level of each 4-simplex, the vertex amplitude reduces to that of the Barrett-Crane model. The error of imposing the constraints in the BG model comes from ignoring the fact that the face normals n_{tf} in two neighbouring 4-simplices should be the same.

8.9.7 The New Model

One imposes the following two constraints on the partition function of the BF theory:

(i) only sum over the simple representations be included in the state sum

and

(ii) instead of integrating over all $SO(4)$ group elements n_{tf} we integrate over the one having the form

$$\mathbf{n}_{tf} = (n_{tf} h_{\phi_t f}, u_t n_{tf} \epsilon h^{-1}_{\phi_t f}).$$

8.9.8 Boundary State Space

$$\mathcal{H}_{(j_1 j_1) \dots (j_4, j_4)} \tag{8.-40}$$

8.9.9 Cubulations

8.10 Linear 2-Cell Spin Foam Models

Standard spin-foam models as formulated up to now are piecewise flat geometries defined on piecewise linear manifolds. For LQG canonical theory to match these models it would have to be restricted to the category of piecewise linear manifolds and piecewise linear spin-networks.

Can a spin foam model be formulated that can match the diffeomorphism invariant framework of loop quantum gravity?

The aim would be to generalise the spin-foam framework so that it can be used to define a spin foam history of an arbitrary spin-network state of LQG. This is achieved by defining spin foams on arbitrary linear 2-cell complexes instead of on just 2-simplicial complexes.

It also allows for a notion of embedded spin foams in which we can consider knotting or linking spin-foam histories.

The main tools of spin foams as described in the previous section can be successfully generalised to this new framework.

8.10.1 Foams

A 2-cell is a convex compact polygon.

Here we redefine a foam as an oriented linear 2-cell complex. Briefly, each foam κ consists of 2-cells (faces), 1-cell (edges), and 0-cells (vertices).

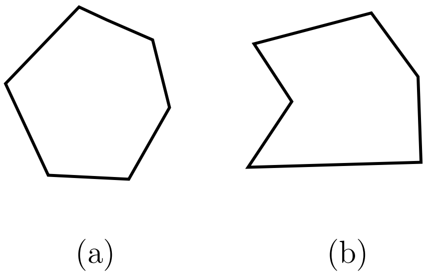


Figure 8.29: (a) A 2-cell (b) Not a 2-cell.

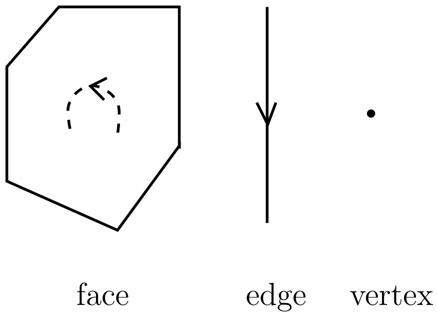


Figure 8.30: .

The faces are polygons.

Obviously triangulated manifolds qualify.

8.11 Group-Field-Theory

A spin foam model can be recast in the form of a rather peculiar field theory over the cartesian product of a group [].

given any spin foam model, there is an algorithm for constructing a field theory whose Feynmann expansion gives back the spin foam model.

$$W[s] = \int \mathcal{D}\Phi \; f_s[\Phi] \; e^{\int \Phi^2 - \lambda \int \Phi^5}. \tag{8.-40}$$

8.12 Semi-Classical Limit

8.13 Reduced Phase Space Path Integral

8.13.1 Introduction

Path Integrals: Covariant formulation of quantum theory expressed as a sum over paths

Formally the path integral can be written as

$$\int \mathcal{D}q \exp^{iS[q]/\hbar} \qquad \text{or} \qquad \int \mathcal{D}q\mathcal{D}p \exp^{iS[q,p]/\hbar} \tag{8.-40}$$

To define a path integral directly involves defining the components of these formal expressions:

1. What is the space of paths $q(t)$ over which we integrate?

2. $\mathcal{D}$: What is the measure on this space of paths?

3. $S[q]$: What is the phase associated with each path.

We hope that each of these components can be determined by constructing the path integral from the canonical theory.

We outline the standard construction of a path integral representation for Non-Relativistic Quantum Mechanics with a polynomial Hamiltonian, $H(q,p)$.

We want path integral representation of the propagator

$$< x'|e^{-iH\Delta t/\hbar}|x> = < x', t'|x, t> \tag{8.-40}$$

Split the exponential into a product of N identical terms.

$$< x'|\prod_{n=1}^{N} e^{-i\hat{H}\epsilon/\hbar}|x> \qquad \text{where} \qquad \epsilon = \frac{\Delta t}{N} \tag{8.-40}$$

Insert a complete basis in x between each exponential.

$$< x'|e^{-iH/\hbar}\int dx_{n-1}|x_{n-1}>< x_{n-1}|e^{-iH/\hbar}\dots|x> \tag{8.-40}$$

Step 2

By defining $x_N = x'$ and $x_0 = x$ this can be written in a simple form.

$$< x'|e^{-iH\Delta t/\hbar}|x> = \prod_{m=1}^{N-1}\left[\int dx_m\right]\prod_{n=1}^{N} < x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}> \tag{8.-40}$$

In the limit $N \to \infty$ $(\epsilon \to 0)$ we can expand each term in the 2nd product of () in ϵ.

$$< x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}> =< x_n|1 - i\hat{H}\epsilon/\hbar|x_{n-1}> +\mathcal{O}(\epsilon^2) \tag{8.-40}$$

This can be written as an integral over momentum p_n.

$$< x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}> \;=\; \frac{1}{2\pi\hbar}\int dp_n e^{ip_n(x_n - x_{n-1})/\hbar}[1 - i\epsilon H(p_n x_{n-1}, x_n)/\hbar + \mathcal{O}(\epsilon^2)] \tag{8.-40}$$

Step 3

Inserting the matrix elements $< x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}>$ into the expression for the full matrix element and simplifying.

$$\begin{aligned} < x_n|e^{-i\hat{H}\Delta t/\hbar}|x_{n-1}> \;=\; & \lim_{N\to\infty}\prod_{m=1}^{N-1}\left[\int dx_m\right]\prod_{n=1}^{N}\left[\frac{1}{2\pi\hbar}\int dp_n\right] \\ & e^{i\epsilon\sum_{n=1}^{N}(p_n(x_n x_{n-1})/\epsilon\hbar)}\prod_{n=1}^{N}[1 - i\epsilon H(p_n, x_n, x_{n-1})/\hbar + \mathcal{O}(\epsilon^2)] \end{aligned} \tag{8.-41}$$

The limit $N \to \infty$ defines the measure of the phase space path integral integral as the integral over the position at each time between t and t' and the integral over all momenta.

Almost in path integral form, except the final product. This can can be replaced according to

$$\lim_{N\to\infty}\prod_{n=1}^{N}[1 - i\epsilon H(p_n, x_n, x_{n-1})/\hbar + \mathcal{O}(\epsilon)] = \exp\left[-i/\hbar \lim_{N\to\infty}\sum_{n=1}^{N}\epsilon H(p_n, x_n, x_{n-1})\right] \tag{8.-41}$$

The final result is

$$< x'|e^{-iH\Delta t/\hbar}|x> = \lim_{N\to\infty}\prod_{m=1}^{N-1}\left[\int dx_m\right]\prod_{n=1}^{N}\left[\frac{1}{2\pi\hbar}\int dxp_n\right]\exp\left(i/\hbar S[x,p,N]\right) \quad (8.\text{-}41)$$

Where we recognise S as the discretised action.

$$S[x,p,N] = \sum_{n=1}^{N}\epsilon(p_n(x_n - x_{n-1})/\epsilon - H[x_n, x_{n-1}, p_n]). \quad (8.\text{-}41)$$

In the limit $N \to \infty$ this is the action

$$S[x,p] = \int dt(p\dot{x} - H(x,p)). \quad (8.\text{-}41)$$

8.13.2 Derivation of the Reduced Phase Space Path Integral

The reduced phase space path integral is formally derived as the generating functional of n-point functions. The path-integral is an integral over the reduced pahse space.

n-point functions

Consider the Heisenberg picture

$$Q_A(t) = e^{-iHt/\hbar}Q^A e^{iHt/\hbar}.$$

$$\tau^{A_1A_2}(t_1,t_2) :=< \Omega, T\{Q^{A_1}(t_1)Q^{A_2}(t_2)\}\Omega >$$

This can be written as

$$\tau^{A_1A_2}(t_1,t_2) = [\theta(t_1 - t_2)]W^{A_1A_2}(t_1,t_2) + [\theta(t_2 - t_1)]W^{A_2A_1}(t_2,t_1)$$

where we have defined the unordered Wightman function

$$W^{A_1A_2}(t_1,t_2) := < \Omega, Q^{A_1}(t_1)Q^{A_2}(t_2)\Omega >$$

$$\begin{aligned}\tau^{A_1A_2A_3}(t_1,t_2,t_3) &= [\theta(t_1-t_2)][\theta(t_2-t_3)]W^{A_1A_2A_3}(t_1,t_2,t_3)\\ &+[\theta(t_1-t_3)][\theta(t_3-t_2)]W^{A_1A_3A_2}(t_1,t_3,t_2)\\ &+[\theta(t_2-t_1)][\theta(t_1-t_3)]W^{A_2A_1A_3}(t_2,,t_1,t_3)\\ &+\cdots\end{aligned}$$

In general we have

$$\tau^{A_1\ldots A_n}(t_1,\ldots,t_n)=\sum_{\pi\in S_n}\prod_{k=1}^{n-1}[\theta(t_{\pi(k)}-t_{\pi(k+1)}]W^{A_{\pi(1)}\ldots A_{\pi(n)}}(t_1,\ldots,t_n) \quad (8.\text{-}45)$$

where

$$W^{A_1\ldots A_n}(t_1,\ldots,t_n):=<\Omega|Q^{A_1}(t_1)\cdots Q^{A_n}(t_n)\Omega> \quad (8.\text{-}45)$$

As $H\Omega=0$ we have

$$e^{-it_+H/\hbar}\Omega=e^{-it_-H/\hbar}\Omega=\Omega$$

for any $t_\pm$. Using this we may write

$$W^{A_1\ldots A_n}(t_1,\ldots,t_n):=<\Omega|e^{iH(t_+-t_1)/\hbar}Q^{A_1}e^{iH(t_1-t_2)/\hbar}Q^{A_2}\cdots Q^{A_n}e^{iH(t_n-t_-)/\hbar}\Omega> \quad (8.\text{-}45)$$

for any $t_\pm$.

Path-integral for matrix elements $<\psi_f|U(t_f-t_i)|\psi_i>$

$$<\psi_f|U(t_f-t_i)|\psi_i> \;=\; \int\{\prod_{n=0}^{N}[dQ_n]\}<\psi_f|Q_n><Q_n|Q_{n-1}>\cdots<Q_1|Q_0><Q_0|\psi_i>$$

write $<\psi_f|Q_n>=\overline{\psi_f(Q_n)}$ and $<Q_0|\psi_i>=\psi_i(Q_0)$. Introduce a complete set of intermediate momentum eigenstates

$$\begin{aligned}<Q_n|Q_{n-1}> &= <Q(t_i+n\epsilon)|Q(t_i+(n-1)\epsilon)>\\ &= \int dP_n<Q(t_i+n\epsilon)|P_n><P_n|Q(t_i+(n-1)\epsilon)> \quad (8.\text{-}46)\end{aligned}$$

then

$$
\begin{aligned}
& < \psi_f | U(t_f - t_i) | \psi_i > \\
= \; & \int \{ \prod_{n=0}^{N} [dQ_n] \} \overline{\psi_f(Q_n)} \psi_i(Q_0) (\int dP_n < Q_n | P_n >< P_n | Q_{n-1} >) \cdots \\
& \cdots (\int dP_1 < Q_1 | P_1 >< P_1 | Q_0 >) \\
= \; & \int \{ \prod_{n=0}^{N} [dQ_n] \} \overline{\psi_f(Q_n)} \psi_i(Q_0)
\end{aligned} \tag{8.-49}
$$

Trotter product formula - when A and B are self-adjoint operators

$$
e^{t(A+B)} = \lim_{n\to\infty} (e^{tA/n} e^{tB/n})^n
$$

It allows us to replace $\exp[\lambda(A+B)]$ by $\exp(\lambda A)\exp(\lambda B)$ when λ is small.

$$
\begin{aligned}
& e^{t(A+B)} - (e^{tA/n} e^{tB/n})^n \\
= \; & e^{t(A+B)} - e^{tA/n} e^{tB/n} \ldots e^{tA/n} e^{tB/n} \\
= \; & e^{t(A+B)} - e^{tA/n} e^{tB/n} e^{t(A+B)} (e^{tA/n} e^{tB/n})^{n-2} + \\
& e^{tA/n} e^{tB/n} e^{t(A+B)} (e^{tA/n} e^{tB/n})^{n-2} - (e^{tA/n} e^{tB/n})^2 e^{t(A+B)} (e^{tA/n} e^{tB/n})^{n-3} + \\
& (e^{tA/n} e^{tB/n})^{n-1} e^{t(A+B)} - (e^{tA/n} e^{tB/n})^n
\end{aligned} \tag{8.-52}
$$

The Baker-Campbell-Hausdorff says

$$
\begin{aligned}
\exp^{(A+B)/n} &= \exp^{A/n} \exp^{A/n} \exp^{[A,B]/2n^2} \\
&= \exp^{A/n} \exp^{A/n} + \mathcal{O}(1/n^2)
\end{aligned}
$$

which implies

$$
\exp^{(A+B)} = (\exp^{A/n} \exp^{A/n})^n + \mathcal{O}(1/n^2)
$$

$$
< Q | P > = \prod_A \frac{\exp(-iQ^A P_A/\hbar)}{\sqrt{2\pi}} \tag{8.-54}
$$

we obtain the formal result

$$< \psi_f|U(t_f - t_i)|\psi_i > \;=\; \int \{\prod_{n=0}^{N}[dQ_n]\}\;\{\prod_{n=0}^{N}[d(P_n/\sqrt{2\pi})]\}\overline{\psi_f(Q_n)}\psi_i(Q_0)$$
$$\exp(-i\frac{\epsilon}{\hbar}\sum_{n=1}^{N}\{[\sum_A \frac{Q_n^A - Q_{n-1}^A}{\epsilon}P_{A_n}] - H(Q_n, P_n)\}). \tag{8.-55}$$

One now takes $N \to \infty$ and formally obtains

$$< \psi_f|U(t_f - t_i)|\psi_i > \;=\; \int [DQ][DP/\sqrt{2\pi}]\overline{\psi_f(Q(t_f))}\psi_i(Q(t_i))\exp(-i\frac{\epsilon}{\hbar}[\sum_A \dot{Q}^A P_A] - H(Q,P)) \tag{8.-55}$$

Path-integral for n-point functions

Note

$$W^{A_1\ldots A_n}(t_1,\ldots,t_n) \;=\; \prod_{k=1}^{n}\int [dQ_k]\;\; < \Omega|U(t_+ - t_1)|Q_1 > Q_1^{A_1} \;\times$$
$$[\prod_{k=1}^{n-1} < Q_k|U(t_k - t_{k+1})|Q_{k+1} > Q_{k+1}^{A_k+1}]\;\; < Q_n|U(t_n - t_-)|\Omega > \tag{8.-56}$$

$$W^{A_1\ldots A_n}(t_1,\ldots,t_n) \;=\; \int [\mathcal{D}Q][\mathcal{D}P/\sqrt{2\pi}]\overline{\Omega(Q(t_+))}\Omega(Q(t_-)) \;\times$$
$$\exp(-\frac{i}{\hbar}\int_{t_-}^{t_+} dt\{[\dot{Q}^A P_A] - H(Q,P)\})) \prod_{k=1}^{n} Q_k^{A_k}(t_k) \tag{8.-56}$$

where

$$[\mathcal{D}Q] = \prod_{t\in[t_-,t_+]}\prod_A dQ^A(t) \tag{8.-56}$$

and similarly for $[\mathcal{D}P]$.

8.13.3 Canonical Path Integral Measures for Holst and Plebanski Gravity

8.13.4 Corrections to the Measure of the "New Spin Foam Models"

We have calculated the appropriate formal path integrals for Holst gravity and Plebanski gravity with Immiriz parameter determined by the canonical analysis.

The main difference between the formal path integral expression for Holst gravity derived above and the "new spin foam Models" that are supposed to be quantisation of Holst gravity are:

1) the appearance of the local measure,

2) the continuum rather than discrete formulation (triangulation)

3) lack of manifest spacetime covariance (if the continuum limit of spin foam models, if one could take it, should be spacetime covariance).

The next step is to propose a discretisation of the path integral derived in this section which takes into account the proper measure factor.

8.14 Covariant Loop Quantum Gravity

Covariant Loop Quantum Gravity (CLQG) is a program to quantize gravity employing loops within a Lorentz covariant canonical formulation [205].

Chapter 9

Extending Standard Quantum Mechanics for Background Independent Theories

- The Problem of Time.
- Covariant quantum mechanics.
- Multiple-Event Probability.
- General boundaries formulation of quantum mechanics.
- Emergence of Temporal Phenomena.
- Consistent discrete quantum gravity.

9.1 Introduction

The general-relativistic revolution of our understanding of space and time has proven extremely effective empirically. Conventional textbook quantum mechanics (QM), and conventional quantum field theory (QFT), however, are formulated in a language which is incompatible with the general relativistic notions of space and, especially, time. Is there a formulation of QM compatible with these notions? Such a formulation should be required, in particular, in order to provide a clear interpretative framework to any attempt to formulate a quantum theory of gravity in a form consistent with the general-relativistic understanding of space and time [??].

It is often claimed that either GR or quantum mechanics most go through a profound, radical change if we are to have a viable theory of quantum gravity. There is tension between the quantum mechanics that was formulated in the non-relativistic setting, this does not necessarily mean that there is tension between the underlying principles of general relativity and quantum mechanics.

Finally, in the opinion of Rovelli and collaberators, the idea that quantum mechanics admits a very simple and straightforward generalization which is fully consistent with general relativity. And therefore that the contradictions between quantum theory and general relativity might be only apparent.

Problems:

• describes mechanics in terms of evolution of observables and states in time.

• time is used to order a sequence of measurements.

• should we basis notions from non-relativist quantum mechanics that don't sit well with special relativistic mechanics and aren't general enough to emcumpass GR.

The fundamental theory is described by an extended Heisenberg-picture canonical quantum mechanics, equipped with the standard probabilistic interpretation.

9.2 The Problem of Time

The label of "the problem of time" is often given to a number of related, but slightly different isues.

Briefly put, the problem of time is as follows: how is one to apply quantum mechanics to general relativity in which a classical non-dynamical background time is missing?

9.3 Rovelli's Formulation of Classical Mechanics

We denote its boundary value $\varphi(s)$

$$\Phi(x(s)) =: \varphi(s). \tag{9.0}$$

The dynamics are governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} m^2 \Phi^2 \tag{9.0}$$

with boundary conditions on Σ.

9.4 Covaraint Quantum Mechanics

A general formulism which gives no privilege to the time variable.

9.4.1 States and Observables

does the measurement of a quantum system necessarily break Lorentz invariance? conventional notions of position and time do not exist.

The basic notions used in the formulation of quantum mechanics were developed for application in nonrelativistic context. They are not easily extendible to relativistic theories. Is there a version of the notions of "state" and "observer" that can be formulated in a naturally relativistic manner?

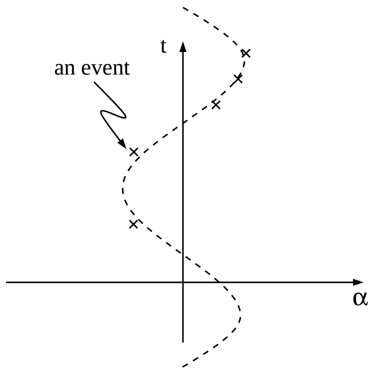


Figure 9.1: Coordinatized by the partial variables α and t. Rovelli calls **useful** measurement is an **event**. Rovelli calls this the **event space** or **relativistic configuration space**.

the motion as a relation between partial observables

$$f(\alpha, t) = \alpha - A\sin(\omega t + \phi) = 0 \tag{9.0}$$

Exact Lorentz invariant detector.

9.4.2 Spacetime States

In a sense, Heisenberg states have be thought of as representative of the history of the system.

In GR an instant in "time" is an abstract 3d spacial surface defined in terms of coordinates. We need to generalise to arbitrary surfaces bounding a region of spacetime.

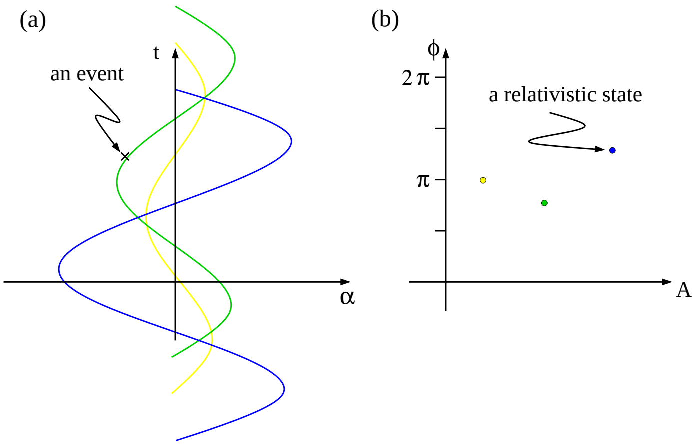


Figure 9.2: space of physical motions or relativistic phase space.

Given any compact support complex function $f(X,T)$

$$|f>= \int dXdT\; f(X,T)\; |X;T> \tag{9.0}$$

These states generalize the conventional wavefunctions for which $f(X,T) = \Psi(X)\delta(T)$. Conventional wavefunctions correspond to results of an instantaneous position measurement.

$$\begin{aligned} W(X,T;X',T') &= <X,T|X',T'> \\ &= \int \frac{d^3p}{2\pi\hbar}\, dE\; e^{(i/\hbar)[p(X-X')-E(T-T')]}\; \delta(E-p^2/2m) \\ &= \int \frac{d^3p}{2\pi\hbar}\; e^{(i/\hbar)[p(X-X')-p^2/2m(T-T')]} \\ &= \left(\frac{2\pi m}{i\hbar(T-T')}\right)^{\frac{1}{2}} \exp\left\{-\frac{m(X-X')^2}{2i\hbar(T-T')}\right\} \end{aligned} \tag{9.-2}$$

When viewed as a function of X and T, with X' and T' fixed, this is a solution of Schrödinger's equation which at time $T=T'$ is a delta function in $X-X'$. Each function $\Psi(X)$ determines a solution of Schrödinger's equation by

$$\Psi(X,T) = \int dX'\; W(X,T;X',0)\; \Psi(X). \tag{9.-2}$$

$$(i\hbar\frac{\partial}{\partial T} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2})W(X,T;X',T') = \delta(X-X')\delta(T-T') \tag{9.-2}$$

$$
\begin{aligned}
\left(i\hbar\frac{\partial}{\partial T} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right)\,\psi(X,T) &= \int dX'\,\left(i\hbar\frac{\partial}{\partial T} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right) W(X,T;X',0)\,\Psi(X') \\
&= \int dX'\delta(X-X')\delta(T)\,\Psi(X') \\
&= 0. \qquad (9.\text{-}3)
\end{aligned}
$$

As is a function $f(X,T)$ with compact support there exists T_0, T_1 such that for $T > T_1$ and $T < T_0$ $f(X,T) = 0$

$$
\begin{aligned}
\left(i\hbar\frac{\partial}{\partial T} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right)\,\psi_f(X,T) &= \int dX'dT'\,\left(i\hbar\frac{\partial}{\partial T} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right)\,W(X,T;X',T')\,f(X',T') \\
&= \int dX'\int_{T_0}^{T_1} dT'\,\delta(X-X')\delta(T-T')\,f(X',T') \\
&= 0 \text{ for } \; T > T_1 \qquad (9.\text{-}4)
\end{aligned}
$$

9.4.3 Review of General Relativistic Classical Mechanics

A Complete set of commuting observables

The outcomes of the measutement of a complete set of commuting observables charaterizes the state.

We can define an orthonormal basis in this Hilbert space by diagonalizing a completeset of commuting self-adjoint operators. Just as in the hydrogen atom where we use the self-adjoint operators J, j_z to form an orthonormal basis.

In a system with a finite number n of degrees of freedom, we choose $n+1$ partial observables (typically the n lagrangian variables plus the time variable), and form the $n+1$ dimensional extended configuration space, or event space, $\mathcal{C}$. The extended configuration space of a relativistic particle is the Minkowski space. The extended configuration space of a homogenous and isotropic cosmological model where a is the volume of the universe and ϕ is the matter density, is coordinatized by a and ϕ. Points in $\mathcal{C}$ are called events and denoted $s, s', s'', \ldots$: for instance, a point in Minkowski space $s = x^\mu = (\tilde{\mathbf{x}}, t)$, or a given value of radius of the universe and matter density $s = (a, \phi)$, define an event. Measuring a complete set of partial observables, that is, determining a point in $\mathcal{C}$ is to detect the happening of a certain event. For instance: a particle is detected in a point of Minkowski space, a certain value of radius of the universe and average energy density are measured, and so on. Each such detection describes an interaction of the system with another system, playing the role of observer.

9.4.4 Conditional Probability Interpretation

The definition of probability does not require any notion of time. The normalization is a prior requirement that follows from the very definition of probability. Similarly, total probability doesn't have to be conserved in time if there is no evaluation in time. Rather, conditional probability has to satisfy some requirement that follow from its physical interpretation, and it can be shown that these requirements reduce to unitary evolution if one is considering the "evolution" of the observables with respect to partial observables (clocks) that have suitable properties, may not exists in a system, exist only in some states, or exist but have these properties only within some approximation (see Peres 1980, Page and Wootters 1983).

Given a complete commuting set of partial variables, we ask if some subset takes the values such and such, then what are the probabilties that the remaining partial variables take the permissible values, say A, B or C? The definition of probability requires the probabilties of measuring A, B or C should add up to 1.

This requirement reduces to the usual one in the case where one of the partial observables forms an ideal clock. The associated transition amplitude A is given by $A =< \eta|U|\psi >$, where $U : \mathcal{H}_1 \to \mathcal{H}_2$ is the time-evolution operator of the system, evolving from t_1 to t_2. The simplest interpretation of $P = |A|^2$ is as expressing the probability of finding the state η at time t_2 given that the state ψ was prepared at time t_1. Thus, we are dealing with a conditional probability.

$$\mathcal{P}_{RR'} = \left| \frac{W(R,R')}{\sqrt{W(R,R)}\sqrt{W(R',R')}} \right|^2 . \tag{9.-4}$$

9.4.5 Time of Arrival in Quantum Mechanics

9.5 Boundary Hilbert Space

$$\mathcal{K}_{t_1,t_2} = \mathcal{H}^*_{t_1} \otimes \mathcal{H}_{t_2} \tag{9.-4}$$

The tensor product of two quantum state spaces describes the ensemble of the measurements described by the two factors. Therefore $\mathcal{K}_{\Sigma_T}$ is the space of the possible results of all measurements performed at $t = 0$ and at $t = T$ [?] Observations at two different times are correlated by the dynamics. Hence $\mathcal{K}_{\Sigma_T}$ is a kinematical state space in the sense that it describes more outcomes than the physically realizable ones. Dynamics is then a restriction on the possible outcome of observations []. It expresses the fact that measurement outcomes are correlated. The state $< 0_{\Sigma_T}|$, seen as a linear functional on $\mathcal{K}_{\Sigma_T}$, assigns an amplitude to any outcome of observations. This amplitude gives the correlation between outcomes at $t = 0$ and outcomes at $t = T$. Therefore the theory can be represented as follows. The Hilbert space $\mathcal{K}_{\Sigma_T}$ describes all possible outcomes of measurements made on σ_T. Dynamics is given by the single linear functional

$$\rho : \mathcal{K}_{\Sigma_T} \to \mathbb{C} \tag{9.-4}$$

$$|\Psi> \mapsto <0_{\Sigma_T}|\Psi> . \tag{9.-4}$$

For a given collection of measurement outcomes described by a state $|\Psi>$, the quantity $<0_{\Sigma_T}|\Psi>$ gives the correlation probability amplitude between these measurements.

9.5.1 General Relativistic Quantum Mechanics

The kernel of H, formed by the (possibly generalized) states in $\mathcal{K}$ satisfying the "Wheeler-DeWitt" equation

$$H\psi = 0 \tag{9.-4}$$

is called the physical state space H. $\mathbb{P}$ is the linear (self-adjoint) operator $P : \mathcal{K} \to \mathcal{H}$, given by:

$$\mathbb{P}\psi = \int dn \; e^{-inH} \; \psi \tag{9.-4}$$

If zero is in the discrete (resp. continuous) spectrum of H, then $\mathcal{H}$ is a proper (resp. generalized) eigenspace of $\mathcal{K}$. On the linear space $\mathcal{H}$ we consider the Hilbert structure

$$<\mathbb{P}s'|\mathbb{P}s>_{\mathcal{H}} := <s'|\mathbb{P}|s> \tag{9.-4}$$

which is well defined when H is a generalized eigenspace. We also write $<s'|s>_{\mathcal{H}}=<\mathbb{P}s'|\mathbb{P}s>_{\mathcal{H}}$. Remark that since in general $\mathbb{P}$ is not a true projector, $<s'|\mathbb{P}|s>$ may very well be different from $<\mathbb{P}s'|\mathbb{P}s>$. In particular, this last quantity is in general divergent in the case in which H is a generalized eigenspace.

9.5.2 Single Event Probability

The probability $\mathcal{P}_{s\Rightarrow s'}$ of observing the event s_2 if the event s was observed (we shall write $\mathcal{P}_{s\Rightarrow s'}$ simply as $\mathcal{P}_{s'}$ when there is no need of indicating the initial state) is given by the modulus square of the amplitude

$$A_{s\Rightarrow s'} = < s|\mathbb{P}|s >, \tag{9.-4}$$

where the states are normalized in $\mathcal{H}$, not in $\mathcal{K}$, that is,

$$< \mathbb{P}s|\mathbb{P}s >=< \mathbb{P}s'|\mathbb{P}s' >= 1. \tag{9.-4}$$

$$\mathcal{P}_{s\Rightarrow s'} = \frac{| < s'|s >_{\mathcal{H}} |^2}{< s'|s' >_{\mathcal{H}}< s|s >_{\mathcal{H}}} = \frac{| < \mathbb{P}s'|\mathbb{P}s >_{\mathcal{H}} |^2}{< \mathbb{P}s'|\mathbb{P}s' >_{\mathcal{H}}< \mathbb{P}s|\mathbb{P}s >_{\mathcal{H}}} \tag{9.-4}$$

Notice that this probability is a standard quantum mechanical probability computed in the physical Hilbert space $\mathcal{H}$, in the following sense. The states $|s >$ and $|s' >$ in $\mathcal{K}$ "project" down to physical states $|\mathbb{P}s >$ and $|\mathbb{P}s' >$ in $\mathcal{H}$. The probability (??) is then simply the standard probability amplitude of measuring the physical state $|\mathbb{P}s' >$ if the physical state $|\mathbb{P}s >$ was measured.

This definition of probability reduces to the conventional one in the non-relativistic case. In the spin system considered above, for instance, the states $\mathbb{P}|S,t >$, are normalized and the amplitude for measuring the spin S' at time t' if S at time t was measured is

$$A_{St\Rightarrow S't'} =< S',t'|\mathbb{P}|S,t >=< S'|e^{-iH_0(t'-t)}|S > \tag{9.-4}$$

in agreement with conventional QM. More in general, in the case in which $H = p_t + H_0$, the definitions above reduce to the standard QM postulates regarding states, observables and probability.

9.6 Multiple-Event Probability

This section is mostly based on [299].

Time ordering does not appear to be a fundamental structure required for the definition of quantum theory and the calculation of its probability amplitudes. This can be seen as reinforcing the hypothesis that the fundamental theory of nature can be formulated in a timeless language [288], and that temporal phenomena could be emergent [289], [290], [291].

We state the problem and we point out a certain number of diffculties that emerge in trying to assign a probability to sets of events in a general relativistic context. In particular, we discuss the diffculties of two apparently "natural" solutions. The first is a direct generalization of the single event probability postulate: the probability of an ensemble of events is determined by the projection on the physical Hilbert space of the subspace of the kinematical Hilbert space associated to this ensemble of events. We show that this postulate is not viable because it does not reduce to the standard QM probabilities in the nonrelativistic case. The second is the use of conditional probability, widely discussed in the literature. We point out certain diffculties with the operational definition of this probability.

We indicate a possible general way for solving the problem. This is based on the observation that a multiple-event probability, such as $\mathcal{P}_{\psi \Rightarrow \psi' \psi''}$ can always be reinterpreted as a single-event probability, once the dynamics and the quantum nature of the apparatus making the measurements are taken into account. If we do so, the time order gets naturally coded into the dynamics of the system. This strategy provides a general way for dealing with multiple-event probabilities in general relativistic quantum mechanics.

We find that the general theory does not say what happens at different times: for every physical situation it gives the probability distribution for all the events, including those that we may wish to view as records of previous events.

We comment on the meaning of the notion of probability in the timeless case. In particular, we clarify the apparent difficulty presented by the fact that probabilities assigned to the possible values of a variable may not sum up to one. In the following section we summarize the results and discuss the issues that remain open.

A typical example is the following.

9.6.1 Multiple-Event Probability

Consider a partial observable A in $\mathcal{K}$ and let a be one of its eigenvalues. If a is nondegenerate, and $|s'>$ is the corresponding eigenstate, then (?) provides the probability amplitude of measuring a. What happens if a is degenerate?

Let us say for simplicity that a is doubly degenerate, and that $|s'>$ and $|s''>$ are two orthogonal eigenstates having eigenvalue a, that is, they span the a-eigenspace $\mathcal{K}_a$. Then, to measure the eigenvalue a, or, equivalently, to measure its associated projection operator

$$\pi_a = |s'><s'| + |s''><s''|,$$

means that we have a measuring apparatus that gives us a Yes answer if either the event s' or the event s'' happen (Yes answer corresponds to the eigenvalue 1 of a). In order to compute the probability of having a Yes answer, we need therefore the probability $\mathcal{P}_{s'\ OR\ s''}$ that the event s' OR the event s' happens.

Alternatively, suppose that we have a measuring apparatus that gives us a Yes answer if both the event s' and the event s'' happen. In order to compute the probability of having a Yes answer, we need therefore the probability $\mathcal{P}_{s'\ AND\ s''}$ AND s'' that the event s' AND the event s'' happen. The solution of one case gives immediately the solution of the other since, clearly

$$\mathcal{P}_{s'\ OR\ s''} = \mathcal{P}_{s'} + \mathcal{P}_{s''} - \mathcal{P}_{s'\ AND\ s''} \tag{9.-4}$$

There are two possibilities: either $\mathcal{P}_{s'\ AND\ s''}$ is always zero, or not. We consider the two cases separately.

(i) **Mutually exclusive events.** If $\mathcal{P}_{s\Rightarrow s'\ AND\ s''} = 0$ for any s, then s' and s'' are alternative events that cannot both happen. That is, if one happens, the probability that the other happens is zero. By (?) and the given interpretation, this is equivalent to

$$< s'|\mathbb{P}|s'' > = 0 \tag{9.-4}$$

In this case, (??) gives

$$\mathcal{P}_{s'\ OR\ s''} = \mathcal{P}_{s'} + \mathcal{P}_{s''}.$$

That is, the probability of s' OR s'' is simply the sum of the probabilities of s' and s''. Observe that this can be written generalizing (??) to

$$\mathcal{P}_{s\Rightarrow a} = < s|\Pi_a|s > \tag{9.-4}$$

A typical example is the following. In the two-state spin system considered in the previous section, let $|s'>=|\uparrow, t>$ and $|s''>=|\downarrow, t>$. In this case, $< s'|P|s''>=<\uparrow|U(0)|\downarrow>=0$. The two events are mutually exclusive. Therefore $\mathcal{P}_{s'\ ANDs''} = 0$. The projector on the a-eigenspace $\mathcal{K}_a$ is

$$\pi_a = |\uparrow, t><\uparrow, t| + |\downarrow, t><\downarrow, t|. \tag{9.-4}$$

The projection $\mathcal{H}_a$ of $\mathcal{K}_a$ in $\mathcal{H}$ is spanned by the two orthogonal states $\mathbb{P}|\uparrow, t>= U^\dagger(t-t_0)|\uparrow>$ and $\mathbb{P}|\downarrow, t>= U^\dagger(t-t_0)|\downarrow>$, therefore

$$\Pi_a = U^\dagger(t-t_0)\big(|\uparrow><\uparrow| + |\downarrow><\downarrow|\big)U(t-t_0). \tag{9.-4}$$

In this two-state system, $\Pi_a = \mathbf{1}$ and the corresponding probability is $\mathcal{P}_a = 1$. Not so, of course, in general.

(ii) **None exclusive events.** The interesting case is when

Let

$$|s'>=|\uparrow, t'> \text{ and } |s''>=|\leftarrow, t''>=\frac{|\uparrow, t''>+|\downarrow, t''>}{\sqrt{2}}.$$

In this case,

$$< s''|\mathbb{P}|s'> = <\leftarrow |U(t''-t')|\uparrow> \neq 0,$$

in general. The question we are asking is: what is the probability of detecting the spin $\uparrow$ at time t' AND the spin $\leftarrow$ at t''? The problem is of course well posed: if a particle is in a certain initial state at t, what is the probability of finding it with a certain spin at time t' AND with another spin at a later time t''? This can be measured by measuring the fraction of a beam that passes through a sequence of two Stern-Gerlach apparatuses.

Now, in ordinary quantum mechanics, these probabilities depends on the time ordering between the events. For instance, let the initial state $|s>$ be the state $|\rightarrow>=\frac{|\uparrow>-|\downarrow>}{\sqrt{2}}$ at time $t=0$, and let $U(t)=\mathbf{1}$ for all t. Then

$$\mathcal{P}_{s\Rightarrow(s'\ AND\ s'')}\begin{cases}\frac{1}{4} \text{ if } t'<t'' \\ 0 \text{ if } t''<t'\end{cases}. \tag{9.-4}$$

Because the sequence

$$|\rightarrow> \quad \Rightarrow \quad |\uparrow> \quad \Rightarrow \quad |\leftarrow> \tag{9.-4}$$

has probability 1/4; while the sequence

$$|\rightarrow> \quad \Rightarrow \quad |\leftarrow> \quad \Rightarrow \quad |\uparrow> \tag{9.-4}$$

cannot happen. The standard way of obtaining these probabilities in conventional quantum mechanics is via the projection postulate. For instance, say $t'<t''$, that is, case (22). We have:

(i) at time t_2 the spin $\uparrow$ is measured with probability $|<\uparrow|\leftarrow>|^2=1/2$;

(ii) the state is hence projected to $|\uparrow>$;

(iii) at time t'' the spin $\leftarrow$ is measured with probability $|<\leftarrow|\uparrow>|^2=1/2$, giving total probability $1/2\times 1/2=1/4$. Standard QM gives also, easily

$$\mathcal{P}_{s\Rightarrow(s'\ OR\ s'')}\begin{cases}\frac{3}{4} \text{ if } t'<t'' \\ \frac{1}{2} \text{ if } t''<t'\end{cases}. \tag{9.-4}$$

Comparing with (??), notice that the probabilities $\mathcal{P}_{s\Rightarrow s'}$ and $\mathcal{P}_{s\Rightarrow s''}$ relevant here (with two detectors) are different from the probabilities $\mathcal{P}_{s\Rightarrow s'}$ and $\mathcal{P}_{s\Rightarrow s''}$ relevant when only one detector is present. For instance, in the first case, we have $\mathcal{P}_{s\Rightarrow s''} = |<\rightarrow|\uparrow><\uparrow|\leftarrow>|^2 + |<\rightarrow|\downarrow><\downarrow|\leftarrow>|^2 = 1/2$, because of the presence of a detector in s'; while in the absence of this, we would clearly have $\mathcal{P}_{s\Rightarrow s''} = |<\rightarrow|\leftarrow>|^2 = 0$. This well known fact will play an important role below.

How do we recover these probabilities in relativistic quantum mechanics, where we do not have a notion of time ordering in t?

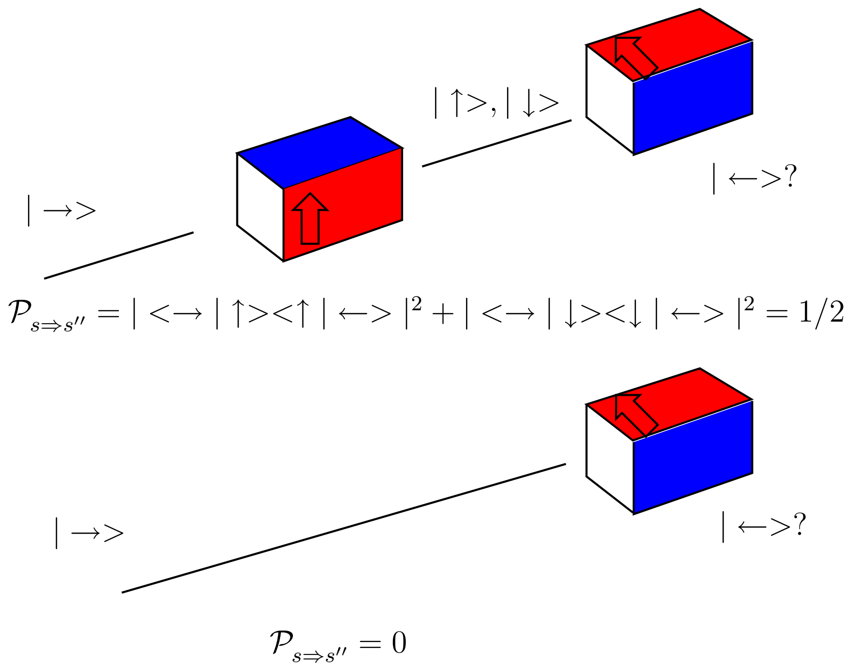


Figure 9.3: multprob2F. The probability $\mathcal{P}_{s\Rightarrow s''}$ is different depending on whever a detector is present or absent in s'. This well known fact plays an important role in the discussion below.

9.6.2 Multi-event probability from the coupling of an apparatus

Consider the experimental question of what is the probability for a spin system to have spin $\uparrow$ at time t' AND spin $\leftarrow$ at a later time t''. This is a statement that gets a precise meaning in an appropriate measurement context. In conventional QM, it can be dealt with by separating the two measurements in time, and using the collapse algorithm to compute joint probabilities.

Can obtain the same probabilities without invoking the collapse postulate? The answer is yes, and follows from an analysis of the experimental situation in volved in the experiment in which we measure the two spins at different times. The key is to bring the appartus that makes the measurement in to the picture.

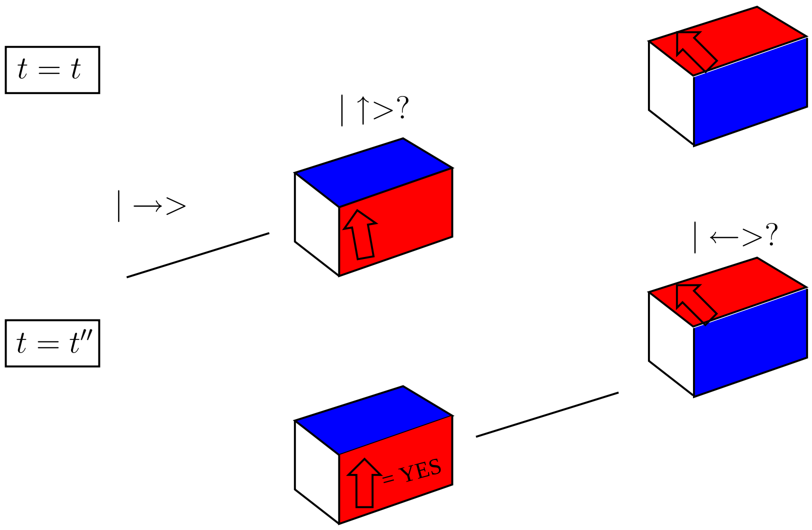


Figure 9.4: multprob3F.

9.6.3 Multi-event probability from the single-event one in non-relativistic QM

Consider an initial state $|\psi>$ at time t. What is the probability of detecting the state $|\psi'>$ at time t' AND the state $|\psi''>$ at time $t''>t'$? This can be computed by projecting on $|\psi'>$ at time t', which gives

$$\mathcal{P}_{\psi\Rightarrow\psi'\psi''} = | < \psi''|U(t''-t')\Pi_{\psi'}U(t'-t)|\psi > |^2. \tag{9.-4}$$

where $\prod_{\psi'} = |\psi' >< \psi'|$ and all states are normailzed $< \psi|\psi >= 1$ .Now, describe the apparatus measuring $|\psi'>$ as a two-state system which is initially in a state $|No>$, which interacts with the system at the time t', jumping to the state $|Yes>$ if and only if the state of the system is in the state $|\psi'>$. The interacting dynamics are given by the unitary evolution operator $U_{\psi',t'}$, which is defined in three parts. Say that evolution begins at t and ends at t''. Evolution which occurs before the interaction, i.e., $t'', t < t'$ and evolution which occurs after the interaction, i.e., $t'', t > t'$ is given by the same unitary operator

$$< \psi'', A''|U_{\psi',t'}|\psi, A >=< \psi''|U_{\psi',t'}|\psi > \delta_{A''A}. \tag{9.-4}$$

Now, consider evolution which starts before the interaction at t' and ends after it, i.e., $t'' < t' < t'$. Say $A = A''$,

$$< \psi''|U(t''-t')(1-\Pi_{\psi'})U(t'-t)|\psi > \tag{9.-4}$$

is the amplitude for the system to evolve up t' at which point and a state is measured which is not ψ', after which the system evolves to the state ψ''. Now, say $A \neq A''$,

$$< \psi''|U(t''-t')\Pi_{\psi'}U(t'-t)|\psi > \tag{9.-4}$$

is the amplitude for the system evolves up t' at which point the state ψ' is measured, after which the system evolves to the state ψ''. This can be expressed as

$$< \psi'', A''|U_{\psi',t'}(t''-t)|\psi, A > \;=\; < \psi''|U(t''-t')\big(\Pi_{\psi'}\mathcal{I}_{A''A} + (1-\Pi_{\psi'})\delta_{A''A}\big)U(t'-t)|\psi > \tag{9.-4}$$

where

$$\mathcal{I}_{AA'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9.-4}$$

is a matrix that flips the apparatus state. Equations (9.6.3) and (9.-4) together define the unitary operator $U_{\psi',t'}$.

Then the question we are interested in can be rephrased as follows: "Given an initial state $|\psi, No >$ at time t, what is the probability of measuring the state $|\psi'', Yes >$ at time t''?" Notice that "the system state is $|\psi'' >$" and "the apparatus state is $|Yes >$", are compatible statements in quantum theory: they refer to orthogonal observables, both at time t'', that are, and can be, measured together! In other words, the question can be captured by the single-event probability amplitude

$$\mathcal{P}_{\psi \Rightarrow \psi'\psi''} = |< \psi'', Yes|U_{\psi',t'}(t''-t)|\psi, No >|^2. \tag{9.-4}$$

9.6.4 The Problem of the "Frozen-time" Formalism

In the generally covariant context, dynamics can be entirely expressed in terms of Dirac observables. Indeed, notice that the probability of a sequence of measurements can be written as in equation (??), namely as the expectation value of the projection operator Π_s defined in (?), or (??). This operator is a Dirac observable of the extended system that includes the measuring devices.

In the present context, this is the answer to the long-standing problem of the description of dynamics in the "frozen-time" formalism; namely in the Dirac's quantization of a system whose dynamics is expressed by constraints [41], [42]. Dynamics is coded into (non-commuting) Dirac observables defined in terms of sets of interactions between (what we call) the system and (what we call) the measuring devices.

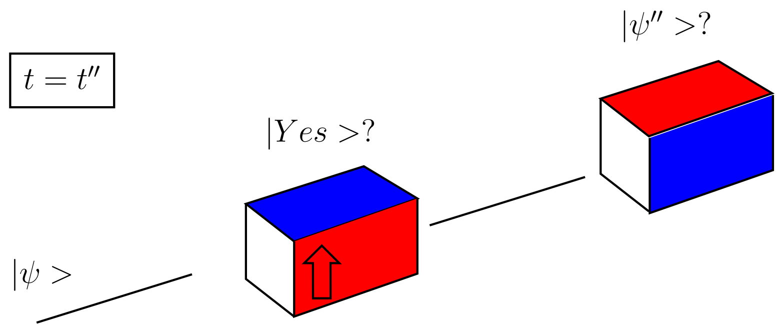


Figure 9.5: multprob4F. "the apparatus state is Yes" while "the system is ψ''" are compatible statements in quantum theory: they refer to orthogonal observables, both at time t''. By moving the quantum/classical boundary, a question such as in fig (9.4) is captured by a single-event probability amplitude.

9.6.5 Future Developments of Multiple-Event Probabilities

The extension of these ideas to field theory, and in particular the connection between this formalism and the boundary formalism [8, 18] which is presently used [19] to compute probability amplitudes in background independent quantum gravity, in the context of loop quantum gravity (next chapter)[8, 20, 21].

9.7 General Boundaries Formulation

In quantum field theory (QFT) on Minkowski space, we can use the Schrödinger picture and have states associated to flat spacelike (hyper-)surfaces. The transition amplitude between an initial state and a final state is obtained by acting with the unitary evolution operator on the former and taking the inner product with the latter. The possibility of a Schrodinger picture has also been considered in QFT on curved spacetime [?]. In this case, states live on arbitrary spacelike Cauchy surfaces forming a foliation of spacetime. Evolution along these surfaces is non-unitary in general, as it does not correspond to a symmetry of the metric.

This restriction is sufficient for ordinary QFT on Minkowski space, but has no meaning in a generally covariant context. It is necessary to consider arbitrary boundary surfaces: How would one encode the fact that initial and final states refer to certain physical times or have a certain physical time interval between them? An element of $\mathcal{H}_{kin}$ or $\mathcal{H}_{diff}$ contains information about quantum 3-geometries, but it does not specify the proper time that is associated to a transition process from one 3-geometry to another 3-geomery.

It has been proposed [??, ??] that this missing information could be encoded by using closed boundaries, in place of the $3d$ hypersurfaces of a foliation (see Fig. N.-19). On a closed boundary, the configuration consists of spatial 3-geometries g_f, g_i, and a timelike

3-geometry g_t. Along the time-like part, we can impose conditions on the proper distance between the space-like surfaces Σ_i and Σ_f. This distance corresponds to the proper time that is measured by clocks with world lines on Σ_t.

In background independent quantum gravity, on the other hand, there is no fixed space-time geometry; in this case, states live on arbitrary Cauchy surfaces and the requirement that the surface is spacelike is encoded in the state itself, which represents a quantum state of a spacelike geometry (see e.g. [??, ??]).

$$A = < \Psi_f | e^{-i\mathcal{H}(t_f - t_i)} | \Psi_i > . \tag{9.-4}$$

$$A = \int \mathcal{D}\varphi_f \int \mathcal{D}\varphi_i \ \Psi_f^*[\varphi_f] W[\varphi_f, t_f; \varphi_i, t_i] \Psi_i[\varphi_i] \tag{9.-4}$$

9.7.1 Preparation - Measurement

We prepare a state ψ at t_1, wait for a time Δt, then measure if we obtain the state η at t_2. The probability for this depends on Δt:

$$P = | < \eta | U(\Delta t) | \psi > |^2$$

Recall properties of U:

(i) Composition: $U(\Delta t) U(\Delta t') = U(\Delta t + \Delta t')$

(ii) Unitarity: $U^\dagger = U^{-1}$

9.7.2 Basic Idea

9.7.3 Basic structures

To prepare the ground for the general boundary formulation let us think of the standard formulation. There are geometric structures: points in time and intervals of time. There are algebraic structures: states and transition amplitudes. To each point $t \in \mathbb{R}$ in time we associate a Hilbert space $\mathcal{H}_t$ of states. These state spaces are simply copies of the usual state space $\mathcal{H}$ and the labeling by a time is only a formality at this point. To each time interval $[t_1, t_2] \subset \mathbb{R}$ we associate a linear map $\rho_{[t_1,t_2]} : \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \to \mathbb{C}$, called amplitude map. This sends a pair of an initial and a final state (ψ, η) to the transition amplitude $\rho_{[t1,t2]}(\psi \otimes \eta) =< \eta | U(t_1, t_2) | \psi >$.

Basic algebraic structures:

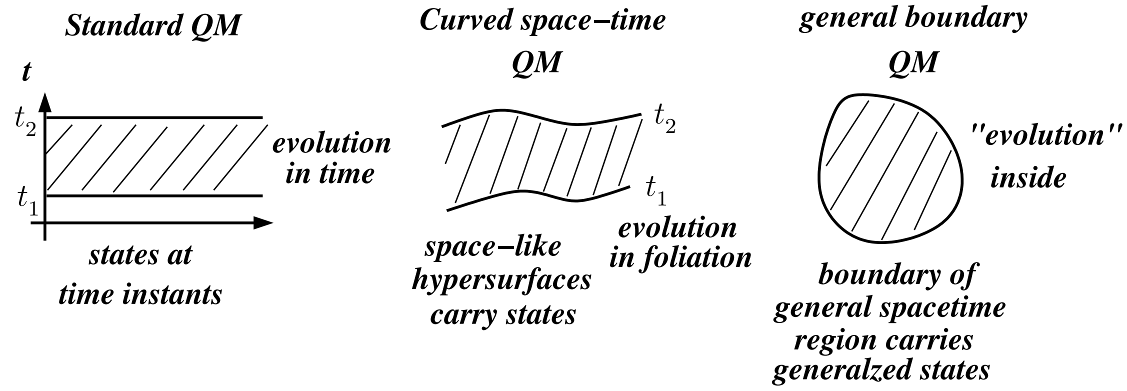


Figure 9.6: GenBoundaF. Basic idea.

- To each hypersurface Σ associate a Hilbert space $\mathcal{H}_\Sigma$ of states.
- To each region $\mathcal{M}$ with boundary Σ associate a linear amplitude map $\rho_\mathcal{M} : \mathcal{H}_\Sigma \to \mathbb{C}$.

The structures are subject to a number of rules. For example:

- $\overline{\Sigma}$ is Σ with the opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}^*_\Sigma$.
- $\Sigma = \Sigma_1 \cup \Sigma_2$ is a disjoint union of hypersurfaces. Then $\mathcal{H} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.

Recovering standard quantum mechanics

Consider the geometry of a stanard transtion amplitude.

- region: $\mathcal{M} = [t_1, t_2] \times \mathbb{R}^3$
- boundary: $\partial\mathcal{M} = \Sigma_1 \cup \overline{\Sigma}_2$
- state space: $\mathcal{H}_{\partial\mathcal{M}} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$

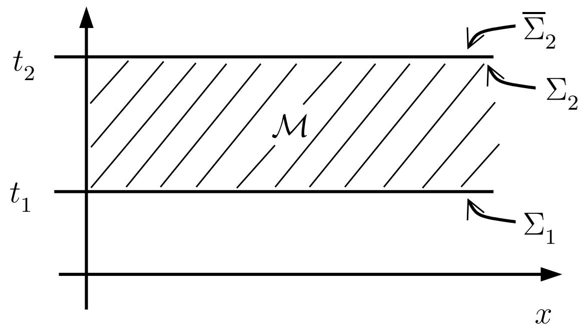


Figure 9.7: GenBoundcF. .

(i) Via time-tranlation symmetry identify $\mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \cong \mathcal{H}$, where $\mathcal{H}$ is the state space of standard quantum mechanics.

(ii) The amplitude map $\rho_{\mathcal{M}} : \mathcal{H} \otimes \mathcal{H}^* \to \mathbb{C}$.

(iii) The relation to the standard amplitude is:

$$\rho_{\mathcal{M}}(\psi \otimes \eta) =< \eta | U(t_2 - t_1) | \psi > \tag{9.-4}$$

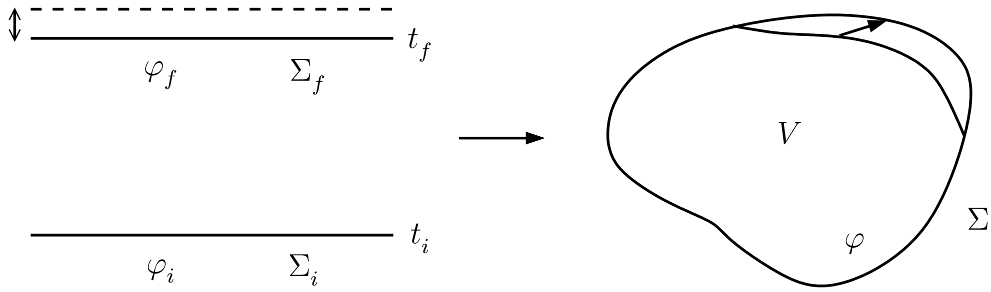


Figure 9.8: GenBound1F.

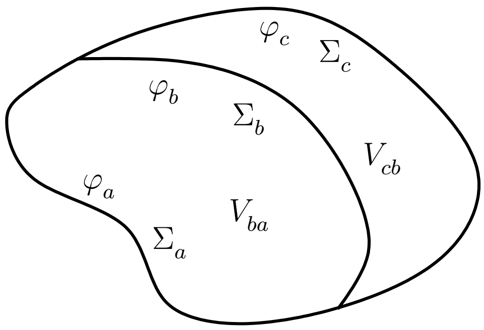


Figure 9.9: GenBound2F. Splitting of V.

9.8 The Vacuum State

Consider a real massive scalar field $\phi(x)$ on Minkowski space. To start with assume it is a free field. We write $x = (t, \vec{x})$. Denote by $\varphi(\vec{x})$ the classical field configuration at time zero: $\varphi(\vec{x}) = \phi(\vec{x}, t = 0)$. The state space at time zero, $H_{t=0}$, is Fock space, where the (distributional) field operator $\varphi(\vec{x})$ and the hamiltonian H are defined. The lowest eigenstate of H is the vacuum state $|0_M >$, and its energy E_0 is zero. Fock space admits countable bases. Choose a basis $|n >$ of eigenstates of H with eigenvalues

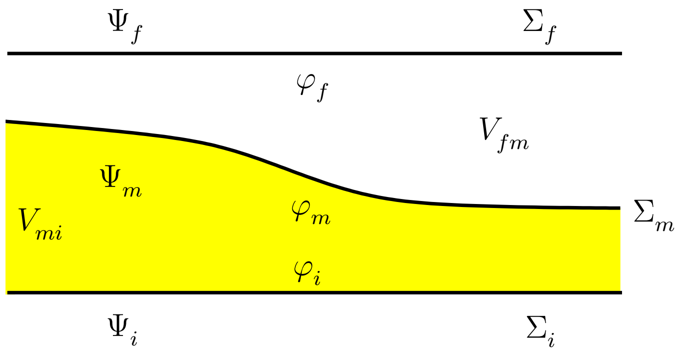


Figure 9.10: GenBound3F. Evolution to a closed surface Σ_m.

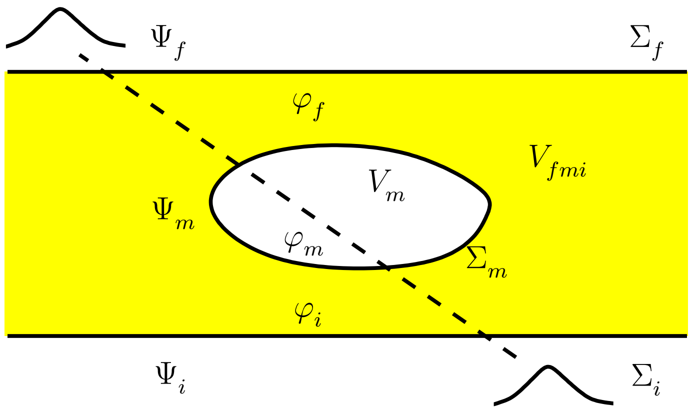


Figure 9.11: GenBound4F. Splitting of V_{fi}

$$W(T) = \sum_n e^{-TE_n} |n >< n|. \tag{9.-4}$$

In the large T limit, this becomes the projector on the

$$\lim_{T \to \infty} W(T) = |0_M >< 0_M|. \tag{9.-4}$$

9.8.1 The Vacuum State

The quantum state $|0_M >$ that describes the Minkowski vacuum is not singled out by the dynamics alone in quantum gravity. Rather, it is singled out as the lowest eigenstate of an energy H_T which is the variable canonically conjugate to a nonlocal function T of the gravitational

This situation has an analogy in the simple quantum system formed by a single a relativistic particle. In the Hilbert space of such a system there is no preferred vacuum state. But we can choose a preferred Lorentz frame, and therefore a preferred Lorentz time x_0. The conjugate variable to x_0 is the momentum p_0, and there is a (generalized) state of minimum p_0

To find the Minkowski vacuum state, we can repeat the very same procedure used above. The only difference is that the bulk functional integral is not over the bulk matter fields, but also over the bulk metric. This difference has no bearing on the above formulas, which regard the boundary metric, which, in the two cases, is an independent variable.

As a first example, a boundary metric can be defined as follows. Consider a three-sphere formed by two polar in and out regions and one equatorial side region. Let the matter+gravity field on the three-sphere be split as

$$\varphi = (\varphi_{out}, \varphi_{in}, \varphi_{side}). \tag{9.-4}$$

Fix the equatorial field φ_{side} to take the special value φ_{RT} defined as follows. Consider a cylindrical surface Σ_{RT} of radius R and height T in $\mathbb{R}^4$, as defined above. Let Σ_{in} in (and Σ_{out}) be a (3d) disk located within the lower (and upper) basis of Σ_{RT}, and let φ_{side} side the part of Σ_{RT} outside these disks, so that

$$\Sigma_{RT} = \Sigma_{in} \cup \Sigma_{out} \cup \Sigma_{insid}. \tag{9.-4}$$

Let g_{RT} be the metric of side and let $\varphi_{RT} = (g_{RT}, 0)$ be the boundary field on side determined by the metric being g_{RT} and all other fields being zero.

Given arbitrary values φ_{out} and φ_{in} of all the fields, included the metric, in the two disks, consider $W[(\varphi_{out}, \varphi_{in}, \varphi_{RT})]$. In writing the boundary field as composed by three parts as $\varphi(\varphi_{out}, \varphi_{in}, \varphi_{side})$ we are in fact splitting $\mathcal{K}$ as

$$\mathcal{K} = H_{out} \otimes H^*_{in} \otimes H_{side} \tag{9.-4}$$

Fixing $\varphi_{side} = \varphi_{RT}$ means contracting the covariant vacuum state $|0_\Sigma >$ in $\mathcal{K}$ with the bra state $< \varphi_{RT}|$ in $\mathcal{H}_{side}$. For large enough R and T, we expect the resulting state in $\mathcal{H}_{out} \otimes \mathcal{H}^*_{in}$ in to reduce to the Minkowski vacuum. That is

$$\lim_{R,T\to\infty} < \varphi_{RT}|0_\Sigma > = |0_M > \otimes < 0_M|. \tag{9.-4}$$

Therefore for a generic in configuration, and up to normalization

$$\Psi_M[\varphi] = \lim_{R,T\to\infty} W[(\varphi, \varphi_{in}, \varphi_{RT}]. \tag{9.-4}$$

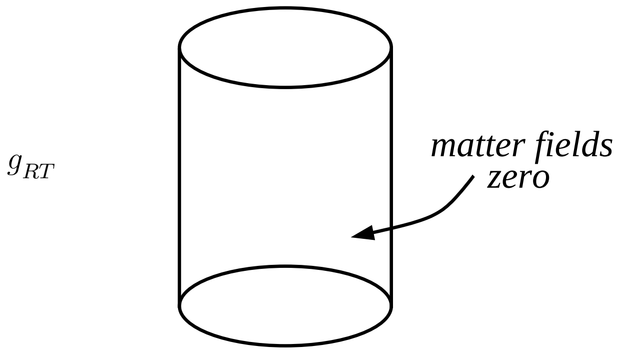


Figure 9.12: VacuumFig.

gives the vacuum functional for large R and T. (Below we shall use simpler geometry for the boundary).

These formulas allow us to extract the Minkowski vacuum state from a euclidean gravitational functional integral. n-particle scattering states can then be obtained by generalizations of the flat space formalism, and, if it is well defined, by analytic continuation in the single variable T. Notice that we are precisely in the case of time independent φ_{side}, where analytical continuation may be well defined.

9.9 Emergence of Temporal Phenomena

The hypothesis that the fundamental theory of nature can be formulated in a timeless language [288], and that temporal phenomena could be emergent [289], [290], [291].

Generally covariant theories and The problem of time

see week 41 beaz

Rovelli wants to use thermodynamics to **define** what we call time as we usualy mean. does this as follows. Given a mixed state with density matrix D, find some operator H such that D is the Gibbs state $\exp(-H/kT)$. In lots of cases this isn't hard; it basically amounts to

$$H = -kT \ln D \tag{9.-4}$$

Of course, H will depend on T, but this is really is just saying that fixing your temperature fixes your units of time!

Operator theorists have pondered this notoin very carefully for a long time and generalised it into the Tomita-Takesaki theorem. This gives a very general way of finding a Hamiltonian (hence a notion of time evolution) from a state of a quantum system! For example, one can use this trick to start with a Robertson-Walkerr universe full of blackbody radiation, and recover a notion of "time".

9.9.1 Gibb's distribution

The quantum state is then given by the Gibbs density matrix

$$\omega = Ne^{\beta H} \tag{9.-4}$$

where H is the hamiltonian, defined on a Hilbert space $\mathcal{H}$, and $N = tr[e^{\beta H}]$.

The KMS condition

$$\omega() = \omega() \tag{9.-4}$$

9.10 Consistent Discretization of Classical and Quantum Gravity

Explicitly working at the physical level.

[326]

Kinematic variables do not have a well defined action as quantum operators on states that are annihilated by the constraints.

Worse, such states are expected to have a distributional nature as a subset of the full space of states. This implies that they do not really admit a probabilistic interpretation (kuchar) [41]

???

[326]

"..it is the presence of the Hamiltonian constraint in general relativity what really complicates the application of this [Page-Wootters] proposal. In the consistent discrete canonical formulation of general relativity the constraints are not present. Therefore it opens the possibility to revisit the Page and Wootters proposal."

As mentioned in [77],

"...One way out could be to look at constraint quantization from an entirely new point of view [here he refers to consistent discretization] which proves useful also in the discrete formulations of classical GR, that is numerical GR. While being a fascinating possibility, such a procedure would be rather drastic step in the sense that it would render most results of LQG obtained so far obsolete."

9.10.1 The problem of Time:

There is no external purely classical time parameter - reference bodies are necessarily part of the system under consideration. They are necessarily a closed systems where everything behaves quantum mechanically. Page and Wootters proposed to treat all variables quantum mechanically and use one of the variables as a clock, as long as it behaves semiclassically.

The clock variable must change during evolution, therefore it cannot commute with the constraints. This implies that it will not be well defined on the physical space of states annihilated by all the constraints.

If one tries to work in the kinematical Hilbert space, these wave functions are distributional and cannot be used to construct a probabilistic interpretation.

Since their discrete theory in constraint free, one can follow the Page Wootters procedure.

$$P_{sim}(\Delta P^{\phi}, \Delta A) = \lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N} \int_{P^{\phi}_{(1)}, A_{(1)}}^{P^{\phi}_{(2)}, A_{(2)}} \Psi^2[A, P^{\phi}, n] dP^{\phi} dA. \tag{9.-4}$$

$$P_{cond}(\Delta P^{\phi}, \Delta A) = \frac{\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N} \int_{P^{\phi}_{(1)}, A_{(1)}}^{P^{\phi}_{(2)}, A_{(2)}} \Psi^2[A, P^{\phi}, n] dP^{\phi} dA}{\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N} \int_{-\infty, A_{(1)}}^{\infty, A_{(2)}} \Psi^2[A, P^{\phi}, n] dP^{\phi} dA}. \tag{9.-4}$$

They have studied the parameterized non relativistic particle with this method: [326] R. Gambini, R. Porto and J. Pulliin gr-qc/0302064

$$S = \int \left[\dot{q} + p_0 \dot{q}^0 - N(p_0 + \frac{p^2}{2m} + \lambda q) \right] d\tau, \tag{9.-4}$$

$$L(n, n+1) = p^n(q_{n+1} - q_n) + p_0^n(q_{n+1} - q_n^0) - N_n(p_0^n + \frac{p_n^2}{2m} \lambda q_n). \tag{9.-4}$$

the conditional probability to obtain $q = x$ given $q^0 = t$

$$(q = x|q^0 = t) = \frac{\sum_{n=-\infty}^{\infty} |\Psi(x,t,n)|^2}{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx |\Psi(x,t,n)|^2} \quad (9.\text{-}4)$$

One can show that this relational description recovers usual quantum mechanics when the discrete approximation approaches the continuum limit and when the clock variable behaves sufficiently to a classical clock.

The procedure can be extended to situations where the simultaneity surfaces are not transverse to the classical orbits of the system.

9.10.2 Realistic clocks, universal decoherence and the black hole information paradox

The system has an unitary evolution in n. At t cannot be perfectly correlated with n, even in the semi-classical regime of the clock, the evolution in t is not perfectly unitary. In fact one can show that the density matrix evolves according to

$$\frac{\partial}{\partial t}\rho_2 = -i[\mathcal{H}_2, \rho_2(t)] - \sigma[\mathcal{H}_2, [\mathcal{H}_2, \rho_2(t)]]. \quad (9.\text{-}4)$$

This equation was first proposed by Milburn based on phenomenological arguments, and is a particular type of non-unitary evolutions considered by Lindblad.

Our derivation allows to estimate σ that is of order of the Planck time.

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0) e^{-i\omega_{nm}t} e^{(-\sigma(\omega_{nm})^2)t} \quad (9.\text{-}4)$$

This equation does not violate the conservation of energy like Hawking proposal for information loss. One could expect to confirm this type of equation by studying some mesesoscopic quantum systems.

Information loss problem in Black Holes

It provides a new and very effective mechanism for treating the information loss problem in Black Holes.

They have shown that for any Black hole bigger than 600 Plank masses the information loss induced by their equation is enough to dissipate all the black hole information before to its evaporation.

For very small black holes, Hawking's semi-classical analysis not valid.

9.10.3 Quantum Cosmology:

9.11 Bearing of Matters of Quantum Gravity on Interpretations of Quantum Mechanics

In some interpretations of quantum mechanics, the wave function is considered a real entity that evolves unitarily, except at measurement time, when it undergoes a sudden change. In particular, some interpretations make the hypothesis that this collapse is a real physical phenomenon whose peculiar nonlocal dynamics is not yet understood. If this is the case, the full freedom of moving the quantum/classical boundary is broken, because once the collapse has happened no more interference between the two branches of a measurement outcome is possible, even in principle. If this is the case, the strategy adopted here is not viable in general, because it assumes, instead, that no true physical collapse happens at anytime.

In some others interpretations, the wave function, or the "quantum state", is not considered as a real entity. Rather, only quantum events are considered real, and probabilities like $| < s'|s > |^2$ are directly interpreted as conditional probabilities for these events to happen. In particular, in [372], [379] these quantum events are assumed to happen at interactions between systems, and to be real only with respect to the interacting systems themselves. From this perspective, there is no specific physical phenomenon corresponding to a quantum collapse, and the strategy considered here is viable. With respect to an external system, what happens at the interaction between system and apparatus is not a sudden change in a hypothetical real state , but simply an entanglement between the probabilities of various outcomes of observations on the system or the apparatus. We refer to [372], [379], and Section 5.6 of [20] for a discussion of this point of view.

Chapter 10

Towards Background-Independent Scattering Amplitudes

10.1 Introduction

Two masses should attract each other in the way that Newton's law specifies. In the quantum field theory this attraction comes about from an exchange of virtual gravitons. It would indicate that at low energies that loop quantum gravity has gravitons.

Why has such a calculation not been done despite the impressive advances made so far in loop quantum gravity? The transition amplitude $W(x,y)$ for a particle starting at the spacetime point x to the spacetime point y. Under an active diffeomorphism α the transition amplitude transforms as

$$W(x,y) \mapsto W(\alpha(x), \alpha(y))$$

Hence, for this transition amplitude to be background independence $W(x,y)$ would have to be constant everywhere. Of course, the reason why it this definition is of little use in a generally covariant context is because the arguments of the transition amplitude $W(x,y)$ are space-time points x, y which have no operational meaning at all in a background independent theory.[1] One obvious strategy would then be to formulate scattering amplitudes operationally, as they are measured in actual laboratory experiments.

To describe local experiments, we will be interested in such observables which (approximately) localize in some spacetime region, and the corresponding operators will need to

[1]Some may argue that $W(x,y)$ can't even be defined because the spacetime points are subject to quantum fluctuations in a quantum theory of spacetime, however, one should be careful not to make arguments based on quantum fluctuations in quantities that have no independent physical reality such as spacetime points.

include physical degrees of freedom which specify this region. Distance and time separation must be extracted from the dynamical variables. In a scattering experiment we measure incoming and outgoing particles as well as distances between instruments and elapsed time. The former are the matter field variables, the latter are gravitational field variables. It is natural to use the apparatus itself to specify the spacetime regime in which the the interaction occurs.

Scattering amplitudes of a background-independent theory can be defined as a function of the mean boundary geometry, instead of a function of the background metric, evaluated in terms of the mean value of the quantum gravitational field itself on a box encircling the interaction region. The mean geometry of the boundary are partial observables whose relation to the readings on the laboratory instruments is determined by the theory.

In the context of curved spacetime there is no Poincare invariant background spacetime... In a background independent context the notion of a global particle is not available. On the viability of the article notion in a finite region.

10.2 Difficulties in Formulating Scattering Amplitudes for BI Theories

Rovelli *et al*, [311], [312], [314].

In the standard formulism on associates A Hilbert space of states with each time-slice of a global foliation of space-time. An evolution takes place between two such time-slices and is represented by a unitary operator. Associated with states in the two such time-slices is a transition amplitude, whose modulus square determines the probability of finding the final state given that the initial one was prepared.

$$\mathcal{A}_{12} =< \psi_{int}|\hat{U}(t_1,t_2)|\psi_{fin} >, \quad \text{where } < \psi_{int}| \in \mathcal{H}^* \text{ and } |\psi_{fin} > \in \mathcal{H}. \tag{10.0}$$

$$\mathcal{A}_{12} = \int d^3x \int d^3y \; \psi^*_{int}(x,t_{int}) W(x,y) \psi_{fin}(y,t_{fin}) \tag{10.0}$$

$$W(x_1,\ldots,x_n) = Z^{-1} \int \mathcal{D}\phi \; \phi(x_1)\ldots\phi(x_n) e^{-iS[\phi]} \tag{10.0}$$

• In a background-independent theory $W(x,y) = \text{const}$.

• Conventional spacelike states don't impose any constraint on the proper time lapsed between the initial and final states. As a result, the transition amplitude will be given by a linear superposition of processes whose duration may range from microscopic to cosmic time scales.

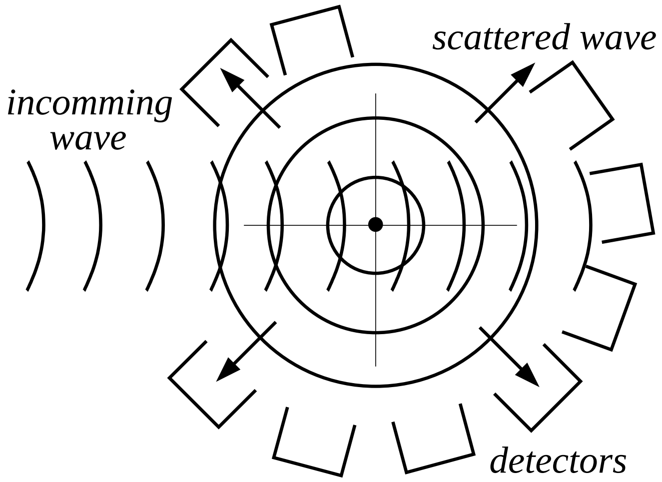


Figure 10.1: scatConven. scattering

• How do we define particles in background-independent quantum gravity where Fock states are not available?

$W[\varphi, \Sigma]$ depends on the field boundary value φ and the 3d surface Σ that bounds $\mathcal{R}$. Say we perform an active diffeomorphism, that is, keep the fields "where they are" but move around the points of the spacetime manifold. This results in a new boundary, Σ'; the field boundary value at this new surface will of course, in general, be different. Whence, if we wish to have a background-independent formulation, $W[\varphi, \Sigma]$ must not depend on Σ, ie. $W[\varphi, \Sigma] = W[\varphi]$.

Fock States and Particles

$$\hat{n}|1, 2, \ldots, N >= N|1, 2, \ldots, N > \tag{10.0}$$

10.2.1 Preparation - Measurement

Measuring Scattering amplitudes

A particle scattering in an electromagnetic field. in (a) the particle arrives at spacetime point y. We perform an active diffeomorphism on (a) and obtain new (b). In new system the particle does not arrive at spacetime point y.

Just as GR doesn't determine the distance between spacetime points, it doesn't determine this probability; the only way to preserve general covariance is if $W(x, y)$ is constant.

The detector and source must be part of the system under study. Their motion and rate are directly effected by the gravitational field. Scattering probability is defined internally.

whether the construction leads, in a first approximation, to the general relativity scattering tree amplitudes.

10.3 Conventional Scattering Theory

The scattering matrix allows the to calculate experimentally observable scattering cross sections.

10.3.1 The Propagator in Conventional Scattering Theory

the notion of a propagator $W[q_2, t_2; q_1, t_1]$. Given a wave function $\psi(q_1, t_1)$ at a time t_1, the propagator gives the corresponding wave function at a later time t_2:

$$\psi(q_2, t_2) = \int W[q_2, t_2; q_1, t_1]\psi(q_1, t_1)d^n q_1. \tag{10.0}$$

$\psi(q_2, t_2)$ is the probability amplitude that the particle is at the point q_2 at the time t_2, so $W[q_2, t_2; q_1, t_1]$ is the probability amplitude for a particle at q_1 at time t_1 to arrive at the point q_2 at time t_2. The probability that it is observed at q_2 at time t_2 is

$$P(q_2, t_2; q_1, t_1) = |W[q_2, t_2; q_1, t_1]|^2. \tag{10.0}$$

Composition

$$W[q_3, t_3; q_1, t_1] = \int d^n q_2 W[q_3, t_3; q_2, t_2]W[q_2, t_2; q_1, t_1] \tag{10.0}$$

$\psi(q_2, t_2)$ obeys Schrödinger's equation

$$\frac{\hbar^2}{2m}\nabla^2_{x_2}\psi(q_2, t_2) + i\hbar\frac{\partial\psi(q_2, t_2)}{\partial t_2} = V(q_2, t_2)\psi(q_2, t_2). \tag{10.0}$$

if

$$\psi(q_2, t_2) = \phi(x_2, t_2) - \frac{i}{\hbar}\int W_0[q_2, t_2; q_1, t_1]V(q, t)\psi(q, t) \tag{10.0}$$

where $\phi(x_2, t_2)$ obeys the free-particle equation (i.e., $V = 0$) and if W_0 obeys

$$\frac{\hbar^2}{2m}\nabla^2_{x_2} W_0[q_2, t_2; q_1, t_1] + i\hbar\frac{\partial}{\partial t_2} W_0[q_2, t_2; q_1, t_1] = i\hbar\delta^3(q_2 - q_1)\delta(t_2 - t_1). \qquad (10.0)$$

the incoming wavefunction $\phi(\vec{x}, t)$ describes a particle from the distant past. We want the wavefunction that results from the interaction with the potential $V(\vec{x}, t)$ in the distant future. We assume an idealization that there is no interaction in the $t \to -\infty$. The initial wavefunction ϕ is therefore a solution of Schrödinger's equation for free particles. The exact wavefunction $\psi(\vec{x}, t)$ then

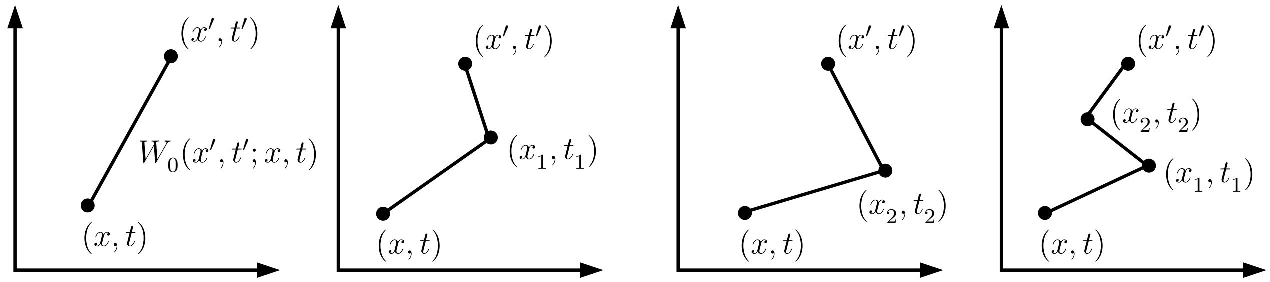


Figure 10.2: scattGraph.

$$\phi_f(\vec{x}', t') = \frac{1}{\sqrt{2\pi\hbar^3}} \exp[i(\vec{k}_f \cdot \vec{x} - \omega_f t')] \qquad (10.0)$$

Scattering matrix S

$$S_{ij} = \lim_{t\to-\infty} \left\langle \psi_f(\vec{x}, t) | \phi_i(\vec{x}, t) \right\rangle \qquad (10.0)$$

Creation Annihilation Operator Formulism

Unitarity of the $S-$Matrix

In background-dependent theory a property of the S matrix is its unitarity, when the Hamiltonian is hermitian. However, in background independent theories there is no "true time" with respect to which everything evolves. There is the time coordinate but this is an unphysical parameter in GR - observables do not depend on it - and there is no reason why evolution with respect to an unphysical parameter should be hermitian.

10.3.2 Conventional Scattering Theory in Quantum Field Theory

In the context of a Schrödinger picture of quantum field theory.

The state space at time t_1, $\mathcal{H}_{t_1}$, is a Fock space, on which the Hamiltonian is defined.

$$W[\varphi_1, t_1; \varphi_2, t_2] = \int_{varphi_1, t_1; \varphi_2, t_2} \mathcal{D}\phi \exp \frac{i}{\hbar} S[\phi], \tag{10.0}$$

or

$$W[\varphi_1, t_1; \varphi_2, t_2] = < \varphi_2 | e^{-iH(t_2 - t_1)} | \varphi_1 > \tag{10.0}$$

inserting sums of eigenstates of the energy,

$$\begin{aligned} W[\varphi_1, t_1; \varphi_2, t_2] &= < \varphi_2 | e^{-iH(t_2 - t_1)} | \varphi_1 > \\ &= \sum_{m,n} < \varphi_2 | \Psi_n >< \Psi_n | e^{-iH(t_2 - t_1)} | \Psi_m >< \Psi_m | \varphi_1 > \\ &= \sum_n \Psi_n[\varphi_2] \Psi_n^*[\varphi_1] e^{-iE_n(t_2 - t_1)} \end{aligned} \tag{10.-1}$$

we know see why the kernel is a field-to-field propagator,

$$\begin{aligned} \Psi[\varphi_2, t_2] &= < \varphi_2 | \Psi > \\ &= \int D\varphi_1 < \varphi_2 | e^{-iH(t_2 - t_1)} | \varphi_1 >< \varphi_1 | \Psi > \\ &= \int D\varphi_1 W[\varphi_1, t_1; \varphi_2, t_2] \Psi[\varphi_1] \end{aligned} \tag{10.-2}$$

Particles' scattering amplitudes can be derived from $W[\varphi_1, \varphi_2, T]$. For instance the 2-point function can be obtained as the analytic continuation of the Swinger function

$$S(x_1, x_2) = \lim_{T \to \infty} \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \, W[0, \varphi_1, T] \varphi_1(\vec{x}) W[\varphi_1, \varphi_2, (t_1 - t_2)] \varphi_2(\vec{x}) W[\varphi_2, 0, T]. \tag{10.-2}$$

this can be generalised to any n-point function where the times $t_1, \ldots, t_n$ are on the $t=0$ and the $t=T$ surfaces; these in turn, are sufficient to compute all scattering amplitudes, since time dependence of asymptotic states is trivial.

it is possible to write down creation and annihilation operators in this representation thus formalizing the particle interpretation of the theory:

$$\begin{aligned} a(\vec{k}) &= \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \left(\omega_k \psi(\vec{x}) + \frac{\delta}{\delta\psi(\vec{x})}\right) \\ a^\dagger(\vec{k}) &= \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \left(\omega_k \psi(\vec{x}) - \frac{\delta}{\delta\psi(\vec{x})}\right) \end{aligned} \tag{10.-2}$$

calculate the extension of the propagation kernel introduced by Feynman in the description of the quantum mechanics of a single particle, that is, the propagation kernel between field configurations defined on infinite spacial hyperplanes at fixed time.

10.4 Primer on the Graviton Propagator

Before we come on to the calculation itself we first review the graviton propagator and some of its properties, including how Newton's law can be derived from it.

to compare the low energy limit of the spin foam to quantities computed by standard QFT perturbative expansion, and the derivation of Newtonian interaction.

First discuss the photon propagator to get the idea in this simpler example.

10.4.1 Perturbation Theory of Scalar QED

The interactions of gravitons with matter can be calculated in the same way as the familiar photon case. Here we develop the quantum theory of a scalar charge coupled to the electromagnetic field.

In electromagnetism the invariance in A_μ comes about because the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is left unchanged by the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$$

We wish to calculate the four-potential $A_\mu(x)$ produced by a source current $J^\nu(x)$ term,

$$\Box A^\nu(x) - \partial^\mu \partial_\mu A^\nu(x) = 4\pi J^\nu(x) \tag{10.-2}$$

We are free to choose the most convenient gauge for the calculation intened to make.

We will choose the Lorentz gauge

$$\partial_\mu A^\mu(x) = 0$$

In momentum space this reads

$$k^\mu A^\mu(k) = 0.$$

$$\Box A^\nu(x) = 4\pi J^\nu(x) \tag{10.-2}$$

The solution of the above equation may be systematically formulated using the appropriate Green's function which we call $D_F(x-y)$, the proporgator for electromagnetism.

$$\Box D_F(x-y)(x) = 4\pi\delta^4(x-y). \tag{10.-2}$$

The Fourier-transformed proporgator is defined by

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp[-iq\cdot(x-y)]D_F(q) \tag{10.-2}$$

Using

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp[-iq\cdot(x-y)] \tag{10.-2}$$

and making comparison we get

$$D_F(q) = -\frac{4\pi}{q^2} \tag{10.-2}$$

10.4.2 Gravitons

As the electromagnetic interaction can be written in terms of a vector current J^μ

the gravitational interaction can be described in terms of the coupling of the energy-momentum tensor $T_{\mu\nu}$ to the gravitational field $h^{\mu\nu}$ with coupling constant κ,

$$\mathcal{L}_{int} = -\frac{1}{2}\kappa T_{\mu\nu} h^{\mu\nu}. \tag{10.-2}$$

$$h'_{ab} = h_{ab} - 2\epsilon\partial_{(a}\xi_{b)}. \tag{10.-2}$$

This is the change in the metric under an infinitesimal active diffeomorphism along the vector field $\epsilon\xi^a$ while maintaining the requirement that the perturbation be small.

polarization states

The line element

$$ds^2 = dt^2 - dx^2 - [1 - \epsilon h_{23}(t-x)]d\overline{y}^2 - [1 - \epsilon h_{23}(t-x)]d\overline{z}^2. \tag{10.-2}$$

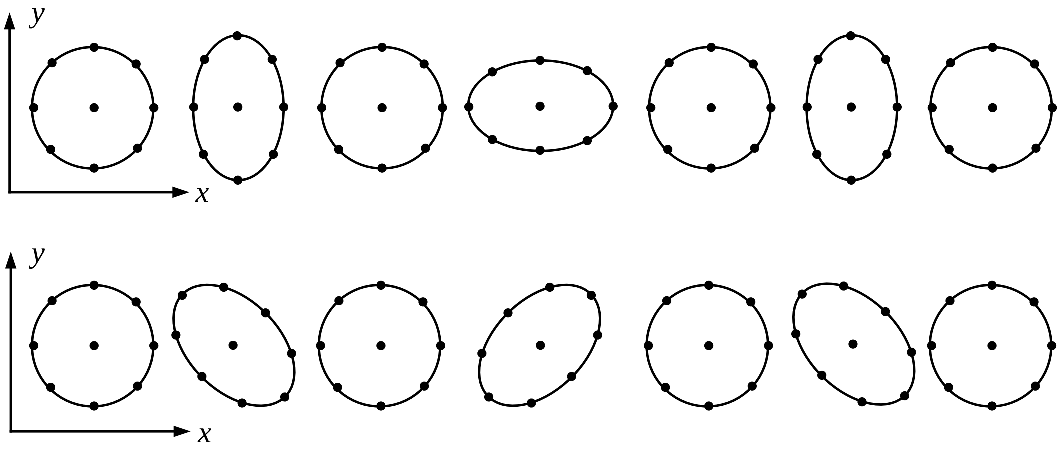


Figure 10.3: gravwavemode.

We now can proceed by perturbation theory and compute Feynmann diagrams to any order in λ.

$$g^{\mu\nu} = (\eta_{\mu\nu} + 2\kappa h_{\mu\nu})^{-1} = \eta^{\mu\nu} - 2\kappa h^{\mu\nu} + 4\kappa^2 h^{\mu}{}_{\beta} h^{\beta\nu} - 8\kappa^3 h^{\mu\beta} h_{\beta\tau} h^{\tau\nu} + \dots \tag{10.-2}$$

We need to calculate the formula for $\sqrt{-g}$

$$\begin{aligned}\sqrt{-g} &= \sqrt{-\eta}\exp\left(\frac{1}{2}tr\ln(\delta^{\beta}{}_{\nu} + 2\kappa h^{\beta}{}_{\nu})\right) \\ &= \exp\left[\frac{1}{2}tr\left(2\kappa h^{\beta}{}_{\nu} - \frac{1}{2}(2\kappa)^2 h^{\beta}{}_{\tau} h^{\tau}{}_{\beta} + \frac{1}{3}(2\kappa)^3 h^{\beta}{}_{\tau} h^{\tau}{}_{\sigma} h^{\sigma}{}_{\beta} + \dots\right)\right] \end{aligned} \tag{10.-2}$$

Plugging in these expressions for $\sqrt{-g}$ and $g^{\mu\nu}$ into the action, we get explicit forms for the coupling of matter and gravity:

$$\begin{aligned}\mathcal{L}(x) &= -\frac{\sqrt{-g}}{2\kappa^2}R + \frac{\sqrt{-g}}{2}(g^{\mu\nu}\partial_\mu\partial_\nu\varphi - m_0^2\varphi^2) \\ &= \frac{1}{2}\int d^4x(\partial^\mu\partial_\mu\varphi - m_0^2\varphi^2) - \kappa \\ +\dots \end{aligned} \tag{10.-3}$$

Any diagram which involves these vertices may now be calculated just as in electrodynamics.

10.4.3 Newton's Law from the Graviton Propagator

Electromagnetism

In electromagnetism the vector potential and current are related by

$$\Box A_\mu(x) = j_\mu(x)$$

or in momentum space

$$A_\mu(k) = -\frac{1}{k^2}j_\mu(k) \tag{10.-3}$$

The calculation of scattering amplitudes is in conventional QED is done with the help of propagators connecting currents - Feynman diagrams!

The dynamics of electromagnetism is contained in specification of the interaction between a current and the field by

$$j^\mu A_\mu.$$

or in terms of sources this becomes two currents:

$$-j'_\mu\frac{1}{k^2}j^\mu. \tag{10.-3}$$

For a particular choice of coordinates, the vector k_μ may be expressed as

$$k^\mu = (\omega, \mathbf{k}, 0, 0) \tag{10.-3}$$

Then the current-current interaction when the exchanged particle has four-momentum k_μ is given by

$$-j'_\mu \frac{1}{k^2} j^\mu = -\frac{1}{\omega^2 - \mathbf{k}^2}(j'_0 j^0 - j'_1 j^1 - j'_2 j^2 - j'_3 j^3). \tag{10.-3}$$

The conservation of charge

$$\partial_\mu j^\mu(x) = 0$$

in momentum space becomes the restriction

$$k_\mu j^\mu = 0. \tag{10.-3}$$

In the particular coordinate systme we have choosen, this restriction connects the third and fourth components of the currents by

$$j^3 = \frac{\omega}{\mathbf{k}} j^4. \tag{10.-3}$$

Substituting this into the amplitude (10.4.3), we find that

$$-j'_\mu \frac{1}{k^2} j^\mu = \frac{j'_4 j^4}{\mathbf{k}^2} + \frac{1}{\omega^2 - \mathbf{k}^2}(j'_1 j^1 + j'_2 j^2) \tag{10.-3}$$

We would wish to take the inverse Fourier transform to convert this to a space-interaction. The first term is independent of the frequency and the inverse Fourier transform of

$$\frac{j'_4 j_4}{\mathbf{k}^2}$$

is

$$\frac{e^2}{4\pi r}\delta(t - t') \tag{10.-3}$$

which represents an instantaneous acting Coulomb potential.

Gravity

$$E_0 = -\frac{1}{32\pi}\int d^3x \int d^3y \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}, \qquad m = \int \rho(\vec{x})\, d^3x. \tag{10.-3}$$

10.4.4 Physics From Spinfoam Kernel

The spinfoam graviton:

$$W_{\mu\nu\rho\sigma}(x,y) = \frac{1}{\mathcal{N}}\sum_s < s|h_{\mu\nu}|s >< s|h_{\rho\sigma}|s > \Psi_q[s]K[s]. \tag{10.-3}$$

Consider the normalized projections

$$W_{ab} := \frac{1}{n_a^2 n_b^2} n_a^\mu(x) n_b^\rho(x) n_b^\sigma(y) W_{\mu\nu\rho\sigma}(x,y). \tag{10.-3}$$

In the linearised continuum theory,

$$W_{ab} = f_\zeta(\varphi_{ab})\frac{1}{|x-y|^2}. \tag{10.-3}$$

where ζ is a gauge-fixing parameter. This result should be reproduced by the spinfoam gravition in the semiclassical limit.

10.5 Background-Independent Strategy

General Boundaries

In the non-general relativity case - A metric is induced on Σ from the background metric g_{ab} of the manifold $\mathcal{M}$. (b) In the general relativistic case the metric of the surface Σ is actively dragged across with the surface Σ.

Observables get defined on a closed finite boundary. Classical dynamics expresses a set of relations between them. The boundary observables are partial obsevables - they are quantities whose measurement can be operationally defined in principle - in scattering experiments the geometry

we recapitulate

In the theoretical analysis of an experiment, one (arbitrary) coordinate system $\vec{x}, x^o$, and then equations of motion, as (N.-19) with the data in the following way. First we have to locally solve the coordinates $\vec{x}, x^o$ with respect to quantities $f_1 \dots f_4$ that represent the physical objects used as clocks and as spatial reference system

$$f_1(\vec{x}, x^o) \dots f_4(\vec{x}, x^o) \rightarrow \vec{x}(f_1, \dots\dots, f_4), x^o(f_1, \dots, f_4) \tag{10.-3}$$

and then express the rest of the remaining fields (f_i $i = 5 \dots N$) as functions of $f_1 \dots f_4$

$$f_i(f_1, \dots, f_4) = f_i(\vec{x}(f_1, \dots, f_4), x^o(f_1, \dots, f_4)). \tag{10.-3}$$

If, for instance, $F(\vec{x}, t)$ is a scalar, then for every quadruplet of numbers $f_1 \dots f_4$, the quantity $F_{(f_1 \dots f_4)} = F(f_1, \dots, f_4)$ can be compared with experimental data. This procedure is routinely performed in any analysis of experimental gravitational data - the physical time f_4 representing the the reading of the laboratory clock.

Distance and time separation must be extracted from the dynamical variables. In a scattering experiment we measure incoming and outgoing particles as well as distances between instruments and elapsed time. The former are the matter field variables, the latter are gravitational field variables.

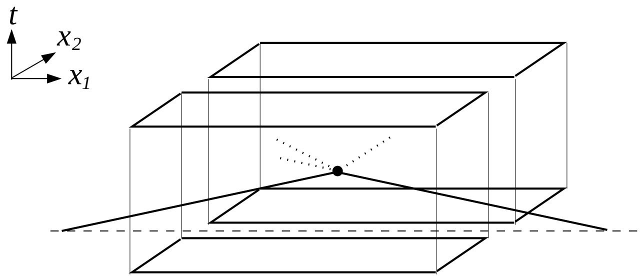


Figure 10.4: scatteringP.

$$W(x, y; \Sigma, \Psi) = \int \mathcal{D}\varphi \; \varphi(x) \; \varphi(y) \; W(\Sigma, \Psi) \; \Psi[\varphi] \tag{10.-3}$$

$$W[\varphi] = \int_{\phi|_\Sigma} \mathcal{D}\phi e^{-\frac{i}{\hbar} S[\phi]}. \tag{10.-3}$$

The boundary value of the gravitational field determine the geometry of the boundary surface Σ.

the boundary values of the gravitational field on the boundary surface Σ determines the the geometry of Σ, and therefore encodes the relative distance of the measuring devices and the proper time elapsed from the bottom to the top of the spacetime region considered.

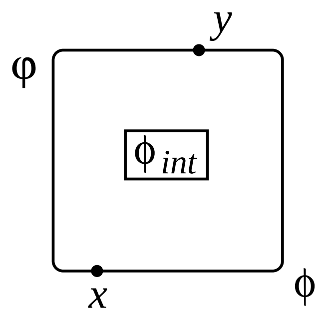


Figure 10.5: Boundfuncxy. Here x and y are points in the 3d metric manifold comprising a boundary with the topology of a 3-sphere S_3 and metric q, i.e. (S_3, q). Metric relations between x and y are determined by q. Under an active diffeomorphism the points x and y change together with q, leaving the metric relations between x and y invariant.

the metric of Σ is not coded in the location of Σ on a manifold: it is coded in the boundary value of the gravitational field on Σ.

Scattering probabilities are determined internally. Scattering amplitude and the spacetime geometry are both encoded in the quantum state. Reflects the fact that there are no *external* reference bodies.

Transition amplitudes are associated with regions of spacetime and states are associated with their boundaries.

But in GR the information on the geometry of a surface is not in Σ. It is in the state of the (gravitational) field on the surface!

Hence, choose $\Psi[\varphi]$ to be a state peaked on a given geometry q of Σ!

Distance and time separations between x and y are now well defined with respect to the mean boundary geometry q.

There is a region, R, of spacetime where the scattering experiment is performed.

expresses the amplitude of having a certain set of initial and final fields, as well as boundary fields, measured by apparatus that are located in spacetime, in the manner described by (the geometry) the surface Σ, this geometry being determined by φ itself.

$$W(x_1, \ldots, x_n) = Z^{-1} \int \mathcal{D}\varphi \; \varphi(x_1) \ldots \varphi(x_n) W[\varphi, \overline{\Sigma}] W[\varphi, \Sigma] \tag{10.-3}$$

$$W_0[\varphi, \overline{\Sigma}] = \int_{\phi|_\Sigma = \varphi} \mathcal{D}\phi_{\overline{R}} e^{-S^{(0)}_{\overline{R}}[\phi]} \equiv \Psi_\Sigma[\varphi]. \tag{10.-3}$$

we may expect an expression of the form

$$W(x_1, \ldots, x_n; q) = Z^{-1} \int_{\phi|_\Sigma = \varphi} \mathcal{D}\varphi\, \varphi(x_1) \ldots \varphi(x_n)\, \Psi_q[\varphi]\, W[\varphi] \qquad (10.\text{-}3)$$

the boundary can be viewed as expressing initial, final as well as boundary values of ϕ and $W[\phi]$ expresses the corresponding amplitude. The quantum dynamics give probabilities for ensembles of boundary measurements.

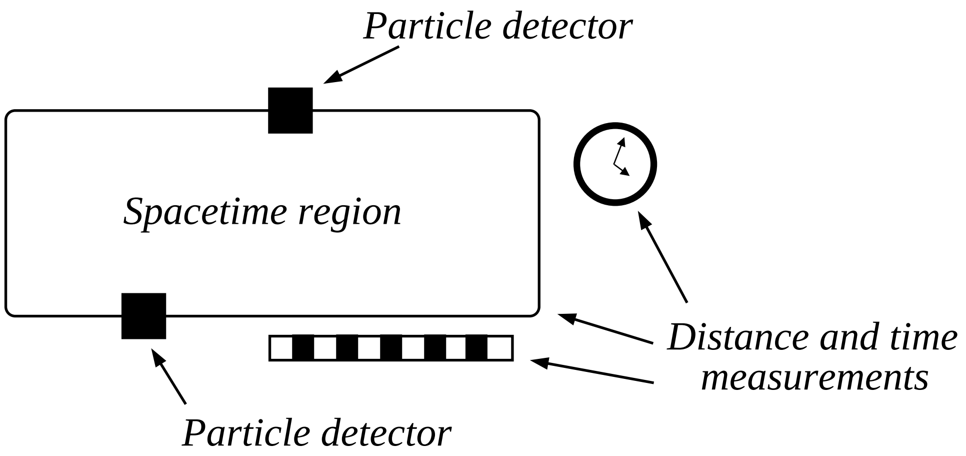


Figure 10.6: RegionPartDec. The boundary geometry is measured by the physical apparatus that surronds a potential interaction region.

In GR distance and time measurements are field measurements like the other ones: they determine the boundary data of the problem.

10.6 Elementary Quantum Mechanics for Comparison Purposes

Consider the two-point function of a single harmonic oscillator with mass m angular frequency ω. This is given by

$$G_0(t_1, t_2) =< 0|x(t_1)x(t_2)|0 >=< 0|xe^{-\frac{i}{\hbar}H(t_1-t_2)}x|0 > \qquad (10.\text{-}3)$$

In the Schrödinger picture, the R.H.S. of (N.-19) reads

$$G_0(t_1, t_2) = \int dx_1 dx_2 \overline{\psi_0(x_1)} x_1 W(x_1, t_1; x_2, t_2) x_2 \psi_0(x_2) \qquad (10.\text{-}3)$$

The following fact is not difficult to prove. $W(x_1, t_1; x_2, t_2)$ is the amplitude to propagate from one point to another in a given time interval. If a particle is initially at position x_2

and we consider W as a function of the final position and time, it is none other than the wave function for a particle with the specific initial condition.

As such, the propagator satisfies the Schrödinger equation at its final point and is the wave function after the elapse of a time $t_1 - t_2$.

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t) e^{i\int_{-\infty}^{t_1} \mathcal{L}(x,dx/dt)} \tag{10.-3}$$

is the functional integral restricted to the interval $(-\infty, t_1)$. As is well known from Euclidean theory this gives the vacuum state.

$$t_2 - t_1 \to -iT. \tag{10.-3}$$

$$1 = \sum_n^{\infty} |n><n|, \qquad H|n>= |n> E_n, \tag{10.-3}$$

$$\begin{aligned} <x_1|e^{-HT}|x_2> &= \sum_n <x_1|e^{-HT}|n><n|x_2> \\ &= \sum_n e^{-E_nT} <x_1|n><n|x_2> \\ &= \sum_n e^{-E_nT} \psi_n(x_1)\psi_n(x_2)^* \end{aligned} \tag{10.-4}$$

$$\psi_0(x) = \lim_{T\to\infty} W[x, 0, T] \tag{10.-4}$$

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t)\ e^{i\int_{-\infty}^{t_1} \mathcal{L}(x,\dot{x})} \tag{10.-4}$$

$$1 = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t)\ e^{i\int_{-\infty}^{\infty} \mathcal{L}(x,\dot{x})} \tag{10.-4}$$

Similarly:

$$\overline{\psi_0(x)} = \int_{x(t_2)=x}^{x(+\infty)=0} \mathcal{D}x(t) e^{i\int_{t_2}^{+\infty} \mathcal{L}(x,dx/dt)} \tag{10.-4}$$

Consider an harmonic oscillator, with mass m compute the 2-point function $G(t_1, t_2) =< 0|x(t_1)x(t_2)|0 >$ in canonical formulism

we get

$$< 0|x(t_1)x(t_2)|0 >=< 0|xe^{iH(t_2-t_1)}x|0 >= \frac{1}{2\omega}e^{i\frac{3}{2}\omega T} \tag{10.-4}$$

with $T := t_2 - t_1$.

Compute it from the propagator kernel $W[x_1, x_2, T]$, as in (N.-19).

We have

$$G(t_1, t_2) = \frac{1}{\mathcal{N}} \int dx_1 dx_2 \; W[x_1, x_2, T] \; \Psi_0[x_1]x_1\Psi_0[x_2]x_2, \tag{10.-4}$$

where the normalisation is $\mathcal{N} = \int dx_1 dx_2 W[x_1, x_2, T]\Psi_0[x_1]\Psi_0[x_2]$, and $\Psi_0[x]$ is the vacuum state. Using expressions (N.-19)

$$\begin{aligned} \Psi_0[x] &= \left(\frac{\omega}{\pi}\right)^{1/4} \exp -\frac{1}{2}\frac{\omega}{\hbar}x^2, \\ W[x_1, x_2, T] &= \sqrt{\frac{\omega}{2\pi i \sin\omega T}} \exp\left(-i\frac{\omega}{2}\frac{(x_1^2 + x_2^2)\cos\omega T - 2x_1x_2}{\sin\omega T}\right) \end{aligned} \tag{10.-4}$$

Substituting (10.6) into this, we obtain

$$G(t_1, t_2) = \frac{1}{\mathcal{N}} \int dx_1 dx_2 \; x_1x_2 e^{-\frac{1}{2}x_iA_{ij}x_j} \tag{10.-4}$$

with

$$A_{ij} = \omega \begin{pmatrix} 1 + i\cot\omega T & -i/\sin\omega T \\ -i/\sin\omega T & 1 + i\cot\omega T \end{pmatrix}$$

This is easily evaluated, we introduce the term $j_1x_1 + j_2x_2$ in the exponent

$$\frac{1}{\mathcal{N}} \int dx_1 dx_2 \; x_1x_2 \exp\left(-\frac{1}{2}x_iA_{ij}x_j + j_1x_1 + j_2x_2\right)$$

$G(t_1, t_2)$ is then

$$G(t_1,t_2)=\frac{\partial^2}{\partial j_2\partial j_1}\left(\frac{1}{\mathcal{N}}\int dx_1dx_2\; e^{-\frac{1}{2}x_iA_{ij}x_j+j_1x_1+j_2x_2}\right)\Big|_{j=0}$$

This Gaussian integral in the brackets is evaluated by completing the square (see appendix O), the answer is

$$\frac{\pi}{\det A^{1/2}}\exp(j_iA^{-1}_{ij}j_j).$$

Finally, we get

$$G(t_1,t_2)=A^{-1}_{12}=\frac{1}{2\omega}e^{i\omega T}.$$

Up to the vacuum energy contribution $e^{i\frac{1}{2}\omega T}$, the result coincides with the canonical evaluation.

$$G_0(t_1,t_2)=\int\mathcal{D}x(t)\;x(t_1)x(x_2)e^{i\int_{-\infty}^{\infty}\mathcal{L}(x,dx/dt)} \tag{10.-4}$$

Lets break it up into

$$G_0(t_1,t_2)=\int dx_1dx_2\;\overline{\psi_0(x_1)}x_1\;W[x_1,x_2;t_1,t_2]\;x_2\psi_0(x_2) \tag{10.-4}$$

where

$$W[x_1,x_2;t_1,t_2]=\int_{x(t_2)=x_2}^{x(t_1)=x_1}\mathcal{D}x(t)\;e^{i\int_{t_2}^{t_1}\mathcal{L}(x,dx/dt)} \tag{10.-4}$$

is the path integral restricted to the open interval (t_1,t_2) integrated over the paths that start at x_2 and end at x_1

The normalisation of the measure in (N.-19) is determined by

$$1=\int\mathcal{D}x(t)\;e^{i\int_{-\infty}^{\infty}\mathcal{L}(x,dx/dt)} \tag{10.-4}$$

Breaking this integral in the same manner as we did above one gets

$$1=\int dx_1dx_2\;\overline{\psi_0(x_1)}\,W[x_1,x_2;t_1,t_2]\;\psi_0(x_2) \tag{10.-4}$$

or equivalently

$$1 =< 0|e^{-\frac{i}{\hbar}H(t_1-t_2)}|0 > \qquad (10.\text{-}4)$$

10.6.1 Covariant Interpretation

$$G_0(t_1, t_2) =< W_{t_1,t_2} \mid \hat{x}_1\hat{x}_2\Psi_0 >, \qquad (10.\text{-}4)$$

in terms of states and operators in the Hilbert space $\mathcal{K}_{t_1,t_2} = \mathcal{H}^*_{t_1} \otimes \mathcal{H}_{t_2}$

(i) "the boundary state"

$$\Psi_0(x_1, x_2) = \overline{\psi_0(x_1)}\psi_0(x_2)$$

represents the joint boundary configuration of the system at the two times t_1 and t_2 if no excitation of the oscillator is present; it describes the joint outcome of a measurement at t_1 and measurement t_2, both detecting no excitations.

(ii) The two operators $\hat{x}_1$ and $\hat{x}_2$ create an ("incoming") excitation at t_2 and an ("outgoing") excitation at t_1; thus the state $\hat{x}_1\hat{x}_2\Psi_0$ can be interpreted as a boundary state representing the joint outcome of a measurement at t_1 and a measurement at t_2, both of them detecting a single excitation.

(iii) The bra $W_{t_1,t_2}(x_1, x_2) = W[x_1, x_2; t_1, t_2]$ is the linear functional coding the dynamics, whose action is on the two-excitation states associates it an amplitude, which can be compared with other similar amplitudes. For instance, observe that

$$< W_{t_1,t_2} \mid \hat{x}_2\Psi_{t_1,t_2} >= 0; \qquad (10.\text{-}4)$$

that is, the probability amplitude of measuring a single excitation at t_2 and no excitation at t_1 is zero.

Finally the

$$1 =< W_{t_1,t_2} |\Psi_0 >; \qquad (10.\text{-}4)$$

which requires that the boundary state Ψ_0 is a solution of the dynamics, in the sense

10.6.2 Regarding Phases of the Boundary State and of the Propagator

There is a phenomenon regarding phase of the boundary state $\Psi_{q_1,p_1,q_2,p_2}(x_1, x_2)$ and the propagator $W_{t_1,t_2}(x_1, x_2)$ that will be important to notice in ascessing the viability of the scattering amplitude calculation in the next section.

We know this from elementary quatum mechanics the phase of the semiclassical state determines where the state is peaked in theconjugate variables (see (7.-12)).

$$S[q'] = S[q+\eta] = S[q] + \int dt\eta(t)\frac{\delta S[q]}{\delta q(t)} + \mathcal{O}(\eta^2). \tag{10.-4}$$

The two paths contribute exp$iS[q]/\hbar$ and exp$iS[q']/\hbar$ to the PI; the combined contribution is

$$A \approx e^{iS[q]/\hbar}\left(1 + \exp\frac{i}{\hbar}\int dt\eta(t)\frac{\delta S[q]}{\delta q(t)}\right), \tag{10.-4}$$

where we have neglected corrections of order η^2. We see that the difference in phase between the two paths, which determines the interference between the two contributions, is

$$\hbar 1 R dt(t)S[q]/q(t).$$

We see that the smaller the value of h, the larger the phase difference between two given paths. So even if the paths are very close together, so that the difference in actions is extremely small, for sufficiently small $\hbar$ the phase difference will still be large, and on average destructive interference occurs.

The exceptional path which extremizes the action, i.e., the classical path, $q_c(t)$. For this path, $S[q_c + \eta] = S[q_c] + \mathcal{O}(\eta^2)$. Thus the classical path and a very close neighbour will have actions which differ by much less than two randomly-chosen but equally close paths (Figure N.-19). This means that for fixed closeness of two paths and for fixed $\hbar$, paths near the classical path will on average interfere constructively (small phase difference) whereas for random paths the interference will be on average destructive.

Thus heuristically, we conclude that if the problem is classical (action $S \gg \hbar$), the most important contribution to the PI comes from the region around the path which extremizes the PI. In other words, the particle's motion is governed by the principle that the action is stationary. This, of course, the principle of least action from which the Euler-Lagrange equations of classical mechanics are derived.

In the classical limit the propagator is approximated by the exponential of a solution to the Hamilton-Jacobi system

Expanding the Hamilton function around q_1 and q_2 to first order

$$S_{t_1,t_2}(x_1,x_2) = S_{t_1,t_2}(q_1,q_2) + \frac{\partial S}{\partial x_1}(x_1 - q_1) + \frac{\partial S}{\partial x_2}(x_2 - q_2) \tag{10.-4}$$

but

$$\frac{\partial S}{\partial x_1} = p_1 \quad \text{and} \quad \frac{\partial S}{\partial x_2} = -p_2. \tag{10.-4}$$

10.7 3d "Nutshell" Model

Implementation in the full 4d quantum gravity is difficult because of the technical complexity of the theory. It is useful to test and illustrate it in a simple context.

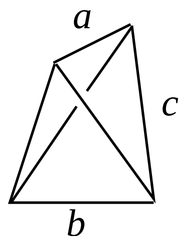


Figure 10.7: equiltetra.

10.7.1 Elementary geometry of equilateral tetrahedra

Consider a tetradhedron embedded in euclidean three-dimension space. Let a be the length of one of the edges (we call it the "top" edge) and b the length of the opposite ("bottom") edge, namely the edge disjoint from the top edge.

Elementary geometry gives (worked examples)

$$\sin\frac{\theta_a}{2} = \frac{b}{\sqrt{4c^2 - a^2}}, \qquad \sin\frac{\theta_b}{2} = \frac{a}{\sqrt{4c^2 - b^2}}, \qquad \cos\theta_c = \frac{ab}{\sqrt{(4c^2 - a^2)(4c^2 - b^2)}} \tag{10.-4}$$

It follows that

$$\cos\theta_c = \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}. \tag{10.-4}$$

For later purpose, we consider also the case in which $c \gg a, b$. In this case, we have, to the first relevant order,

$$\theta_a = \frac{b}{c}, \qquad \theta_b = \frac{a}{c}, \qquad \theta_c = \frac{\pi}{2} - \frac{ab}{4c^2} \tag{10.-4}$$

and

$$\theta_a = \frac{\pi}{2} - \frac{\theta_a\theta_b}{4} \tag{10.-4}$$

We consider also the three external angles at the edges

$$k_a(a,b,c) = \pi - \theta_a(a,b,c), \qquad k_b(a,b,c) = \pi - \theta_b(a,b,c), \qquad k_c(a,b,c) = \pi - \theta_c(a,b,c). \tag{10.-4}$$

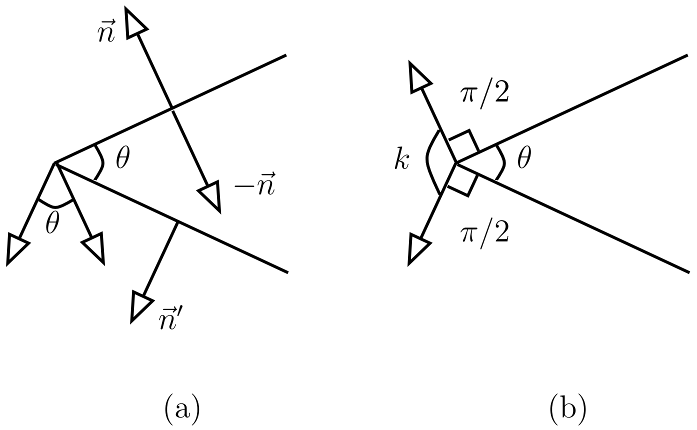


Figure 10.8: The $\vec{n}$ and $\vec{n}'$ are the outward normls to the triangles. (a) The internal angle θ is obtained from $\cos\theta = (-\vec{n}) \cdot \vec{n}'$. (b) The external angle k is obtained from $\cos k = \vec{n} \cdot \vec{n}'$, and as can be seen from the diagram $k = \pi - \theta$.

Notice that these are the discretized (discretized due to triangulation) extrinsic curvature of the surface of the tetrahedron. We have denoted them with the letter k, as it is often used for the extrinsic curvature. Using (10.7.1) and the relation (10.7.1), the relation between the edge lengths a, b, c and the externl angles k_a, k_b, k_c can be written in the form

$$
\begin{aligned}
a &= \sqrt{4c^2 - b^2}\cos\frac{k_b}{2}, \\
b &= \sqrt{4c^2 - a^2}\cos\frac{k_a}{2}, \\
ab &= -\sqrt{(4c^2 - a^2)(4c^2 - b^2)}\cos k_c \; ; \qquad (10.\text{-}5)
\end{aligned}
$$

(where we used $\sin(\pi/2 + x) = \cos x$ and $\cos(\pi + x) = -\cos x$) while (10.7.1) reads

$$\cos k_c = -\cos\frac{k_a}{2}\cos\frac{k_b}{2.} \qquad (10.\text{-}5)$$

10.7.2 Classical theory

Regge action

Consider the action of general relativity, in the case of a simply connected finite region of spacetime $\mathcal{R}$. In the presence of a boundary $\Sigma = \partial\mathcal{R}$ we have to add a boundary term to the Einstein-Hilbert action, in order to have well defined equations of motion. The full action reads

$$S_{GR}[g] = \int_{\mathcal{R}} d^n x \sqrt{\det g} R + \int_{\Sigma} d^{n-1}x \sqrt{\det q}\; k. \qquad (10.\text{-}5)$$

Since the bulk action vanishes on a vacuum solution of the equations of motion, the Hamilton function of GR reads

$$S[q] = \int_{\Sigma} d^{n-1}x \sqrt{\det q}\; k[q] \qquad (10.\text{-}5)$$

Let i be the index labelling the links of the triangulation and denote the length of the link i by l_i. In three dimensions, the bulk Regge action is

$$S_{Regge} = -\sum_i l_i \left(2\pi - \sum_t \theta_{i,t}(l)\right), \qquad (10.\text{-}5)$$

where $\theta_{i,t}(l)$ is the dihedral angle of the tetrahedron t at the link i, and the angle in the paranthesis is therefore the deficit angles at i. The boundary term is

$$S_{boundary} = - \sum_{boundary} l_i \left(\pi - \sum_t \theta_{i,t}(l) \right), \tag{10.-5}$$

We choose the minimalist triagulation formed by a single tetrahedron, and furthermore, consider the case in which the tetrahedron is equilaterial. Then there are no internal links, the Regge action is the same as the Regge Hamiltonian function, and is given by

$$\begin{aligned} S_{boundary} &= - \sum_{boundary} l_i \left(\pi - \theta_i(l)\right) \\ &= - \sum_{boundary} l_i k_i(l) \end{aligned}$$

or exlicitly

$$S(a,b,c) = -ak_a(a,b,c) - bk_b(a,b,c) - 4ck_c(a,b,c). \tag{10.-7}$$

The expression for the dihedral angles as functions of the edge length, for a flat interior geometry, is given in (10.7.1) and (10.7.1). Substituting these into (10.7.2)

$$\begin{aligned} S(a,b,c) &= a \left(2 \arcsin \frac{b}{\sqrt{4c^2 - a^2}} - \pi \right) + b \left(2 \arcsin \frac{a}{\sqrt{4c^2 - b^2}} - \pi \right) \\ &\quad + 4c \left(\arccos \frac{ab}{\sqrt{4c^2 - a^2}\sqrt{4c^2 - b^2}} - \pi \right) \end{aligned} \tag{10.-7}$$

The dynamical model and its physical meaning

Define the momenta

$$p_a(a,b,c) = \frac{\partial S(a,b,c)}{\partial a}, \qquad p_b(a,b,c) = \frac{\partial S(a,b,c)}{\partial b}, \qquad p_c(a,b,c) = \frac{\partial S(a,b,c)}{\partial c} \tag{10.-7}$$

and equate them to constants

$$p_a(a,b,c) = p_a, \qquad p_b(a,b,c) = p_b, \qquad p_c(a,b,c) = p_c. \tag{10.-7}$$

The action $S(a,b,c)$, given in (10.-7), is a homogeneous function of degree one meaning

$$S(\rho a, \rho b, \rho c) = \rho S(a, b, c). \tag{10.-7}$$

Define $a' = \rho a$, $b' = \rho b$, $c' = \rho c$. Then by differentiating both sides of (10.7.2) with respect to ρ we find

$$\begin{aligned} S(a,b,c) &= \frac{\partial S}{\partial a'}\frac{\partial a'}{\partial \rho} + \frac{\partial S}{\partial b'}\frac{\partial b'}{\partial \rho} + \frac{\partial S}{\partial a'}\frac{\partial c'}{\partial \rho} \\ &= a\frac{\partial S}{\partial a'} + b\frac{\partial S}{\partial b'} + c\frac{\partial S}{\partial c'} \\ &= a\frac{\partial S}{\partial(\rho a)} + b\frac{\partial S}{\partial(\rho b)} + c\frac{\partial S}{\partial(\rho c)}. \end{aligned}$$

Setting $\rho = 1$ gives

$$S(a,b,c) = a\frac{\partial S(a,b,c)}{\partial a} + b\frac{\partial S(a,b,c)}{\partial b} + c\frac{\partial S(a,b,c)}{\partial a}. \tag{10.-10}$$

This observation simplfies the calculation of the momenta as it allows us to identify immediately, via (10.7.2), that

$$p_a(a,b,c) = -k_a(a,b,c), \qquad p_b = -k_b(a,b,c), \qquad p_c = -4k_c(a,b,c). \tag{10.-10}$$

Inserting the explicit form (10.7.1) of the angles, we obtain the evolution equation

$$\begin{aligned} a &= \sqrt{4c^2 - b^2}\cos\frac{p_b}{2}, \\ b &= \sqrt{4c^2 - a^2}\cos\frac{p_a}{2}, \\ ab &= -\sqrt{(4c^2 - a^2)(4c^2 - b^2)}\cos\frac{p_c}{4} \end{aligned} \tag{10.-11}$$

10.7.3 Time evolution

10.7.4 Quantum theory

10.7.5 Time evolution in the quantum theory

10.8 Geometry of a 3-Simplex and a 4-Simplex

10.8.1 3-Simplex

10.8.2 4-Simplex

Let us consider five 4d unit vectors $\hat{N}_I \in \mathcal{S}^3$, $I = 1 \dots 5$, and introduce the ten angles defined by their scalar products, $\cos\phi_{IJ} = \hat{N}_I \cdot \hat{N}_J$, with the convention $\phi_{II} = 0$. Finally, we define the 5×5 Gram matrix, $G_{IJ} = \cos\phi_{IJ}$. These five vectors are not linearly independent, so we can find $v_I \in \mathbb{R}^5$ such that:

$$\sum_{I=1} v_I \hat{N}_I = 0. \qquad (10.\text{-}11)$$

This means that the 5-vector v_I is a null vector for the Gram matrix G_{IJ} . In particular, we get a constraint on the angles ϕ_{IJ}:

$$\sum_{I=1} v_I \hat{N}_I \cdot \hat{N}_J = 0 \quad \Rightarrow \quad \text{for all } J. \qquad (10.\text{-}11)$$

This constraint can be interpreted geometrically as follows. The five unit vectors define a unique geometric 4- simplex (embedded in $\mathbb{R}^4$) up to a global scale (4-volume of the simplex). They are the (outward) normals to the five tetrahedra of the 4-simplex. The closure condition of the 4-simplex reads exactly as (53) with the v_I being the (oriented) 3-volumes of the tetrahedra. Furthermore, we can differentiate the equation (54) and contract it with the null vector. This gives:

$$\sum_{I,J} v_I v_J \sin\phi_{IJ}\, d\phi_{IJ} \qquad (10.\text{-}11)$$

10.9 Quantum Gravity - The Four Ingredients

Giving meaning to the expression

$$W(x,y;q) = \int \mathcal{D}\varphi \; \varphi(x)\varphi(y) \; W[\varphi] \; \Psi_q[\varphi] \tag{10.-11}$$

(i) $\int \mathcal{D}\phi \to \sum_{s-knots}$;

(ii) $W[\phi] \to W[s]$ defined by GFT spinfoam model;

(iii) $\Psi_q \to$ a suitable coherent state on the geometry q;

(iv) $\phi(x) \to$ graviton field operator from LQG.

(i) a proper definition of the space of 3d fields φ integrated over and a well posed definition of the integration measure.

(ii) An explicit expression for the boundary propagator $W[\varphi]$.

(iii) An explicit expression for the boundary state $\Psi_q[\varphi]$.

(iv) A definition of the field operator $\varphi(x)$.

10.9.1 Space of 3d Fields

$$W(x_1,\dots,x_n;q) = Z^{-1} \sum_s \; c(s) \; \varphi(x_1)\dots\varphi(x_n) \; \Psi_q[s] \; W[s] \tag{10.-11}$$

10.9.2 The Boundary Propagator $W[\varphi]$ from Group-Field-Theory

A spin foam model can be recast in the form of a rather peculiar field theory over the cartesian product of a group [].

$$W[s] = \int \mathcal{D}\Phi \; f_s[\Phi] \; e^{\int \Phi^2 - \lambda \int \Phi^5}. \tag{10.-11}$$

$$W[s] = \sum_{\partial\sigma} \prod_{faces} A_{faces} \prod_{verties} A_{vertex} \tag{10.-11}$$

which has a nice interpretation as a discretization of the Misner-Hawking sum over geometries

$$W(^3g) = \int_{\partial g = ^3g} \mathcal{D}g \; e^{iS_{Eintein-Hilbert}[g]} \tag{10.-11}$$

with background triangulations summed over as well.

To first order in λ the only nonvanishing connected term in $W[s]$ is for

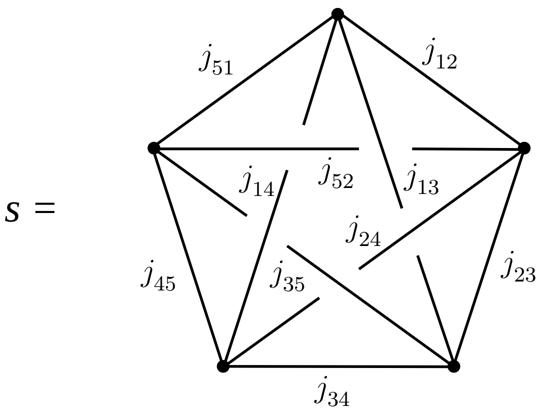


Figure 10.9: tenjsymbRov.

And the dominate contribution for large j is given by the spinfoam σ dual to a *single* 4-simplex. This is

$$W[s] = \frac{\lambda}{5!} \left(\prod_{n<m} dim(j_{nm}) \right) \; A_{vertex}(j_{nm}) \tag{10.-11}$$

Regge action

Consider the action of general relativity, in the case of a simply connected finite region of spacetime $\mathcal{R}$. In the presence of a boundary $\Sigma = \partial \mathcal{R}$ we have to add a boundary term to the Einstein-Hilbert action, in order to have well defined equations of motion. The full action reads

$$S_{GR}[g] = \int_{\mathcal{R}} d^n x \sqrt{\det g} R + \int_{\Sigma} d^{n-1} x \sqrt{\det q} \; k. \tag{10.-11}$$

Since the bulk action vanishes on a vacuum solution of the equations of motion, the Hamilton function of GR reads

$$S[q] = \int_{\Sigma} d^{n-1} x \sqrt{\det q} \; k[q] \tag{10.-11}$$

$$S_{Regge} = \tag{10.-11}$$

10.9.3 The Boundary State $\Psi_q[\varphi]$

Linearized quantum gravity gives us a crucial hint, and provides us with a straightforward way to *interpret* semiclassical boundary states. Indeed, consider linearized quantum gravity, namely the well defined theory of a noninteracting spin-2 graviton field $h_{\mu\nu}(x)$ on a flat spacetime with background metric $g^0_{\mu\nu}$. This theory has a preferred vacuum state $|0>$.

can be obtained from the analysis of the coherent states in LQG. Here q is the intrinsic and extrinsic (a coherent state depends on both classical position and momentum) geometry of the closed 3d surface

Choose a boundary geometry q: Let q be the geometry of the 3d boundary (Σ, q) of a spherical 4d ball, with linear size $L \gg \sqrt{\hbar G}$.

Interpret s as the (dual) of a triangulation of this geometry. Choose a *regular* triangulation of (Σ, q); interpret the spins as the areas of the corresponding triangles, using the standard LQG interpretation of spin networks.

This determines the "background" spins $j^{(0)}_{nm} = j_L$. $\Psi_q(s)$ must be peaked on these values. Choose a Gaussian state around these values with α, to be determined.

A Gaussian can have an arbitrary *phase*:

$$\Psi_q[s] = \exp\left\{-\frac{\alpha}{2}\sum_{n<m}(j_{nm} - j^{(0)}_{nm})^2 + i\sum_{n<m}\Phi^{(0)}_{nm}j_{nm}\right\}. \tag{10.-11}$$

must be a coherent state, determined by coordinate and *momentum*, namely by the intrinsic 3-geometry and the *extrinsic* 3-geometry q!!

The $\Phi^{(0)}_{nm} = \phi$ are the background **dihedral angles**.

The phase factor in this state is important. As we know from elementary quantum mechanics (see equation (7.-12)), it determines where the state is peaked in the variables conjugate to the spins j_{mn}. The form of the Regge action is $S_{Regge} = \sum_{n<m}\Phi_{nm}(j_{mn})j_{nm}$, where $\Phi_{nm}(j_{mn})$ are dihedral angles at the triangles, which are function of the areas themselves and that $\partial S_{Regge}/\partial j_{nm} = \Phi_{nm}$. It is then easy to see that these dihedral angles are the variables conjugate to the spins.

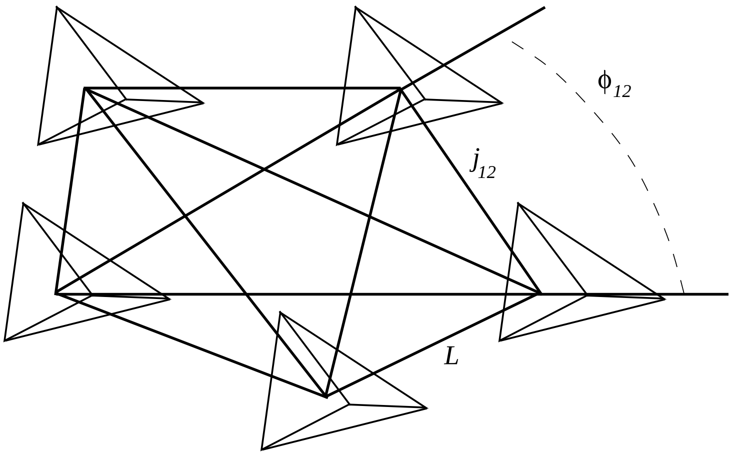


Figure 10.10: DihedAngRov.

10.9.4 Definition of the Field Operator $\varphi(x)$

$$G_q^{abcd}(x,y) = \frac{\sum_s W[s]\ \hat{h}^{ab}(x)\hat{h}^{cd}(y)\ \Psi_q[s]}{\sum_s W[s]\ \Psi_q[s]} \tag{10.-11}$$

the fluctuation of the metric over the flat metric

$$h_s^{ab}(x) = (q_{S(s)}^{ab}(x) - \delta^{ab}) = E^{ai}(x)E^{bi}(x) - \delta^{ab} \tag{10.-11}$$

We know the action of the operator $\hat{E}^{ai}$ on a spin network state $|s>$,

$$E^{Ii}(n)E_i^I(n)|s >= (8\pi\hbar G)^2\ j_I(j_I+1)|s>, \tag{10.-11}$$

whether or not the action is diagonal depends on the orientation of the surface associated to the area operator.

Define

$$W^{abcd}(x,y;q) = \mathcal{N}\sum_{ss'} W[s'] < s'|h^{ab}(x)h^{cd}(y)|s > \Psi_q[s]. \tag{10.-11}$$

$$W(L) = W^{abcd}(x,y;q)n_a n_b n_c n_d \tag{10.-11}$$

Standard perturbation theory gives

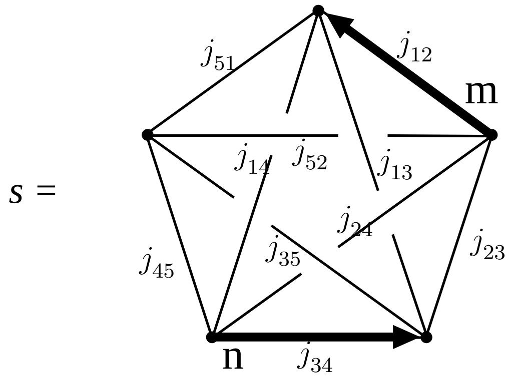


Figure 10.11: tenjsymbRov2.

$$W(L) = i\frac{8\pi}{4\pi^2}\frac{1}{|x-y|_q^2} = i\frac{8\pi\hbar G}{4\pi^2}\frac{1}{L^2} \tag{10.-11}$$

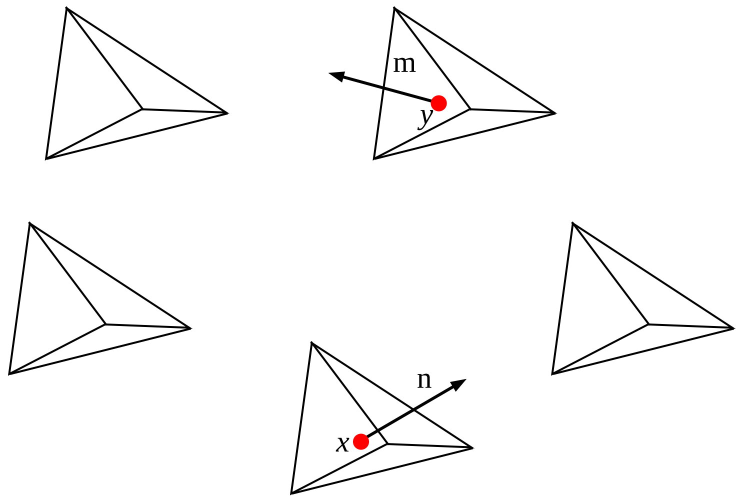


Figure 10.12: DihedAngRov2.

$$\begin{aligned} W(L) &= W^{abcd}(x,y;q)n_a n_b n_c n_d = \\ &= N\frac{\lambda(\hbar G)^4}{5!}\sum_{j_{nm}}(j_{12}(j_{12}+1)-j_L^2)\,(j_{34}(j_{34}+1)-j_L^2) \\ &\quad A_{vertex}(j_{nm})\,e^{-\frac{\alpha}{2}\sum_{n,m}(j_{nm}-j_l)^2-i\Phi\sum_{n,m}j_{nm}} \end{aligned} \tag{10.-13}$$

$$A_{vertex} \sim e^{iS_{Regge}} + e^{-iS_{Regge}} + D \tag{10.-13}$$

$e^{-i\Phi \sum_{n,m} j_{nm}}$ is a rapidly oscillating phase.

But since

$$S_{Regge}(j_{nm}) = \sum_{n<m} \Phi_{nm}(j) j_{nm} \tag{10.-13}$$

and

$$S_{Regge}(j_{nm}) \sim \Phi \sum_{nm} j_{nm} + \frac{1}{2} G_{(nm)(kl)} \delta j_{nm} \delta j_{kl} \tag{10.-13}$$

only the "good" component of A_{vertex} survives!

This is the "forward propagating" component of A_{vertex}.

The Gaussian "integration" gives finally

$$W(L) = \frac{4i}{\alpha^2} G_{(12)(34)} \tag{10.-13}$$

where $G_{(nm)(kl)}$ is the ("discrete") derivative of the dihedral angle, with respect to the area (the spin').

$$G_{(nm)(kl)} = \frac{\partial \Phi_{mn}(j_{ij})}{\partial j_{kl}}\Big|_{j_{ij}=j_L} \tag{10.-13}$$

It can be computed from geometry, giving $G_{(12)(34)} = \frac{8\pi\hbar G k}{L^2}$, where k is a numerical factor ~ 1

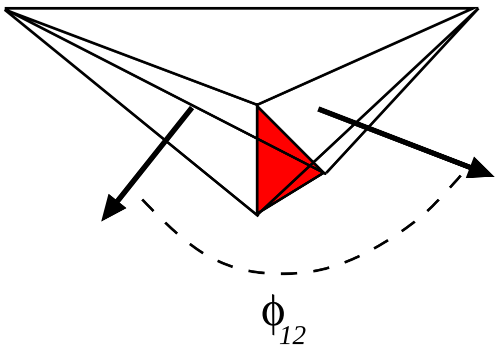


Figure 10.13: TetphiRov.

Cfr the "nutshell" dynamics in 3d gravity

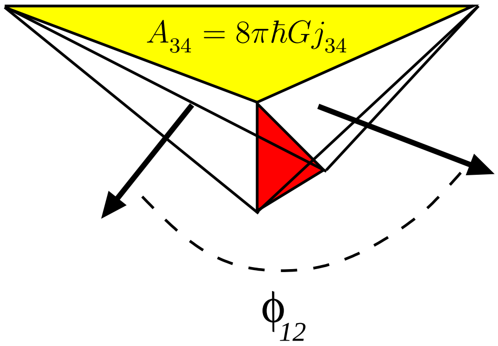


Figure 10.14: TetphiRov2.

Adjusting numerical factors $\alpha^2 = 16\pi^2 k$, this gives

$$W(L) = i\frac{8\pi\hbar G}{4\pi^2}\frac{1}{L^2} = i\frac{8\pi}{4\pi^2}\frac{1}{|x-y|_q^2} \tag{10.-13}$$

which is the correct graviton 'propagator-component'.

→ This is only valid for $L^2 \gg \hbar G$.

For small L, the propagator is affected by quantum gravity effects, and is given by the $10j$ symbol combinatorics.

→ This is equivalent to the Newton law.

10.9.5 Bringing it Together

$$W^{a_1 b_1 \ldots a_n b_b}(x_1, \ldots, x_n) = Z_{LT}^{-1} \sum c(s)\, h^{a_1 b_1}(x_1) \ldots h^{a_n b_n}(x_n) \tag{10.-13}$$

$$W^{a_1 b_1 \ldots a_n b_b}(x_1, \ldots, x_n) = Z^{-1} \int \mathcal{D}g\; g^{a_1 b_1}(x_1) \ldots g^{a_n b_n}(x_n)\, e^{-S_{EH}[g]} \tag{10.-13}$$

10.10 The Complete LQG Graviton Propagator

Some components of the graviton two-point function were computed in the last section using the spinfoam Barrett-Crane vertex. In [286] Rovelii *at el* complete the calculation of the remaining components. They find that, under the above assumptions, the Barrett-Crane vertex does *not* yield the correct long distance limit. They argue that the problem is general and can be traced to the intertwiner-independence of the Barrett-Crane vertex, and therefore to the well-known mismatch between the Barrett-Crane formalism and the standard canonical spin networks.

Alternatives to the Barrett-Crane model have been proposed which are more physically reasonable from which the correct graviton propagator could be recovered: Engle, Pereira and Rovelli [202], Livine and Speziale [201] as well as Freidel and Krasnov [287].

10.10.1 Problems with the Non-diagonal Matrix Elements of the Propagator

$$G^{abcd}(x,y) = < 0|h^{ab}(x)h^{cd}(y)|0 > \tag{10.-13}$$

10.10.2 Graviton Propagator from the New Spin Foam Model

Now with the simplicity constraints appropriatly implemented, we need to check the new spin foam model yeilds a better behaved graviton propragator than the Barrett-Crane model does.

10.11 Which is more Fundamental: Particles or Quantum Fields

We investigate the analogy between then otion of "quanta" in the first-quantized theory of "particles" in QFT.

Consider a quantum particle moving in one dimension, having the SHO Hamiltonian

$$\hat{H} = \frac{\hat{p}}{2m} + V(\hat{x}). \tag{10.-13}$$

we introduce the operators

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{x} + i\frac{\hat{p}}{\sqrt{m\omega}}\right) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{x} - i\frac{\hat{p}}{\sqrt{m\omega}}\right) \end{aligned} \tag{10.-13}$$

Using the commutation relation $[\hat{x}, \hat{p}] = i$, we obtain

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

This, together with the trivial commutation relations $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$, shows that $\hat{a}^\dagger$ and $\hat{a}$ are the raising and lowering operator, respectively. As we speak of one particle, the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ now cannot be called the number of particles. Instead, we use a more general terminology (applicable to (??) as well) according to which $\hat{N}$ is the number of "quanta". But quanta of what?

$$\mathcal{L}(\phi, \partial_\alpha \phi) = \frac{1}{2}[(\partial^\mu \phi)(\partial_\mu \phi) - m^2 \phi^2] \tag{10.-13}$$

from the Lagrangian density (10.11) turns out to be

$$\hat{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\hat{N}_{\mathbf{k}} + \frac{1}{2} \right),$$

with $\hat{N}_{\mathbf{k}} \equiv \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}$, which is an analog of the first term in (93??). This analogy is related to the fact that (82??) represents a relativistic-field generalization of the harmonic oscillator. (The harmonic-oscillator Lagrangian is quadratic in $\mathbf{x}$ and its derivative, while (82??) is quadratic in ϕ and its derivatives). The Hamiltonian (97??) has a clear physical interpretation; ignoring the term 1/2 (which corresponds to an irrelevant ground-state energy $\sum_{\mathbf{k}} \omega_{\mathbf{k}} 1/2$), for each $\omega_{\mathbf{k}}$ there can be only an *integer* number $n_{\mathbf{k}}$ of quanta with energy $\omega_{\mathbf{k}}$, so that their total energy sums up to $n_{\mathbf{k}} \omega_{\mathbf{k}}$. These quanta are naturally interpreted as "particles" with energy $\omega_{\mathbf{k}}$.

10.11.1 QFT Notion of a Particle in a Background-Independent Context

definition of quantities that can be interpreted as particle transition amplitudes.

Particle physicists who hold that QFT is fundamentally a formalism for describing processes involving particles, such as scattering or decays.

In fact, it is well known that the Poincare group plays a central role in the particle interpretation of the states of the field: Wigner's celebrated analysis [313] has shown that the particle states are the irreducible representations of the Poincare group in the QFT state space. The defining properties of the particles, mass and spin (or helicity), are indeed the invariants of the Poincare group.

One of the main problems regarding quantum field theory in curved space-time is how to introduce the concept of particles.

The problem becomes even more severe when we go to full quantum gravity as there is no background spacetime.

However, particles are what we observe in experiments.

particle states that can be defined in background independent formulation and the usual particle states of quantum field theory, defined on infinite spacelike region.

Global and Local Particles [312].

particles described by the n-particle Fock states are idealizations that do not correspond to the real objects detected in the detectors. There is no reason for interpreting the Fock basis as "more physical" than any other basis in the state space of QFT.

10.11.2 Normal Modes and Global Particles

$$H_0 = H_1 + H_2 + V = \frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) + \lambda q_1 q_2, \tag{10.-13}$$

where p_1, p_2 are the momenta conjugate to q_1, q_2 and $\lambda \ll \omega^2$.

$$q_a = \frac{q_1 + q_2}{\sqrt{2}}, \qquad q_b = \frac{q_1 - q_2}{\sqrt{2}}, \tag{10.-13}$$

with eigenvalues

$$\omega_a^2 = \omega^2 + \lambda, \qquad \omega_b^2 = \omega_2^2 - \lambda$$

In terms of these

$$H = H_a + H_b = \frac{1}{2}(p_a^2 + \omega^2 q_a^2) + \frac{1}{2}(p_b^2 + \omega^2 q_b^2)$$

10.11.3 Local Particles

We can define an orthonormal basis in this Hilbert space by diagonalizing a complete set of commuting self-adjoint operators.[2] Let us choose the set formed by H_1 and H_2 where

$$H_1|n_1, n_2 >= E_1^{(n_1)}|n_1, n_2 >, \qquad H_2|n_1, n_2 >= E_2^{(n_2)}|n_1, n_2 >$$

[2]Just as in the hydrogen atome where we use the self-adjoint operators J, j_z to form an orthonormal basis.

10.11.4 Open Issues

• Is it just chance?

• Other components? Full tensorial structure? (Modesto, Speziale)

• Do higher order terms in λ change the result? (Modesto)

• Other models GFT/C seems to give the same result (Modesto)

• $n-$point functions? Computing the undetermined constants of the non-renormalizable perturbative QFT?

• ...

10.11.5 Inclusion of Matter - The Standard Model

10.11.6 Conclusion

i. **Low energy limit.** 'One component of' the graviton propagator or the Newton's law appears to be correct, to 1st order in λ.

ii. **Barret-Crane vertex.** Only the 'good' component of the $10j$ symbol survives, the others are suppressed by the rapidly oscillating phase in the vacuum state that peaks the state on its correct extrinsic geometry. The BC vertex works.

iii. **Scattering amplitudes.** A technique to compute n-point functions within a background formalism exists.

10.12 Scattering Amplitudes from the Master Constraint Programme

10.13 Biblioliographical notes

In this chapter I have relied on the following references:

D. Colosi: *On some aspects of canonical and covariant approaches to quantum gravity.*

H. Nikolic: *Quantum mechanics: Myths and facts.*

10.14 Worked Exercises

Given the equilateral tetrahedron,

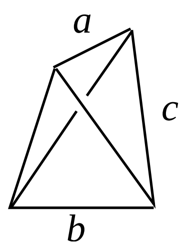


Figure 10.15: The equilateral tetrahedron.

Prove

(a)

$$\sin\frac{\theta_a}{2} = \frac{b}{\sqrt{4c^2 - a^2}}$$

(b)

$$\sin\frac{\theta_b}{2} = \frac{a}{\sqrt{4c^2 - b^2}}$$

(c)

$$\cos\theta_c = \frac{ab}{\sqrt{(4c^2 - a^2)(4c^2 - b^2)}}$$

Proof:

(a) We take the bottom left-hand corner as the origin. Denote the distance from the origin to the midpoint of the top edge by x, then

$$x^2 = c^2 - \frac{a^2}{4}$$

and

$$\frac{1}{2}\sqrt{4c^2 - a^2}\sin\theta_a = \frac{1}{2}b.$$

(b)

By symmetry $a \leftrightarrow b$

$$\frac{1}{2}\sqrt{4c^2 - b^2}\sin\theta_b = \frac{1}{2}a.$$

(c)

We obtain the last expression for the "internal" angle from the scalar product of the normals to two adjacent triangles, by working in the orthonormal basis determined by the top and bottom edges.

The bottom edge is taken to be the $x-$axis, the top edge parallel to be the z-axis. A normal to these $x-$ and $z-$axis, pointing vertical, is taken to be the $y-$axis.

The bottom edge is described by $b\hat{e}_x$. The vertical height, h, from the mid-point of the bottom edge to the midpoint of the top edge is first obtained by calculating the distance from the origin to the midpoint of the top edge:

$$\sqrt{c^2 - \frac{a^2}{4}} = \frac{1}{2}\sqrt{4c^2 - a^2}$$

then

$$h = \sqrt{\left(\frac{1}{2}\sqrt{4c^2 - a^2}\right)^2 - \frac{b^2}{4}} = \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}$$

We define $\vec{c}$ as the vector from the origin to the "out of page" end of the top edge, and $\vec{c'}$ as the vector from the origin to the "in of page" end of the top edge.

$$\begin{aligned}
\vec{c} &= \frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y + \frac{a}{2}\hat{e}_z \\
\vec{c'} &= \frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y - \frac{a}{2}\hat{e}_z
\end{aligned} \tag{10.-13}$$

The normal to the triangle formed between the origin and two ends of the top edge is proportional to

$$
\begin{aligned}
\vec{c} \times \vec{c}' &= \left(\left\{ \frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y \right\} + \frac{a}{2}\hat{e}_z \right) \times \left(\left\{ \frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y \right\} - \frac{a}{2}\hat{e}_z \right) \\
&= a\hat{e}_z \times \left(\frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y \right) \\
&= -\frac{a}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_x + \frac{ab}{2}\hat{e}_y. \qquad (10.\text{-}14)
\end{aligned}
$$

This has norm:

$$
\begin{aligned}
|\vec{c} \times \vec{c}'| &= \sqrt{\frac{a^2}{4}(4c^2 - a^2 - b^2) + \frac{a^2b^2}{4}} \\
&= \frac{a}{2}\sqrt{4c^2 - a^2} \qquad (10.\text{-}14)
\end{aligned}
$$

The normal to the triangle formed between the origin and "out of page" end of the top edge and the right-hand end on the bottom edge is proportional to

$$
\begin{aligned}
\vec{b} \times \vec{c} &= b\hat{e}_x \times \left(\frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y + \frac{a}{2}\hat{e}_z \right) \\
&= -\frac{ab}{2}\hat{e}_y + \frac{b}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_z \qquad (10.\text{-}14)
\end{aligned}
$$

This has norm:

$$
\begin{aligned}
|\vec{b} \times \vec{c}| &= \sqrt{\frac{b^2}{4}(4c^2 - a^2 - b^2) + \frac{a^2b^2}{4}} \\
&= \frac{b}{2}\sqrt{4c^2 - b^2} \qquad (10.\text{-}14)
\end{aligned}
$$

We then obtain an expresion for the dihedral angle θ_c,

$$
\begin{aligned}
\cos\theta_c &= \left(-\frac{\vec{c}\times\vec{c}'}{|\vec{c}\times\vec{c}'|}\right)\cdot\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|} \\
&= \frac{1}{\frac{a}{2}\sqrt{4c^2-a^2}}\frac{1}{\frac{b}{2}\sqrt{4c^2-b^2}}\times \\
&\quad \left(\frac{a}{2}\sqrt{4c^2-a^2-b^2}\hat{e}_x-\frac{ab}{2}\hat{e}_y\right)\cdot\left(-\frac{ab}{2}\hat{e}_y+\frac{b}{2}\sqrt{4c^2-a^2-b^2}\hat{e}_z\right) \\
&= \frac{1}{\frac{a}{2}\sqrt{4c^2-a^2}}\frac{1}{\frac{b}{2}\sqrt{4c^2-b^2}}\frac{a^2b^2}{4} \\
&= \frac{ab}{\sqrt{(4c^2-a^2)(4c^2-b^2)}}. \qquad (10.\text{-}17)
\end{aligned}
$$

□

Final Summary:

• The early rough findings motivated a lot of interest in this approach. (But not outside of the general relativist's community).

• Subsequent activity has put the field on a very strong mathematical footing.

• Black hole entropy from first principles. Loop quantum cosmology resolve the initial singularity, initial conditions predicted rather that guessed at, a new mechanism for inflation, possible signature in CMB.

• Many avenues of research have sprung from Loop quantum gravity: Path integrals (spin foams), consistent discrete quantum Einstein equations; some that were not mentioned: ,...

• The Master Constraint programme has the potential of putting us in strong position to start addressing in a rigorous way physical issues - most notable being the existence of the correct semiclassical limit.

• Formulating background independent scattering amplitudes. Can explicitly demonstrate that LQG, (or at least certain versions of it), has the correct semiclassical limit.

• Technical and conceptual problems are being addressed. These issues all interrelated and so progress in one will help in the progress of the other.

• Experimental tests of Quantum gravity for example, time of arrival of higher frequency radiation from gamma ray bursts. In fact, present day cosmological observations have already put limitations on the extent of quantum gravitation effects.

Appendix A

Physics Glossary

This discussion is intended to provide only a passing familiarity with definitions to allow the unfamiliar reader to follow certain calculations and to have a general understanding of the results.

• **absolute space and time:** Newton's view of space and time according to which they exist, independent of whether anything is in the universe or not.

• **abstract spin network:** They are the equivalence classes of spin networks under spatial diffeomorphisms. GR tells us that the spacetime manifold is a convenient mathematical device devoid of physical meaning. How a spin network is embedded in the spacial three-slice. Once we factorize out the spatial diffeomorphism invariance the only remaining information in the state is the knotedness of the loop. These are objects that don't live in the manifold anymore - they are not embedded in a space. They are not quantum excitations in space, but quantum excitations of space - if there is no spinor network there is no space!

• **achronal:** A subset S of spacetime $\mathcal{M}$ is said to be achronal provided no two of its points can be joined by a timelike curve.

• **active diffeomorphism:** transformation relates different objects in $\mathcal{M}$ in the same coordinate system. This means that the diffeomorphism f is viewed as a map associating one point of the manifold to another one.

Invariants with respect to active diffeomorphisms are obtained by solving away the coordinates x from solutions to the equations of motion.

is a solution of Einstein's equations, namely active diffeomorphisms are dynamical symmetries of Einstein's equations sending solutions to solutions.

• **ADM:** In the Hamiltonian formulation of GR the generator of time translations in an asymptotically flat spacetime.

(1) ADM energy

(2) ADM angular momentum

• **Alexandrofff topology:** For a spacetime $\mathcal{M}$ and any two spacetime points a, b. The chronological sets $\{I^+(a) \cap I^-(b) : a, b \in \mathcal{M}\}$ are open and and form a base for a topology on $\mathcal{M}$, which is called the Alexandroff topology. (Also see maths glossary).

• **algebraic quantum theory:** In the Heisenbergs picture of quantum mechanics the observables play the dominant role, they are governed by an equation of motion while the states are time-independent. In algebraic quantum theory we formulate more absract version of Heisenberg's approach, in which the observables are the relevent objects and the quantum states are secondary.

There are two main ingredients: firstly, a $C\star$-algebra of observables often denoted $\mathcal{A}$, and secondly, states (different meaning from above) ω which act on an observable $A \in \mathcal{A}$ giving a complex number i.e. $\omega : \mathcal{A} \to C$ (In this report we use the notation E_ψ instead of ω),

$$E_\psi(.) := \frac{< \psi, \ . \, \psi >}{\|\psi\|} \quad \Longleftrightarrow \quad \omega(.) \tag{A.0}$$

These states are positive linear functionals ($\omega(A^*, A) \geq 0$ for all $A \in \mathcal{A}$) such that $\omega(I) = 1$, or in our notation

$$E_\psi(A) = \frac{< \psi, A^* \hat{A} \psi >}{\|\psi\|} \geq 0 \text{ for all } A \in \mathcal{A}, \quad E_\psi(I) = \frac{< \psi, \hat{I} \psi >}{\|\psi\|} \equiv 1. \tag{A.0}$$

The value of the state ω acting on an observable A can be interpreted as the expectation value of the operator A on the state ω, i.e. $< A >= \omega(A)$, or

$$E_\psi(A) = \frac{< \psi, \hat{A} \psi >}{\|\psi\|} \tag{A.0}$$

• **affine quantum gravity:** [403]

• **algebraic quantum field theory (AQFT):** Reformulating QFT on an axiomatic basis: that is, starting from what seem to be physically necessary and mathematically precise principles which any QFT would have to satisfy, and then finding QFTs which actually satisfy them, (see [114], and references therein, for extensive discussion of it).

Fields at a point. In the formulation of quantum field theory one traditionally starts with a particle picture and investigate the scattering theory.

takes the collection of all operators localized in a particular region.

Operators of observables localized in an open region of spacetime $\mathcal{O}$ form an algebra $\mathcal{A}(\mathcal{O})$ of bounded operaotrs. It is possible to encode all physically relevant properties in terms of these algebras and their transformation behaviour under Poincare group.

Algebraic quantum field theory (see) is a general approach to quantum field theory based on algebras of local observables, the relations among them, and their representations.

Haag-Kastler axioms

(i) powerful structural analysis proven useful for rigorous treatment of models,

(ii) revision of the particle concept,

(iii) algebraic approach has shed new light on the treatment of quantum field theories on curved spacetime.

the physical concept of locality of nets of local algebras $\mathcal{O} \mapsto A(\mathcal{O})$. For each open region $\mathcal{O}$ one assigns a C^*-algebra $A(\mathcal{O})$. These are mutually (anti)commuting for spacelike separated regions.

In AQFT one cleanly separates two states of quantizing a field theory, namely first to define a suitable algebra A and then study its representations in a second step.

The axiomatic nature of AQFT makes it ideal for foundational study, since we can specify with mathematical precision exactly what the entity is that we are studying.

• **algebraic quantum gravity:** Algebraic qunatum gravity, proposed by Thiemann [??], is a new approach canonical quantum gravity suggeted by loop quantum gravity.

It has been shown that algebraic quantum gravity admits a semiclassical limit whose infintesimal gauge symmetry agrees with that of general relativity.

• **algebra of kinematic observables:** kinematic observables because they are not gauge invariant. These separate points of the unconstrained classical phase space of the system - that is . More complicated composite functions can be expressed in terms of (of limits) them. Dirac constraint approach: We want to ultimately construct a physical Hilbert space of solutions to the constraints. The kinematic observables can be projected onto the physical observables.

In the reduce phase space scheme we first find the physical Dirac observables and then atempt to quantize our system by finding a representation of the Poison algebra of these observables as linear operators on a Hilbert space.

• **almost periodic functions:**

• **alternative theories of quantum gravity:**

Well developed background independent approaches to quantum gravity are causal sets, causal dynamical triangulations and quantum Regge calculus.

• **anomilies:** When we quantize a classical system it is possible that a symmetry of the classical theory is lost. That is, an anomily is a classical symmetry not carried over to the quantum theory. Anomalises arise when the classical brackets cannot be fully realized by the quantum commutators.

• **apparent horizon:** See trapped surface.

i) S, the expantion $\Theta_{(\ell)}$ of the one null normal, ℓ^a, is everywhere zero and that

ii) that of the other null normal, n^a, is negative.

• **Arnowitt-Deser-Misner (ADM):** together, (as well as Dirac), they laid much of the groundwork for canonical quantum gravity. They expressed GR in Hamiltonian form by splitting spacetime into $R \times \sigma$ where R (the real line) represents "time" and σ is an three-dimensional space at a given "time". As their "configuration" variable they used the 3-metric that the spacial slice picks up from the metric of the spacetime, and as their "conjugate momentum" the extrinsic curvature(-to do with the time derivative of the 3-metric). This Hamiltonian construction is known as the ADM formulism.

• **Ashtekar, Abhay:**

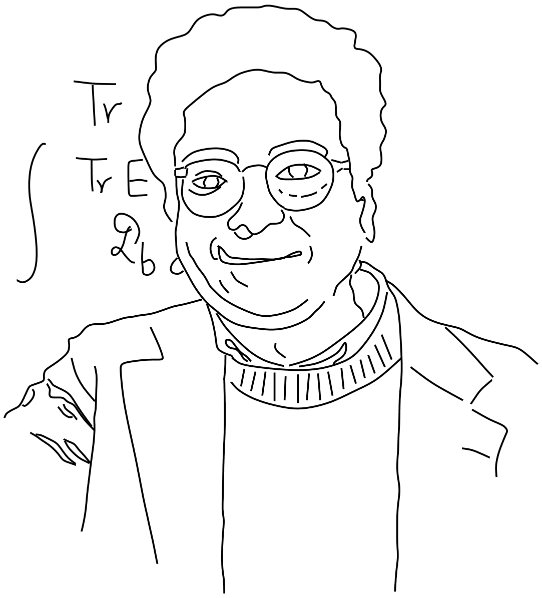


Figure A.1: Abhay Ashtekar.

• **Ashtekar's new variables:** A major obstacle to progress in the canonical approach had been the complicated nature of the field equations in the traditional variables, (q_{ab}, p^{ab}). This obstacle was removed in 1984 with the introduction of new canonical variables - Astekar's new variables. In terms of these, *all* equations of the theory become polynomial, in fact, at worst quartic, [93].

Self-dual connection. This odd choice leads to constraint equations that are polynomial in the basic variables - the connection A and the “electric” field E. The Hamiltonian constraint, that was non-polynomial in the ADM treatment, is quadratic in A and E.

• **Ashtekar-Isham-Lewendowski state:**

Haar measure on cylindrical functions.

• **asymtotically flat gravitational field:**

an asymtotically flat gravitational field: in this case the Hamiltonian is given by suitable boundary terms and the observables at infinity evolve in the Lorentz time of the asymptotic metric.

• **asymptotic flatness:** There exists coordinates $x = (x1, x2, x3)$ defined outside a compact set on Σ such that, for some $\delta > 1$,

$$h_{ij} - (1 + \frac{2M}{r})\delta_{ij} = \mathcal{O}(r^{-\delta}),)$$

as $r \to \infty$ with $M \geq 0$ (positive mass theorem).

• **automrphisms:** generalized Heisenberg picture one-parameter group of automorphisms of the algebra of the algebra of observables. Take any $A \in \mathcal{A}$an automorhism is

$$\gamma_t A = e^{it\hat{\mathcal{H}}/\hbar} A e^{-it\hat{\mathcal{H}}/\hbar} \tag{A.0}$$

where $\gamma_t A \in \mathcal{A}$. Obviously

$$\gamma_s \gamma_t A = \gamma_{s+t} A \tag{A.0}$$

$$\frac{d}{dt} A(t) = [A(t), \mathcal{H}]. \tag{A.0}$$

• **background independent:** There are various ways of expressing this property - it means that how a particle or a field is localization with respect to a spacetime manifold has no physical significance.

The theroy does not depend on a background metric, thus all distances are gauge equivalent.

There is no fixed geometric component to the Einstein-Hilbert action.

• **back-reaction:** in some background spacetime affect the dynamics of the background. This back-reaction can be described in terms of an effective energy-momentum tensor τ_{ab}.

$$G_{\mu\nu} = \kappa < \hat{T}_{\mu\nu} > \tag{A.0}$$

Gravitat

Dynamical theory over curved spacetime not a dynamical theory of spacetime.

see quantum field theory on curved spacetime.

• **Baez, John**

Figure A.2: John Baez.

• **Barbour, Julian:** Reinterpreted Einstein's general theory of relativity as a relational theory in which space and time are nothing but a system of relations. Since receiving his doctorate in 1968 from the University of Cologne, has never held an academic job but became an honored member of the quantum gravity community as it was Barbour who taught people what it means to make a background-independent theory.

• **Bargmann representation:**

• **Barrett, John**

• **Barrett-Crane model:** The Barrett-Crane model is one of the most extensively studied spin foam models for quantum gravity. It can be viewed as a spin foam quantization of the $SO(4)$ Plebanski formulation of general relativity. Specifically, one quantizies the Topological BF theory then imposes the simplicity constraints (that convert BF theory into GR) at the quantum level.

Figure A.3: Julian Barbour.

The original strategy followed by Barrett and Crane the starting point was with the quantization of a 4-simplex.

There are problems with the Barret-Crane models from its ultra-locality which follows from the way in which the simplicity constraints are imposed. Various proposals have been made recently to address these problems, [201], [287] . From [287]: "... that a new exciting period of the development of the subject of spin foam models may be opening."

• **baryonic matter:** Baryonic matter or simply ordinary matter is anything made of atoms and their constituents, and this includes all of stars, planents, gas and dust in the universe. Ordinary baryonic matter, it turns out, is not enough to account for the observed matter density; Baryons are present in the amount preredicted by the Big Bang Nucleosynthesis, some percent of the density required to close the universe.

• **Benny Hill effect:** There is a negative force between a particle of negative mass and a particle of positive mass, whereas by $F_n = m_n a_n$ (F_n is the force exerted on the negative mass particle and a_n is the acceleration of the negative particle mass particle) the negative mass particle accelerates towards the positive mass particle, while the positive particle mass accelerates away.

• **Bergmann-Komar group:** Transformations between different covariant gauge fixings form a group - Bergmann-Komar group.

• **beta function:**

• **Bekenstein bound:**

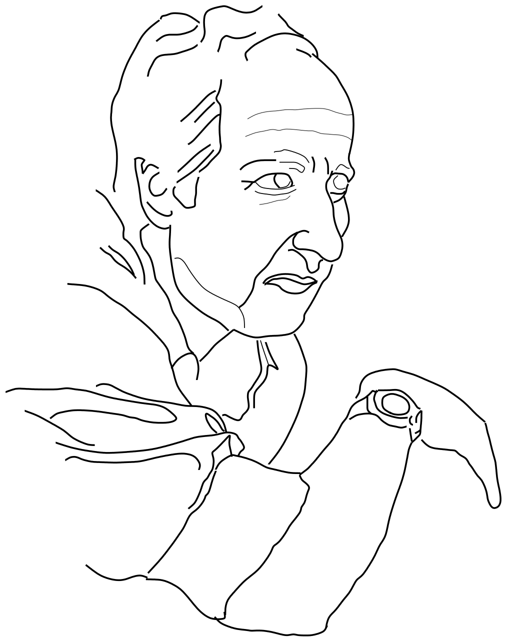


Figure A.4: John Barrett.

$$S < 8\pi\zeta RE \tag{A.0}$$

• **Big bang:**

journal Advances in Theoretical and Mathematical Physics:

"The question of whether the universe had a beginning at a finite time is now 'transcended'. At first, the answer seems to be 'no' in the sense that the quantum evolution does not stop at the big-bang. However, since space-time geometry 'dissolves' near the big-bang, there is no longer a notion of time, or of 'before' or 'after' in the familiar sense. Therefore strictly, the question is no longer meaningful. The paradigm has changed and meaningful questions must now be phrased differently, without using notions tied to classical space-times."

• **Biot-Savart law:**

$$B = -\frac{1}{4\pi} \int d^3y \, \epsilon_{\alpha\beta\gamma} \, j^\beta(y) \, \frac{(x-y)^\gamma}{|x-y|^3} \tag{A.0}$$

• **black-body radiation:** Thermal radiation emmited by black-bodies, black-bodies being perfect emmiters and absorbers of radiation. photons moving around randomly, without any discernible source.

• **black hole entropy:**

• **black hole:** colapsing star forming an object so dense that nothing can escape from it. A region of spacetime hidden behind a horizon.

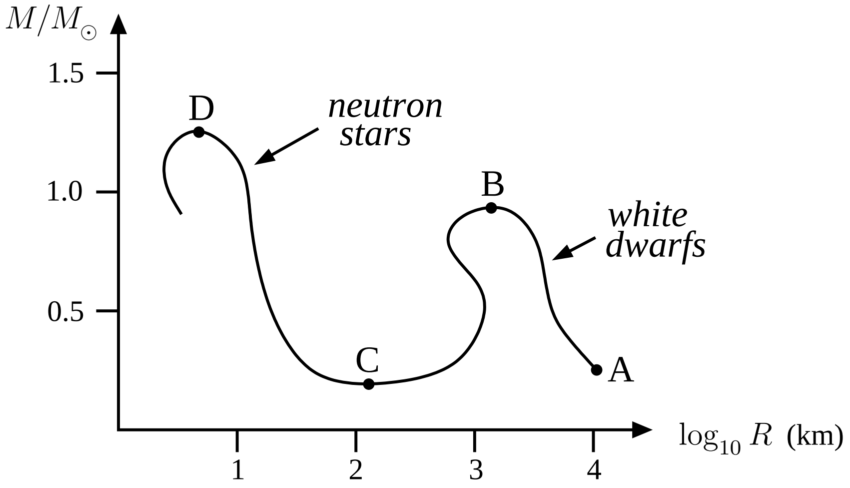


Figure A.5: LebRien.

• **black hole uniqueness theorems:** If a black hole is stationary, then it must be either static or axially symmetric. This is the Kerr family of black holes

• **Bochner-Minlos theorem:** Fourier transform corresponds a unique positive-definite measure.

• **Bogoliubov transforms:** Two different Fock space representations are related by a transformation to new operators.

In ordinary quatum field theory in flat spacetime, one quantizes ϕ by first decomposing the field into Fourier modes,

$$\phi = \sum_{\mathbf{k}} \left(a_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(t, \mathbf{x}) \right)$$

with

$$u_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{\mathbf{k}} t}, \qquad \omega_{\mathbf{k}} = (|\mathbf{k}|^2 + m^2)^{1/2}$$

and replacing the coefficients $a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}$ by annihilation and creation operators respectively. The Fourier modes $u_{\mathbf{k}}$ are a set of orthonormal functions satisfying

$$(\Box + m^2) u_{\mathbf{k}}(t, \mathbf{x}) = 0, \qquad \partial_t u_{\mathbf{k}}(t, \mathbf{x}) = -i\omega_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}),$$

where the second condition determines what one means by positive and negative frequency, and thus allows to distinguish creation and annihilation operators.

In a curved spacetime, or noninertial reference frame in flat spacetime, standard Fourier modes are no longer available. However, with a choice of time coordinate t one can still find a decomposition to obtain creation and annihilation operators. Given two different reference frames with time coordinates t and $\overline{t}$, two such decompositions exist:

$$\phi = \sum_i \left(a_i u_i + a_i^\dagger u_i^* \right) = \sum_i \left(\overline{a}_i \overline{u}_i + \overline{a}_i^\dagger \overline{u}_i^* \right)$$

and since (u_i, u_i^*) are a complete set of functions, we can write

$$\overline{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*).$$

This relation is known as a Bogoliubov transformation, and the coefficients α_{ji} and β_{ji} are called Bogoliubov coefficients.

• **Bojowald, Martin:**

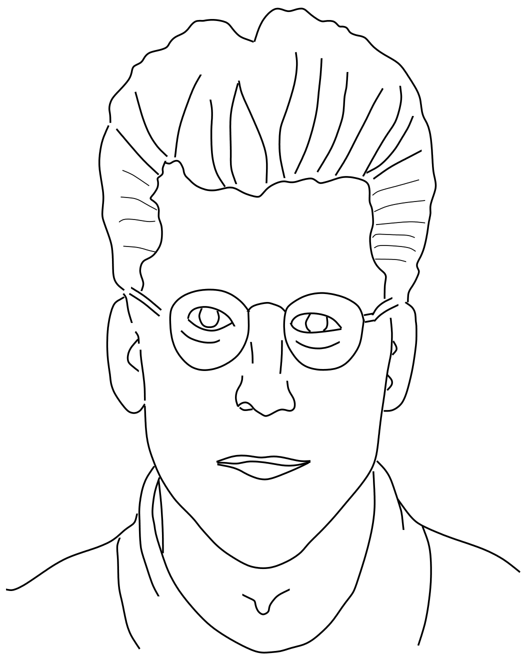


Figure A.6: Martin Bojowald.

• **Boltzmann statistical mechanics:**

• **Bondi coordinates:** $\mathcal{I}^+$ can be foliated into a one parameter family of two dimensional non-intersecting spacelike surfaces, each difeomorphic to S^2, parametrized by a variable

u. On each of these surfaces one can introduce complex stereographical coordinates ζ and $\bar{\zeta}$. The coordinates $(u, \zeta, \bar{\zeta})$ on $\mathcal{I}^+$ are called the Bondi coordinates on $\mathcal{I}^+$.

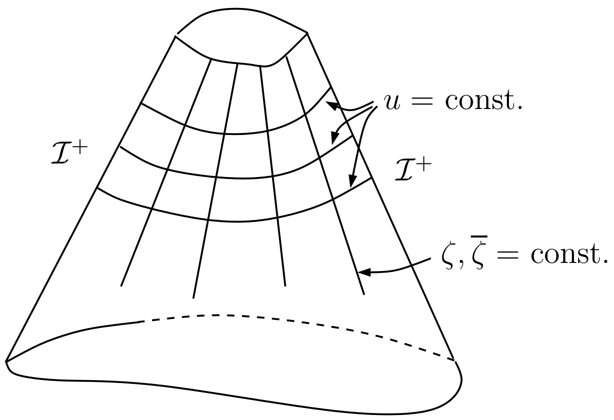


Figure A.7: Bondi coordinates at future null infinity.

• **Bondi-Sachs mass:** Both the energy of the radiation and the energy of the leftover system are included in the total ADM energy. The Bondi energy is the gravitating mass as seen by light rays propagating out to infinity in the lightlike direction, rather than the spacelike direction. For a spacetime of an isolated body emitting graviational the Bondi-Sachs mass decreases, while the ADM mass stays constant.

• **Born-Oppenheimer approximation:**

• **bottom-up:** 'bottom-up' approach one attempts to formulate the full theory and then make some approximation to get the low energy description. Hopeful, in doing so we recover the correct classical theory. When there are an infinite number of degrees of freedom, the passage from the quantum theory to the classical regime does not always have to give the classical theory that was quantized!

• **BRST:** The central idea of BRST theory is to replace the original gauge symmetry by a rigid fermionic symmetry acting on an appropriately extended space containing new variables, "ghosts". Ghosts first appeared in quatum field theory - they ensured that the theory would be unitary and independent of the gauge choice. They did this through contributions to virtual process only - they have the wrong spin statistics to correspond to physical particles. They violate the spin statistic relation which requires physical fields with half-integer spin to be fermionic and fields with integer spin to be bosonic.

• **bubble formulism:** Kuchař's bubble time formulism

• **canonical:** Simplest or standard form.

• **canonical quantisation:** Classical mechanics admits two different kinds of formulation: Hamiltonian and Lagrangian. So does quantum mechanics. The one corresponding to the Hamiltonian formulation of classical mechanics is called canonical quantisation and involves a Hilbert space and operators.

The basic idea is

(i) to take the states of the system to be described by wavefunctions $\Psi(q)$ of the configuration variables,

(ii) to replace each momentum variable p by differentiation with respect to the conjugate configuration variable, and

(iii) to determine the time evolution of Ψ via the Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{\mathcal{H}}\Psi,$$

where $\hat{\mathcal{H}}$ is an operator corresponding to the classical Hamiltonian $\mathcal{H}(p, q)$.

General relativity can be cast in Hamiltonian form, however the Hamiltonian is a constraint on phase space. Dirac developed the general method for the canonical quantisation of systems with constraints. It involves the imposition of the constraints as an additional condition on the Hilbert space.

• **canonical transformations:** A canonical transformation in a classical theory relates descriptions by different 'generalised coordinates' on phase space with the same sympletic structure. These classical theories are equivalent, but the resulting quantum theories need not be equivalent.

• **Cartan's structure equations:** See maths glossary.

• **Cauchy Horizon:** $D^-(p)$

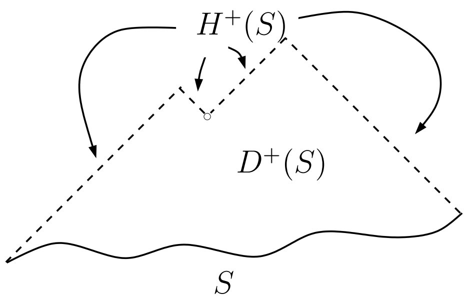


Figure A.8: Cauchy Horizon.

• **Cauchy surface:** A Cauchy surface is a spacelike or null surface that intersects every timelike curve in a spacetime $\mathcal{M}$ once and only once. (Not every spacetime has a Cauchy surface! - see global hyperbolicity).

• **causal** As the very concepts of time and spacial location are not fundamental at all in quantum gravity and only emerges in a semiclassical approximation, causality should too only emerge and be applicable only in a semi-classical limit.

??but no background causal structure in the classical theory - the notion of causality only emerges when the dynamics of gravity can be neglected??

• **causal future:** events are influenced by those in their past.

• **causality conditions:** Avoid paradoxes traveling into the past. Heierarky of conditions:

1) Chronology: no closed timelike curves;

2) Causality: no closed causal curves;

3) Strong causality: no "almost closed" causal curves;

4) Stably causality: close metrics are causal.

• **causal set theory:** The causal approach to quantum gravity is the proposal that the structure of spacetime at the quantum scale should be modelled after a locally finite partially ordered set (poset) of elements, the *causal set*, whose partial order is the relation that defines past distinction between events.

See quantum causal histories.

[gr-qc/0507078]. This approach does not attempt a quantization of classical theory. Rather, it is a construction of a quantum theory of causal sets based on two features of classical genral relativity that it taakes as fundamental: the causal structure, which is replaced by discrete causal set;

• **causal spin networks:**

Do not necessarily need to build in causality as it could be an emergent property??

• **CBR, cosmic background radiation:** Thermal radiation. photons moving around randomly, without any discernible source, corresponding to a temperature of about $2.7K$.

• **charge spin networks:** $U(1)$ gauge group labelled by integers.

• **Chern-Simons:**

Relevant in condensed-matter physics - description of excitations with fractional statistics, *anyons*, in the fractional Hall effect (FQHE) and a proposal for a mechanism of superconductivity based on anyons Y-H. Chen, F. Wilczek, E. Witten, and B.I. Halerin, Int. J. Mod. Phys. B **3**, 1001 (1989).

• **Chern-Simons-Witten gravity:** The local Poincare $ISO(2,1)$ symmetry is, on shell, equivalent to the usual diffeomorphism [388]. This $(2+1)$ dimensional gravity is based on the non-degenerate, invariant and associative inner product on $iso(s,1)$ on which the conventional Killing metric is degenerate [388]:

How do we define the trace operation in Chern-Simons define

$$< P^a, J^a >= \eta^{ab}, \quad < J^a, J^b >=< P^a, J^b >= 0, \tag{A.0}$$

$$[P^a, J^a] = \epsilon_{abc} J^c, \quad [J^a, J^b] = \epsilon_{abc} P^c \quad [P^a, P^b] = 0, \tag{A.0}$$

where $\eta^{ab} = diag(-1, 1, 1)$ and P^a, J^a are the generators of $iso(2, 1)$ (here Roman indices are for internal coordinates and Greek indices for spacetime coordinates). The gauge connection is an $iso(2, 1)$ valued one form

$$A = \omega_a J^a + e_a P^a, \tag{A.0}$$

where ω and e are the homogenious (Lorentz) and the inhomogeneous (translational) part of the connection components respectively.

ISO(2,1) Gauge theory with the gravitational Chern-Simons term

If we define Tr(. .) :=< ., . > so that (A)

$$\mathrm{Tr}(P^a J^a) = \eta^{ab}, \quad \mathrm{Tr}(J^a J^b) = \mathrm{Tr}(P^a J^b) = 0, \tag{A.0}$$

$$\int \mathrm{Tr} \left\{ A \wedge dA + \frac{3}{2} A \wedge A \wedge A \right\} \tag{A.0}$$

• **chromodynamics:**

• **closed systems:** The universe as a whole. while the standard Copenhagen interpretation works well in the laboratory, it cannot be applied to closed systems.

• **classical configuration space:** The classical configuration is generally taken to be the space of all smooth functions which dcay rapidly at infinity an a t = constant slice.

• **clock variable:** A clock variable is a function on the extended phase space of a system which strictly monotonous along the evolution orbits...

• **closed trapped surface:** See trapped surfaces.

• **coherent states:** They provided the closest approximation to classical physics as uncertainty is minimum and the states are peaked in both configuration and momentum representations.

• **collapse of the wave function:**

• **Coleman-Mandula theorem:** The theorem states that a group that non-trivially combines both the Lorentz group and a compact Lie group cannot have finite dimensional, unitary representations. This presens a problem to constructing unified theory of gravity and the other forces. A way to evade the Coleman-Mandula theorem is to use super symmetry.

• **comparative properety:**a comparative property and to express it we need a way to introduce scale into our system.

• **constraints:**

we can take these constraints to say that small automorphisms of the bundle $P|_S$

• **constructive quantum field theory:**

Geared to scalar fields propagating on Minkowski space. Key axioms are called the Osterwalder-Schrader axioms.

• **common eigenstate:**

• **complete set of commuting operators:** In three-dimensions, three coordinate operators $\hat{x}, \hat{y}$ and $\hat{z}$. As they commute with each other we can measure one then measure the next without changing the . and uniquely specifies the state of the system.

The outcomes of the measurement of a complete set of commuting observables characterizes the state.

When we have spherical symmetry potentials in three-dimensions, the set $\hat{\mathcal{H}}$, $\hat{L}^2$ and $\hat{L}_z$

We can define an orthonormal basis in this Hilbert space by diagonalizing a complete set of commuting self-adjoint operators.

Measuring A first, we would find a value a_n and would leave the system in an eigenstate of A.

$$A|\psi_i>= a_i|\psi_i>, \quad B|\psi_i>= b_i|\psi_i> \tag{A.0}$$

$$(AB - BA)|\psi_i>= 0 \quad \Longrightarrow \; AB|\psi_i>= a_i B|\psi_i> \tag{A.0}$$

$$<\psi_i|(AB - BA)|\psi_j>= (a_i - a_j) <\psi_i|B|\psi_j>= 0, \tag{A.0}$$

So $<\psi_i|B|\psi_j>$ vanishes unless $a_i = a_j$.

$|\psi_i>$ are **common** eigenstates of $\hat{A}$ and $\hat{B}$. We denote them as $| >$

Measuring A first we find it has the value a_i and the system would be left in the eigenstate of A. As shown above, this eigenstate is also an eigenstate of B. Thus a measurement of B yields a value b_i

Definition: A set of self-adjoint operators is called a CSCO if the operators all commute with each other and if the set of their common eigenstates is complete and not degenerate.

• **complexifier:** The "complexifier" was introduced by Thiemann in [244] as a powerful tool that can give rise to kinematical coherent states for a general quantuum system.

the complexification of $SU(2)$ is $SL(2, \mathbf{C})$.

$$u = \exp[i\tau_j^j/2] \tag{A.0}$$

• **Compton scattering:** When a photon scatters off a free electron, in the relativistic case $\hbar f >> m_e c^2$.

• **conformally flat spacetimes:**

$$g_{ab} = \Omega \eta_{ab} \tag{A.0}$$

• **conformal transformations:**

• **congruence:** A congruence is a family of curves such that precisely one curve of the family passes through each point. It is a geodesic congruence if the curves are geodesics. A vector field v^a determines a congruence whose curves are tangent to v^a. We call such curves streamlines of v^a.

• **conjugate momentum:**

$$p_i := \frac{\partial \mathcal{L}}{\partial q^i}. \tag{A.0}$$

$$\{q, p\} = 1 \tag{A.0}$$

In the Hamilton formulation of GR, the conjugate momenta of the induced metric is related to the extrinsic curvature K^{ab} by

$$\pi^{ab} = q^{-1/2}(K^{ab} - Kq^{ab}). \tag{A.0}$$

• **connection:** self-consistent way of comparing vectors in two different vector spaces.

We define connection and curvature variables $A^{IJ}_{(\gamma)\mu}$ and $F^{IJ}_{(\gamma)\mu\nu}$

$$A^{IJ}_\mu = \omega^{IJ}_\mu + \frac{\gamma}{2}\epsilon^{IJ}{}_{KL}\omega^{IJ}_\mu \tag{A.0}$$

$$F^{IJ}_{\mu\nu} = R^{IJ}_{\mu\nu} + \frac{\gamma}{2}\epsilon^{IJ}{}_{KL}R^{IJ}_{\mu\nu} \tag{A.0}$$

If one chooses the value of γ such that $\gamma^2\sigma = 1$, then the variables A^{IJ}_{μ} and $F^{IJ}_{\mu\nu}$ are self-dual with respect to the internal indices.

• **Connes-Kreimer's Hopf algebra of Feynman's diagrams:** renormalization via Hopf algebra of rooted trees. Applied by Fontini Markopoulou to renormalization of spin foams background-independent models of quantum gravity.

• **consistent discretizations:** Rodolfo Gambini Jorge Pullin based on approximating the continuum theory by a discrete theory. It allows to bipass difficult conceptual problems turning them into technical ones.

relational quantum theories unfortunately one would have to disregard all the results of LQG

consistent discrete quantum gravity *a la* Pullin, Gambini *et al*: A discrete approach to quantum gravity that is free of constraints translating difficult conceptual problems into problems of computational nature.

• **consistent-histories approach to quantum theory** The consistent-histories approach to quantum theory was originally introduced to provide a novel way of re-interpreting standard quantum theory, particularly in regard to the role played by measurement. However, because of the novel way in which time is handled, consistent-history theory also has the potential for providing new and powerful ways of studying quantum theories of gravity. Most recently, in [2] the formalism was applied to construct a history version of the canonical form of classical general relativity. The possibility also arises to use this formalism in the context of generalised ideas of time and space: for example, in models where spacetime is not represented by a differentiable manifold.

[**?**] 28 Oct 2001, *Quantising the Foliation in History Quantum Field Theory C.J. Isham, K. Savvidou*, [quant-ph/0110161].

• **constants of motion:**

$$\frac{d}{dt} < \hat{A} >= \frac{1}{i\hbar} < [\hat{\mathcal{H}}, \hat{A}] > + < \frac{\partial \hat{A}}{\partial t} > . \tag{A.0}$$

If $\hat{A}$ commutes with the Hamiltonian and does not explicitly depend on time, the observable A is said to be a constant of motion.

$$[\hat{\mathcal{H}}, \hat{A}] = 0 \tag{A.0}$$

• **constraint:** [89]: In addition to the Hamiltonian equations of motion, the theory will exhibit constraint equations. The constraints play a double role in a Hamiltonian theory.

On the one hand they generate a group of symmetries of the phase space referred to as the gauge transformations, on the other hand the set of solutions of the constraints defines the constraint surface of the phase space.

The simplest example for such a kind of theory is certainly Maxwell theory,where the structure group is $U(1)$. A more general example is Yang Mills theory, where the structure group may be an arbitrary compact Lie group G. In this case the group of the gauge transformations is the group of the fiber preserving automorphisms of the given bundle, homotopic to the identity. The group is often referred to as the Yang-Mills gauge transformations.

• **constraint algebra:**

quantum constraint algebra - is mathematically consistent but seems to fail to reproduce the classical constraint algebra - however in calculating the righthand side of the classical Hamilton-Hamilton Poisson bracket involves... So not obvious if it does fail??

• **constructive quantum field theory:** The program of constructive quantum field theory was initiated in the early seventies. This approach has had remarkable success in certain 2 and 3 dimensional models. From a theoretical physics perspective, the underlying ideas may be summarized roughly as follows. Consider, for definiteness, a scalar field theory. The key step then is that of giving meaning to the Euclidean functional integrals by defining a rigorous version $d\mu$ of the heuristic measure "$[\exp(S(\phi))]\prod_x d\phi(x)$" on the space of histories of the scalar field, where $S(\phi)$ denotes the action governing the dynamics of the model. The appropriate space of histories turns out to be the space $\mathcal{S}'$ of (tempered) distributions on the Euclidean space-time and regular measures $d\mu$ on this space are in one to one correspondence with the so called generating functionals, which are functionals on the Schwarz space $\mathcal{S}$ of test functions satisfying certain rather simple conditions. (Recall that the tempered distributions are continuous linear maps from the Schwarz space to complex numbers.)

Thus, the problem of defining a quantum scalar field theory can be reduced to that of finding suitable measures $d\mu$ on $\mathcal{S}'$, or equivalently, appropriate functionals χ on $\mathcal{S}$. Furthermore, there is a succinct set of axioms due to Osterwalder and Schrader which spells out the conditions that χ must satisfy for it to be appropriate, i.e., for it to lead to a consistent quantum field theory which has an associated

i. Hilbert space of states,
ii. a Hamiltonian,
iii. a vacuum and
iv. an algebra of observables.

(from Integration on the space of Connections Modulo Gauge Transformations)

For introduction see for example [398].

• **coordinate singularity** Are locations in coordinate space where calculations become degenerate or divergent, due solely to "bad coordinates". An example of this is the

Schwarzschild radius $r = 2M$ in Schwarzschild coordinates. A radially-infalling observer takes an infinite amount of time in the Schwarzschild time coordinate to cross the Schwarzschild radius. The proper time of the observer would measure although he would see the outside universe speed up??????

One can avoid the detection of unreal coordinate singularities by maximally extending the space-time.

• **Copenhagen interpretation:** ??? (quant-ph/9803052) an a priori classical part of the world is assumed to exist from the outset. Such a structure is there thought to be necessary for the "coming into being" of observed measurement results. ???

In the absence of a background spacetime geometry, the Copenhagen interpretation is of little use - it is definitely physically incomplete.

[339] "The usual well defined quantum operators like the area and volume will exist and be well defined, except that in the discrete theory the total volume of the slice will be an observable since there are no further constraints."

no external observer of the universe as a whole throws into doubt the instrumentalism of the Copenhagen interpretation; as does an attempt to construct a quantum gravity theory with no background spacetime in which an 'observer' could be placed. [363].

standard Copenhagen divide the physical world into the microscopic regime that obeys the laws of quantum mechanics and the macroscopic regime which obeys the laws of classical mechanics e.g. Maxwell's equations, Einstein's equations. Observers are classical. artificial partition not clear where to put this quantum/classical border. (quant-ph/0103161)

• **Corichi, A:**

• **cosmic censorship:** The principle that singularities are never "naked", that is, do not occur unless surrounded by a shielding event horizon.

• **cosmological constant:** A constant introduced into Einstein's field equations of general relativity in order to supplement to gravity. If positive (repulsive), it counteracts gravity, while if negative (attractive), it augments gravity. It can be interpreted physically as an energy density associated with space itself.

• **cosmological constant problem:** zero point energy of quantum fields - why is Λ not infinite? With a Plankian cut-off the value obtained from calculations performed in quantum field theory on the vacuum energy density corresponding to quantum fluctuations of the fields we observe in nature, imply values that are over 120 orders of magnitude greater than the observed value.

In his model, he was able to show he cosmological constant paradox appears only if spacetime is regarded as fundamental rather than emergent, [267]. The excitations are part of the pre-geometry and cannot curve the effective spacetime they simulate. The fields we observe in nature emerge as in the low energy, course grained limit of a more fundamental theory.

• **cosmological principle:** this states that the universe must appear the same whatever the point from which it is observed. It implies that the universe is homogeneous and isotropic on the largest scales and forms the foundations of all of present day cosmology.

• **course graining:** A nice example of a Hopf algebra.

background-independent course graining *a la* Fontini-Markopoulou [122], [**?**]

• **covariant quantum statistical mechanics:**

intensive thermodynamical quantities extensive thermodynamical quantities

• **Covariant Loop Quantum Gravity:** (CLQG) is a loop quantization of a Lorentz covariant formulation. It uses the full Lorerentz gauge group and produces results independent of the Immirzi parameter.

• **Crane, L:**

• **crossing-symmetry:** In gravity this has a relation to the following: [151], [23]. 4d-diff invariant spinfoam. A problem of the canonical formalism is that it is necessary to make a $(3+1)$ decomposition of spacetime which breaks the manifest 4-diffeomorphism invariance. The vector constraint, generated space diffeomorphisms, and the scalar constraint, generating time reparametrization, arise from the underlying 4-diffeomorphism invariance. It reduces the full, 4-dimensional invariance - the canonical formalism preserves spatial diffeomorphism invariance within each hypersurface of the foliation.

There is a suggestion for curing the slicing dependence.

the spacetime covariant formalism in [151] naturally suggest a "covariantisation" of the operator, described in [151] under the name of "crossing symmetry".

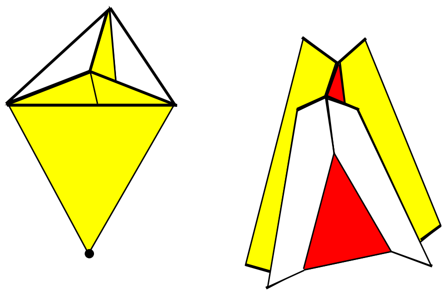


Figure A.9: crossing. The crossing symmetry vertices.

Actually, a well known property of standard QFT is that the vertex amplitutde does not depend on which states are "in" and which are considered "out" - this is related to the crossing symmetry mentioned here.

• **curvature:**

• **curvature tensor:** The vector X defines a curve through the point p via parael transport, The vector Y defines another curve through p. We can form and attempted parallelagram.

$$\epsilon^2 X^a Y^b Z^c R_{abc}{}^d$$

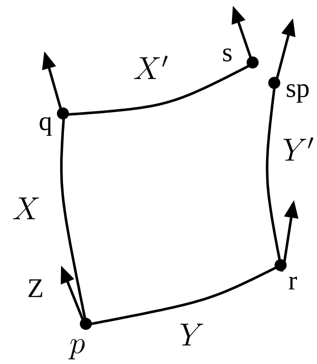


Figure A.10: We display the geometric interpretation of the curvature tensor. Carry a third vector Z, by parallel transport from p to s via q, comparing this with transporting this from p to s' via r we find a discrepancy between the two vectors given in terms of the curvature tensor components $R_{abc}{}^d$ by the formula $\epsilon^2 X^a Y^b Z^c R_{abc}{}^d$.

• **cut-off coherent states:** The coherent state ψ_m, produced by the complexifier method, with respect to a finite graph as a graph dependent coherent state in $\mathcal{H}_{Kin}$ is called a cut-off coherent state. These are then normalizable, graph dependent states, however they can not be used to test the semiclassical properties of graph changing operators, such as the Hamiltonian constraint, as their expectation values with respect to these cut-off states is always zero.

• **dark matter:** A kind of particle that may exist in our universe. This may not be part of the standard Model of particle physics.

conclusion comes from galaxy rotation curves,

[?] A proper General relativist calculation shows a large departure from Newtonian approximation,

A galaxy is modelled as a stationary axially symmetric pressure free-fluid in general relativity. For weak gravitational fields under consideration, the field equations and the equations of motion ultimately lead to one linear and one non-linear equation relating the angular velocity to the fluid density. It is shown that the rotation curves for the Milky Way, NGC 3031, NGC 3198 and NGC 7331 are consistent with the mass density distribution of the visible matter cocentrated in flattened disks. Thus the need for massive *halo of exotic dark matter is reamovedd. For these glaxies we determine the mass density for the luminous threshold as* $10^{-21.75} kg{\cdot}m^{-3}$.

"It is understandable that the convectional gravity approach has focused upon Newtonian theory in the study of galactic dynamics as the galactic field is weak (apart from the deep core regions where black holes are said to reside) and the motions are non-relativistic ($v \ll c$)."

"We have seen that the non-linearity for the computation of density inherent in the Einstein field equations for a stationary axially-symmetric pressure-free mass distribution, even in the case of weak fields, leads to the correct galactic velocity curves as opposed to the incorrect curves that had been derived on the basis of Newtonian gravitational theory."

• **decoherence:** what decoherence does for you is make any discrepencies minute so we don't have to worry about conceptual difficulties. aleviates a practical problem - it is not clear if it elevates conceptual difficulties.

kent dowker (1995): one might at least hope to predict the gross features: the existence and persistence of (what appears to us to be) a classical world with very large-scale structure; the fact that matter clumps and fields in this classical world generally obey classical equations of motion to a very good degree of approximation;

interacting with the classical world, follow the probabilistic laws of Copenhagen quantum theory.

• **decoherent histories:** Taken from - Quantum logic and decohering histories, [quant-ph/9506028]:

In recent years, much attention has been devoted to the so-called decoherent histories approach to quantum theory. A major motivation for this scheme is a desire to replace the traditional Copenhagen interpretation of quantum theory with one that avoids any fundamental split between observer and system and the associated concept of state-vector reduction induced by a measurement. The key ingredient of the new approach is an assertion that, under certain conditions, a probability can be ascribed to a complete history of a quantum system without invoking any external state-vector reductions in the development of the history. Any such scheme would clearly be particularly attractive in quantum cosmology where a fundamental observer-system split seems to be singularly inappropriate.

Dowker and Kent 1 have raised some serious doubts in the context of their penetrating analysis. [366], see also L Smolin [13].

the possibility that the decoherent histories programme could provide a framework for solving certain technical or structural problems that arise in quantum gravity. An example is the problem of time that features prominently in canonical quantum gravity (see [365]).

• **decoupling:** this is the instant in the history of the universe where matter (baryons and electrons) and the electromagnetic radiation cease to be coupled. After this point . This took place when the temperature was about 4000 K.

• **deformed special relativity:**

Deformed special relativity provides an arena in which to build an effective theory for quantum gravity. [257].

• **density matrix:** mixed state described by a density matrix ρ. Expectation values of observables are given by

$$\rho[A] = \mathrm{tr}\,[A\rho]\,. \tag{A.0}$$

• **de Sitter universe:** The de Sitter universe plays an important role in cosmology as it can be used as a good approximation for the stage of accelerated expansion - inflation.

• **DeWitt, Bryce:** Made majour contributions to classical and quantum field theory, in particular, to the theory of gravitation. Through application of Dirac's theory of constraints in development of a canonical approach to quantum gravity, DeWitt was led to what became to be known as the Wheeler-DeWitt equation (the Hamilton constraint in 'metric variables'), which has since been applied many times to problems in quantum cosmology.

By the end of 1965 he had found the rules for quantising the gravitational and non-Abelian gauge fields to all orders nearly two years befor a paper by Fad'deev and Popov deriving the same rule was published.

See BRYCE SELIGMAN DEWITT 1923-2004 - A Biographical Memoir by STEVEN WEINBERG.

• **Dirac conjecture:** All first class constraints generate gauge transformations at a given time.

• **Dirac, Paul:** Developed the hamiltonian formulation of GR independently of Bergmann and his group, with the long term goal to quantize gravity. The main tool for this, the hamiltonian theory of contsrained systems, was developed for this purpose.

• **Dirac-Bergman algorithm:** Procedure for generating a finite set of constraints consistent under evolution.

• **Dirac observable:** An observable is a function on the constraint surface that is invariant under gauge transformations generated by *all* the first class constraints. Equivalently, an observable is a function on the phase space that has weakly vanishing Poisson brackets with the first class constraints.

It is often stated that there are no Dirac observables known for General Relativity, except for the ten Poincar?e charges at spatial infinity in situations with asymptotically at boundary conditions.

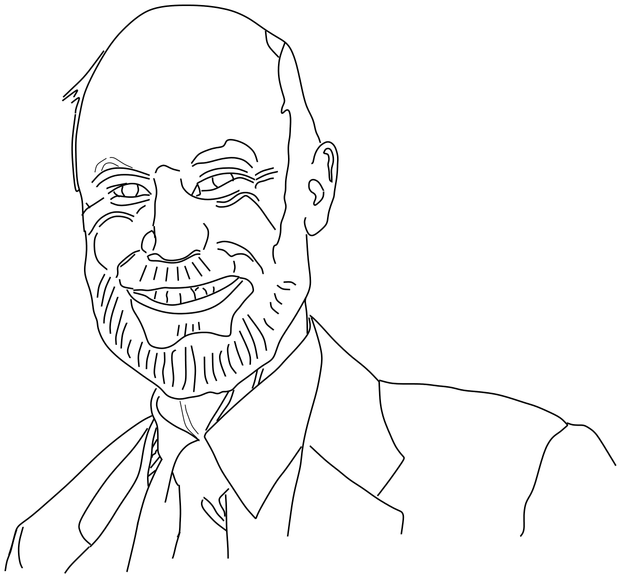


Figure A.11: Bryce DeWitt.

• **dispersion relations:** The form of the dispersion relations is dictated by Lorentz invariance.

• **Dittrich, Bianca:**

• **domain of dependence:** The future domain of dependence, $D^+(\Sigma)$, is the set of points p in $\mathcal{M}$ for which every past-inextendable causal curve through p intersects Σ.

• **Doplicher, Haag, Roberts DHR:** for classification of available representations in order to know which features are tied to a specific representation and which are not.

• **double harmonic oscillator model:** The double harmonic oscillator was first studied by Rovelli [278] as a toy model to help understand the "problem of time".

The Hamiltonian for the double harmonic oscillator

$$\mathcal{H} = \lambda \left(\frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) - E \right) \tag{A.0}$$

One of the oscillators can be thought of as a "clock" for the other oscillator.

Other toy models that share some essential features of GR are .

• **doubly special relativity:** Doubly special relativity theories are relativistic theories in which the transformations between inertial observers are characterized by two observer-independent scales - the light speed and the Planck length l_p. The Planck length seems to have a crucial role to play in quantum theories of gravity as a threshold to quantum

Figure A.12: Paul Dirac.

effects of spacetime. It is argued that the planck length must have the same value in all inertial frames which is in contradiction with special relativity. Double special relativity is a proposal to solve this perceived problem.

A phenomenological description of the signatures of Lorentz invariance violation could be doubly special relativity (DSR) [??], a theory in which not only the speed of light but also the Planck energy is (inertial) frame independent.

• **Dreyer, Olaf:** [19] "...argues that the incompatibility between quantum theory and general relativity can be resolved only if we give up the idea that space is fundamental. He proposes that space itself emerges from a more fundamental description that is quite different."

• **dust:** Matter that does not interact on one another by mechanical forces but interacts with itself only via the gravitational field.

• **dynamical horizons:** Isolated horizons provide a quasi-local description of black holes in equilibrium. There has also been developed a framework for the quasi-local treatment of black holes that are not in equilibrium (because of infalling matter or interaction with external bodies), namely *dynamical horizons*.

The formal definition of a dynamical horizon is as follows,

i) S, the explanation $\Theta_{(\ell)}$ of the one null normal, ℓ^a, is everywhere zero and that

ii) that of the other null normal, n^a, is negative.

The second condition merely says that n^a is inward pointing null normal. none of the

Figure A.13: Bianca Dittrich.

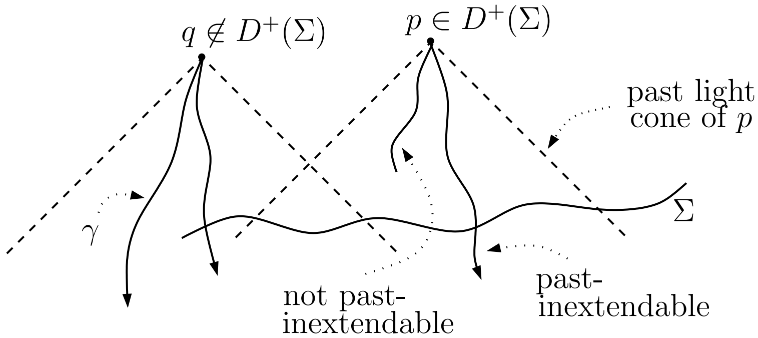


Figure A.14: The future domain of dependence, $D^+(\Sigma)$, of Σ. p is in $D^+(\Sigma)$, q isn't because ther are past-inextendable causal curve through q that don't intersect Σ, e.g. the curve γ.

light rays emerging from *any* point on S are directed towards the 'outside' region.

how black-holes grow

• **dynamical symmetry:** e.g. invariance under active diffeomorphisms inGR.

• **dynamically triangulated quantum gravity:** dynamical triangulations which edge lengths are fixed and the path integral is represented as a sum over triangulation.

A weak coupling phase with quantum fluctuations around a "semiclassical" dynamically generated background geometry despite the fact that the formulation is explicitly background-independent, [265].

• **edge:** Let S be a closed achronal set. The edge of S is defined as a set of points $x \in S$ such that every neihbourhood of x contains $y \in I^+(S)$ and $z \in I^-(S)$ with a timelike curve from z to y which does not meet S.

Link between vertices of a graph.

• **Ehernfest's theorem:**

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}_x\rangle}{m}, \tag{A.1}$$

$$\frac{d\langle\hat{p}_x\rangle}{dt} = -\left\langle\frac{\partial\hat{V}}{\partial x}\right\rangle. \tag{A.2}$$

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{x}], \quad \frac{d\hat{p}_x}{dt} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{p}_x] \tag{A.2}$$

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{p_x}^2 + \hat{p_y}^2 + \hat{p_z}^2) + \hat{V}(x, y, z) \tag{A.2}$$

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}_x}{m}, \quad \frac{d\hat{p}_x}{dt} = -\frac{\partial\hat{V}}{\partial x} \tag{A.2}$$

$$\frac{d}{dt}\int\psi^*\hat{x}\psi dx = \frac{1}{m}\int\psi^*\hat{p}_x\psi dx, \qquad \frac{d}{dt}\int\psi^*\hat{p}_x\psi dx = -\int\psi^*\frac{\partial\hat{V}}{\partial x}\psi dx \tag{A.2}$$

• **Einstein, Albert-(1878-1955):** Began his scientific work at the beginning of the 20th century. During the years 1905-1906 he published three articles: (1) The special theory of relativity; (2) an explanation of the photo-electric effect; (3) Brownian motion. The three corner stones of modern physics. Considered to be very clever.

• **Einstein's hole argument:** As explained by Rovelli *"Assume the gravitational field equations are covariant. Consider a solution of these equations in which the gravitational field is* e [basically the metric] *and there is a region H of the universe without matter (the "hole"). Assume that inside H there is a point A where* e *is flat and the point B where it is not flat. Consider a smooth map $\phi : \mathcal{M} \to \mathcal{M}$ which reduces to the identity outside H, and such that $\phi(P) = Q$, and let $\tilde{e} = \phi_* ei$ be the pull back of* e *under ϕ* [the metric induced by the diff transformation ϕ]. *Notice that* e *is flat around the event A while $\tilde{e}$ is not. The two fields* e *and $\tilde{e}$ have the same past, are both solutions of the field equations, but have different properties at the event A, therefore they are not deterministic. But we know that (classical) gravitational physics is deterministic, therefore, either the field equations can or cannot be generally covariant, or there is no meaning to the physical event A."*

• **eikonal approximation:** an approximation for the scattering amplitude valid in the high energy, low momentum transfer regime,

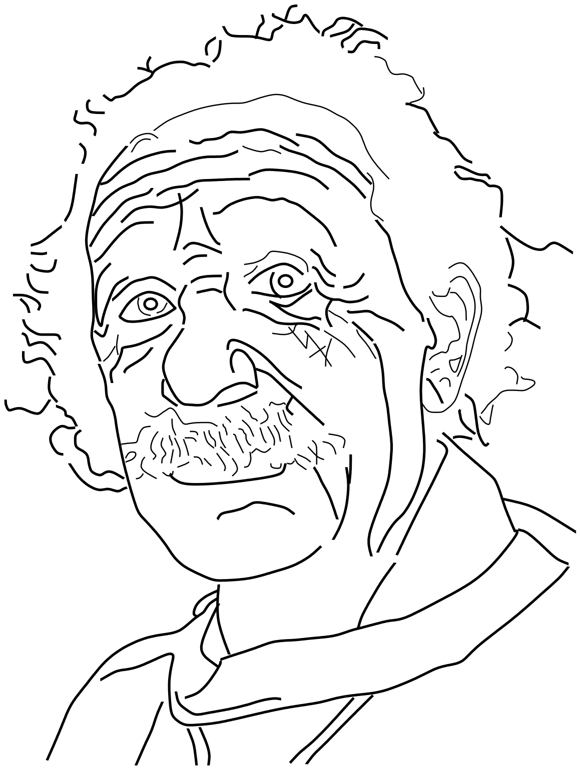


Figure A.15: Albert Einstein (1879-1955).

$$\mathbf{p}' = \mathbf{p} - \frac{\mathbf{k}}{2}, \;\; \mathbf{p}'' = \mathbf{p} + \frac{\mathbf{k}}{2}, \quad \mathbf{p}^2 \to \infty, \;\; \mathbf{p}^2 > \mathbf{k}^2 \tag{A.2}$$

The dominant contributions come from classical trajectories.

• **eikonal equation:**

• **electroweak theory:** combing theory of QED and weak-interactions.

• **energy conditions:**

The weak energy condition:

$$T^{ab} u_a u_b \geq 0, \tag{A.2}$$

for all timelike vectors u^a. This condition requires that the local energy density be non-negative in every observer's frame. Again this seems to be a very reasonable condition in classical theory.

The strong energy condition:

$$((T^{ab} - \frac{1}{2} g^{ab} T) u_a u_b \geq 0, \tag{A.2}$$

for all timelike vectors u^a. The frame in which T^{ab} is diagonal, this condition implies that the local energy density ρ plus the sum of local pressures p^i is non-negative: $\rho + \sum_i p^i \geq o$, and that $\rho + p^i$ for each ^{i}. This condition certainly holds for ordinary forms of matter, although it can be violated of classical physics.

The null energy condition:

$$T^{ab} k_a k_b \geq 0, \tag{A.2}$$

This condition is implicit in the weak energy condition. That is, if we assume the weak energy condition, then the null energy condition follows by continuity as u^a approaches a null vector.

The dominant energy condition:

The dominant energy condition implies that matter cannot travel faster that the speed of light.

$$R_{ab} W^a W^b \geq 0 \tag{A.2}$$

for any null vector $\mathbf{W}$. The importance of the weak energy condition is that it implies that in presence of matter a congruence of null geodesics that are initially diverging will begin to .

$$\frac{d}{d\lambda} \hat{\theta} = -R_{ab} K^a K^b - 2\hat{\sigma}^2 - \frac{1}{2} \hat{\theta}^2 \leq 0. \tag{A.2}$$

$\hat{\theta}$ monotonically decreases along the null geodesics if the first term on the right hand side if the weak energy condition holds.

• **energy conservation:** Notion of energy and the law of energy conservation have played a key role in analyzing behaviour of physical theories. However, it is not an independent physical requirement, but rather a consequence of time translation symmetry of the background metric and the dynamical equations.

Energy conservation is just a consequence of invariance under time translations, which, it turn is a feature of the homogeneity of Minkowski spacetime. However, when we come to general relativity the homogeneity of the Minkowski solution is not feature invariant under active diffeomorphisms. Hence there is no fundamental energy conservation in nature - there is nothing sacred about energy conservation!

When we restrict possible spacetimes to be those that are asymptotically flat, diffeomorphisms at infinity are just those of time and space translation we recover our notions of energy and momentum conservation.

In the absence of fixed a spacetime we must learn how to do physics in the absence of energy or momentum.

Define $E = v^a K_a$, where K_a is a killing vector,

$$\begin{aligned} \nabla_v E &= v^a \nabla_a (v^c K_c) \\ &= v^a v^c \nabla_a K_c \quad \text{since } v \text{ is the tangent of a geodesic,} \\ &= v^a v^c \nabla_{(a} K_{c)} = 0 \end{aligned} \tag{A.1}$$

• **Euclidean geometry:** Flat geometry based upon the geometries axioms of Euclid.

• **energy momentum tensor T_{ab}:**

• **entropy:** A quantitative measure of disorder of a system. The greater the disorder, the higher the entropy. It is defined as the amount of information about the microscopic motion of the constituents making up the system which are not determined by a description of the macroscopic state of that system.

• **epistemology:** Philosophic subject concerned with the significance of what we can know.

• **EPR:**

Rovelli's book [20]

The EPR apparent paradox might be among these. The two observers far from each other are physical systems. The standard account neglects the fact that each of the two is in a quantum superposition with respect to the other, until the moment they physically communicate. But this communication is a physical interaction and must be strictly consistent with causality.

• **entanglement entropy:** Black hole entanglement entropy is a measure of the information loss due to separation between

inside and outside the event horizon.

• **equations of motion (EOM):** They are a prescription for calculating the acceleration from the state of the system (from initial position and velocity). The initial state yields the initial acceleration, and the integration leads from the initial acceleration to later sates.

• **event:** A point in four-dimensional: a location in both space and time.

• **Event horizon:** A surface that divides spacetime into two regions: that which can be observed and that which can not. The Schwartzschild radius of a nonrotating black hole is an event horizon.????????

• **evolving Constants of motion:** Introduced by rovelli in [280] (an idea that goes back to DeWitt, Bergmann nad Einstein). The Hamiltonian constraints generate the dynamics, dynamics is pure gauge, and the observables of the theory are constant along the gauge orbits - these are the evolving constants of the motion. They are formed as follows: in a totally constrained theory, the values of fields are not physically observable. On the other hand, if one chooses a one-parameter family of observables such that their value coincides with the value of a dynamical varaible when the parameter takes the value of another dynamical variable, which one uses to characterize the evolution.

• **exceptional edges:** Exceptional edges are characterized by not being effected by action of the Scalar constraint. gr-qc/0409061

Application of the Hamiltonian constraint on a vertex v will produce a new edge a_{12} fig.(A.16) attached to (two) the edges e_1 and e_2. The edges incident to the vertices v_1 and v_2 only have two linearly independent tangent directions, namely those of e_1 and a_{12}. An exceptional edge, here a_{12}, verticies are annihilated by a second application of the Hamiltonian constraint. A failure to 'propagate'.

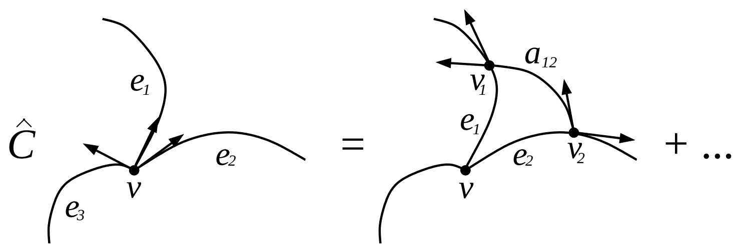


Figure A.16: exceptional. The Hamiltonian constraint's action on the trivalent node v with linearly independent tangent vectors produces two new vertices v_1 and v_2 - each of which has only two linearly independent tangent vectors.

• **Extended Master Constraint Operator $\hat{M}_E$:**

$$\hat{\mathbf{M}}_{\mathbf{E}} := \int_\sigma d^3x \frac{C^2 + q^{ab}C_aC_b}{\sqrt{\det(q)}} \tag{A.1}$$

Both constraints are spatially diffeomorphism invariant. However, $\hat{\mathbf{M}}_{\mathbf{E}}$ allows us to implement both the Hamiltonian and the spatial diffeomorphism constraint on $\mathcal{D}_{kin}$ provided we implement the corresponding operators in a non graph changing fashion. In [77] it was shown how to do that using the notion of a minimal loop which is a loop (average) within the graph on which the constraint acts. It follows that instead of using dual operators we can directly work with operators on $\mathcal{H}_{kin}$ and their adjoints so that the construction of the quadratic form can be sidestepped.

There is the Extended Master Constraint Operator $\hat{\mathbf{M}}_{\mathbf{EE}}$ which also includes the Gauss constraint in addition:

$$\hat{\mathbf{M}}_{\mathbf{EE}} := \int_\sigma d^3x \frac{C^2 + q^{ab}C_aC_b + C_jC_j}{\sqrt{\det(q)}}. \tag{A.1}$$

See page ?? of the Feynman lectures.

• **extentive thermodynamical quantities:**

• **extremal black holes** Black holes having the maximum possible charge for a given mass and angular momentum. Thought to be astronomically extremely rare. Ruled out by the third law of black hole mechanics.

• **extrinsic curvature:** We can describe this curvature by looking at how the normal vectors to the surface change along the surface. One uses the derivative operator of the larger space to measure the changes in the normal vector field, and then project back into the surface.

extrinsic curvature of spacetime $\mathcal{M}$ to measure the changes in the timelike normal vector field of the slice Σ. Roughly speaking, it is to do with the time derivative of the 3-metric.

Contains the information on how a hypersurface is embedded in the enveloping spacetime.

• **faithful:**

faithful representation:

faithful state:

• **false vacuum:** A metastable state in which a quantum field is zero, but its corresponding vacuum energy density is not zero.

• **falsifiable:** The property of a scientific hypothesis that it is not possible to perform an experiment that would disprove, or falsify, the hypothesis.

• **Fell's theorem:** (From [11] Quantum Field Theory in Curved Sapcetime and Black Hole Thermodynamics, Robert M. Wald) For an algebra of observables $\mathcal{A}$ of a field theory, although two representations of $\mathcal{A}$ may be unitarily inequivalent, the determination of a finite number of expectation values of observables in $\mathcal{A}$ - each made with fininte accuracy - cannot distinguish between different representations.

Quote:

whether the resulting physical Hilbert space admits a sufficient number of physical semi-classical states - in some sense this is granted by Frell's theorem [114] once we have a representation of the physical Hilbert space of the C$*$ algebra of our preferred algebra of bounded Dirac observables. [209]

• **Fermions:** The metric formulation cannot incorporate fermionic equations of motion. The reason for this is there is no finite-dimensional spinorial representation of the 4×4 general-linear real matrix group, however, there is a finite-dimensional spinorial representation of the Lorentz group - which is what Dirac discovered. We use a tetrad frame to establish an instantaneous inertial frame, in which the laws of special relativity apply. It can be introduced in a generally covariant way. When we fix the tetrad to Minkowski spacetime we recover Dirac's equation.

• **Fermi-Walker derivative:**

• **Fermi-Walker propagated:** propagation along a non-geodesic curve.

• **Feynmann diagrams:**

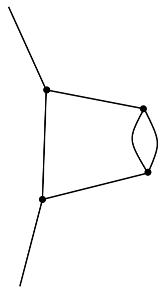


Figure A.17: The charge renormalization.

Feynman graphs yield integrals that diverge when momentum becomes indefinitely large (or, equivalently, when distances become indefinitely small).

one-particle irreducible (1PI) diagrams

• **Feynmann-Kac formula:** Instead of considering matrix elements of the unitary operator $U(t) = \exp(itH/\hbar)$, consider the analytic continuation $S^{A_1 \dots A_n}(t_1, \dots, t_n)$, of the Wightman functions $W^{A_1 \dots A_n}(t_1, \dots, t_n)$ is

$$S^{A_1 \dots A_n}(t_1, \dots, t_n) := W^{A_1 \dots A_n}(t_1, \dots, t_n).$$

These are the so-called Schwinger functions. A path integral for the Schwinger functions have better chance to be rigorourslу defined because the measure has a damping factor rather than an oscillating one and so one can give a rigorours defination is by backwards analytic continuation (when possible). The path integral for the Schwinger functions (when it can be proved) is called the Feynmann-Kac formula.

• **first class constraints:** restricting the fields to the constraint surface.

infinitesimal “gauge transformation”

$$(1 + a\frac{\partial}{\partial x})f(x) = f(x + a) + \mathcal{O}(a^2) \tag{A.1}$$

Functions invariant under this gauge transformation satisfy the constraint equation,

$$\frac{d}{dx}f(x) = 0. \tag{A.1}$$

The first class property is that

$$\{C_m, C_n\} = C_{mn}{}^l C_l \tag{A.1}$$

where the $C_{mn}{}^l$ are called the structure constants if they are independent of on the phase space coordinates and structure functions if they are functions over phase space; that is the first class constraints satisfy, in general, a “non-Lie” algebra

$$C_{mn}{}^l = C_{mn}{}^l(q^i, p_i) \tag{A.1}$$

as it is possible that from the canonical variables.

The first class property of the constraints guarantees, that the flow of the constraints is integrable to an n dimensional surface - the gauge orbit.

they generate gauge transformations.

in the Hamiltonian approach based on Dirac’s conjecture [72], where a suitable combination of the first class constraints is shown to to be a generator of local symmetries of the Lagrangian.

as apposed to a structure *constant* we can have a structure *function* depending on the phase space coordinates.

Dirac’s conjecture in the Hamiltonian approach the symmetries are generated by the first class constraints.

In the Lagrangean approach local symmetries are reflected in the existence of so called gauge identities.

• **first class constraints:** When all the constraints are first class we then have a closed constraint algebra.

• **flatness problem:** The observed fact that the geometry of of the universe is nearly very nearly flat, a very special condition, without an explanation of why it should be.

• **Fleischhack, Christian:**

Figure A.18: Christian Fleischhack.

• **Fock representation:** The most commonly used representations of canonical commutation relations/canonical anti-commutation relations are the so called Fock representations, defined in bosonic/fermioinic Fock spaces. These spaces have distinguished vector Ω called the vacuum killed by the annihilation operators and cyclic with repsect to creation operators (i.e., repeated applications and complex linear combinations of creation operators on the Ω generates a dense subset of the Hilbert space). They were introduced to describe systems of many particle systems with Bose/Fermi statistics.

Fock space is considered the prototypical QFT state space by some

fundamentally the Fock representation is not a valid representation for interacting quantum field theories.

States in LQG are fundamentally different from Fock states of Minkowskian quantum field theories. The main reason is the underlying diffeomorphism invariance: In the absence of a background geometry, it is not possible to introduce the familiar Gaussian measures and associated Fock spaces.

• **foliation of space time:** We have a spacial three-space stacked one above the other, one for every "moment in time".

• **form factor:** Given a loop γ, its form factor $F^a(\gamma, \vec{x})$ is defined by

$$F^a(\gamma, \vec{x}) := \oint_\gamma ds \dot{\gamma}^a \delta^3(\vec{x}, \vec{\gamma}(s)) \tag{A.1}$$

• **focussing theorem:** It is easy to show that the Raychaudhuir equation implies the following inequality for the expansion $\theta(\lambda)$

$$\theta^{-1}(\lambda) \geq \theta^{-1}(0) + \frac{\lambda}{4} \tag{A.1}$$

From this inequality we see that if the congruence is initially converging ($\theta(0) < 0$), then $\theta(\lambda) \to -\infty$ within an affine parameter $\lambda \leq 2/|\theta(0)|$. Demonstrates attractive instability of gravity. This result is employed in the area increase law of black holes and in the singularity theorems.

• **frame of reference:** The coordinate system to which a particular observer refers his or her measurements.

• **free-fall:** a particle is said to be in free-fall when its motion is affected by no forces except gravity. Unrestrained motion under the influence of the gravitational field.

or

A body in free fall does not feel any acceleration, and hence behaves locally as an inertial reference system of SR.

It is meaningless to say whether or not the gravitational field is flat around some point p of $\mathcal{M}$. It is meaningful, however, to ask whether or not the gravitational field is flat around the place where two particles coincide.

• **free data:** isolated horizons. reconstruct from...

• **Friedmann models:** A class of cosmological models that are isotropic and homogeneous, contain specified matter-energy density, conserve matter, and admit no cosmological constant. Also called standard models.

[308]

All textbooks on classical GR incorrectly describe the Friedmann equations as physical evolution equations rather than what they really are, namely gauge transformation equations. The true evolution equations acquire possibly observable modifications to the gauge transformation equations whose magnitude depends on the physical clock that one uses to deparametrise the gauge transformation equations.

• **functional analysis:** is the branch of mathematics which is concerned with the study of spaces of functions.

• **functional integration:**

In the Hilbert space language, this is the origin of field theoretic infinities, e.g., the reason why we can not naively multiply field operators.

• **future-distinguishing spacetime:** A spacetime is said to be future-distinguishing if $a \neq b$ implies $I^+(a) \neq I^+(b)$. (past-distinguishing is similarly defined: if $a \neq b$ implies $I^-(a) \neq I^-(b)$). (See also chronology condition, strongly causality, stable causality).

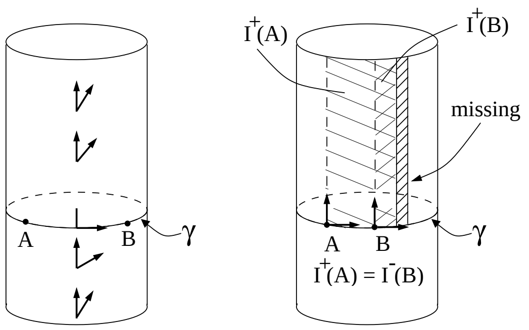


Figure A.19:

• **Gambini, Rodolfo:**

Figure A.20: Rodolfo Gambini.

• **gamma ray burst:** A $\gamma-$ray burst is a light signal of extremely high energetic photons (up to 1 TeV) that travells over cosmological distances ($\approx 10^9$ years). Could have applications in quantum gravity phenomenology.

They are enormous explosions that for a few seconds can produce as much light as that emitted by a whole galaxy. Signals from these explosions reach Earth on average once about once a day.

• **gauge fixing:** One method to eliminate the superfluous degrees of freedom of a gauge theory is through the introduction of a new set of constraints.

leads to second class constraints. null components? Poisson bracket is replaceds by the Dirac bracket

$$\{f, g\}^* := \{f, g\} - \{f, C_i\} B_{ij}^{-1} \{C_j, g\} \tag{A.1}$$

where

$$B_{ij} := \{C_i, C_j\} \tag{A.1}$$

The geometric interpretation of this is: the symplectic structure induced on the hypersurface, defined by the condition $C_j = 0$??, from the symplectic structure of the phase space $\mathcal{M}$ is given by the Dirac bracket $\{\cdot, \cdot\}^*$.

Interpretation of gauge invariant variables is based on the coincidence of the variables in an appropriate gauge choice. For example, blah defined by blah describes blah , because the variable blah coincides with blah, defined in blah in the gauge choice blah.

• **gauge invariance:**

• **gauge invariant coherent states:** Projections of the complexifier coherent states onto the gauge-invariant subspace, [254], [255].

• **gauge potential:** Gauge potentials can be identified as the connection on a principal fibre bundle in a certain gauge.

• **gauge symmetry:**

A gauge symmetry is a redundancy in the description of the system and not a symmetry of a system. For example in a situation with rotational symmetry, such as a planet moving about the sun, although there is a symmetry that changes the angle in the plane of motion there is still physically a difference between two different values of the angular variable. In a gauge theory this is not the case. If two field configurations are related by a gauge transformation then they are physically identical.

Dirac's definition of a gauge symmetry is when two solutions with the same initial data set separate at some point.

• **gauge transformations:** Gauge transformations are maps from solutions of the equations of motion into other solutions which are physically equivalent.

There is a choice of a gauge potential in a certain chart but there is also a choice of local charts. This other flexibility means that a gauge does not just refer to a particular gauge potential.

In the physics literature gauge transformations often refer to passing from one chart to another in a overlapping region - a passive gauge transformation. In the physics literature gauge transformations always refer to passing from one gauge potential to a physically equivalent one in the same chart - an active gauge transformation. It is active gauge transformations that have physical meaning.

electromagnetism the gauge transformation of the gauge potential is

$$\delta A_{\mu} = \{A_{\mu}, G\} = \partial_{\mu}\Lambda, \tag{A.1}$$

for an arbitrary function Λ.

In general relativity the gauge transformations (diffeomorphisms) for the metric field

$$\delta g_{\mu\nu} = \epsilon^{\rho}\partial_{\rho} + g_{\mu\rho}\partial_{\nu}\epsilon^{\rho} + g_{\rho\nu}\epsilon^{\rho}, \tag{A.1}$$

for some arbitrary function ϵ^{ρ}.

• **Gauss-Codaza equations:** Gauss-Codacci equation relating the space-time curvature to the intrinsic curvature of a sub-manifold. The relation can be written:

$$R^{p}_{qrt} = \Gamma^{p}_{qr|t} - \Gamma^{p}_{qt|r} + \Gamma^{s}_{qr}\Gamma^{p}_{st} - \Gamma^{s}_{qt}\Gamma^{p}_{sr}. \tag{A.1}$$

See maths glossary.

This is not the same as saying that "General covariance is the idea that the laws of nature must be the same in all reference frames, and hence all coordinate systems".

Gelfand triple: In LQG the nice quantum states are typically taken to be finite linear combinations of spin network states whose space is denoted by Cyl^{*} (the space of cylindrical functions of connections). Solutions to the quantum Einstein equations belong to its dual, Cly^{*}.

• **generating functional:**

$$W[J] = \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}[\mathcal{L}(\phi, \partial_{\mu}\phi) + J\phi]d\tau\right) \tag{A.1}$$

$$N^{-1}\exp(i\int\int \mathcal{L}[\phi])\prod_{x}[d\phi(x)] \tag{A.1}$$

in QFT the generating functional (A) is introduced as formal extrapolation of measures in infinite-dimensional spaces

$$\prod_{x}[d\phi(x)]. \tag{A.1}$$

This fails to be a true measure because the Lebesgue measure in infinite-dimensional spaces fails to be defined in general.

Given a connection A all its gauge invariant information is contained in all graphs with a finite number of degrees of freedom. We have a genuine field theory with an infinite number of degrees of freedom.

called the vertical polarization.)

• **general boundary formulation:** The so-called *general boundary formalism* extends the usual formulation of dynamics in terms of space-like hypersurfaces to one with closed boundary surfaces.

Homepage of Robert Oeckl http://www.matmor.unam.mx/ robert/

"Quantum gravity is the search for a theory that may unify general relativity with quantum mechanics and ultimately with the standard model of elementary particle physics. This problem has motivated me to think about the foundations of quantum theory. In particular, I am trying to extend the standard formulation of quantum mechanics in such a way as to make it more compatible with general relativistic concepts. It turns out that this is best done directly in the context of quantum field theory rather than non-relativistic quantum mechanics. I call this extension tentatively the general boundary formulation."

• **general covariance:** The principle that the laws of nature must be the same in all reference frames. From [17] (p.469): there should not be special coordinates that have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates.

• **general relativity:** Often described as a theory that conceives gravity not as a force between masses but as a change in the geometry of spacetime due to the presence of the matter and energy. However, the full content of GR is that it *discards the very notion of spacetime*. Instead, "spacetime geometry" is conceived as certain aspects of the relationships that exist between physical objects that live in the world (this what general relativists are referring to when they say that GR is a *background independent* theory). This is the context for Einstein's remark *"Beyond [my] wildest expectations"* (see Einstein's hole argument, active diffeomorphisms)

• **generalized fields:** in the passage from the classical field to the quantum field.with classical fields there is no classical particle picture space of distributional fields - smooth classical fields have measure zero in the quantum configuration space.

In quantizing a finite dimensional system one defines states as functions on a configuration space and then defines an inner product of two such functions ψ and ϕ through

$$(\psi, \phi) = \int_{\mathcal{Q}} d\mu \psi^* \phi \tag{A.1}$$

In the quantum theory systems with a finite number of degrees of freedom - the classical and quantum configuration spaces agree. In field theory the quantum configuration space is a substantial enlargement of the classical; in scalar field theories, for example, although the classical configurations are smooth functions on a t =const slice, the quantum configuration space consists of all tempered distributions.

The set of smooth configurations is of zero measure with respect to the Gaussian measure that determines the inner product! That is, for two smooth wavefunctions Ψ and Φ

$$\int_{\mathcal{Q}} d\mu \Psi^* \Phi = 0 \tag{A.1}$$

The regularization problems of quantum field theory can be traced back to this fact.

It is the distributional field fluctuations at the horizon that are the microscopic source for black hole entropy.

• **generalized Wick transformation:** which is arising by relating simpler Riemannian quantum theory to the more complicated physical Lorentzian theory. Allows one to obtain physical states in Lorentzian gravity from Wick transforming solutions of Riemannian quantum theory.

• **generic condition:**

(i) The strong energy condition holds,

(ii) Every timelike or null geodesic contains a point where

$$\ell_{[a} R_{b]dc[e} \ell_{f]} \ell^c \ell^d \neq 0, \tag{A.1}$$

(ℓ_a being the tangent vector of the geodesic). The condition serves to exclude certain pathological spacetimes ... I think?

see energy conditions - strong and weak.

• **geodesic:** A geodesic is the closest thing there is to a straight line curved space time. The shortest distance between two points. A more general definition of a geodesic is its velocity vector is parallel transported along the curve it traces out in spacetime. In other words, the parallely propagated vector at any point of the curve is parallel, that is, proportional to the vector at this point:

$$\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = \lambda \frac{dx^a}{du}. \tag{A.1}$$

• **geodesically convex:** A set U of spacetime $\mathcal{M}$ is geodesically convex if any two points of U may be joined by a unique geodesic lying entirely in U.

• **geometric quantization:** Geometric quantization provides a geometric, general framework for the quantization of a sympletic manifold M which is not neseccarily a cotangent bundle, for example when M it is compact.

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Roughly a polarization is a choice of d coordinates on the $2d-$phase space $\mathcal{M}$, with the idea that the functions in our quantum Hilbert space will be independent of these d variables.

For example if $\mathcal{M} = \mathbb{R}^{2d}$, then we may take the usual position and momentum variables $x_1, \ldots, x_d, p_1, \ldots, p_d$ and then consider functions that depend only on $x_1, \ldots, x_d$ and are independent of $p_1, \ldots, p_d$. (This is called the vertical polarization.)

Alternatively, one may consider complex variables $z_1, \ldots z_d, \overline{z}_1, \ldots, \overline{z}_d$ and then consider the functions that are independent of $?z_k$, that is, holomorphic. (This is a complex polarization.) In that case our Hilbert space is the Segal-Bargmann space. To be more precise, in geometric quantization the elements of the quantum Hilbert space are not functions, but rather sections of a certain complex line bundle with connection. The sections are required to be covariantly constant in the directions corresponding to the polarization.

Apparent dependence of the act of quantization on the choice of coordinates.

• **geons:**

Lines of force can disappear at one point and reappear at the other - one points acts as sink and the other a source. The field equations of gravity would determine the motion of the sources and sinks and therefore how the charged particles represented by them. Purely gravitational effects.

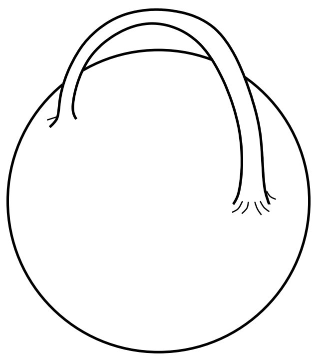

Figure A.21: geons.

• **Gibbs state:**

$$\exp(\mathcal{H}/kT) \tag{A.1}$$

• **Giesel, Kristina:**

Figure A.22: Kristina Giesel.

• **Gleason's theorem:**

• **globally hyperbolic:** Globally hyperbolicity is a proper of some spacetimes with the topology $\mathbf{R} \times \Sigma$, it ensures that there exists a universal time function whose gradient is everywhere timelike.

The physical significance of global hyperbolicity comes from the fact that there is a family of Cauchy surfaces $\Sigma(t)$ for a region of spacetime $\mathcal{M}$. One can predict what will happen in from data on the Cauchy surface.

In a globally hyperbolic region of spacetime, there is a geodesic of maximum length joining any pair of points that can be joined by a timelike or null curve. This fact is used in the singularity theorems: one establishes there is a region of spacetime which is globally hyperbolic, then one demonstrates that there is a geodesic which has a pair of conjugate points and hence is not a of doesn't maximum length (involves focussing theorem). The way out of this contradiction is that this geodesic comes to a full stop indicating the presence of a singularity.

Equivalent definition of global hyperbolicity: A spacetime $\mathcal{M}$ is said to be globally hyperbolic if the sets $J^+(x) \cap J^-(y)$ are compact for all x, y in $\mathcal{M}$

The global hyperbolicity of $\mathcal{M}$ is closely related to the future or past development of initial data from a given spacelike hypersurface.

• **"global time problem"**

"multiple choice problem" - that one could use different time functions and one must prove the quantum theories are equivalent.

[?] Arlen Anderson, *Evolving Constants of Motion*, [gr-qc/95070708].

• **Gowdy spacetime:** an example of a that can be quantized as a midisuoerspace model.

• **gravitational backgrounds:**

$$\mathcal{L} = \sqrt{-g}\{g^{\mu\nu}\nabla\phi^*\nabla_\nu\phi - m^2\phi^*\phi\} \tag{A.1}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + m^2\phi = 0 \tag{A.1}$$

and the conserved current density is:

$$J_\mu = i(\phi^*\nabla_\mu\phi - (\nabla_\mu\phi^*)\phi) \equiv i\phi^* \overleftrightarrow{\nabla}_\mu \phi \tag{A.1}$$

the inner product:

$$< \psi|\phi >= i\int_\Sigma \sqrt{-g}\psi^* \overleftrightarrow{\nabla}_\mu \phi \, d\Sigma^\mu \tag{A.1}$$

is independent of Σ.

• **"gravitational bound":** the gravitational boound proble in GR is an intrinsically non-linear problem even when the conditions are such that the field is weak and the motions are non-relativistic, at least in the time-dependent case. [**?**]

• **gravitational wave:** A gravitational wave is a linear perturbation of the gravitational field characterised by an oscillating curvature tensor which causes the immediate neighbourhood of space-time region through which it passes to oscillate.

• **gravitons:** Gravitons don't play a fundamental role in the loop quantum gravity. Instead, they only serve as a kind of approximate way of dealing with small perturbations of a weave state in which a superposition of spin networks mimic a given classical spacetime metric.

"There is a general belief, reinforced by statements in standard textbooks, that: (i) one can obtain the full non-linear Einstein's theory of gravity by coupling a massless, spin-2 field h_{ab} self-consistently to the total energy momentum tensor, including its own; (ii) this procedure is unique and leads to Einstein-Hilbert action and (iii) it only uses standard concepts in Lorentz invariant field theory and does not involve any geometrical assumptions. After providing several reasons why such beliefs are suspect and critically re-examining several previous attempts we provide a detailed analysis aimed at clarifying the situation. First, we prove that it is impossible to obtain the Einstein-Hilbert (EH) action, starting from the standard action for gravitons in linear theory and iterating repeatedly. Second, we use the Taylor series expansion of the action for Einstein's theory, to identify the tensor $\mathcal{S}^{ab}$, to which the graviton field h_{ab} couples to the lowest order. We show that the second

rank tensor $\mathcal{S}^{ab}$ is not the conventional energy momentum tensor T^{ab} of the graviton and provide an explanation for this feature. Third, we construct the full nonlinear Einstein's theory with the source being spin-0 field, spin-1 field or relativistic particles by explicitly coupling the spin-2 field to this second rank tensor $\mathcal{S}^{ab}$ order by order and summing up the infinite series. Finally, we construct the theory obtained by self consistently coupling h_{ab} to the conventional energy momentum tensor T^{ab} order by order and show that this does not lead to Einstein's theory. (condensed)." There is more to gravity than gravitons and this will be elaborated in a separate publication, [in preparation].

Fock states are not available in the absence of background spacetime - so what are particles? ????

• **Groenwald-van Hove theorem:** A collection f_i of functions on the symplectic space $(\mathcal{M}, \omega)$ which Poisson commute with each other (are in involution) is said to be complete if the vanishing of $\{f_i, g\}$ for all i implies that g is a function of the form $g(x) = h(f_1(x), \ldots, f_n(x))$. A collection of operators $\{A_j\}$ is said to be complete if any operator B which commutes with each A_j is a multiple of the identity. Consider a linear map ρ from the space of observables $C^\infty(\mathcal{M})$ on the symplectic space $(\mathcal{M}, \omega)$ such that

(1) $\rho(1) = \mathbb{I}$,

(2) $\rho(\{f, g\}) = i[\rho(f), \rho(g)]/\hbar$,

(3) for some complete set of functions $f_1, \ldots, f_n$ in involution, the operators $\rho(f_1), \ldots, \rho(f_n)$ form a complete commuting set.

The Groenwald-van Hove theorem states that such a map does not exist in general.

• **group averaging:** Projecting it onto the physical Hilbert space by gauge-orbit smearing. We are using the finite - rather than the infinitesimal - form of the constraints.

you have to always first of all you should define a subspace in the Hilbert space which should have stronger topology that the topology of the Hilbert space.

• **Gribov problem:** An obstruction occurs when the gauge fixing term can not be defined globally or that the gauge fixing function intersects a gauge orbit more than once.

• **Haag-Kastler axioms:** [32] [394] [395] Axioms of algebraic quantum field theory.

• **Haag-Ruelle theory:** In standard scattering theory one makes the physical assumption that in the far future $t_f \to \infty$ and far past $t_i \to -\infty$ any outgoing and ingoing particles respectively do not interact. This is not really true. However, using the methods of of local quantum physics, assuming that te theory has a mass gap (the four momentum squared operator should have a pure point spectrum which is separeted from the continuum) one can prove that the vacuum correlators of the asymptotic fields reduce to those of free field, where the vacuum really means the interacting vacuum.

• **Haag's theorem:** In quantum field theory, as apposed to quantum mechaincs, there exist unitary inequivalent representations (that is, representations which cannot be connected by a unitary transformation) of the canonical commutation relations (CCR). Haag's theorem shows that the representation of the interacting theory differs from that of the free theory. This poses problems for perturbative techniques used in standard textbooks on quantum field theory. The free part is used to define an orthonormal basis of states to which the interaction applies. Haag's theorm says that this is not possible. Haag's theorem forbids us from appling perturbation theory we learned in quantum mechanics to quantum field theory. We are left with the question of why perturbation theory works as well as it does.

• **habitat:** Let $\mathcal{D}$ be the finite linear span of SNWF's which is dense in the kinematic Hilbert space and let $\mathcal{D}^*$ denote its algebraic dual. A habitat is a subspace of $\mathcal{D}^*$ containing $\mathcal{D}^*_{Diff}$ with the minimal requirement that is preserved by the dual action of the Hamiltonian constraints.

raises questions about the Hamiltonian constraints introduced by Thiemann see [].

habitat, 1784.

• **Hamilton constraint:** Hamilton constraint generates "time" translations and hence encodes the dynamics of the theory.

$$\hat{\mathcal{H}}_\epsilon(N)|f>= 0 \tag{A.1}$$

$< \Psi| \,.\, > \leftrightarrow \; \Psi(\,.\,)$

$$\Psi(\,.\,) \in (\mathcal{H}^*_{Kin})_{Diff} \tag{A.1}$$

$$< \Psi|\hat{\mathcal{H}}_\epsilon(N)f > \leftrightarrow \; \Psi(\hat{\mathcal{H}}_\epsilon(N)f) = 0 \tag{A.1}$$

it is independent of the value of ϵ! The underlying reason is understood to be the diffeomorphism covariance of the graph-dependent triangularization prescription. Proved in section ??? The same reasoning applies to the matter coupled case. The limit $\epsilon \to 0$ is therefore already performed; hence the theory, including coupling to the standard model, is manifestly finite and does not require renormalization!

• **Hamilton function:** The Hamilton function of a finite dimensional dynamical system is the value of the action of a solution of the equations of motion, viewed as a function of the initial and final coordinates.

The

gr-qc/0408079

• **Hamilton vector field:** Hamiltonian vector field - Given a manifold X with a symplectic structure ω, any smooth function $f : X \to R$ can be thought of as a "Hamiltonian", meaning physically that we think of it as the energy function and let it give rise to a flow on X describing the time evolution of states. Mathematically speaking, this flow is generated by a vector field $v(f)$ called the “Hamiltonian vector field” associated to f

$$\omega(.,v(f)) = df??? \tag{A.1}$$

In other words, for any vector field u on X we have

$$\omega(u,v(f)) = df(u) = uf??? \tag{A.1}$$

The vector field $v(f)$ is guaranteed to exist by the fact that ω is nondegenerate.

• **Hamilton-Jacobi equation:**

$$\begin{aligned}\int_0^t \left(p\frac{dq}{d\tau} - \mathcal{H}\right) d\tau &= \int_0^t \left(P\frac{dQ}{d\tau} - \mathcal{H}' + \frac{dF(q,P)}{dt}\right) d\tau \\ &= \int_0^t \left(P\frac{dQ}{d\tau} - \mathcal{H}'\right) d\tau \\ &+ F(q'',P'',t) - F(q',P',0)\end{aligned} \tag{A.0}$$

$$\frac{\partial(P,Q)}{\partial p,q} = 1 \tag{A.0}$$

consider the transformation for which the new Hamiltonian is zero.

one gets the Hamilton-Jacobi equation

$$H\left(q,\frac{\partial F}{\partial x},t\right) + \frac{\partial F}{\partial t} = 0 \tag{A.0}$$

Since $\mathcal{H}' = 0$, the new coordinates Q and P are constants of motion.

• **Hawking, Stephen:**

• **Hawking radiation:** $T = \kappa$

• **Heisenberg representation:** A Heisenberg state is can be viewed as the value Schroödinger state at some fixed time.

• **Heisenberg uncertaiinty principle:** From “THE LIFE OF THE COSMOS” by Lee Smolin p.306:

Figure A.23: Stephen Hawking.

“Let us imagine that we have a friend who lives in Quantumland, where big things can be simple. To tempt us with the beauty of her world, she has seny us a present. We go to the airport to receive it and are given a sealed box with a door on each end. On top of is written, ‘QUANTUM PET CARRIER - ONLY OPEN END AT A TIME.’ In Quantumland no one would have t be told that you can only open one end of a box at a time, but we will see that the reason to be grateful for this advice.

Bringing the box home, we hurriedly open it to see what is inside. Opening one end, we see the head of a cat peer out! Lovely, but the cat stays inside. It seems that one property of pets in Qunatumland is that they can never come out of their box, one can only interact with them by opening one of the doors. OK, we can live with this, but we become curious to at least know the sex of our cat. Well, we can use the door at the other end for this. We try to open it, but we find it is closed tight. Remembering what was written on top, we close the first door. Immediately the back door comes open. By looking in, we are able ascertain that our pet is a boy.

This done, we go back to the first door to play with our cat. To do this we must first close the back door. We then open the front door to find a jolly looking puppy gazing back at us!

After some trials and examinations, we discover that we are in an interesting situation. When we open the first door we discover that our quantum pet is either a cat or a dog. If we open the second door, we discover that our pet is either male or female. Howeveer, by the peculiar properties of the box, our vision is obscured so we cannot be sure, when gazingin the front, what sex our cat or dog is. And when we look in the back door we can ascertain the sex, but we cannot judge reliably whether it is a dog or a cat.

We cannot have both doors open at once, so we can never be sure of both species of our pet and its sex. Ascertaining one destroys knowledge of the other. If all we do is look ot the front end, then once we have seen a cat there, we will always see a cat. If we wish, we may at any time close that door and peer in the other side to learn the pet's sex. Whether it is a cat or a dog, we discover that there is a fifty percent probability that it is male and a fifty percent probability that it is female. But, once we have done that, if we go back to the front, we will not necessarily find a cat, as we did before. For once we have ascertained the sex, the species again gets scrambled, and half the time it will be a cat, and half the time it will be a dog.

What we are experiencing is exactly the Heisenberg Uncertainty Principle. It is happeneing because a complete description of our quantum pet would include its species and its sex. According to classical science, we ought to be able to take the animal out of the box and see what it is. But a quantum pet can never be removed from its box and, for reasons that are perhaps mysterious, we can only ever observe one aspectg at a time.

Perhaps the reader thinks I'm being facetious, or teasing. But no, I am describing what we believe is the general situation we are in when we observe any physical system. The Heisenberg Uncertainty Principle limits the information we can have about any system to always exactly half of the information we need to have a complete description. We always have some choice of which inforamtion we would like to have. But, try as we may, we cannot exceed this limit."

• **holographic principle:**

• **holonomic equivalence:** two loops α and β are said to be holonomic equivalent if they have the same holonomy with respect to every connetion, i.e.

$$H_A(\alpha) = H_A(\beta) \quad \text{for all } A \text{ in the space of connections } \mathcal{A}. \tag{A.0}$$

The set of all holonomic equivalence classes is called the **holonomic loop space** and is denoted $\mathcal{HL}$.

• **holonomy algebra:** denoted $\mathcal{HA}$.

• **holonomy-flux $*$-algebra:** Holonomies of connections lead to well defined operators on $\mathcal{H}^0$.

$$h_\alpha(A) = \mathcal{P}\ \exp\left(\oint_\alpha A_a ds^a\right) \tag{A.0}$$

The connection is smeared in *1-dimensions.* In LQG the classical electric flux $E_k(S)$ through a surface S is the integral of the densitised triad $\tilde{E}^a_k$ over a two surface

, where n^S_a is the conormal vector with respect to the surface S.

The conjugate momentum is dual to a 2-form

$$e_{abi} := \frac{1}{2}\eta_{abc}\tilde{E}^c_i$$

geometrically it is natural to smear in 2-dimensions. It again turns out that these 2-dimensionally smeared fields lead to well defined operators on $\mathcal{H}^0$.

$$[E_i]_f := \int_S e_{abi} f^i dS^{ab}$$

$$\begin{aligned} \frac{\delta h_\alpha(A)}{\delta A_i^a(x)} &= \int_0^1 \delta^3(x^j - \alpha^j(t)) \frac{d\alpha^i}{dt} (U_\alpha)_0^t(A)\ \tau_a\ (U_\alpha)_t^1(A)\ dt & \text{(A.1)} \\ \frac{\delta E(S)}{\delta E_a^i(x)} &= \int_S dudv\ n_i \delta(x^j - X^j(u,v))\ \tau_a, & \text{(A.2)} \end{aligned}$$

By virtue of the canonical commutation relations, the Poisson brackets turn out to be

$$\{E_a(S), h_\alpha(A)\} = \int d^3x \frac{\delta E_a(S)}{\delta E_b^i} \frac{\delta h_\alpha(S)}{\delta A_i^b}. \tag{A.2}$$

From these last three relations we see that if S and α have no common points the integration vanishes. The same happens if α belongs to S since in this case vectors tangent to the graph are orthogonal to n^i. We are led to consider loops with a finite number of intersections with the surface. Consider the case of one edge e intersecting the surface S

$$\begin{aligned} \{[E_S]_f, (h_e)_{mn}\} &= \lim_{\epsilon \to 0} [(h_{e_1}^\epsilon)_{ml}\ \{[E_S]_f, (h_{e(\epsilon)})_{lp}\}\ (h_{e_2}^\epsilon)_{pn}] \\ &= \lim_{\epsilon \to 0} \left[(h_{e_1}^\epsilon)_{ml} \left\{ \int_S f_b(y) \tilde{E}^b(y), \int_{e(\epsilon)} A^a(x) \tau_{lp}^a \right\} (h_{e_2}^\epsilon)_{pn} \right] \\ &= \mp 8\pi G \gamma f_a(P) (h_{e_1}\ \tau_a\ h_{e_2})_{mn} \end{aligned}$$

$$\{E_a(S), h_\alpha(A)\} = 8\pi G \gamma \tau_a \sum_{e \subset \alpha} h_e(A) o(e,S) \tag{A.-1}$$

where $o(e,S)$ is $+1$, -1 i the orientations of e and S agree or disagree, respectively.

Hence one can define the action of $E(S)$ ona generic cylindrical function by the expression

$$E_a(S)[f] = \{E_a(S), f(h_\alpha(A))\} \tag{A.-1}$$

with $E_a(S)$ acting as a vector field.

The Poisson bracket between the strip functionals vanishes unless the two strips intersect. If they do, the bracket is given by a sum of slightly generalized strip functionals.

If one attempts to smear connections and triads in 3-dimensions the resulting operators fail to be well defined on $\mathcal{H}^0$.

The new algebra provides a distributional extension of the old one.

There is uniqueness theorem on this algebra, see section.

• **Holst action:** The Palatini action plus an extra term multiplied by the inverse of the Barbero-Immirzi parameter which does change the classical equations of motion:

$$S_H(e, A) = \frac{1}{2\kappa} \int_{\Sigma \times \mathbb{R}} (\frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} + \frac{1}{\gamma} e^I \wedge e^J \wedge F^{KL})$$

• **hoop group:** the quotient of the loop group with respect to holonomic equivalence.

• **horismos:** The future horismos of a set S is defined to be $E^+[S] := J^+[S]/I^+[S]$.

• **Husain, :**

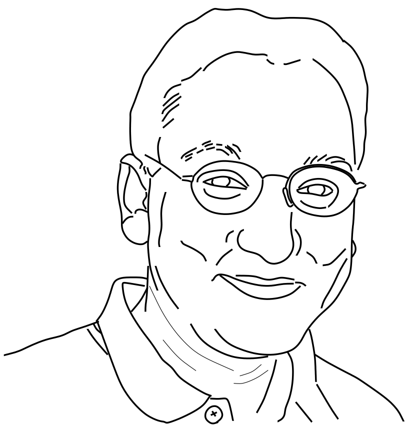

Figure A.24: Husain.

• **ideal clock limit:**

• **Immirzi-Barbero parameter:** A certain free parameter, γ, in the canonical quantization of gravity. This parameter is not present in the classical theory - In the classical theory, different values of the Immirzi parameters label equivalent classical theories. However, in the quantum thoery, representations with different Immirzi parameters are not unitarily equivalent - arises from an ambiguity in the quantization procedure... It can be fixed by the requirement that the quantum gravity computation reproduce the Hawking value for the entropy of a black hole.

A conflicting assertion Alexandrov *et.al*: [153] [154]. *"It was thought that the anomaly in question is a physical one so that the Immirzi parameter becomes a new fundamental constant. However, in a series of works [153] it was shown that this is not the case because* *it is actually a consequence of a diffeomorphism anomaly, whereas there is a quantization which preserves all classical symmetries and leads to results independent of the Immirzi parameter.".*

• **induced metric:** The induced metric the metric q a hypersurface picks up from the metric g of the spacetime the hypersurface is embedded. The induced metric of a hypersurface Σ is obtained by restricting the line element to displacements within the hypersurface Σ.

The induced metric is also called the **first fundamental form**.

Some jargon: The induced metric q is the pullback of the spacetime metric g, denoted $\phi^* g$, where ϕ is the map from the spacetime manifold $\mathcal{M}$ to the hyperspace manifold N, i.e., $\phi : \mathcal{M} \rightarrow N$.

• **inertial frames:** The set of all inertial systems is the set that can be derived from a given one by a Lorentz transformations.

• **inextendable curve:**

• **inflation:** An accelerated expansion of the very early universe. Eliminates the need for contrived and highly special initial conditions.

• **inner product for quantum gravity:** In LQG the inner product on spacially diffeomorphism invariant states is known in closed form. One now wishes to have an inner product on the subset of states that satisfy all the constraints. Such a can be written formally in terms of a "projection operator" by means of spin foams - one of the promises of the Master constraint programme is the construction of a well defined physical inner product. - i.e., the physical inner product, ??from which we can calculate physical expectation values??.

• **integrable:**

integrability of of the system in the sense of Lioville:

(1) the number of independent conserved quantities equals the number of degrees of freedom, and that

(2) these conserved quantities are in involution.

are in involution means F_i

$$\{F_i, F_j\} = 0 \tag{A.-1}$$

There are standard methods available for solving the system completely (Hamilton-Jacobi theory, action-angle varaibles, ...).

• **intensive thermodynamical quantities:**

• **interaction picture:** In the Schr?odinger picture, time evolution is governed by the states and their equations. In the Heisenberg picture, time evolution is governed by the observables and the equation $\frac{dA}{dt} = \{B, \mathcal{H}\}$.

In the interaction picture we divide the time dependence between the states and the observables. This is suitable for systems with a Hamiltonian of the form B H = B H0 + B H1 where $A_H 0$ is time independent. The interaction picture has many uses in perturbation theory.

• **interpretations:** Some seek to interpret quantum mechanics in a way which accords more with our intuition derived from classical mechanics.

reduction of the wave-packet of a microscopic system and a specification of its quantum state by a macroscopic instrument

• **interval:** In the general theory of relativity, the square of the interval is given by:

$$ds^2 = g_{ab}\, dx^a\, dx^b, \tag{A.-1}$$

where g_{ab} are the components of the metric tensor, and dx^a is the differential of the coordinate x^a. Time-like interval $ds^2 < 0$; Space-like interval $ds^2 > 0$.

• **inverse loop transformation:**

• **irreduciable representation:**

• **Chris Isham:** In a seminal paper with Ashtekar rigorouly constructed the quantum configuration space of distributional connections using GNS theory for C^* algebras. Also later apllied topos methods to develop of a theory of quantum gravity.

• **ISO (3) group:** $ISO(3)$ denotes the Euclidean group in three dimensions.

The translation group is an abelian group.

Obviously

$$\hat{R}\hat{T}\hat{R}^{-1} = \hat{T}' \tag{A.-1}$$

consequently the translation group is an invariant abelian subgroup of the Euclidean group, (translation-rotation group). $SO(3)$ is another subgroup such that the only element they have in common with the translation group is the identity element $\mathbf{I}$. Every element of $ISO(3)$ can be written in a unique way as $g = RT$, $R \in SO(3), T \in \mathbf{R}^3$ then $ISO(3)$ is the semi-direct product of $\mathbf{R}^3$ and $SO(3)$,

$$SO(3) \otimes_S \mathbf{R}^3. \tag{A.-1}$$

Figure A.25: Chris Isham.

We introduce translation generators P_a, $a = 1, 2, 3$, which satisfy

$$[J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = 0, \tag{A.-1}$$

- **ISO (2,1) group:** $ISO(2,1)$ denotes the Poincare group in three dimensions.

- **isolated horizons:**

Isolated horizons are generalizations of the event horizon of stationary black holes to physically more realistic situations. The generalization is in two directions. First, while one needs the entire spacetime history to locate an event horizon, isolated horizons are defined using properties of spacetime *at the horizon*. Second, although the horizon itself is stationary, the outside space-time can contain time-dependent fields and geometry.

Restriction to spacetimes which admit an internal boundary which has certain boundary conditions.

Type **I** the isolated horizon geometry is spherical

Type **II** the isolated horizon geometry is axi-symmetric. This class includes rotating isolated horizons as well as distorted (due to exterior matter), see [136].

Type **III** Most general include arbitrary distortions

Note that the symmetries refer *only* of the horizon geometry. This weak set of boundary conditions give rise to the zeroth and first laws of black hole mechanics, in fact, isolated horizons can be characterized this way.

Black hole entropy [132].

The action principle and the Hamiltonian description is well defined and the resulting phase space has an infinite number of degrees of freedom. The full spacetime metric need not admit any geometric symmetries even in a neighbourhrood of the horizon. They quantize this sector of the full phase space using techniques developed for the full theory.

The problem was reduced to counting the number of ways to puncture a two-dimensional sphere, representing the horizon, which gave the horizon area close to a given value.

Isolated Horizons in Numerical Relativity [131].

• **Kaluza-Klein:** The radius of the extra dimension is taken to be fixed and non-dynamical and not subject to Einstein's field equations. As such there are no processes in which gravitational and electromagnetic effects convert into each other. For the thoery to be a true unification, gravity and electro-magnetism should be aspects of one phenomenon.

Another objection is that the theory cannot incorporate ferminons of the gauge theory.

• **Kauffman, Louis:**

• **kernel:** the set of (eigenstates) solutions of an operator which have zero as their eigen-value. or

• **Killing horizon:**

$$\xi^a \xi_a = -1 - \frac{2M}{r} \tag{A.-1}$$

Thus outside the horizon ($r > 2M$) ξ^a is timelike, at the horizon ($r = 2M$) ξ^a is null.

• **Killing vector field:** Geometry does not change along a Killing vector field. An isometry of the spacetime. A Killing vector field satisfies the Killing equation

$$\nabla_a K_b + \nabla_b K_a = 0. \tag{A.-1}$$

• **Kinematic Hilbert space:**

There are several complementary characterisations of the Kinematic Hilbert space $\mathcal{H} := \mathcal{H}_\omega$ which are useful in different contexts.

1. Positive linear functional characterisation

2. C^*- algebraic characterisation

3. Projective limit characterisation

• **KMS condition:** Kubo-Martin-Swinger condition for thermal equilibrium.

$$< 0,\beta|A(t)B(t)|0,\beta >=< 0,\beta|B(t')A(t+i\beta)|0,\beta > \tag{A.-1}$$

$$\omega((\gamma_t A)B) = \omega(B(\gamma_{t+i\beta}A)) \tag{A.-1}$$

Haag *et al* [53] have shown that this KMS condition reduces to the well known Gibbs condition

$$\omega(.) = \frac{\sum_n e^{-\beta E_n}|m\psi_n >< \psi_n|}{(\sum_n e^{-\beta E_n} < \psi_n|\psi_n >)} \tag{A.-1}$$

• **KMS states:** The relevant states describing thermal equilibrium are the so-called KMS-states.

• **Kochen-Specker theorem:** [364] [362] Implies it is not possible to assign actual, *possessed*, values to all quantum mechanical physical quantities of an individual system. (See section contextual interpretations)

It is the one of the no-go theorem against hidden variables theories, (the other being Bell's theorem which states that, given the premise of locality of physical theories, a hidden variables theory cannot match the statistical predictions of quantum mechanics).

better version: [363] Kochen-Specker theorem asserts the impossibility of associating real values $V(\hat{A})$ to all physical quantities in a quantum theory (if $\dim\mathcal{H} > 2$) whilst preserving the '$FUNC$' rule that $V(f(\hat{A})) = f(V(\hat{A}))$ - i.e., the value of a function f of a physical quantity A is equal to the function of the value of the quantity. Equivalently, it is not possible to assign true-false values to all the propositions in a quantum theory in a way that respects the structure of the associated lattice of the projection operators.

Before we can state the theorem we first need to introduce the idea of the valuation of an operator $\hat{A}$ representing a physical quantity, $V_s(A)$. Valuation is assigning truth-values to propositions '$A \in \Delta$' asserting that the quantity A lies in the Borel subset Δ of the spectrum of the operator $\hat{A}$ that represents A.

The Kocher-Specker theorem asserts that, in the case when the dimension of Hilbert space is greater than 2, one *cannot* assign real numbers to all quantum quantities in a way that preserves functional relations between them. By 'preserves functional relations' is meant for any operator $\hat{A}$ and any function of it $f(\hat{A})$ (f a function from $\mathbb{R}$ to $\mathbb{R}$), the value of $f(\hat{A})$ is the corresponding function of the value of $\hat{A}$, i.e., $V_s(f(A)) = f(V_s(A))$.

(1)

??? (2) ideal measurement of an observable $\hat{A}$ in a state $\psi \in H$ yields an eigenvalue of $\hat{A}$ and, immediately after the measurement, the state is thrown into corresponding eigenstate. ???

• **Komar integral:** An expression for the conserved mass and angular momentum for stationary spacetime evaluated when the spacial boundary is pushed to infinity.

$$E = \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x\sqrt{q}n_a\sigma_b\nabla^a K^b, \quad J = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{q}n_a\sigma_b\chi^a K^b \tag{A.-1}$$

where $\partial\Sigma$, σ_b (timelike and future pointing) and n^a timelike orthogonal unit normal vectors to $\partial\Sigma$. where K^b time translations for energy.

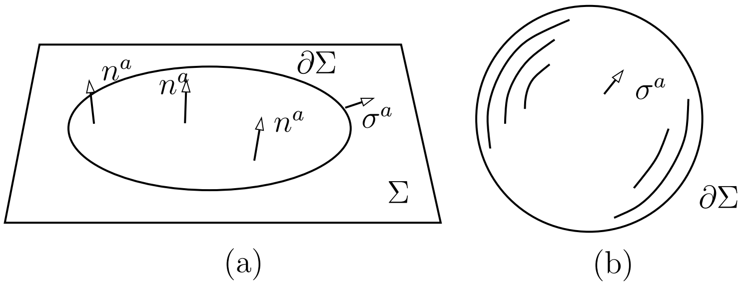

Figure A.26: An expression for the conserved mass, evaluated when the spacial boundary is pushed to infinity. (b) $\partial\Sigma$ is a closed spacelike two-surface surrounding the source.

• **Krasnov, Kirill:**

Figure A.27: Kirill Krasnov.

• **Kruskal coordinates:** These corrdinates cover the entire spacetime manifold of a maximally extended solution of classical general relativity.

• **Kuchar's "reduction ad absurdum":** We require a "global time coordinate"

• **the landscape:**

• **lapse function** N**:** N, together with the shift function N^a, only serves to specify the foliation of $\mathcal{M}$ and as such it is not itself a dynamical field.

• **lattice gauge theories:**

The difference between lattice approaches in Minkowskian field theories and background independent field theories that use floating graphs. An approximation to the continuum theory limit . Whereas in background independent field theories

continuum limit used in lattice approaches in Minkowskian field theories, whereas in background independent field theories allow all possible graphs in $\mathcal{M}$.

lattice gauge theory formulations applied to GR conflict with its invariance under diffeomorphisms as it introduces a fixed background geometric structure.

• **Laws of black hole mechanics:**

The zeroth law of black hole mechanics stationary black holes have constant surface gravity on the entire horizon.

The first law of black hole mechanics tells us how the area of a black hole increases when we make a transition from an initial stationary solution to a nearby stationary solution.

$$\delta M = \frac{\kappa}{8\pi G}\delta a + \Omega \delta J, \tag{A.-1}$$

where κ is the surface gravity.

The second law of black hole mechanics states that the area of an event horizon can never decrease.

A system in equilibrium will have settled to a stationary state.

Isolated horizons: all quantities which enter the statement of the law now refer to the horizon itself.

• **Laws of thermodynamics:**

The zeroth law of thermodynamics states that in thermal equilibrium the temperature is constant throughout the system.

The first law of thermodynamics tells us how the

$$\delta E = T\delta S + \delta W. \tag{A.-1}$$

The second law of thermodynamics states that the entropy of an isolated system can never decrease.

• **Lax pairs:**

$$\frac{dL}{dt} \equiv \dot{L} = [M, L]. \tag{A.-1}$$

$$L(t) = g(t)L(0)g(t)^{-1} \tag{A.-1}$$

where the invertible matrix is determined by the equation

$$M = \frac{dg}{dt}g^{-1}. \tag{A.-1}$$

• **Lewandowski, Jerzy:**

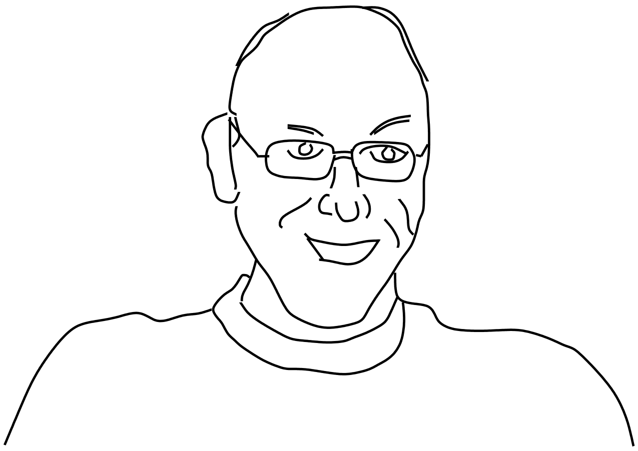

Figure A.28: Jerzy Lewandowski.

• **Lie derivative:** Lie derivative.

see maths glossary.

• **linear 2-cell spin-foam models:**

• **Livine, Etera:**

• **local particles:** Realistic particle detectors are finitely extended. Physical states detected by a localized detector (eigenstates of local operators describing detection) are local particles states. Local particles have a chance of retaining meaning in the fully general relativistic context where Fock states are not available.

• **Loll, Renate:** Loll works on developing a theory of quantum gravity, reconciling the beautiful geometric description of space and time laid out in Einstein's theory of General Relativity with the insight that all of physics at its most fundamental level must

Figure A.29: Etera Livine.

be described by quantum laws of motion. She is one of the pioneers of a new approach to the nonperturbative (background independent) quantization of gravity, that of Causal Dynamical Triangulations which recently has produced a number of remarkable results. These include a dynamical derivation of the fact that space-time is four-dimensional (something that can be taken for granted only in classical gravity) and that it has the shape of a de Sitter Universe (like our own universe in the absence of matter), and of the so-called wave function of the universe which plays an important role in understanding the quantum behaviour of the very early universe. Remarkably, one also finds that the dimensionality of spacetime reduces smoothly to two at short distances, indicative of a highly nonclassical behaviour of spacetime geometry near the Planck scale. These results are obtained by superposing elementary quantum excitations of geometry which have a notion of causality ("cause preceding effect") built into them at the very smallest scale.

• **loop algebra:**

given a connection the identities satisfied by Wilson loops may be implemented on the free vector space of formal sums of *single* loops by requiring that if

$$\sum_i c_i T[\alpha_i, A] = 0 \qquad \text{(A.-1)}$$

for *all* connections, then these loops are linearly dependent

$$\sum_i c_i \alpha_i = 0 \qquad \text{(A.-1)}$$

on the free vector space of single loops.

Figure A.30: Renate Loll.

When the product on the free vector space is defined by

$$\left(\sum_j c_j \alpha_j\right) \cdot \left(\sum_k d_k \alpha_k\right) = -\sum_{j,k} c_j d_k \left(\alpha_j \# \beta_k + \alpha_j \# \beta_k^{-1}\right) \tag{A.-1}$$

these Mandelstam relations define an ideal. The quotient space by the ideal defines an algebra, which is called the holonomy algebra...

[gr-qc/9512020]

• **loop quantum cosmology:** Mini-superspace but which appropriate incorporation of the discrete nature of quantum geometry. Very good accessible account can be found in [242].

As yet it makes no attempt to address the problem of time or interpretation of the wavefunction of the universe.

• **loop quantum gravity LQG:**

A background independent approach to quantum gravity. In many ways conservative based only on the experimentally well confirmed principles of general relativity and quantum mechanics.

a canonical quatization of a Hamiltonian description of classical general relativity, whereby spacetime is sliced into a stack of 3-d spacial hypersurfaces Arnowitt-Deser-Misner (ADM). In LQG the basic variable used is, rather than a metric, a connection and its "conjugate momentum" a triad field (can be thought of as the square root of the 3-d spatial metric).

The spacial 3-metric on a hypersurface is constructed from the triad field and the connection contains information about extrinsic curvature of the slice of the slice sitting in the spacetime, establishing contact with the metric ADM formulation.

Wilson loops, polymer nature

• **loop representation:** import machinery from particle physics... [54].

• **loop transformation:** the Fourier transformation [54]

$$\tilde{\psi}(k) = \int dk e^{ik \cdot x} \psi(x), \tag{A.-1}$$

which relates the position representation with the momentum representation.

$$\psi(\gamma) := \int_{\mathcal{A}/\mathcal{G}} d\mu(A) T^0(\gamma)[A] \Psi(A), \tag{A.-1}$$

where the Wilson loop functionals T^0 plays the role of an integral kernal, (as does $e^{ik \cdot x}$ in the Fourier transformation), and wave functions $\psi(\gamma)$ in the loop representation are labelled by closed spacial curves γ. Once the loop transform is rigourously defined the loop representation is unitarily equivalent to the connection representation.

• **Lorentz covariant loop quantum gravity (CLQG):** Standard loop quantization is based on the Astekar-Barbero formulism and leads to the theory in which $SU(2)$ gauge group in the tangent space. It starts with the first order formulation of general relativity in 3+1 dimensions with the Lorentz gauge group in the tangent space. Then as a result of partial gauge fixing, the canonical formulation possesses, besides the usual diffeomorphism invariance, only a local $SU(2)$ symmetry.

• **LOST uniqueness theorem:** Named after the people who developed it: Lewandowski, Okolow, Sahlmann, and Thiemann. In LQG once one decides to base quantization on the holonomy-flux algebra and have the spacial diffeomorphism group implememnted unitarily, there is only one choice for the Kinematical Hilbert space. Once the Kinematical representation has been choosen in which the constraints can be defined as operators, the physical representation follows by a rather tight procedure. Hence, the whole quanization programme is put on a very theoretical robust footing leading to a falsifiable theory of quantum gravity. One predicition of the theory being the discreteness of the eigenvalues of geometric operators.

• **luminosity distance:**

• **LSZ reduction formulas:** using the LSZ reduction formulas it is more easily demonstrated that the a scattering amplitude is unchanged by making in states $|in>$ into out states $|out>$ and vice versa. so we have a symmetry that implies we can exchange preparation of states and the measurement proper.

• **Mach's principle:**

• **macroscopics:**

• **Major, Seth:**

• **marginally trapped surfaces:** See trapped surface.

• **Markopoulou, Fontini:**

Figure A.31: Fontini Markopoulou.

• **Master constraint:** The multiplicity of smeared Hamiltonian constraints combined into one constraint.

since only a non-anomalous constraint algebra usually leads to a sufficiently large enough physical Hilbert space $\mathcal{H}_{Phys}$

the question of anomaly freeness is translated into the size of the physical Hilbert space $\mathcal{H}_{Phys}$.

the infinite dimensional constraint algebra, whether it closes with structure functions or structure constants, is replaced by a one-dimensional Abelean Lie algebra.

• **Master equation:** Any complete observable $\mathcal{O}$ must be constant under a transformation generated by the Hamilton constraint $C_j(x)$ i.e.,$\{\mathcal{O}, C_j(x)\}|_{C=0} = 0$. This condition is equivalent to

$$\{\mathcal{O}, \{\mathcal{O}, \mathbf{M}\}\}_{\mathbf{M}=0} = 0 \iff \{\mathcal{O}, C_j(x)\}|_{\mathcal{C}=0} = 0. \tag{A.-1}$$

see Eq (O.1).

• **Master Constraint Operator:** Replaces the Hamiltonian constraints $\hat{C}_a$ by a single Master constraint $\hat{\mathbf{M}}$ which is a spacially diffeomorphism invariant integral of squares of the individual Hamiltonian constraints

$$\hat{\mathbf{M}} := \frac{1}{2}\hat{C}_a Q^{ab} \hat{C}_b \tag{A.-1}$$

which encodes all the necessary information about the constraint surface and the associated invariants (Dirac observables).

• **Master Constraint Programme:**

Extended Master Constraint Programme:

• **maximal extension of a spacetime:** in the context of the Schwarzschild spacetime coordinate systems that allow the continuation of a metric across the event horizon. There also is an inner horizon. is maximally extended if the coordinate system allow for continuation of the metric across every horizon.??

• **matrix models:** field theories over groups.

• **maximally extended curve:** A timelike curve is maximally extended in the past if it has no past endpoint. (Such a curve is also called past-inextendible). The idea behind this is that such a curve is fully extended in the past direction, and not merely a segment of some other curve.

• **mesoscopics:** gravity experiments quantum mesoscopics and structural foundations of quantum mechanics.

• **microscopics:**

• **moments:**

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \frac{1}{2r^5}(3(\mathbf{x}\cdot\mathbf{x}') - r^2 r'^2) + \ldots \tag{A.-1}$$

With this expansion, we can rewrite $\Phi(\mathbf{x})$ as

$$Q_{ij} = \int d^3x' \rho(\mathbf{x}')(3x'_i x^{j\prime} - r'^2 \delta_{ij})$$

• **Montevideo intepretation:** In this interpretation enviromental decoherence is supplemented with loos of coherence due to the use of realistic clocks to measure time to solve the measurement problem. The resulting formulation is framed entirely in terms of quantum objects without having to invoke the existence of measurable classical quantities like time in ordinary quantum mechanics. The formulation eliminates any privileged role to the measurment process giving an objective definition of when an event occurs in a system.

• **M-theory:** background independent M-theory for strings

does not presume that they are vibrating in an preexisting spacetime. a formulation in the absence of spacetime in which instead spacetime emerges from the collective behaviour of the strings.

• **M-theory conjecture:** There is a background independent formulation of string theory which unifies all the known string theories, 11 dimensional supergravity and the 11 dimensional supermembrane theory.

C. M. Hull and P.K. Townsend, Unity of superstring dualities, Nucl. Phys. B348 (1995) 109; P. Townsend, (M)embrane theory on T 9, Nucl. Phys. (Proc. Suppl) 68 (1998) 11-16; hep-th/9507048, in Particles, Strings and Cosmology, ed J. B agger et al (World Scienti c,1996); hep-th/9612121; I. Bars, hep-th/9608061; hep-th/9607122; Petr Horava, MTheory as a Holographic Field theory, hep-th/9712130, Phys.Rev.D59 (1999)046004.

E. Witten, String theory in various dimensions, Nucl. Phys. B443 (1995) 85.

Once (or if) one finds this theory, one must find the classical solutions to this theory and show that the different perturbative string theories arise from expansions around them.

• **multipoles:**

field multipoles

source multipoles

• **naked singularities:** . In recent paper [399] Harada …*We propose the concept of 'effective naked singularity', which will be quite helpful because general relativity has the limitation of its application for the high-energy end. The appearance of naked singularities is not detestable but can open window for new physics of strongly curved spacetime.* - which of course includes quantum gravity.

• **negative mass:** (the so-called Benny Hill effect).

• **noiseless subsystems:** The framework of noiseless subsystems has been developed as a tool to preserve the fragile quantum information against decoherence []. In brief, when a quantum register (a Hilbert space) is subjected to decoherence due to an interaction with an external and uncontrollable environment, information stored in the register is, in general, degraded. It has been shown that when the source of decoherence exhibits some symmetries, certain subsystems of the quantum register are unaffected by the interactions with the environment and are thus noiseless. These noiseless subsystems are therefore very natural and robust tools that can be used for processing quantum information.

• **net of operator algebras:** an open region of sacetime $\omega \subset \mathcal{M}$

$$\omega \subset \omega' \implies \mathcal{A}_{\omega} \subset \mathcal{A}_{\omega'} \qquad \text{for all } \omega, \omega' \subset \mathcal{M}, \;\; \mathcal{A}_{\omega}, \mathcal{A}_{\omega'} \subset \mathcal{A}, \tag{A.-1}$$

• **Newmann-Penrose null tetrads:** frames that consisted of two real null vectors and two "complex null vectors". seemingly strange choice well suited for analysis of null geodesics. Intimately related to spinor basis

• **Noether's theorem:** If the equations of motion possesses some continuous symmetry, then there will be a conservation law associated with that symmetry. The most familiar examples of conservation laws are:

(i) if there is invariance under some *spatial translation*, then *momentum* is conserved;

(ii) if there is invariance under *time translation*, then *energy* of the system is a conserved quantity;

(iii) if there is invariance under *rotation* about some axis, then *angular momentum* about the same axis is conserved.

Noether's gave a proof of the general result. Most physical theories implement dynamics as a result of a variation principle, by means of a Lagrangian. Continuous transformations that leave the action invariant - except for boundary terms.

(i) Spatial translation: using the Lagrangian $\mathcal{L}(x,t)$.

$$\mathcal{L}(x+\delta x, t) - \mathcal{L}(x,t) = 0 \;\; \Rightarrow \frac{\partial \mathcal{L}}{\partial x}(x,t) = 0 \tag{A.-1}$$

using this in the Euler-Lagrange equations (N.-19),

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,t) = 0 \;\; \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}}(x,t) =: p = Const'. \tag{A.-1}$$

(ii) Time translation: That the Lagrangian does not explicitly depend on t,

$$\frac{\partial \mathcal{L}}{\partial t}(x,t) = 0 \tag{A.-1}$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \dot{x}\frac{\partial \mathcal{L}}{\partial x} + \ddot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{A.-1}$$

Making use of the Euler-Lagrange equations

$$\frac{d\mathcal{L}}{dt} = \dot{x}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \ddot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\left(\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right), \tag{A.-1}$$

$$\frac{d}{dt}\left(\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L}\right) = 0. \tag{A.-1}$$

$$E = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x}) - \mathcal{L}(x,\dot{x}) \tag{A.-1}$$

• **non-commutative geometry:** The idea is to describe geometry in algebraic terms. One deals with a function algebra over the space one is interested in and the geometry of a manifold is reformulated in terms of this algebra. A topological space $\mathcal{M}$ is completely described by the commutative algebra $\mathcal{C}^0(\mathcal{M})$ of continuous functions over it. Once the algebraic description of the geometry is obtained, the structure algebra can be made noncommutative.

Noncommutative geometry is the study of noncommutative algebras as if they were algebras of functions on spaces. noncommutative algebras should be studied as if they were the function algebras of "noncommutative spaces"

See for example [402] for introduction to mathematical constructions.

Connes proved that there is an equivalence between a spinorial compact manifold and a Spectral Triple $(\mathcal{C}(\mathcal{M}), \mathcal{D}, \Gamma)$, which contains the algebra, the Dirac operator $\mathcal{D}$ encoding the differential structure and the metric, and finally the chiral operator Γ.

Connes. quantum mechanics does not permit the measurement of distance smaller than the Plank length and time intervals smaller than the Plank time. Non-commutative spacetimes are adopted as *background* spacetimes.

• **non-commutative spacetime:** non-commutative spacetime can be derived from a quantum deformation which replaces the enveloping algebra of the Poinare Lie algebra by a non-commutative Hopf algebra(quantum group).

[]

• **non-commutative theories:** hepth/9808192

"One success of noncommutative geometry is that the standard model of particle physics of particle physics. As Alain and his colleagues discovered, if you take Maxiwell's theory of electrodynamics and write down the simplest possible noncommutative geometry, out pops the Weinberg-Salaman model unifying electromagnetism with the weak nuclear force. In other words, the weak interactions, together with the Higgs field, show up automatically and correctly."

Non-commutative geometry is a third approach to quantum gravity. It is by definition a background independent approach, as the basic idea is to replace the background manifold by algebraic generalisations of a certain set of diffeomorphism invariant observables, which are the spectrum of the Dirac operator. []

• **non-expending horizon (NEH):** A horizon is a compact, spacelike 2-surface (usually a sphere) expanding at the speed of light, however, not changing its are element.

The zeroth and first law of Black Hole thermodynamics still hold

A three dimensional sub-manifold Δ of spacetime is said to be non-expending horizon (NEH) if it satisfies the following conditions:

(i) Δ is topologically $S^2 \times \mathcal{I}$ and null where $\mathcal{I}$ is an interval on the real line;

(ii) The expansion $\theta_{(\ell)} := q^{ab}\nabla_a \ell_b$ of ℓ vanishes on Δ, where ℓ is any null normal to Δ and q_{ab} is the degenerate metric on Δ;

(iii) All equations of motion hold at Δ.

• **non-perturbative:**doe not describe dynamics of linear perturbations around some fixed background spacetime.

compare with background independence: does not distinguish any particular background metric from the outset.

• **non-renormalizability:**

Part of the reason why some people consider non-renormalizability unacceptable as a physical theory may lie in what one considers the meaning of “quantization” to be.

• **null-tetrads:** Newman-Penrose null-tetrads.

$$\ell^a := \frac{1}{\sqrt{2}}(r^a + t^a), \quad n^a := \frac{1}{\sqrt{2}}(r^a + t^a) \tag{A.-1}$$

$$\ell^2 = 0, \quad n^2 = 0, \tag{A.-1}$$

complex null vectors

$$m := \frac{1}{\sqrt{2}}(e_1 + ie_2), \quad \overline{m} := \frac{1}{\sqrt{2}}(e_1 - ie_2) \tag{A.-1}$$

• **normal region:** A normal region

• **observables:** Observables in Hamiltonian mechanics are real functions defined on phase space, $\mathbb{R}^{2n}$. Some examples of observables are position, momentum, energy, angular momentum.

Observables is a property of a physical system that can in principle be measured. In quantum mechanics, an observable is represented by a *self-adjoint operator*.

In quantum mechanics, however, an observable is represented by a self-adjoint operator defined on a dense subspace of a Hilbert space. Work out mean values, mean deviations - the operator to any power should be a finite quantity for it to have a physical meaning.

In the classical GR we compare observations with coordinate-independent quantities. The coordinate time (as well as the spatial coordinates) can in principle be discarded from the formulation of the theory without loss of physical content, because results of real gravitational experiments are always expressed in coordinate-free form.

• **Oeckl, Robert:** In particular, he works on an extension of the standard formulation of quantum mechanics, precisely to make it compatible with general relativistic concepts. This program is called the general boundary formulation of quantum theory.

Figure A.32: Robert Oeckl.

• **off-shell:**

• **one-partial irreducable (1PI):** See Feynmann diagrams.

• **on-shell:** Equations of motion are imposed.

off-shell

• **operator:** An operator is a linear map taking vectors to vectors

$$\mathbf{A} : |\psi > \rightarrow \mathbf{A}|\psi >$$

linear meaning

$$\mathbf{A}(a|\psi > +b|\phi >) = a\mathbf{A}|\psi > +b\mathbf{A}|\phi >$$

• **operator algebra:**

$$[\hat{q}, \hat{p}] = i. \tag{A.-1}$$

Define

$$U = 1 + i\delta q'\hat{p}, \tag{A.-1}$$

where $\delta q'$ is an infinitesimal parameter. To infinitesimal order this is a unitary operator,

$$U^\dagger U = (1 - i\delta q'\hat{p})(1 + i\delta q'\hat{p}) = 1 + \mathcal{O}((\delta q')^2), \tag{A.-1}$$

Hence U preserves the norm of operators it acts on. It is easily seen that U is the operator which performs infinitesimal translations on the coordinate operator $\hat{q}$:

$$UqU^{-1} = q + i[p, q]\delta q' = q + \delta q', \tag{A.-1}$$

therefore

$$< q'|Uq =< q'|UqU^{-1}U =< q'|(q + \delta q')U = (q + \delta q') < q'|U, \tag{A.-1}$$

which implies that

$$< q'|U =< q + \delta q'|. \tag{A.-1}$$

• **operator ordering ambiguity:** Hamiltonian $\mathcal{H}(q, p) = p^2 q$. The classical variables replaced by their operators $p \rightarrow \hat{p} = -i\hbar\frac{\partial}{\partial q}$ and $[\hat{p}, \hat{q}] = i\hbar$. There are two possible hermitian Hamiltonian operators corresponding to the classical Hamiltonian,

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{1}{2}\left(\hat{p}^2\hat{q} + \hat{q}^2\hat{p}\right) \quad \text{and} \quad \hat{\mathcal{H}}' = \hat{p}\hat{q}\hat{p}, \\ \hat{\mathcal{H}} &= \hat{\mathcal{H}}' - i\hbar\hat{p}. \end{aligned}$$

Operator ordering ambiguities is an important issue in LQG as it relates to the size of the physical Hilbert space, which in turn relates to the question of whether LQG has the correct semi-classical limit.

• **operator valued distribution:**

$$\hat{A}^a(x) = \int_{-\infty}^{\infty} dk e^{ix\cdot k} \hat{a}_k \tag{A.-3}$$

A quntumm field $\Phi(x)$ at a point cannot be a proper observable. Physically this appears evident because a measurement at a point would necessitate infinte energy. It may be regarded as a sesquilinear form on a dense domain $\mathcal{D} \subset \mathcal{H}$; this means that the matrix element $< \Psi_2|\Phi(x)|\Psi_1 >$ is a finite number when both Ψ_1 and Ψ_2 are in $\mathcal{D} \subset \mathcal{H}$ characterized by the property that the probability amplitudes for particle configurations decrease fast with increasing momenta and increasing particle number.

and that it depends linearly on Ψ_1, conjugate linearly on Ψ_2. To obtain an operator defined on the vectors in $\mathcal{D}$ one has to average ("smear out") Φ with a smooth function f in Minkowski space i.e. take

$$\Phi(f) = \int \Phi(x) f(x) d^4x.$$

• **Osterwalder-Schrader Reconstrusction:** An integral of the form "$\int \mathcal{D}\phi e^{iS[\phi]/d}$" ocsilates wildly. We instead first consider the integration "$\int \mathcal{D}\phi e^{iS[\phi]/d}$"; If the Lagrangian is positive then the integral will fall off rapidly. If one can define this and it satisfies certain conditions one can go back to the Lorentzian theory with the Osterwalder-Schrader reconstrusction.

, most importantly, requires that the measure $d\mu(\phi)$ be invariant with respect to Euclidean transformations and that

reflection with respect to the hyperplane

$$(x^0 = 0, x^1, \ldots, x^d). \tag{A.-3}$$

The Osterwalder-Schrader Reconstrusction theorem allows one to obtain the Hilbert space of the canonical quantum field theory from the Euclidean one.

The Osterwalder-Schrader reconstruction of Euclidean QFT was generalized to diffeomorphism invariant theories. Replacing the Euclidean group with the diffeomorphism group. [251]

M. Rainer, *Is Loop Quantum Gravity a QFT?*, [gr-qc/9912011].

• **outer marginally trapped surface** See trapped surface.

• **Page-Wooters constrauction:** Proposed by Page-Wooters [47] to deal with an aspect of the 'problem of time". It consists in promoting all variables of the system to quantum operators and choosing one of the variables to be a "clock" and computing conditional probabilities for the other variables to take certain values when the "clock" takes a certain value. This proposal runs into technical difficulties when applied in detail to constrained systems, as emphasized by Kuchar [41]. In constrained systems the physically observable variables are those that have vanishing Poisson brackets with the constraints. If one of the constraints is the Hamilonian then this candidte for a clock does not evolve. Page-Wooters tried circumvent this difficulty by considering "kinematical" variables, i.e. variables that dont commute with the constraints and hence appaer to "evolve". This brings about other problems. Such variables can be promoted to quantum operators acting on the space of wavefunctions that are not necessarily annihilated by the contraints (the kinematic Hilbert space). When the zero eigenvalue of a constraint operator lies in the continuous spectrum (as is the case with GR), the states that are annihilated by the contraints are distributional. Therefore it is difficult to see how they can admit a probablistic interpretation.

• **Palentini first order formulism of GR:**

This condition, which is imposed *a priori* in Einstein's formulation of general relativity, is seen to seen to be part of the equations of motion.

$$\mathcal{D}e_a = \partial_b e_a + \epsilon_{abc} A^b e^c = 0. \tag{A.-3}$$

• **partial Cauchy surface:** A partial Cauchy surface, Σ, for a spacetime $\mathcal{M}$ is a hypersurface which no causal curve intersects more than once. An important example is that of the hypersurface Σ extending from a black hole horizon to spacial infinity.

• **partial observables:** These are measurable quantities not determined by the theory, however, correlations between them are. These correlations constitute the dynamics of the theory. We understand dynamics as the relative evolution of partial variables with respect to one another - in which all these variables are treated on an equal footing.

By choosing one of these variables to be an *independent* ("time") and the rest to be *dependent* variables.

e.g. area as described in chapter 1.

doesn't commute with constraints Kuchar.

see complete observables.

• **partial truths:** cosmology. When it will be in the causal past of the point p, once it is it is remains in the past, there are some points that will never be in the past of p.

• **particle:** Fock states: with no interactions present we can expand $\hat{\psi}(x)$ as a complete set of normal modes

$$\{u_i(x)a_i + v_i(x)b_i^\dagger\} \tag{A.-3}$$

particles are objects that are trigger readings of detector or that produce bubbles in bubble chambers.

• **particle horizon:** Two different regions of the universe that are separated from each other by a distance greater than $2R_H(t)$ at epoch t can not by causally related. This boundary is called the "particle horizon".

• **partition function:** The partition function is

$$Z(\beta,\mu) = \mathrm{Tr} e^{-\beta(\hat{\mathcal{H}}-\mu\hat{N})} \tag{A.-3}$$

where $\hat{\mathcal{H}}$ is the many-body Hamiltonian, $\hat{N}$ the particle number operator, and μ the chemical potential.

$$\mathcal{E} = -\frac{1}{V}\frac{\partial \log Z}{\partial \beta} = \frac{\partial \beta \mathcal{F}}{\partial \beta}, \quad p = \frac{1}{\beta}\frac{\partial \log Z}{\partial V} = -\mathcal{F} \tag{A.-3}$$

• **passive diffeomorphism transformation:** Refers to a change of coordinates, i.e. the same object is represented in a different coordinate system.

• **peeling theorem:** asymptotic expansions.

• **Penrose abstract tensor notation:**

• **Penrose inequality:** The Penrose inequality relates the area of cross-sections of an event horizon A_E on a Cauchy surface with the ADM mass M_{ADM} at infinity:

$$\sqrt{A_E/16\pi} \leq M_{ADM}. \tag{A.-3}$$

The Penrose inequality has been one of major open questions in classical general relativity, closely tied to an even bigger open question, the cosmic censorship conjecture. It was attempts at finding violations of this that led Penrose to formulate this inequality. The derivation relies so heavily on the cosmic censorship conjecture that a violation of this inequality would go a long way towards contradicting it.

• **Penrose-Carter diagram:** Let $\tilde{\mathcal{M}}$ denote physical space-time with metric $\tilde{g}_{ab}$. The idea was to construct another "unphysical" manifold $\mathcal{M}$ with boundary ζ and metric g_{ab}, such that $\tilde{\mathcal{M}}$ is conformal to the interior of $\mathcal{M}$ with $g_{ab} = \Omega^2\tilde{g}_{ab}$, and so that the "infinity" of $\tilde{\mathcal{M}}$ is represented by the finite hypersurface ζ. We realise the whole physical spacetime $\tilde{\mathcal{M}}$ as a subset of the larger spacetime $\mathcal{M}$.

Asymptotic properties of $\tilde{\mathcal{M}}$ and of fields in $\tilde{\mathcal{M}}$ can be investigated by studying ζ, and the local behaviour of the fields at ζ provided the relevant information is conformally invariant.

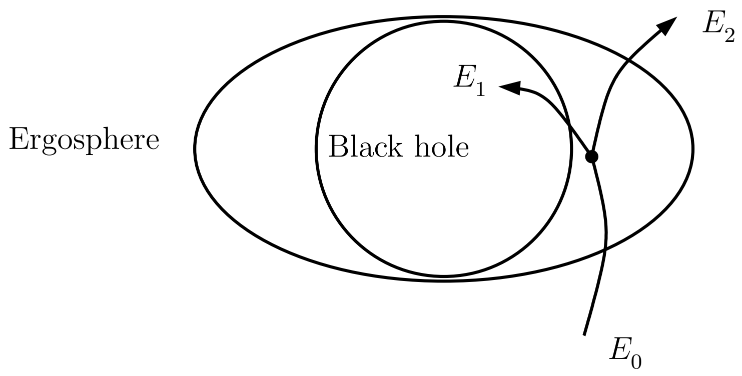

Figure A.33: Penrose process.

• **Penrose process:** extraction of energy from a rotating black hole.

• **Sir Roger Penrose:** modern techniques in GR e.g. use of spinors plus elegant topological methods. Invented twistor theory as a theory of quantum gravity. Ingredients spinor networks which were later found an very important place in LQG by Smolin and Rovelli.

Figure A.34: Sir Roger Penrose.

• **perennial:** Perennials are classical variables that are observables, that is, have vanishing Poisson brackets with all the constraints.

One of the constraints, the Hamiltonian constraint, generates you can not expect any of them to work as a clock??

• **Petrov classification:** number of principle spinors

• **perturbative quantum gravity:**

"However, even if it is accepted that we have finite amplitudes for each fixed topology, we are far from finished. The expressions then have to be summed up. Now there is a problem that the sum apparently actually diverges. The intended finite theory is actually not finite after all! This particular divergence does not seem to worry the string theorists, however because they take this series as an improper realization of the total amplitude. This amplitude is taken to be some analytic quantity, with the power series attempting to find an expression for it by 'expanding around the wrong point', i.e. about some point which is singular for the amplitude ....????? This could be OK, although the divergence encountered here has been shown to be of a rather uncontrollable kind ('not Borel-summable")....... Moreover, if string-theoretic (perturbational) calculations are actually expansions 'about the wrong point', then it is unclear what trust we may place in all these perturbative calculations in any case! Thus we do not yet know whether or not string QFT is actually finite, let alone whether string theory, with all its undoubted attractions, really provides us with a quantum theory of gravity.

The amplitudes they are calculating are not invariant under active diffeomorphisms and hence cannot be an observables of the theory. So can't have any meaning beyond the semiclassical limit. Cannot be applied to extreme situations that we would like to access in quantum gravity.

• **phenomenology of quantum gravity:** Gamma-ray bursts,

• **physical coherent states:** Physical coherent states are needed in order to test the classical limit of gauge-invariant observables.

• **Plebanski action:** Used in a construction of spinfoam models.

• **point holonomies:** Scalar fields e.g. Higg's particle, quantum cosmology transforms as a scalar so not well defined on an edge - instead we consider the exponential at a point

$$\exp\left(i\mu A^{\varphi}(x)\right) \tag{A.-3}$$

• **point-identification:** If one manifold is a replica of another we need a point-identification map which relates points in the two manifolds which are to be regarded as the "same". The correspondence of the points of these two spacetime manifolds is arbitrary and so is a "gauge freedom".

• **Poisson bracket:**

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \tag{A.-3}$$

• **polymer particle representation:** [242] If one drops the requirement of weak continuity we can circumvent the Stone-von Neumann theorem, one finds that there are unitarily inequivalent representation of CCR algebra (Weyl algebra).

The standard Schrödinger representation representation of the Weyl algebra plays the role of the Fock representation of low energy quantum field theories and the new, unitarily inequivalent representation -called the polymer particle representation- in which states are mathematically analogoous to the polymer-like excitations of quantum egometry.

It mimics the loop representation - e.g. it has discrete - momentum not a well defined operator in this representation.

Appears in the mathematics of loop quantum cosmology.

• **Ponzano-Regge:** A partion function for 3d GR.

$$Z_{PR} = \sum_j \prod_f \dim(j_f) \prod_v$$

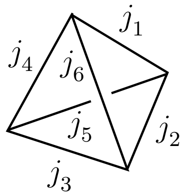

Figure A.35: PonRegg.

$$W(\mathbf{j}) = \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \tag{A.-3}$$

Witten's $ISO(3)$ Chern-Simons theory [E. Witten, Nucl. Phys. **B311**, 46 (1988).]

• **pre-geometry:** More fundamental substratum from which our familiar continuum physics is assumed to emerge as a course grained and secondary model theory. The dynamics of pre-geometry in the appropriate physical regime, give rise to the appearance of a background spacetime with matter fields or particles whose dynamics are played out over it. Examples of pre-geometric theories, spinfoam of LQG,...

• **preons:** Quarks, leptons and heavy vector bosons are suggested to be composed of a stable spin$-1/2$ preons, existing in three flavours, combined according to simple rules.

• **prequantization:** A prequantisation of a sympletic manifold (M, ω) is a pair (L, D) where L is complex Hermitian line bundle over M and D a compatible connection with curvature $(1/\hbar)\omega$.

• **primary constraints:**

• **principal fibre bundle:** Loosely speaking, in the context of gauge theory in spacetime, it is a manifold that locally looks like cartesian product of spacetime manifold and the parameter space of the gauge group. A fibre bundler whose fibre is a Lie group in a certain special way.

Principal bundles are equivalent, if the fibres are transform $g \to hg$. This corresponds to a gauge transformation for the connection 1-form, $A_\mu(x)$. The curvature of this bundle connection then turns out to be the Maxwell field tensor F_{ab}.

• **principle of complementarity:** The predictions of general relativity will coincide with the Newtonian gravity when there particles have small mass and speeds are small compared to the speed of light c.

• **problem of time:** The problem of time in quantum gravity is a bit tricky to describe, since it takes different guises in different approaches to quantum gravity. (beaz week 41)

In a time reparameterization invariant system of General Relativity there is no time evolution - "nothing happens" in the theory in obvious contradiction to what we observe.

Newton's absolute time is an apparently indispensable notion in standard quantum mechanics. But GR teaches us that there is no external time - no true time by which mechanics evolves. There is only relative evolution of variables. Time must be promoted to the same status with other variables.

In the first chapter we saw that reference bodies are necessarily part of this dynamical system under study - so system is closed and all dynamical variables behaves quantum mechanically - all variables at the same level. In the context of closed systems where everything behaves quantum mechanically. also in quantum cosmology close to the big bang.

• **projected spin network:**

• **proper length:** a proper length of an object is its length as measured in the rest system of the object????????

• **proper time:**The label of "the problem of time" is often given to a number of related, but slightly different isues.

Briefly put, the problem of time is as follows: how is one to apply quantum mechanics to general relativity in which a classical non-dynamical background time is missing?

the proper time of an object is the time measured by a clock located in the rest system of the object.

• **Pullin, Jorge**

• **qef representation (quantized electric Flux representation):** Analogue of spin network states for background independent Maxwell theory. Electric field flux is quantized. The analogous result in $SU(2)$ LQG theory , with its geometric interpretation, corresponds to quantized area.

• **quadrupole moment:** Einstein already in 1917 worked out for quadrupole

Figure A.36: Jorge Pullin.

$$Q_{ij} = \int d^3x' \rho(\mathbf{x}')(3x'_i x^{j'} - r'^2 \delta_{ij})$$

See moments.

• **quantization:** [arXiv: quant-ph/ 0412015]. Quantisation is the problem of deriving the mathematical framework of a quantum mechanical system from the mathematical framework of the corresponding classical mechanical system. A method of quantisation must contain a map $\mathcal{A}$ from the set of classical observables to the set of quantum observables with the following properties:

The theories exist in their own right and perturbation methods serve as approximation techniques to extract answer to "physically interesting questions."? hepth9408108

• **quantum causal histories:** [368]

???Sorkin. Postulating that the fundamental underlying structure is a discrete causal. Attempts to recover the continuum picture by suitable coarse graining.???

• **quantum configuration space:** In quantum mechanics systems with a finite number of degrees of freedom, states are represented by functions on the classical configuration space. However, in field theories quantum states are on a larger space - the *quantum configuration space.*

For systems with a finite number of degrees of freedom, the classical configuration space also serves as the space of independent variables of the wavefunction of the quantum theory; we say that the configuration space is then the quantum configuration space.

For systems with an infinite number of degrees of freedom, on the other hand, the quantum configuration space is an enlargement of the classical configuration space. The free field theory in Minkowski spacetime while the classical configuration space can be built from suitably continuous fields, the quantum configuration space includes all tempered distributions.

$$\int [\mathcal{D}\varphi] e^{iS} \Phi(x)\Phi(x) \tag{A.-3}$$

Typical measures used in quantum field theories are. The classical configuration spaces are contained in sets of measure zero. As a result, the action S inside the functional integral is ill-defined

In the Hilbert space language, this is the origin of field theoretic infinities, e.g., the reason why we can not naively multiply field operators.

The presence of an infinite number of degrees of freedom causes only one major modification: the classical configuration space $\mathcal{C}$ of smooth fields is enlarged to the quantum configuration space $\mathcal{S}'$ of (tempered) distributions. Quantum theoretical difficulties associated with defining products of operators can be directly traced back to this enlargement [34].

This enlargement from $\mathcal{A}$ to $\overline{\mathcal{A}}$ which occurs in the passage to the quantum theory is very similar to the enlargement from $\mathcal{C}$ to $\mathcal{S}'$ in the case of scalar fields. This enlargement plays a key role in the quantum theory (especially in the discussion of surface states of the quantum horizon).

• **quantum cosmology:** A theory that attempts to describe the whole universe in quantum-mechanical terms.

• **quantum field theory: Background *dependent* quantum field theory:** In spite its many practical successes where predictions from calculations have been shown to agree with experiment with phenomenal accuracy, QFT is mathematically a problematic construction.

very naturally defined, that we are however unable to calculate without incurring infinities.

can be reformulated in terms of classical statistical field theory.

Background *independent* quantum field theory:

• **quantum field theory on curved spacetime:** There is often confusion between a dynamical theory *on* a given curved spacetime with the dynamic theory *of* the spacetime, which is what GR is really about.

From [305]:

“Approximation methods for gauge invariant observables provide an explicit way to construct observables and may also help to test (quantum) interpretation of these as well as to discuss phenomenological implications [8, 9]. The approximation scheme developed in this work shows explictly how to express local observables in a gauge invariant way and therefore indicates how one could understand quantum field theory on curved space time as an approximation to the full theory of quantum gravity.”

• **quantum gauge fixing:** An approximation scheme

all the degrees of freedom are fluctiating but the unphysical ones’ fluctuations are suppressed by the semiclassical state.

• **quantum mechanics:**

• **quantum logic reconstruction:** Quantum logic reconstruction attempts

From the single premise that the “experimental propositions” associated with a physical system are encoded by projections in the way indicated above, one can reconstruct the rest of the formal apparatus of quantum mechanics. The first step, of course, is Gleason’s theorem, which tells us that probability measures on $L(H)$ correspond to density operators.

Taken directly from http://plato.stanford.edu/entries/qt−quantlog/ (1) implication ($Q_1 \Rightarrow Q_2$),

(2) union ($Q_3 = Q_1 \vee Q_2$), and

(3) intersection ($Q_3 = Q_1 \wedge Q_2$).

(1) always false (Q_0) and always true (Q_∞)

(2) negation of question ($?Q$)

(3) orthogonality: if $Q_1 \Rightarrow ?Q_2$ then Q_1 and Q_2 are orthogonal denoted ($Q_1 \perp Q_2$).

• **quantum tetrahedron:** A quantum tetrahedron is a single tetrahedron whose geometric variables are considered as dynamical and represented by operators on a quatum state space $\mathcal{H}$.

• **quintessentian dark matter:**

“Reinterpreting quintessential dark energy through averaged inhomogeneous cosmologies”, [astro-ph/0609315].

“...Einstein General Relativity: the cosmic quintessence emerges in the process of interpreting the real Universe in a homogeneous context.”

“...It consists in defining cosmologies that are homogeneous on large scales without supposing any local symmetry, thanks to a spatial averaging procedure. It results in equations for a volume scale factor that not only include an averaged matter source term, but also additional terms that can be interpreted as the effects of the coarse-grained inhomogeneities on the large scales dynamics. These additional terms are commonly named backreaction."

• **RAQ:** Refined algebraic quantization.

• **radar time:** Popularized by Bondi, *radar time* is used to define hypersurfaces of simultaneity in terms of physical clocks and light pulses, as such it should be independent of the choice of coordinates.

Consider an observer travelling on path $\gamma : x^a = x^a_{(\gamma)}(\tau)$ with proper time τ. Define

$\tau^+(x)$:= (earliest possible) proper time at which light ray (null geodesic) leaving the point x could intercept γ.

$\tau^-(x)$:= (latest possible) proper time at which a light ray could leave γ, and still reach the point x.

$\tau(x) := \frac{1}{2}(\tau^+(x) + \tau^-(x))$ is the *radar time.*

$\tau(x) := \frac{1}{2}(\tau^+(x) - \tau^-(x))$ is the *radar distance.*

$\Sigma_{\tau_0} := \{x : \tau(x) = \tau_0\}$ is the observer's *hypersurface of simultaneity at time* τ_o.

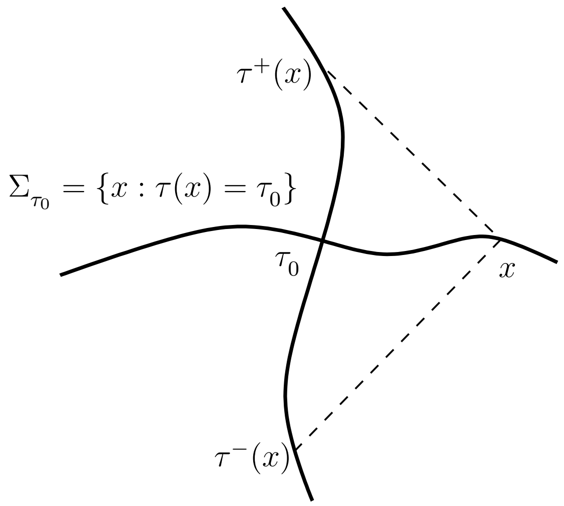

Figure A.37: radartimeF. Schematic of the definition of radar time $\tau(x)$.

• **Raychauduri equation:** Equation that governs the expansion of an infinitesimal bundle of geodesics. There are separate versions for time-like and null geodesics.

Play a central role in the proof of the singularity theorems.

• **real formalism:** With the introduction of a generalized Wick transform to map the constraint equations of Riemannian general relativity to those of the Lorentzian theory.

• **recombination:** This is the moment, in the history of the universe, when free electrons recombine with protons to form hydrogen atoms. Matter ceases to be ionized, and decouples from electromagnetic radiation. This defines the period of decoupling, a million years or so after the Big Bang.

• **reconstruction problem:** how do we extract local geometry and dynamics from the global nature of observables. Observables are to be invariant under active diffeomorphisms. local from non-local observables.

• **reconstruction of quantum mechanics:** See quantum logic reconstruction.

• **reduced phase space:**

• **reduced phase space quantization:** Solve the constraints at the classical level. Quantize without any constraints. Given two 3-metrics one would need to know if one is a time-evolved version of the other, this requires solving for the dynamics. A reduced phase space quantization of gravity would require finding the general solution of the classical field equations.

In the reduced phase space quantization one first performs a gauge fixing and then quantizes. That is one has to represent teh algebra of gasuge fixed functions equipped with teh Dirac bracket as (self-adjoint) operators on a Hilbert space.

"In her recent work, Dittrich generalized Rovelli's idea of partial observables to construct Dirac observables for constrained systems to the general case of an arbitrary first class constraint algebra with structure functions rather than structure constants. Here we use this framework and propose how to implement explicitly a reduced phase space quantization of a given system, at least in principle, without the need to compute the gauge equivalence classes."

[302] based on the framework constructed in [301]

the ultimate Hilbert space must carry a representation of the Poisson algebra of Dirac observables.

• **refined algebraic quantization (RAQ):**

RAQ, as it so far formulated, cannot rigorously deal with constraints which involve nontrivial structure functions since the group averaging really requires an honest (Lie) group structure.

RAQ has actually two implementations: A heuristic version, called group averaging, and a rigorous version, using Rigged Hilbert Spaces.

• **reflection positivity:** Let θ denote the action of the time reflection, i.e. reflection with respect to the hyperplane

$$(x^0 = 0, x^1, \ldots, x^d), \tag{A.-3}$$

then θg is the time reflection of the function g. Consider functions of the form

$$f(\phi) = \sum_{j=1}^{N} c_j e^{i\phi(e_j^+)}, \tag{A.-3}$$

where c_j is a complex number and e_j^+ are arbitrary Schartz functions that are non-zero in the $x^0 > 0$ half-space. Then reflection positivity of the measure $\mu(\phi)$, is the condition that

$$< \theta f, f >_{L_2} = \int_{S'} (\theta f)(\phi)]^\dagger f(\phi) d\mu(\phi) \geq 0. \tag{A.-3}$$

It is the axiom of Osterwalder-Schrader theorem that allows one to construct a physical Hilbert space of the canonical quantum field theory from the Euclidean one with a non-negative Hamiltonian acting on it.

• **Regge manifold:** A d-dimensional Regge manifold is a metric space composed by glued flat d-simplices, with matching geometry at the intersections.

• **Regge model:** One approximates a curved spacetime by a large collection of simplices glued together. The interior of every simplex is flat, the spacetime curvature resides in how the simplices are glued to each other.

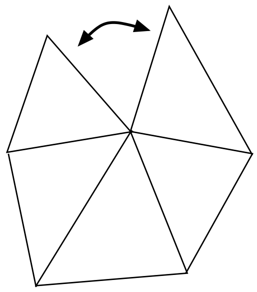

Figure A.38: ReggeGlos. Gluing flat simplices to get a surface with curvature.

Regge Lagrangian:

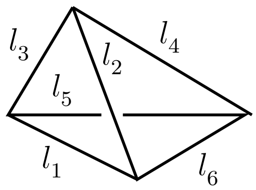

Figure A.39: Regge Lagrangian.

denote the lengths of its edges by l_i $(I = 1, 2, \ldots, 6)$ (Fig N.-19). The Regge action for the tetrahedron is

$$S_{Regge} = \sum_{I=1} l_i \theta_i, \tag{A.-3}$$

where θ_i is the angle between the outward normals of two faces sharing the I-th edge.

• **reheating:** Process whereby the period of inflationary expansion gives way to the standard hot big bang scenerio.

• **relational notion of space:** Aristotle to Decartes picture of space where described by the conjunuity of objects: there is no notion of space without matter.

• **relational interpretation of quantum mechanics:** Rovelli's relational quantum mechanics. resolves the paradox of superpositions by making it a consequence of one' point of view.

It does not try to append a reasonable interpretation to the quantum mechanics formalism, nor to postulate as yet unknown new physics, rather it attempts to uncover the physical meaning quantum mechanics. One drops the assumption that quantum states are observer independent.

In the relational interpretation, the very notion of state of a physical system should be considered meaningless unless it is understood to be relative to another system that plays the role of one possible observer. No implication that the observer is human or has any other a priori special properties only that it has the possibility of interacting with the 'observered ' system.

It is the specification of such an observer that allows the ascription of a state to a system to make sense. So the description the theory provides is complete because it exhausts everything that can be said about the world.

A physical quantity has value with respect to certain observer.

The cat finds himself in one or other of the possible outcomes while at the very same time according to the person outside the box the cat is in a quantum superposition. People say the two can't be true at the same time so we have a paradox. However this discord does not impinge on you because when to different observers can ask the same question they get the same answer.

That the photon has been absorbed (or not) according to the cat isn't to say that it has been (or has not been) absorbed as far as anyone else is concerned - it is not observer independent, that is it is not an absolute property of the system. In this interpretation there is no need for special systems that collapse the wavefunction for everyone else's benefit.

There is a long history in physics of progress being made by abandoning the belief that there are preferred systems and by 'relationalizing' notions that were previously thought to be absolute. The description provided by the theory exhausts everything that can be said about the world.

What is stopping us from taking this as a description of reality is a philosophical prejudice and nothing more. We should adjust our philosophical prejudices to what nature is telling us. We shouldn't be trying to fit reality into our preconceived notions about the world.

http://plato.stanford.edu/entries/qm-relational/

consistent discretizations

context created by the presence of an observer

different descriptions as there are as there are contexts, but with interrelations between them so that if two observers can ask the same question they get the same answer.

• **Relational quantization *a la* Page-Wootters:** [47]

• **relational quantum cosmology:**

[29] Slogan: "Many quantum states to describe one universe, not one state describing many universes".

• **relativistic spin networks:**

• **renormalization:**

renormalized mass and charge values are inserted into the theory as unexplained parameters. Coupling constants of various kinds and masses of the basic quarks and leptons, the Higgs particle, etc. need to be specified.

Any acceleration of a charged particle entails the acceleration of part of the charged particles surrounding it, so the mass m that you observe is the mass m_0 of the particle plus a correction Δm, due to its interaction with the cloud of charged particles. Procedure of replacing infinities with experimentally determined values. a quantum field theory whose Lagrangian has a few free parameters - masses and charges and so. These quantities are refered to as the "coupling constants".

with obvious physical meaning cannot be calculated without incurring divergencies.

Renormalization occurs in evaluating physical observable quantities which in simple terms can be written as formal functional integrals of the form

$$\int [\mathcal{D}\phi] e^{\mathcal{L}(\phi,\partial\phi)} A(\phi, \partial\phi) \tag{A.-3}$$

where $\mathcal{L}$ is composed by two parts, a part quadratic in ϕ and $\partial\phi$ term, $\mathcal{L}_0$, and an 'interaction' term, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$. Computing such an integral in perturbative terms leads to a formal power series, each term G of which is obtained by integrating a polynomial under a Gaussian $e^{\mathcal{L}_0}$:

$$\int [\mathcal{D}\phi] e^{\mathcal{L}_0(\phi,\partial\phi)} (1 + g\mathcal{L}_{int}(\phi,\partial\phi) + \frac{g^2}{2!}[\mathcal{L}(\phi,\partial\phi)]^2 + \dots) A(\phi,\partial\phi) \tag{A.-3}$$

Such Feynmann diagrams G are given by multiple integrals over spacetime or, upon Fourier transformation, over momentum space, and are typically divergent in either case. The renormalizatoin technique consists in adding counterterms $\mathcal{L}_G$ to the original Lagrangian $\mathcal{L}$, one for each diagram G, whose role is to cancel the undesired divergencies.

In the case of a renormalizable theory, all the necessary counterterms can be obtained by modifying the numerical parameters of the original Lagrangian.

considered to be ad-hoc

Kreimer there is a Hopf algebra underlies quantum field theory renormalization, and has led to a new approach to perturbative renormalization under development.

The antipode of a diagram is related to the counterterms necessary to subtract its divergencies.

renormalization - background independent: The regularized Hamiltonian operator is independent of the regularization parameter ϵ to first order - the regularization parameter has disappeared from the expresion! Hence no renormailization is involved! The reason why the regularization parameter does not appear is, roughly speaking, because the shrinking process, $\epsilon \to 0$, can be compensated for by an active spacial diffeomorphism.

Tentative ideas have been formulated by Markopoulou [122], [123] and Oeckl [124].

renormalization *a la* Fontini-Markopoulou [122], [123]

$\gamma_1 \subset \gamma_2$ $\gamma_2 \subset \gamma_1$ $\gamma_1 \cap \gamma_2 = \emptyset$

let γ denote a proper sublattice of Γ, namely $\gamma \neq e$ and $\gamma \neq \Gamma$. We call the lattice that is left after we "cut out" γ the **remainder** and denote it Γ/γ.

$$\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p \tag{A.-3}$$

These are the **primitive elements** of the Hofp algebra.

The **counit** is an operation which annihilates every lattice except e.

renormalization *a la* R. Oeckl [124]

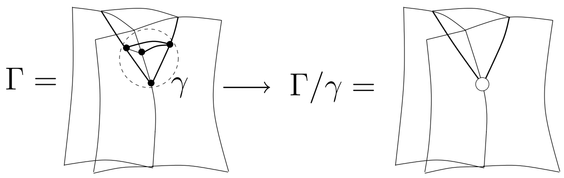

Figure A.40:

• **renormalization group:** A method designed to describe how the dynamics of a system change as we change the scale at which we probe it.

• **renormalized charge:** bare charges divergent, absorbtion of infinites into a redefinition of the charges. screening effects (see effective charges).

• **Ricci identity:**

$$2\nabla_{[a}\nabla_{b]}V_c = R_{abcd}V^d \tag{A.-3}$$

• **rigged Hilbert space - also called the Gel'fand triple:** In infinite dimensional Hilbert space there can be operators that have a continuous spectrum, (the most simple example would be that of a free electron which has a continuous momentum spectrum). The corresponding eigenvectors are not normalizable, and hence cannot lie inside the Hilbert space $\mathcal{H}$, (the eigenvector of the momentum operator of a free electron is a plane wave e^{ikx}, which isn't normalizable). The rigged Hilbert space approach to dealing with such operators on or a Gel'fand triple $\{\Phi, \mathcal{H}, \Phi'\}$, in which the Hilbert space $\mathcal{H}$ appears as a linear subspace Φ' which is not itself a Hilbert space. Eigenvectors associated with a continuous spectrum belong to Φ' but are no members of its subspace $\mathcal{H}$. The space Φ' is constructed as a type of dual of another vector space Φ which is a subspace of $\mathcal{H}$. Thus we have $\Phi \subset \mathcal{H} \subset \Phi'$.

• **Riemannian geometry:** A generalized geometry that has the property of being locally flat; that is, in a sufficiently small region, a Riemann geometry can be approximated by a Euclidean or Minkowski geometry.

• **Robertson-Walker metric:** The metric that describes an isotropic and homogeneous cosmological space time.

• **Rovelli, Carlo:**

• **Sandwich Conjecture:** The conjecture that, given two spatial metrics q and q' on two hypersurfaces then there is unique metric that will interpolate between them, up to gauge.

• **scale factor:** The quantity that describes how length change in the expanding (or contracting) universe.

Figure A.41: Carlo Rovelli.

• **scattering amplitudes:**

There is no given *a priori* geometry here, as is the case in background dependent theories. The geometry is encoded in the quantum state itself, not given a priori as in conventional QFTs.

$$\int \mathcal{D}\varphi_I \int \mathcal{D}\varphi \qquad \text{(A.-3)}$$

how can any physics possibly come out of the setup here if the amplitude doesn't change when deforming the boundary surface - that is, until you realize that the metric is one of the boundary fields!

$$< \Psi_1 | \Psi_2 >= \int \mathcal{D}\varphi_m \Psi_1^*[\varphi_m] \Psi_2[\varphi_m], \quad |\Psi_1 >, |\Psi_2 > \in \mathcal{H}_{\Sigma_m}. \qquad \text{(A.-3)}$$

The physical information is contained in the spectrum of the boundary operators of partial observables.

• **scattering matrix:** The scattering matrix allows the to calculate experimentally observable scattering cross sections.

• **Schödinger equation:** The equation that describes the evolution of a nonrelativistic wavefunction.

• **Schödinger picture:**

• **Schwarzschild radius:** The radius of the event horizon of a nonrotating black hole of mass M, equal to $2GM/c^2$.

• **secondary constraint:** Dirac bracket substituting for the Poison bracket

• **second-quantization:** The passage from a one particle system with an arbitrary number of particles subject to the Bose/Fermi statistics is called second-quantization.

$\hat{\mathbf{\Psi}}(\mathbf{x})$ does not represent the actual quantum state it is the operator that creates a new particle having the one-particle wavefunction $\psi(\mathbf{x})$, by application on the ground state $|0>$ (the state representing the absence of any particles).

• **self organized criticality:** Self organized critical systems are statistical systems that naturally evolve without fine tuning to critical states in which correlation functions are scale invariant. The earliest example of such a system is the sandpile of Bak, Tang and Wiesenfeld [259]. Since then, many such systems have been studied, including models of phenomena as diverse as biological evolution, earthquakes, astrophysical phenomena and economics [260].

It has been suggested that the emergence of classical spacetime from a discrete microscopic dynamics may be a self-organized critical process, [258].

• **semi-classical quantum gravity:** Semi-classical quantum gravity refers to the setting in which a classical background geometry is coupled to the quantum matter field through the semiclassical Einstein equations:

$$G_{\mu\nu} = -8\pi < T\mu\nu > .$$

It is a perturbative approach where there is a well-defined classical background geometry about which the quantum fluctuations are occuring.

• **separating:** Functions of phase space $\mathbb{R}^{2n}$ - observables. the energy and momentum functions on phase space is sufficient know to find out a which particular point in phase space the system is at.

these functionals form a separating set on $\mathcal{A}/\mathcal{G}$: if all the T_α assume the same values at two connections, they are necessarily gauge related.

• **shadow states:** It would help in assesing the viability of LQG providing the correct semi-classical limit if we closed the gap between its one dimension excitation description with the usual Fock space description, in the low energy limit cutoff.

The problem with the polymer represenation is that, Cly^*, the space containing the solutions to the quantum constraint equations, does not have a physically justified inner product; a definite Hilbert space structure is not yet available. While the kinematic states belong to Cyl, the physical states belong to Cly^*. Therefore, semi-classical states, capturing the low energy physics, should also be in Cly^*. Can we still test a candidate state in Cyl^* for semi-classicality at this stage without acces to expectation values? One uses the notion of *shadow states* to do semiclassical analysis in the polymer represenation.

Shadow states use graphs as 'probes' to extract physical information from elements $(\Psi|$ of Cly^*. More precisely, one 'projects' each $(\Psi|$ to an element $|\Psi_\gamma^{shad}>$ in Cyl_γ, a so-called shadow state, and analyses properties of $(\Psi|$ in terms of its shadows $|\Psi_\gamma^{shad}>$. Each shadow captures only a part of the information contained in our state, but the collection of shadows can be used to determine the properties of the full state in Cly^*.

• **simplicial spin-foam:** Standard spin-foam models as formulated up to now are piecewise flat geometries defined on piecewise linear manifolds. For LQG canonical theory to match these models it would have to be restricted to the category of piecewise linear manifolds and piecewise linear spin-networks. A suitable generalisation of simplicial spin-foam models to lift this restriction is linear 2-cell spin-foam models.

• **simplicity constraints:** Imposing the simplicity constraints turns the topological BF theory into (non-topological) 4-dimensional gravity.

• **simple regions:** A simple region of spacetime $\mathcal{M}$ is a geodesically convex set with compact closure, whose boundary is diffeomorphic to the three shpere S^3.

• **singularity:** the inevitable presence of singularities in general relativity of the classical gravitational being always attractive.

A singularity is a point in a spacetime which the curvature becomes infinite. Robrtson-Walker, Schwarschild, Reissner-Nordstrom and Kerr are examples of solutions contain points of inifinite curvature. The presence of infinite curvature lets itself know by timelike or null geodesics coming to a full stop. In the context of the singularity theorems, the working definition of a singularity is a kind of incompleteness of the space-time under consideration, more precisely, it is an obstruction of some sort to timelike or null geodesics from being indefinitely extendible. Investigate the nature of singularities.

We require a smooth Lorentz metric. You cannot have point on the manifold with singular metric. So the points where this can happen are excised.

• **singularity theorems:** Proved by Hawking and Penrose. The singularity theorems are based on very powerful indirect arguments which show that black hole and cosmological singularities are generic in classical general relativity. Gives us confidence in the big bang. The classical theory predicts its own breakdown. One of the main motivations for attempting to quantize general relativity.

A feature of the singularity theorems is that they do not directly show the existence of black holes. Instead they show that spacetime is geodesically incomplete so that the worldline of an observer comes to an end and cannot be extended. The obstruction to extending the worldline is some kind of singularity, but this might be rather mild and need not correspond to a black hole.

If there is generically in spacetime a closed, trapped region, energy certain positivity and if gravity remains attractive a singularity is inevitable. the conclusion is that there is to be some some sort of obstruction to timelike or null geodesics from being indefinitely extendible.

• **slow-roll approximation:**

from gr-qc/0511007:

The mechanism, proposed in [], considers so called slow-roll approximation, when the inflation rolls down its potential hill towards the potential minimum, but, at the same time, remaining far away from the minimum. It is exactly in the vicinity of the minimum of the potential, where the inflation stops.

• **Smarr's formula:** Smarr's formula translates black hole mass M to its angular momentum J, angular velocity Ω_H, surface gravity κ, and surface area A:

$$M = 2\Omega_H J + \frac{\kappa A}{4\pi}. \tag{A.-3}$$

• **Smolin, Lee:**

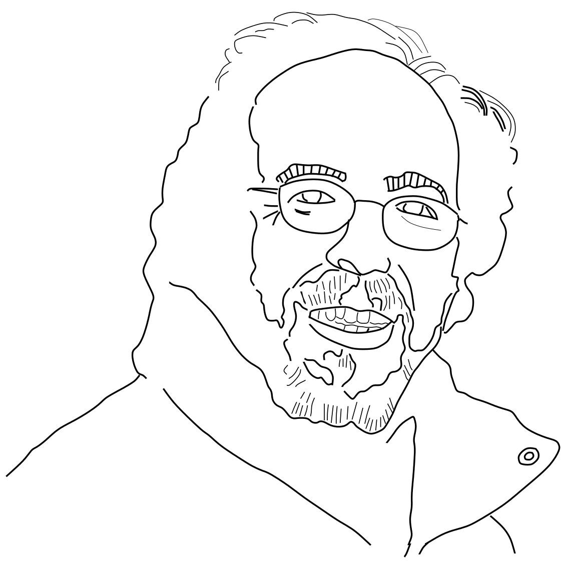

Figure A.42: Lee Smolin.

• **Smolin's Darwinian hypothesis:** Smolin put forward the proposal that the universe fine tunes the values of its physical constants through a Darwinian selection process [184]. Every time a black hole forms, a new universe is developed inside it that has different values for its physical constants from the one in its progeitor.

[125] not just based on *"authors intuition"*

• **space of solutions:** The space of solutions are taken to be the phase space or space of states. This definition is independent of any special time choice so that it is manifestly covariant.

• **spacial averaging problem:**

• **spacial diffeomorphism invariance:**

• **spacetime singularity** The working definition of a spacetime singularity is a point in spacetime beyond which null or timelike geodesics cannot be extended. We mean the singularity

• **special relativity:**

(i) **The principle of relativity**. The laws of physics are the same for all inertial reference frames.

(ii) **The constancy of the speed of light.** The speed of light in a vacuum is the same for all inertial observers irrespective of the motion of the source.

• **Speziale, Simone:** Work on spin foams and the graviton propagator in loop quantum gravity.

Figure A.43: Simone Speziale.

• **spin foam:** Spin foam models are an attempt to describe the geometry of spacetime at the quantum level. They give a construction of transition amplitudes between initial and final spacial geometries labelled by spin networks.

???the spin foam is independent of any special time choice so that it is manifestly covariant???

Spin foams can be understood to be dual to triangularizations.

In this sense the physical state is associated with an entire history of the system, rather than any point on that history.

• **spin network:** A spin network is a generalization of a knot or link. It is a graph embedded in space, with edges carrying labells corresponding to an irreducible representation of a Lie group, and vertices with labells corresponinding to interwining operators.

• **spin network state:** They diagonalize geometric operators and describes quantized three geometry. They provide a complete orfonormal basis for the kinematic Hilbert space. Using spin networks we are working with finite dimensional spaces that have the structure of Hilbert space of spin systems - the same mathematics at work as in ordinary quantum mechanics of angular momentum but with completely different interpretation. They are natural candidates for cylindrical functions in the construction of well defined measures of integration. The scalar product has a very simple form $< \Phi_\alpha \Phi_\beta >= \delta_{\alpha\beta}$. This make easy at the time of calculations.

Spin networks states provide a natural decomposition of gauge-independent kinematic Hilbert space, $\tilde{\mathcal{H}}^0$, into finite dimensional subspaces each of which can be identified with the space of states of a spin-system. This simplifies various constructions and calculations enormously.

Using spin networks we are working with finite dimensional spaces that have the structure of Hilbert spaces of spin systems.

• **spinor:** It is well known that the Pauli matrices satisfy the commutation relations of the rotation algebra,

$$[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k \tag{A.-3}$$

They therefore are a representation of the rotation algebra; they generate infinitesimal rotations on a two component vector - this is the 2-component spinor. Finite rotations are obtained by the exponentiating of (N.-19),

$$\begin{aligned} \hat{U}_R &= \exp(-i\phi \cdot \tau) = \exp(-i\frac{1}{2}\phi \mathbf{n} \cdot \tau) \\ &= \mathbf{I}\cos(\frac{1}{2}\phi) - i\mathbf{n} \cdot \tau \sin(\frac{1}{2}\phi). \end{aligned} \tag{A.-3}$$

The defining property of a spinor is the way it transforms under rotations (or Lorentz transformations depending on the context).

• **Standard model:** particle physics. $SU(3) \times SU(2) \times U(1)_Y$. The Standard Model is not a mathematically consistent theory. It has had undeniable success perturbative calculations of collision cross section, the numerical analysis of particle spectra etc.

• **State:**

There is a second use of this term. A collection 'observables' and is capable of being in certain 'states'. We can define the state of a system as knowledge of the expected values of the observables, that is, a state is an assignment of an expected value of each observable.

A second use of this term is *an expectation value functional.* This is to be chosen at some point in the quantization.

$$F_{\mathrm{Sch}}(W(\zeta)) := e^{-\frac{1}{2}|\zeta|^2}. \tag{A.-3}$$

• **state sum model:** [412]

• **Stone-von Neumann uniqueness theorem:**

$$\begin{aligned} \{q,p\} &= 1, \\ \{q,q\} = \{p,p\} &= 0. \end{aligned} \tag{A.-3}$$

$$[q,p] = i\hbar,\ [q,q] = [p,p] = 0. \tag{A.-3}$$

Now we find concrete repesentation of this algebraic relation. Scrödinger's wavefunction representation

$$\hat{q}\psi(q) = q\psi(q), \quad \hat{p}\psi(q) = -i\hbar\frac{d\psi(q)}{dq} \tag{A.-3}$$

$$[\hat{q},\hat{p}]\psi(q) = (-i\hbar q\frac{d}{dq} + i\hbar\frac{d}{dq}q)\psi(q) = i\hbar\psi(q) \tag{A.-3}$$

Heinsenberg's matrix mechanics every observable physical quantity, that is energy, position, momentum, angular momentum, is described by an operator represented as a matrix.

Scrödinger's wavefunction representation and Heinsenberg's matrix mechanics describe the same physics. Are essentially unitary equivalent.

In quantum field theory there is no analog of the Stone-von Neumann uniqueness theorem. Hence, there are an infinite number of inequivalent representations of the Poisson bracket algebra. These inequivalent representations can be thought of as different "phases", which have different physics.

The Fock representation the natural generalization of the Schrödinger representation to infinitely many degrees of freedom. Quantum field theories written in the Fock representation by theoretical physicists.

• **string theory:** all known particles and their interactions are supposed to emerge as certain modes of excitation of and interactions between quantized strings.

String theory attempts to produce a theory of everything, including quantum gravity that will have general relativity as part of the classical limit.

a little "far-out". String theory is a perturbative approach depending on the choice of background metric. background independent M-theory for strings , Witten using twister space methods in which physics happens in twister space and spacetime is a secondary construction.string field theory.

We find that the massless closed quantum states include one-particle graviton states, making string theory a quantum theory of gravity!

Except string theory can only be defined over stationary spacetimes, which have measure zero in the space of solutions to Einstein's equations - a timelike Killing vector field is needed to have spacetime suppersymmetry, with out it you get unphysical tachions in the spectrum. dynamical theory over spacetime confused with a dynamical theory of spacetime.

A necessary condition for a perturbative theory to be consistent is that the two dimensional world sheet quantum field theory that defines the theory be conformally invariant. This means that the conformal anomaly on the two dimensional worldsheet vanishes. To leading order in l_{string} this condition is equivalent to the Einstein equations of the background manifold.

The so called Pohlmayer string can be quantized in any number of dimensions, including four dimensional Minkowski space [**?**].

Background dependency: (arXiv:hep-th/9808192)

in which the definitions of the states, operators and inner product of the theory require the specification of a classical metric geometry. The quantum theory then describes quanta moving on this background. The theory may allow the description of quanta fluctuating around a large class of backgrounds, but nevertheless, some classical background must be specified before any physical situation can be described or any calculation can be done. All weak coupling perturbative approaches are background dependent, as are a number of non-perturbative developments. In particular, up to this point, all successful formulations of string theory are background dependent.

String theory only studies semiclassical effects in fixed classical spacetime backgrounds.

• **strongly causal:** There are several types of causality violations that can be are considered, other than the existence of closed timelike curves. Strong causality is an important example. Roughtly speaking, strong causality is said to fail if an arbitrarily small perturbation of the metric will result closed timelike curves, that is, the chronology condition is almost violated. Strong causality plays a key role in the Hawking-Penrose singulairt theorem. Important there is the result: Let $\mathcal{M}$ be a spacetime satisfying: (1) There are no closed trips. (2) Every endless null geodesic in $\mathcal{M}$ contains a pair of conjugate points. Then $\mathcal{M}$ is strongly causal.

see future-distinguishing spacetimes, stablely causal

• **supergravity:**

Bosonic and fermionic Feynman diagrams cancel each other out.

• **superselection:** A representation is irreducible if the Hilbert space does not split into orthogonal subspaces preserved by the action of $\mathcal{A}$. Equivalently, there must exist in the Hilbert space a cyclic vector whose image under the action of $\mathcal{A}$ is dense in the Hilbert space. Physically this requirement rules out the existence of superselection rules: superselection sectors can be identified with the irreducible sectors of a reducible representation.

The representation should be irreducible on physical grounds (otherwise we have superselection sectors implying that the physically relevant information is already contained in a closed subspace).

• **superspace:** the collection of all possible three-metrics (nothing to do with supersymmetry)

• **supersymmetry:**

Supersymmetry requires fermions and bosons to come in pairs consisting of one of each, with the same mass. This is not observed in nature.

• **surface gravity: Surface gravity** is the acceleration of a static particle near the horizon as a measured at spacial infinity.

$$\kappa = -\frac{1}{2}(\nabla^a \xi^b)(\nabla_a \xi_b)|_{\mathcal{N}} \tag{A.-3}$$

• **symmetry:** In quantum mechanics group transformations that leaves invariant key structures in the theory - Hilbert space, algebra of observables, etc.

An unitary operator on Hilbert space preserves all scalar products, i.e.,

$$< \hat{U}\psi, \hat{U}\phi >=< \psi, \phi > \tag{A.-3}$$

• **symplectic form:** $\omega_{\mu\nu}$ is antisymmetric and also has the properties

$$\omega^{\mu\nu} = \{\xi^\mu, \xi^\nu\} \tag{A.-3}$$

(i) It is antisymmetric $\omega_{\mu\nu} = -\omega_{\nu\mu}$,

(ii) is non-degenerate, i.e., $\omega_{\mu\nu}v^\nu = 0$ implies $v^\mu = 0$, that means that the inverse matrix $\omega^{\mu\nu}$ exists,

(iii) it is closed , i.e., $\partial_\mu\omega_{\nu\gamma} + \partial_\nu\omega_{\gamma\mu} + \partial_\gamma\omega_{\mu\nu} = 0$ which is equivalent to the fact that the Poisson bracket satisfies the Jacobi identity.

$$\omega = \frac{1}{2}\omega_{\mu\nu}d\xi^\mu \wedge d\xi^\nu \tag{A.-3}$$

• **tempered distributions:** they are continuous (or equivalently - bounded) linear maps from the Schwarz space of functions to complex numbers. In other words, it is the topological dual space to the of $\mathcal{S}(\mathbf{R}^n)$

• **tensors:** The point about tensors is that when we want to make statement we do not wish it to hold in just one coordinate system but rather all coordinate systems.

• **tensor calculus:**

• **test particle:** is an idealized point particle with energy and momentum so small that its effects on spacetime are negligible.

• **thermal clocks:** Clocks that change linearly with respect to thermal time. (Rovelli)

• **thermal field theories:**

• **thermal time hypothesis:** The dynamical laws of generally covariant systems determine correlations observables, but they don't single out any of the observables as that we would call "time".

two equivalent ways of describing the time flow: either the flow in state space (Schrödinger picture), or (generalized Heisenberg picture)

So we say time flow is as a one-parameter group of automorphisms of the algebra of observables.

the physical basis of time [291] [292].

$$\mathcal{H} = -kT\ln D \tag{A.-3}$$

The thermal states of a field theory are characterized by have a correlation function, the KMS condition (A)

• **thermodynamics:** Thermodynamics deals with large systems in terms of macroscopic observables alone.

know as the zeroth, first and second laws

zeroth:

first:

second:

Bekenstein suggested that the second law of thermodynamics should be extended in the presence of black holes.

• **Thiemann, Thomas:**

Figure A.44: Thomas Thiemann.

• **Thomson scattering:** When a photon scatters off a free electron, in the nonrelativistic case $\hbar f << m_e c^2$.

• **tidal force:**

• **top-down:** the is required. Loop quantum gravity is an example. One of the biggest challenges for the theory, most be established before it can be claimed to be a viable theory of quantum gravity.

• **time:** ??

• **timelike membranes:** Their definition is the same as a dynamical horizons except that they are timelike instead of spacelike. [149]

• **topological quantum field theories:**

They have a finite number of degrees of freedom so are really mechanical systems disguised as field theories. they posses certain features one expects of a quantum theory of gravity.

• **trace class operator:** nuclear operator.

• **trapped surfaces:** A trapped surface $\theta_{(\ell)} < 0$ and $\theta_{(n)} < 0$

marginally trapped surface: A marginally trapped surface $\theta_{(\ell)}, \theta_{(n)} \leq 0$

An **outer marginally trapped surface** $\theta_{(\ell)} \leq 0$. is the boundary of a three-dimension volume whose expansion of the outgoing family of null geodesics orthogonal to S is everywhere non-positive, $\theta_{(\ell)} \leq 0$.

An apparent horizon

A closed, spacelike, two-surface is an apparent horizon if it is the outermost marginally trapped surface.

trapped boundary a la Hayward:

trapped surface a la Penrose:

closed trapped surface:

• **twisted geometry:** A basis of states for LQG is given by the spin network states. A spin network state has support on a graph Γ and determines a 3d "quantum geometry"; they are eigenvectors of geometric operators which determine the intrinsic geometry. The extrinsic gemetry (analogous to the conjugate momentum) is completely "spread out", due to the Heisenberg uncertainty principle. To make contact with a semiclassical description of space, we can consider coherent states peaked (but not sharp) on both the intrinsic and exstrinsic geometry. A twisted geometry is a specific choice of "interpolating geometry", chosen among distcontinuous metrics. To any graph and any holonomy-flux configuration, we can associate a twisted geometry: a discrete discontinuous geometry on a cellular decomposition space into polyhedra. Thanks to this result, the phase space of LQG on a graph can be visualised not only in terms of holonomies and fluxes, but also in terms of a simple geometric picture of adjacent flat polyhedra.

• **twister theory:** invented by Sir Roger Penrose. : entire light-rays are represented as points, and events by entire Riemann spheres. Twistors defined in terms of a pair of spinors. Twisters are the coordinates of twister space. In twister theory spacetime is a secondary concept. Witten is keen on twister theory in developing background-independent string (M-)theory. Witten found that a lot of the beautiful results of twistor theory did not extend from four dimensions to higher dimensions. If Thiemann's LQG-string theory stands up to the critical examination, in four dimensions there could be possibilty of aplying the powerful results of twister theory to a background independent string theory?? [16]. Seminar given by Penrose (2004) *Goals and Achievements of Twistor Theory*, [445].

• **2+1 quantum gravity:** Witten Chern-Simons. See D.9.

In a vacuum in three-dimensions the vanishing of the Ricci tensor implies the Reinmann tensor vanishes

$$R_{ab} = 0 \implies R^a{}_{bcd} = 0 \tag{A.-3}$$

so the solution of the equations of motion is that curvature vanishes.

This condition, which is imposed *a priori* in Einstein's formulation of general relativity, is seen to be part of the equations of motion.

$$\mathbf{R}^3 \otimes_S SO(3). \tag{A.-3}$$

• **uncertainty relations:**

$$\Delta A \Delta B \geq \frac{1}{2} | < [\hat{A}, \hat{B}] > |, \tag{A.-3}$$

where $\Delta A = \sqrt{< \hat{A}^2 > - < \hat{A} >^2}$.

• **unification:** String theory.

Could degrees of freedom corresponding to matter arise in some natural way from those of geometry?

Chakraborty-Peldan unified model [339]

"...If this difficulty could be overcome and the model could be made to include Ferimons via supersymmetry it would become a viable, elegant and mathematically well defined way of having a unified theory of quantum fundamental interactions."

Quantum Gravity and the Standard Model

Discrete quantum gravity: a mechanism for selecting the value of fundamental constants [329].

• **uniform distretization:** A particularly promising version of "consistent discretizations".

Several approaches to the dynamics of loop quantum gravity involve discretizing the equations of motion. Discretized theories are problematic since the first class algebra of constraints of the continuum theory become second class upon discretization. If one treats the second class constraints normaly, the resulting theories have very different dynamics and number of degrees of freedom than those of the continuum theory. It is therefore questionable how these theories could be considered an appropriate starting point for quantization and the definition of a continuum theory through a continuum limit. The uniform discretization approach is a proposal for the quantization of constrained systems which could overcome these difficulties and construct the correct quantum continuum limit.

• **uniqueness theorems:** See black hole uniqueness theorems.

• **unitarity:** to preserve probabilities

Unitarity implies that expectation values of gauge invariant observables does not depend on the gauge or frame of reference.

• **universe:** That which contains and subsumes all the laws of nature, and everything subject to those laws; the sum of all things that exists physically, including matter, energy, physical laws, spacetime.

• **Unruh effect:** An observer who accelerates uniformly through flat empty space will observe a thermal bath of particles, at a temperature given by their acceleration. This means that a state which is empty according to one observer will not be empty according to an accelerating observer, and hence demonstrates that the concept of "vacuum" must be observer dependent. Discovered by Unruh [386] and Davies [387].

• **vacuum:**

$$|0> \tag{A.-3}$$

define the vacuum state as follows

$$a(k)|0>= 0 \tag{A.-3}$$

where

$$W(q_1, q_2, \ldots q_n) =< q_1, q_2, \ldots q_n|0> \tag{A.-3}$$

where $< q_1, q_2, \ldots q_n|$ are eigenstates of observable quantities.

$$\lim_{t\to\infty} W(\alpha, -it, \alpha', 0) \to H_0(\alpha)e^{-E_0 t}\overline{H_0(\alpha')} \tag{A.-2}$$

$$\lim_{t\to\infty} e^{E_0}|0_{-it}>= |0_M>< 0_M| \tag{A.-2}$$

• **vacuum expectations:**

$$G(x,y) =< 0|a(y)a^\dagger(x)|0 > \tag{A.-2}$$

• **Vaidya:** The spherically solution to Einstein's equations with a null fluid as source.

describe the formation of a black hole through infalling null dust.

Simplest examples of spacetimes admitting dynamical horizons.

• **Varadarjan, :**

• **Varadarjan isomorphism:**

$$\sum_i a_i W_{\alpha_i}[A] = 0. \tag{A.-2}$$

$$\sum_i a_i \alpha_i = 0. \tag{A.-2}$$

• **Vidotto, Francesca:** Work inhomogeneous quantum cosmology and spin foam cosmology.

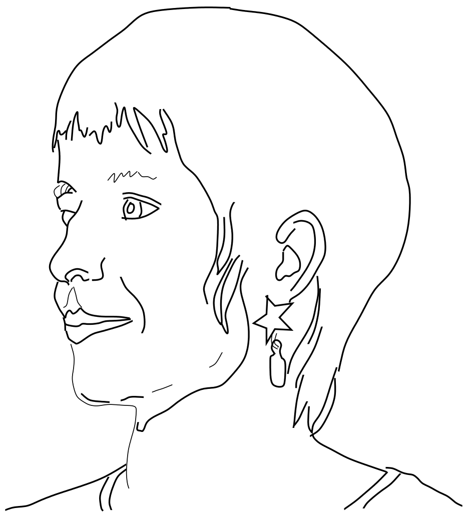

Figure A.45: Francesca Vidotto.

• **volume operator:** $\hat{\mathcal{V}}_{AL}$ $\hat{\mathcal{V}}_{RS}$

• **von Neumann, John:** Developed rigourous mathematical foundations to quantum mechanics. Improtant in quantum gravity where experimental guidence, so far, is missing. There is the Stone-von Neumann uniqueness theorem - a version of which was developed for LQG to prove the uniqueness of the kinematic representation of the holonomy-flux

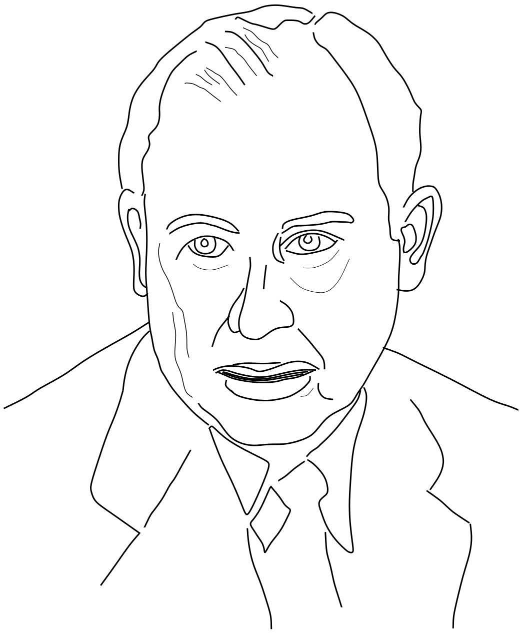

Figure A.46: John von Neumann.

algebra. Developed the infinite tensor product Hilbert space, applied by Thieman et al to quantum gravity in the semiclassical analysis and AQG.

[99], [100]

• **weak anthropic principle:** The weak anthropic principle is the principle whereby the existence of life explained by random selection from an emsemble of universes with differing properties.

• **weave states:** A superposition of spin networks that mimic a given classical spacetime metric. Spin network states are eigenstates of E, so they are like "momentum" eigenstates. So a weave state is maximally spread in the configuration variable $A^a(x)$. Hence they can not represent semiclassical states, just as $e^{ik\cdot x}$ can not represent a semiclassical wavefuntion of a particle because measuring momentum precisely results in complete uncertainty in position.

• **weak cosmic censorship:**

1. $\mathcal{I}^+$ and $\mathcal{I}^-$ are complete;

2. $I^-(\mathcal{I}^+)$ is globally hyperbolic.

• **Weiner-measure:** Brownian

• **Weyl algebra:** Translation $V(\mu)$

• **Weyl tensor:** The Weyl tensor is the conformaly independent part of the curvature tensor. It is the part of the curvature tensor describes the local degrees of freedom of the gravitational field.

• **Wheeler, John:**

Figure A.47: John Wheeler.

• **Wheeler-DeWitt equation:** the Hamiltonian constraint

all the dynamics is contained in constraints.

• **Wick:**

We take spacetime (or manifold?) to Riemannian, but physically, the spacetime is Lorentzian. We pass from one to the other through a Wick rotation.

See the generalized Wick rotation.

• **Wightman axioms:** A list of axioms taken as a blue-print of the general concept of a Quantum Field Theory ([gr-qc/0512053]). Here for special relativistic field theory. Generalised by Thiemann *et al* in [] to generally covariant theories such as Quantum gravity.

i) There exists a Hilbert space $\mathcal{H}$ with a dense domain $\mathcal{D} \subset \mathcal{H}$, so that all $\Phi(F)$ are well defined operators on $\mathcal{D}$, and $\Phi(F)^* = \Phi(\overline{F})$.

ii) $F \mapsto \Phi(F)$ is complex linear and suitably continuous.

iii) Covariance:
There is on $\mathcal{H}$ a unitary representation

$$U(L)\Phi(F)\Phi(L)^* = \Phi(F \circ L^{-1}) \tag{A.-2}$$

with $U(L)\mathcal{D} \subset \mathcal{D}$.

iv) Locality, or relativistic causality:
If the *supports* of the test function F_1 and F_2 are causally separated, the corresponding field operators commute

$$[\Phi(F_1), \Phi(F_2)] = 0. \tag{A.-2}$$

v) spectrum condition/positivity of the total energy:
Writing $U(1, a) = e^{iP_\mu a^\mu}$, it holds (for expectation values) that

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 \geq 0, \qquad P_0^2 \geq 0. \tag{A.-2}$$

vi) Existence and uniqueness of the vacuum:
There exists $\Omega \in \mathcal{D}$, $\|\Omega\| = 1$, so that $U(L)\Omega = \Omega$ and this vector is uniquely determined up to phase factor.

vii) Cyclicity of the vacuum:
The domain $\mathcal{D}$ is spanned by vectors of the form

$$\Omega,\ \Phi(F)\Omega,\ \Phi(F_1)\Phi(F_2)\Omega,\ \ldots. \tag{A.-2}$$

• **Wigner function:** The Wigner function of a semi-classical state is peaked along the phase-space trajectory.

• **Wilson loop:** The trace of the holonomy.

• **Wilson renormalization:**

Can be used even if theory not renormalizable.

$$\int \mathcal{D}[\phi] \tag{A.-2}$$

• **Winkler, Oliver**

• **world function:** The world function $\Omega(x_A, x_B)$ is defined as half the squared geodesic distance between two points x_A and x_B.

• **Winicour solutions:** Winicour J 1975 *J. Math. Phys.* **16** 1805

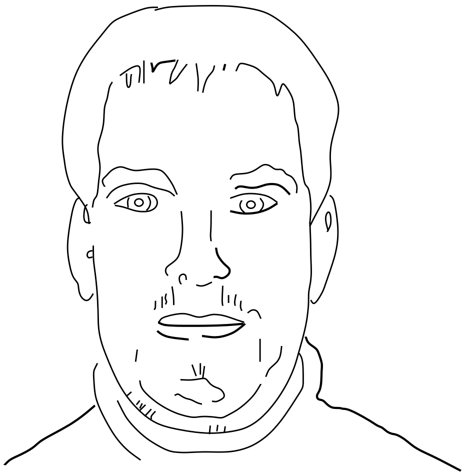

Figure A.48: Oliver Winkler.

• **Witten model:**

[388] E. Witten, "(2 + 1)-dimensional gravity as an exactly solvable system", Nucl. Phys. **B311**: 46, (1988).

• **Yang-Mills:** There is a \$1,000,000 mathematical prize for the consolidation of Yang-Mills and special relativity - does consolidation within a background-independent theory with quantum gravity coupled to the standard model count? Or specifically, background dependent, Minkowski spacetime Yang-Mills theory?

Smolin said

"someone might earn a Clay prize by rigorously constructing quantum Yang-Mills within LQG. It will certainly not be me, but there are people working on exactly that program. The conjecture is that background independent QFTs are more likely to exist rigorously in 3+1 dimensions than Poincare invariant QFTs."

$$\emptyset,\ \circ\bigcirc,\ \forall,\ \exists,\ \ell,\ \wp,\ \Rightarrow,\ \uparrow,\ \cong, \star, \simeq, \propto, \subseteq, \gg, \ll, \supset, \not\subset, \imath, \jmath, \wp \tag{A.-2}$$

$$\|, \prod, \aleph, \surd, \ddagger, \mapsto, \ni \tag{A.-2}$$

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Appendix B

Mathematics Glossary

• **absolute continuous functions:** These are functions of the form:

$$u(x) = \int_{x_0}^{x} v(y)dy + c$$

where $v(y)$ is considered as of u and the fundamental formula

$$u(x) = u(x_0) + \int_{x_0}^{x} \frac{d}{dy} u(y)dy + c$$

still holds. [from http://www.math.ku.dk/ grubb/distribution.htm]

• **absolute continuous measure:** We say that a Borel measure μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ if there is a function, f, locally L^1 (that is, $\int_a^b |f(x)|dx < \infty$ for any finite interval (a, b)) so that

$$\int g d\mu = \int g f dx$$

for any Borel function $g \in L^1(\mathbb{R}, d\mu)$. This generalizes to to measures on topological spaces.

• **absolute continuous operator:**

• **absolute convergence:** The series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

• **additive groups:** There are additive groups. Examples are:

$$
\begin{aligned}
&G = \mathbb{Z}, \quad \text{the group of integers} \\
&G = \mathbb{R}, \quad \text{the group of real numbers} \\
&G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}, \quad \text{the group of integers mod 2} \\
&G = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}, \quad \text{the group of integers mod p.}
\end{aligned} \tag{B.-2}
$$

The notation $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ means that the group $\mathbb{Z}$ of integers we shall indentify any two integers that differ by an even integer, that is, an element of the subgroup $2\mathbb{Z}$. Thus $\mathbb{Z}_2$ consists of two elements, $[\tilde{0}, \tilde{1}]$, where:

$$
\begin{aligned}
&\tilde{0} \text{ is the equivalence class of } 0, \pm 2, \pm 4, \ldots \\
&\tilde{1} \text{ is the equivalence class of } 1, \pm 3, \pm 5, \ldots
\end{aligned} \tag{B.-2}
$$

with addition defined by $\tilde{0} + \tilde{0} = \tilde{0}$, $\tilde{0} + \tilde{1} = \tilde{1}$, and $\tilde{1} + \tilde{1} = \tilde{0}$. Likewise, one can consider the group $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, the group of integers modulo the integer p, where two integers are indentified if their difference is a multiple of p. This group has the elements, written $0, 1, \ldots, p-1$.

• **algebra:** An algebra is simply a vector space over $\mathbb{C}$ (or more generally over a field k) in which there is defined a distributive and, (in a certain sense), associative multiplication:

(i) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(ii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every scalar α, (a complex number).

Note that an algebra need not be associative in the sense $x \cdot (y \cdot z) \neq (x \cdot y) \cdot z$ where x, y, z are elements of the algebra.

Example An example of a **non-associative algebra** is the vector space of 3-d Euclidean vectors with the multiplication relation taken to be the crossed product $\times$ of two vectors

$$
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}). \tag{B.-2}
$$

An **associative** algebra is an algebra whose multiplication also satisfies;

$$
x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{B.-2}
$$

A **commutative** algebra is an algebra whose multiplication also satisfies the condition:

$$
x \cdot y = y \cdot x. \tag{B.-2}
$$

It is unital if there is defined a unit **1** which satisfies

$$a\mathbf{I} = \mathbf{I}a = a, \quad \text{for all } a \in A. \tag{B.-2}$$

Intro into $*$algebras.

A **$*$-algebra** if there is defined an **involution** satisfying

$$(xy)^* = y^* x^* \quad \text{and} \quad (x^*)^* = x \tag{B.-2}$$

which reduces to complex conjugation on the scalars $\alpha \in \mathbb{C}$, i.e.,

$$(\alpha x)^* = \alpha^* x^*. \tag{B.-2}$$

A **Banach algebra** is an algebra with norm $\|a\| \geq 0$ which satisfies the conditions $\|x + y\| \leq \|x\| + \|y\|$, $\|xy\| \leq \|x\| \; \|y\|$, $\|\alpha x\| = |\alpha| \; \|x\|$, $\|x\| = 0 \Leftrightarrow x = 0$ and with respect to which it is complete; it contains its limit points$\|x_n - x\| \rightarrow 0$.

A **C*-algebra** is a Banach $*$-algebra whose norm satisfies the C^*-property:

$$\|a * a\| = \|a\|^2, \quad \text{for all } a \in A. \tag{B.-2}$$

A familiar example of a C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$.

Example An example would be an associative algebra of the space real-valued of continuous functions formed by ordinary pointwise addition $(f + g)(x) := f(x) + g(x)$ and multiplication $fg(x) := f(x)g(x)$. It is obvious that the addition of two continuous functions is continuous. The product of two continuous functions is continuous. To prove that $fg(x)$ is continuous we must show that given any $\epsilon > 0$ there exists $n > N$ such that

$$|fg(x) - fg(x_n)| < \epsilon. \tag{B.-2}$$

Given any $\epsilon > 0$ define $\epsilon' := (|g(x)| + |f(x_n)|)/\epsilon$. As $f(x)$ and $g(x)$ are continuous for $\epsilon' > 0$ there exists $n > N$ such that

$$|f(x) - f(x_n)| < \frac{\epsilon'}{2} \quad \text{and} \quad |g(x) - g(x_n)| < \frac{\epsilon'}{2}. \tag{B.-2}$$

continuity of $fg(x)$ follows from

$$\begin{aligned}
|fg(x) - fg(x_n)| &= |f(x)g(x) - f(x_n)g(x) + f(x_n)g(x) - f(x_n)g(x_n)| \\
&\leq |f(x) - f(x_n)||g(x)| + |f(x_n)||g(x) - g(x_n)| \\
&< \epsilon'(|g(x)| + |f(x_n)|) \\
&= \epsilon
\end{aligned} \tag{B.-4}$$

• **adjoint representation:** $[t_a, t_b] = C^c_{ab} t_c$. The commutation constants C^c_{ab} form a representation of the group algebra.

• **algebraic dual** The algebraic dual $\mathcal{D}^*$ is the space of linear functionals on $\mathcal{D}$ without continuity assumptions. See dual spaces.

• **algebraic graph theory:** http://www.utm.edu/departments/math/graph/glossary.html

• **almost everywhere:**

• **almost periodic functions:** Almost periodic functions are functions that can be written as

$$f(x) = \sum_j f_j e^{\frac{\mu_i x}{2}} \tag{B.-4}$$

where the sum contains a finite number of terms ($j = 1, 2, \ldots, N$ and $N < \infty$), $f_j \in \mathbb{C}$, and $\nu_j \in \mathbb{R}$.

For loop quantum cosmology models of spaces that have homogeneity and isotropy the holonomy algebra consists of the set of almost periodic functions.

The fundamental theorem for almost periodic functions is a generalisation of the Parseval identity for Fourier series. This result lead Bohr to a result on the uniform approximation to almost periodic functions by exponential functions.

• **analytic function:** A function is called real analytic at a point if it possesses derivatives of all orders and given by a convergent power series locally. For example, a function on the real line $\mathbb{R}$ is analytic at the point p if there exists an interval (a, b) containing p such that in this interval the function can be expanded as a convergent series

$$f(x) = a_0 + a_1(x-p) + a_2(x-p)^2 + a_3(x-p)^3 + \ldots, \tag{B.-4}$$

where

$$a_0 = f(p),\ a_1 = f'(p),\ a_2 = \frac{f''(p)}{2!},\ a_3 = \frac{f'''(p)}{3!}, \ldots \tag{B.-4}$$

A function is analytic if it is analytic at each point in its whole domain. The set of all analytic functions is contained in the set of smooth functions. Analytic functions are also referred to as C^ω-smooth functions.

• **analytic continuation:** If we have two complex functions $f(z)$ and $g(z)$ satisfying the following properties:

(a) $f(z)$ is defined on a set U of the z complex plane $\mathbb{C}$;

(b) $g(z)$ is analytic in the domain V containing U;

(c) $g(z)$ coincides with $f(z)$ on U;

then $g(z)$ is said to be the analytic continuation of $f(z)$ to the domain V.

• **analytic curve:** A curve in Euclidean space $\mathbb{R}^n$ is piecewise analytic if it can be expanded as a Taylor series locally. A curve in a manifold $\mathcal{M}$ is analytic if and only if its image under a chart is an analytic curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval (a, b) to $\mathbb{R}^n$ in Fig.(C.7.1) is a analytic map.

• **analytic structure:** a covering homeomorhic to open sets in a fixed Euclidean space, C^ω The coordinate transforms are analytic in both directions, i.e.,

$$\begin{aligned}(\phi_1 \circ \phi_2^{-1})(x_1, x_2, \ldots, x_n) &= a_0 + a_1^{(i)} x_i + a_2^{(ij)} x_i x_j + a_3^{(ijk)} x^3 + \ldots, \\ (\phi_2 \circ \phi_1^{-1})(y_1, y_2, \ldots, y_n) &= b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \ldots.\end{aligned} \tag{B.-4}$$

in some interval containing p.

• **associated bundles:** Given a particular principal bundle $(P, \pi, \mathcal{M})$ with structure group G, we can form a fibre bundle F for each space F on which G acts as a group of transformations.

• **atlas:** Two charts ϕ_1, ϕ_2 are $C^\infty-$related if both the map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and its inverse are $C^\infty-$related. A collection of related charts such that every point of $\mathcal{M}$ lies in the domain of at least one chart forms an atlas.

• automorphisms: An isomorphism $X \to X$ is called an automorphism of X. see group homomorphism.

[404] **Automorphisms of groups:** Let G be a group. An isomorphism $G \to G$ is called an automorphism of G. Let $x, y \in X$ and $g \in G$,

$$g(xy)g^{-1} = (gxg^{-1})(gyg^{-1}), \quad i_g(xy) = i_g(x) i_g(y) \tag{B.-4}$$

The set $\mathrm{Aut}(G)$ of such automorphisms becomes a group under composition:

(i) the composite of two automorphisms is again an automorphism;

(ii) composition of maps is always associative;

(iii) the identity map $g \mapsto g$ is an identity element;

(iv) an automorphism is one-to-one and onto, and therefore has an inverse (we just change the direction of the arrows), which is again an automorphism.

Automorphisms of algebras: Let $\mathcal{A}$ be an algebra. An isomorphism $\mathcal{A} \to \mathcal{A}$ is called an automorphism of $\mathcal{A}$. Let $a, b \in G$

$$\begin{aligned} g(ab)g^{-1} &= (gag^{-1})(gbg^{-1}), \quad i_g(ab) = i_g(a)i_g(b) \\ g(a+b)g^{-1} &= gxg^{-1} + gyg^{-1}, \quad i_g(a+b) = i_g(a) + i_g(b). \end{aligned} \tag{B.-4}$$

$*-$automorphisms:

Time automorphisms of operator algebras in quantum mechanics: $\hat{\mathcal{O}}_i \in \mathcal{A}$ where $\mathcal{A}$ is the quantum operator algebra of a system and $e^{i\hat{\mathcal{H}}t} \in G$ where G is the group of time-evolution operators of parameterized by time t, $-\infty < t < \infty$.

$$e^{-i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_i(t_0)e^{i\hat{\mathcal{H}}t} = \hat{\mathcal{O}}_i(t_0 + t) \tag{B.-4}$$

it is an algebra homomorphism because

$$\begin{aligned} e^{i\hat{\mathcal{H}}t}(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2)e^{-i\hat{\mathcal{H}}t} &= (e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_1e^{-i\hat{\mathcal{H}}t})(e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_2e^{-i\hat{\mathcal{H}}t}) \\ e^{i\hat{\mathcal{H}}t}(\hat{\mathcal{O}}_1 + \hat{\mathcal{O}}_2)e^{-i\hat{\mathcal{H}}t} &= e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_1e^{-i\hat{\mathcal{H}}t} + e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_2e^{-i\hat{\mathcal{H}}t} \end{aligned} \tag{B.-4}$$

it is one-to-one and onto because

$$e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_i(t_0 + t)e^{-i\hat{\mathcal{H}}t} = \hat{\mathcal{O}}_i(t_0) \tag{B.-4}$$

is an inverse (obtained by simply replacing t with $-t$).

is an example of an inner automorphism of the algebra of observables. two equivalent ways of describing the time flow: either the flow instate space(Schrödinger picture), or (generalized Heisenberg picture). So we say time flow is as a one-parameter group of automorphisms of the algebra of observables. A mathematic theorem called the Tomita-Takikey that states in quantum field theory there is a unique one-parameter automorphism of the operator algebra of - the physical basis of time - this is the thermal time hypothesis [291] [292].

• **axiom of choice:** Say we have a family of nonempty sets $\{X_\alpha\}$. Then there is a set X which contains exactly one element from each set $\{X_\alpha\}$. For finite collection of sets, this is obvious and isn't really an axiom. It is when you It applies as an axiom when there are an infinite number (countable and uncountable) number of sets.

• **Banach algebra:** A Banach algebra is a complex Banach space which is also an algebra with identity 1, and in which

(i) $\|xy\| \leq \|x\| \, \|y\|$,

(ii) $\|1\| = 1$.

Also a normed vector ring.???????

If the norm satisfies the parallelogram law, its is also a Hilbert space.

A **Banach subalgebra** is a closed subalgebra of A which contains 1; they are precisely those subsets of A which themselves are Banach algebras with the same identity, and the same norm.

• **Banach space:** A Banach normed space complete normed space which is a complete metric space with respect to the metric induced by norm, $d(x-y) := \|x-y\|$.

A unital Banach space: A banach space containing the identity with respect to multiplication.

• **Bargmann transform:** The Bargmann transform is the unitary transformation that takes an $L^2(\mathbb{R}^n)$ function of the coordinates q_i to a holomorphic square integrable (with respect to a certain measure) function of n complex variables. It can be interpreted as the transform that takes

$$f \rightarrow \text{ Analytic continuation of } e^{-t\Delta/2} f,$$

that is, one obtains the representatioon by heat kernel evolution followed by analytic continuatiooon from the usual position space representation.

• **bijective:** A function is bijective if it is one-to-one and onto, that is injective and surjective.

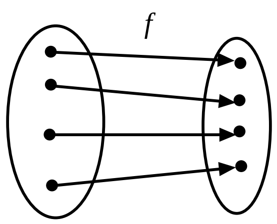


Figure B.1: bijective

See injective, surjective.

bilinear form: A bilinear form (or sesquilinear form) $F(u, v)$ on a Hilbert space H is an assignment of a scalar to each pair of vectors u, v of a subspace $D(F)$ of H in such a way that

(i) $F(\alpha u + \beta v, w) = \overline{\alpha}F(u, w) + \overline{\beta}F(v, w)$, and
(ii) $F(u, \alpha v + \beta w) = \alpha F(u, v) + \beta F(u, w)$

for $u, v, w \in D(F)$. The subspace $D(F)$ is called the domain of the bilinear form.

Bohr compactification: Bohr compactification $\overline{\mathbb{R}}$ of the real line $\mathbb{R}$ is the dual group of $\mathbb{R}$ equipped with the "discrete" topology in which the real line is totally atomized as if no point is near any other point.

The dual of the dual of a group G is the same group back again But suppose you take the dual of G and replace the usual topology with the discrete topology (the power set, same as saying that singletons are open) so you say H is equal to G' as a group but atomized as a topological space and all functions from it are continuous. *Then* you take the dual of H, and that is the Bohr compactification of G.

Bohr group: Dual group to the discrete line. Is compact.

Boolean algebra: A distributive lattice in which every element has a complement is a Boolean lattice or a *Boolean algebra.* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Boolean logic: Formulises in algebraic terms for statements that are either true or false.

See non-Boolean logic

Borel function:

Borel sets: One wants to define a probability distribution on the real line $\mathbb{R}$, that is, assign a real number to any subset Δ of $\mathbb{R}$ that is the probability of some quantity taking its value in Δ. It turns out that it is impossible to construct a probability measure defined on all subsets. one must restrict oneself to a particular family of sets - the Borel subsets, $\mathcal{B}(\mathbb{R})$. This is the smallest family of subsets of $\mathbb{R}$ that includes all the open sets and which is closed under complements and countable intersections.

Borel $\sigma-$algebra: Let X be a topological space. The smallest $\sigma-$algebra on X that contains all open sets of X is called Borel $\sigma-$algebra of X.

• **Borel-summable:** Borel-summation does not pick up on non-perturbative effects.

$$\sum_n a_n x^n \tag{B.-4}$$

with $a_n \approx Cn!$

$$\int_0^\infty e^{-t/x}\, t^n dt = n!\, x^{n+1} \tag{B.-4}$$

$$\int_0^\infty e^{-t/x} \left(\sum_n \frac{a_n}{n!} x^n \right) \frac{dt}{x} = \sum_n a_n x^n \tag{B.-4}$$

$$g(t) := \sum_n \frac{a_n}{n!} x^n \tag{B.-4}$$

• **bounded:**

(i) bounded linear operator:

$$\|A\| := \sup_{x \in X} \frac{\|Ax\|}{\|x\|} \equiv \sup_{\|x\|=1} \|Ax\| \tag{B.-4}$$

then operator A is bounded if $\|A\| \leq C$. Note: boundedness and continuity are equivalent for linear operator.

- bounded set:

We call a nonempty subset $M \subset X$ a bounded set if its diameter

$$\delta(M) := sup_{x,y \in M}\, d(x, y) \tag{B.-4}$$

is finite.

• **bounded contraction semigroup:** imaginary-time path integral has kernel

$$e^{\frac{-\hat{\mathcal{H}}t}{\hbar}} \tag{B.-4}$$

• **bounded convergence theorem:**

If the sequence $\{f_n\}$ of measurable functions is uniformally bounded and if $f_n \to f$ in measure as $n \to \infty$, then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

• **bounded inverse theorem:** The range of a bounded operator A is closed in $\mathcal{H}$ implies that $A - \lambda\mathbb{I}$ has in inverse in $\mathcal{B}(\mathcal{H})$.

• **bounded linear functional:** A linear functional satisfying

$$|F(x)| \leq M\|x\|, \quad x \in X.$$

• **bounded linear operator:** An operator A is called bounded if its domain is the whole of X, i.e. $D(A) = X$, and there is a constant M such that

$$\|Ax\| \leq M\|x\|, \quad x \in X.$$

The norm of such an operator is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

• **bounded variation:** A function $f(x)$ defined on an interval $[a, b]$ is said to be of bounded variation if it satisfies

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < C \tag{B.-4}$$

for any partition $a = x_0 < x_1 < \cdots < x_n = b$.

• **braid:**

• **braided moinodal category:** A 3-category with one object.

• **braid group B_n:** a restriction of the permutation group.

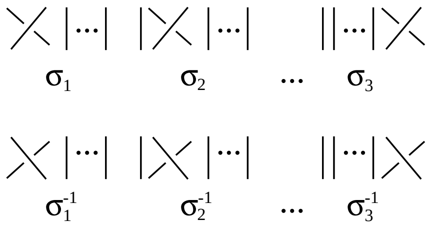


Figure B.2: Braid group generators

$$\begin{aligned} \sigma_i \sigma_i^{-1} &= 1, & i &= 1, \ldots, n-1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_i. & i &= 1, \ldots, n-2 \\ [\sigma_i, \sigma_j] &= 0 & |i-j| &> 1. \end{aligned} \tag{B.-5}$$

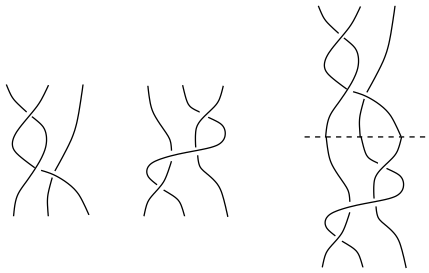


Figure B.3: Multiplication of braids.

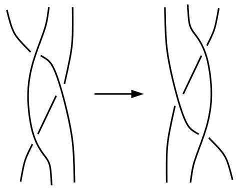


Figure B.4: A Reidermeister move on a three-string braid.

Quantum groups were invented largely to provide solutions for the Yang-Baxter equation and hence solvable models in 2-dimensional statistical mechanics and one-dimensional quantum mechanics. why i-d quantum???????????

• **Caley transform:**

$$a \mapsto u := \frac{a-i}{a+i}$$

• **canonical local trivialisation:** There are local sections $s_i : U_i \to \pi^{-1}(U_i)$ canonically associated to the trivialisation, eefined so that for every $p \in U_i$, $\phi_i(s_i(p)) = (p, e)$. In other words, the map from U_i to G is the constant function sending every point to the identity.

• **Cantor Set:** The Cantor set is

$$C = \bigcap_{k=1}^{\infty} C_k. \tag{B.-5}$$

where

$$
\begin{aligned}
C_0 &= [0,1] \\
C_1 &= [0,\frac{1}{3}]\bigcup[\frac{2}{3},1]. \\
C_2 &= [0,\frac{1}{9}]\bigcup[\frac{2}{9},\frac{1}{3}]\bigcup[\frac{2}{3},\frac{7}{9}]\bigcup[\frac{8}{9},1],
\end{aligned}
\tag{B.-6}
$$

and in general C_{n+1} consists of the intervals of C_n with their open middle thirds removed.

• **cardinality:**

• **Cartan's structure equations:**

$$
d\theta_i = -\frac{1}{2}c^i_{kl}\theta_k \wedge \theta_l. \tag{B.-6}
$$

• **category:** A category $\mathcal{C}$ consists of "objects", C, and "morhisms", f, between them, such that

(i) If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morhisms, then there exists a morphism $g \circ f : C_1 \to C_3$.

(ii) It is assumed that the identity map for C, $\mathrm{id} : C \to C$, is a morphism for every object C of $\mathcal{C}$.

The set of objects is usually denoted ob$\mathcal{C}$.

Examples:

The category of open sets in Euclidean spaces, where the morphisms are the smooth maps.

The category of abelian groups, where the morphisms are homomorphisms.

n-category:

n-categorical group: A is by definition a group-object of the category od groupoids.

• **category theory:** TAKEN DIRECTLY FROM mathworld.wolfram.com

Category Theory

The branch of mathematics which formalizes a number of algebraic properties of collections of transformations between mathematical objects (such as binary relations, groups, sets, topological spaces, etc.) of the same type, subject to the constraint that the collections contain the identity mapping and are closed with respect to compositions of mappings. The objects studied in category theory are called categories.

http://mathworld.wolfram.com/

an abstraction of many concrete concepts in diverse branches of mathematics.

• **CAR:** stands for canonical anti-commutation relations. CAR algebra....

• **Cat:** The category of all small categories.

• **Cauchy-Kowalewski:**

• **Cauchy sequence:** A sequence $\{a_l\}$ satisfying

$$\|a_n - a_m\| \to 0 \quad \text{ as } m, n \to \infty.$$

• **CCR:** stands for canonical commutation relations.

• **Cech compactification:**

• **center:** An abelian subgroup of a ring R is the **center**

$$X = Z(R) \;\; X = \{a \in R : b \in R, \; ab = ba \;\text{ for all } b\} \tag{B.-6}$$

The center of a Lie group is the set of commuting elements $\{x \in \mathbf{g} : [x, y] = 0$, for all $y \in \mathbf{g}\}$.

X is the centralizer of Y $X = C_R(Y)$

$$X = Z(R) \;\; X = \{a \in R : b \in Y, \; ab = ba \;\text{ for all } b\} \tag{B.-6}$$

Alternative

The kernel of the $G-$action is the subgroup of G defined by

$$K := \{g \in G : gp = p \text{ for all } p \in \mathcal{M}\}.$$

The kernel measures the part of the group that is not represented at all in the $G-$action on $\mathcal{M}$. An example is given by the adjoint action of G on itself in which

$$\mathrm{Ad}_g(g') := gg'g^{-1}.$$

The kernel of this action is the centre $C(G)$ of G.

• **central extension:** A central extension of a Lie algebra $\mathrm{Lie}(G)$ is a Lie algebra E together with a homomorphism $\pi : E \to \mathrm{Lie}(G)$ such that $\mathrm{Ker}(\pi) \subset Z(E)$ where $Z(E) = \{A \in E : [A, B] = 0 \text{ for all } B \in E\}$ is the centre of E.

• **characters:**

Characters of a finite group:

$$\chi(A) = \mathrm{tr}A = \sum_i A_{ii}. \tag{B.-6}$$

These characters are invariant under simultaneity transformations because of the cyclic symmetry of matrices

$$\chi(U^{-1}AU) = \mathrm{tr}(U^{-1}AU) = \mathrm{tr}A \tag{B.-6}$$

Characters of algebras:

Let $\mathcal{U}$ be an abelian C^*-algebra. A **character** χ, of A, is a nonzero linear map, $\omega; A \in \mathcal{U} \mapsto \chi(A) \in \mathbb{C}$, of $\mathcal{U}$ into the complex numbers $\mathbb{C}$ such that

$$\chi(AB) = \chi(A)\chi(B) \tag{B.-6}$$

for all $A, B \in \mathcal{U}$. As it preserves multiplication (B) it is a homomorphism. See the spectrum $\sigma(\mathcal{U})$.

• **chart:** Given a topological space $\mathcal{M}$, a chart on $\mathcal{M}$ is a one-to-one map ϕ from an open subset $U \subset \mathcal{M}$ to an open subset $\phi(U) \subset \mathbb{R}^n$, i.e., a map $\phi : \mathcal{M} \to \mathbb{R}^n$. A chart is often called a coordinate system.

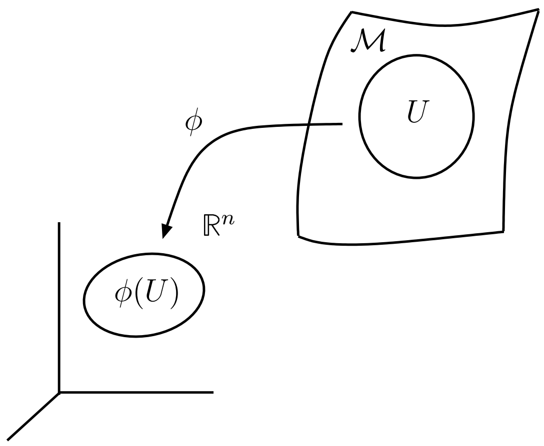


Figure B.5: DiffClass0. A chart on $\mathcal{M}$ comprises an open set U of $\mathcal{M}$, called a coordinate patch, and a map $\phi : U \to \mathbb{R}^n$.

• **Chern classes:** cohomology

$$\det(t\mathbf{I} + a_c T^c) \tag{B.-6}$$

substituting

second *Chern class*

$$c_2(P) = \frac{1}{4\pi^2}\left(\frac{1}{2}(TrF) \wedge (TrF) - \frac{1}{2}Tr(F \wedge F)\right) \tag{B.-6}$$

• **chromatic evaluation:** The chromatic evaluation of the spin network is equivalent to the Temperley-Lieb trace of the spin network, or, equivalent to the Kauffman bracket.

• **class function:** A function of elements $x, y, z, ..$ of a group G is said to be a class function if

$$f(x, y, \ldots, z) = f(g^{-1}xg, g^{-1}yg, g^{-1} \ldots zg) \tag{B.-6}$$

where $g \in G$.

• **closable operator:** Let A be a linear operator from a normed vector space X to a normed vector space Y. It is called closable if for $\{x_k\} \subset D(A)$, $x_k \to 0$, $Ax_k \to y$ imply that $y = 0$.

• **closed bilinear form:** A bilinear form $F(u, v)$ is called closed if $\{u_n\} \subset D(F)$, $u_n \to u$ in H, $F(u_n - u_m) \to 0$ as $m, n \to \infty$ imply that $u \in D(F)$ and $F(u_n - u) \to 0$ as $n \to \infty$.

• **closed graph theorem:** The closed graph theorem implies that if $A : X \to Y$ is closed and has the domain of A is equal to X then A is bounded.

See bounded inverse theorem.

• **closed surface:** Examples of closed surface are the sphere, the torus, the Klein bottle. They are classified by the genus and their orientability. An examples of a non-closed surface is a disk which is a sphere with a puncture.

• **closure property:** If certain set of operations on sets in $\mathcal{F}$ again produce sets in $\mathcal{F}$, we say $\mathcal{F}$ is closed under these operations.

• **cobordism:** see Fig.(**??**) - referring to wrong figure for some reason!

• **co-cycle:**

• **cocycle Radon-Nikodym theorem:** The Cocycle Radon-Nikodym [] theorem states that two modular automorphisms defined by two states of a von Neumann algebra are inner-equivalent.

see rovelli thermal time hypothesis

• **codimension:** $\text{codim}X = \text{dim}X^{\perp}$.

• **cohomology:** Roughly speaking, the cohomology $H^p(M, \mathbb{R})$ counts the number of noncontractable p-dimensional surfaces in M.

we consider all closed forms modded out by exact forms. In other words two forms are said to be equivalent if

$$\lambda_1 = \lambda_2 + d\Phi \quad \rightarrow \quad [\lambda_1] = [\lambda_2] \tag{B.-6}$$

for any Φ. two closed forms are called cohomology.

• **commutant:** The commutant of a collection of operators $\mathcal{A}$ is defined as

$$\mathcal{A}' := \{A \in \mathcal{A} : [A, B] = 0 \text{ forall } B \in \mathcal{A}\}. \tag{B.-6}$$

• **compact:** Every open cover has a finite subcover.

• **compactification:** The process of adding points to a given topological space in order to make it compact. The simplist compactifiction is adding just one point, for exmple the process of adding a point to a plane to make a sphere.

• **compact Lie group:**

Each group element is uniquely related by the parameters.

• **compact manifold:** A manifold is compact every open cover has a finite subcover. "A town is compact when it can be policed by a finite number of arbitrary short-sighted policemen".

• **compact operators:** An operator A is said to be a compact operator if there exists a sequence of finite rank operators that converge to A in the operator norm.

Compact operators can be alternatively defined as the closure of the set of finite-rank operators in the operator norm topology.

• **comparable:** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms defined on the same linear space Φ. These two norms are called **comparable** if if for every $\varphi \in \Phi$ there exists a constant $C > 0$ such that

$$\|\varphi\|_1 \leq \|C\varphi\|_2, \quad \text{for all } \varphi \in \Phi. \tag{B.-6}$$

• **compatible:** Let Φ be a vector space on which are defined two inner products. Two norms are called compatible if and only if every sequence $(\varphi_n)_{n=1}^{\infty} \subset \Phi$ which is Cauchy with respect to both norms and which converges to $\varphi \in \Phi$ with respect to one of them, also converges to φ with respect to the other. It suffices to check that the above condition only for sequences which converge to zero.

• **completely continuous operator** A degenerate operator A can be written in the for $A = UT$, where U is an isometric operators and T is a positive deinite operator. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T. the space of continuous operators is obtained from the space of degenerates operators by completeing the latter in the norm $\|A\| = \max_k \lambda_k$.

Alternative definition: A linear operator A which maps a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$ is called completely continuous if it maps any bounded set into a set whose closure is compact.

• **completely regular topological space:** Let X be a topological space. If for each open neighbourhood $\mathcal{U}$ of x, there is a continuous function $0 \leq f(x) \leq 1$ such that $f(x) = 0$ and f is identically one on the complement $X - \mathcal{U}$ of $\mathcal{U}$ in X, i.e.

$$f(x) = \begin{cases} 0 & \text{for } x \in X \\ 1 & \text{for } x \in \mathcal{U} - X \end{cases} \tag{B.-6}$$

(Completely regular spaces are also called Tychonoff spaces). are able to support sufficiently many continuous functions: for two distinct points x and y of a completely regular space X, there is a continuous function on X taking distinct values at x and y.

Third class of spaces contains, for example, all normal and all Hausdorff spaces.

• **complete set:** A collection f_i of functions on the symplectic space $(\mathcal{M}, \omega)$ which Poisson commute with each other (are in involution) is said to be complete if the vanishing of $\{f_i, g\}$ for all i implies that g is a function of the form $g(x) = h(f_1(x), \ldots, f_n(x))$.

A collection of operators $\{A_j\}$ is said to be complete if any operator B which commutes with each A_j is a multiple of the identity. This condition is equivalent to the irreducibility of $\{A_j\}$, that is, there is no non-trivial subspace that is invariant under each A_j.

• **complete space:** a space is complete if all Cauchy sequences (defined wrt the distance formula) converge to an element of the space. The distance formula can be a metric space. A vector space equipped with a norm is complete if all sequences which are Cauchy with respect to the distance function $d(x, y) = \|x - y\|$ converge to an element in the space. A vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ induces a norm $\|x\| = \sqrt{\langle x, x \rangle}$. It is complete if all sequences are Cauchy with respect to the distance function $d(x, y) = \sqrt{\langle x, y \rangle}$ converge to an element in the space.

• **completion:** complicates matters: it would be like doing real analysis in the rational numbers instead of the real line $\mathbb{R}$.

Completeness is an important property since it allows us to perform limit operations which arise frequently in our constructions.

• **complexification of a Lie group:** The complexification of a Lie group $\mathbf{g}$ denoted $\mathbf{g}_{\mathbb{C}}$.

Recall the defining properties of a Lie algebra: antisymmetric $[X,Y] = -[Y,X]$, bilinearity $[\alpha X, Y] = \alpha[X,Y] = [X, \alpha Y]$ for any real number α, and satisfies the Jacobi identity $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] \equiv 0$. $X_1 + iX_2$ where $X_1, X_2 \in \mathbf{g}$

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1]). \tag{B.-6}$$

It is clear that (B) is *real* bilnear and skew-symmetric. If we prove that it is *complex* linear in the first factor, it will be complex linear in the second because of the skew-symmetry. As we already know it is real linear in the first factor, it suffices to show that it is *imaginary* linear. This is not difficult to prove, all we need to do is verify that

$$[i(X_1 + iX_2), Y_1 + iY_2] = i[X_1 + iX_2, Y_1 + iY_2] \tag{B.-6}$$

is true. This is easily done by expanding each side and seeing they are indeed equal.

It remains to check the Jacobi identity. First consider Y and Z to be in $\mathbf{g}$ but take X to be in $\mathbf{g}_{\mathbb{C}}$. Now, $X = X_1 + iX_2$ is linear in the Jacobi identity and the Jacobi identity holds separately for X_1 and X_2,

$$[[X_1, Y], Z] + [[Z, X_1], Y] + [[Y, Z], X_1] + i([[X_2, Y], Z] + [[Z, X_2], Y] + [[Y, Z], X_2]) \equiv 0, \tag{B.-6}$$

and so the Jacobi identity holds for $X \in \mathbf{g}_{\mathbb{C}}$ and $Y, Z \in \mathbf{g}$. Similarly for Y and Z. Therefore we have shown that the elements of the complexification $\mathbf{g}_{\mathbb{C}}$ satisfy the Jacobi identity.

• **complexification:** - We can tensor a real vector space with the complex numbers and get a complex vector space; this process is called complexification. For example, we can complexify the tangent space at some point of a manifold, which amounts to forming the space of complex linear combinations of tangent vectors at that point. (from Geometric Quantization John Baez August 11, 2000).

• **complex manifold:** $\mathcal{M}$ is a complex manifold if

(i) $\mathcal{M}$ is a topological space;

(ii) $\mathcal{M}$ is provided with a family of pairs $\{(U_i, \varphi_i)\}$;

(iii) $\{U_i\}$ is a family of open sets which covers $\mathcal{M}$, φ_i is a homomorphism from U_i to an open subset U_i' of $\mathbb{C}^m$;

(iv) Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_j \varphi_j^{-1}$ is holomorphic.

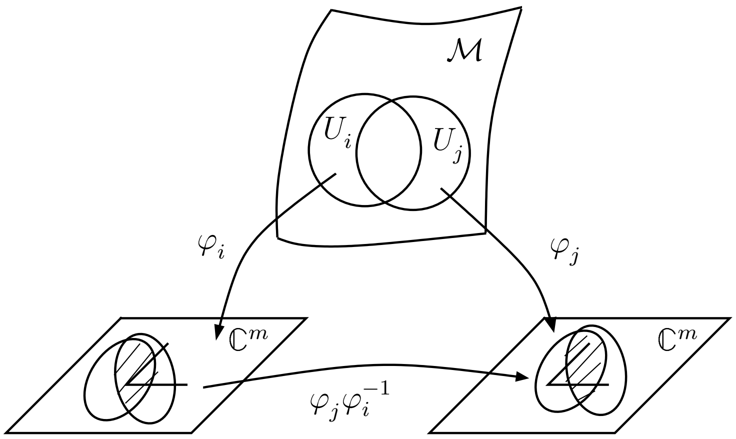


Figure B.6: Complex manifold.

• **congruence:** A congruence is a set of curves which fill a manifold, or part of it, without intersecting. Through every point there passes one and only one curve.

• **congruence relation:**

• **connected:** In the topological sense: a topological set not able to be partitioned into non-empty open subsets each of which has no points in common with the closure of the other.

• **connection:** Roughly, comparison of objects in two different spaces is made by a prescribed mapping, and the mappings that connect the various spaces are called connections.

A connection on a principal bundle is an assignment to each local trivialisation $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ (a choice of gauge in physics terms) a Lie algebra one-form ω_i on U_i which satisfies the following rule between different local trivialisations, (for simplicity here in the case of matrix groups G):

$$\omega_j = t_{ij}\omega_i t_{ij}^{-1} + t_{ij}^{-1} dt_{ij}.$$

A connection is an example of a gauge field.

• **continuity:**

− uniformaly continuous when for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x, x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$

− pointwise continuous when for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. A function which is pointwise continuous at every point is pointwise continuous.

– sequentially continuous when it preserves limits of convergent sequences: if $(a_i)_{i\in N}$ converges to a in X then $(f(a_i))_{i\in N}$ converges to $f(a)$ in Y.

• **continuous function:** A function f is continuous at a point p if whenever we can force the distance between $f(x)$ and $f(p)$ to be as small as desired by taking the distance between x and p to be small enough.

A function is said to be absolutely continuous if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon \tag{B.-6}$$

whenever (a_i, b_i), $i = 1, 2, \ldots, n$, are non-overlapping subintervals of I with $\sum_{i=1}^{n} |b_i - a_i| < \delta$

• **contractible open cover:** A collection $\{U_i\}$ is a contractible open cover of M if each U_i and each non-empty finite intersection $U_i \cap U_j \cap \cdots$ is contractible to a point.

• **contraction:** Suppose there is a map A from the space M to itself, this is a contraction mapping if there is a positive constant $K < 1$ such that $\|Ax - Ay\| < K\|x - y\|$ for all $x, y \in M$.

• **convergence:**

(x_n) converges to x if and only if the sequence (x_n) is eventually in every neighbourhood of x.

(From Weak convergence of inner superposition operators) modes of convergence:

The precise definitions. Let $A_\nu : X \to Y$, $A : X \to Y$, where $\nu \in \mathbb{N}$ are the mappings between two Banach spaces X and Y . One says that the sequence A converges to A

(S) strongly (pointwise), if $A_\nu x \to Ax$ in Y for all $x \in X$;

(C) continuously, if $A_\nu x \to Ax$ in Y for any norm converging sequence $x_\nu \to x$ in X;

(W) weakly (pointwise), if $A_\nu x \rightharpoonup Ax$ in Y for all $x \in X$;

(CW) continuously weakly, if $Ax \rightharpoonup Ax$ in Y for any weakly converging sequence $x_\nu \rightharpoonup x$ in X.

• **countable:** (denumerable) A set is countably infinite if there is a one-to-one onto function from the set A to the set $\{1, 2, 3, \ldots, \}$.

• **countably Hilbert space:** is a complete linear topological space whose topology is defined by a countable family of Hilbert spaces, Φ_n. The topology. Countable set of norms $\|\varphi\|_n$ where the norms are associated with a scalar product - $\|\varphi\|_n := \sqrt{(\varphi, \varphi)_n}$. First property is that Φ_n is the Cauchy completion of Φ in the norm $\|\cdot\|_n$???. Then for any m, n it is required that if (ϕ_k) is both a $\|.\|_m$ convergent sequence and an $\|\cdot\|_n$ Cauchy sequence in Φ then is also$\|\cdot\|_n$ convergent.

Second definition:

A Frechét space Φ is a **countably Hilbert space** if its topology can be induced by a countable system of pairwise compatible inner products.

Third definition

A space Φ is a countably Hilbert space (or a countably scalar product space) if an increasing denumerable number of scalar products

$$(\varphi, \varphi)_1 \leq (\varphi, \varphi)_2 \leq \cdots \leq (\varphi, \varphi)_p \leq \cdots \tag{B.-6}$$

are defined on Φ such that the norms

$$\|\varphi\|_p := \sqrt{(\varphi, \varphi)_p}, \ \ p = 0, 1, 2, \ldots \tag{B.-6}$$

are comparable and compatible.

• **countably neighbourhood base:**

• **cover:** Given a set X with $A \subseteq X$ is said to be regular if, for a **cover** for A is a family of subsets $\mathcal{U}$ =$\{U_i{:}\in I\}$ of X such that $A \subseteq \cup_{i \in I} U_i$. A subcover $\mathcal{U}$ for A is a subfamily $\mathcal{V} \subset \mathcal{U}$ which still forms a cover for A.

• **cubulations:** Cubic triangulations of a four manifold are called cubulations. The first motivation for considering cubulations in spin foam models is that current semiclassical states used in LQG do not assign good classical behaviour to the volume operator of LQG (which plays a pivitol role in the dynamics) unlessthe underlying graph has cubic topology.

Cubulations also nicely fit in with the framework of Algebraic Quantum Gravity which in its minimal version is also formulated in terms of algebraic graphs of cubic topology only.

• **curve:** We refer to Fig.(C.7.1). A curve in a manifold $\mathcal{M}$ is a map λ of the open interval $I = (a, b) \in \mathbb{R} \to \mathcal{M}$ such that for every coordinate system of $\mathcal{M}$ $\phi \circ \lambda : I \to \mathbb{R}^n$. We say the curve smooth if $\phi \circ \lambda : I \to \mathbb{R}^n$ is smooth. The set of curves is denoted $\mathcal{C}^\infty$.

Alternative kinds of curves such as piecewise analytic, continuous, oriented, an embedding (does not come arbitrarily close to itself) are defined in the obvious way.

See piecewise-analytic

• **cyclic state:** GNS “vacuum state”. For example for the SHO $|0>$ for which $\hat{\mathcal{H}}|0>= \frac{1}{2}\hbar|0>$. $\hat{a}|0>=0$. This, with $|n>:=(\hat{a}^\dagger)^n|0>$.

$$< 0|[\hat{a},\hat{a}^\dagger]|0>= \left(\frac{1}{2}+1\right)\hbar \tag{B.-6}$$

Any representation with a cyclic state, can be constructed from the “vacuum” expectation value of the algebra of operators.

the precise definition of a cyclic vector is: a vector Φ is cyclic for a C^*algebra A acting in a Hilbert space $\mathcal{H}$ if and only if the linear space $\mathcal{A}\Phi=\{A\Phi, A\in\mathcal{A}\}$ is dence in $\mathcal{H}$.

(In particular the vacuum) is cyclic and separating for the algebras $\mathcal{A}$, i.e. $\mathcal{A}\Phi$ is dense in $\mathcal{H}$ (Φ is cylic) and $A\Phi=0$, $A\in\mathcal{A}$ implies $A=0$ (Φ is separating).

More precisely, a vector Φ is separating for $\mathcal{A}$ acting in $\mathcal{H}$ if and only if it is cyclic for the commutant $\mathcal{A}'$ of $\mathcal{A}$; the commutant is the set of operators in $\mathcal{H}$ which commute with all $A\in\mathcal{A}$. If $[A,A']=0$ then

Let $\mathcal{R}$ be a von Neumann algebra generated by $\pi_\omega(\mathcal{A})$, i.e., $\mathcal{R}=()''$ A cyclic and separating vector in the Hilbert space $\mathcal{H}_\omega$ (ξ is separating in $\mathcal{A}$ if it is cyclic in $\mathcal{A}'$).

• **cylindrical function:** Fake infinite functions: although they depend on a field, say a scalar field $\phi(x)$, they only really depend on a finite number of variables. One begins by introducing a space $\mathcal{S}$ of ‘probes’. Elements of $\mathcal{S}$ probe the structure of the scalar field $\phi\in\mathcal{C}$, ($\mathcal{C}$ being the classical configuration space of scalar functions), through linear functions h_e on $\mathcal{C}$:

$$h_e(\phi)=\int_{\mathcal{M}} d^3x\ e(x)\phi(x)$$

which capture a small part of the scalar field. Given a set α of probes, $h_{e_1},\ldots,h_{e_n}$ and a (suitably regular) complex-valued function ψ of n real variables, we can now define a more general function Ψ on $\mathcal{C}$,

$$\Psi_\alpha(\phi):=\psi(h_{e_1}(\phi),\ldots,h_{e_n}(\phi))$$

These are examples of cyclindrical functions. Their linear span is usually denoted Cyl_α.

???term cylindrical borrowed from free fields in Minkowski???

Kolmogorov used this special class of functions to define measures for infinite dimensional integration theory.

• **cylindrical measure theory:** The idea of cylindrical measure theory is to reduce the integration over infinite dimensional spaces to a series of finite dimensional subspaces by making use of a special class of function(-al)s - cylindrical function(al)s. However, because of nesting, overlapping, (as well as freedom in choice of basis) of the finite dimensional subspaces, integration over them are subject to certain non-trivial consistency conditions. When these consistency conditions are satisfied, we say the infinite dimensional function space is said to be equipped with a cylindrical measure.

• **Darboux theorem:** Let $\mathcal{Q}$ be a configuration space (manifold). In local coordinates $(q^1, \ldots, q^n)$ one may identify the canonical momentum variables $(p_1, \ldots, p_n)$ with the cotangent vectors in the coordinate basis of $(q^1, \ldots, q^n)$. The Darboux theorem asserts there is a sympletic form Ω on the cotangent space $T^*\mathcal{Q}$ (phase space Γ) given by

$$\Omega_{ab} = \sum_{\mu=1}^{n} 2\nabla_{[a} dp_\mu \nabla_{b]} dq^\mu \tag{B.-6}$$

and the pair (Γ, Ω) is a symplectic manifold.

• **Degenerate operator:** An operator is called degenerate if it maps the entire space onto a finite-dimensional subspace.

A theorems says that every degenerate operator A can be written in the form $A = UT$, where U is an isometric operator and T is a positive degnerate operator

• **Dehn surgery:** Dehn surgery is a procedure by which one can construct all three-manifold topologies. One begins with drawing a knot or link (i.e., a set of knots) on a given manifold, then we thicken the knot into a tube. One then removes this tubular region, give it a twist and then glue it back.

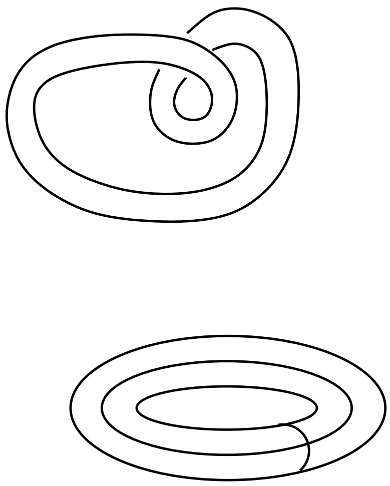


Figure B.7: Dehn surgery.

The three-sphere is two torus glued together. It can all be reduced to working with Chern-Simons on a torus. Once we solve Chern-Simons on a torus we have the exact closed solution for Chern-Simons on any three-manifold. Chern-Simons is related to $2+1$ quantum gravity, hence Witten showed that $2+1$ quantum gravity is exactly solvable, [388].

• **DeMorgan'a Laws:** Let A and B be subsets of X then

1) $X\backslash(A \cup B) = (X\backslash A) \cap (X\backslash B)$ and

2) $X\backslash(A \cap B) = (X\backslash A) \cup (X\backslash B)$.

• **dense:**

• **De'Ram cohomology:**

• **dihedral angles:** Angle subtended where to planes intersect???

• **diffeomorphic:** Two manifolds connected by a diffeomorphism are said to be diffeomorphic. From the differential-geometric point of view, diffeomorphic manifolds not distinguished by some othe structure (e.g. a metric) are effectively the same.

• **diffeomorphism:** A $f : M \to N$ from one manifold M to a manifold N is a smooth map whose inverse is also smooth.

• **differentiable manifold:** If $\mathcal{M}$ is a space and Φ its maximal atlas, the set $(\mathcal{M}, \Phi)$ is a differentiable manifold. We can have C^∞, C^k, analytic, and semianalytic manifolds.

• **differential forms:** A $p-$form is defined to be a completely antisymmetric tensor of type $\begin{pmatrix} 0 \\ p \end{pmatrix}$. A one-form is a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor and a scalar function is a zero form. The number p is the degree of the form.

• **Dirac operator:** a "square root" of the D'Alambert operator in flat Minkowski spacetime

• **direct sum:**

$$\underline{\underline{A}} \oplus \underline{\underline{B}} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{B}} \end{pmatrix}, \quad \vec{v} \oplus \vec{w} = \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \tag{B.-6}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \oplus \begin{pmatrix} p & q \\ r & t \end{pmatrix} = \begin{pmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & r & t \end{pmatrix} \tag{B.-6}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} \tag{B.-6}$$

• **discrete counting measure:**

• **discrete topology:** Let $\mathcal{T}$ be a topological space. A poit $p \in \mathcal{T}$ is called isolated if $\{p\}$ is open in $\mathcal{T}$. The unique topology in which every point is isolated is called the discrete topology. All functions are contiuous in the discrete topology.

• **distribution:** A distribution V on a manifold M is a choice of a subspace V_x of each tangent space $T_p(M)$, where the choice depends smoothly on x.

See integrable distribution.

• **division ring:** A ring with identity is called a *division ring* if all its non-zero elements are regular (invertible).

• **domain of an operator:** $\mathcal{D}(A) \subset \mathcal{H}$ such that $||\hat{}\psi > |^2 < \infty$

• **dominating convergence theorem:** The (Lebesgue) dominating convergence theorem is concerned with when the integral of a limit functions is equal to the limit of integrals, i.e. , when

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n. \tag{B.-6}$$

• **dual complex:** Let us fix a triangulation Δ of a $d-$dimensional spacetime manifold $\mathcal{M}$. This triangulation defines another decomposition of $\mathcal{M}$ into cells called a dual complex. There is a one-to-one correspondence between $k-$simplicies of the triangulation and the $(d-k)-$cells in the dual complex.

• **dual group:** The dual group $\hat{G}$ is the set of characters $\{\gamma\}$ of G, i.e., homorphisms of G into the circle group $\{z \in \mathbb{C},\ |z| = 1\}$, with group multiplication defined by

$$< \gamma_1\gamma_2, g > = < \gamma_1, g > < \gamma_2, g > \tag{B.-6}$$

[157]

$$(\lambda_1 g_1 + \lambda_2 g_2) \cdot g_3 = \lambda_1 g_1 \cdot g_3 + \lambda_2 g_2 \cdot g_3 \tag{B.-6}$$

group algebra $\mathbb{R}(G)$.

As this has the structure of a vector space we may form a dual vector space via some inner product Fun(G) of functions on G. This bilinear inner product will take values from $< .,. >$: $\mathbb{R}(G)\otimes$Fun(G) $\to$ **R**. A natural choice is simply as functions on the group space $< g, f >:= f(g)$.

given the inner product between these two vector spaces and given an operator on one of them we can define its dual action, that is its adjoint, acting on the other.

generators of the group vector space e_i and the generators of the dual space e^i.

• **dual space:**

the algebraic dual The set of all functionals defined on a vector space X is can itself be made into a vector space. This is called the alghebraic dual of X and is denoted X^*.

the topological dual The set of all linear bounded (that is, continuous) functionals on X is called the topological dual of X and is denoted X'. It is called topological dual because ... continuous transformation preserves the topology

the topological dual [209]:

Choose a dense and invariant domain Φ for a .is defined as the algebraic dual of Φ, i.e. the set of all linear functionals on Φ equipped with the weak *topology of pointwise convergence.

• **embedding:** doesn't come arbitrarily close to itself.

• **empty set:** Set which has no elements, denoted $\emptyset$.

• **entire function:** A complex function $f(z)$ analytic at all points of any open set of the complex plane.

OR

A complex function $f(z)$ analytic everywhere in the complex plane within a finite distance of the origin, it is an entire function. For example polynomials $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ are entire functions. They diverge at infinity...

• **enveloping algebra:** see universal enveloping algebra

If I is an ideal of a Lie algebra $\mathcal{G}$, the equivalence classes in $\mathcal{G}$

$$[A] = [A + I], \quad A \in \mathcal{G}, I \in I, \tag{B.-6}$$

forms a Lie algebra - the quotient algebra, $\mathcal{G}/I$.

$$[[A], [B]] = [[A, B]] \tag{B.-6}$$

$$[A + I_1, B + I_2] = [A, B] + [i_1, B] + [A, i_2] + [i_1, I_2] \tag{B.-6}$$

$\psi : \mathcal{G}$

the sums and products in $im(\psi) \subset$ form an algebra called the **enveloping algebra** of $\mathcal{G}$

• **epimorphism:** A surjective group homomorphism.

• **epsilon net:** A finite or infinite number of points on a metric space such that each point on the space is with a distance of ϵ of some point on the net.

• **equivalence classes:** We transfer our attention from objects and products of objects to consideration of equivalence classes of objects and the induced multiplication between these classes.

• **essentially self-adjoint:** An operator which has a unique self-adjoint extension is said to be essentially self-adjoint, having a canonically defined Friedrichs extension and canonical functional calculus.

• **ergodic:**

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \omega(\tau^t(A)B)dt = \omega(A)\omega(B), \tag{B.-6}$$

and weak-mixing

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} |\omega(\tau^t(A)B) - \omega(A)\omega(B)|^2 dt = 0. \tag{B.-6}$$

• **Euler's theorem:**

$$F - E + V \tag{B.-6}$$

• **extension of an operator:** An opeator B is an extension of an operator A if $D(A) \subset D(B)$ and $Bx = Ax$ for $x \in D(A)$.

• **extended diffeomorphisms:** Consider maps $\phi : \Sigma \to \Sigma$, that are continuous, invertible, and such that the map and its inverse are smooth everywhere, except, possibility, at a finite number of isolated points. The group formed by these maps is usually denoted $Diff^*$. (A strict diffeomorphism is a map that is smooth everywhere and whose inverse is also smooth everywhere).

See diffeomorphisms.

• **15-j symbols:** An elementary vertex of BF theory. A 4-simplex with 15 quantum numbers, the 10 irreducible representations associated with the 10 edges and the five interwiner labels of the 5 nodes (tetdrahedra) of the 4-simplex. See fig (B.8).

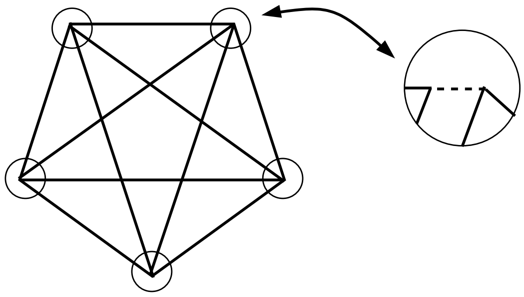


Figure B.8: 15JSymbol.

• **faithful:**

faithful representation: A representation is said to be faithful if Kerπ = $\{0\}$ and non-degenerate if $\pi(a)\psi = 0$ for all $a \in A$ implies $\psi = 0$.

The representation () is said to **faithful** if and only if π is a $*$-morphism between $\mathcal{A}$ and $\pi(\mathcal{A})$, i.e., if and only if ker $\pi = \{0\}$.

• **Fell's theorem:** From [11]: Let $(\mathcal{F}_1, \pi_1)$ and $(\mathcal{F}_2, \pi_2)$ be possibly unitary inequivalent representations of the Weyl algebra $\mathcal{A}$. Let $A_1, \ldots, A_n \in \mathcal{A}$ and let $\epsilon_1, \ldots, \epsilon_n > 0$. Let ω_1 be an algebraic state corresponding to a density matrix on $\mathcal{F}_1$. Then there exists a state ω_2 corresponding to a density matrix $\mathcal{F}_2$ such that for all $i = 1, \ldots, n$ we have

$$|\omega_1(A_i) - \omega_2(A_i)| < \epsilon_i. \tag{B.-6}$$

See physics glossary.

• **fibre bundles:** A bundle is a triplet $(E, \pi, \mathcal{M})$, where E and $\mathcal{M}$ are manifolds of some differentiability class and $\pi : E \to \mathcal{M}$ is the projection map. The inverse image $\pi^{-1}(x)$, $x \in \mathcal{M}$, is the fibre, F_x, over x. Fibre bundles are those bundles whose fibres over all of $\mathcal{M}$ are homoemorphic to a common space F, the typical fibre. They are the proper mathematical notion for introducing internal symmetries in a field theory.

• **fibre metric:** Let s and s' be sections over U_I. The inner product between s and s' at p is defined by

$$(s, s')_p := h_{IJ}(p) s^I(p) s'^J(p)$$

if the fibre is in $\mathbb{R}^k$. If the fibre is $\mathbb{C}^k$ we define

$$(s, s')_p := h_{IJ}(p)\overline{s^I(p)}s'^J(p).$$

• **field:** (b) A commutative division ring. Are the "number systems" in maths. *over a field* means $\alpha\mathbf{x} + \beta\mathbf{v}$.

• **filter:** A filter in a set X is a system $\mathcal{F}$ of non-empty subsets of X satisfying the conditions:

i) $A \cap B \in \mathcal{F}$ for all A and B in $\mathcal{F}$,
ii) if $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

• **first countable:** A topological space is first countable space if it has a countable if it has a countable base at each point.

• **first homotopy group:**

• **foliation:** A foliation consists of an integrable sub-bundle of a tangent bundle.

• **folium:** A state's **folium** is the set of states ω_ρ on $\mathcal{U}$ defined by

$$\omega_\rho := \frac{\mathrm{tr}_{\mathcal{H}_\omega}(\rho\pi_\omega(a))}{\mathrm{tr}_{\mathcal{H}_\omega}(\rho)} \tag{B.-6}$$

where ρ is a positive trace class operator on the GNS Hilbert space $\mathcal{H}_\omega$.

• **folium of states:** The folium of states is the set of states determined by the density matrices on the Hilbert space of the given representation.

• **forgetful functor:** MOSTLY TAKEN DIRECTLY FROM mathworld.wolfram.com

In many categories the objects are sets equipped with some kind of additional structure. For example the category of groups, where an object is a group, and an arrow is a group homomorphism $f : G_1 \to G_2$ from G_1 to G_2. However a category need not have structured sets as its objects.

In a mere set elements are either the same or different; and that's all.

A forgetful functor (also called underlying functor) is defined from a category of algebraic gadgets (groups, Abelian groups, modules, rings, vector spaces, etc.) to the category of sets. A forgetful functor leaves the objects and the arrows as they are, except for the fact they are finally considered only as sets and maps, regardless of their algebraic properties.

Other forgetful functors neglect only part of the algebraic properties, e.g., the commutative law when passing from Abelian groups to groups, or multiplication when passing from rings to Abelian groups.

• **free action:** The group action on a manifold (moving one point of the manifold to another in a way that has the structure of a group G) is said to be free if the every element that is not the identity of G has no fixed points.

• **free group:** see ???

• **free associative algebra:**

• **Fréchet space:** complete metric space???

• **Friedrichs extension:** quadratic form $Q_{\hat{\mathbf{M}}}$ whose closure is the quadratic form of a unique self-adjoint operator $\overline{\hat{\mathbf{M}}}$, called the Friedrichs extension of $\hat{\mathbf{M}}$.

Master constraint

The Friedrichs extension is a self-adjoint extension of a non-negative densely defined symmetric operator

Let A be a semi-bounded symmetric operator, that is,

$$q_A(\psi) = < \psi, A\psi > \geq \gamma \|\psi\|^2, \qquad \gamma \in \mathbb{R}.$$

Then there is a self-adjoint extension $\tilde{A}$ whihc is also bounded from below by γ and which satisfies $D(\tilde{A}) \subset = \mathcal{H}_{A-\gamma}$.

• **functional calculus:**

allows one to construct all kinds of operators.

finite dimensional Hilbert space $\mathcal{H}$

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

Define $\Psi : C(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ by

$$\Psi(f) = f(\lambda_1)P_1 + \cdots + f(\lambda_n)P_n.$$

It is not hard to see that Ψ is an isometric $*-$isomorphism into its range: i.e.,

$$\begin{array}{ll} (a) & \|\Psi(f)\| = \|f\|_\infty \\ (b) & \Psi(f+g) = \Psi(f) + \Psi(g) \\ (c) & \Psi(fg) = \Psi(f)\Psi(g) \\ (d) & \Psi(\overline{f}) = \Psi(f)^*. \end{array} \qquad \text{(B.-8)}$$

The process of passing from $f \in C(\sigma(T))$ to $f(T)$ is called functional calculus. It allows us to construct, for example, square roots, logs, and exponentials of operators.

• **functionals:**

Positive linear functionals correspond precisely to the set of density matrices.

• **functions:**

functions of compact support: function is non-zero only within a compact region - a function whose domain is a compact space. Interesting things about these are....

• **function spaces:**

Examples. Let $U \subset \mathbb{R}^n$ be an open set of $\mathbb{R}^n$.

(1) $P(U)$ is the space of all polynomials of n variables as functions on ω;

(2) $C(U)$ the space of all continuous functions on U;

(3) $C^k(U)$ the space of all functions with continuous partial derivatives of order k on U;

(4) $C^\infty(\omega)$ the space of all smooth (infinitely differentiable) functions on U.

• **functor:** Functors are the structure preserving assignments between categories.

A **contravariant functor** $F : \mathcal{C} \to \mathcal{V}$ between two categories

(i) maps every object $C \in \text{ob}\mathcal{C}$ to an object $F(C) \in \text{ob}\mathcal{V}$, and

(ii) every morphism $f : C_1 \to C_2$ in $\mathcal{C}$ to a morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathcal{V}$, such that

$$F(g \circ f) = F(f) \circ F(g), \quad F(\text{id}_C) = \text{id}_{F(C)}. \tag{B.-8}$$

A **covariant functor** $F : \mathcal{C} \to \mathcal{V}$ that maps every morphism in $\mathcal{C}$ to $F(f) : F(C_1) \to F(C_2)$, and

$$F(g \circ f) = F(g) \circ F(f), \quad F(\text{id}_C) = \text{id}_{F(C)}. \tag{B.-8}$$

Contravariant ones change the direction of arrows, the covariant ones preserve directions.

Examples:

Let A be a vector space and $F(C) = \text{Hom}(C, A)$, the linear maps from C to A.

• **Gauss-Bonnet theorem:**

Figure B.9: A contravariant functor.

$$\int_{\mathcal{M}} K dS = 2\pi\chi(M) \qquad \text{(B.-8)}$$

• **Gauss-Codaza equations:** The torsion is the anti-symmetric part of the connection and the curvature in terms of the connection:

• **Gaussian curvature:** a disk D_ϵ centered at the point $\mathbf{p}$ with area $A(D_\epsilon)$

$$K(\mathbf{p}) = \frac{12}{\pi} \lim_{\epsilon \to 0} \frac{\pi\epsilon^2 - A(D_\epsilon)}{\epsilon^4} \qquad \text{(B.-8)}$$

• **Gauss linking number:**

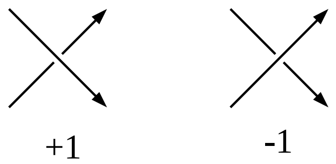


Figure B.10: Computing the Gauss linking number.

$$L = \frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\nu \epsilon_{\mu\nu\beta} \frac{(z^\beta - y^\beta)}{|z-y|^3} \qquad \text{(B.-8)}$$

Consider two closed loops γ and γ', as for example in Fig N.-19 . If we think of first loop γ to be a wire carrying a current I, then by law it will generate a magnetic field $\mathbf{B}$ around the closed curve γ'.

$$B[\alpha] = \int_{S^\alpha} B^a d^2 S_a = \oint_\alpha A_a dl^a, \qquad \text{(B.-8)}$$

n is the number of times that the current passes through the closed loop. Let us set $I = 1$, we have

$$L = n = \frac{1}{4\pi} \oint_{\gamma'} \mathbf{B}(x') \cdot d\mathbf{x}. \qquad \text{(B.-8)}$$

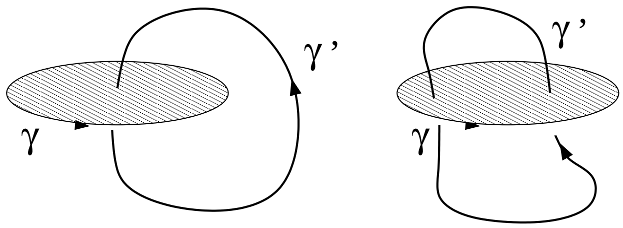


Figure B.11: Computing the Gauss linking number.

We can calculate the magnetic field $\mathbf{B}(x')$ produced by wire γ with the Biot-Savart law, Eq(B). Substituting this into N.-19, we get an explicit equation for the Gauss linking number:

$$L = \frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\nu \epsilon_{\mu\nu\beta} \frac{(z^\beta - y^\beta)}{|z-y|^3} \tag{B.-8}$$

started off knot theory. knot theory of atoms. Irony that that the basic constituents of the fundamental physical description of nature put forward by LQG are knots (well actually links - a link being a set of knots).

We say that two spin networks are **isotopic**. So by abstract spin networks we mean the isotopic type rather than a particular way of realizing the spin network in "space".

• **Gauss self-linking number:**

$$\frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_i} dy^\nu \epsilon_{abc} \frac{(z^c - y^c)}{|\gamma(s) - \gamma(t)|} \tag{B.-8}$$

Figure B.12: Gauss self-linking number.

• **Gel'fand map:** A map from the commutative $C*$ −algebra $\mathcal{A}$ onto the space of continuous functions on the spectrum of the algebra, $\mathcal{C}(\Delta(\mathcal{A}))$.

$$\begin{array}{rcl} Alg & \rightarrow & C(\Delta) \\ a & \mapsto & \overset{\vee}{a} \end{array} \tag{B.-8}$$

• **Gel'fand-Naimark-Segal construction:**

• **Gel'fand-Neimark theorem:** Every C^*-algebra with identity is isomorphic to the C^*-algebra of all continuous bounded functions on a compact Hausdorff space called the *spectrum* of the algebra.

• **Gel'fand spectral theorem:**

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with C.

• **Gel'fand topology:**

$$\begin{array}{rcl} Alg & \to & C(\Delta) \\ a & \mapsto & \check{a} \end{array} \tag{B.-8}$$

• **Gel'fand spectral theorem:**

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with C.

• **Gel'fand topology:**

weak* convergence of a weak* convergence of a sequence (rather, a generalization of a sequence called a net) of functionals. Then weak* convergence of (f_n) means that there is an $f \in X'$ such that $f_n(x) \to f(x)$ for all $x \in X$.

Every character is a bounded linear functional on $\mathcal{A}$, that is, $\Delta(\mathcal{A}) \subset \mathcal{A}'$. The Gel'fand topology on the spectrum of a unital, Abelean Banach algebra is the weak * topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$.

• **Gel'fand transformation:** Let $\mathcal{A}$ be a Banach algebra?? Given any $A \in \mathcal{A}$ we can define a function $\hat{A} : \Delta \to \mathbb{C}$ by

$$\hat{A}(h) = h(A). \tag{B.-8}$$

$\hat{A}$ is called the Gel'fand transform of A.

• **generalized functions:** distributions

• **generalized knot theory:** knots with intersections...

• **germs:** Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds and $x \in \mathcal{M}$. Consider all smooth mappings $f : \mathcal{U}_f \to \mathcal{N}$, where $\mathcal{U}_f$ is some open neighbourhood of x in $\mathcal{M}$. We say two such functions f, g are equivalent and we put $f \sim_x g$ if there exists an open neighbourhood $\mathcal{V}$, which contains x, such that $f|\mathcal{V} = g|\mathcal{V}$. This is an equivalence relation on the set of mappings considered. The equivalence class of a mapping f is called the germ of f at x, sometimes denoted by $\mathrm{germ}_x\, f$. The set of all these germs is denoted by C.

We may also consider the composition of germs: $\mathrm{germ}_{f(x)} \circ \mathrm{germ}_x\, f = \mathrm{germ}_x\, (g \circ f)$.

Germ of an edge: Let $x \in \Sigma$ be given. The germ $[e]_x$ of an entire analytic edge e with $b(e) = e(0) = x$ is defined by the infinite number of Taylor coefficients $e^{(n)}(0)$ in some parameterisation. The germ $[e]_x$ encodes the orientation of e and its knowledge allows us to reconstruct $e(t)$ from x up to reparametrisation due to analyticity.

• **Gleason's theorem:** Let H have dimension greater than 2. Then every countably additive probability measure on the lattice $L(H)$ has the form $\mu(P) = Tr(WP)$, for a density operator W on H.

Stanford encyclopedia of philosophy.

• **GNS construction:**

• **graph**

For a linear operator $A : X \to Y$ the set of points $\{x \in X, Ax \in Y\}$ is called the graph of the operator A.

See partial function.

• **greatest lower bound:** the greatest lower bound is usually called its **infimum** and denoted inf A.

• **Green's function:** Roughly, a Green's function $G(\vec{r}, t)$ is the solution of a differential equation subject to the initial condition $G(\vec{r}, t) = \delta(\vec{r})$.

• **Groenewold and van Howe theorem:**

• **group:** A collection G of objects g_i upon which we associate a 'multiplication' operation (techniqually know as a binary operation) which we write as " $\cdot$ ". By definition a group satisfies the following properties

(i) identity e such that $e \cdot g = g \cdot e = g$;
(ii) closed under multiplication i.e. for any $g_1, g_2 \in G$ their product is also an element of the group i.e., $g_1 \cdot g_2 \in G$;
(iii) every element has an inverse;
(iv) associativity $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$;

• **group homomorphism:** A homomorphism between two groups is, roughly speaking, a function between them which preserves the (respective) group operations. Let G_1, G_2 be groups. A function f that maps the elements of G_1 in to G_2 ($f : G_1 \to G_2$) is a homomorphism if and only if, for all $a, b \in G_1$ we have

$$f(ab) = f(a)f(b). \tag{B.-8}$$

$f : G_1 \to G_2$ is an isomorphism if it is a bijection (one-to-one and onto). We write $G_1 \cong G_2$. An isomorphism from a group G to itself is called an automorphism.

• **groupoid:** closure applying the binary operator to two elements of a given set S returns a value which is itself a member of S. Associativity, existence of identity, and inverse of each element are not required. An example of a groupoid is for a, b positive definite real numbers

$$a \star b = \sqrt{(a^2 + b)} \tag{B.-8}$$

There is no identity element because we require $a, b > 0$, and correspondingly no inverse. It is easy to check that associativity does not always hold:

$$\begin{aligned} a \star (b \star c) &= a \star (\sqrt{(b^2 + c)}) \\ &= \sqrt{(a^2 + \sqrt{(b^2 + c)})} \end{aligned}$$

and

$$\begin{aligned} (a \star b) \star c &= \sqrt{((a \star b)^2 + c)} \\ &= \sqrt{(a^2 + b + c)}, \end{aligned}$$

for these to be equal requires

$$\sqrt{(b^2 + c)} = b + c$$

or

$$b^2 + c = (b + c)^2 = b^2 + 2bc + c^2$$

or

$$c[c + (2b - 1)] = 0$$

or

$$c = 0 \quad or \quad c = 1 - 2b$$

so unless $b < 1/2$, c wont be positive definite.

• **group theory:**

• **Haag's theorem:** Two significant consequences of Haag's theorem are:

i) Scalar filed theories with different masses correspond to unitarily inequivalent representations of the Weyl algebra.

ii) Representations of interacting and free field theories are unitarily inequivalent and hence means that the interaction picture underlying perturbative QFT of Wightman fields strictly speaking does not exist, it only exists if there are no interactions.

The formal statement of Haag's theorem is:

Suppose that

(1) two weakly continuous and irreducible representations $(\pi_I, \mathcal{H}_I), I = 1, 2$ of the Weyl algebra $\mathfrak{A}$ of a scalar field theory are given,

(2) the Euclidean group E of spacial transaltions and rotations is implemented unitarily and weakly continuously by representations u_I on $\mathcal{H}_I$ such that $u_I(e)\pi_I(a)u_I^{-1}(e) = \pi_I(\alpha_e(a))$ for all $e \in E, a \in \mathfrak{A}$ and

(3) there is a unique Euclidean invariant state $\Omega_I \in \mathcal{H}_I$, that is, $u_I(e)\Omega_I = \Omega_I$.

If the two representations of the Weyl algebra are unitary equivalent, that is, there exista unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that $W\pi_1(a)W^{-1} = \pi_2(a)$ for all $a \in \mathfrak{A}$, then $Wu_1(a)W^{-1} = u_2(e)$ for all $e \in E$ and $V\Omega_1 = c\Omega_2$ where c is a complaex number of modulus one.

• **Haar measure:** For any compact group G the Haar measure is the (unique) measure dU that is group invariant.

Let G be a locally compact abelian group. A Haar measure on G is a positive regular Borel measure μ having the following two properties:

(1) $\mu(E) < \infty$ if E is compact;
(2) $\mu(E + x) = \mu(E)$ for all measurable $E \subset G$ and all $x \in G$.

One can prove that the Haar merasure always exists and that it is unique up to multiplication by a positive constant.

• **Hahn-Banach theorem:** The Hahn-Banach theorem is a central tool in functional analysis. It allows for the extending bounded linear functionals defined on a subspace of some vector space to the full space, in a norm-preserving manner, and shows that there are "enough" continuous linear functionals defined on every normed vector space to make the study of dual spaces interesting.

• **Hall transform:**

• **Harmonic polynomials:** homomgeneous polynomials P annihilated by the Laplace operator on $\mathbb{R}^d$:

$$\Delta P(x^1,\ldots,x^d)=0, \tag{B.-12}$$

for

$$\Delta=\partial_1^2+\cdots+\partial_d^2 \tag{B.-12}$$

where $\partial_i=\partial/\partial x_i$.

• **Hausdorff (or T_2):** A topological space is called Hausdorff if and only if for any two distinct points x and y, $x\neq y$, there are open sets O_1, O_2 such that $x\in O_1$, $y\in O_2$, and the two open sets do not overlap, i.e., $O_1\cap O_2=\emptyset$.

• **heat kernal:**

the coherent states of the simple harmonic oscillator coherent states can be obtained as analytic continuation of the heat kernal on $\mathbb{R}^n$:

$$\psi_z^t(x)=e^{-t\Delta}\delta_{x'}(x)\Big|_{x'\to z}\quad x\in,z\in\mathbb{C}, \tag{B.-12}$$

the Laplacian Δ playing the role of a *complexifier*???

It was shown by Hall [243] that coherent states on a connected compact Lie group G can analogously be defined as an analytic continuation of the heat kernel

$$\psi_g^t(x)=e^{-t\Delta_G}\delta_{h'}^{(G)}(h)\Big|_{h'\to u}, \tag{B.-12}$$

to an element u of the complexification $G^{\mathbb{C}}$ of G.

• **heat kernal measure:**

$$\frac{d\mu}{dt}=\frac{1}{4}\Delta K_C\mu_t \tag{B.-12}$$

let μ_t denote the associated heat kernel measure

$$d\mu_t(g):=\mu_t(g)dg. \tag{B.-12}$$

• **Heisenberg group:** A matrix representation of the Heisenberg Lie algebra is

$$m(p,q,t) = \begin{pmatrix} 0 & p_1 & \cdots & p_n & t \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It is easily verified that

$$m(p,q,t)m(p',q',t') = m(0,0,pq')$$

and so

$$[m(p,q,t), m(p',q',t')] = m(0,0,pq' - qp').$$

Using

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

we have

$$\exp m(p,q,t) \exp m(p',q',t') = \exp m\big(p+p', q+q', t+t' + \frac{1}{2}(pq' - qp')\big).$$

One identifies a point $X \in \mathbb{R}^{2n+1}$ with the matrix $e^{m(X)}$, and makes $\mathbb{R}^{2n+1}$ into a group with group law

$$(p,q,t)(p',q',t') = \big(p+p', q+q', t+t' + \frac{1}{2}(pq' - qp')\big).$$

This is called the Heisenberg group and is denoted $\mathbf{H_n}$. The element $(0,0,0)$ is the identity and the inverse of the element (p,q,t) is $(-p,-q,-t)$.

• **Hellinger-Toeplitz theorem:** If A is a symmetric operator whose domain of dependence $D(A)$ is the whole of the Hilbert space $\mathcal{H}$, $D(A) = \mathcal{H}$, then A is bounded. Note that symmetric everywhere defined operators are self-adjoint, so this theorem can also be stated that a self-adjoint operator is bounded. This shows that unbounded operators are not everywhere defined on the Hilbert space.

• **Heyting algebra:** ??? [368] Internal observables satisfy a Heyting algebra, which is a weak version of the Boolean algebra of ordinary observables.

Open sets are the standard example of a Heyting algebra. Given an open set $U \subset O$, the complement of U, which we call U^c, is a closed set. If we want an algebra of open sets, we need, instead of U^c, to use the interior of the set complement of U, $Int(U^c)$. But then, clearly, $U \cap Int(U^c) \subset O$ since the closure of U has been left out. (example comes from [367]).

A poset $\mathbf{H}$ is a Heyting algebra if we have elements

$$T \in \mathbf{H}, \qquad \perp \in \mathbf{H} \tag{B.-12}$$

and operations

$$T \in \mathbf{H} \tag{B.-12}$$

See sections **??**.

• **higher-dimensional algebra:** [401]

• **Hilbert-Schmidt operators:** The space of Hilbert-Schmidt operators consists of those compact operators A such that the trace $\mathrm{Tr}(A^*A)$ exists. These operators naturally appear in Bogol'ubov transformations.

Alternative definition: A degenerate operator A can be written in the for $A = UT$, where U is an isometric operators and T is a positive deinite operator. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T. the space of continuous operators is obtained from the space of degenerates operators by completeing the latter in the norm

$$\|A\|_2 = \sqrt{\sum_{k=1}^{n} \lambda_k^2}$$

• **Hilbert space:** A complete inner product space which is a complete metric space with respect to the metric induced by its inner product (compare to a Banach space).

(see also a pre-Hilbert space).

Hilbert space completion: Consider the completion of an inner product space V as the metric space completion, $\mathcal{H}$, of V by taking equivalence classes of Cauchy sequences in V. It can be shown that the inner product structure of V naturally extends to $\mathcal{H}$ in such a way as to provide $\mathcal{H}$ with the structure of a Hilbert space, with V naturally identified with a dense subspace of $\mathcal{H}$. See Reed and Simon.

• **Hölder inequality:** integral (or sum) inequality

$$\left(\int |fg|\right) \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q} \tag{B.-12}$$

where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$.

$$\left(\sum_i |f_i g_i|\right) \leq \left(\sum_i |f_i|^p\right)^{1/p} \left(\sum_i |g_i|^q\right)^{1/q}. \tag{B.-12}$$

• **homeomorphic:** Related by a homeomorphism.

• **homeomorphism:** A one-to-one correspondence that is continuous in both directions between the points of two topological spaces.

• **homogeneous function:** A function $f(x_1, \ldots, x_n)$ is a homogeneous function of degree D if

$$f(\rho x_1, \ldots, \rho x_n) = \rho^D f(x_1, \ldots, x_n) \tag{B.-12}$$

Define $x'_1 = \rho x_1, \ldots x'_n = \rho x_n$. Then differentiating both sides of (B) by ρ we find

$$\begin{aligned} D\rho^{D-1} f(x_1, \ldots, x_n) &= \frac{\partial f}{\partial x'_1}\frac{\partial x'_1}{\partial \rho} + \cdots + \frac{\partial f}{\partial x'_n}\frac{\partial x'_n}{\partial \rho} \\ &= \left[x_1 \frac{\partial}{\partial(\rho x_1)} + x_2 \frac{\partial}{\partial(\rho x_2)} + \cdots + x_n \frac{\partial}{\partial(\rho x_n)}\right] f(\rho x_1, \ldots, \rho x_n) \end{aligned}$$

Setting $\rho = 1$ gives

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) = D f(x_1, \ldots, x_n). \tag{B.-14}$$

• **homeomorphism:** A homeomorphism is a one-to-one correspondance that is continuous in both directions between two topological spaces. It is an equivalence relation and preserves topological properties. If it also preserves distances it is an isometry. See also diffeomorphism.

• **Hom(A,B):** $\mathrm{Hom}(A, B)$ denotes the collection of morphisms from A to B.

• **homogeneous simultaneous equations:**

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z + d_1 &= 0 \\
a_2 x + b_2 y + c_2 z + d_1 &= 0 \\
a_3 x + b_3 y + c_3 z + d_1 &= 0
\end{aligned} \tag{B.-15}
$$

$$
\begin{aligned}
a_1 \tilde{x} + b_1 \tilde{y} + c_1 \tilde{z} &= 0 \\
a_2 \tilde{x} + b_2 \tilde{y} + c_2 \tilde{z} &= 0 \\
a_3 \tilde{x} + b_3 \tilde{y} + c_3 \tilde{z} &= 0
\end{aligned} \tag{B.-16}
$$

$$
X = \frac{\tilde{x}}{\tilde{z}}, \quad Y = \frac{\tilde{y}}{\tilde{z}} \tag{B.-16}
$$

$$
\begin{aligned}
a_1 X + b_1 Y + c_1 &= 0 \\
a_2 X + b_2 Y + c_2 &= 0 \\
a_3 X + b_3 Y + c_3 &= 0
\end{aligned} \tag{B.-17}
$$

These have the trivial solution $X = 0, Y = 0, Z = 0$. If the condition

$$
\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0 \tag{B.-17}
$$

is met there exist non-trivial solutions to the homogeneous equations. In this case we can start with a solution to the non-homgeneous system of equations equations and obtain another one by adding arbitrary linear combination of the solutions to the homogeneous system of equations:

$$
\begin{aligned}
x &\to x + v_x X \\
y &\to y + v_y Y \\
z &\to z + v_Z Z
\end{aligned} \tag{B.-18}
$$

• **homomorphism:** A homomorphism θ is a mapping from one algebraic structure to another under which the structual properties of its domain are preserved in its range in the sense that if $*$ is te operation on the domain, and $\circ$ is the operation on the range, then

$$\theta(x * y) = \theta(x) \circ \theta(y).$$

In particular, a group homomorphism is a mapping θ such that both domain and range are groups, and

$$\theta(xy) = \theta(x)\theta(y)$$

for all x and y in the domain. A ring homomorphism is a mapping θ from one ring to another such that

$$\theta(x+y) = \theta(x) + \theta(y) \quad \text{and} \quad \theta(xy) = \theta(x)\theta(y).$$

• **hoop group:** 'holonomy equivalence class of a loop based at x_0'.

• **Hopf algebra:** [191], [193], [192]

Algebra

multiplication $\mu : A \otimes A \to A$ $m(a \otimes b) = ab$. m is **associative**, $(ab)c = a(bc)$.

$$[m(m \otimes \mathbf{1})](a \otimes b \otimes c) = [m(\mathbf{1} \otimes m)](a \otimes b \otimes c) \tag{B.-18}$$

mathematicians write this in the short-hand notation

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m) \tag{B.-18}$$

and depict it as a diagram (**??**(a)).

unit η sends every number to the same number times the identity element, i.e.

$$\eta(k) := \mathbf{1}k, \quad \text{for all } k \in \mathbb{C}. \tag{B.-18}$$

$$m(a \otimes \eta(k)) = ak = ka = m(\eta(k) \otimes a), \quad \text{for all } a \in A, \text{ and for all } k \in \mathbb{C}. \tag{B.-18}$$

$$m \circ (\eta \otimes \mathrm{id})(a) = a = m \circ (\mathrm{id} \otimes \eta)(a) \tag{B.-18}$$

Coalgebra

comultiplication $\Delta : C \to C \otimes C$. Δ is **coassociative**.

$$(\mathbf{1} \otimes \Delta)\Delta = (\Delta \otimes \mathbf{1})\Delta \tag{B.-18}$$

counit

Coalgebra

A bialgebra structure on a vector space $A = C$ is a quadruple of objects $(\mu, \eta, \Delta, \epsilon)$ which satisfy all of the commutative diagrams as well as the following compatibility equations:

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1) = 1 \otimes 1, \quad \epsilon(hg) = \epsilon(h)\epsilon(g), \quad \epsilon(1) = 1, \quad \text{for all } g, h \in A. \tag{B.-18}$$

linear antipode $S : H \to H$

$$\mu(S \otimes \mathrm{id}) \circ \Delta = \mu(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{B.-18}$$

the antipode is a kind of inverse.

In component form using generators:

$$\begin{aligned} e_i e_j &= m^k_{ij} e_k \\ \Delta(e_k) &= \mu^{ij}_k e_i \otimes e_j \end{aligned} \tag{B.-18}$$

$$\epsilon_i \mu^{ij}_k = \delta^j_k \tag{B.-18}$$

Definition: Let C and D be coalgebras, with comultiplication Δ_C and Δ_D, and counits ϵ_C and ϵ_D respectively.

(i) A map $f : C \to D$ is a **coalgebra morphism** if $\Delta_D \circ f = (f \otimes f)\Delta_C$.

(ii) A subspace $I \subseteq C$ is a **coideal** if $\Delta \subseteq I \otimes C + C \otimes I$ and if $\epsilon(I) = 0$.

if I is a coideal then, the quotient C/I is a coalgebra with comultiplication induced from Δ. Proof in Appendix Q.

• **Hopf fibration:**

Hopf fibration of S^3. The base space is a 2-dimensional sphere S^2 and fibres are circles S^1

• **horizontal distribution:** Given a principal fibre bundle a horizontal distribution is the assignment of a subspace $V_p(P)$ to the tangent space $T_p(P)$ at each point p of P tat is tangent to the fibre.

• **horizontal vector fields:** Horizontal vector fields are fields whose flow lines move from one fibre into another.

• **hyperbolic differential equations:** wave equation. Consider the partial differential equation

$$g^{ab}\nabla_a\nabla_b\phi + A^a\nabla_a + B\phi + C = 0 \tag{B.-18}$$

where A^a is an arbitrary smooth vector field, B and C are arbitrary smooth functions, and g_{ab} is an arbitrary smooth Lorentz metric such that the spacetime $(\mathcal{M}, g_{ab})$ is globally hyperbolic. A second order partial differential equation is said to be hyperbolic if and only if it can be expressed in the form

• **hyphs:** Hyphs are a type of collections of loops which have the advantage of being independent of the differentiablity category of the graphs under consideration and in particular includes the analytical and smooth category.

• **ideal:** An **ideal** a in A is a subset such that

(i) a is a subgroup of A regarded as a group under addition;

(ii) $a \in a,\ r \in A \Rightarrow ra \in A$. ??????????????? left-ideal

right-ideal

two-sided ideal

Ideals play the same role in Lie algebras as the normal subgroups play in Lie group theory.

The set of commutators of a Lie algebra, $\mathcal{G}$, denoted by $[\mathcal{G}, \mathcal{G}]$, is a subalgebra of $\mathcal{G}$.

It is also a two-sided ideal of $\mathcal{G}$, for any $A, A_1, A_2 \in \mathcal{G}$,

$$[[A_1, A_2], A] = [A_3, A] \in [\mathcal{G}, \mathcal{G}], \tag{B.-18}$$

where $A_3 = [A_1, A_2]$.

$$[A, [A_1, A_2]] = [A, A_3] \in [\mathcal{G}, \mathcal{G}], \tag{B.-18}$$

The set $[\Delta A, \Delta A_2]$. Two-sided coideal

The ideal of a Lie algebra $[h_i, g_k] = \sum a_{ikl}h_l$ for all $g_i \in \mathcal{L}(G)$

• **immersion:** A differentiable map $\phi : \mathcal{M} \to \mathcal{N}$ between finite-dimensional manifolds $\mathcal{M}, \mathcal{N}$ is called an immersion when ϕ has everywhere rank $\dim(\mathcal{M})$. An immersion need not be injective (se fig B.13) but when it is,it is called an embedding.

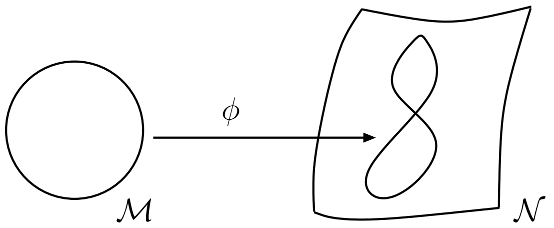


Figure B.13: .

• **implicit function theorem:** The simplest version of the implicit function theorem can be stated as follows. Let f be a continuous real-valued function on an open subset of $\mathbb{R}^2$ that contains the point (a, b), with $f(a, b) = 0$. Suppose that $\partial f/\partial y$ exists and is continuous on the given open subset and that $\partial f/\partial y(a, b) \neq 0$. Then there exist open intervals $U, V \in \mathbb{R}$, with $a \in U$ and $b \in V$, such that there exists a unique function $\rho : U \to V$ such that

$$f(x, \rho(x)) = 0$$

for all $x \in U$, and such that this function is continuous.

• **inclusion map:** IF $U \subseteq V$, the inclusion map i sends U to V, i.e., $i : U \to$.

• **if and only if:** This means necessary and sufficient. A statement is made of the form: "A is true if and only B is true". To establish A is true only if B is true, we assume A and then show it follows that B must be true. To establish A is true if B is true, we assume B and then show it follows that A must be true.

• **Infeld-van der Waerden symbols:**

• **infinite tensor product:**

• **injective:** A function that is one-to-one. Equivalently, a function is injective when no two distinct inputs give the same output.

• **inner product space:** vector space equipped with an inner product

• **insertion operator:** Given an $r-$form

$$\omega = \frac{1}{r!}\omega_{\nu_1\nu_2\ldots\nu_r} x^{\nu_1} \wedge x^{\nu_2} \cdots \wedge x^{\nu_r}, \tag{B.-18}$$

the insertion operator is defined by the operation

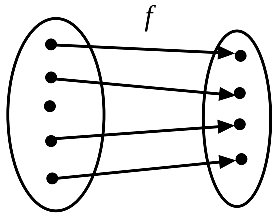


Figure B.14: injective

$$i_X\omega := \frac{1}{(r-1)!}\omega_{\nu\nu_2\ldots\nu_r}X^{\nu}\wedge x^{\nu_2}\cdots\wedge x^{\nu_r}. \tag{B.-18}$$

• **integrable distribution:** A distribution on a manifold is said to be integrable if at least locally, there is a foliation of M by submanifolds such that V_x is the tangent space of the submanifold containing the point x.

• **interal subgroup:** Let G be a Lie group. An integral subgroup of G is a subgroup H with a connected Lie group structure such that the canonical injection of H into G is an immersion.

• **interchange of limit operations:** If a sequence of Riemann integral real-valued functions $f_1(x), f_2(x), \ldots$ converges to the function $f(x)$, can we assert that

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx \tag{B.-18}$$

is true?

If a series of

$$\frac{d}{dx}\lim_{n\to\infty}\sum_{n=1}^{N} e_n(x) = \lim_{n\to\infty}\sum_{n=1}^{N}\frac{d}{dx}e_n(x) \tag{B.-18}$$

• **interwiner:** A trivalent interwiner is a map $I : V^i\otimes V^j\otimes V^k \to \mathbb{C}$ invarinat under the diagonal action of the group on the tensor product.

• **intuituinistic logic:** Comes from constructive mathematics: No proof by contradiction allowed, i.e. *"for all x, either x or not x*

• **invariant polynomial:** consider polynomials in n^2 variables. Let us label these n^2 variables as the entries of an $n-$dimensional matrix A_{ij}, We write for the polynomial: $P(A)$. The polynomial $P(A)$ is said to be **invariant** when

$$P(gAg^{-1}) = P(A) \tag{B.-18}$$

for all $g \in GL_n(C)$.

For example

$$\det\left(\mathbf{1} + \frac{\lambda}{2\pi i}A\right) = \sum_{n=0}^{N} \lambda^n P_n(A) \tag{B.-18}$$

the coefficients $P_n(A)$ are invariant polynomials of degree n in A.

$$\begin{aligned}
\sum_{n=0}^{N} \lambda^n P_n(gAg^{-1}) &= \det\left(\mathbf{1} + \frac{\lambda}{2\pi i}gAg^{-1}\right) = \det\left(g(\mathbf{1} + \frac{\lambda}{2\pi i}A)g^{-1}\right) \\
&= \det(g)\det\left(\mathbf{1} + \frac{\lambda}{2\pi i}A\right)\det(g^{-1}) \\
&= \sum_{n=0}^{N} \lambda^n P_n(A).
\end{aligned} \tag{B.-19}$$

• **invariant subspace:** a subspace $\mathcal{M}$ is invariant under a set of operators if $A\psi$ is in $\mathcal{M}$ for every operator A in the set and every vector ψ in $\mathcal{M}$.

• **inverse function theorem:**

• **irreducible representation:** An representation of operator relations on $\mathcal{M}$, for example the Weyl relations, is irreducible if no proper subspace of $\mathcal{M}$ is invariant under a set of operator relations. Equivalently, given any $\Psi \in \mathcal{M}$, the span of all vectors under the operator relations forms a dense subspace of $\mathcal{M}$. The representation should be irreducible on physical grounds otherwise we have superselection sectors implying that the physically relevant information is already contained in a closed subspace.

• **ISO:**

• **isometrically isomorphic:** Let N and N' be normed linear spaces. These spaces are said to be isometrically isomorphic if the linear transformation T is a one-to-one from N to N' is a one-to-one such that

$$\|x\| = |\ T(x)\| \tag{B.-19}$$

for all x in N. So that the normed linear space N' is essentially the same as N.

• **isometric operator:** An operator V defined on the whole of a Hilbert space $\mathcal{H}_1$ and mapping $\mathcal{H}_1$ on to the whole of another Hilbert space $\mathcal{H}_2$ is said to be isometric if, for all $f, g \in \mathcal{H}_1$,

$$(Vf, Vg)_2 = (f, g)_1,$$

where $(.\,,.\,)_1$ and $(.\,,.\,)_2$ denote the inner product on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

In particular, $\mathcal{H}_1$ and $\mathcal{H}_2$ may be subspaces of a single space (the subscrips on the inner products are then unnecessary).

A unitary operator is a particular case of an isometric operator; this case occurs if $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide.

• **isomorphic:** Two groups G and G' are said to isomorphic if their elements can be put into one-to-one correspondence which is preserved under multiplication.

• **isomotopic:** If two objects can be deformed into each other they are said to be isomotopic. For example knots that can be deformed into each other are called isomotopic.

• **isomotopy:** There are two kinds regular and ambient isotopy.

• **Jones polynomial:** The Jones polynomial is defined by:

its value on the unknot,
its value on the disjoint union of a knot with the unknot,
and a "Skein relation".

value on the unknot is chosen to be 1:

$$p(U) = 1. \tag{B.-19}$$

Let $U \sqcup K$ be the disjoint union of U and K

$$p(U \sqcup K) = -(x + x^{-1})p(K). \tag{B.-19}$$

$$x^{-2}p(K^+) - x^2 p(K^{-1}) + (x^{-1} - x)p(K^0) = 0. \tag{B.-19}$$

• **Jones-Witten invariant:** won Witten his Fields medal.

• **Kauffman bracket:**

• **kernel:** The kernel of a group homomorphism $\varphi : G \to H$ is defined by

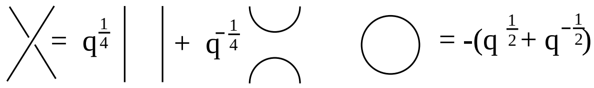


Figure B.15: Skien relation for the Kauffman bracket.

$$\ker\varphi := \varphi^{-1}(\{e_H\}) = \{x \in G : \varphi(x) = e_H\}.$$

• **Killing vector field:** isometries of the spacetime.

• **Kirby calculus:** must be related through sequence of operations. reminiscent of the Reiemeister moves.

• **knots and knot theory:** ... The formal statement of this intuitive idea is: a knot is a smooth embedding of a circle in three-dimensional Euclidean space. Two knots are said to be topologically equivalent if one can be deformed continuously into the other, without crossings.

Knots can be displayed as projections onto a 2-d plain surface.

Two knots or links are topologically equivalent if they can be transformed into each other by repeated use of the three Reidemeister moves.

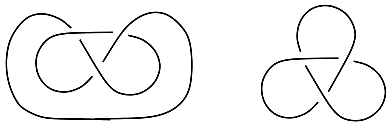


Figure B.16: knots Reidemeister moves.

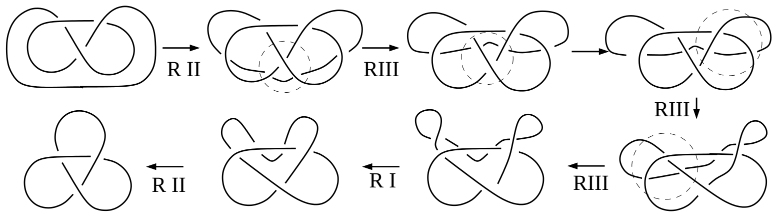
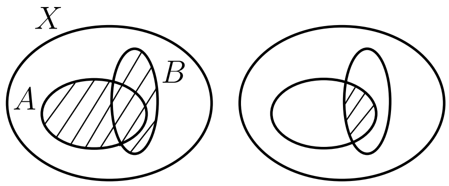


Figure B.17: knots Reidemeister moves.

The aim of knot theory is to characterize knots by a topological invariant.

• **lattice:** A lattice (a mathematical term) is a partially ordered set $(L, \leq)$ such that any two elements a and b possess a minimum in L, denoted $a \wedge b$ ("meet") and a maximum in L, denoted $a \vee b$ ("join"). That is, there exists respectively an element $a \wedge b$ satisfying $a \wedge b \leq a$, $a \wedge b \leq b$ such that any other element c which smaller than both a and b satisfies

$c \leq a \wedge b$, and one element satisfying $a \vee b \geq a$, $a \vee b \geq b$ and any other element c with the same property is greater, i.e., $a \vee b \geq c$. For examples of lattices see section **??**.



(a) (b)

Figure B.18: If $A \subseteq B$ then $A \leq B$. partially ordered set of subsets of X is a lattice. (a) Their union $A \cap B$ is the l.u.b. (b) $A \cup B$ is the g.l.b.

a lattice is called complete if and

Orthomodular lattice

$$(a \vee (a^{\perp} \wedge (a \vee b))) = (a \vee b) \tag{B.-19}$$

• **least upper bound:** 3 is an upper bound and its least upper bound the least upper bound is usually called its **supremum** and denoted sup A.

• **Lebesgue integral:** The Reimann integral doesn't deal with all functions we need for our purposes in formulating quantum mechanics and quantum field theory. Disturbing fact that the limit function $f(x) := \lim n \to \infty f_n(x)$ of a sequence of continuous functions $\{f_n(x)\}$ can be discontinuous. A function $\mu(x)$ we would hope has the following properties:

$$\begin{aligned} \mu((a,b]) &= b - a \\ &= \end{aligned} \tag{B.-19}$$

Instead of splitting the integration domain into small parts, we decompose the range of the function (see Fig(J.14)).

$$\sum_{n}^{N} c_n \mu(f^{-1}(J_n)) \tag{B.-19}$$

maximal invariant subspace:

• **left invariant vector field:**

$$g(t+s) = g(t)g(s) \tag{B.-19}$$

Differentiating with respect to s and setting $s = 0$,

$$g'(t) = g(t)g'(0) \tag{B.-19}$$

$$g(t) := L_g \exp(tX_e) = g \exp(tX_e) \tag{B.-19}$$

Or the push-forward of the vector $X(g_e)$ at g_e by multiplication on the left by any g produces a vector $g_*[X(g_e)]$ at gg_e that coincides with the vector $X(gg_e)$ already at that point. So it is the natural definition of a 'constant' vector field on G.

• **left invariant form:** A differential $p-$form ω is called left invariant provided

$$L_x^* \omega = \omega.$$

If ω_0 is any given $p-$form at e, then a left invariant form is defined by

$$\omega_x = L_{x^{-1}}^* \omega_0.$$

One of the most important applications of left invariant forms is in the theory of connections in fibre bundles - featuring in theoretical physics in relation to the general mathematical framework for Yang-Mills theories.

• **Lie derivative:** Generates infinitesimal active difffeomorphisms.

• **Lie Group:**

one definition: A **Lie group** G is a group which is also a smooth manifold such that the multiplication $G \times G \to G$, $(a, b) \mapsto ab$,and the inverse $G \to G$, $a \mapsto a^{-1}$, are smooth.

• **limit curve lemma:** Introduce a background Riemannian (positive definite) metric on spacetime $\mathcal{M}$. A future inextendible causal curve will have infinite length to the future, as measured in the metric h, parameterization defined on the interval $[0, \infty)$. The limit curve lemma can then be stated:

Let $\gamma_n : [0, \infty) \to \mathcal{M}$ be a sequence of future inextendible causal curves, parameterized with respect to h-arc length, and suppose that $p \in \mathcal{M}$ is an accumulation point of the sequence $\{\gamma_n(0)\}$. Then there exists a future inextendible C^0 causal curve $\gamma : [0, \infty) \to \mathcal{M}$ such that $\gamma(0) = p$ and a subsequence $\{\gamma_m\}$ which converges to γ uniformaly with respect to h on compact subsets of $[0, \infty)$.

There are analogous versions of the limit curve lemma for past inextendible, and (past and future) inextendible causal curves.

• **linear:**

(i) $\alpha(x+y) = \alpha x + \alpha y$;

(ii) $(\alpha+\beta)x = \alpha x + \beta x$;

(iii) $(\alpha\beta)x = \alpha(\beta x)$;

(iv) $1 \cdot x = x$

A linear space is called a **real** linear space or a complex linear space according to whether the scalars are the real numbers or complex numbers.

Some jargon: A complex vector space will often be referred to as a vector space over *over* the complex numbers. This phrase isn't just reserved for vector fields, and will also be used in reference to say rings, groups. Also, the scalars may not just refer to numbers, there are more general "number systems" used in mathematics called *fields*. The real and complex numbers are special cases of fields.

see physics glossary.

• **linear analysis:**

Uniform boundedness

interior mapping principle

Hahn-Banach theorem

• **line bundle:** A line bundle L is a complex vector bundle with one-dimensional fibres.

• **links:**

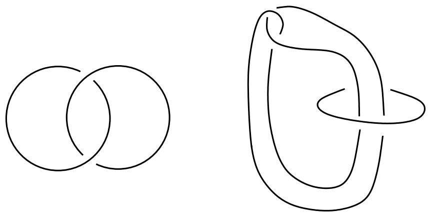


Figure B.19: links Whitehead link.

• **link invariants:**

invariants for three-manifolds Edward Witten topological field theory Chern-Simons

$$\int \mathcal{D}A e^{\int \mathcal{L}_{CS}} W_\alpha[A] \tag{B.-19}$$

Knots and links can be obtained by the closure of braids (Alexander theorem)

• **linear independence:**

$$\sum_{i=1}^{N} a_i \mathbf{e_i} = 0 \implies \tag{B.-19}$$

• **Lioville's theorem:** A bounded entire function is constant.

• **locally compact:** A topological space is said to be *locally compact* if every point $x \in X$ has an open neighbourhood whose closure is compact.

• **locally finite:**

• **locally trivial:** Given a fibre bundle P with typical fibre F and base space $\mathcal{M}$. Locally trivial means that for each $x \in \mathcal{M}$, there is a neighbourhood U of x and an isomorphism,

$$\phi : \pi^{-1}(U) \to U \times F,$$

sending each fibre $\pi^{-1}(x)$ to $\{x\} \times F$. Intuitively, a bundle looks locally as a product of the base manifold and the fibre. We call ϕ a local trivialisation. If the bundle space E is globally $\mathcal{M} \times F$ the bundle is said to be trivial.

• **logic:**

Conjunction

$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

$$\frac{p}{p \vee q} \qquad \frac{q}{p \vee q} \qquad \frac{q \vee p \qquad p \to r \qquad q \to r}{r} \tag{B.-19}$$

Addition

$$\begin{array}{l} p \\ \hline p \vee q \end{array}$$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

Conjunction

$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$

$$\frac{p \qquad q}{p \wedge q} \tag{B.-19}$$

Modus ponens

$$\begin{array}{l} p \\ p \to q \\ \hline q \end{array}$$

Modus tollens

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \neg p \end{array}$$

Hypothetical syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

See intuituinistic logic. See sections **??** and **??**.

• **Mandelstam identity:** $SL(2,\mathbb{C})$ A satisfies $\det A = 1$. for $A, B \in SL(2,\mathbb{C})$

$$\mathrm{tr}A\mathrm{tr}B = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) \tag{B.-19}$$

$SU(2) \subset SL(2,\mathbb{C})$ so the identity. Holonomies H_α are $SU(2)$ matrices.

• **manifold:** In simply terms, a manifold is a space $\mathcal{M}$ which locally looks like an n-dimensional Euclidean space $\mathbb{R}^n$.

Formally, a topological space $\mathcal{M}$ is an $n-$dimensional manifold if there is a locally finite open cover, $\{U_\lambda : \lambda \in \Lambda\}$, of $\mathcal{M}$ such that, for each λ, there is a map ϕ_λ that maps U_λ homeomorphically onto an open subset of $\mathbb{R}^n$.

(i) differentiable (or smooth manifolds, on which one can do calcus;

(ii) Riemannian manifolds, on which distances and angles can be defined;

(iii) symplectic manifolds, which serve as the phase space of dynamical systems;

(iv) 4D pseudo-Riemanniam manifolds which are used in general relativity .

• **maximal atlas:** We take as an example the definition of a maximal atlas with the $C^\infty-$property. Two charts ϕ_1, ϕ_2 are $C^\infty-$related if both the map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and its inverse are $C^\infty-$related. A collection of related charts such that every point of $\mathcal{M}$ lies in the domain of at least one chart forms an atlas. The collection of all such $C^\infty-$related charts forms a maximal atlas.

• **maximal ideal:** A *maximal left ideal* in A is a proper left ideal which is not properly contained in any other proper left ideal.

• **measure:** A measure

• **measurable functions:** Let X, Y be metric spaces, and let $f : X \to Y$ be a function.

A function $f : E \to \mathbb{R}$ is (Lebesgue-)measurable if for any interval $I \subseteq \mathbb{R}$

$$f^{-1}(I) = \{x \in \mathbb{R} \ : \ f(x) \in I\} \in \text{ collection of measurable sets.????} \tag{B.-19}$$

• **measure space:** A measure space $(\Omega, \mathcal{B}, \mu)$ is a set Ω together with a $\sigma-$algebra $\mathcal{B}$ of subsets of Ω and μ is $\sigma-$additive, that is,

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) = \sum_{n=1}^{\infty} \mu(U_n) \tag{B.-19}$$

for all disjoint measurable sets U_n.

• **minimal loop:** Given a graph γ and a vertex v of this graph, and two different edges e and e' outgoing from v, a loop $\alpha(\gamma, v, e, e')$ within γ with outgoing along e and incoming along e' is said to be minimal if there is no other loop within γ with the same properties and fewer edges tranaversed.

• **Minkowski's inequality:**

$$\left(\sum_i |x_i + y_i|^2\right)^{1/2} \leq \left(\sum_i |x_i^2\right)^{1/2} \left(\sum_i |y_i|^2\right)^{1/2} \tag{B.-19}$$

• **module:** Modules are also referred to as representations: for instance, representations of a group are essentially the same as modules over the group algebra. Even if you have

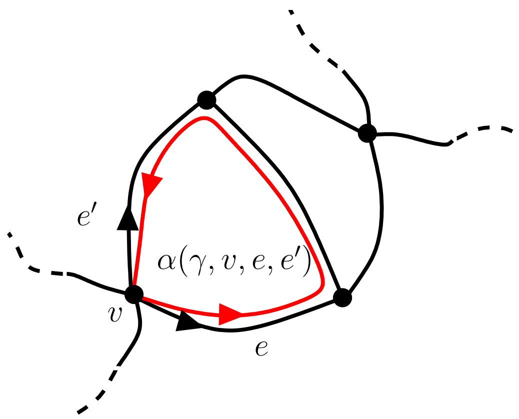


Figure B.20: minimal loop.

not come across the term "module" you surely have come across some examples. Vector spaces are (rather simple) examples, as are abelian groups. We elaborate on some of the examples below after giving the formal definition of a module.

A commutative group on which there is defined an *exterior multiplication* (left or right) by elements of a *ring* R, such that multiplication is associative and distributive, and a group element multiplied by an element of the ring is a group element.

1) $(u + v) + w = u + (v + w)$ for all $u, v, w \in M$

2) $u + v = v + u$ for all $u, v \in M$

3) There exists an element $0 \in M$ such that $u + 0 = u$ for all $u \in M$

4) For any $u \in M$, there exists an element $v \in M$ such that $u + v = 0$

5) $a \cdot (b \cdot u) = (a \cdot b) \cdot u$ for all $a, b \in R$ and $u \in M$

6) $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$ for all $a \in R$ and $u, v \in M$

7) $(a + b) \cdot u = (a \cdot u) + (b \cdot u)$ for all $a, b \in R$ and $u \in M$

A vector space is a module where complex numbers (so the ring is the field of complex numbers). A more complex example would be a set of operators acting on a vector space. This is a left module where the ring is the collection of operators.

$$Lu = x_1 L_{1i} + x_2 L_{2i} + \cdots + x_2 L_{ni} = \sum_{m=1}^{n} x_m L_{mi}$$

The action of L on a basis can be represented by a matrix L_{ij}. Any vector, u say, can be represented in the basis

$$u = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{m=1}^{n} a_m x_m$$

$$Lv = L(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) = a_1 L x_1 + a_2 L x_2 + \cdots + a_n L x_n$$

so the coefficients of Lv are

$$(Lv)_m = \sum_{m'=1}^{n} L_{mm'} a_{m'}$$

• **moniod:** A set M with a binary operation $\cdot$ that is

(i) associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and

(ii) there is a unit element $\mathbf{1}$ in M such that $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in M$.

A moniod is a semigroup that has an *identity*.

A monoid is a category with one object.

• **moniodal category:** A moniodal category is a 2-category with one object.

• **morphism:**

A **$*$-morphism** between two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is a mapping $\pi : A \in \mathcal{A} \in \mathcal{A} \mapsto \pi(A) \in \mathcal{B}$, defined for all $A \in \mathcal{A}$ such that

$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,

$\pi(AB) = \pi(A)\pi(B)$,

$\pi(A^*) = \pi(A)^*$.

• **neighbourhood:** A set N is a neighbourhood of a point x in a topological space $(X, \mathcal{T})$ if and only if there is a open set U in $\mathcal{T}$ such that $x \in U \subseteq N$. Note that N need not be open itself.

• **neighbourhood base:** a collection of neighbourhoods, so that every open set of the topology can be expressed as a union of some of these neighbourhoods.

• **net:** Generalization of a the concept of a sequence to permit talk of convergence in non-meterizable topological spaces. Nets are often called Moore-Smith sequences.

• **normal operator:** A normal operator commutes with its self-adjoint.

If N is normal then,

$$N = \sum_{i=1}^{N} \lambda_i |f_i >< f_i|,$$

where $\lambda_1, \ldots, \lambda_N$ are igenvalues and $\{|f_i >\}$ are orthonormal eigenvectors of N. This is known as a strectral decomposition.

• **normal topological space:** A topological space is normal when it is T_1 (see separation conditions), and given any two disjoint closed sets V_1 and V_2, there are disjoint open sets U_1 and U_2 such that $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$.

• **non-Boolean logic:** Non-Boolean logic does not assume every statement can be judged true or false, there are some statements upon which one cannot decide. An example of a physical situation where the underlying logic would be non-Boolean is in cosmology where observers can only make judgements upon statements that have to do with their backwards light cone.

relational type quantum mechanics.

see topos , Heyting algeras

• **norm:**

(i) $\|zx\| = |z|\|x\|$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

• **normed space:** A normed linear space in which every vector x there corresponds a **norm** $\|x\|$ of x, such that

(i) $\|x\| \geq 0$, and $\|x\| = 0 \;\Rightarrow\; x = 0$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$,

(iii) $\|x\| = |\alpha|\|x\|$.

• **normed space:** A vector space endowed with a norm.

• **nuclear operator** A degenerate operator A can be written in the for $A = UT$, where U is an isometric operators and T is a positive deinite operator. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T. the space of nuclear operators is obtained from the space of degenerates operators by completeing the latter in the norm $\|A\|_1 = \sum_{k=1}^{n} \lambda_k$.

• **nuclear spectral theorem:**

• **nuclear topology:**

• **open algebras:**

The smeared Hamiltonian Poisson brackets are an example, see Eq(N.-19). Very little is known about the representation theory of open algebras.

• **open mapping theorem:** A fundamental result of functional analysis which states that if a continuous linear operator between Banach spaces is surjective (onto) then it is an open map (i.e. if U is an open set in the intial Banach space then $A(U)$ open set in the Banach space which the operator A maps to).

• **open sets:** Intuitively a set U is open if any point x in U can be moved in any "direction" and still be in the set. For example the set (a, b) of the real line $\mathbb{R}$ is open in $\mathbb{R}$, any point in (a, b) is arbitrarily close to the boundaries a or b, but cannot be a or b, as such if moved a sufficiently small amount are still contained in (a, b).

Subets understood with the properties

(1) Let A be a topological space and $\{\mathcal{U}_\alpha\}$, be any collection (possibly infinite) of open sets in A.

Then

$$\cup_\alpha \mathcal{U}_\alpha \tag{B.-19}$$

is open

(2) Let be a finite collection of open sets in A

$$\mathcal{U}_1 \cap \mathcal{U}_2 \cap \cdots \cap \mathcal{U}_n \tag{B.-19}$$

is open.

The concept of an open set is fundamental to amny areas of mathemetics, we see them in operator theory as well as in point-set topolgy (singularity theorems for example).

• **operator topologies:** The set of all bounded, linear operators acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. A subset $S \subset \mathcal{B}$ is called an algebra if $\alpha A + \beta B \in S$ and $AB \in S$

Topology needs a definition of what we mean by a neighbourhood of an element.

The norm topology: the ϵ-neighbourhood of A is the set of operators B with $\|A - B\|$. **norm topology** or **uniform topology** in $\mathcal{B}(\mathcal{H})$.

Other important topologies in $\mathcal{B}(\mathcal{H})$ are defined by means of seminorms.

For topologies that which are defined by seminorms it is not enough to consider only the convergence of sequences. The closure of a set is obtained by adding the limit points of em nets.

generally, if a subset is closed in one topology it is automatically closed in every stronger topology one gets even more limit points.

weak operator topology: topology such that are all continuous.

the weak topology is the topology such that all functionals on X^{**} are continuous.

weak star operator topology: The weak star topology is obtained if we use the absolute values of $| < \Phi|A|\Psi > |$ between arbitrary state vectors as a system of seminorms. Thus a sequence of operators converges weakly if all matrix elements converge.

• **oriented manifold:** Intuitively, in the case of a 2-manifolds, a surface is oriented if it is two-sided, and non-oriented if it is 1-sided. The cylinder is oriented, but the Möbius strip is not.

• **orbit:** Let G act on a set X. A subset $\subset X$ is said to be **stable** under the action of G if

$$g \in G \quad x \in S \Rightarrow gx \in S. \tag{B.-19}$$

• **outer measure:** Let a be an algebra of subsets of X and μ a measure on it. For $A \subset X$ is defined by

$$\mu^*(A) = \inf \sum_{i=1}^{\infty} \mu(E_i) \tag{B.-19}$$

where the infimum is taken over all $\mathcal{U}$-coverings of the set A all collections (E_i), $E_i \in \mathcal{U}$ with $\cup_i E_i \supset A$. to extend μ to as many elements of the powerset as possible.

• **Pachner moves:** Two finite triangulations related by a finite sequence of local modifications, called Pachner moves. Sequences of Pachner moves are the combinatorial equivalent of applying spacetime diffeomorphisms.

?????????is the lattice as being embedded in some continuum manifold or whether one regards the lattice itself as existing independently of any background embedding.????????

• **paracompact:** A topological space is said to be paracompact if evry open cover admits an open locally finite refinement.

• **parallelogram law:** A normed space

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{B.-19}$$

• **parallel transport:** Parallel transport of a vector is defined as transport without change.

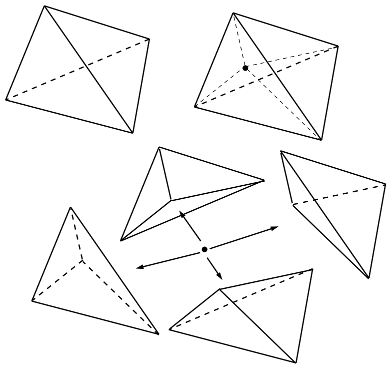


Figure B.21: Pachner move in d=3. (a) the $1 \to 4$ move subdivides.

• **parseval's formula:** The generalized Fourier series

$$f(x) = \sum_{-\infty}^{\infty} f_n e_n(x)$$

with Fourier coefficients

$$f_n = \int_a^b e_n^*(x) F(x) dx$$

$$\begin{aligned} E_{min} &= \int_a^b |F(x) - f(x)|^2 dx \\ &= \int_a^b F^2(x) dx - \sum_{-\infty}^{\infty} f_n^2 \geq 0. \end{aligned} \qquad \text{(B.-19)}$$

gives an average measure of convegence. The Fourier series $f(x)$ of a function $F(x)$ is said to converge in the mean to $F(x)$ if $E_{min} = 0$. When this happens, the Bessel inequality becomes the Parseval equation

$$\int_a^b F^2(x) dx = \sum_{-\infty}^{\infty} f_n^2. \qquad \text{(B.-19)}$$

• **partial function:** A partial function is the triple $f = (A, G, B)$ where A and B are sets (possibly empty) and G is a functional relation (possibly empty) between them, called the **graph** of f.

$f : A \to B$ is a **total function** if and only if domain$(f) = A$. A total function is often just referred to as a function.

• **partial isometry:** An operator U in the space of linear operators $\mathcal{L}(\mathcal{H})$ from a separible Hilbert space $\mathcal{H}$ to itself is called a partial isometry if $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$ when restricted to the closed subspace $(\text{Ker}U)^\perp$ (Ker$U$ being the set of $x \in \mathcal{H}$ for which $Ux = 0$).

See polar decomposition.

• **partially ordered set:** Let P be a non-empty set. A partial order relation in P is a relation denoted by $\leq$ which has the following properties:

(i) $a \leq a$ for every a (reflexivity);

(ii) $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry);

(iii) $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

• **partition of unity:** Take an open covering $\{U_i\}$ of $\mathcal{M}$ such that each point of $\mathcal{M}$ is covered by a finite number of U_i (if this is always possible, $\mathcal{M}$ is called paracompact, which we assume to be the case). If a family of differentiable functions $\epsilon_i(p)$ satisfies

(i) $0 \leq \epsilon_i(p) \leq 1$

(ii) $\epsilon_i(p) = 0$ if $p \neq U_i$

(iii) $\epsilon_1(p) + \epsilon_2(p) + \cdots = 1$ for any point $p \in \mathcal{M}$

the family $\{\epsilon_i(p)\}$ is called a partition of unity subordinate to the covering $\{U_i\}$. It follows from (iii) that

$$f(p) = \sum_i f(p)\epsilon_i(p) = \sum_i f_i(p)$$

where $f_i(p) \equiv f(p)\epsilon_i(p)$ vanishes outside of U_I by (ii).

• **path ordered:** $\mathcal{P}\left(A(x(s_2))\; A(x(s_1))\; A(x(s_4)) \ldots\right) = A(x(s_1))\; A(x(s_2))\; \ldots\; A(x(s_n))$, where $s_1 \geq s_2 \geq \cdots \geq s_n$.

• **Penrose's abstract index notation:** In this notation, the index 'a' of a vector v^a is to be seen as a label indicating that v is a vector (very much like the arrow in $\vec{v}$), and it does not take values in any set. That is, 'a' is not the component of v on any basis.

• **permutation group:**

• **Peter-Weyl Theorem:** The Peter-Weyl theorm asserts that the representaion matries, $D^J_{mn}(g)$, form a complete set of orthogonal functions on the group manifold.

$$f(g) = f(UgU^{-1}) \Rightarrow \quad f(g) = \sum_i a_i \chi_i(g) \tag{B.-19}$$

where

$$a_i = \sum \tag{B.-19}$$

The *Peter-Weyl Theorem* applied to $U(1)$ gives the Fourier series theory:

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \tag{B.-19}$$

where $f(\theta) \in L^2(U(1))$.

• **p-form:** An anti-symmetric covariant tensor field over a manifold $\mathcal{M}$.

• **piecewise:** We say something piecewise, with respect to some property, if it is made up of a finite number of pieces, each of which shares this property.

• **piecewise-analytic curve:** A curve is piecewise analytic if it is made up of a finite number of pieces, each of which is analytic.

See semianalytic curve.

piecewise-smooth curve: A curve is piecewise smooth if it is made up of a finite number of pieces, each of which is smooth.

• **piecewise-linear (PL-) manifolds:** The transition functions are maps between polyhedra which map simplices to simplices.

• **Poincare lemma:** Say we have a $p-$form ω. Recall the exterior derivative, denoted d. If $d\omega = 0$, then ω is said to be closed. If $\omega = d\alpha$, then ω is said to be exact. Exactness implies closure, since $\omega = d\alpha \Rightarrow d\omega = d^2\alpha = 0$. The converse is in general not true. The Poincare lemma states that every closed form is locally exact. That is, if $d\omega = 0$, then $\omega = d\alpha$ in some local region. In general, this will not hold globally.

• **Poincare algebra in $(2+1)$ dimensions $iso(2,1)$:** Poincare algebra $iso(2,1)$ $(a, b, c = 0, 1, 2)$:

$$[J_a, J_b] = \epsilon^c{}_{ab} J_c, \quad [J_a, P_a] = \epsilon^c{}_{ab} P_c, \quad [P_a, P_b] = 0. \tag{B.-19}$$

• **Poisson resummation formula:** Let $f(x)$ be a continuous and continuously differentiable function of x such that the series $\sum_{n=-\infty}^{\infty} f(x+n)$ and $\sum_{n=-\infty}^{\infty} f'(x+n)$ converge absolutely and uniformally for all x in the interval $0 \leq x \leq 1$. The second series is then the derivative of the first.

The function

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

is periodic in x with period 1, therefore we can take its Fourier series expansion:

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} a_k \exp(2\pi ikx)$$

where

$$\begin{aligned}
a_k &= \int_0^1 F(x) e^{-2\pi ikx} dx = \int_0^1 \sum_{m=-\infty}^{\infty} f(x+m) e^{-2\pi ikx} dx \\
&= \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m) e^{-2\pi ikx} dx \\
&= \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m) e^{-2\pi ik(x+m)} dx \\
&= \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(x) e^{-2\pi ikx} dx \\
&= \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} dx
\end{aligned}$$

Therefore

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \exp(2\pi kx) \int_{-\infty}^{\infty} dy f(y) \exp(-2\pi iky).$$

and

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dy f(y) \exp(-2\pi iky).$$

This is Poisson's formula.

$$\sum_n e^{-\epsilon(n-N)^2} f(n) = \sum_n e^{-\epsilon(y-N)^2} f(y) e^{2\pi I yn} \tag{B.-24}$$

• **polar decomposition:** A polar decomposition is the analogue to the decomposition

$$z = \left(\frac{z}{|z|}\right)\sqrt{\bar{z}z}$$

if z has the analogy with the operator A. We have an analogy between $|z| = \sqrt{\bar{z}z} = 1$ with isomteris which satisfy $W^*W = I$. This corresponds to the term $z/|z|$. This ananolgy suggests that we guess that $T = W\sqrt{A^*A}$. This guess r=turns out to be correct.

Let $\mathcal{H}$ be a Hilbert space. If $A \in \mathcal{B}(\mathcal{H})$ then (by the polar decomposition theorem) it can be uniquely expressed as $A = W|A|$ where $|A|$ is the positive square root of A^*A and W is a partial isometry with initial space equal to the closure of the range of $|A|$ and final space equal to the range of A. The expression $W|A|$ iscalled the polar decomposition of A.

left polar decomposition

• **polyhedron:** A subset $X \subseteq \mathbb{R}^n$ is said to be a polyhedron if every point $x \in X$ has a neighbourhood in X of the form

$$\{\alpha x + \beta y : \alpha, \beta \geq 0, \alpha + \beta = 1, y \in Y\}$$

where $Y \subseteq X$ is compact.

• **Pontyagin duality:**

• **positive elements:** An element of a C^*algebra $\mathcal{A}$ is said to be positive if $A = A^*$ and $\sigma(A) \geq 0$, or equivalently, $A = B^*B$ for some $B \in \mathcal{A}$.

• **positive operator:** An operator $B \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called positive if $(Bx, x) \geq 0$ for all $x \in \mathcal{H}$

• **power set:** Let X be a set. The power set $\mathcal{P}(X)$ is the collection of all subsets of X.

• **presheaf:** TAKEN DIRECTLY FROM mathworld.wolfram.com

For X a topological space, the presheaf $\mathcal{F}$ of Abelian groups (rings, ...) on X is defined such that

1. For every open subset $U \subseteq X$, an Abelian group (ring, ...) $\mathcal{F}(U)$, and

2. For every inclusion $V \subseteq U$ of open subsets of X, a morphism of Abelian groups (rings, ...) $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$

subject to the conditions:

1. If $\emptyset$ denotes the empty set, then $\mathcal{F}(\emptyset) = 0$,

2. ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and

3. If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

In the language of categories, let $\mathbf{Top}(X)$ be the category whose objects are the open subsets of X and the only morphisms are the inclusion maps (these map V to U if $V \subseteq U$). Thus, $\mathbf{Hom}(V, U)$ is empty if $V \not\subseteq U$ and $\mathbf{Hom}(V, U)$ has just one element if $V \subseteq U$. Then a presheaf is a contravariant functor from the category $\mathbf{Top}(X)$ to the category $\mathbf{Ab}$ of Abelian groups (Ring of rings, ...).

As a terminology, if $\mathcal{F}$ is a presheaf on X, then $\mathcal{F}(U)$ are called the sections of the presheaf over the open set U, sometimes denoted as $\Gamma(U, \mathcal{F})$. The maps ρ_{UV} are called the restriction maps. If $s \in \mathcal{F}(U)$, then the notation $\rho_{UV}(s)$ is usually used instead of $s|_V$.

Sheaf: Important in twistor theory and other applications of algebraic geometry and topology in physics. A presheaf $\mathcal{F}$ is called a *sheaf* if for every collection U_i of open subsets of X with $U = \cap_{i \in I} U_i$ the following conditions hold:

1. If $s, t \in \mathcal{F}(U)$ and $\rho_{UU_i}(s) = \rho_{UU_i}(t)$ for all i, then $s = t$;

2. If $s_i \in \mathcal{F}(U_i)$ and if for $U_i \cap U_j \neq \emptyset$ we have

$$\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j) \tag{B.-24}$$

for all i, then there exists an $s \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s) = s_i$ for all i.

The example of germs of functions on a differentiable manifold is a familiar example of a sheaf.

(definition taken from Topi and computation)

patching condition: Let an open set U be covered by open subsets U_i; then any given sections over the U_i's, such that the sections over U_i and U_j have the same restriction to $U_i \cap U_j$, are restrictions of a single section over U. (for example letting $\mathcal{F}(U)$ be the set of all continuous real-valued functions on U). The crucial thing about this definition is that the space X enters the picture only through its poset of open subsets and the notion of a cover.

- **primative elements of a Hopf algebra:**

- **principal bundle:** A principal bundle is a fibre bundle $\pi : P \to E$ with fibre F equal to the structure group G and having the property that for all U_a and U_b with $U_a \cap U_b \neq \emptyset$,

$$\varphi_{ba} : U_a \cap U_b \to Left(F) \subset Diff(F),$$

where $Left(F) = \{L_g | L_g(h) = gh, \text{ for all } h \in G, g \in G\}$. In other words, changing coordinates corresponds to multiplying the fibre on the left by some element of G.

As an example consider the frame bundle $B(\mathcal{M})$. The total space consists of the set of basis vectors v_i^a of the tangent space for all points over the manifold $\mathcal{M}$. Here i is an index $i = 1, \ldots, n$ labeling the n basis vectors v^a. There is a natural free right action by $GL(n, \mathbb{R})$ on $B(\mathcal{M})$: If we have a basis $\{v_i^a, i = 1, \ldots, n\}$, we know that

$$(v_1^a, v_2^a, \ldots, v_n^a)g := (v_j^a g^j{}_1, v_j^a g^j{}_2, \ldots, v_j^a g^j{}_n),$$

where g in $GL(n, \mathbb{R})$. We can also see the array of vectors $(v_1^a, v_2^a, \ldots, v_n^a)$ as a $n \times n$ matrix with non-zero determinant.

The frame bundle is an example of a principal bundle. In a principal bundle $(P, \pi, \mathcal{M}, G)$ each fibre is diffeomorphic to the structure group G. Principal bundles are in a sense more fundamental than vector bundles, since one can always regard vector bundles as associated bundles to a particular principal bundle. In this example, the tangent bundle $T\mathcal{M}$ is the associated vector bundle to the frame bundle $B(\mathcal{M})$.

• **product spaces:** A Cartesian product equipped with a product topology is called a product space.

• **product topology:** Let $\mathcal{S}$ and $\mathcal{T}$ be topological spaces, and form the product $\mathcal{S} \times \mathcal{T} = \{(u, v) : u \in \mathcal{S} \text{ and } v \in \mathcal{T}\}$ of the two sets $\mathcal{S}$ and $\mathcal{T}$. The product topology on $\mathcal{S} \times \mathcal{T}$ consists of all subsets that are unions of sets of the form $U \times V$, where U is open in $\mathcal{S}$ and V is open in $\mathcal{T}$. Thus these open rectangles form a basis for the product topology.

• **projective family:** Let L be a partially ordered directed set with an pre-order relation denoted $\preceq$ for $S, S', S'' \cdots \in L$. A *projective* family $(\chi_S, p_{SS'})_{S,S' \in L}$ consists of sets χ_S indexed by elements of L, together with a family of onto *projections*,

$$p_{SS'}\, \chi_{S'} \to \chi_S, \tag{B.-24}$$

such that

$$p_{SS} = id_{\chi_S}, \tag{B.-24}$$

and

$$p_{SS'} \circ p_{S'S''} = p_{SS''} \tag{B.-24}$$

for ay triple $S \preceq S' \preceq S''$.

• **projective limit:** Unfortunately the projective family itself does not have a largest element from which one can project to any other. However, such an element can in fact be obtained by a standard procedure called the "projective limit". It emerges as an appropriate limit, defined as follows. The projective limit $\overline{\chi}$ of a projective family $(\chi_S, p_{SS'})_{S,S' \in L}$ is the subset of the Cartesian product $\times_{S \in L} \chi_S$ that satsfies certain consistency conditions:

$$\overline{\chi} = \{(x_S)_{S \in L} \in \times_{S \in L} \chi_S : S \preceq S' \text{ implies } p_{SS'} x_{S'} = x_S\}. \tag{B.-24}$$

This limit is naturally equipped with a family of projections:

$$p_S : \overline{\chi} \to \chi_S, \quad p_S((x_{S'})_{S' \in L}) := x_S. \tag{B.-24}$$

• **projective limit construction:**

A graph is a collection of edges such that if two edges meet. Consider the space A_γ, each element of which assigns to every edge in γ

The projective limit of these spaces are precisely the spaces $\mathcal{A}_\gamma$, $\mathcal{G}_\gamma$ and $\mathcal{A}_\gamma / \mathcal{G}_\gamma$.

What about operators that act on this Hilbert space? All operators that are well defined on that Hilbert space arise from consistent family of operators. These operators on each of these individual finite Hilbert spaces fit together in a certain way. If it is well defined on here then it can be shown that that they come from something that fits together in this way.

• **projection mappings:**

$$\hat{P}_k^2 = \hat{P}_k \tag{B.-24}$$

$$\sum_{k=1}^{N} \hat{P}_k = \mathbf{I}. \tag{B.-24}$$

group averaging Rovelli projection onto physical states not strictly projection operators.

• **pull-back:** The diffeomorphism ϕ maps points in $\mathcal{M}$ to points in $\mathcal{N}$. The push-forward $\phi^*|_p$ is the natural map between the co-tangent spaces $T_p^* \mathcal{M}$ and $T_{\phi(p)}^* \mathcal{N}$ induced by the diffeomorphism ϕ.

$$[\phi^* \omega](X_1, X_2, \ldots, X_p) = \omega(\phi_* X_1, \phi_* X_2, \ldots, \phi_* X_p). \tag{B.-24}$$

• **pure point spectrum:** Let a be a self-adjoint operator. The pure point spectrum $\sigma_{pp}(a)$ is the set of eigenvalues of a.

• **pure states:** Those states which cannot be written as convex linear combinations of other states, i.e. states which cannot be written as $\alpha a + (1-\alpha)b$ for some pair of states a, b.

• **push-forward:** map head-to-head and tail-to-tail. If the vector has components X^μ and the map takes the point with coordinates x^μ to one with coordinates $\xi(x)$, the vector ϕ_* has components

$$(\phi_* X)^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} X^\nu. \tag{B.-24}$$

This looks like the transformation formula for contravariant vector components under a coordinate transformation. We are doing an active transformation, changing a vector into a different one.

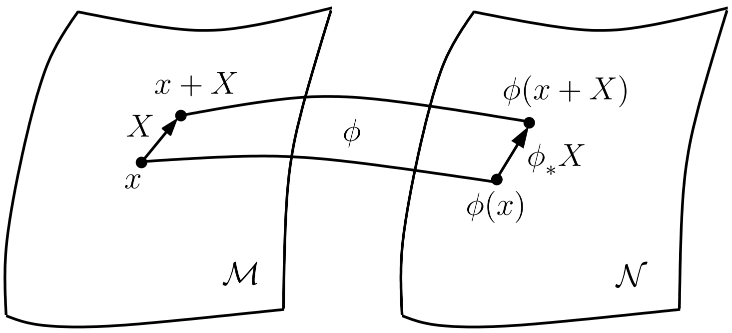


Figure B.22: pullbackDef0. Pushing forward a vector X from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$.

The diffeomorphism φ maps points in $\mathcal{M}$ to points in $\mathcal{N}$. The push-forward $\varphi_*|_p$ is the natural map between the tangent spaces $T_p\mathcal{M}$ and $T_{\varphi(p)}\mathcal{N}$ induced by the diffeomorphism φ.

• **q-deformation:** Introduces a non-commutative to a mathematical structure that was absent in the first place. The non-commutativity is measured by a deformation q parameter. The limiting case gives back the results afforded by the ordinary situation. Some examples are $su_q(2)$ which defines a q-analogue of angular momentum.

• **q-series:**

by L'Hopital's rule

$$\lim q \to 1 \frac{1 - q^a}{1 - q} = a \tag{B.-24}$$

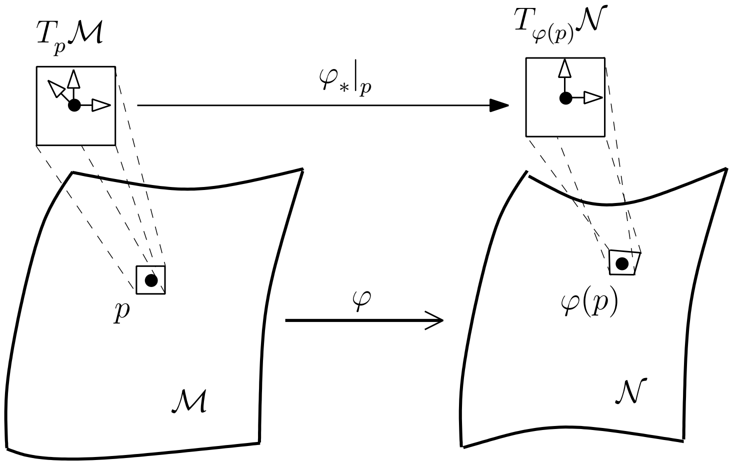


Figure B.23: pullbackDef. $\varphi_*|_p : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$

q-shifted factorial

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)(1-aq^{n-1}), & n = 1, 2, \dots \end{cases} \tag{B.-24}$$

q-binimial expansion

$$\sum_{m=0}^{n} y^m q^{\frac{m(m+1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix} \tag{B.-24}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-m+1})}{(1-q)(1-q^2)\dots(1-q^m)}, \tag{B.-24}$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1. \tag{B.-24}$$

• **quantum groups:** non-co-commutative quasi-triangular Hopf algebras. deform any classical Lie algebra. The initial Hopf algebra is the universal enveloping algebra of the Borel subalgebra of a Lie algebra.

The limiting case - as q goes to 1, the quantum algebra $su_q(2)$ Quantum $SL(2)$ group and quantum $SU(2)$ group.

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B.-24}$$

$$U\tilde{\epsilon}U^T = \tilde{\epsilon}, \quad \text{where } \tilde{\epsilon} = \begin{pmatrix} 0 & -A \\ A^{-1} & 0 \end{pmatrix} \tag{B.-24}$$

Figure B.24: links.

$$\begin{aligned}
&ba = qab \;\; dc = qcd \\
&ca = qac \;\; db = qbd \\
&bc - cb = 0 \\
&ad - da = -(q - q^{-1})bc \\
&ad - q^{-1}bc = 1
\end{aligned} \tag{B.-27}$$

where $q = aA^2$. complex non-commuting components a, b, c, d. $q = e^{in}$

$su_q(2)$ algebra

$$\begin{aligned}
[j_3, j_\pm] &= \pm j_\pm, \\
[j_+, j_-] &= \frac{q^{2j_3} - q^{-2j_3}}{q - q^{-1}}
\end{aligned} \tag{B.-27}$$

In a similar way we can define the quantum group $GL_q(2)$,

$$UQU^T = z_1 Q, \;\; U^T QU = z_2 Q, \quad z_1, z_2 \in \mathbf{C}^* \tag{B.-27}$$

$z_1 = z_2 = z$

• **quasilinear differential equations:** Differential equations linear in the highest derivative terms. For second order quasilinear differential equations, many of the results on linear systems apply locally.

• **quasi-triangular Hopf algebras:** See Hopf algebras. A non-co-commutative quasi-triangular Hopf algebra is called a quantum group.

$$\Delta'(a) = \sigma \circ \Delta(a) \tag{B.-27}$$

It is non-co-commutative, but the non-co-commutativity is controlled by a matrix $\mathcal{R}$,

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \quad \text{for all } a \in A. \tag{B.-27}$$

and

$$(\mathbf{1} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} = \sum_{i,j} A_i A_j \otimes B_j \otimes B_i,$$
$$(\mathbf{1} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} = \sum_{i,j} A_i \otimes A_j \otimes B_i B_j \tag{B.-28}$$

and

$$(\gamma \otimes \mathbf{1})\mathcal{R} = (\mathbf{1} \otimes \gamma^{-1})\mathcal{R} = \mathcal{R}^{-1} \tag{B.-28}$$

Where is called the universal $\mathcal{R}$-matrix. We write $\mathcal{R} = \sum_i A_i \otimes B_i$ and let

$$\mathcal{R}_{12} = \sum_i A_i \otimes B_i \otimes \mathbf{1},$$
$$\mathcal{R}_{13} = \sum_i A_i \otimes \mathbf{1} \otimes B_i, \tag{B.-29}$$
$$\mathcal{R}_{23} = \mathbf{1} \otimes \sum_i A_i \otimes B_i. \tag{B.-29}$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{B.-29}$$

- **quotient map:**
- **quotient topology:** Let

$$\mathcal{T}' = \{V : V \subseteq Y \text{ and } p^{-1}(V) \text{ is open in } X\}.$$

It is immediate that $\emptyset, Y \in \mathcal{T}'$ as $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(Y) = X$.

- **radical:** A *radical* R of an algebra A is the intersection of all its maximal left ideals. R itself is obviously a proper left ideal.
- **Randon-Nikodym derivative:**
- **Randon-Nikodym theorem:**

• **(real infinite) sequence** is a map a $a : N \to R$

Of course if is more usual to call a function f rather than a; and in fact we usually start labelling a sequence from 1 rather than 0; it doesn't really matter. What the definition is saying is that we can lay out the members of a sequence in a list with a first member, second member and so on. If $a : N \to R$, we usually write a_1, a_2 and so on, instead of the more formal $a(1)$, $a(2)$, even though we usually write functions in this way.

• **reconstruction Theorem:** Does it? We say it *separates* points in $\mathcal{A}/\mathcal{G}$ in that , if A_1 and A_2 are not related by a gauge transformation, there exists a loop γ such that: $T_\gamma(A_1) \neq T_\gamma(A_2)$.

Suppose Σ is a connected manifold with basepoint x_0 and the map $H : \Omega_{x_0} \to G$ satisfies the following conditions:

(i) H is a homomorphism of the composition law of loops, $H(\gamma_1 \circ \gamma_2) = H(\gamma_1)H(\gamma_2)$,

(ii) H takes the same values on thinly equivalent loops: $\gamma_1 \sim \gamma_2$ if $\gamma_1 \circ \gamma_2^{-1}$ is thin??,

(iii) For any smooth finite-dimensionale family of loops $\bar{\psi} : U \to \Omega_{x_0}\Sigma$, the composite map $H\psi : U \to \Omega_{x_0}\Sigma \to G$ is smooth.

Then there exists a differentiable principle fibre bundle ..R. Lool hep-th/9309056

• **regular Borel measure:** A non-negative countably additive set function μ defined on $\mathcal{B}$ is called a *regular Borel measure* if for every Borel set B we have:

$$\begin{aligned} \mu(B) &= \inf\{\mu(O) \ : \ O \text{ open}, O \supset B\}, \\ \mu(B) &= \sup\{\mu(F) \ : \ F \text{ closed}, F \subset B\}. \end{aligned} \tag{B.-29}$$

taken from Measure, Integral and Probability, M Capiński and E. Kopp, [**?**].

Notice that regularity of μ on a compact Hausdorff space X reduces to the fact that the measure of every measurable set can be approximated arbitrarily well open or compact (and hence closed since in a Hausdorff space every compact subset is closed, see ??) sets respectively.

• **regular measure:**

• **regular embedding:** For an embedding $\phi : \mathcal{M} \to \mathcal{N}$, the map $\phi : \mathcal{M} \to \phi(\mathcal{M})$ is a bijection and the manifold structure induced by ϕ on $\phi(\mathcal{M})$ is given by the atlas $\{\phi(U_I), \varphi_I \circ \phi^{-1}\}$ where $\{U_I, \varphi_I\}$ is an atlas of $\mathcal{M}$. This structure need not be equivalent to the submanifold structure of $\phi(\mathcal{N})$ which is given by the atlas $\{V_J \cap \phi(\mathcal{M}), \phi_J\}$ where $\{V_I, \phi_I\}$ is an atlas of $\mathcal{N}$. When both differential structures are equivalent the embedding is called regular.

• **regular representation of a C^*-algebra:** If we wish to require that a Weyl algebra is represented weakly continuously, states whose GNS representation have this property are said to be regular.

• **regular representation of a finite group:** It is a matrix representation of the group. We construct the matrices as follows:

	e	a	b	c
e	e			
a^{-1}		e		
b^{-1}			e	
c^{-1}				e

To establish that this is a representation we must prove that

$$A^{(i)}_{jk} = \begin{cases} 1 \text{ if } a_j^{-1}a_k = a_i \\ 0 \text{ otherwise.} \end{cases} \tag{B.-29}$$

Proof: It is easily established that

$$\sum_k A^{(i)}_{jk} A^{(m)}_{kl} = \begin{cases} 1 \text{ if and only if } a_k = a_j a_i = a_l a_m^{-1} \\ 0 \text{ otherwise.} \end{cases} \tag{B.-29}$$

• **regular representation of a Lie group:** Provides a systematic procedure to construct irreducible representations of a group.

$$\int d\mu(c) \mathcal{D}^{\text{reg}}(a; b, c) f(c) \tag{B.-29}$$

$$\mathcal{D}^{\text{reg}}(a; b, c) = \frac{\delta^p(c - \phi(a, b))}{\rho(c)}, \tag{B.-29}$$

• **Reidermeister Moves:** G.59

• **Reiz lemma:** A vector in the normed space uniquely defines a continuous functional via

$$F_f(V) =< f|V > \tag{B.-29}$$

• **Riesz representation theorem:**

Application: [96] - since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, the Riesz representation theorem ensures that there is a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma(f) = \tag{B.-29}$$

• **Reiz representation theorem:**

An application ([96]): Now, since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, the ensues that there is a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu([A])\; \tilde{f}([A]) \tag{B.-29}$$

where $\tilde{f} \in C^0(\overline{\mathcal{A}/\mathcal{G}})$ corresponds to f in $\overline{\mathcal{HA}}$.

• **relative topology:** (or induced topology) Let X be a topological space, and let Y be a non-empty subset of X. The relative topology on Y is defined to be the class of all intersections with Y of open sets in X.

• **repeated bisection argument:**

• **representation:** A representation of a group G over a field k (often. $\mathbb{R}$ or $\mathbb{C}$) is a homomorphism $\pi : G \to GL(V)$, where V is a (usually finite dimension) vector space over k. That is, a representation π of G "compares" the "abstract" group G with a concrete group $GL(V)$. There are similar defintions for C^*-algebras and other such ab.

• **representation theory:** Representation theory studies how any given abstract group can be realized as a group of matrices.

operators on the vector space??? concrete example often matrices and operators on a Hilbert space.

• **resolvent set:** The resolvent set $\rho(A)$ is defined as the subset of $\mathbb{C}$ defined as $\{z \in \mathbb{C} : A - z\mathbf{1}$ has bounded inverse $\}$.

• **Riemann integral:**

$$\overline{\int} f d\alpha = \underline{\int} f d\alpha \tag{B.-29}$$

• **Reimann's crition:** We first give the crition of the most elementary case of a real function f of a closed interval, $[a, b]$, of the real, $f : [a, b] \to \mathbb{R}$. Such Reimann-integrable if and only if for every $\epsilon > 0$ there exists a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

The Reimann integral doesn't deal with all functions we need for our purposes in formulating quantum mechanics and quantum field theory in a mathematically rigorous way.

• **Riemann-Stieltjes integral:**

$$s(P_n) = \sum_{j=1}^{n} x(t_j)[w(tj) - w(t_{j-1})] \tag{B.-29}$$

$$\int_a^b x(t)dw(t) \tag{B.-29}$$

• **ring:**

In the set algebraic sense:

If groups are roughly thought of as collections of elements that can be added together, then rings are collections of elements for which there is addition and multiplication. To be more specific, a ring is an additive abelian group whose elements can be multiplied as well as added, and in which multiplication is

(i) associative, that is, if $x,\ y,\ z$ are any three elements in R, then $x(yz) = (xy)z$; and

(ii) is distributive, that is, if $x,\ y,\ z$ are any three elements in R, then $x(y+z) = xy+xz$ and $(x+y)z = xz+yz$.

If an element x of R has an inverse, then x is said to be **regular** (or invertible).

a division ring: A ring with identity is called a division ring if all its non-zero elements are regular.

In the set theoretical sense:

Let X be a set, let $R \subseteq \mathcal{P}(X)$. Then R is a ring of subsets if

(i) $\emptyset \in R$;

(ii) if $A, B \in R$ then $A \cap B$, $A \cup B$ and $A \backslash B$ are all in R.

An an alternative definition of a ring, equivalent to the above, R is a ring if

(i) $\emptyset \in R$;

(ii) if $A, B \in R$ then $A \cap B$ is in R, and $A \triangle B \in R$, (where $A \triangle B$ is the symmetric difference $(A\backslash B) \cup (B\backslash A)$).

With operations $\cap$ as multiplication, $\triangle$ as addition, R is a ring in the algebraic sense.

• **right action of a group:**

$$R_g(g') := g'g. \tag{B.-29}$$

$g = e^{tX_g}\big|_{t=1}$

$$\frac{d}{dt}(g' e^{tX_g}) \tag{B.-29}$$

Vectors of the tangent space at g, $T_g(G)$, are related to vectors in the tangent space at e, $T_e(G)$. This map is denoted $(R_{g^{-1}})_*$ and is called the **pull-back** of the right action $R_g(G)$, the notation and name will be clarified in a moment. We can find this relationship. For clarity we will derive the matrix element of this map

There is a map which acts on the tangent space $T_g(G)$ and takes it to $T_e(G)$

$$(R_{g^{-1}})_* : T_g(G) \rightarrow T_e(G) \tag{B.-29}$$

• **rooted tress:**

• **saddle-point approximation:** The idea of the saddle point method (or method of steepest descent) is to deform the contour in such a way that the main contribution to the integral comes from a neighborhood of a single point, (or finite number of points).

Let C be a contour in the complex plane and the functions g and S are holomorphic in a neighborhood of this contour. We will consider the asymptotics as $\lambda \rightarrow \infty$ of the Laplace integrals

$$G(\lambda) = \int_C g(z) \exp[\lambda S(z)] dz : \tag{B.-29}$$

$$f(z) = \int d\tau g(\tau) e^{f(\tau)} \tag{B.-29}$$

$$f(z) \approx \sum_i g(\tau_i) e^{f(\tau_i)} \tag{B.-29}$$

τ_i determined by

$$\left.\frac{df(\tau)}{d\tau}\right|_{\tau=\tau_i} = 0. \tag{B.-29}$$

• **Schwartz space** $\mathbb{S}(\mathbb{R}^n)$**:** - also called the space of **test functions**. The set of functions that are infinitely differentiable and whose derivatives

$$\frac{\partial^k \varphi(x)}{\partial x^k} \rightarrow 0 \text{ as } x \rightarrow \infty \tag{B.-29}$$

faster than any power of $1/|x|$, for $k = 0, 1, 2, \ldots$.

natural duality between Schwartz space and space of tempered distributions??

• **Schwarz's inequality:** For two square-integrable functions $f(x), g(x) \in L^2[dx]$

$$\int |f(x)g(x)|^2 dx \le \int |f(x)|^2 dx \int |g(x)|^2 dx \qquad \text{(B.-29)}$$

holds. Easily follows from

$$\int |f(x) + \alpha g(x)|^2 dx \ge 0 \qquad \text{(B.-29)}$$

where α is real, as this implies (when $\int |g(x)|^2 dx \neq 0$)

$$\left(\alpha + \frac{\int |f(x)||g(x)|dx}{\int |g(x)|^2 dx}\right)^2 + \frac{\int |f(x)|^2 dx}{\int g(x)^2 dx} - \frac{\int |f(x)g(x)|dx}{(\int |g(x)|^2 dx)^2} \ge 0. \qquad \text{(B.-28)}$$

Used in derivation of the uncertainty principle. For a Banach space X with norm $\|\cdot\|$ for any $A, B \in \mathcal{B}(X)$

$$\|AB\| \le \|A\| \cdot \|B\|. \qquad \text{(B.-28)}$$

• **section:** It is a smooth assignment to each point in the base space of a point in the fibre over it. As an example, the section of the tangent bundle of a manifold $\mathcal{M}$ is a vector field. Note that a section is globally defined and will not alway exist. In the case of a principal fibre bundle, such as a frame bundle, it has a section if and only if it is trivial. This is not necessarily the case for general fibre bundles, such as a tangent fibre bundle. Notice, however, that local sections always exist as bundle spaces are locally trivial.

• **Segal-Bargmann transformation:** The Segal-Bargmann transformation B_t in ordinary quantum mechanics on the phase space $\mathbb{R}^2$ gives a representation in which wave functions are holomorphic, square integrable (with respect to the Liouville measure) functions of the complex variable $z = q - ip \in \mathbb{C}$, given by the convolution of f with a Gaussian,

$$B_t f(z) = \frac{1}{(2\pi t)^{-d/2}} \int_{\mathbf{R}^d} e^{-(z-x)^2/2t} f(x) dx, \quad z \in \mathbb{C}. \qquad \text{(B.-28)}$$

• **self-adjoint:**

An Unbounded operator are self-adjoint if it is a densely defined operator a with domain $D(A)$ is called self-adjoint if $a^\dagger = a$ and $D(a^\dagger) = D(a)$ where

$$D(a^\dagger) := \{\psi \in \mathcal{H};\ \sup_{0 \neq \psi' \in D(a)} | < \psi, a\psi' > | / \|\psi'\| < \infty\} \qquad \text{(B.-28)}$$

..... definition not finished this taken from Thiemann Intro to Modern...

• **self-adjoint extensions:**

• **semianalytic function:** A function $f : U \to \mathbb{R}^m$, where U is some open subset of $\mathbb{R}^n$, is called semianlytic

• **semianalytic partition:** Let $U \subset \mathbb{R}^n$ be open and $h := \{h_1, \ldots, h_N\}$ be a system of real-valued analytic functions h_k defined on a neighbourhood of U. The subsets defined by the 3^N sequences of inequalities/inequalities

$$\begin{aligned} &h_1(x) > 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0; \\ &h_1(x) < 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0; \\ \\ &h_1(x) > 0, \quad h_2(x) = 0, \ldots, h_N(x) < 0; \\ \\ &h_1(x) = 0, \quad h_2(x) = 0, \ldots, h_N(x) = 0 \end{aligned} \tag{B.-32}$$

satisfy the conditions of a partition (i.e. the union of these subsets covers U and the intersection between any two such distinct subsets is empty), this is called the semianalytic partition of U subordinate to h. These subsets can vary in dimension from $n-$dimensional to zero-dimensional.

• **semianalytic structures:** Semianalytic structures are objects such as paths or surfaces which are analytic except possibly on lower-dimensional subsets. On these subsets there is again a notion of semianalyticity. These objects, unlike analytic ones, have a local character since knowing the object on an arbitrary open subset only determines it up to where it fails to be analytic.

• **semi-continuous function:**

• **semi-direct product:** A group G is said to be a semidirect product of the subgroups N and Q, denoted $N \otimes_S Q$, if

(i) $N \triangleleft G$ ($\triangleleft$ denoting that N is a normal subgroup);

(ii) $NQ = G$; (meaning that every element of G can be written as a product nq where n is some element of N and q is some element of Q);

(iii) $N \cap Q = \mathbf{1}$. (Milne Group theory)

For example the translation-rotation group G, for Q as the rotation group and N the translation group. As

$$\hat{R}\hat{T}\hat{R}^{-1} = \hat{T}',$$

where $\hat{R}$ is a rotation and $\hat{T}$ a translation, is again a translation. Hence, translations form an (abelean) normal subgroup of G. (iii) is obvious.

Equivalent condition: and $G \to G/N$ induces an isomorphism $Q \overset{\approx}{\to} G/N$.

+ + + + + + + ++

When a group B acts on another group A as a subgroup of the automorphisms of A, a larger group $A \triangleleft B$ can be constructed, whose elements are all pairs $\{(a, b) \; : \; a \in A, b \in B\}$,

• **semi-group:** elements with an associative multiplication, which is closed under multiplication.

• **semi-norm:** same as norm except that we do not demand that only the zero element has zero norm.

in the study of topology defined by a system of semi-norms requires nets instead of sequences??

• **semi-semianalytic partition:** A semi-semianalytic partition of an open set is analagous to a semianalytic partition, except that the functions h are not required to be analytic, just semianalytic.

• **semisimple group:**

Any compact Lie algebra is semisimple.

• **separating:** A collection of functions $\mathcal{C}$ on a (topological) space X is said to separate its points if and only if for any $x_1 \neq x_2$ we can find $f \in \mathcal{C}$ such that $f(x_1) \neq f(x_2)$.

The only if part of the definition says given the values assumed by each and every function in the collection $\mathcal{C}$ exists a unique point $p \in X$ for which the functions take their given values.

They encode all the information about the (topological) space X. This is the starting point for non-commutative geometry...

An important example is that of gauge theory when connection representation to the loop representation. It separates points of $\mathcal{A}/\mathcal{G}$ in the sense that, if $[A_1] \neq [A_2]$, there exists a loop α such that: $T_\alpha(A_1) \neq T_\alpha(A_2)$. The set of configuration variables is sufficiently large in that they encode all the information that is contained in a connection

these functionals form a separating set on $\mathcal{A}/\mathcal{G}$: if all the T_α assume the same values at two connections, they are necessarily gauge related.

• **separation conditions:** There is a whole hierarchy of separation conditions, here we mention a few of them. Let $\mathcal{T}$ be a topological space, and let P and Q be two distinct points of $\mathcal{T}$. $\mathcal{T}$ is called

(i) T_0 if at least one of the points has a neigbourhood excluding the other,

(ii) T_1 if each point has a neigbourhood excluding the other,

(iii) T_2 is the Hausdorff condition holds.

One more separation condition of note is a normal topological space.

• **sesquileanr forms:**

(*i*) $F(\alpha u + \beta v, w) = \overline{\alpha} F(u, w) + \overline{\beta} F(v, w)$, and
(*ii*) $F(u, \alpha v + \beta w) = \alpha F(u, v) + \beta F(u, w)$.

sesquileanr form with....

• **sets:** a **set** is a collection of "things".

Standard notation for often-used sets

$\emptyset = \{\}$ = set with no elements

$\mathbb{Z}$ = the integers

$\mathbb{Q}$ = the rational numbers

$\mathbb{R}$ = the real numbers

$\mathbb{C}$ = the complex numbers

• **shadow state:** A shadow state $|\Psi_\gamma^{shad}>$ is an element $(\Psi|$ of Cyl^* projected to the subspace Cyl_γ by the projection operator $\hat{p}_\gamma$:

$$(\Psi|\hat{p}_\gamma : \sum_{\vec{x}_i \in \gamma} \Psi(\vec{x}_i)|\vec{x}_i > \equiv |\Psi_\gamma^{shad} >, \tag{B.-32}$$

with $(\Psi| = \sum_{\vec{x}} \overline{\Psi}(x_i)(\vec{x}|$.

See physics glossary.

• **sheaf:** See presheaf.

• **sieve:** closed under post composition.

• $\sigma-$**additive:** the measure of a *countable* union of non-intersecting measurable sets is equal to the sum of their measures:

$$\mu\left(\bigcup_i^\infty A_i\right) = \sum_i^\infty \mu(A_i) \quad \text{where} \;\; A_i \cap A_j = \emptyset \text{ for any} i \neq j. \tag{B.-32}$$

• $\sigma-$**algebra:**

The word "sigma" refers to sum, meaning union, while the word "algebra" indicates that $\mathcal{M}$ is defined in terms of certain operations, in this case unions and complements of sets

• **simple functions:** Let X be a non-empty set. Then a *simple function* s is a mapping from X to the real line, i.e. $s :\rightarrow R$, such that s only takes finitely many different values.

• **simple representations:** The representation π of G in the vector space V over k is said to be simple if no proper subspace of V is stable under G. That is, π is simple if the following property holds: if U is a subspace of V such that

$$g \in G, \quad u \in U \Rightarrow gu \in U$$

then either $U = 0$ or $U = V$.

• **simplex:** the most elementary geometric figure of a given dimension. For zero dimension it is a point, in two dimensions it is the line, in three it is the tetrahedron in, the 4-simplex in four dimensions, etc. B

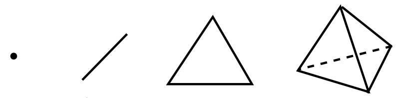


Figure B.25: Simplices in 3d.

The formal definition of a $p-$simplex is the following: a $p-$simplex $\sigma^{(p)} = [\nu_0, \nu_1, \ldots, \nu_p]$ embedded in $\mathbb{R}^d$ is the convex hull of $p+1$ affinely independent vertices $\nu_0, \nu_1, \ldots, \nu_p \in \mathbb{R}^d$. That is, $\sigma^{(p)}$ is the set of points

$$\sigma^{(p)} := \left\{ \sum_{i=0}^{p} t_i \nu_i | \sum_{i=0}^{p} t_i = 1 \text{ with } t_i \geq 0 \forall i \right\},$$

which spans a $p-$dimensional vector space.

• **simplicial complex:**

• **Skien relation:** A recursive relation which allows us to assign a polynomial to a knot which is a topological invariant of the knot. A skien relation is an equation that relates the polynomial of links obtained by changing the crossings in the projection of the original link. Skien relations allows us to calculate the invariants by decomposing the link step by step to a union of unkotted, unlikned loops.

If two knots have different polynomials then they are topologically inequivalent. Two different knots can have the same polynomial.

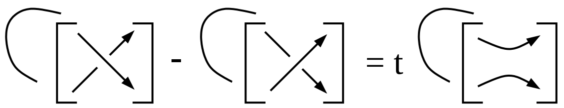


Figure B.26: Skien relation for the Conway polynomial.

We want a way of obtaining the bracket polynomial of a link in terms of the bracket polynomials of simpler links. Skein relations.

The holy grail of knot theory is to have a recursive relation that distinguishes between all topologically inequivalent knots. The standard example is evaluation of the Conway polynomial for the Trefoil knot

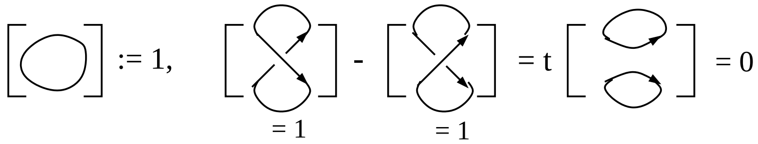


Figure B.27: Skien relation for the Conway polynomial.

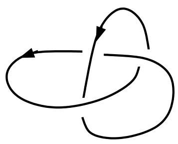


Figure B.28: The Trefoil knot.

(see knot invariants, Kauffman bracket, knot polynomials,...)

• **smooth curve:** A curve in Euclidean space $\mathbb{R}^n$ is smooth if and only if it is infinitely differentiable. A curve in a manifold $\mathcal{M}$ is smooth if and only if its image under a chart is a smooth curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval (a, b) to $\mathbb{R}^n$ in Fig.(C.7.1) is a analytic map. These curves are denoted as $C^\infty-$curves.

Finite differentiable curves (C^r-curves) are defined in the obvious way.

• **smooth function:** A smooth function is a function that is infinitely differentiable, that is, it does not matter how many times you differentiate the function, the resulting functions are always continuous. Such functions are denoted C^∞.

• **Sobolev embedding theorem:** It shows that $C^k(\omega) \subset H^s(\omega)$ (with continuous inclusion - i.e. the C^k-norm is bounded in terms of the H^s-norm) provided that $s > k + n/2$.

• **Sobolev inequalities:** One can relate the Sobolev norm to more usual norm via Sobolev inequalities.

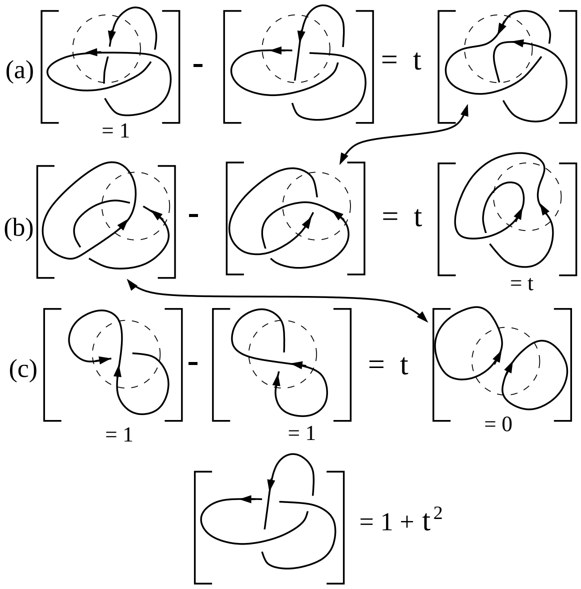


Figure B.29: Evaluation of the Conway polynomial for the Trefoil knot $1 + t^2$.

• **Sobolev norm:** The Sobolev norm is used in the standard formulation of well-posed initial value problems in a general globally hyperbolic spacetime [7] - stability of closed trapped surfaces away from spherical symmetry (I think). Energy inequalities. Relate initial conditions to the solution at a time t later. Can relate the Sobolev norm to more usual norm via Sobolev inequalities.

E.g. Consider the Klein-Gordon equation in $(1+1)$- dimensional Minkowski spacetime, $(-\partial_t^2 + \partial_x^2)f = 0$ defined on a suitable region of spacetime. (Who's Afraid of Naked Singularities? gr-qc/9907009)

$$\|f\| := \left(\frac{q^2}{2} \int dx |f|^2 + \frac{1}{2} \int dx \left| \frac{df}{dx} \right|^2 \right)^{\frac{1}{2}}, \tag{B.-32}$$

where q^2 is a positive constant.

$$\|\psi, \mathcal{N}\|_m = \left[\sum_{i=0}^{m} \int_{\mathcal{N}} |D^p \psi|^2 d\sigma \right]^{1/2} \tag{B.-32}$$

$$\|\psi\| < Const \times \|\psi, \tilde{\mathcal{N}}\|_m \tag{B.-32}$$

• **Sobolev space:** The **Sobolev space or** H^1 is the function space $\mathcal{H} = \{f \| f\| < \infty\}$. with inner product

$$(f,g) := \left(\frac{q^2}{2} \int dx f^* g + \frac{1}{2} \int dx \frac{df^*}{dx} \frac{dg}{dx} \right)^{\frac{1}{2}}, \tag{B.-32}$$

so that $\|f\|^2 = (f,f)$.

• **span:** For a nonempty subset $M \subset X$ the set of all linear combinations of vectors of $x_i \in M$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, \tag{B.-32}$$

is called the span of M, written span M. The completion of span M is denoted $\overline{\text{span}\ M}$.

• **spectral mapping theorem:** If a $\mathcal{A}$ is a C^*-algebra and if $A \in \mathcal{A}$ is normal then $\sigma_{\mathcal{A}}(f(A)) = f(\sigma_{\mathcal{A}}(A))$ for all continuous functions f. This result is known as the spectral mapping theorem. Can be generalized to f measurable function.

• **spectral measure:**

• **spectral radius:**

$$\sigma(A) = \lim_{n\to\infty} \|A^n\|^{1/n} \tag{B.-32}$$

• **spectral representation:** A self-adjoint operator in a (finite) Hilbert space $\mathcal{H}$ has a spectral representation - it's eigenstates form a complete orthonormal basis in $\mathcal{H}$. We can express a self-adjoint operator $\mathbf{A}$ as

$$\mathbf{A} = \sum_n a_n \mathbf{P_n},$$

where a_n is an eigenvalue of $\mathbf{A}$ and $\mathbf{P_n}$ is the corresponding orthogonal projection onto the space of eigenvectors with eigenvalues a_n.

• **spectral theorem:** Every Hertitian matrix is unitarily equivalent to a diagonal one. The spectral theorem is the generalisation of this assertion to operators on Hilbert space.

We first review the spectral theorem for $\mathcal{H}$ finite dimensional:

Suppose that $\mathcal{H}$ is finite dimensional. If $T \in \mathcal{B}(\mathcal{H})$, then $\sigma(T) \neq \emptyset$. Furthermore, $\lambda \in \sigma(T)$ if and only if λ is an eigenvalue of T. Let $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$. Let M_i the corresponding eigenspaces and P_i be the orthogonal projections on these eigenspaces. The spectral theorem then states that the following three conditions are equivalent:

(a) The M_i's are pairwise orthogonal and span $\mathcal{H}$;

(b) The P_i's are pairwise orthogonal

$$I = P_1 + \cdots + P_n.$$

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n;$$

(c) T is normal.

This decomposition is usually called the Spectral Theorem in finite dimensions.

In general, the classical Spectral Theorem says that each normal $T \in \mathcal{B}(\mathcal{H})$ is associated to a (unique) $\mathcal{H}-$projection valued measure on $\sigma(T)$, so that

$$T = \int_{\sigma(T)} d\lambda P(\lambda).$$

• **spectrum:** Infinite dimensional spaces there are operators that have no eigenvalues. For example, $(x_1, x_2, x_3, \dots)$ the operator that translates the components by one place - $(0, x_1, x_2, \dots)$ has no eigenvectors. The next best thing that is useful... is the spectrum of an operator A. The spectrum is defined as the set of values λ for which the operator $(A - \lambda\hat{I})$ is not invertible.

Let $\mathcal{U}$ be a C^*-algebra. The spectrum $\sigma(\mathcal{U})$, of $\mathcal{U}$ is the set of all characters on $\mathcal{U}$.

Any Abilian C*-algebra with identity is naturally isomorphic with the C*-algebra of all continuous functions on a compact, Hausdorff space, called the spectrum of the algebra.

• **spinor group:** The spinor group Spin(n) is a particular double cover of the rotation group $SO(n)$.

$Sp(2n)$ is generated by $2n \times 2n$ matrices X that satify

$$XG + GX^T = 0, \tag{B.-32}$$

where $G = -G^T$ is some non-degenerate antisymmeteric matrix. We can write G, and the generators X, as tensor products 2×2 and $n \times n$ matrices. One takes

$$G = \sigma_2 \otimes \mathbf{1}, \tag{B.-32}$$

where $\mathbf{1}$ is the $n \times n$ unit matrix, and σ_2 the second Pauli matrix. Explicitly G is

$$G = \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix}$$

The generators X are Hermitian matrices, and in addition they must satisfy (B). With G given by (B), the set of all X can be obtained from the following sets of matrices:

$$\mathbf{1} \otimes A, \quad \sigma_1 \otimes S_1, \quad \sigma_2 \otimes S_2, \quad \sigma_3 \otimes S_3.$$

Here A is an arbitrary $n \times n$ imaginary antisymmetric matrices, S_1, S_2 and denote S_3 denote arbitrary $n \times n$ real symmetric matrices. Explicitly one has

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} 0 & S_1 \\ S_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -iS_2 \\ iS_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} S_3 & 0 \\ 0 & -S_3 \end{pmatrix}.$$

• **stable:** Let G act on a set X. A subset $S \subset X$ is said to be **stable** under the action of G if

$$g \in G \quad x \in S \Rightarrow gx \in S. \tag{B.-32}$$

• **stablizer:** The **stabalizer** of an element $x \in X$ is

$$Stab\ (x) = \{g \in G : gx = x\}. \tag{B.-32}$$

It is a group - It's obviously associative and closed. $\mathbf{1} \in Stab\ (x)$ as $\mathbf{1}x = x$. If $gx = x$ then $g^{-1}gx = g^{-1}x$ so that $g^{-1}x = x$, hence $g^{-1} \in Stab\ (x)$.

• **states:** A state on a $^*-$algebra is a linear functional $\omega : \mathbb{U} \to \mathbb{C}$ which is positive, that is, $\omega(A^*A) \geq 0$ for all $A \in \mathbb{U}$. The states that physicists are most familiar with are vector states, that is, if we are given a representation $(\mathcal{H}, \pi)$ and an element ψ in the common domain of all the $A \in \mathbb{U}$ then $a \mapsto < \psi, \pi(A)\psi >_{\mathcal{H}}$ evidently defines a state.

• **stratification:** [90] or [102] Let $\mathbb{X}$ be a manifold of differentiability p, and U be a subset of $\mathbb{X}$. Then:

$\mathcal{M}$ is called a **stratification** of $\mathbb{X}$ is a locally finite, disjoint decomposition of $\mathbb{X}$ into connected embedded C^p manifolds $\mathbb{X}_i$ of $\mathbb{X}$, such that:

$$\mathbb{X}_i \cap \mathbb{X} \neq \emptyset \Rightarrow \mathbb{X}_i \subseteq \partial\mathbb{X}_j \text{ and } dim\mathbb{X}_i < dim\mathbb{X}_j. \tag{B.-32}$$

for all $\mathbb{X}_i, \mathbb{X}_j \in \mathcal{M}$.

The elements of the decomposition are called **strata**. $\mathcal{M}$ is called a stratification of U, if and only if U is the union of some elements of $\mathcal{M}$.

• **stratified diffeomorphism:** a stratified map f is a stratified diffeommorphism if and only if $f|_{\mathbb{X}_i}$ is a injective and the restriction of each f_i to the restriction of each f_i to the respective U_i is a C^p-diffeomorphism.

• **stratified isomorphism:** A map f is a stratified isomorphism if and only if in addition to being a stratified mononmorphism, f is a homeomorphism and each $f_i : U_i \to f_i(U_i)$ is a C^p diffeomorphism.

• **stratified map:** Let f be a continuous map from C^p-manifold $\mathbb{X}$ to a $\mathbb{Y}$-manifold. The map f is called a stratified map if and only if:

There is a pair of stratifications $\mathcal{M}$,$\mathcal{N}$ of $\mathbb{X}$ respectively $\mathbb{Y}$, and for each stratum $\mathbb{X}_i$ there exists an open neighbourhood U_i and a C^p map $f_i : \mathbb{X}_i \subseteq U_i \to \mathbb{X}$ with:

$$\overline{\mathbb{X}_i} \subseteq U_i, \quad f_i|_{\mathbb{X}_i} = f|_{\mathbb{X}_i}, \quad \mathbb{X}_i \in \mathcal{N} \quad \text{and} \quad rankf|_{\mathbb{X}_i} = dimf(\mathbb{X}_i).$$

• **stratified monomorphism:** A map f is a stratified monomorphism if and only if in addition to being a stratified map, $f|_{\mathbb{X}_i}$ is injective.

• **strongly continuous one-parameter unitary group:** An operator valued function satisfying

(i) For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$.

(ii) If $\varphi \in$ and $t \to t_0$, then $U(t)\varphi \to U(t_0)\varphi$,

is called a strongly continuous one-parameter unitary group.

• **strongly convergence:** A sequence of elements $h_1, h_2, \dots$ of a Hilbert spaceis said to coverge strongly to an element h if

$$\lim_{n\to\infty} \|h_n - h\| = 0.$$

• **subcover:** See cover.

• **subgroup:** A subgroup of G is a subset $H \subset G$ such that

(a) $E_G \in H$

(b) $xy \in H$ for all $x, y \in H$

(c) $x^{-1} \in H$ for every $x \in H$.

• **submanifold:**

This class of subsets of a manifold should exclude anything with sharp corners, such as the surface of a cube or of a cone, or anything whose dimensioon could be said to vary.

• **subobject:** A subobject is the analogue of the set-theoretical idea of a subset.

• **subobject classifier:** [361] encodes the possible answers to natural multiple choice question one can ask of the 'elements' of which objects in the topos are built.

While the category **Set** has just the two truth-values , $\{0, 1\}$, so also in any topos the collection of truth-values is an object in the topos - the subobject classifier, written Ω.

• **subring:** Let A be a ring. A **subring** of A is a subset containing I that is closed under addition, multiplication, and the formation of negatives.

• **sujective:** A function is surjective if it is onto.

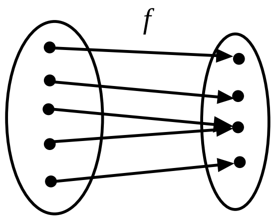


Figure B.30: surjective

See injective, bijective.

• **symmetric operator:** An operator A is called symmetric if its domain of dependence $D(A)$ is contained in the domain of dependence, $D(A^\dagger)$, of its adjoint operator $A^\dagger$, i.e. $D(A) \subset D(A^\dagger)$, and if $A^\dagger = A$ for all $A \in D(A)$.

A linear operator on a Hilbert space H is calledsymmetris if the condition

$$< Ax|y =< x|Ay > \tag{B.-32}$$

for all elements x and y in the domain of A. A symmetric every-where- defined operator s self-adjoint.

• **sympletic manifold:** A sympletic manifold (M, ω) is a smooth real $n-$dimensional manifold M without boundary, equipped with a closed non-degenerate two-form ω. The most familiar example of a sympletic manifold is a cotangent bundle $M = T^*Q$. This is nothing but the traditional phase space, Q being the configuration space.

• **tangles:**

• **Temperly-Lieb algebra:**

• **tempered distributions:** The tempered distributions are continuous linear maps from the Schwarz space to complex numbers.

• $10j$ **symbol:** The $10j$ symbol depends on ten spins and appears as the vertex amplitude in the Barrett-Crane model.

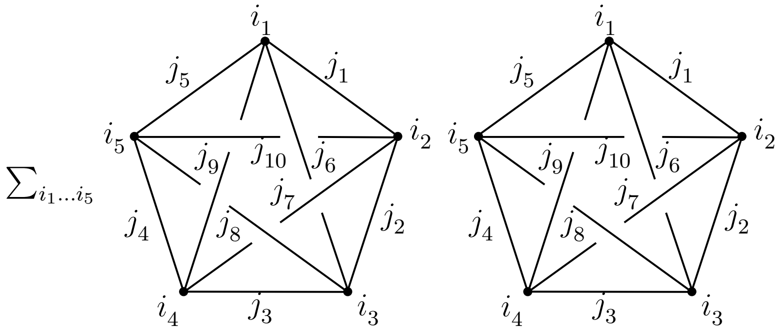


Figure B.31: tenjsymbol. This depends on ten spins and is called the $10j$ symbol.

• **tensor:** covariance etc.

• **tensor algebra:** It is the direct product $\otimes$ that is the binary operation or product that make the tensor algebra an algebra,

The tensor product of any number of elements can be taken as many times as one likes The tensor algebra $T(\mathcal{L})$ is the collection of (including up to an infinite number) collectively as direct summation:

$$T(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \ldots . \tag{B.-32}$$

• **Theta function:**

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \tag{B.-32}$$

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right). \tag{B.-32}$$

• **three-manifolds:** All three-dimensional topologies can be constructed by starting with a three-sphere S^3, drawing all possible links on it, and performing all possible surgeries on them, (although one may need to perform an infinite number of surgeries to construct some manifolds). The classification of three-manifolds is related to the classification of all links.

Theorem (Lickorish and Wallace): Every closed, orientable, connected three-manifold, M^3 can be obtained by surgery on an unoriented framed knot or link $[L, f]$ in S^3. [hep-th/99071119] topological quantum field theories - a meeting ground for physicists and mathematicians

• **Tomita theorem:** Given any state ρ over a von Neumann algebra, there is always a flow α_t, called the Tomita flow of ρ, such that

$$\rho_0[\alpha_t(A)B] = \rho_0[\alpha_{(-t-i\beta)}(B)A] \tag{B.-32}$$

• **topological dual:** See dual spaces.

• **topological vector space:** A topological vector space is a vector space with a topology defined on it such that the operations of addition and scalar multiplication are continuous.

• **topological structure:** In order to have a notion of convergence of points in a space X and a notion of continuity for functions $f : X \to Y$ into some space Y, one has to give X a topological structure.

• **topology:** A very far from being a comprehensive account of the subject of topology.

topology as determined by which functions are continuous. Weakest topology so that a given set of mappings are continuous.

Example: projection mappings $p_\gamma \to$ Hausdorff topology

• **topology of pointwise convergence:** Topology which arises from the seminorm given by

$$\|f\|_x = |f(x)|.$$

The space of functions with this topology is called the space of pointwise convergence.

• **topological space:** A topological space X is a set, with a specified family of subsets...

• **topos:** Very roughly, a topos is a category which behaves much like the category of sets; in fact this category, which is denoted **Set**, is itself a topos.

OR: topos is a categorical model of constructive set theory.

a branch of category theory.

applications to interpretive problems in quantum theory and quantum gravity, see [362].

Non-Boolean logic - that does not assume every statement can be judged true or false, there are statements upon which one cannot decide. Instead a Boolean algebra one has a Heyting algebra. The notion of a topos plays provides a good framework for making sense of the idea of partial truth.

Fotini Markopoulou should be modified to take into account the fact that observers can only give truth values to observables that have to do with their backwards light cone.

topos: To find out more about the subject to John Baez homepage (http://math.ucr.edu/home/baez/README.html) "fun stuff" and then to the link topos.

1. an initial object (an object like the empty set)
2. a terminal object (an an object like a set with one element)
3. binary coproducts (something like the disjoint union of two sets)
4. binary products (something like the Cartesian product of two sets)
5. equalizers (something like the subset of X consisting of all elements x such that $f(x) = g(x)$, where $f, g : X \to Y$)
6. coequalizers (something like the quotient set of X where two elements $f(y)$ and $g(y)$ are identified, where $f, g : Y \to X$)

(3) analogy of $\{0, 1\}$ denoted Ω, intuitively, the elements of Ω are the answers to a natural 'multiple-choice questions' about objects in the objects in the topos, just as "$x \in X$" is a natural for sets.

coming from [362]

See section **??**.

• **total function:** See partial function.

• **totally bounded:** A set X for which all $\epsilon > 0$ there are a finite number of points $x_1, \ldots, x_n$ such that

$$X \subset \bigcup_{i=1}^{n} B_\epsilon(x_i). \tag{B.-32}$$

• **trace:** Let $\{e_n\}_{n\in\mathbb{N}}$ be a basis for a Hilbert space $\mathcal{H}$ and let A be an operator on $\mathcal{H}$. The trace $\mathrm{Tr}(A)$ is defined as $\sum_{n=1}^{\infty} < Ae_n, e_n >$, whnever this limit exists.

• **trace class operators:** Let $\mathcal{L}_1$ denote the space of, necessarily compact, operators A such that $\mathrm{Tr}(|A|)$ exists. Then $A \mapsto \mathrm{Tr}(|A|)$ defines a norm on $\mathcal{L}_1$. The space $\mathcal{L}_1$ is called the space of trace class operators. The name trace class comes from its property that if A is trace class, then for any orthononormal basis $\{\varphi_n\}$

$$tr(A) = \sum_n < e_n, Ae_n > \tag{B.-32}$$

is finite and independent of the orthononormal basis.

• **transition functions:** Whenever we define an object by use of local coordinates, it must have the same meanig in all coordinate systems. For it to have a basis-free significance, one must require the object's coefficients to have a special transformation property under a chgange of basis, this is acheived by the transition functions. As an example, the components of a vector field in one coordinate system must be related to those in another overlapping coordinate system by the Jacobian matrix which is the corresponding transition function. In practical situations transition functions are the gauge transformations required for pasting local charts together.

• **transitively:** given any two points x, y, in a group G there is at least one $g \in G$ that takes x to y, i.e. $gx = y$.

• **triangulable:** A space X is said to be triangulable if there exists a simplicial complex K_X that is exactly homeomorphic to X.

• **triangulation:** When a space X is triangulable the pair (X, K_X) is called a triangulation of X. The triangulation of a space is not unique.

• **trivialisation:** Say P is a fibre bundle with typical fibre F and base space $\mathcal{M}$. Trivialisations are global maps and only apply to trivial bundles. For each $x \in \mathcal{M}$, a trivialisation is an isomorphism,

$$\phi : P \to \mathcal{M} \times F$$

sending each fibre $\pi^{-1}(x) \in P$ to $\{x\} \times F$.

• **twists and writhe:**

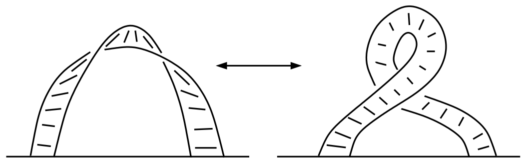


Figure B.32: Twists going to writes.

$$L = Tw + Wr \tag{B.-32}$$

• **Tychonov topology:** The Tychonov topology on the direct product $X_\infty = \prod_{l \in \mathcal{L}} X_l$ of topological spaces X_l is the weakest topology such that all projections

$$p_l : X_\infty \to X_i;\ (x_{l'})_{l' \in \mathcal{L}} \to x_l \tag{B.-32}$$

are continuous, that is, a net $x^\alpha = (x_l^\alpha)_{l \in \mathcal{L}}$ converges to $x = (x_l)_{l \in \mathcal{L}}$ if and only if $x_l^\alpha \to x_l$ for every $l \in \mathcal{L}$ pointwise (not necessarily uniformally) in $\mathcal{L}$.

• **uniform boundedness theorem:** A fundamental theorem of functional analysis. Let X be a Banach space and Y a normed space. Let $\Phi \subseteq B(X, Y)$ be the set of bounded operators from X to Y which is pointwise bounded, in the sense that, for each $x \in X$ there is some $c \in \mathbb{R}$ so that

$$\|Tx\| \leq c$$

for all $T \in \Phi$. Then Φ is uniformaly bounded: There is some constant C with

$$\|T\| \leq C$$

for all $T \in \Phi$.

• **uniform convergence:** A series of functions $\{f_n(x)\}$ is said to converge uniformally to $f(x)$ if when we put an $\epsilon-$tube around the function $f(x)$, the functions $f_n(x)$ eventually fit inside this.

• **uniform Rovelli-Smolin topology:** The Hamiltonian constraint operator $\hat{S}^\epsilon(N)$ does not converge with respect to the weak operator topology in $\mathcal{H}_{kin}$ when $\epsilon \to 0$. The convergence of $\hat{S}^\epsilon(N)$ holds with respect to the uniform Rovelli-Smolin topology, where one defines $\hat{S}^\epsilon(N)$ to converge if and only if $\Psi_{Diff}[\hat{S}^\epsilon(N)\phi]$ converge for all $\Psi_{Diff} \in Cyl^*_{Diff}$ and $\phi \in Cly(\overline{\mathcal{A}/\mathcal{G}})$

• **uniqueness theorems:**

Some important examples:

Stone-von Neumann theorem

• **unitary transformations:** We must make a distinction between unitary and isometric transformations which do not arise in finite-dimensional vector spaces. An isometry is defined in vector spaces of any dimension as a linear transformation U satisfying

$$(Uf, Ug) = (f, g)$$

for all in the vector space. This implies that $U^\dagger$ is a left inverse of U:

$$U^\dagger U = I.$$

now in finite-dimensional space, the existence of a left inverse guarantees the existence of a right inverse, but on an infinite-dimensional space this is not always the case. If a right inverse also exists, U is said to be unitary. This right inverse must be equal to the left inverse, since if $BA = I = AC$, then $B = C$. Thus for unitary transformations, we write

$$U^\dagger = U^{-1}.$$

• **universal cover:**

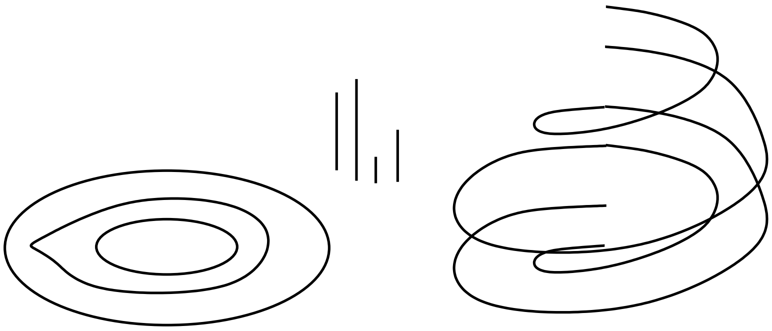


Figure B.33: unicoverEx.

• **universal enveloping algebra:**

$U_q(\mathbf{g})$ denotes the universal enveloping algebra - i.e., all formal powers and linear combinations of the elements of the deformed algebra modulo the standard Lie algebra relations.

Let g be a Lie algebra.

We know $T(\mathcal{L})$ is an associative algebra.

$$[x, y] - (x \otimes y - y \otimes x) \tag{B.-32}$$

this collection forms a two-sided ideal I. $z \in I$

described as "universal" because any Lie algebra homomorphism $\psi\mathbf{g} \to A$, where A is a *unital* associative algebra, extends uniquely to a unital algebra homomorphism.

$$z(x \otimes y - y \otimes x) \tag{B.-32}$$

$$T(\mathcal{L})/I \equiv U(\mathcal{L}) \tag{B.-32}$$

$$\begin{aligned} \Delta\xi &= \xi \otimes 1 + 1 \otimes \xi \quad \text{coroduct} \\ \epsilon\xi &= 0 \quad \text{counit} \\ S\xi &= -\xi \quad \text{antipode} \end{aligned} \tag{B.-33}$$

It is a **biideal** The quotient B/I is a bialgebra. An element of a general Hopf algebra which has this linear form

$$\Delta\xi = \xi \otimes 1 + 1 \otimes \xi \tag{B.-33}$$

is called **primitive**.

$$\Delta(a) = a \otimes a + b \otimes c$$
$$\Delta(b) = a \otimes b + b \otimes d$$
$$\Delta(c) = c \otimes a + d \otimes c$$
$$\Delta(d) = c \otimes b + d \otimes d \quad \text{(B.-35)}$$

• **universal home:** a certain completion, $\overline{\mathcal{A}/\mathcal{G}}$, of $\mathcal{A}/\mathcal{G}$ the universal home for measures

• **universal net:**

• **upper semi-continuous:** A function $f(x)$ is said to be upper semi-continuous if for any x in the domain of f and for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(x) - f(x_0) < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

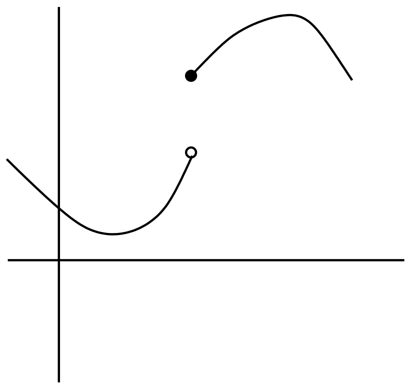


Figure B.34: An upper semi-continuous function.

• **Vassiliev Invariants:**

• **weakly continuous:**

$$(\dot{f}, x) \quad \text{(B.-35)}$$

• **weak convergence:** A sequence of elements $h_1, h_2, \ldots$ of a Hilbert spaceis said to coverge strongly to an element h if

$$\lim_{n\to\infty}(h_n, g) = (h, g)$$

for all $g \in H$.

• **weakness of a topology:**

Roughly, one topology is weaker that another if it has fewer open sets, and stronger than another if it has more open sets. Let X be a non-empty set. $\emptyset, X$ is the weakest topology and the discrete topology is the strongest topology on X. The more open sets there are, the more continuous functions the space has.

• **weak operator topology:** The *weak operator topology* on $\mathcal{B}(H)$ is the weak topology generated by all functions of the form $T \to (Tx, y)$; that is, it is the weakest topology with respect to which all these functions are continuous. It is easy to see from the inequality $|(Tx, y) - (T_0 x, y)| \leq \|T - T_0\| \, \|x\| \, \|y\|$ that this topology is weaker than the usual norm topology, so that its closed sets are also closed in the usual sense. A C^*-algebra with the further property of being closed in the weak operator topology is called a W^*-algebra. Algebras of this kind are also called *rings of operators, or von Neumann algebras.*

(from intro topology and modern analysis)

• **weak topology:** The weak topology on X^* is the topology such that all functionals on X^{**} are continuous.

• **weak star operator topology:** The weak $*$topology with respect to a Hilbert space $Y = X$: this is similar to the weak topology, however, insted of $X' = X$ we now take a subspace $\mathcal{D}$ of X equipped with a finer topology and as $\mathcal{D}'$ teh topological dual of that topological space. Physical applications are the topology in which the Hamitonian constraint converges and Refined algebraic quantization (RAQ).

The weak star topology is obtained if we use the absolute values of $| < \Phi | A | \Psi > |$ between arbitrary state vectors as a system of seminorms. Thus a sequence of operators converges weakly if all matrix elements converge.

• **webs:** From [arXiv:math-ph/0304002]: For technical purposes, one assumed in the very beginning that these graphs are formed by piecewise analytic paths only. Namely, only in this case two finite graphs are always both contained in some third, bigger graph being again finite. This restriction has the drawback that only analyticity preserving diffeomorphisms can be implemented into that framework. In order to guarantee the inclusion of all diffeomorphisms, at least, piecewise smooth and immersive paths have to be considered as well. For the first time, this has been done by Baez and Sawin [q-alg/9507023] introducing so-called webs. A web is defined as a piecewise smooth graph determined by the union of a finite number of smooth curves that intersect in a controlled way, albeit possibly a countably infinite number of times.

• **Weierstrass approximation theorem:** Any real-valued continuous function f on $[a, b]$ can be arbitrary well approximated by a finite polynomial: given any $\epsilon > 0$, there is a polynomial P such that $\|f - P\| < \epsilon$:

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x,\ a \leq x \leq b. \tag{B.-35}$$

We can restate the theorem as: polynomials are dense in the space of real-valued continuous functions on $[a, b]$ (dense in the same kind of way that the rationals $\mathbf{Q}$ are dense in $\mathbb{R}$).

• **well-posedness of initial value problem:** An initial value problem compose by a differential equation together with initial conditions on an suitable boundary. Well-posedness of an initial value problem requires

(i) existence of solutions,

(ii) uniqueness of solutions,

(iii) continuous dependence of solutions on initial conditions.

• **Whitehead's theorem:** a smoothing. For each smooth manifold M, there exists a PL-manifold M_{PL}, called its Whitehead triangulation, so that M is diffeomorphic to a smoothing of M_{PL}. [397]

• **Whitehead's triagulation:** Whitehead's triagulation provide us with a way of "discretizing" spacetime which is not merely some approximation nor introduces a physical cut-off, but [397]

• **Wigner's $6-j$ symbol:**

$$W(j) = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \tag{B.-35}$$

• **Wigner transform:**

• **Von-Neumann algebra:** A von Neumann algebra is a *-algebra of bounded operators on a Hilbert space which is closed in the weak operator topology, or more explicitly:

A bounded operator B is a **weak limit** of a set of bounded operators if for each choice of positive number ϵ, positive integer n, and vectors $\psi_1, \psi_2, \ldots \psi_n$ and $\phi_1, \phi_2, \ldots \phi_n$ there is an operator A in the set such that

$$|(\phi_k, A\psi_k) - (\phi_k, B\psi_k)| < \epsilon \tag{B.-35}$$

for $k = 1, 2, \ldots, n$. The extension of a set of bounded operators to its weak limits is its **weak closure**. A set of bounded operators is **weakly closed** if it contains its weak limits. A symmetric ring of bounded operators which is weakly closed is a von Neumann algebra.

Applications are:

(i) in Algebraic Quantum Field theory. In algebraic Quantum Field theory associates a von Neumann algebra to each causally complete region of spacetime.

(ii) Tomita-Takesaki theorem - thermal states and thermal time hypothesis.

(iii) Quantum causal histories

(iv) Noiseless subsystems

(v) Infinite tensor product Hilbert spaces.

• **Voronoi Diagram:**

http://www.lepp.cornell.edu/spr/2001-06/msg0033516.html

• **Zorn's lemma** Let $X \neq \emptyset$ be a partially ordered set with the property that every linearly ordered subset $Y \subset X$ (i.e., $y \leq y'$ or $y' \leq y$ for all $y, y' \in Y$) has an upper bound $x_Y \in X$ (i.e., $y \leq x_Y$ for all $y \in Y$). Then X has a maximal element $m \in X$ (i.e., $m \leq x$ for $x \in X$ implies $x = m$) which is a common upper bound for all linearly ordered subsets.

Appendix C

Basic Mathematics

C.1 Introduction

This appendix contains a miscellaneous assortment of mathematical ideas and results. those topics of relevance to the subject of the report.

C.1.1 Summary of Tensor Calculus

- Linear spaces.
- Curvilinear coordinates
- Curved spaces.
- Vector and tensors: covariant and contravriant.
- Affine and metric connections.
- Differential manifolds.

C.1.2 Linear operators and Matrices

A linear operator T maps a vector space V onto itself which obeys linearity

$$T(a\hat{x} + b\hat{y}) = aT(\hat{x}) + bT(\hat{y}). \tag{C.0}$$

If we have a basis $\{\hat{e}_i, i = 1, \ldots, n\}$ for V, then

$$\hat{x} = \sum_{i=1}^{n} a_i \hat{e}_i \tag{C.0}$$

$$\begin{aligned} T(\hat{x}) &= T\left(\sum_{i=1}^{n} a_i \hat{e}_i\right) = \sum_{i=1}^{n} a_i T(\hat{e}_i) \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j T_{ij} \hat{e}_j, \end{aligned} \tag{C.0}$$

where we have replaced each vector $T(\hat{e}_i)$ by its component form $\sum_{j=1}^{n} T_{ij}\hat{e}_j$.

Two successive linear transformations T and U acting on a space V produce the transformation UT:

$$\begin{aligned} UT(\hat{x}) &= U(T(\hat{x})) = U\left(\sum_{j=1}^{n} a_i T_{ij} \hat{e}_j\right) \\ &= \sum_{j=1}^{n} a_i T_{ij} U(\hat{e}_j) \\ &= \sum_{j=1}^{n} a_i T_{ij} U_{jk} \hat{e}_k \\ &= \sum_{j=1}^{n} a_i \left(T_{ij} U_{jk}\right) \hat{e}_k \end{aligned} \tag{C.-2}$$

From which it follows that the components of UT are

$$\sum_{j=1}^{n} T_{ij} U_{jk}. \tag{C.-2}$$

We see that this sum is just the matrix product of the respective matrices.

C.2 Group Theory

A group G is defined as a set of objects or operations (called elements) that may be combined or multiplied to form a well defined product and that satisfy the following four conditions. If we label the elements $a, b, c, \ldots$, then the conditions are:

(1) If a and b are any two elements, then the product ab is an element.

(2) The defined multiplication is associative, $(ab)c = a(bc)$

(3) There is a unit element I, with $Ia = aI = a$ for all elements a.

(4) Each element has an inverse $b = a^{-1}$, with $aa^{-1} = a^{-1}a = I$ for all elements a.

In physics, these abstract conditions will take on a physical meaning, for example in terms of transformations of vectors and tensors.

C.2.1 Examples of Groups

Translation in time: $t' = t + a^0$
Translation in space: $\vec{r}' = \vec{r} + \vec{a}$
Rotation in space : $\vec{r}' = R\vec{r}$

Lie groups

Roughly, a Lie group is a group with an infinite number of elements but which can be parametrized by one or several real numbers (the total of which is refered to as the *dimension* of the Lie group).

The simplest example of a Lie group is $SO(2)$, the group of rotations in the plane. Each element $R(\theta)$ is labelled by rotation angel θ, with multiplication acting as $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$. Because the angle θ is defined only modulo 2π, the manifold of $SO(2)$ is a circle S^1.

interesting properties of Lie groups is that in neighbourhood of the identity element they can be expressed in terms of a set of generators T^a as

$$D(g) = \exp(-\alpha_a T^a) := \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \alpha_{a_1} \ldots \alpha_{a_n} T^{a_1} \ldots T^{a_n}, \tag{C.-2}$$

where $\alpha_a \in \mathbb{C}$ are a set of coordinates of M in a neighbourhood of $\mathbb{1}$.

Internal symmetries

for e.g. $SU(2)$.

finite-dimensional Lie groups of imporance relativity translations, rotations, Lorentz, Poincare and special unitary group $SU(2)$. Apart from these finite-dimentional Lie groups, the infinie-dimensional ones play an important role in GR. These are the Lie group of *diffeomorphisms* of the spacetime manifold.

two groups are isomorphic if there is a one-to-one correspondence between their elements and if they have exactly the same structure.

The neighbourhood of a group element is chararterized by the neighbourhood of the corresponding parameter set.

Cosets

Let G be a group and H be a subgroup of G. The set of the form

$$gH = \{gh : h \in H\} \tag{C.-2}$$

for $g \in G$ is called the (left) coset of H in G. There is one coset for each $g \in G$.

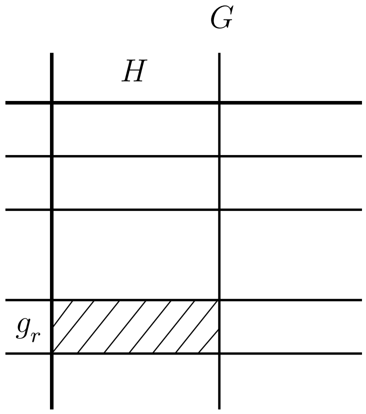


Figure C.1: coset. Suppose H is a subgroup of a finite group G, here the elements of H are listed first. The shaded box are the elements of the (left) coset of $g_r \in G$

As we will see now, cosets define an equivalence on G, where $g \sim g'$ if the set gH is equal to the set $g'H$, in other words there is an $h \in H$ with $gh = g'$.

Cosets have the property that if cosets overlap they are the same coset, i.e., that for $g_1, g_2 \in G$, $g_1 \neq g_2$

$$g_1 H = g_2 H \tag{C.-2}$$

or that different cosets do not overlap

$$g_1 H \cap g_2 H = \emptyset. \tag{C.-2}$$

Proof: Suppose there is an overlap between the two cosets corresponding to the elements g_1 and g_2, that is, for some $h_1, h_2 \in H$ we have

$$g_1 h_1 = g_2 h_2.$$

Therefore

$$g_1 = g_2 h_2 h_1^{-1}.$$

If h is *any* element of H then

$$g_1 h = g_2 h_2 h_1^{-1} h = g_2 h'$$

where $h' = h_2 h_1^{-1} h$. Since H is a subgroup h' is also an element of H. Now since no element of G appears more than once in each row, if cosets do overlap at all they are in fact the same coset.

proving (C.2.1).

□

Right cosets are defined similarly, they are the sets

$$Hg = \{hg : h \in H\}. \tag{C.-2}$$

These also define an equivalence realtion on the group G.

The equivalence class of a particular representative g is denoted

$$[g]. \tag{C.-2}$$

Now the set of cosets gH, under the binary operator

$$[g] \cdot [g'] = [gg']. \tag{C.-2}$$

forms a group if and only if H is what's known as a normal subgroup.

Definition Normal subgroups.

A normal N subgroup of a group G is

$$h \in H \text{ if } ghg^{-1} \in H \text{ for all } g \in G. \tag{C.-2}$$

We write $N \triangleleft G$ to indicate that N is a normal subgroup of G.

We need to establish that induced multiplication between equivalence classes is independent of the representative chosen, i.e., that $[g_1'][g_2'] = [g_1][g_2]$, where g_1 is any element of $[g_1']$ and similarly for g_2.

$$[g_1'][g_2'] = [g_1h_1][g_2h_2] = [g_1h_1g_2h_2] = [g_1g_2hh_2] = [g_1g_2h] = [g_1g_2] = [g_1][g_2]. \tag{C.-1}$$

$$[hg] = [h][g] = [e][g] = [eg] = [g] = [ge] = [g][e] \tag{C.-1}$$

$[e] = [h]$

$$[e] = [gg^{-1}] = [g][g^{-1}] = [g][g]^{-1} \tag{C.-1}$$

$[g]^{-1} = [g^{-1}]$

This makes G/N into a group, called the quotient group.

Definition The **Quotient group** is the subgroup is the collection of cosets, each being considered an element. The quotient group is denoted G/H.

We will be interested in other mathematical things such as algebras, rings, categories amongt others. There are analogous notions of the normal subgroups for each of these goining by the names of ideals, , quotient categories.

Simple and Semi-simple Lie Groups

A Lie group is called simple if it does not possess an invariant subgroup.

A Lie group is called semi-simple if it does not possess an abelian invariant subgroup. Note that a semi-simple subgroup can possess a non-abelian invariant sugroup.

An invariant subset is an ideal if

$$[A_i, G_j] = \sum a_{ijk} A_k \tag{C.-1}$$

Cartan killing form k_{ij} built from the structure constants

$$k_{ij} = C_{iab}C_{jba} \tag{C.-1}$$

Cartan killing form

The killing form k_{ab} is nondegenerate if the Lie algebra is semisimple

$$\det(k_{ij}) \neq 0. \tag{C.-1}$$

Proof:

we have to demonstrate that $\det(k_{ij}) = 0$ if there is an abelian ideal. suppose that the Lie algebra has an abelian ideal. generators belonging to the ideal will be labelled marked indices. Then for the b' column of the killing form,

$$\det(k_{ij'}) = C_{iab}C_{j'ba} = C_{ia'b}C_{j'ba'} \tag{C.-1}$$

as by (C.2.1) $C_{j'ba} = 0$ for those values of a which do not belong to the ideal. it follows from (C.2.1) that

$$\det(k_{ij'}) = -C_{a'ib}C_{j'ba'} = -C_{a'ib'}C_{j'b'a'} \tag{C.-1}$$

since $C_{a'ib} = 0$ for all $b \neq b'$ the same reason. $[A_i, A_j] = 0$ for an abelian ideal so

$$C_{j'b'a'} = 0$$

holds. As the j' column of the killing form vanishes it follows that

$$\det(k_{ij}) = 0. \tag{C.-1}$$

Cartan's condition can be stated that if a Lie algebra is semisimple its killing form is invertible, that is, there is an inverse $k^{ij} := (k_{ij})^{-1}$ for which

$$k^{ik}k_{kj} = \delta^i_j$$

holds.

C.2.2 Unitary Representations of Groups

R	Group elements.
$[G]$	The collection of group elements
R_a	An indexed group element.
$\alpha, \beta, \ldots$	Labels irreducible representations.
D	Matrix representation.
$D^{(\alpha)}$	Indexed irriduceable matrix representation.
$i, j, \ldots$	Labels components of matrix representations
h	Dimension of group.
χ	Character

Proof Let $\{A_a\}$ be a representation of the group G. Construct the Hermitian matrix $H = \sum_{a=1}^{h} A_a A_a^\dagger$

$$H^\dagger = [\sum_a^h (A_a A_a^\dagger)]^\dagger = \sum_a^h (A_a A_a^\dagger)^\dagger = \sum_a^h A_a A_a^\dagger = H \tag{C.0}$$

From the earlier discussion of matrix algebra, any Hermitian matrix can be diagonalized by a similarity transformation. Let

$$d = \sum_a^h U^\dagger A_a A_a^\dagger U = \sum_\alpha (U^\dagger A_a U)(U^\dagger A_a^\dagger U) = \sum_\alpha (U^\dagger A_a U)(U^\dagger A_a U)^\dagger \tag{C.0}$$

Hence, $d = \sum_a \tilde{A}_a \tilde{A}_a^\dagger$ where $\tilde{A}_a = U^\dagger A_a U = U^{-1} A_a U$. The elements of the diagonal d matrix are real and positive.

$$d_{jj} = \sum_a^h \sum_k (\tilde{A}_a)_{jk} (\tilde{A}_a^\dagger)_{kj} = \sum_a^h \sum_k (\tilde{A}_a)_{jk} (\tilde{A}_a)_{jk}^* = \sum_a^h \sum_k |(\tilde{A})_{jk}|^2 \tag{C.0}$$

for all $j = 1, \ldots, h$. Since d is diagonal, we can define its square-root $d^{1/2}$

$$d^{1/2} = \begin{pmatrix} d_{11}^{1/2} & 0 & \cdots & 0 \\ 0 & d_{11}^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{hh}^{1/2} \end{pmatrix} \tag{C.0}$$

and $d^{-1/2}$, which is given by an analogous expression. Evidently, $(d^{1/2})^2$ and $d^{1/2} d^{-1/2} = 1$, where I is the identity matrix. Diagonal matrices commute with each other, and we can write for the identity I

$$I = d^{-1/2} d d^{-1/2} = d^{-1/2} \sum_{a}^{h} \tilde{A}_a \tilde{A}_a^\dagger \, d^{-1/2}. \tag{C.0}$$

By the rearangement theorem, $\{\tilde{A}_b \tilde{A}_a\}$ all a, and any one b is equal to $\{\tilde{A}\}$ for all a. Hence,

$$\begin{aligned} I &= d^{-1/2} \sum_a (\tilde{A}_b \tilde{A}_a) \; (\tilde{A}_b \tilde{A}_a)^\dagger d^{-1/2} \\ &= d^{-1/2} \sum_a (\tilde{A}_b \tilde{A}_a \; \tilde{A}_a^\dagger \tilde{A}_b^\dagger) d^{-1/2} \\ &= d^{-1/2} \tilde{A}_b d^{1/2} \left(d^{-1/2} \sum_a \tilde{A}_a \tilde{A}_a^\dagger d^{-1/2} \right) d^{1/2} \tilde{A}_b^\dagger d^{-1/2} \end{aligned} \tag{C.-1}$$

using (C.2.2) in this gives

$$\begin{aligned} I &= (d^{-1/2} \tilde{A}_b d^{1/2})(d^{-1/2} \tilde{A}_b d^{1/2})^\dagger \\ &= (d^{-1/2} U^{-1} A_b U d^{1/2})(d^{-1/2} U^{-1} A_b U d^{1/2})^\dagger. \end{aligned} \tag{C.-1}$$

Let us define the matrix $B_a := d^{-1/2} U^{-1} A_a U d^{1/2}$. It is easy to see that B_a has the same multiplication table as A_α. If we have $A_a A_b = A_c$, then

$$\begin{aligned} B_a B_b &= (d^{-1/2} U^{-1} A_\alpha U d^{1/2})(d^{-1/2} U^{-1} A_b U d^{1/2}) \\ &= (d^{-1/2} U^{-1} A_a) \; (U d^{1/2} d^{-1/2} U^{-1}) \; (A_b d^{1/2} U) \\ &= (d^{-1/2} U^{-1} \; A_a A_b \; U d^{1/2}) \\ &= (d^{-1/2} U^{-1} A_c U d^{1/2}) = \; B_c \end{aligned} \tag{C.-3}$$

Therefore $\{B_a\}$ is a unitary reprsentation of the group G.

Representation of finite groups can always be taken to be unitary. It is essential that the sum over $g \in G$ converge. This is guaranteed for a finite group, but may not work for infinite groups. In particular, non-compact Lie groups, such as the Lorentz group, have no finite dimensional unitary representations.

C.2.3 Schur's First Lemma

Theorem C.2.1 *A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix.*

Proof:

We can take these matrices to be unitary without loss of generality. Suppose there is a matrix M that commutes with all of the A_a but which is not a constant multiple of the unit matrix:

$$MA_a = A_a M \tag{C.-3}$$

for all $a = 1, 2, \ldots, |G|$. By taking the edjoint of each of these equations, we obtain

$$A_a^\dagger M^\dagger = M^\dagger A_a^\dagger. \tag{C.-3}$$

Since A_a are unitary

$$A_a^{-1} M^\dagger = M^\dagger A_a^{-1}. \tag{C.-3}$$

Multiplying both this from left and right we obtain

$$M^\dagger A_a = A_a M^\dagger \tag{C.-3}$$

So that, if M commutes with every matrix of the representation, then so does $M^\dagger$. As such the Hermitian matries

$$H_1 = M + M^\dagger, \quad H_2 = i(M - M^\dagger) \tag{C.-3}$$

also commute with every matrix of the representation. We prove the statement of the theorem for Hermitician matrices. We start with

$$HA_a = A_a H. \tag{C.-3}$$

Since a hermitian matrix H can be diagonalized by some U, $D = U^{-1}HU$.

$$U^{-1}HUU^{-1}A_a U = U^{-1}A_a UU^{-1}HU, \tag{C.-2}$$

Let $\tilde{A}_a = U^{-1}A_aU$

$$D\tilde{A}_a = \tilde{A}_aD. \tag{C.-2}$$

A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix.

$$(\tilde{A}_a)_{ij}(D_{ii} - D_{jj}) = 0. \tag{C.-2}$$

(i) Suppose all the diagonal elements of D are distinct: $D_{ii} \neq D_{jj}$ if $i \neq j$. Then (C.2.3) implies that

$$(\tilde{A}_a)_{ij} = 0, \qquad i \neq j,$$

i.e., the off-diagonal elements of $\tilde{A}_\alpha$ must vanish. They form a reducible representatoin composed of d one-dimentional representations. Since the $\tilde{A}_a$ are obtained from the A_a by a similarity transformation, the A_a are a reducible representaion.

(ii) Suppose p of the diagonal elements of D are equal, but the remaining entries are distinct from these and from each other. Using similarity transfromations, (for example (C.2.3)), we can arrange the diagonal elements so that the first p are equal.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{C.-2}$$

Hence, we can assume this is the case without loss of generality, i.e. $D_{11} = D_{22} = \cdots = D_{pp}$, $D_{mm} \neq D_{nn}$, otherwise. The $(\tilde{A}_i)_{mn}$ must vanish for any pair of unequal diagonal entries.

$$\tilde{A}_i = \begin{pmatrix} B_1 & 0 \\ 0 & B_2, \end{pmatrix} \tag{C.-2}$$

where B_1 is a $p \times p$ matrix and B_2 is a $(p-d) \times (p-d)$ diagonal matrix.

We have shown that if a Hermtian matrix is not a multiple of the unit matrix and commutes with all the matrices of a representation, then that representation is necessarily reducible. Thus, if a non-zero Hermtian matrix which commutes with all the matrices of an irreducible representation that matrix must be a multiple of the unit matrix.

Given that $M+M^\dagger$ and $i(M-M^\dagger)$ are both Hermitician matrices that commute with all of the matrices of an irreducible representation they are a constant multiple of the unit matrix. we have $H_1 = M + M^\dagger = c_1 I$ and $H_2 = i(M - M^\dagger) = c_2 I$. We have that

$$M = H_1 - iH_2 = (c_1 - ic_2)I$$

and hence must be proportional to the unit marix.

□

C.2.4 Schur's Second Lemma

Theorem C.2.2 *Let $\{A_1, A_2, \ldots, A_{|G|}\}$ and $\{B_1, B_2, \ldots, B_{|G|}\}$ be two irreducible representations of a group G of dimensionalities d an d' respectively. If there is a matrix M such that*

$$MA_a = B_a M$$

for $a = 1, 2, \ldots, |G|$, then if $d = d'$, either $M = 0$ or the two representations are related by a similarity transformation. If $d \neq d'$, then $M = 0$.

As

$$MA_a = B_a M$$

then

$$A_a^\dagger M^\dagger = M^\dagger B_a^\dagger$$

using unitary

$$A_a^{-1} M^\dagger = M^\dagger B_a^{-1}. \tag{C.-2}$$

Multiplying on the right by M, we get

$$A_a^{-1} M^\dagger M = M^\dagger B_a^{-1} M. \tag{C.-2}$$

By the group properties B_a^{-1} is some B_b and A_a^{-1} is the corresponding A_b, and so by the posulate of the theorem, $MA_b = B_b M$, or

$$MA_b^{-1} = B_b^{-1}M$$

substituting this into (C.2.4), gives

$$A_a^{-1}M^\dagger M = M^\dagger M A_a^{-1}. \tag{C.-2}$$

Thus, the $d' \times d'$ matrix $MM^\dagger$ commutes with all the matrices of an irreducible representation. By Schur's first lemma, $MM^\dagger$ must therefore be a constant multiple of the unit matrix,

$$MM^\dagger = cI. \tag{C.-2}$$

(i) Say $d = d'$. If $c \neq 0$, equation (C.2.4) implies that

$$M^{-1} = \frac{1}{c}M^\dagger.$$

plus (C.2.4) can be rearranged

$$A_a = M^{-1}B_a M,$$

so the two representations are related by a similarity transfromation and are, therefore, equivalent and since $M^\dagger M = c^d$,it follows that $c^{-d'/2}M$ is a unitary matrix. If $c = 0$ then

$$(MM^\dagger)_{ij} = \sum_k M_{ik}(M^\dagger)_{kj} = \sum_k M_{ik}M^*_{jk} = 0. \tag{C.-2}$$

By setting $i = j$, we obtain

$$\sum_k M_{ik}M^*{}_{ik} = \sum_k |M_{ik}|^2 = 0 \tag{C.-2}$$

which implies that $M_{ik} = 0$ for all i and k, i.e., that M is the zero matrix.

(ii) Say $d \neq d'$. We take $d < d'$. Them M is a rectangular matrix with d columns and d' rows,

$$\begin{pmatrix} M_{11} & \cdots & M_{1d} \\ M_{21} & \cdots & M_{2d} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ M_{d'1} & \cdots & M_{d'd}. \end{pmatrix} \tag{C.-2}$$

We add $d' - d$ columns of zeros

$$N = \begin{pmatrix} M_{11} & \cdots & M_{1d} & 0 & \cdots & 0 \\ M_{21} & \cdots & M_{2d} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & & \vdots & & \vdots \\ M_{d'1} & \cdots & M_{d'd} & 0 & \cdots & 0 \end{pmatrix} \tag{C.-2}$$

Taking the adjoint of this matrix yields

$$N^{\dagger} = \begin{pmatrix} M_{11}^{*} & M_{21}^{*} & \cdots & M_{d'1} \\ M_{12}^{*} & M_{22}^{*} & \cdots & M_{d'2} \\ \vdots & & \cdots & \vdots \\ M_{d1}^{*} & M_{2d}^{*} & \cdots & M_{d'd}^{*} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{C.-2}$$

We have

$$NN^{\dagger} = MM^{\dagger} = cI.$$

Taking the determinant

$$\det(NN^{\dagger}) = \det(N)\det(N^{\dagger}) = c^{d'} = 0.$$

so $c = 0$, so that, as before we arrive at

$$\sum_{k} |N_{ik}|^2 = 0. \tag{C.-2}$$

which implies that $N_{ik} = 0$ for all i and k, i.e. N is the zero matrix. Hence, $M = 0$.

□

C.2.5 Orthogonality relations

Let $D^{(\alpha)}(R)_{ij}$ be the (i,j) matrix elements of the α^{th} irreducible unitary representation of the element R of G. The range of R will be the h elements of G. The range of α will be the number of inequivalent irreducible representations. The range of i and j will be the dimension d_α of the α^{th} representation.

Theorem:

$$\sum_R D^{(\alpha)}(R)^*_{ij} D^{(\beta)}(R)_{i'j'} = \frac{h}{d_\alpha}\delta_{\alpha\beta}\delta_{ii'}\delta_{jj'} \tag{C.-2}$$

Proof:

Let $D^{(1)}(R)$ not be equivalent to $D^{(2)}(R)$. We define the matrix M,

$$M = \sum_R D^{(2)}(R) X D^{(1)}(R^{-1}). \tag{C.-2}$$

Then

$$\begin{aligned} D^{(2)}(S)M &= \sum_R D^{(2)}(S)\left[D^{(2)}(R)XD^{(1)}(R^{-1})\right]D^{(1)}(S^{-1})D^{(1)}(S) \\ &= \sum_R \left[D^{(2)}(S)D^{(2)}(R)\right]X\left[D^{(1)}(R^{-1})D^{(1)}(S^{-1})\right]D^{(1)}(S) \end{aligned} \tag{C.-2}$$

If $SR = T$, then

$$D^{(\alpha)}(S)D^{(\alpha)}(R) = D^{(\alpha)}(SR) = D^{(\alpha)}(T)$$

and

$$D^{(\alpha)}(R^{-1})D^{(\alpha)}(S^{-1}) = D^{(\alpha)}(R^{-1}S^{-1}) = D^{(\alpha)}(T^{-1})$$

So by the rearrangment theorem, (C.-2) becomes

$$
\begin{aligned}
D^{(2)}(S)M &= \sum_T D^{(2)}(T)XD^{(1)}(T^{-1)})D^{(1)}(S) \\
&= MD^{(1)}(S) \qquad \text{(C.-2)}
\end{aligned}
$$

for all S.

Hence by Schur's second lemma, $M = 0$. Now choose X to be a matrix every element of which is zero except one, which we shall call $X_{j'i}$, and $X_{j'i} = 1$.

$$
M = 0 = \sum_R D^{(2)}(R)XD^{(1)}(R^{-1}) \qquad \text{(C.-2)}
$$

or

$$
\begin{aligned}
0 &= \sum_{kl}\sum_R D^{(2)}(R)_{i'l}X_{kl}D^{(1)}(R^{-1})_{lj} \\
&= \sum_R D^{(2)}(R)_{i'j'}D^{(1)}(R^{-1})_{ij} \\
&= \sum_R D^{(1)}(R)^*_{ji}D^{(2)}(R^{-1})_{i'j'} \qquad \text{(C.-3)}
\end{aligned}
$$

The last step follows from the unitarity of the representation.

$$
D^{(\alpha)}(R^{-1})_{ij} = [D^{(\alpha)}(R^{-1})]^{-1}_{ij} = D^{(\alpha)}(R)^*_{ji} \qquad \text{(C.-3)}
$$

This proves the theorem for $\alpha \neq \beta$

$$
\sum_i D^{(\alpha)}_{ii}(R)\sum_j D^{(\beta)}_{jj}(R)^* = g\delta_{\alpha\beta} \qquad \text{(C.-3)}
$$

case(b): $\alpha = \beta = 1$

We define $M = \sum_R D^{(1)}(R)D^{(1)}(R^{-1})$. Then just as in case (a),

$$
MD^{(1)}(S) = D^{(1)}(S)M \qquad \text{(C.-3)}
$$

for all S. Hence $M = cI$. Let $X_{jj'} = 1$ for a particular j and j' and choose all other elements of X equal to zero.

$$M = cI \sum_R D^{(1)}(R) X D^{(1)}(R^{-1})$$

or in component form

$$c\delta_{ii'} = \sum_R D^{(1)}(R)_{ij} D^{(1)}(R^{-1})_{j'i'}. \tag{C.-3}$$

Let $j = j'$ and sum over j:

$$c \sum_{i=1}^{d_1} 1 = \sum_R \sum_i D^{(1)}(R^{-1})_{j'i} D^{(1)}(R)_{ij}. \tag{C.-3}$$

$$\begin{aligned} cd_1 &= \sum_R \sum_i D^{(1)}(R^{-1}R)_{jj'} \\ &= \sum_R \sum_i D^{(1)}(I)_{jj'} \\ &= \sum_R \delta_{j'j} = h\delta_{j'j} \end{aligned} \tag{C.-4}$$

Therefore

$$c = \frac{h}{d_1} \delta_{jj'}. \tag{C.-4}$$

We replace c in (C.2.5) by this value and rewrite $D^{(1)}(R^{-1})_{j'i'}$ as $D^{(1)}(R)^*_{i'j'}$ to get

$$\sum_R D^{(1)}(R)^*_{i'j'} D^{(1)}(R)_{ij} = \frac{h}{d_1} \delta_{ii'} \delta_{jj'}. \tag{C.-4}$$

This establishes the theorem for general $\alpha = \beta$.

□

C.2.6 The Characters of a Representation

The characters are invariant under similarity transformations, all elements of one class have the same character.

First Orthogonal Relation

If α and β are irreducible and N_k is the number of elements in c_k, then

$$\sum_k \chi^{(\alpha)}(c_k)^* \chi^{(\beta)}(c_k) N_k = h\delta_{\alpha\beta} \tag{C.-4}$$

Proof:

Starting with the orthogonality relatioins (C.2.5), theorem and setting $i = j$ and $i' = j'$ summing over i and i', we get

$$\sum_R \chi^{(\alpha)*}(R)\chi^{(\beta)}(R) = \frac{h}{d_\alpha}\delta_{\alpha\beta} \underbrace{\sum_{i,i'} \delta_{ii'}\delta_{ii'}}_{=d_\alpha}$$

or

$$\sum_R \chi^{(\alpha)}(c_k)^* \chi^{(\beta)}(c_k) N_k = h\delta_{\alpha\beta}. \tag{C.-4}$$

□

The Second Orthogonality Relation If $D^{(\alpha)}$ is irreducible, then

$$\sum_\alpha \chi^{(\alpha)}(c_k)^* \chi^{(\alpha)}(c_l) = \frac{h}{N_l}\delta_{kl} \tag{C.-4}$$

Proof:

$$\chi = \sum_\alpha a_\alpha \chi^{(\alpha)} \tag{C.-4}$$

Defining the weighted inner product

$$
\begin{aligned}
\chi \cdot \lambda &= \sum_{\alpha} a_{\alpha} \chi(c_k) \cdot \chi(c_{\beta}) \\
&= \sum_{\alpha} a_{\alpha} h \delta_{ij} = h a_j
\end{aligned} \tag{C.-4}
$$

Hence,

$$
\begin{aligned}
\chi &= \sum_{\alpha} \frac{\chi \cdot \chi^{(\alpha)}}{h} \chi^{(\alpha)} \\
&= \chi \cdot \sum_{\alpha} \frac{\chi^{(\alpha)} \chi^{(\alpha)}}{h}
\end{aligned} \tag{C.-4}
$$

$$
\chi(c_l) = \sum_{k,\alpha} N_k \chi(c_k) \chi^{(\alpha)}(c_k) \chi^{(\alpha)}(c_l) \frac{1}{h} \tag{C.-4}
$$

$$
0 = \sum_{k} \chi(c_k) \left[\frac{N_k}{h} \sum_{\alpha} \chi^{(\alpha)}(c_k) \chi^{(\alpha)}(c_l) - \delta_{kl} \right] \tag{C.-4}
$$

□

C.2.7 Direct Products

Details Direct Product of matrices and vectors.

By definition the vector product of two 2-by-2 matrices is

$$
A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B \\ a_{21} B & a_{22} B \end{pmatrix} \tag{C.-4}
$$

A more explicit form

$$
A \otimes B = \begin{pmatrix}
a_{11} b_{11} & a_{11} b_{12} & a_{12} b_{11} & a_{12} b_{12} \\
a_{11} b_{21} & a_{11} b_{22} & a_{12} b_{21} & a_{12} b_{22} \\
a_{21} b_{11} & a_{21} b_{12} & a_{12} b_{11} & a_{22} b_{12} \\
a_{21} b_{21} & a_{21} b_{22} & a_{22} b_{21} & a_{22} b_{22}
\end{pmatrix} \tag{C.-4}
$$

$$
C = \begin{pmatrix} c_{11;11} & c_{11;12} & c_{12;11} & c_{12;12} \\ c_{11;21} & c_{11;22} & c_{12;21} & c_{12;22} \\ c_{21;11} & c_{21;12} & c_{12;11} & c_{22;12} \\ c_{21;21} & c_{21;22} & c_{22;21} & c_{22;22} \end{pmatrix} \tag{C.-4}
$$

In general the direct product C of two matrices A and B is defined in terms of matrix elements by

$$
a_{ij} b_{kl} = c_{ik;jl}. \tag{C.-4}
$$

The row and column labels of C are ***composite*** labels:

$$
M_{i1}, M_{i,2}, \ldots, M_{1,n} \tag{C.-4}
$$

whereas the as one goes along a row the read

$$
c_{ik;1,1},\ c_{ik;1,2},\ \ldots,\ c_{ik;1,n},\ c_{ik;2,1},\ c_{ik;2,2},\ \ldots,\ c_{ik;2,n,},\ \ldots,\ c_{ik;n,1},\ c_{ik;n,2},\ \ldots,\ c_{ik;n,n,}. \tag{C.-4}
$$

the row label, is obtained from the

We prove these direct products have the same operations as matries.

Products

$$
(A \otimes B)(C \otimes D) = AC \otimes BD \tag{C.-4}
$$

$$
\begin{aligned}
(A \otimes B)(C \otimes D) &= \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \begin{pmatrix} c_{11}D & c_{12}D \\ c_{21}D & c_{22}D \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}c_{11} + a_{12}c_{21})BD & (a_{11}c_{12} + a_{12}c_{22})BD \\ (a_{21}c_{11} + a_{22}c_{21})BD & (a_{21}c_{12} + a_{22}c_{22})BD \end{pmatrix} \\
&= AC \otimes BD
\end{aligned} \tag{C.-5}
$$

$$
D \times D \tag{C.-5}
$$

$$\chi^{(\alpha\times\beta)} = \sum_{ij} D^{(\alpha\times\beta)}_{(ii)(jj)}(G_a) = \sum_i D^{(\alpha)}_{ii}(G_a) \sum_j D^{(\beta)}_{jj}(G_a) \tag{C.-5}$$

The character of the representation of the direct product is equal to the product of the characters of the original representations α and β, which implies that

$$\chi^{(\alpha\times\beta)} = \chi^{(\alpha)} \cdot \chi^{(\beta)}. \tag{C.-5}$$

Thr representations resulting from (C.2.7) is in general reducible, that is

$$D^{(\alpha\times\beta)}(G_a) = \oplus_\gamma m_\gamma D^\gamma(G_a) \tag{C.-5}$$

$$m_\gamma = \frac{1}{g} \sum_p c_p \chi^{(\gamma)*} \chi^{(\alpha)}_p \chi^{(\beta)}_p. \tag{C.-5}$$

C.3 Continuous Groups, Lie Groups and Lie algebras

We are only interested in symmetery transformations are all based on *contiuous* quantities determing how summations over group elements are carried out.

The neighbourhood of a group element is characterized by the neighbourhood of the corresponding parameter set.

symetries e.g. rotational each point in three spacial manifold).

Instead of having to consider the group as a whole, for many purposes it is sufficient to consider the an infinitesimal transformation around the identity. Any finite transformation can then be constructed by the reapeted application of this infinitesimal transformation.

We will the properties of Lie groups and algebras in terms of specific examples, especially the two and three dimensional rotation groups and the Lorentz group.

C.3.1 Infinitesmal Generating Technique

A function

$$f(x+\delta x) = f(x) + \delta x \frac{df(x)}{dx} \tag{C.-5}$$

$$f(x+2\delta x) = f(x+\delta x) + \delta x\frac{df(x+\delta x)}{dx} = f(x) + 2\delta x\frac{df(x)}{dx} + \delta x\frac{d^2f(x)}{dx^2} \tag{C.-5}$$

Binomial type expansion

$$f(x+N\delta x) = \sum_{r=0}^{N} \frac{N!}{r!(N-r)!}\delta x^r \frac{d^r f(x)}{dx^r} \tag{C.-5}$$

In the limit $N \to \infty$ the factor $N!/(N-r)$ can be replaced by N^r. We put $N\delta x = a$ where, in the limit $N \to \infty$ a is a finite number.

$$f(x+a) = \sum_{r=0}^{\infty} \frac{a^r}{r!}\frac{d^r f(x)}{dx^r} = f(x) + a\frac{df(x)}{dx} + \frac{a^2}{2!}\frac{d^2 f(x)}{dx^2} + \ldots \tag{C.-5}$$

which we can formally write

$$f(x+a) = \exp\left(a\frac{d}{dx}\right) f(x) \tag{C.-5}$$

Rotation operators

$$R(\delta\theta) = \begin{pmatrix} 1 & d\theta \\ -d\theta & 1 \end{pmatrix} = 1 + id\theta J \quad \text{where} \;\; J = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{C.-5}$$

$$J^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{C.-5}$$

$$\begin{aligned} R(\theta) &= \left(1+\frac{i\theta}{N}J\right)^N = \exp{(i\theta J)} = I + iJ + \frac{(i\theta)^2}{2!}J^2 + \frac{(i\theta)^3}{3!}J^3 + \\ &= I\left[1-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\ldots\right] + iJ\left[1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\ldots\right] = I\cos\theta + iJ\sin\theta \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned} \tag{C.-6}$$

$$\mathbf{r}' = \mathbf{r} + \delta\phi \times \mathbf{r} \tag{C.-6}$$

expressed in component form

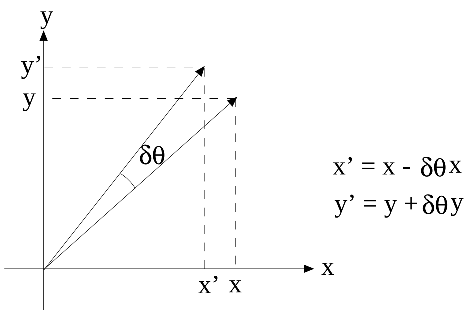


Figure C.2: infintesmal rotation.

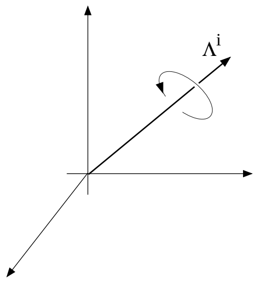


Figure C.3: infintesmal rotation.

$$x'_I = x_i + \epsilon_{ijk} x_k = (\delta_{ik} + \epsilon_{ijk}) x_k \tag{C.-6}$$

$$\hat{R} = \begin{pmatrix} 1 & \delta\phi_z & \delta\phi_y \\ -\delta\phi_z & 1 & \delta\phi_x \\ \delta\phi_y & -\delta\phi_x & 1 \end{pmatrix} \tag{C.-6}$$

$$\left(\mathbf{I} - \frac{i}{\hbar} \delta\phi \cdot \hat{\mathbf{L}} \right) \tag{C.-6}$$

where

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{C.-6}$$

Thus, given two vector angles α and β, there must exist a third one, γ, such that

$$\exp(i\alpha \cdot \mathbf{J}) \exp(i\beta \cdot \mathbf{J}) = \exp(i\gamma \cdot \mathbf{J}). \tag{C.-6}$$

For this to be so the matrices must have to satisfy some condition, which can be found by considering the angles α^i and β^i as small.

$$U(\delta\alpha_\mu) = I + i\alpha^i J_i - \frac{1}{2}\sum_{i,j}\alpha^i\alpha^j J_i J_j + \dots \quad , \tag{C.-6}$$

as J_i are *unitary*, i.e. $U^\dagger(\alpha^i) = U^{-1}(\alpha^i)$, then from() we obtain

$$U^\dagger(\alpha^i) I - i\alpha^i J_i^\dagger = U^{-1}(\alpha^i) = I - i\alpha^i J_i. \tag{C.-6}$$

from this follows the *Hermiticity of the generators*

$$U_i^\dagger = U_i \tag{C.-6}$$

Next we calculate the inverse operator to second order in α^i

$$U^{-1}(\alpha^i) = I - i\alpha^i J_i - \frac{1}{2} \tag{C.-6}$$

$$I + i(\alpha^i J_i + \beta^i J_i) - \frac{1}{2}(\alpha^i J_i + \beta^i J_i)^2 = I + i(\alpha^i J_i + \beta^i J_i) - \frac{1}{2}(\alpha^i\alpha^j + \alpha^i\beta^j + \alpha^j\beta^i + \beta^i\beta^j) J_i J_j \alpha^i\beta^i [J_i, J_j] + \dots \tag{C.-6}$$

$$+ \dots = i(\alpha^i + \beta^i) J_i - \frac{1}{2} \tag{C.-6}$$

the commutator must be a linear combination of the J_i, that is

$$[J_i, J_j] = C_{ij}^k J_k. \tag{C.-6}$$

The generators of *any* Lie group must have such commutation relations. The coefficients C_{ij}^k are called the *structure constants*

The product of two rotations $\exp(-iT_y)\exp(-iT_z)$ can always be written as a single exponential, say $\exp(-i\alpha \cdot T)$ where $\alpha \cdot T := \alpha_x T_x + \alpha_y T_y + \alpha_z T_z$. Suppose we set $\exp(-i\alpha \cdot T)\exp(-i\beta \cdot T) = \exp(-i\gamma \cdot T)$ and try to calculate γ in terms of α and β. If we expand the exponentials we find

$$
\begin{aligned}
[1 - i\alpha \cdot t - \frac{1}{2}(\alpha \cdot t)^2 &+ \dots][1 - i\beta \cdot t - \frac{1}{2}(\beta \cdot t)^2 + \dots] \\
&= \exp(-i(\alpha + \beta) \cdot t - \frac{1}{2}[\alpha \cdot t, \beta \cdot t] + \dots) \qquad \text{(C.-6)}
\end{aligned}
$$

(and this is known as the Campbell-Baker-Hausdorff theorem - see Appendix S). It is for this reason that we can learn all we need to know about Lie groups by studying the commutation algebras of the generators

C.3.2 General Structure of Lie Groups

An infinite group is a group that contains an infinite nuber of elements. The rotation group is an example of such a group.

$$SO(3) = \exp(-i\hat{\phi} \cdot J). \qquad \text{(C.-6)}$$

The fact that these matrices are functions of only three fundamental matrices $\{\hat{J}_k\} = \{\hat{J}_1 \hat{J}_2 \hat{J}_3\}$ allows us to represent them in a simple way.

The set of group elements are characterized by a set of continuous real parameters.

A continuous group G is said to be *compact* if the parameter space is finite and non-compact if it is infinite. The rotation group $SO(3)$ is an example of a compact group with the Lorentz group $SO(3,1)$ as an example of a non-compact group.

$$
\begin{aligned}
[A,[B,C]] &= A[B,C] - [B,C]A \\
&= ABC - ACB - BCA + CBA \qquad \text{(C.-6)}
\end{aligned}
$$

We now add and subtract the quantities BAC and CAB on the right-hand side of this equation and rearange the resulting expression

$$
\begin{aligned}
[A,[B,C]] &= ABC - ACB - BCA + CBA \\
&\quad -BAC + BAC + CAB - CAB \\
&= -C(AB - BA) + (AB - BA)C \\
&\quad +B(AC - CA) - (AC - CA)B \\
&= -[C,[A,B]] - [B,[C,A]] \qquad \text{(C.-9)}
\end{aligned}
$$

A simple rearrangement yields the *Jacobi identity*:

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \tag{C.-9}$$

The important properties of the structure constants are the following

(i) They are antisymmetric in their lower indices

$$C_{ij}^{k}=-C_{ji}^{k} \tag{C.-9}$$

(ii) The Jacobi identity defined by the infinitesimal generators (e.g. for the rotation group: $A=J_k,\ B=J_l,\ C=J_m$) leads to the condition on the scructure constants

$$C_{kl}^{n}C_{mn}^{p}+C_{lm}^{n}C_{kn}^{p}+C_{mk}^{n}C_{ln}^{p}=0 \tag{C.-9}$$

C.3.3 Rotations SO(3) and SU(2)

By a *representation* we mean a set of matrices T_x, T_y , and T_z with the same commutation relations as the t's. The T's of Eqs() and () are an examples in which the matrices are 3×3 and the representation is said to be of dimension three.

We recall the construction in standard quantum mechanics lectures.

$$[J_1,J_2]=iJ_3,\quad [J_2,J_3]=iJ_1,\quad [J_3,J_1]=iJ_2 \tag{C.-9}$$

$$\hat{J}^2\psi_{jm}=j(j+1)\psi_{jm},\quad \hat{J}_3\psi_{jm}=m\psi_{jm} \tag{C.-9}$$

It is convenient to define

$$J_+=iJ_1-J_2,\ \text{ and its complex conjugate }\ J_-=-iJ_1-J_2 \tag{C.-9}$$

the commutation relations become

$$[J_z,J_+]=J_+,\ \ [J_z,J_-]=-J_-,\ \ [J_+,J_-]=2J_z. \tag{C.-9}$$

$$J_3\psi_1=J_3J_+\psi=[J_3,J_+]\psi+J_+J_3\psi \tag{C.-9}$$

$$J_+J_- = J^2 - J_3(J_3 - 1) \tag{C.-8}$$
$$J_-J_+ = J^2 - J_3(J_3 + 1) \tag{C.-7}$$

$$J_+J_-\psi_{jm} = [j(j+1) - m(m-1)] \tag{C.-6}$$
$$J_-J_+\psi_{jm} = [j(j+1) - m(m+1)] \tag{C.-5}$$

$$U = e^{-i\theta_i\sigma_i} \tag{C.-5}$$

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.-5}$$

$$\tau^{\mathbf{i}}\tau^{\mathbf{j}} - \tau^{\mathbf{j}}\tau^{\mathbf{i}} = \epsilon_{ijk}\tau^{\mathbf{k}} \tag{C.-5}$$

$SU(2)$ is the universal cover of $SO(3)$

By definition $SU(2)$ is the group of 2×2 special unitary matricies with determinant one. If

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$ then the requirements $P{-1} = P^*$ and $\det P = 1$ translates to

$$\overline{a} = d \text{ and } \overline{b} = -c.$$

That is

$$P = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

and the condition that $\det P = 1$ gives the condition

$$a\overline{a} + b\overline{b} = 1$$

This says that P is fully determined by a vector $(a, b) \in \mathbb{C}^2$ of length one. If we write a, b in terms of their real and imaginary parts, then the above condition becomes

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

that is the unit sphere S^3 in $\mathbb{R}^4$, which is simple connected as the unit sphere S^2 in $\mathbb{R}^3$ is.

Details The Pauli matrices form a vector space

$$\tau_i = \frac{\hat{\sigma}_i}{2} \tag{C.-5}$$

They form linear independent complete vector space for 2×2 matrices

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \hat{\sigma}^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{C.-5}$$

$$\begin{aligned} & a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + bI = \mathbf{a} \cdot \sigma \\ &= \begin{pmatrix} b + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & b - a_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \mathbf{A} \end{aligned} \tag{C.-5}$$

$$TrI = 2, \qquad Tr\sigma_i = 0 \tag{C.-5}$$

Hence,

$$A = \sum_i \hat{\sigma}_i \frac{1}{2} Tr(A\hat{\sigma}_i) + (\frac{1}{2} TrA)I \tag{C.-5}$$

C.3.4 Spin Direct Products

A subspace V is said to be **invariant** if it is mapped into itself by application of each matrix element of the group.(refine definition) A less trival example of a reduciable representation is the "addition of angular momentum" in quantum mechanics

We combine two irreduciable representations by forming the product space $V = V_1 \otimes V_2$.

$$
\begin{aligned}
\hat{\sigma}_1^x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \\
\hat{\sigma}_1^y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & & -i & \\ & & & -i \\ i & & & \\ & i & & \end{pmatrix}, \\
\hat{\sigma}_1^z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},
\end{aligned} \tag{C.-6}
$$

and

$$
\begin{aligned}
\hat{\sigma}_2^x &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \\
\hat{\sigma}_2^y &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{pmatrix}, \\
\hat{\sigma}_1^z &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.
\end{aligned} \tag{C.-7}
$$

Note that

$$\left[\hat{\sigma}_1^i, \hat{\sigma}_2^j\right] = 0 \tag{C.-7}$$

for any $i, j \in \{x, y, z\}$ as can be shown directly or better still using the rule for multiplying direct products of matrices :$(A \otimes B)(C \otimes D) = (AC \otimes BD)$,

$$\left[\hat{\sigma}_1^i, \hat{\sigma}_2^j\right] = (\hat{\sigma}^i \otimes I)(I \otimes \hat{\sigma}^j) - (I \otimes \hat{\sigma}^j)(\hat{\sigma}^i \otimes I) = (\hat{\sigma}^i \otimes \hat{\sigma}^j) - (\hat{\sigma}^i \otimes \hat{\sigma}^j) = 0 \tag{C.-7}$$

These operators act on the direct product space $V \otimes V$, with, elements

$$\eta \otimes \omega = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \otimes \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \eta_1\omega_1 \\ \eta_1\omega_2 \\ \eta_2\omega_1 \\ \eta_2\omega_2 \end{pmatrix} \tag{C.-7}$$

$$\begin{aligned}
|\psi_{(1,1)}> &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
|\psi_{(1,0)}> &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
|\psi_{(1,-1)}> &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
|\psi_{(0,0)}>= &= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned} \tag{C.-9}$$

This result

$$S := \frac{1}{2}(\hat{\sigma} \otimes I + I \otimes \hat{\sigma}) \tag{C.-9}$$

are generators of a *reducible* 4-dimensional reperentation of $\int\sqcap(2)$. There exists a unitary matrix U such that

$$USU^{\dagger} = \begin{pmatrix} S_{(S=1)} & \\ & S_{(S=0)} \end{pmatrix}, \tag{C.-9}$$

where $S_{(S=1)}$ and $S_{(S=0)}$ representaions of dimension 3 and 1, respectively.

$$
\begin{aligned}
S^2 &= \frac{1}{4}(\hat{\sigma} \otimes I + I \otimes \hat{\sigma})^2 \\
&= \frac{1}{4}(\hat{\sigma}^2 \otimes I + 2\hat{\sigma} \otimes \hat{\sigma} + I \otimes \hat{\sigma}^2) \\
&= \frac{1}{2}\hat{\sigma} \otimes \hat{\sigma} + \frac{3}{2} I \otimes I,
\end{aligned} \tag{C.-10}
$$

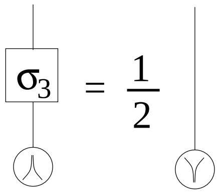
since $\hat{\sigma}^2 = 3I$.

$$\vec{S}^z = \hat{\sigma}^z \otimes I + I \otimes \hat{\sigma}^z \tag{C.-10}$$

$$\vec{S}^z|S, S^z >= S^z|S, S^z > \tag{C.-10}$$

$$\vec{S}^2|S, S^z >= S(S+1)|S, S^z >, \quad S^z = -S, \ldots, S; \;\; S = 0, 1. \tag{C.-10}$$

I will first go through the explicit calculation of the action of the z-component of the angular momentum, $\hat{J}_3$, and the total angular momentum, $\hat{J}^2$, so the reader may have a better feel for what is going on in the abstract diagrammatic calculation (and to appreciate the simplicity of the diagrammatic version). Moreover, I wish to go through the calculation for the total angular momentum operator both ways because of its importance in finding the (main sequence of) eigenvalues of the area operator (see appendix on geometric operators).



Details

$$
\begin{aligned}
\omega_{AB\ldots F}(i=0) &= \delta_{A0}\delta_{B0}\ldots\delta_{E0}\delta_{F0} && \text{(C.-9)}\\
\omega_{AB\ldots F}(i=1) &= \delta_{A1}\delta_{B0}\delta_{C0}\ldots\delta_{E0}\delta_{F0}\\
&\quad+\delta_{A0}\delta_{B1}\delta_{C0}\ldots\delta_{E0}\delta_{F0}\\
&\quad+\delta_{A0}\delta_{B0}\delta_{C1}\ldots\delta_{E0}\delta_{F0}+\ldots\\
&\ \vdots\\
\omega_{AB\ldots F}(i=2s) &= \delta_{A1}\delta_{B1}\ldots\delta_{E1}\delta_{F1} && \text{(C.-12)}
\end{aligned}
$$

$$
\hat{\sigma}_z=\sum_{k=1}^{2s+1} 1\otimes\cdots\otimes 1\left(\frac{\hat{\sigma}^3}{2}\right)\otimes\cdots\otimes 1 \qquad \text{(C.-12)}
$$

which written in component form reads

$$
\begin{aligned}
\frac{1}{2}\hbar\hat{\sigma}^3_{AA'BB'\ldots EE'FF'} &:= \frac{1}{2}\hbar\hat{\sigma}^3_{AA'}\delta_{BB'}\ldots\delta_{FF'}\\
&\ \vdots\\
&\quad \frac{1}{2}\hbar\hat{\sigma}^3_{FF'}\delta_{AA'}\delta_{BB'}\ldots\delta_{EE'}\\
&\quad (AA';BB';\ldots;EE';FF'=0,1). && \text{(C.-14)}
\end{aligned}
$$

We verify that the "states" $\omega(i)$ fulfill the eigenvalue equation

$$
\begin{aligned}
\hbar\frac{1}{2}\hat{\sigma}^3\omega(i) &= \hbar(s-i)\omega(i)\\
i &= 0,1,\ldots,2s. && \text{(C.-14)}
\end{aligned}
$$

Using

$$\hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.-14}$$

and (C.-14) we get

$$\begin{aligned}
\left(\frac{1}{2}\hbar\,\omega(i=0)\right)_{AB...F} &= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'BB'...EE'FF'}\;\omega_{A'B'...E'F'}(i=0) \\
&= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'}\delta_{A'0}\delta_{B0}\ldots\delta_{E0}\delta_{F0} + \\
&\quad + \delta_{A0}\hbar\frac{1}{2}\hat{\sigma}^3_{BB'}\delta_{B'0}\ldots\delta_{E0}\delta_{F0} + \ldots \\
&\quad + \delta_{A0}\delta_{B0}\ldots\delta_{E0}\hbar\frac{1}{2}\hat{\sigma}^3_{FF'}\delta_{F'0} \\
&\quad = \frac{\hbar}{2}2s(\delta_{A0}\delta_{B0}\ldots\delta_{F0}) = \hbar s\omega_{AB...F}(i=0)
\end{aligned} \tag{C.-17}$$

$$\begin{aligned}
\left(\frac{1}{2}\hbar\,\omega(i=1)\right)_{AB...F} &= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'}\delta_{A'1}\delta_{B0}\ldots\delta_{E0}\delta_{F0} + \delta_{A1}\frac{\hbar}{2}\hat{\sigma}^3_{BB'}\delta_{B'0}\ldots\delta_{F0} + \\
&+ \cdots + \delta_{A1}\delta_{B0}\ldots\frac{\hbar}{2}\hat{\sigma}^3_{FF'}\delta_{F'0} + \frac{\hbar}{2}\hat{\sigma}^3_{AA'}\delta_{A'0}\delta_{B1}\ldots\delta_{F0} \\
&+ \delta_{A0}\frac{\hbar}{2}\hat{\sigma}^3_{BB'}\delta_{B'1}\ldots\delta_{F0} + \ldots \\
&+ \delta_{A0}\delta_{B1}\ldots\frac{\hbar}{2}\hat{\sigma}^3_{FF'}\delta_{F'1} + \ldots \\
&= \frac{\hbar}{2}\left[-\delta_{A1}\delta_{B0}\ldots\delta_{F0} + (2s-1)\times\delta_{A1}\delta_{B0}\ldots\delta_{F0} + \right. \\
&+ \delta_{A0}\delta_{B1}\ldots\delta_{F0} - \delta_{A0}\delta_{B1}\ldots\delta_{F0} + \\
&+ \left.(2s-3)\times\delta_{A0}\delta_{B1}\ldots\delta_{F0} + \ldots\right] \\
&= \hbar(s-1)\omega_{AB...F}(i=1),
\end{aligned} \tag{C.-23}$$

$$\vdots$$

$$\begin{aligned}
\left(\frac{1}{2}\hbar\,\omega(i=2s)\right)_{AB...F} &= \\
&= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'}\delta_{A'1}\delta_{B1}\ldots\delta_{F1} + \delta_{A1}\frac{\hbar}{2}\hat{\sigma}^3_{BB'}\delta_{B'1}\ldots\delta_{E1}\delta_{F1} + \ldots \\
&+ \cdots + \delta_{A1}\delta_{B1}\ldots\frac{\hbar}{2}\hat{\sigma}^3_{FF'}\delta_{F'1} \\
&= -\hbar s\omega_{AB...F}(i=2s).
\end{aligned} \tag{C.-26}$$

Details K1.3

$$\hat{\sigma}^2 = \sum_{k=1}^{2s+1} \sum_{k'=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes \left(\frac{\hat{\sigma}^{k'}}{2}\right) \otimes \cdots \otimes 1 \tag{C.-26}$$

$$(\hbar^2/4)\hat{\sigma}^2_{AA'BB'\ldots FF'} \tag{C.-26}$$

has eigenvalue $\hbar^2 s(s+1)$

$$\left(\frac{1}{4}\hat{\sigma}^2\right)_{AA'BB'\ldots FF'} = \frac{1}{4} \tag{C.-26}$$

$$\hat{\sigma}^2 = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k \hat{\sigma}^k}{4}\right) \otimes \cdots \otimes 1 \tag{C.-26}$$

$$\hat{\sigma}^2_{AA'} = (\hat{\sigma}^1)^2_{AA'} + (\hat{\sigma}^2)^2_{AA'} + (\hat{\sigma}^3)^2_{AA'} = 3\delta_{AA'} \tag{C.-26}$$

$$\begin{aligned} \left(\frac{1}{4}\hat{\sigma}^2\right)\omega(i=0) &= \left[\frac{3}{2}s + \frac{1}{4}(2s-1)2s\right]\omega(i=0) \\ &= s(s+1)\omega(i=0). \end{aligned} \tag{C.-26}$$

$$\begin{aligned} \left(\frac{1}{4}\hat{\sigma}_{AA'}\hat{\sigma}_{BB'}\right)\delta_{A'1}\delta_{B'0} &= \\ &= \left(\hat{\sigma}^1_{AA'}\hat{\sigma}^1_{BB'} + \hat{\sigma}^2_{AA'}\hat{\sigma}^2_{BB'} + \hat{\sigma}^3_{AA'}\hat{\sigma}^3_{BB'}\right)\delta_{A'1}\delta_{B'0} \\ &= \frac{1}{4}\left((+1)\delta_{A0}\delta_{B1} + (+1)\delta_{A0}\delta_{B1} + (-1)\delta_{A1}\delta_{B0}\right) \end{aligned} \tag{C.-27}$$

$$\begin{aligned} \left(\frac{1}{4}\hat{\sigma}_{AA'}\hat{\sigma}_{BB'}\right)(\delta_{A'1}\delta_{B'0} + \delta_{A'0}\delta_{B'1}) &= \\ \frac{1}{4}(\delta_{A0}\delta_{B1} + \delta_{A1}\delta_{B0}) \end{aligned} \tag{C.-27}$$

C.3.5 Direct Products and Clebsch-Gordan Coefficients

$$\tau^{\mathbf{i}}_{(\mathbf{j})} = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes 1 \qquad \text{(C.-27)}$$

We wish to calculate $\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})}$. The terms of (N.-19) for $k \neq k$ wont contribute to the commutator as the order of multiplication irrelevant,

$$\sum_{k \neq k'} \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes 1\right) \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^{k'}}{2}\right) \otimes \cdots \otimes 1\right) - (k \leftrightarrow k') = 0 \quad \text{(C.-27)}$$

$$\begin{aligned}
\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})} &= \\
&= \sum_{k=1}^{2s+1} \left[1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i}{2}\right)\left(\frac{\hat{\sigma}^j}{2}\right) \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^j}{2}\right)\left(\frac{\hat{\sigma}^i}{2}\right) \otimes \cdots \otimes 1 \right] \\
&= \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i\hat{\sigma}^j}{4} - \frac{\hat{\sigma}^j\hat{\sigma}^i}{4}\right) \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk} \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \frac{\hat{\sigma}^k}{2} \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk}\tau^{\mathbf{k}}_{(\mathbf{j})} \qquad \text{(C.-30)}
\end{aligned}$$

$A \otimes B$

$$a_{ij}b_{kl} = c_{ik;jl}. \qquad \text{(C.-30)}$$

the row and column labels of the matrix elements of C are *composite* labels: the row label ik, is obtained from the row labels of the matrix elements of A and B and the column label, jl is obtained from the corresponding column labels.

$$\begin{aligned}
\Delta_j(\theta) &= \sum_{m=-j}^{j} e^{-im\theta} \\
&= e^{ij\theta}\frac{e^{-i(2j+1)\theta} -}{e^{-i\theta} - 1} \\
&= \frac{e^{i(j+\frac{1}{2})\theta} - e^{-i(j+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}} \\
&= \frac{\sin\left[(j+\frac{1}{2})\theta\right]}{\sin(\frac{1}{2}\theta)} \qquad \text{(C.-32)}
\end{aligned}$$

$$\int_0^{2\pi} \overline{\Delta_k(\theta)} \Delta_j(\theta) \frac{1-\cos\theta}{2\pi}\, d\theta = \delta_{kj}. \tag{C.-32}$$

Since the spinor indices take only one of two values an antisymmetrizable over three indices of a multivalent spinor $\tau_{...ABC...}$ is zero. In particular we have the *Jacobi identity*

$$\epsilon_{A[B}\epsilon_{CD]} = 0 = \epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC} \tag{C.-32}$$

$$\tau_{...AB...} = \tau_{...(AB)...} + \frac{1}{2}\epsilon_{AB}\tau_{...C}{}^{C}{}_{...}. \tag{C.-32}$$

The proof

$$\epsilon_{AB}\tau_{C}{}^{C} - \tau_{AB} + \tau_{BA} = 0 \tag{C.-32}$$

$$\tau_{[AB]} = \frac{1}{2}\epsilon_{AB}\tau_{C}{}^{C}. \tag{C.-32}$$

Since $\tau_{AB} = \tau_{(AB)} + \tau_{[AB]}$ the result follows

$$\tau_{AB} = \tau_{(AB)} + \frac{1}{2}\epsilon_{AB}\tau_{C}{}^{C} \tag{C.-32}$$

It should be evident that this result will also apply to the more general case where we consider only two particular indices of a multivalent spinor $\tau_{...ABC...}$ of valence > 2, hence we have the result (C.3.5).

The same is true for the followings cases.

(a) Case τ_{AB} we already have

(b) Case τ_{ABC}

$$\begin{aligned} 3\tau_{(ABC)} &= \tau_{A(BC)} + \tau_{B(AC)} + \tau_{C(AB)} \\ &= 3\tau_{A(BC)} - (\tau_{A(BC)} - \tau_{B(AC)}) - (\tau_{A(BC)} - \tau_{C(AB)}) \\ &= 3\tau_{A(BC)} - \epsilon_{AB}\sigma_C - \epsilon_{AC}\sigma_B \end{aligned} \tag{C.-33}$$

where $\sigma_C = \epsilon^{AB}(\tau_{A(BC)} - \tau_{B(AC)})$.

This rearanged gives

$$\tau_{A(BC)} = \tau_{(ABC)} + \frac{1}{3}\epsilon_{AB}\sigma_C + \frac{1}{3}\epsilon_{AC}\sigma_B. \tag{C.-33}$$

$$\tau_{A(BC)} = \tau_{ABC} - \frac{1}{2}\epsilon_{BC}\tau_{AD}{}^{D} \tag{C.-33}$$

Using (C.3.5) in (C.3.5) we have the desired expansion for τ_{ABC}

$$\tau_{ABC} = \tau_{(ABC)} + \frac{1}{2}\epsilon_{BC}\tau_{AD}{}^{D} + \frac{1}{3}\epsilon_{AB}\ \sigma_C + \frac{1}{3}\epsilon_{AC}\ \sigma_B. \tag{C.-33}$$

or more generally

$$\tau_{...ABC...} = \tau_{...(ABC)...} + \frac{1}{2}\epsilon_{BC}\tau_{...AD}{}^{D}{}_{...} + \frac{1}{3}\epsilon_{AB}\ \sigma_{...C...} + \frac{1}{3}\epsilon_{AC}\ \sigma_{...B...}. \tag{C.-33}$$

where $\sigma_{...C...} = \epsilon^{AB}(\tau_{...A(BC)...} - \tau_{...B(AC)...})$

(c) Case $\tau_{A...F}$:

Proof is by induction

Any spinor $\tau_{(A...F)}$

$$n\tau_{(ABC...F)} = \tau_{A(BC...F)} + \tau_{B(AC...F)} + \cdots + \tau_{F(BC...A)}. \tag{C.-33}$$

$$\tau_{A(BCD...F)} - \tau_{B(ACD...F)} = \frac{1}{2}\epsilon_{AB}\sigma_{(CD...F)} \tag{C.-32}$$

where $\sigma_{(CD...F)} = \epsilon^{AB}(\tau_{A(BCD...F)} - \tau_{B(ACD...F)})$

$$n\tau_{(ABC...F)} = n\tau_{A(BC...F)} - (\tau_{A(BC...F)} - \tau_{B(AC...F)}) - \cdots - (\tau_{A(BC...F)} - \tau_{F(BC...A)}) \tag{C.-32}$$

$$\tau_{A(BC...F)} = \tau_{(ABC...F)} + \frac{1}{n}\epsilon_{AB}\rho_{(C...F)} + \cdots + \frac{1}{n}\epsilon_{AF}\rho_{(B...F)} \tag{C.-32}$$

where

$$\rho_{(C...F)} = \epsilon^{AB}(\tau_{A(BC...F)} - \tau_{B(AC...F)}) \tag{C.-32}$$

C.3.6 Recoupling Theory

Recoupling Theory of Three Angular Momenta - 6-j-Symbols

$$|j_{12}(j_1, j_2), j(j_{12}, j_3) > \;\; = \;\; | \tag{C.-31}$$

C.3.7 SO(3,1) and SL(2,C)

$$x' = \frac{x + vt}{\sqrt{1 - c^2 t^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t + vx/c^2}{\sqrt{1 - c^2 t^2}} \tag{C.-31}$$

$$\gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi, \tag{C.-31}$$

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{C.-31}$$

$$K_x = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.-31}$$

$$K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.-31}$$

$$J_x = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_y = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.-31}$$

$$\begin{aligned}
\left[K_x, K_y\right] &= -iJ_z \\
[J_x, K_x] &= 0 \\
\left[J_x, K_y\right] &= iK_z
\end{aligned} \tag{C.-32}$$

$$\begin{aligned}
A &= \frac{1}{2}(J + iK), \\
B &= \frac{1}{2}(J - iK).
\end{aligned} \tag{C.-32}$$

$$\begin{aligned}
\left[A_x, A_y\right] &= iA_z \text{and cyclic perms,} \\
[B_x, B_x] &= iB_z \text{and cyclic perms,} \\
\left[A_x, B_y\right] &= 0 \ \ (i, j = x, y, z).
\end{aligned} \tag{C.-33}$$

This shows that A and B each generate a group $SU(2)$ $SU(2) \otimes SU(2)$

$$X(x) := \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{C.-33}$$

unique vector x^μ

$$x^\mu = \frac{1}{2}\mathrm{tr}(\sigma_\mu X) \tag{C.-33}$$

$$\begin{aligned}
\det X(x) &> 0 \quad \text{if } x \text{ is a timelike vector} \\
&= 0 \quad \text{if } x \text{ is on the light cone} \\
&< 0 \quad \text{if } x \text{ is a spacelikevector}
\end{aligned} \tag{C.-34}$$

$$A = \exp\left(-i\frac{\theta}{2}(\vec{n} \cdot \sigma)\right) = \cos\frac{\theta}{2}I - i(\vec{n} \cdot \sigma)\sin\frac{\theta}{2} \tag{C.-34}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C) \tag{C.-34}$$

$$A = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta e^{i\gamma} \\ -\sin\beta e^{-i\gamma} & \cos\beta \end{pmatrix} B \tag{C.-34}$$

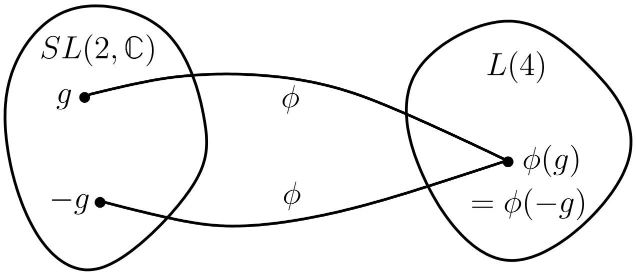


Figure C.4:

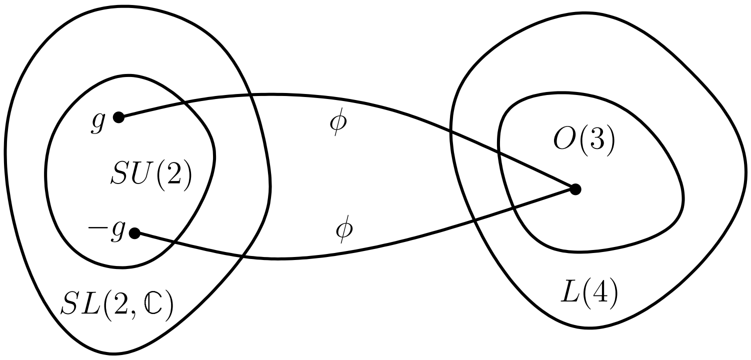


Figure C.5:

C.3.8 SO(4)

$$U(g) = 1 - \frac{1}{2} J_i^j \epsilon_j^i + \mathcal{O}, \tag{C.-34}$$

where the J_i^j are $N \times N$ matrices

$$[J_i^j, J_k^l] = \delta_k^j J_i^l - \delta_l^i J_k^j + \delta_k^i J_j^l - \delta_l^j J_i^k \,, \quad i, j, kl = 1, \ldots, n. \tag{C.-34}$$

$$SO(4) = SO(3) \oplus SO(3). \tag{C.-34}$$

$$L^{\pm} := \frac{1}{2}(L \pm M') \tag{C.-34}$$

The structure constants with respect to this new basis are given by

$$[(L^{\pm})^i, (L^{\pm})^j] = i\hbar \epsilon^{ij}{}_k (L^{\pm})^k \,, \quad [(L^+)^i, (L^-)^j] = 0 \tag{C.-34}$$

C.3.9 Conformal Group

In Minkowskian spacetime a *conformal transformation* are coordinate transformations $x \to x'(x)$ which are such that the induced change in the metric is a positive rescaling by a positive function:

$$ds'^2 = \Omega(x)^2\, ds^2 \tag{C.-34}$$

where $\Omega(x)$ is a real-valued function. Geometrically conformal transformations leaves thae *angles* but changes distances; it involves dilations (rescalings). Tranformations with this property are used in the design of geographic maps. Penrose diagrams bring infinity onto a page.

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin\theta d\phi^2) \tag{C.-34}$$

We introduce double null coordinates

$$\begin{aligned} u &= t + r && \text{(C.-33)} \\ v &= t - r && \text{(C.-32)} \end{aligned}$$

$dudv = (dt + dr)(dt - dr) = dt^2 - dr^2$ and $(u - v)^2 = (t + r - t + r)^2 = 4r^2$ so the line element becomes

$$ds^2 = dudv - \frac{1}{4}(u - v)^2(d\theta^2 + \sin\theta d\phi^2). \tag{C.-32}$$

$$p = \tan^{-1} u, \quad q = \tan^{-1} v. \tag{C.-32}$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{4}\sec^2 p \sec^2 q[4dpdq - \sin^2(p - q)(d\theta^2 + \sin\theta d\phi^2)] \tag{C.-32}$$

$$d\bar{s}^2 = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta = dpdq - \sin^2(p - q)(d\theta^2 + \sin\theta d\phi^2) \tag{C.-32}$$

$$\Omega = \frac{1}{4}\sec^2 p \sec^2 q \tag{C.-32}$$

Now finally we introduce the coordinates

$$t' = p + q \tag{C.-31}$$
$$r' = p - q \tag{C.-30}$$

with the coordinate range

$$-\pi < t' + r' < \pi, \tag{C.-29}$$
$$-\pi < t' - r' < \pi, \tag{C.-28}$$
$$r' > 0 \tag{C.-27}$$

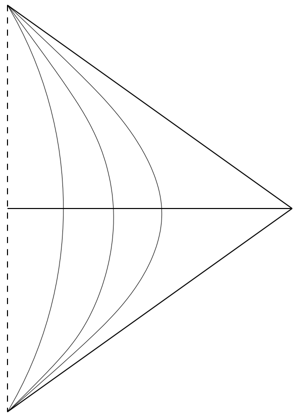


Figure C.6: Penrose diagram for Minkowskian spacetime.

C.3.10 Group Integration: The Haar Measure

For a compacct group, we can replace the sums over group elements that occur in the representation theory of finite groups, by convergent integrals over the group elements using the invariant Haar measure, which is usually denoted by $d[g]$. The invariance property is expressed by $d[g_1 g] = d[g]$. This allows us to make a change of variables transfromation, $g \rightarrow g_1 g$, identaical to that which played such an important role in deriving the finite group theorems. Consequently, all the results from finite groups, such as the existence of an invariant inner product and the orthoganality theorems, acn be taken over by the simple replacement of a sum by an integral.

As a vector space, $R(G)$ has a distinguished basis given by the characters $\chi_i(g)$ of the irreducible representations. Recall that the orthog-onality relations for characters are essentially the same as in the finite group case, with the sum over group elements replace by an integral

$$\int \overline{\chi_i(g)}\chi_j(g)dg = \delta_{ij} \tag{C.-27}$$

where i and j are labels for irreducible representations and dg is the standard Haar measure, normalized so that the volume of g is 1.

For the case of finite group one important property of these groups is the rearrangement theoremis given for fixed S,

$$\sum_R f(R) = \sum_R f(SR) \tag{C.-27}$$

$$\int f(R)dR = \int f(SR)dR \tag{C.-27}$$

To clarify the relation to the finite group rearangement theorem we write the integral on the left-hand side of (C.3.10) as an integral over the parameters. the density of group elements is arranged so that the density of points at SR is the same as that at R.

$$\int f(R)\,dR = \int f(R)\mu(R)\,da, \tag{C.-27}$$

where $\mu(R)$ is the density of group elements in the parameter space in the neighborhood of R.

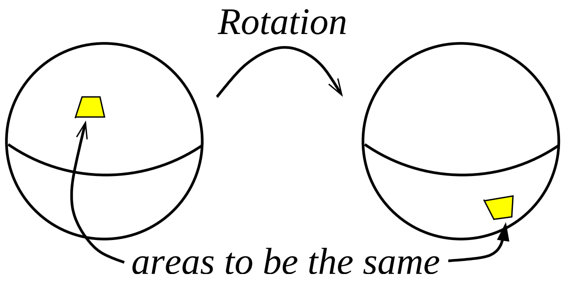


Figure C.7: Haarmeas1.

uniform measure over a 3-sphere S^3

Of course the measure is

$$dU = d\theta \cos\theta d\phi \tag{C.-27}$$

as under a rotation

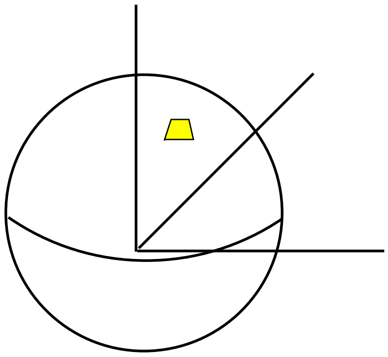


Figure C.8: HaarmeasSO3.

$$d\theta \cos\theta d\phi \mapsto d\theta' \cos\theta' d\phi' \tag{C.-27}$$

Approach that is easily extended to other Lie groups.

$$dU = \pi^{-2} da_1 da_2 da_3 \delta\left(\sum_{i=1}^{2} a_i^2 - 1\right) \tag{C.-27}$$

$SU(2)$ a 3-sphere in 4-dimensional Euclidean space.

$$dU = \pi^{-2} da_0 da_1 da_2 da_3 \delta\left(\sum_{i=0}^{3} a_i^2 - 1\right) \tag{C.-27}$$

$$\mu(C) = \mu(0) / \left(\frac{\partial c}{\partial a}\right)_{a=0} \tag{C.-27}$$

$$\begin{aligned} \int dU U_{ij} &= 0 \\ \int dU U_{i_1 j_1} U_{i_2 j_2} &= \frac{1}{2} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \\ \int dU U_{ij} (U^{-1})_{kl} &= \frac{1}{2} \delta_{jk} \delta_{il} \end{aligned} \tag{C.-28}$$

$$\int dU |\mathrm{Tr} U|^2 = 1 \tag{C.-28}$$

Details

$$\int dU U_{ij}(U^{-1})_{kl} = A\delta_{jk}\delta_{il} \tag{C.-28}$$

The constant A is easily found by setting $j = k$ summing over the index

$$\int dU \delta_{il} = A\delta_{jj}\delta_{il} = 2A\delta_{il}, \tag{C.-28}$$

noting $\int dU = 1$ implies $A = 1/2$.

$$\int dU|\mathrm{Tr}U|^2 = \int dU U_{ij}(U^{\dagger})_{kl} = \int dU U_{ij}(U^{-1})_{kl} = \frac{1}{2}\delta_{ji}\delta_{ij} = \frac{1}{2}\delta_{jj} = 1 \tag{C.-28}$$

For any compact group G the Haar measure is the unique measure dU on G which obeys invarianc (C.3.10) and normalization: $\int_G dU = 1$.

$$dc = J\left[\frac{\partial c}{\partial a}\right] da \tag{C.-28}$$

where $J[\partial c/\partial a]$ is the *Jacobian* defined by

$$J\left[\frac{\partial c}{\partial a}\right] = \frac{\partial(c_1, c_2, \ldots, c_h)}{\partial(a_1, a_2, \ldots, a_h)} \equiv det\left|\frac{\partial c_i}{\partial a_i}\right| \tag{C.-28}$$

Example 1: $SO(2)$ $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$. Thus, from $R(\theta') = R(\theta)R(\epsilon) = R(\theta + \epsilon)$, where $R(\epsilon)$ is the infintesimal transformation, we have

$$\theta' = \theta + \epsilon \tag{C.-28}$$

Accordingly $d\theta'/d\epsilon = 1$, so we have that $\mu(\theta) = \mu_0$. Through normalization

$$\int_{-\pi}^{\pi} \mu(\theta)d\theta = 2\pi\mu_0 = 1 \tag{C.-28}$$

we obtain $\mu(\theta) = 1/(2\pi)$.

Unitary Group $U(2)$

The unitary group $U(N)$ has N^2 generators

$$[\mathcal{C}_{im}, \mathcal{C}_{jn}] = \delta_{jm}\mathcal{C}_{in} - \delta_{in}\mathcal{C}_{jm}. \tag{C.-28}$$

a representation of $U(2)$ is

$$(\mathcal{C}_{im}) = (\delta_{im}) \tag{C.-28}$$

where $(\delta_{im}$ denotes the matrix which has a "1" at the intersection of the ith row and the mth column and zeros everywhere else i.e.

$$\begin{aligned} \mathcal{C}_{11} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{C}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{C}_{21} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{C}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{C.-28}$$

An arbitrary group element of $U(2)$ is given by

$$\exp\left(-i\sum_{k,l=1}^{2}\theta_{kl}\mathcal{C}_{kl}\right) \tag{C.-28}$$

where the "angles" θ parametrize the group tranformation.

The transition to $SU(2)$ can be made by constructing traceless matrices from the $\mathcal{C}_{kl}$

All in all

$$\begin{aligned} \tilde{\mathcal{C}}_{11} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \tilde{\mathcal{C}}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \tilde{\mathcal{C}}_{21} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\mathcal{C}}_{22} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{aligned} \tag{C.-28}$$

Finally for $\mu(c)$ we obtain

$$\mu(c)1/J[\partial c/\partial a]_{a=0} \tag{C.-28}$$

U(2), SU(2) The measure is given by

$$
\begin{aligned}
\mu(U(2)) &= |\epsilon_1 - \epsilon_2|^2 = |\exp(-i\theta_{11}) - \exp(-i\theta_{22})|^2 \\
&= |\exp\left(-\frac{1}{2}i(\theta_{11}+\theta_{22})\right)|^2 \times |\exp\left(-\frac{1}{2}i(\theta_{11}-\theta_{22})\right) \\
&\quad |\exp\left(\frac{1}{2}i(\theta_{11}-\theta_{22})\right)|^2 \\
&= 4\sin^2\left(\frac{1}{2}(\theta_{11}-\theta_{22})\right). \qquad \text{(C.-30)}
\end{aligned}
$$

restriction to $SU(2)$ yields $\theta_{11}+\theta_{22}=0$. With $\theta_{11}-\theta_{22}=\phi$ then

$$\mu(SU(2)) = 4\sin^2\frac{1}{2}\phi. \qquad \text{(C.-30)}$$

Normalization $C\int_0^{4\pi}\mu(SU(2))\,d\phi = 1$ determines constant C;

$$C\int_0^{4\pi} d\phi\sin^2\frac{1}{2}\phi = \frac{C}{2}\int_0^{4\pi} d\phi\,[1-\cos\phi] = \frac{C}{2}\int_0^{4\pi} d\phi = 2\pi \qquad \text{(C.-30)}$$

C.3.11 Peter-Weyl theorem

The irreducible representation functions $D^l(R)^{m'}_{m}$ satisfy orthogonality and completeness relations. In fact, they form a complete basis in the space of square integrable functions defined on the group parameter space. This is the Peter-Weyl theorem. That the irreducible representation functions form a complete basis for square-integrable functions $f(R)\in L^2$ can be expressed as

$$f(R) = \sum_{\alpha ab} f^{ab}_{\alpha} D^{(\alpha)}_{ab}(R) \qquad \text{(C.-30)}$$

Moreover the Peter-Weyl theorem states that the χ_r form a basis of the space of square-integrable class functions on the group G. That means that every square-integrable function $f(U)$, obeying

$$f(U) = f(VUV^{-1}), \qquad \text{(C.-30)}$$

can be expanded into characters:

$$f(U) = \sum_{r\in G} f_r\chi_r(U) \qquad \text{(C.-30)}$$

$$f_r = \int dU \overline{\chi_r}(U) f(U). \tag{C.-30}$$

(compare with discrete case: $\lambda(c_l) = \sum_\alpha a_\alpha \chi^{(\alpha)}(c_l)$ where $a_\alpha = \sum_k N_k \chi^{(\alpha)}(c_k)^* \lambda(c_k)$) the completeness relation can be written as

$$\sum_{r\in G} \chi_r(U)\overline{\chi_r}(V) = \delta(UV^{-1}). \tag{C.-30}$$

$$\int dU \overline{\chi_r}(U)\chi_s(U) = \delta_{rs}. \tag{C.-30}$$

The invariant $\delta-$function on G is defined by means of

$$\int dU f(U)\delta(UV^{-1}) = f(V), \tag{C.-30}$$

and obeys

$$\delta(U) = \delta(U^{-1}). \tag{C.-30}$$

The charactres are

$$\chi_j(U) = \frac{\sin\left(j+\frac{1}{2}\right)\phi}{\sin\frac{1}{2}\phi}. \tag{C.-30}$$

$$f_j = \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \sin\frac{1}{2}\phi\ \sin\left(j+\frac{1}{2}\right)\phi\ f(\phi). \tag{C.-30}$$

As an example consider the group $U(1)$

$$D_{(\mu)}(\theta) = e^{i\mu\theta}, \tag{C.-30}$$

where $\mu = 0, \pm 1, \pm 2, \ldots$.

$$D_{(\mu)}(\theta + 2\pi) = D_{(\mu)}(\theta). \tag{C.-30}$$

The Haar measure is

$$\Omega = \frac{1}{2\pi}\, d\theta. \tag{C.-30}$$

Theorem applied to $U(1)$ gives

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i\mu\theta} e^{i\nu\theta} = \delta_{\mu\nu} \tag{C.-30}$$

where $0, \pm 1, \pm 2, \dots$.

The *Peter-Weyl Theorem* applied to $U(1)$ gives the Fourier series theory:

$$f(\theta) = \sum_{-\infty}^{\infty} f_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \tag{C.-30}$$

where $f(\theta) \in L^2(U(1))$.

C.3.12 Analogies

$$\int dx\delta(x)f(x) = f(0) \quad \Longleftrightarrow \quad \int_G dU\delta(U)f(U) = f(I) \tag{C.-29}$$

$$\int dy\delta(y-a)f(y) = f(a) \quad \Longleftrightarrow \quad \int_G dU\delta(U\gamma^{-1})f(U) = f(\gamma) \tag{C.-28}$$

$$f(x) = \sum_n a_n e_n(x) \quad \Longleftrightarrow \quad f(g) = \sum_j a_j \chi_j(g) \quad (f(g) = f(\gamma g \gamma^{-1}))$$

$$a_n = \int \overline{e_n(x)} f(x) dx \quad \Longleftrightarrow \quad a_j = \int_G \overline{\chi_j(g)} f(g) dg \tag{C.-28}$$

$$\delta(x-y) = \sum_n \overline{e_n(x)} e_n(y) \quad \Longleftrightarrow \quad \delta(gx^{-1}) = \sum_j \overline{\chi_j(g)} \chi_j(x) \tag{C.-27}$$

$$\delta(x+a-a) \quad \Longleftrightarrow \quad \delta(xgx^{-1}) = \delta(g) \tag{C.-26}$$

$$\int dx\overline{e_m(x)} e_n(x) = \delta_{mn} \quad \Longleftrightarrow \quad \int D^j_{mn}(g)\overline{D^{j'}_{m'n'}(g)} dg = \delta_{jj'}\delta_{mm'}\delta_{nn'} \tag{C.-26}$$

$$\int dy\overline{e_m(y)} e_n(x-y) = \delta_{mn} e_n(x) \quad \Longleftrightarrow \quad \int_G \overline{\chi_{j'}(g)} \chi_j(gx) dg = V_G \delta_{jj'} \frac{\chi_j(x)}{d_j} \tag{C.-25}$$

Exercise

(a):

Recall how we prove $\int dx\delta(x)f(x) = f(0)$ implies $\int dy\delta(y-a)h(y) = h(a)$ where $h(y) = f(y-a)$

make the substitution $x = y - a$

$$h(a) = f(0) = \int dy\delta(y-a)f(y-a) = \int dy\delta(y-a)h(y) \qquad \text{(C.-25)}$$

prove

$$\int_G dU\delta(U)f(U) = f(I) \quad \Rightarrow \int_G dU\delta(U\gamma^{-1})f(U) = f(\gamma) \qquad \text{(C.-25)}$$

define $h(U\gamma^{-1}) = g(U)$

$$h(\gamma) = f(1) \qquad \text{(C.-25)}$$

(b):

$$f(g) = \sum_{\Lambda} \phi_{\Lambda}^{\alpha\beta} D_{\alpha\beta}^{(\Lambda)}(g) \qquad \text{(C.-25)}$$

$$f(g_1, g_2) = \sum_{\Lambda_1\Lambda_2} \phi_{\Lambda_1\Lambda_2}^{\alpha_1\alpha_2\beta_1\beta_2} D_{\alpha_1\beta_1}^{(\Lambda_1)}(g_1) D_{\alpha_2\beta_2}^{(\Lambda_2)}(g_2) \qquad \text{(C.-25)}$$

$$f(g_1, g_2 = C) = \sum_{\Lambda_1} a_{\Lambda_1}^{\alpha_1\beta_1}(g_2 = C) D_{\alpha_1\beta_1}(g_1) \qquad \text{(C.-25)}$$

$$a_{\Lambda_1}^{\alpha_1\beta_1}(g_2) = \sum_{\Lambda_2}^{\alpha_2\beta_2} D_{\alpha_2\beta_2}(g_2) \qquad \text{(C.-25)}$$

$$a_{\Lambda_1}^{\alpha_1\beta_1} =: \phi_{\Lambda_1\Lambda_2}^{\alpha_1\alpha_2\beta_1\beta_2} \qquad \text{(C.-25)}$$

Using $D_{ml}^{j}(g)D_{ln}^{j}(x) = D_{mn}^{j}(gx)$

$$\int \overline{D_{m'n'}^{j'}(g)} D_{mn}^{j}(gx) dg = \delta_{jj'}\delta_{mm'} D_{nn'}^{j}(x) \qquad \text{(C.-25)}$$

contracting m, m' and nn' we egt (C.-25).

Riesz-Fishcher theorem

Parseval's equation :

$$\sum_{-\infty}^{\infty} |c_n|^2 = \int |f(x)|^2 dx. \tag{C.-25}$$

$$f_n(x) = \sum_{k=-n}^{n} c_k e_k(x) \tag{C.-25}$$

converge to the vector f in the sense of L_2:

$$||f - f_n|| \to 0. \tag{C.-25}$$

limit in the mean of the f_n's.

$$||f_m - f_n||^2 = \sum_{|k|=n+1}^{m} |c_k|^2. \tag{C.-25}$$

if c_n are given complex numbers for which $\sum_{-\infty}^{\infty} |c_n|^2$ converges, then there exists a function f in L_2. If we grant the completeness of L_2 as a metric space this is easy to prove.

This tells us that the f's form a Cauchy sequence in L_2; and since L_2 is complete, there exists a function f in L_2 such that $f_n \to f$.

It is apparent that

C.3.13 Clebsch-Gordan

$$D_{(\mu)} \times D_{(\nu)} = \sum_{\otimes\sigma} n_{\mu\nu}^{(\Lambda)} D_{(\Lambda)} \tag{C.-25}$$

where $n_{\mu\nu}^{(\Lambda)}$ is the number of times that the Λ-th irreducible representation occurs in the product representation (**??**).

$$\chi_{(\mu\times\nu)}(g) = \chi_{(\mu)}(g)\chi_{(\nu)}(g). \tag{C.-25}$$

Thus by

$$n_{\mu\nu}^{(\Lambda)} = \int_G dg \overline{\chi_{(\Lambda)}(g)} \chi_{(\mu)}(g)\chi_{(\nu)}(g) \tag{C.-25}$$

C.3.14 Semi-direct Products

A full Lorentz transformation can be decomposed into an ordinary spacial rotaion, followed by a boost, followed by a further ordinary rotation.

Definition (First definition)

Suppose N is a normal subgroup of G and H is another subgroup of G such that $N \cap H = E$ (the identity of G) and every element of G can be written in a unique way as

$$g = ba, \quad b \in H, \ a \in N,$$

then G is said to be a **semi-direct product** of H and N, written $G = H \otimes_S N$.

Definition (Second definition)

We form a new group whose elements are the elements of $H \times N$ and multiplication given by

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1 h_2 \ , \ n_1 \rho_{h_1}(n_2)) \quad \text{with} \quad \rho_{h_1}(n_2) = h_1 n_2 h_1^{-1}. \tag{C.-25}$$

Note $h_1 n_2 h_1^{-1} \in N$ as N is a normal subgroup of G.

Check it forms a group

The identity element of this group is (E, E):

$$(h, n) \cdot (E, E) = (h, h \ hEh^{-1}) = (h, n),$$
$$(E, E) \cdot (h, n) = (h, E \ EnE^{-1}) = (h, n).$$

The inverse of (h, n) is (h^{-1}, n') where $n' = h^{-1} n^{-1} h$:

$$(h, n) \cdot (h^{-1}, n') = (E, n \ hn'h^{-1}) = (E, E),$$
$$(h^{-1}, n') \cdot (h, n) = (E, n' \ h^{-1} nh) = (E, E).$$

Associativity $[(h_1, n_1) \cdot (h_2, n_2)] \cdot (h_3, n_3) = (h_1, n_1) \cdot [(h_2, n_2) \cdot (h_3, n_3)]$:

$$\begin{aligned} [(h_1, n_1) \cdot (h_2, n_2)] \cdot (h_3, n_3) &= (h_1 h_2, n_1 \ h_1 n_2 h_1^{-1}) \cdot (h_3, n_3) \\ &= (h_1 h_2 h_3, n_1 \ h_1 n_2 h_1^{-1} \ h_1 h_2 n_3 (h_1 h_2)^{-1}) \end{aligned}$$

$$
\begin{aligned}
(h_1, n_1) \cdot [(h_2, n_2) \cdot (h_3, n_3)] &= (h_1, n_1) \cdot (h_2 h_3, n_2\ h_2 n_3 h_2^{-1}) \\
&= (h_1 h_2 h_3, n_1\ h_1 (n_2\ h_2 n_3 h_2^{-1}) h_1^{-1})
\end{aligned}
$$

$\square$

Equivalence of the two defintions

for example $H \cap N$ might be empty.

for each $h \in H$ the inner automorphism $x \rightarrow hxh^{-1}$ takes N to N and defines an automorphism

$$\rho_h(n) = hnh^{-1}$$

Moreover,

$$
\begin{aligned}
\rho_{h_1}(\rho_{h_2}(n)) &= \rho_{h_1}(h_2 n h_2^{-1}) = h_1 h_2 n h_2^{-1} h_1^{-1} \\
&= \rho_{h_1 \cdot h_2}(n)
\end{aligned}
$$

Thus $h \rightarrow \rho_h$ is a homomorphism of H into the group of automorphisms of N, we write $\rho \in Hom(H, Aut(N))$.

Sub-groups

The group G is the semigroup product of N by H with homomorphism ρ.

Recall $\rho(h)(n) = hnh^{-1}$. Note that if $\rho \in Hom(H, Aut(N))$

Obviously there is the subgroup of G composed of elements of the form:

$$(h_i, E) \quad \text{for all } h_i \in H.$$

$$H \otimes_S I \simeq H$$

there is the subgroup of G composed of elements of the form:

$$(E, n_i) \quad \text{for all } n_i \in N.$$

as

$$(E, n_1) \cdot (E, n_2) = (E, E, n_1\ Eh_2E) = (E, n_1 n_2)$$

$$I \otimes_S N \simeq N$$

Conversely, suppose that we are given two groups N and H and a homomorphism $h \to \rho_h$ of H into the group of all automorphisms of N. We may then define a group $H \otimes_S N$ with respect to ρ as follows.

$\square$

Example The group RT is a semi-direct product of the rotation group $R(2)$ and the group of all trnaslations $T(2)$

Proof:

Example The group ?? is a semi-direct product of the rotation group $R(3)$ and the group of all trnaslations $T(3)$

Proof:

The group of all translations and rotations has six generators $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{J}_1, \hat{J}_2, \hat{J}_3$. More concisely, it is called the transaltion-rotaion group and has the translations $(\hat{p}_\nu)$ as an abelian subgroup. It is obvious that

$$\hat{R}\hat{T}\hat{R}^{-1} = \hat{T}',$$

where $\hat{R}$ is a rotation and $\hat{T}$ a translation, is again a pure translation $\hat{T}'$ (see fig (N.-19)). Consequently the translation group is an invariant abelian subgroup of the translation-rotation group.

The group consists of pairs of the pairs (r, t) with $t \in T$, $r \in O(3, \mathbb{R})$ and multiplication rule

$$(r_1, t_1) \cdot (r_2, t_2) = (r_1 r_2, t_1 r_1 t_2 r_1^{-1}). \tag{C.-31}$$

Example The Poincare group P includes the abelian subgroup $T(4)$ of all translations in Minkowski spacetime in addition to the Lorentz transformations. The Poincare group P is a semi-direct product of the Lorentz group $L(4)$ and the group of all translations $T(4)$ on Minkowski spaectime, that is,

$$P = L(4) \otimes_S T(4). \tag{C.-31}$$

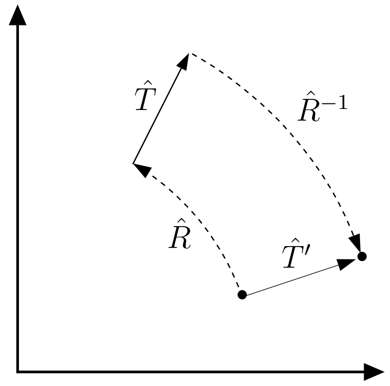


Figure C.9: TranRotGrF.

Proof:

$T(4)$ is a normal abelian subgroup of P as for $t \in T(4)$ and any $p \in P$, $ptp^{-1} \in T(4)$.

An element $p \in P$ is then denoted $p = (\Lambda^\mu_\nu, a^\mu)$.

Examples from Quantum Gravity

(i) Three diemnsional gravity:

where $ISU(2)$ is the (universal cover of the) group of Euclidean transformations. It is the semigroup:

$$ISU(2) = SU(2) \otimes_S \mathbb{R}^3 \tag{C.-31}$$

Its elements are written as $(u, \vec{a})$ where u is an element of $SU(2)$ and $\vec{a}$ is a vector in $\mathbb{R}^3$. We have

$$(u_1, \vec{a}_1) \cdot (u_2, \vec{a}_2) = (u_1 u_2, u_1 \vec{a}_2 + \vec{a}_1), \tag{C.-31}$$

the notation $u\vec{a}$ means $U(u)\vec{a}$ where U is the vectorial representation of $SU(2)$.

(ii) black hole gauge group is the semi-direct product of ....

(iii) Gauge group of LQG

The group of these symmetries is the semi-direct product the group of smooth local gauge transformations with the group of smooth diffeomorphisms on Σ.

C.4 Infinite-Dimensional Group Representations

C.4.1 Group Actions

If G is a permutation group, then every element of the group permutates elements of the set $\{1, 2, \cdots n\}$. We say that the group G acts on the set $\{1, 2, \cdots n\}$.

Group action on a set Let G be a group and let A be a some set. Suppose that we have a map $T : G \times A \to A$ such that for every fixed $g \in G$ the map $a \to T(g, a)$ is a permutation of the set A.

We require the compatibility conditions:

(i) $T(g_1, T(g_2, a)) = T(g_1 g_2, a)$ for all $g_1, g_2 \in G$ and $a \in A$.

(ii) $T(e, a) = a$ for all $a \in A$

where $e = e_G$ the identity of G.

define what is called T is a **group action** and that G acts on A by the action of T.

We can shorten the notion by writing $g \cdot a$ instead of $T(g, a)$.

These conditions are designed so that the map $G \to S_A$ is a group homomorphism.

C.4.2 Countable and Locally Compact Topological Groups

Let S be a set on which some group acts as a group of transformations. Let us write sx for the transfrom by x of $s \in S$. Then

$$(sx)y = s(xy) \qquad \text{for all } s \in S, x, y \in G.$$

and

$$se = s$$

where e is the identity element of G.

We assume that G is a topological group which is separable and locally compact and S comes with a 'volume element'. We assume that S is a Borel space with a measure μ; that is, that we are given a family $\mathcal{B}$ of subsets of S, closed with respect to completions and countable uniions, and measure μ assigning a nonnegative real number or ∞ to each subset $E \in \mathcal{B}$ so that

$$\mu(E_1 + E_2 + \cdots) = \mu(E_1) + \mu(E_2) + \cdots$$

whenever $E_i \cap E_j = 0$ for all $i \neq j$. The members of $\mathcal{B}$ are called *Borel sets*. We assume S is a union of countably many sets of finite measure and that

If S_1 and S_2 are Borel spaces, then a function from S_1 to S_2 is a Borel function if $g^{-1}(E)$ is a Borel set in S_1 whenever E is a Borel set in S_2. Any topological space becomes a Borel space if we define the Borel sets to be the sets one can obtain from closed sets by the taking of complements, countable unions, and countaale intersections. We extend theorems already proven to much more general context as the notion of Borel space and Borel function are quite general. A function can be widly discontinuous and still be a Borel function.

We shall say that the measure μ is *invariant* if

$$\mu(Ex) = \mu(E) \qquad \text{for all Borel sets } E \text{ and all } x \in G. \tag{C.-31}$$

C.4.3 Haar Measure

Given an $n-$dimensional manifold and a nowhere-vanishing orieneted $n-$form η, we can make a measure on $\mathcal{M}$ by defining the integral of f against μ to be the integral of the $n-$form $f\eta$.

It is not hard to show that on an $n-$dimensional Lie group G, there exists a nowhere-vanishing $n-$form that is invariant under left translations and that this form is unique up to a constant. Integrating fucntions against this form gives a letf-invariant measure (i.e., a left Haar measure).

C.4.4 Summary of Group theory

- Unitrary representations.
- Irreduciable representations.
- Schur's lemmas.
- Orthoganality theorems.

C.5 Manifolds and Elementary Topology

Roughly, a manifolds are sets on which, at least around each point, everything looks Euclidean.

Organized set of points with a structure - a division into convenient family of subset families.

It is this need for care, to ensure we can rely on calculations we do, that motivates much of this course, illustrates why we empathize accurate argument as well as getting the "correct" answers, and explains why in the rest of this section we need to revise elementary notions.

C.5.1 Sets and Mappings Between Sets

We need to be able to talk easily about certain subsets of R. We say that I is an **open interval** if

$$I = (a, b) = x \in R : a < x < b. \tag{C.-31}$$

Thus an open interval excludes its end points, but contains all the points in between. x is always separated from F by

In contrast a **closed interval** contains both its end points, and is of the form

$$I = [a, b] = \{x \in R : a \leq x \leq b\}. \tag{C.-31}$$

It is also sometimes useful to have **half-open intervals** like $(a, b]$ and $[a, b)$. It is trivial that $[a, b] = (a, b) \cup_{\{a\}} \cup_{\{b\}}$.

The two end points a and b are points in R. It is sometimes convenient to allow also the possibility $a = -\infty$ and $b = +\infty$; it should be clear from the context whether this is being allowed. If these extensions are being excluded, the interval is sometimes called a *finite interval*, just for emphasis.

Of course we can easily get to more general subsets of R. So $(1, 2) \cup [2, 3] = (1, 3]$ shows that the union of two intervals may be an interval, while the example $(1, 2) \cup (3, 4)$ shows that the union of two intervals *need not* be an interval.

An open subset V of X is a subset where if $x \in V$, then there is a $\delta > 0$ such that an open ball $B_\delta(x)$ is entirely contained in V.

Notation	Meaning
$a \in A$	a belongs to A
$A \subset B$	A is included in B
$A = B$	A is identical to B
$A \cup B$	The union of A and B ($\cup$ is for **u**inion)
$A \cap B$	The intersection A and B ($\cap$ is for **in**tersection)
$A - B$	The set of elements of A not included in B

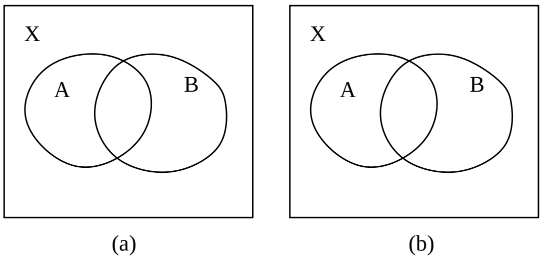


Figure C.10: .

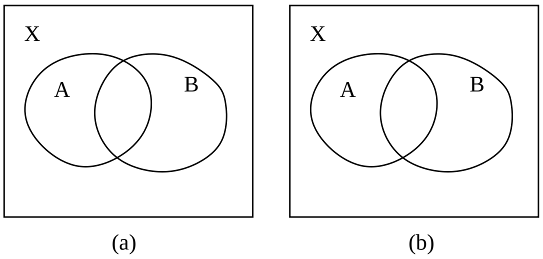


Figure C.11: .

C.5.2 Continuity

A function f is continuous at a point p if whenever we can force the distance between $f(x)$ and $f(p)$ to be as small as desired by taking the distance between x and p to be small enough.

The definition of continuity of a function f on the real line: $f : R \mapsto R$ is continuous at x_0 if for any positive number ϵ, there exists a positive number δ such that if $|y - x_0| < \delta$, then $|f(y) - f(x_0)| < \epsilon$. The setting for this definition is the real line, a Euclidean space.

We can generalize this definition a little by considering mapping between spaces with a metric. Let X, Y be two spaces with metrics d_1 and d_2, respectively. Then, $f : X \to Y$ is continuous at $x_0 \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that if $d_1(x, x_0) < \delta$, then $d_2(f(x), f(x_0)) < \epsilon$.

However, the notion of continuity does not depend on a metric.

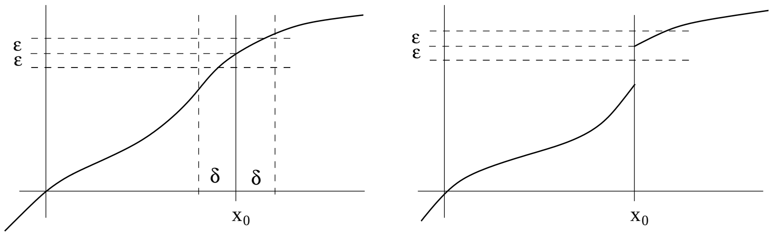


Figure C.12: (a) (b) f is discontinuous at p_1.

We need to be able to talk about a function near a point a. If we want to look at the points a distance less than d for a. we are looking at an interval $(a - d, a + d)$. We call such an interval a neighbourhood of a.

Definition A subset $\mathcal{U}$ is **open** if given $a \in \mathcal{U}$, there is some $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq \mathcal{U}$.

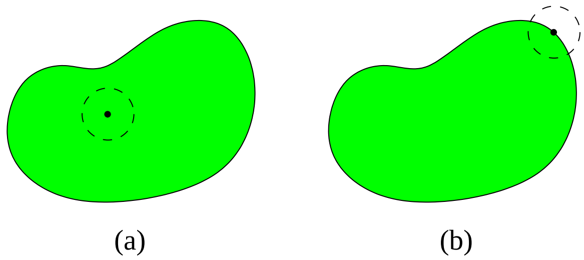


Figure C.13: Open sets interior points (b) .

In fact this is the same as saying that given $a \in \mathcal{U}$, there is some open interval containing a which lies in $\mathcal{U}$ - a set is open if it contains a neighbourhood of each of its points. This definition has the effect that if a function is defined on an open set that its behaviour near point a of interest from both sides.

We can transfer such things as limits and calculus from Euclidean space. A ***topology*** is a structure added to an arbitrary point set which enables one to define a convergent sequence and to define a continuous function in a general setting.

A ***topology***, $\mathcal{T}$, on a set X is defined to be a specified family of open subsets on X satisfying the following 3 properties:
(i) The empty set, $\emptyset$, and the space X belong to $\mathcal{T}$.
(ii) The union of any number (possibly infinite) of open subsets belonging to $\mathcal{T}$ is also in $\mathcal{T}$.
(iii) The intersection of any finite number (nor infinite) open subsets in $\mathcal{T}$ also belongs to $\mathcal{T}$.

The set X together with a topology $\mathcal{T}$ is called a topological space.

Let $f : x \rightarrow Y$ be a function between two topological spaces $(X, \mathcal{T}_X)$, and $(Y, \mathcal{T}_Y)$. If $x \in X$, then $f(x) \in Y$ is the image of x under f. Let U be an open set in X (i.e. $U \in \mathcal{T}_X$). The image of U under f is the subset $V = f(U) \subset Y$, the range of f with domain U. If V is a subset of Y, then the inverse image of V by f. The mapping f is defined to be *continuous* when the inverse image of any open set is open. That is, f is continuous if $U = f^{-1}(V) \in \mathcal{T}_X$ when $V \in \mathcal{T}_Y$.

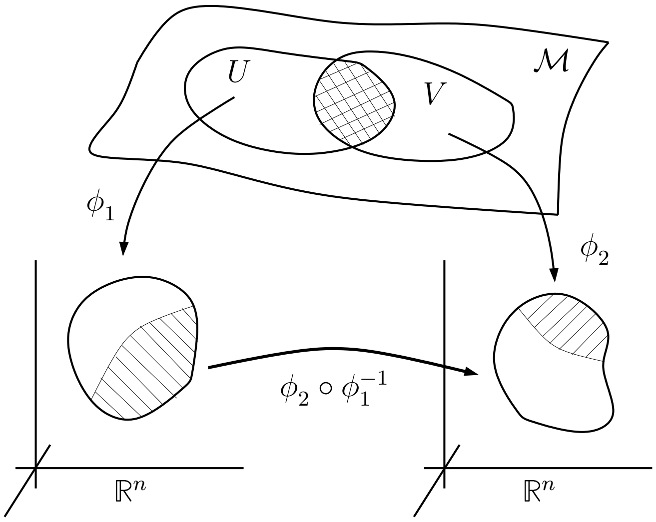


Figure C.14: (U, ϕ_1) and (V, ϕ_2) are two coordinate patches on X. Transition functions, $\phi_2 \circ \phi_2^{-1}$, are ordinary functions that go from points of one R^n space onto another, i.e. $\phi_2 \circ \phi_2^{-1} : R^n \mapsto R^n$. The domain and range of the transition function are the shaded regions in R^n.

The transition functions transform the coordinates of one overlapping patch into another.

The usual Jacobian of ψ_{ij} is

$$
\begin{aligned}
(\Lambda^a{}_b) &= \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} \\
&= \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^n} \\ \cdots & \cdots & \\ \frac{\partial x'^n}{\partial x^1} & \cdots & \frac{\partial x'^n}{\partial x^n} \end{pmatrix}
\end{aligned} \tag{C.-31}
$$

C.6 Elementary Tensor Analysis

One often needs to do a change of coordinates, either because we prefer to use a different choice of coordinates valid in some patch, or because we need to transform to new patch which covers a different portion of the manifold.

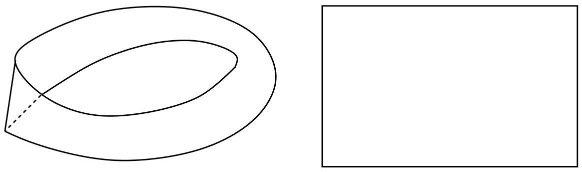


Figure C.15: Möbius.

Einstein summation convention

$$\sum_a (..)^a\,(..)_a \rightarrow (..)^a\,(..)_a \tag{C.-31}$$

Contravariant Vectors

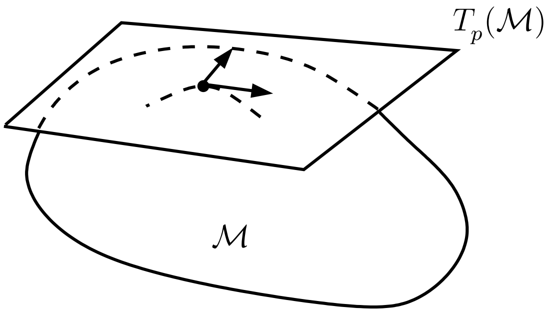


Figure C.16: tangtoM.

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{C.-31}$$

the infinitesimal displacement in two different coordinate systems x^a and x'^a is related by

$$dx'^a = \sum_b \frac{\partial x^{a\prime}}{\partial x^b}dx^b \tag{C.-31}$$

where $x^{a\prime}/x^b$ is evaluated at the point $p = x^a = x'^a(x^a)$. **Einstein summation convention**

$$dx^{a\prime} = \frac{\partial x^{a\prime}}{\partial x^b}dx^b \tag{C.-31}$$

We set

$$\Lambda^a_b = \frac{\partial x^{a\prime}}{\partial x^b} \tag{C.-31}$$

The infinitesimal displacement is the prototype of a geometric object which is called a contravariant vector. A set of quantities X_a are said to be the components of a contravariant vector if they transform, under a change of coordinates, as

$$X'^a = \frac{\partial x^{a\prime}}{\partial x^b} X^b \tag{C.-31}$$

Covariant Vectors

$$df = \frac{\partial f}{\partial x^a} dx^a = 0 \tag{C.-31}$$

$$\frac{\partial f'}{\partial x^{a\prime}} = \frac{\partial f}{\partial x^b} \frac{\partial x^b}{\partial x^{a\prime}} \tag{C.-31}$$

$f = f(x^a) = f(x^{a\prime}) = f'$.

A set of quantities X_a are said to be the components of a covariant vector if they transform as

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b \tag{C.-31}$$

Covariant vectors can be interpreted as linear functionals mapping vectors to $\mathbb{R}$.

$$\mathbf{V} = V^a \mathbf{e}_a = V'^a \mathbf{e}'_a \tag{C.-31}$$

It is evident that if the vector $\mathbf{V}$ is to be invariant the basis vectors $\{\mathbf{e}_a\}$ must transform under coordinate transformations as the components of a covector. Similarly, basis covectors $\{\mathbf{e}^a\}$ must change under a coordinate transform as the components of a contravariant vector.

Tensors

A set of quantities T_{ab} are said to be the components of a covariant tensor of second?? if they transform as

$$T'_{ab} = \frac{\partial x^c}{\partial x'^b} \frac{\partial x^d}{\partial x'^a} T_{cd} == \tag{C.-31}$$

Contractions of Tensors

$$\frac{\partial x^{a\prime}}{\partial x^c}\frac{\partial x^c}{\partial x^{b\prime}} = \delta^b_a \tag{C.-31}$$

C.6.1 Affine Connection

$$\begin{aligned} dX^a &= X^a(Q) - X^a(P) \\ &= X^a(x^b + dx^b) - X^a(x^b) \\ &= \frac{\partial X^a}{\partial x^b}dx^b \end{aligned}$$

is not tensorial. Under a coordinate transformation a partial derivative of a vector X^a transforms as

$$\frac{\partial X'^a}{\partial x'^b} = \frac{\partial x^c}{\partial x'^b}\frac{\partial}{\partial x^c}\left(\frac{\partial x'^a}{\partial x^d}X^d\right) = \underbrace{\frac{\partial x^c}{\partial x'^b}\frac{\partial x'^a}{\partial x^d}\frac{\partial X^d}{\partial x^c}}_{tensorial} + \underbrace{\frac{\partial^2 x'^a}{\partial x^c \partial x^d}\frac{\partial x^c}{\partial x'^b}X^d}_{inhomogeneous} \tag{C.-34}$$

This is not a tensorial transformation because of the inhomogeneous term.

have

$$dx^b \nabla_b X^a = \frac{\partial X}{\partial x^b}dx^b + \Gamma^a_{cb}X^c dx^b \tag{C.-34}$$

For $\nabla_b X^a$ to transform as a tensor of type $(1,1)$ then the connection must transform as

$$\Gamma'^{\alpha}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial x^{\sigma}}{\partial x'^{\beta'}}\frac{\partial x^{\tau}}{\partial x'^{\gamma}}\Gamma^{\delta}_{\sigma\tau} + \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial^2 x^{\delta}}{\partial x'^{\beta}\partial x'^{\gamma}} \tag{C.-34}$$

Any quantity that transforms as (C.6.1) is called an affine connection. If the second term on the right-hand side were absent, then this would be the transformation law of a tensor of type (1,2).

$$T^{\alpha}_{\beta\gamma} = \Gamma'^{\alpha}_{\beta\gamma} - \Gamma'^{\alpha}_{\gamma\beta} \tag{C.-34}$$

is a tensor called the *torsion tensor*. If the torsion tensor vanishes then the connection is said to be *symmetric* or (*torsion-free*), i.e.

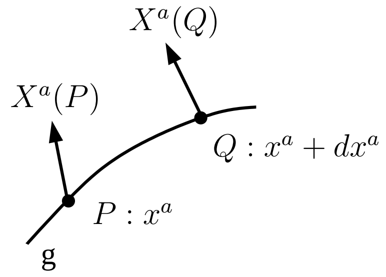


Figure C.17: connection.

$$\Gamma'^{\alpha}_{\beta\gamma} = \Gamma'^{\alpha}_{\gamma\beta} \tag{C.-34}$$

Covariant differentiation can be extended to other types of tensors by demanding that the covariant derivative obeys the product rule of differential calculus. The covariant derivative of a scalar field is the same as its partial derivative,

$$\nabla\phi = \partial_a \phi. \tag{C.-34}$$

Demanding that the covariant derivative obeys the Leibniz rule, then we find

$$\nabla_b X_a = \partial_b X_a - \Gamma^c_{ab} X_c \tag{C.-34}$$

$$\nabla_c T^{a\ldots}_{b\ldots} = \partial_c T^{a\ldots}_{b\ldots} + \Gamma^a_{dc} T^{d\ldots}_{b\ldots} + \cdots - \Gamma^d_{bc} T^{a\ldots}_{d\ldots} - \ldots. \tag{C.-34}$$

The covariant derivative contracted with X.

$$\nabla_X T^{a\cdots}_{b\cdots} := X^c \nabla_c T^{a\cdots}_{b\cdots} \tag{C.-34}$$

$$\frac{D}{Du} T^{a\cdots}_{b\cdots} = \nabla_X T^{a\cdots}_{b\cdots} \tag{C.-34}$$

C.6.2 Affine Geodesic

A geodesic is the closest thing there is to a straight line curved space time. In we are given a metric we can define a geodesic as the shortest distance between two points. However, there is a more general definition of a geodesic is its velocity vector is parallel transported along the curve it traces out in spacetime (it follows its own nose, so to speak). In other words, the parallely propagated vector at any point of the curve is parallel, that is, proportional to the vector at this point:

$$\frac{dx^b}{du}\nabla_b\left(\frac{dx^a}{du}\right) = \lambda(u)\frac{dx^a}{du} \tag{C.-34}$$

The L.H.S. becomes

$$\begin{aligned}\frac{dx^b}{du}\nabla_b\left(\frac{dx^a}{du}\right) &= \frac{dx^b}{du}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{du}\right) + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^a}{du} \\ &= \frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du}.\end{aligned}$$

We have the geodesic equation

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \lambda(u)\frac{dx^a}{du}. \tag{C.-36}$$

Note that this definition did not involve a metric, it only refers to a connection. Other geometric notions can also be defined using a connection only. Such description are desirable when we come to formulate the quantum theory where there is background metric. In fact LQG is formulated with a connection of a frame field (which we will be coming to presently). In the next section we shall verify that the differential equation derived from the principle of shortest distance is as in (C.6.2) but with the connection Γ^a_{bc} as a particular function of the metric. This connection is called the metric connection.

C.6.3 The Metric Connection

The vanishing of the covariant derivative of the metric is equivalent to requiring that the length of a vector is unchanged under parallel propagation. To see this first consider the covariant derivative of the length squared l^2 of a arbitrary vector l^μ,

$$\begin{aligned}0 = \partial_\sigma(l^2) = \nabla_\sigma(l^2) &= \nabla_\sigma(g_{\mu\nu}l^\mu l^\nu) \\ &= l^\mu l^\nu \nabla_\sigma g_{\mu\nu} + g_{\mu\nu}l^\nu\nabla_\sigma l^\mu + g_{\mu\nu}l^\mu\nabla_\sigma l^\nu \\ &= l^\mu l^\nu\nabla_\sigma g_{\mu\nu} + 2l_\mu\nabla_\sigma l^\mu \end{aligned} \tag{C.-37}$$

By assumption the vector vector l^μ is parallel propagated i.e. $\nabla_\sigma l^\mu = 0$

$$l^\mu l^\nu \nabla_\sigma g_{\mu\nu} = 0 \tag{C.-37}$$

since l^μ was arbitrary this implies,

$$\nabla_\sigma g_{\mu\nu} = 0 \tag{C.-37}$$

(It should be noted that the vanishing of the covariant derivative was motivated by a physical requirement and not a mathematical one). This condition is sufficient to determine the connection as a function of the metric. We show this by using

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} + \Gamma^\rho_{\sigma\mu} g_{\nu\rho} + \Gamma^\rho_{\sigma\nu} g_{\mu\rho} = 0 \tag{C.-37}$$

$\Gamma_{\mu\nu,\sigma} = g_{\sigma\lambda}\Gamma^\lambda_{\mu\nu}$. Written out explicitly, with cyclic rotating of indices,

$$\begin{array}{rcl} \partial_a g_{bc} + \Gamma_{ab,c} + \Gamma_{ac,b} = 0 & \leftrightarrow & (\nabla_\sigma g_{ab} = 0) \\ \partial_\nu g_{\lambda\mu} + \Gamma_{\nu\lambda,\mu} + \Gamma_{\lambda\nu,\mu} = 0 & \leftrightarrow & (\nabla_\nu g_{\lambda\mu} = 0) \\ \partial_\lambda g_{\mu\nu} + \Gamma_{\lambda\mu,\nu} + \Gamma_{\nu\mu,\lambda} = 0 & \leftrightarrow & (\nabla_\lambda g_{\mu\nu} = 0) \end{array} \tag{C.-38}$$

These three equations are identical. But by adding the first two equations and subtracting the last (and remembering that the Christoffel symbol is symmetric in the lower case indices), we find:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\{\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\} \tag{C.-38}$$

Covariant Derivative

$$\nabla_a V_b = \partial_a V_b + \Gamma^c_{ab} V_c \tag{C.-38}$$

Metric Geodesic

$$\left(\frac{ds}{du}\right)^2 = g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}. \tag{C.-38}$$

$$s = \int_{p_1}^{p_2} ds = \int_{p_1}^{p_2} \frac{ds}{du} du = \int_{p_1}^{p_2} \left(g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}\right)^{1/2} du. \tag{C.-38}$$

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \left(\frac{d^2s}{du^2}\Big/\frac{ds}{du}\right)\frac{dx^a}{du}, \tag{C.-38}$$

$$u = \alpha s + \beta, \tag{C.-38}$$

where α and β are constants, then the righthand side vanishes.

we assume $ds \neq 0$

equations for a metric geodesic

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{C.-38}$$

and

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = -1, \tag{C.-38}$$

where Γ^a_{bc} is given by (C.6.3).

Proof.

$$\frac{d}{du}\frac{\partial\mathcal{L}}{\partial\dot{x}^a} - \frac{\partial\mathcal{L}}{\partial x^a} = 0 \tag{C.-38}$$

We instead minimize this instead. So we use $\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b$ in

$$2\mathcal{L}\left[\frac{d}{du}\frac{\partial\mathcal{L}}{\partial\dot{x}^a} - \frac{\partial\mathcal{L}}{\partial x^a}\right] = 0 \tag{C.-38}$$

which can be rewritten as

$$\frac{d}{du}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right) - \frac{\partial\mathcal{L}^2}{\partial x^a} = 2\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\frac{d\mathcal{L}}{du} \tag{C.-38}$$

Substituting for $\mathcal{L}^2$, the left hand side gives

$$
\begin{aligned}
\frac{d}{du}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right)-\frac{\partial\mathcal{L}^2}{\partial x^a} &= \frac{d}{du}\left[\frac{\partial}{\partial\dot{x}^a}(g_{bc}\dot{x}^b\dot{x}^c)\right]-\frac{\partial}{\partial x^a}(g_{bc}\dot{x}^b\dot{x}^c)\\
&= \frac{d}{du}(2g_{ab}\dot{x}^b)-(\partial_a g_{bc})\dot{x}^b\dot{x}^c\\
&= 2g_{ab}\ddot{x}^b+2\partial_c g_{ab}-\partial_a g_{bc}\dot{x}^b\dot{x}^c\\
&= 2g_{ab}\ddot{x}^b+2\dot{x}^b\dot{x}^c[\frac{1}{2}(\partial_c g_{ba}+\partial_b g_{ca}-\partial_a g_{bc})] \qquad \text{(C.-40)}
\end{aligned}
$$

$$
\begin{aligned}
2\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\frac{d\mathcal{L}}{du} &= 2\frac{\partial}{\partial\dot{x}^a}(g_{bc}\dot{x}^b\dot{x}^c)\frac{d}{du}\frac{ds}{du}\\
&= 2(g_{bc}\dot{x}^b\dot{x}^c)^{-1/2}g_{ad}\dot{x}^d\frac{d^2s}{du^2}\\
&= 2\left(\frac{d^2s}{du^2}\Big/\frac{ds}{du}\right)g_{ab}\dot{x}^b. \qquad \text{(C.-41)}
\end{aligned}
$$

multiplying through by g^{ad}

Lighlike inertial motion cannot be characterized with reference to proper time parameterization since the proper time along a null curve vanishes. However, this does not prevent us from characterizing such motion in the same manner as in the timelike case. To this end we go back to the more general definition of a geodesic as a curve $x^a(\lambda)$ with the property that the coordinate acceleration $d^2x^a/d\lambda^2$ at every point p is parallel to the tangent vector.

It is independent of the parameterization of the curve

C.6.4 Curvature

Let ξ^a be a contravariant vector field with

$$\dot{\xi}^c(x(\tau)) = 0. \qquad \text{(C.-41)}$$

We take the curve to be small so that we can write

$$\xi^c(x) = \xi^c + \xi^c_{,b}x^b + \mathcal{O}(x^2). \qquad \text{(C.-41)}$$

In an affine space without metric the term 'small' and 'large' appear to be meaningless. However, since differentiability is required, the small size limit *is* well defined. Thus, it is more precise to state that the curve is *infinitesimally* small.

If there is a strong gravitational field the contravariant vector may not return to its original value going around the loop once and have deviation $\delta\xi^a$. We find:

$$\begin{aligned}
\oint d\tau \dot{\xi} &= 0 \\
\delta\xi^c &= \oint d\tau \frac{d}{d\tau}\xi^c(x(\tau)) = -\oint \Gamma^c_{ab}\frac{dx^b}{d\tau}\xi^a(x(\tau))d\tau \\
&= -\oint d\tau \big(\Gamma^c_{ab} + \Gamma^c_{ab,d}x^d\big)\frac{dx^b}{d\tau}\big(\xi^a + \xi^a_{,d}x^d\big). \qquad \text{(C.-42)}
\end{aligned}$$

where we chose the function $x(\tau)$ to be v. small, so that terms $\mathcal{O}(x^2)$ can be neglected. We have for a closed curve,

$$\oint d\tau \frac{dx^b}{d\tau} = 0 \quad \text{and} \quad \nabla_a \xi^c \approx 0 \to \xi^c_{,a} \approx -\Gamma^c_{ab}\xi^b, \qquad \text{(C.-42)}$$

so that (N.-19) becomes

$$\delta\xi^c = \frac{1}{2}\left(\oint x^d \frac{dx^b}{d\tau} d\tau\right) R^c_{abd}\xi^a + \mathcal{O}(x^2). \qquad \text{(C.-42)}$$

covariant derivative of $T_{\nu\gamma}$ is given by

$$\nabla_\mu T_{\nu\gamma} = \partial_\mu T_{\nu\gamma} + \Gamma^\delta_{\mu\nu} T_{\delta\Gamma} + \Gamma^\delta_{\mu\gamma} T_{\nu\delta} \qquad \text{(C.-42)}$$

$$\begin{aligned}
\nabla_c(\nabla_d V^a) &= \partial_c(\nabla_d V^a) + \Gamma^e_{cd}(\nabla_e V^a) - \Gamma^a_{ce}(\nabla_b V^e) \qquad \text{(C.-41)} \\
&= \partial_c(\partial_d V^a + \Gamma^a_{df}V^f) + \Gamma^e_{cd}(\partial_e V^a + \Gamma^c_{df}V^f) - \Gamma^a_{ce}(\partial_b V^d + \Gamma^d_{bf}V^f)
\end{aligned}$$

$$\begin{aligned}
\nabla_c\nabla_d V^a - \nabla_d\nabla_c V^a &= \partial_c\partial_d V^a - \partial_d\partial_c V^a \ \text{(we take} = 0) \\
&+ \partial_c\Gamma^a_{df}V^f - \partial_d\Gamma^a_{cf}V^f \\
&+ \Gamma^e_{cd}\partial_e V^a - \Gamma^e_{dc}\partial_e V^a \quad (= 0 \ \text{as } \Gamma^e_{cd} = \Gamma^e_{dc})
\end{aligned} \qquad \text{(C.-44)}$$

$$\nabla_c\nabla_d V_a - \nabla_d\nabla_c V^a = \qquad \text{(C.-44)}$$

where $R^{\alpha}_{\beta\gamma\delta}$

$$R^{\alpha}_{\beta\gamma\delta} \tag{C.-44}$$

$$R^{\rho}_{\mu\nu\sigma} = \partial_{[\alpha}\Gamma_{\ \alpha} \tag{C.-44}$$

C.6.5 Gaussian Normal Coordinates

$$x'^a = x^a + Q^a_{bc}\frac{x^b x^c}{2} \text{ where } Q^a_{bc} = Q^a_{cb} \tag{C.-44}$$

$$\frac{\partial x'^a}{\partial x^d} = \delta^a_d + Q^a_{bd}x^b \tag{C.-44}$$

$$\frac{\partial^2 x'^a}{\partial x^d \partial x^e} = Q^a_{de} \tag{C.-44}$$

$$x^a \stackrel{*}{=} 0 \tag{C.-44}$$

$$\left[\frac{\partial x'^a}{\partial x^b}\right]_P = \delta^a_b, \quad \left[\frac{\partial x^b}{\partial x'^a}\right]_P = \delta^a_b \tag{C.-44}$$

$$\Gamma'^a_{\ bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\Gamma^d_{\ ef} - \frac{\partial x^d}{\partial x'^b}\frac{\partial x^e}{\partial x'^c}\frac{\partial^2 x'^a}{\partial x^d \partial x^e} \tag{C.-44}$$

$$\left[\Gamma'^a_{\ bc}\right]_P = [\Gamma^a_{bc}]_P - Q^a_{bc} \tag{C.-44}$$

choose $Q^a_{bc} = [\Gamma^a_{bc}]_P$

Mathematically this says that there exists a coordinate system the space time manifold is locally Minkowskian - one can always find a local coordinate system in which the metric tensor takes the pseudo-Euclidean form and the connection Γ^a_{bc} vanish at a point. In geometric terms, such a freely-falling frame of reference represents a local coordinate system.

Defined only in the immediate vicinity of a physically defined location (for example the place where two particles intersect), whose coordinate axes are very close to straight line and mutually perpendicular.

The physical statement is the gravitational field can be made to vanish at the place where two particles intersect, by going into free-fall.

C.6.6 Bianchi Identities

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} \tag{C.-44}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\beta\alpha\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{C.-44}$$

Cyclic Bianchi identity

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} \equiv 0 \tag{C.-44}$$

$$\nabla_{\alpha} R_{\delta\sigma\beta\gamma} + \nabla_{\gamma} R_{\delta\sigma\alpha\beta} + \nabla_{\beta} R_{\delta\sigma\gamma\alpha} \equiv 0. \tag{C.-44}$$

Define the tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R, \tag{C.-44}$$

the so-called Einsten tensor (its name comming from that it appears on the left-hand side of Einstein's field equations of general relativity). We have the contracted Bianchi identity

$$\nabla_{\alpha} G^{\alpha\beta} = 0. \tag{C.-44}$$

which follows from the Bianchi identity,

$$\nabla_{\sigma} R^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu} R^{\alpha}{}_{\beta\nu\sigma} + \nabla_{\nu} R^{\alpha}{}_{\beta\sigma\mu} = 0$$

Contracting α and μ.

$$\nabla_{\sigma} R^{\mu}{}_{\beta\mu\nu} + \nabla_{\mu} R^{\mu}{}_{\beta\nu\sigma} + \nabla_{\nu} R^{\mu}{}_{\beta\sigma\mu} = 0$$

then contracting with $g^{\sigma\beta}$ gives

$$\nabla_{\sigma} R^{\mu\sigma}{}_{\mu\nu} + \nabla_{\mu} R^{\mu\sigma}{}_{\nu\sigma} - g^{\sigma\beta} \nabla_{\nu} R^{\mu}{}_{\beta\mu\sigma} = 0$$

or

$$\nabla_{\mu} (R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}) = 0.$$

C.6.7 Conformal Tensor, Ricci tensor and Ricci Scalar

C.6.8 The Weyl Tensor

The conformal tensor describes the components of the Riemann tensor, that are not contained in the Ricci tensor. The Ricci tensor being the contraction of the Riemann tensor, the rest of the information of the curvature is contained in the trace free part of the Riemann tensor, called the Weyl tensor,

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{bd} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R. \quad \text{(C.-44)}$$

Constructed to have the same symmetries of the curvature tensor:

$$\begin{aligned} &C_{abcd} = -C_{abdc} = -C_{bacd} = C_{cdab}, \\ &C_{abcd} + C_{adbc} + C_{acdb} \equiv 0. \end{aligned} \quad \text{(C.-44)}$$

$$C^a{}_{bad} \equiv 0 \quad \text{(C.-44)}$$

C.6.9 Index Free Formulism

necessary to exploit coordinate systems. In this approach, tensor quantities are defined in terms of their components and the transformation rules for the latter under coordinate changes.

Physical quantities are (coordinate free) geometric objects scalar fields, vectors (not there components) and so on. The laws of physics are expressible as geometric relationships between these geometric objects.

$$\begin{aligned} X[f] &= X^a \frac{\partial}{\partial x^a} f \\ &= \\ &= X'^a \frac{\partial}{\partial x'^a} f \\ &= X'[f] \end{aligned} \quad \text{(C.-46)}$$

the torsion $T(X,Y) = T^a_{\ bc}X^bY^c$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{C.-46}$$

$R(X,Y)Z = R^a{}_{bcd} X^c Y^d Z^b$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \tag{C.-46}$$

An element $\xi \in T_p\mathbb{R}^n$ is identified with a mapping which takes every smooth function f, defined on any neighbourhood of p, to its directional derivative at p along ξ, denoted $\xi[f]$; that is

$$\xi[f] = \frac{d}{dt} f(p + t\xi)\Big|_{t=0}. \tag{C.-46}$$

Notation for vector and covector coordinates basis

Suppose we have a vector ξ and a coordinate system x^a with basis vectors $\{\mathbf{e_a}\}_x$. Recall that for the vector ξ to remain unchanged under a coordinate transformation the basis vectors $\{\mathbf{e_a}\}_x$ should transform as the components of a covector, this suggests the following alternative notation

$$\{\mathbf{e_a}\}_x \quad \equiv \quad \{\frac{\partial}{\partial \mathbf{x^a}}\} \tag{C.-46}$$

so that we would write

$$\xi = \xi^a \frac{\partial}{\partial \mathbf{x^a}},$$

and under a transformation to the new coordinates x' with new basis vectors we have

$$\xi = \xi^a \frac{\partial}{\partial \mathbf{x^a}} = \xi^b \Big(\frac{\partial x'^a}{\partial x^b} \frac{\partial}{\partial \mathbf{x'^a}} \Big) = \xi'^a \frac{\partial}{\partial \mathbf{x'^a}}$$

and hence

$$\xi'^a = \xi^b \frac{\partial x'^a}{\partial x^b}$$

which corresponds to the definition of the components of a contravariant vector. For covector ω and a coordinate system with basis covectors $\{\mathbf{e^a}\}_x$. For the covector ω to remain unchanged under a coordinate transformation the co-basis vectors $\{\mathbf{e^a}\}_x$ should transform as the components of a vector, this suggests the following alternative notation

$$\{\mathbf{e^a}\}_x \quad \equiv \quad \{\mathbf{dx^a}\} \tag{C.-46}$$

so that we would write

$$\omega = \omega_a \mathbf{dx^a}.$$

$$\big(\mathbf{e^a}, \mathbf{e_b}\big) = \delta^a_b \quad \equiv \quad \big(\mathbf{dx^a}, \frac{\partial}{\partial \mathbf{x^b}}\big) = \delta^a_b \tag{C.-46}$$

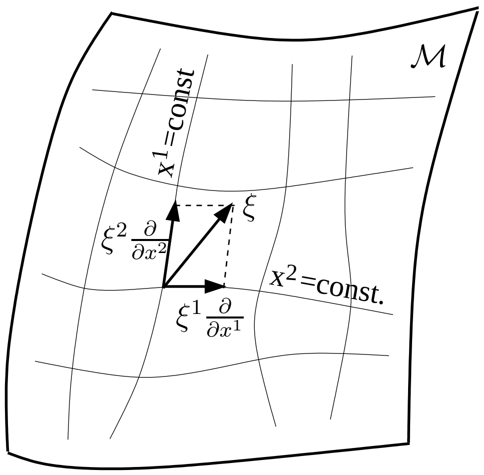


Figure C.18: coordbasevec. The vector ξ may be thought of as being composed of $\xi = \xi^1\partial/\partial\mathbf{x^1} + \xi^2\partial/\partial\mathbf{x^2}$. ξ^1 and ξ^2 are the components of ξ in the $(x^1, x^2)-$coordinate system.

The contraction of the vector and covector would go as

$$\begin{aligned}\big(\omega, \xi\big) &= \omega_a \xi^b \big(\mathbf{dx^a}, \frac{\partial}{\partial \mathbf{x^b}}\big) \\ &= \omega_a \xi^b \delta^a_b \\ &= \omega_a \xi^a\end{aligned} \tag{C.-47}$$

A tensor with raised indices has contravariant components and is therefore expanded in terms of basis vectors, i.e.

$$\mathbf{T} = T^{ab} e_a \otimes e_b$$

A tensor with lowered indices has covariant components and is therefore expanded in terms of basis one-forms, i.e.

$$\mathbf{T} = T^{ab}\mathbf{e_a} \otimes \mathbf{e_b}$$

We have already seen an example of a tensor with lowered indicies, the metric tensor

$$\mathbf{g} = g_{ab}\omega^{\mathbf{a}} \otimes \omega^{\mathbf{b}}.$$

We have notation

$$\mathbf{T}(\omega_a, \omega_b) = T^{ab}$$

In the $x-$coordinates the metric tensor $\mathbf{g}$ would be written as

$$\mathbf{g} = g_{ab}\mathbf{dx^a} \otimes \mathbf{dx^b}.$$

In this notation an arbitrary $(p, q)-$tensor T is expanded in the above notation as

$$\mathbf{T} = T^{a_1 \ldots a_p}_{b_1 \ldots b_q} \frac{\partial}{\partial \mathbf{x^{a_1}}} \otimes \cdots \otimes \frac{\partial}{\partial \mathbf{x^{a_p}}} \otimes \mathbf{dx^{b_1}} \otimes \cdots \otimes \mathbf{dx^{b_q}} \tag{C.-47}$$

C.7 Differential Geometry

C^r-function f if all its representatives $\psi \circ f \circ \phi^{-1} : R^m \to R^n$: (**??**)

In many undergraduate texts, one fixes a covering and coordinate patches and writes any tensor in terms of its value in some coordinate system. This approach is convenient an teaching of elementary GR, but it can obscure the coordinate independent meaning of important concepts. A more preferable formulation of the principle is based on modern differentiable geometry: such a formulation is coordinate free. Physical quantities are (coordinate free) geometric objects scalar fields, vectors (not there components) and so on. The laws of physics are expressible as geometric relationships between these geometric objects.

More advanced texts tend to use a coordinate free formulation, which is what we will present in the next few sections. We will see how the coordinate free approach replaces the 'tensor component' description.

Derive coordinate-free form of equations. These equations will involve objects such as vector fields, one-forms and scalar functions and geometric operations such as . The geometric objects belong to the manifold itself, be it a space-time or phase space or others, rather than any coordinate system on it. We will often use formulation in derivations employing local coordinate systems, but the definition used will hold on every chart, and hence they hold globally, making them chart independent. So we will have shown that they can be written in a consistent coordinate-independent manner.

C.7.1 Tangent Vectors

There are different ways of defining tangent vectors. Smooth manifolds embedded in Euclidean space

We don't want a definition dependent on embedding out space into a larger space. We can define tangent vectors in a way that is defined intrinsically to the manifold. a tangent vector being tangent to a curve in the manifold.

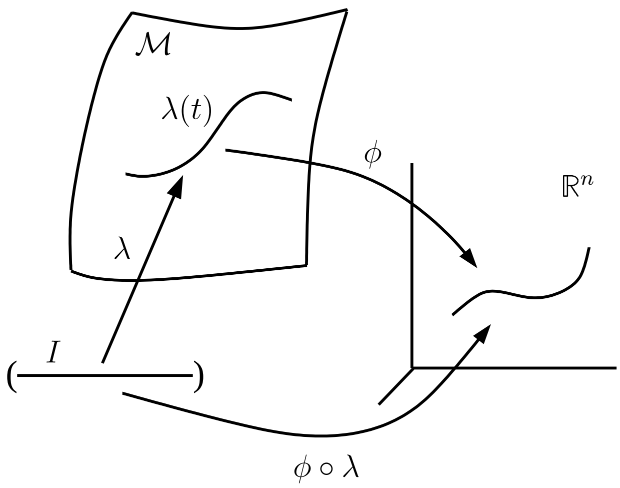


Figure C.19: tangvector. maps the tangent spaces of $\mathcal{M}$ *linearly* into the .

We will generalize the notion of a tangent vector to manifolds in a coordinate free way.

consider a curve in $\mathbb{R}^n$ $\lambda : (0, 1) \to \mathbb{R}^n$. The tangent vector to a point $p = \lambda(t_1)$ is

$$\left(\frac{d\lambda^1}{dt}\bigg|_{t=t_1}, \dots, \frac{d\lambda^n}{dt}\bigg|_{t=t_1} \right) \tag{C.-47}$$

where $\lambda(t) = (\lambda^1(t), \dots, \lambda^n(t)) \in \mathbb{R}^n$.

Basis Vectors

The rate of change of $f(\lambda(t))$ at $t = 0$ along the curve is

$$\frac{d}{dt}f(\lambda(t))\Big|_{t=0} \tag{C.-47}$$

In terms of a coordinate system, this becomes

$$\left[\frac{\partial}{\partial x^a}f(x^a)\right]\frac{dx^a}{dt}(\lambda(t))\Big|_{t=0} \tag{C.-47}$$

In other words, $f(\lambda(t))$ at $t = 0$ is given by applying the differential operator X to f, where

$$\frac{d\lambda(t)}{dt} = X^a\frac{\partial f}{\partial x^a} =: X[f]. \tag{C.-47}$$

Thus are the components of X_p in the basis

$$\frac{\partial}{\partial x^a}\Big|_p. \tag{C.-47}$$

The coordinate transformation

$$y^b = y^b(x^1, \ldots, x^n), \quad b = 1, \ldots, n.$$

If we have a coordinate system in a neighbourhood U of P, then the coordinate basis $\{\frac{\partial}{\partial x^a}\}$.

$$\hat{V} = V^a\frac{\partial}{\partial x^a} = V'^b\hat{e}_b. \tag{C.-47}$$

The numbers $\{V^a\}$ are the components of $\hat{V}$ on $\{\frac{\partial}{\partial x^a}\}$. The numbers $\{V^a\}$ are the components of $\hat{V}$ on $\{\frac{\partial}{\partial y^a}\}$.

$$\frac{\partial}{\partial x^1} = \frac{\partial y^1}{\partial x^1}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^1}\frac{\partial}{\partial y^2} + \cdots + \frac{\partial y^n}{\partial x^1}\frac{\partial}{\partial y^n}$$

and similarly for other x^as.

$$X = X^a\frac{\partial}{\partial x^a} = X'^a\frac{\partial}{\partial y^a} \tag{C.-47}$$

This shows that X^a and X'^a are related by

$$X'^a = X^b \frac{\partial y^a}{\partial x^b}. \tag{C.-47}$$

the components of the vector transform in such a way that the vector itself is left invariant.

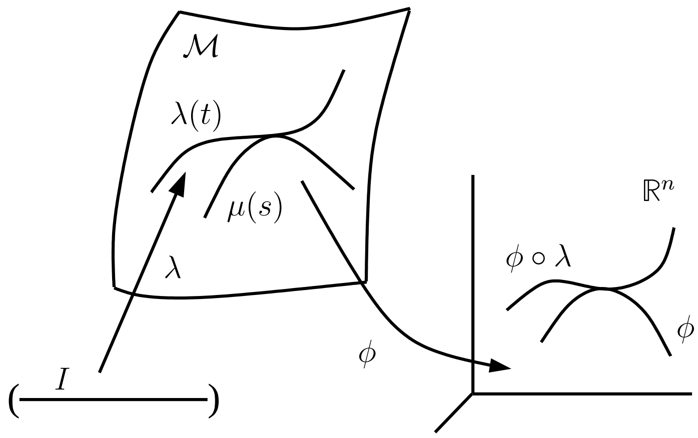


Figure C.20: Two curves $\lambda(t)$ and $\mu(t)$ are tangent at p if and only if their images are tangent at $\phi(p)$ in $\mathbb{R}^n$.

The basis need not be $\{e_a\}$, we can form linear combinations $\hat{e}_i := E^a_i e_a$, where $E = (E^a_i)$ are matrices that . These are referred to as **non-coordinate basis** or a **frame**. Note that they transform as scalars under coordinate transformations. These are very important as frames fields will be basic variables in a formulation of GR that resembles Gauge field theories that are adopted for the quantum theory

C.7.2 Covectors

it is natural to regard $\{dx^a\}$ as a basis of $T^*_p\mathcal{M}$.

$$< dx^b, \partial/\partial_a >= \delta^b_a. \tag{C.-47}$$

An arbitrary covector can be written

$$\omega = \omega_a dx^a \tag{C.-47}$$

where ω_a are the components of ω.

Mixed tensors

The set of type (p, q). A tensor is written in terms of the coordinate basis as

$$T = T^{a_1 \ldots a_p}_{b_1 \ldots b_q} \frac{\partial}{\partial x^{a_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{a_p}} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_q}. \tag{C.-47}$$

The various connections are defined by index structure, i.e. how they transform, and by restrictions placed on them.

C.7.3 Induced Metric and Other Objects on Sub-manifolds

If we denote the normal to the surface as n^a ($n^a n_a = 1$) then the induced metric can be written as

$$h_{ab} = g_{ab} - n_a n_b \tag{C.-47}$$

So that h_{ab} projects out the components of a vector normal to the hypersurface. Say $\xi^b = Cn^b$

$$h^a{}_b \xi^b = \delta^a{}_b \xi^b - n^a n_b \xi^b = 0. \tag{C.-47}$$

Let $\mathcal{N}$ be a $n-$dimensional manifold of an $m-$dimensional manifold . The hypersurface on which we have coordinates y^α. If $\mathcal{N}$ is a hypersurface in $\mathcal{M}$, for the point labelled by y^α corresponds to the point of labelled by $\mathcal{M}$. Thus the hypersurface is described by the equations

$$x^a(y^\alpha) \tag{C.-47}$$

tangent vector e^α_a, and induced metric $h_{\alpha\beta}$

$$e^\alpha_a = \frac{\partial y^\alpha}{\partial x^a} \tag{C.-47}$$

ds^2 for an infinitesimal curve lying in the hypersurface:

$$ds^2 = h_{\alpha\beta} dy^\alpha dy^\beta = h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^a} \frac{\partial y^\beta}{\partial x^b} dx^a dx^b \tag{C.-47}$$

$$g_{ab} = h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^a} \frac{\partial y^\beta}{\partial x^b} \tag{C.-47}$$

or

$$g_{\mathcal{M}ab}(x) = g_{\mathcal{N}\alpha\beta}(f(x))\,\frac{\partial f^\alpha}{\partial x^a}\frac{\partial f^\beta}{\partial x^b} \tag{C.-47}$$

where f^α denote the coordinates of $f(x)$.

Definition $g_{\mathcal{N}\alpha\beta}(f(x))$ is said to be the pull-back of $g_{\mathcal{M}ab}(x)$, denoted $f_* g$

$$\begin{aligned} g_{ab} dx^a \otimes dx^b &= \delta_{\alpha\beta}\,\frac{\partial f^\alpha}{\partial x^a}\frac{\partial f^\beta}{\partial x^b}\,dx^a \otimes dx^b \\ &= d\theta \otimes \theta + \sin^2\theta\; d\phi \otimes \phi. \end{aligned} \tag{C.-47}$$

where

$$h_{\alpha\beta} = g_{ab} e^a_\alpha e^b_\beta \tag{C.-47}$$

is the induced metric or the first fundamental form of the hypersurface. It is a scalar with respect to coordinate transformations $x^a \to x'^a$ on $\mathcal{M}$. behaves like a tensor under coordinate transformations of the manifold $\mathcal{N}$.

C.8 Active Diffeomorphisms and the Lie Derivative

Up until now we have only considered coordinate transformation, that is, passive diffeomorphisms. We now move onto active diffeomorphisms. As their formulas look very alike the two are easily mixed up. But as we have seen in chapter 1 they are quite different, active diffeomorphisms relate *distinct* spacetime geometries, whereas a coordinate transformation merely represents the same spacetime geometry in a different coordinate system.

In chapter 1 we defined an active diffeomorphism as simultaneously dragging the metric and matter fields over the spacetime manifold while keeping the coordinate lines 'attached' (fig C.8). This is called a **pushforward**.

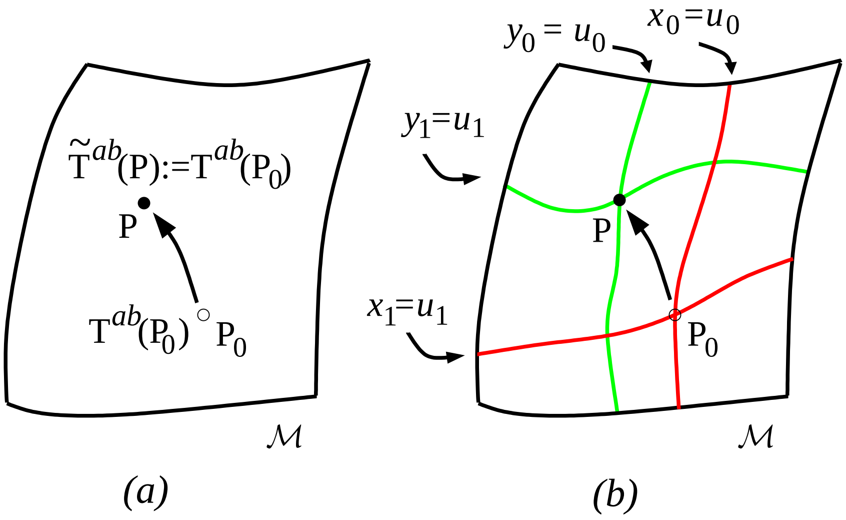


Figure C.21: activeDiffGeom. A pushforward of the tensor $T_{ab}(x)$, i.e. $T_{ab}(x) \to \tilde{T}_{ab}(y)$.

Let us slightly modify the definition of an active diffeomorphism by requiring that after we have dragged the fields across the manifold we perform a coordinate transformation back to the original coordinates. An active diffeomorphism defined this way then relates different space-time geometries and matter field configurations in the same coordinate system.

They relate $g_{ab}(x)$ to $\tilde{g}_{ab}(h(x))$ by the Jacobian matrix of the coordinate transformation $x \mapsto h(x)$,

$$\tilde{g}_{ab}(h(x)) = \Lambda_c^a \Lambda_d^b g_{ab}(x) \tag{C.-47}$$

Two metrics related by an active diffeomorphism, viewed in the same coordinate system, compared at the same point also have 'transformation matrices', however, these have a different geometric interpretation!

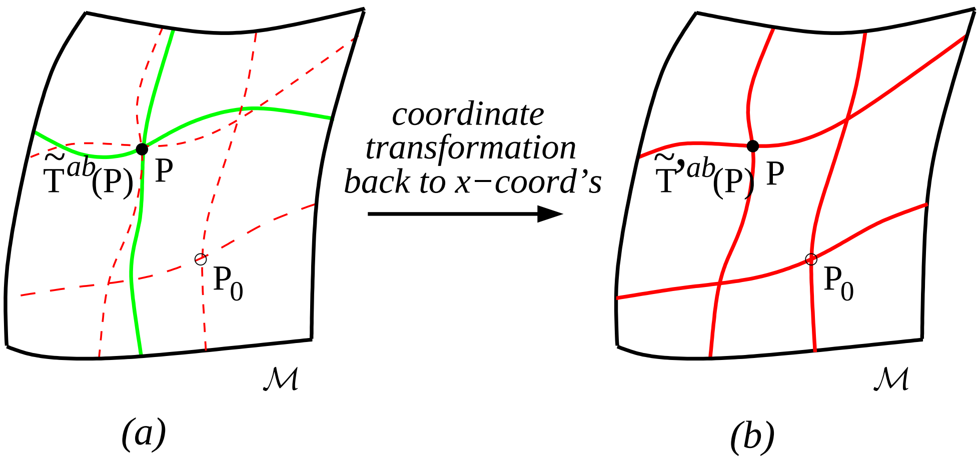


Figure C.22: activeDiffGeom1. The red dashed lines in (a) are the $x-$coordinate lines of the point P. We perform a coordinate transformation back to the original coordinate system. The pushed-forward tensor $\tilde{T}_{ab}(y)$ transforms to $\tilde{T}'_{ab}(x)$, i.e. $\tilde{T}_{ab}(y) \to \tilde{T}'_{ab}(x)$.

$$\tilde{g}_{ab}(x) = \frac{\partial h^c(x)}{\partial x^a}\frac{\partial h^d(x)}{\partial x^b} g_{cd}(h(x)) \tag{C.-47}$$

The fact that the coordinate values do not change, while the tensor fields do, distinguishes the active diffeomorphism from a simple coordinate transformation.

Passive diffeomorphism invariance refers to invariance under change of coordinates, i.e. the same object represented in different coordinate systems. Choose a (local) coordinate system for S in which the metirc $g_{ab}(x)$. (If the map h sends each point to the same point of the manifold $\mathcal{M}$, then in the second system S' the metric given by $\tilde{g}_{ab}(h(x))$, $f(x)$ being the coordinates on $\mathcal{M}$ of the second system.)

Any theory can be made invariant under passive diffeomorphisms because a dynamical system doesn't care which coordinate system you use to describe it. However, general relativity is the only theory invariant under active diffeomorphisms and this invariance is a property of the dynamical theory itself.

Maths Tools for Manifold Without a Metric

In the previous section we reviewed metrics on manifolds, these are important in classical general relativity and are what a physicists is most likely to be familiar with. As we have learned, in reality it is only **geometry up to active diffeomorphisms** that has physical meaning. As we have empathized, in formulating the quantum theory we prefer not to employ metrics with its direct relation to the notion of distance.

There is a rich geometric structure of the manifold without a metric defined on it. Important tools of the Lie derivative and differential forms which have nothing to do with metrics. These will be important in the quantum theory where we will avoid introducing a background metric whenever possible.

C.8.1 Mapping a Manifold to Itself Along Integral Curves

We start by considering a congruence of curves defined such that only one curve goes through each point in the manifold. Then, given any one curve of the congruence,

$$x^\mu = x^\mu(u), \tag{C.-47}$$

we can use it to define the tangent vector field dx^μ/du along the curve. If we do this for every curve in the congruence, then we end up with a vector field X^μ (given by dx^μ/du at every point) defined over the whole manifold, then this can be used to define a congruence of curves in the manifold called the orbits or trajectories of X^μ.

a smooth, non-intersecting family of curves on a manifold then the tangent vectors at each point can be taken together to form a vector field on the manifold.

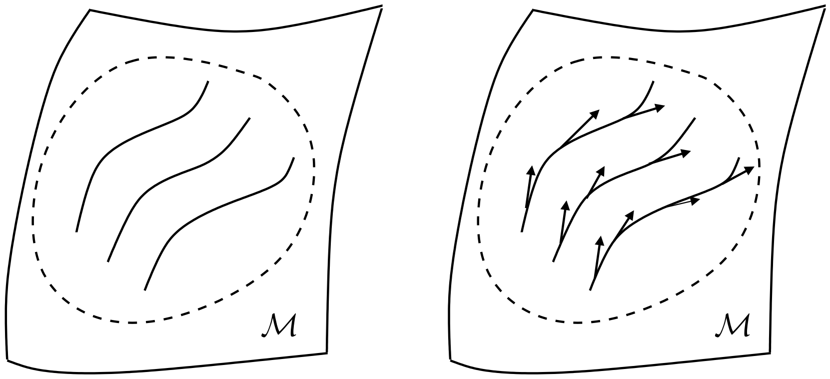


Figure C.23: The tangent vector field resulting from a congruence of curves.

These curves are obtained by solving differential equations

$$\frac{dx^\mu}{du} = X^\mu(x(u)) \tag{C.-47}$$

Let x^i be a local coordinate system and let x^i_p be the coordinates of p. The equation of the integral curve is

$$\frac{d}{dt}x^i(t) = X^i(x^m(t)),$$

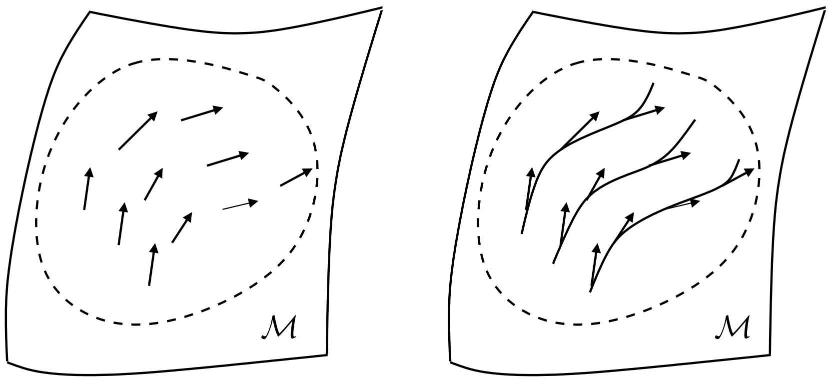


Figure C.24: The local congruence of curves resulting from vector field.

with initial conditions $x^i(0) = x^i_p$. Provided X is smooth the theory of ordinary differential equations guarantees the existence and uniqueness, (at least locally, i.e., for small t), of a solution. Uniqueness implies that no two curves in the congruence intersect (at least locally).

Definition A congruence of curves is a family of curves such that precisely one curve of the family passes through each point. It is a geodesic congruence if the curves are geodesics.

Active Diffeormorphisms

C.8.2 The Lie Derivative

This is called an *active transformation.* The *passive transformation* is a coordinate transformation.

A contravariant vector flow determines a local congruence of curves,

$$x^a = x^a(u),$$

where the tangent vector field to the congruence is

$$\frac{dx^a}{du} = X^a.$$

at least locally, a vector field generates a unique integral flow about any given point p. We use this flow to take a tensor to a nearby point and hence form a derivative. This derivative is called the Lie derivative.

$$\sigma^b(\epsilon, p) = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.-47}$$

The Lie derivative of a scalar field $f \in \mathcal{C}^\infty(\mathcal{M})$. Let X be a vector field on $\mathcal{M}$ we define the Lie derivative of f along X to be

$$\mathcal{L}_X f(p) = \lim_{\epsilon \to 0} \frac{f(\sigma(\epsilon, p)) - f(p)}{\epsilon} \tag{C.-47}$$

which is the usual directional derivative along X.

point transform

$$x'^b = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.-47}$$

We generate a new vector (with vector components in the x' coordinates). By definition its components are related to $T^a(x)$ by a pushforward

$$\tilde{T}'^a(x') := T^a\left(x^c + \epsilon X^c(x)\right) = T^a(x) + \epsilon X^c(x)\partial_c T^{ab}(x) + \mathcal{O}(\epsilon^2). \tag{C.-47}$$

We now wish to transform this tensor to the $x-$coordinates so we can compare it with the original tensor $T^{ab}(x)$. Using (C.8.2) we have

$$\frac{\partial x^a}{\partial x'^c} = \delta^a_c - \epsilon \partial_c X^a + \mathcal{O}(\epsilon^2) \tag{C.-47}$$

The parameter distance derivative of an object along the vector field is the *Lie derivative.*

$$\begin{aligned}
\tilde{T}^a(x) &= \frac{\partial x^a}{\partial x'^c}\tilde{T}'^c(x') \\
&= (\delta^a_c - \epsilon\partial_c X^a)(T^c(x) + \epsilon X^e \partial_e T^c) + \mathcal{O}(\epsilon^2) \\
&= T^a(x) + [X^e\partial_e T^a - \partial_c X^a T^c(x)]\epsilon + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{C.-48}$$

$$\mathcal{L}_X T^a = \lim_{\epsilon \to 0} \frac{\tilde{T}^a(x) - T^a(x)}{\epsilon} \tag{C.-48}$$

$$\mathcal{L}_X T_a(x) = X^c \partial_c T_a + T_b \partial_c X^a \tag{C.-48}$$

What is the Lie derivative for a tensor $T^{ab}(x)$? We generate a new tensor (with tensor components in the x' coordinates). By definition its components are related to $T^{ab}(x)$ by a pushforward

$$\tilde{T}'^{ab}(x') := T^{ab}\left(x^c + \epsilon X^c(x)\right) = T^{ab}(x) + \epsilon X^c(x)\partial_c T^{ab}(x) + \mathcal{O}(\epsilon^2). \tag{C.-48}$$

We now wish to transform this tensor to the $x-$coordinates so we can compare it with the original tensor $T^{ab}(x)$. Using (C.8.2) again. The parameter distance derivative of an object along the vector field is the *Lie derivative*.

$$\begin{aligned}\tilde{T}^{ab}(x) &= \frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}\tilde{T}'^{cd}(x') \\ &= (\delta^a_c - \epsilon\partial_c X^a)(\delta^b_d - \epsilon\partial_d X^b)(T^{cd}(x) + \epsilon X^e\partial_e T^{cd}) + \mathcal{O}(\epsilon^2) \\ &= T^{ab}(x) + [X^e\partial_e T^{ab} - \partial_c X^a T^{cb}(x) - \partial_d X^b T^{ad}(x)]\epsilon + \mathcal{O}(\epsilon^2)\end{aligned} \tag{C.-49}$$

$$\mathcal{L}_X T^{ab} = \lim_{\epsilon\to 0}\frac{\tilde{T}^{ab}(x) - T^{ab}(x)}{\epsilon} \tag{C.-49}$$

$$\mathcal{L}_X T^{ab} = X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a. \tag{C.-49}$$

$$\mathcal{L}_X T_a(x) = X^c\partial_c T_a + T_b\partial_c X^a, \qquad \mathcal{L}_X T_{ab}(x) = X^c\partial_c T_{ab} + T_{cb}\partial_a X^c + T_{ac}\partial_b X^c$$

The first term of the Lie derivative, $X^c\partial_c$, corresponds to the pushforward, shifting the tensor to another point in the manifold. The remaining terms arise from the coordinate transformation back to the original coordinates. Is it coordinate invariant? Does it have the same form in all coordinate systems? In fact (C.8.2) is equivalent to:

$$X^c\nabla_c T^a - T^c\nabla_c X^a \tag{C.-49}$$

since

$$\begin{aligned}\mathcal{L}_X T^a(x) &= X^c\partial_c T^a - T^c\partial_c X^a \\ &= X^c(\partial_c T^a + \Gamma^a_{dc}T^d) - T^c(\partial_c X^a + \Gamma^a_{dc}X^d) \\ &= X^c\nabla_c T^a - T^c\nabla_c X^a\end{aligned} \tag{C.-50}$$

where we have used that the connection is symmetric in its lower indices. Similarly, (C.8.2) is equivalent to:

$$
\begin{aligned}
& X^c\nabla_c T^{ab} - T^{ac}\nabla_c X^b - T^{cb}\nabla_c X^a \\
= \; & X^c(\partial_c T^{ab} + \Gamma^a_{dc}T^{db} + \Gamma^b_{dc}T^{ad}) - T^{ac}(\partial_c X^b + \Gamma^b_{dc}X^d) - T^{cb}(\partial_c X^a + \Gamma^a_{dc}X^d) \\
= \; & X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a + T^{db}X^c(\Gamma^a_{dc} - \Gamma^a_{cd}) + T^{ad}X^c(\Gamma^b_{dc} - \Gamma^b_{cd}) \\
= \; & X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a \qquad \text{(C.-52)}
\end{aligned}
$$

In general, the partial derivatives appearing in Lie derivatives can be replaced by covariant derivatives. Hence, the combination of pushback and coordinate transformation make the Lie derivative a tensor in the tangent space at x^a.

$$\tilde{T}'(q) = T(h_\epsilon(p)) \qquad \tilde{T}(p) = h_{\epsilon *}[T(h_\epsilon(p))] \qquad \text{(C.-52)}$$

We can write down the coordinate free equation

$$(\mathcal{L}_X T)(p) = \frac{h_{\epsilon *}[T(h_\epsilon(p))] - T(p)}{\epsilon} \qquad \text{(C.-52)}$$

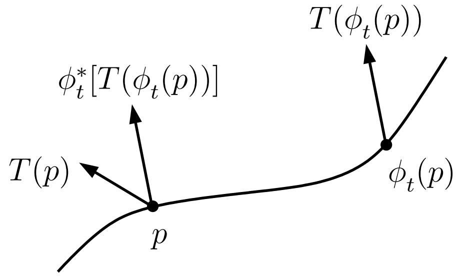


Figure C.25: .

the original tensor components at a different point. distinguishes the Lie derivative from the directional derivative.

the curve passing through P is given by x^1 varying, with x^2, x^3, x^4 all constant along the curve, and such that

$$X^\alpha \stackrel{*}{=} \delta^\alpha_1 = (1, 0, 0, 0) \qquad \text{(C.-52)}$$

along this curve. The notation used in means that the equation holds only in a particular coordinate system. Then it follows that

$$X = X^\alpha \partial_\alpha = \partial_1, \qquad \text{(C.-52)}$$

and equation reduces to

$$L_X T_{\alpha\beta} \stackrel{*}{=} \partial_1 T_{\alpha\beta} \tag{C.-52}$$

Thus, in this special coordinate system, Lie differentiation reduces to ordinary differentiation.

If we have a map ϕ from a manifold $\mathcal{M}$ to another manifold $\mathcal{N}$, and we choose a point $x \in \mathcal{M}$, we can *push forward* a vector from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$, by a head-to-head and tail-to-tail map. If the vector has components X^μ and the map takes the point with coordinates x^μ to one with coordinates $\xi(x)$, the vector $\phi_* X$ has components

$$(\phi_* X)^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} X^\nu. \tag{C.-52}$$

This looks like the transformation formula for contravariant vector components under a coordinate transformation, but we are doing an active transformation, changing a vector into a different one.

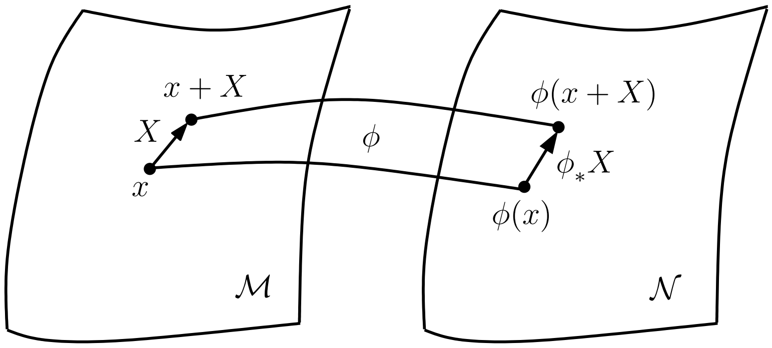


Figure C.26: pullbackDef0. Pushing forward a vector X from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$.

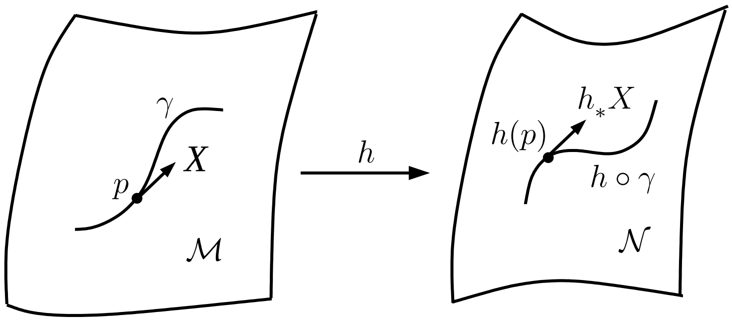


Figure C.27: The **push-forward map** h_* that maps the tangent spaces of $\mathcal{M}$ *linearly* into the tangent spaces of $\mathcal{N}$.

pushforward ϕ_* (**??**)

Recall that a one-form maps a vector to a number. Given a one-form ω on $\mathcal{N}$, we define $\phi^*\omega$ as a one-form on $\mathcal{M}$ by specifying what we get when we plug the vector X at $x \in \mathcal{M}$ into it. This we do by pushing the X forward to $T\mathcal{N}_{\phi(x)}$, plugging it into ω, and declaring the result to be the evaluation of $\phi^*\omega$ on the X. Symbolically

$$[\phi^*\omega](X) = \omega(\phi_* X). \tag{C.-52}$$

or in components

$$[\phi^*\omega]_a X^a = \omega_a [\phi_* X]^a. \tag{C.-52}$$

We work a coordinate system $(x^1, \ldots, x^m)$, such that x^1 is the parameter along the integral curves and the other coordinates are choosen any way. In this coordinate system and the components of the tensor pulled back from $\phi_t(p)$ to p are simply

$$\phi_{t\star}[T^{a_1 \ldots a_k}_{b_1 \ldots b_l}(\phi_t(p))] \stackrel{*}{=} T^{a_1 \ldots a_k}_{b_1 \ldots b_l}(x^1 + t, x^2, \ldots, x^n). \tag{C.-52}$$

In this coordinate system the Lie derivative becomes

$$\mathcal{L}_V T^{a_1 \ldots a_k}_{b_1 \ldots b_l} \stackrel{*}{=} \frac{\partial}{\partial x^1} T^{a_1 \ldots a_k}_{b_1 \ldots b_l}, \tag{C.-52}$$

Coordinate-Free Description

We will prove

$$\mathcal{L}_X Y = [X, Y]. \tag{C.-52}$$

$$\sigma^b(\epsilon, p) = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.-52}$$

for any f

$$
\begin{aligned}
Y_{\sigma(\epsilon)} f &= \sum_a Y^b(\sigma(\epsilon)) \frac{\partial}{\partial x^a}\Big|_{\sigma(\epsilon)} f \\
&= \sum_a (Y^b + \epsilon X^c \partial_c Y^a)\Big|_{\sigma(\epsilon)} f + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b + \epsilon X^c \partial_c Y^a)(\partial f + \epsilon X^c \partial_a \partial_c f) + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b \partial_a + \epsilon X^c \partial_c Y^a \partial_a + \epsilon Y^a X^c \partial_c \partial_a) f + \mathcal{O}(\epsilon^2) \qquad \text{(C.-54)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sigma(-\epsilon)_* Y_{\sigma(\epsilon)} f &= Y_{\sigma(\epsilon)}(f \circ \sigma(-\epsilon)_*) \\
&= \sum_a Y^a \, (h \circ \sigma(\epsilon)) \frac{\partial}{\partial x^a}\Big|_p (h \circ \sigma(-\epsilon)) + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b \partial_a + \epsilon X^c \partial_c Y^a \partial_a + \epsilon Y^a X^c \partial_c \partial_a)(f - \epsilon \partial_b f X^b) + \mathcal{O}(\epsilon^2) \\
&= \sum_a Y^b \partial_a f + \epsilon(\partial_c Y^a X^c - Y^c \partial_c X^a) \partial_a f + \mathcal{O}(\epsilon^2) \qquad \text{(C.-56)}
\end{aligned}
$$

From which

$$
\begin{aligned}
\frac{\sigma(-\epsilon)_* Y_{\sigma(\epsilon)}(f) - Y(f)}{\epsilon} &= (\partial_c Y^a X^c - Y^c \partial_c X^a) \partial_a f + \mathcal{O}(\epsilon^2) \\
&= [X, Y]^a \partial_a f + \mathcal{O}(\epsilon^2) \\
&= [X, Y]^a (f) + \mathcal{O}(\epsilon^2) \qquad \text{(C.-57)}
\end{aligned}
$$

The Lie derivative of a covariant tensors

$$
L_X Y^\alpha = X^\beta \partial_\beta Y^\alpha - Y^\beta \partial_\beta X^\alpha \qquad \text{(C.-57)}
$$

The Lie derivative of a covariant vector field Y_α is given by

$$
L_X Y_\alpha = X^\beta \partial_\beta Y_\alpha + Y^\beta \partial_\alpha X^\beta \qquad \text{(C.-57)}
$$

The Lie Derivative

there is a coordinate system in which

$$\mathcal{L}_{\vec{N}}\vec{M} = N^a \partial_a M_b - M^a \partial_a N_b \tag{C.-57}$$

It satisfies the Leibniz rule

$$L_X(Y^a Z_{bc}) = Y^a (L_X Z_{bc}) + (L_X Y^a) Z_{bc}. \tag{C.-57}$$

It is type-preserving; that is, the Lie derivative of s tensor of type(p,q) is again a tensor of type (p,q).

The Lie derivative of a scalar field ϕ is simply an ordinary derivative in the direction of X

$$L_X \phi = X\phi = X^a \partial_a \phi \tag{C.-57}$$

Now, given the Lie derivative of a vector and a scalar, we can apply the Leibniz rule to deduce the Lie derivative of a covariant vector field Y_a: consider the Lie derivative of the scalar formed by the contraction of an arbitrary vector Z^a with an arbitrary covector Y_a.

$$L_X(Y_c Z^c) = X^b \partial_b (Y_c Z^c) = Z^c X^b \partial_b Y_c + Y_c X^b \partial_b Z^c \tag{C.-57}$$

whereas the Leibniz rule gives

$$L_X(Y_c Z^c) = Y_c L_X Z^c + (L_X Y_c) Z^c. \tag{C.-57}$$

$$Z^c L_X Y_c = Z^c X^b \partial_b Y_c + Z^c Y_b \partial_a X^c. \tag{C.-57}$$

but as Z^c is arbitrary this means

$$L_X Y_a == X^b \partial_b Y_a + Y_b \partial_a X^b. \tag{C.-57}$$

$$\bar{T} := \phi_{\Delta\lambda} T(p_0) \quad \overline{x}^a(P) = x^a(P_0) \tag{C.-57}$$

$$\mathcal{L}_\xi T := \lim_{\Delta\lambda} \frac{\phi_{\Delta\lambda} T(x) - T(x)}{\Delta\lambda} \tag{C.-57}$$

$$\mathcal{L}_X T^{a\dots}_{b\dots} = X^c \partial_c T^{a\dots}_{b\dots} - T^{c\dots}_{b\dots} \partial_c X^a - \cdots + T^{a\dots}_{c\dots} \partial_b X^c + \cdots \tag{C.-57}$$

C.8.3 Pull-back and Lie Derivative of a co-vector

The pullback of the function f by ϕ, denoted $\phi^* f$, is defined by

$$\phi^* f = (f \circ \phi). \tag{C.-57}$$

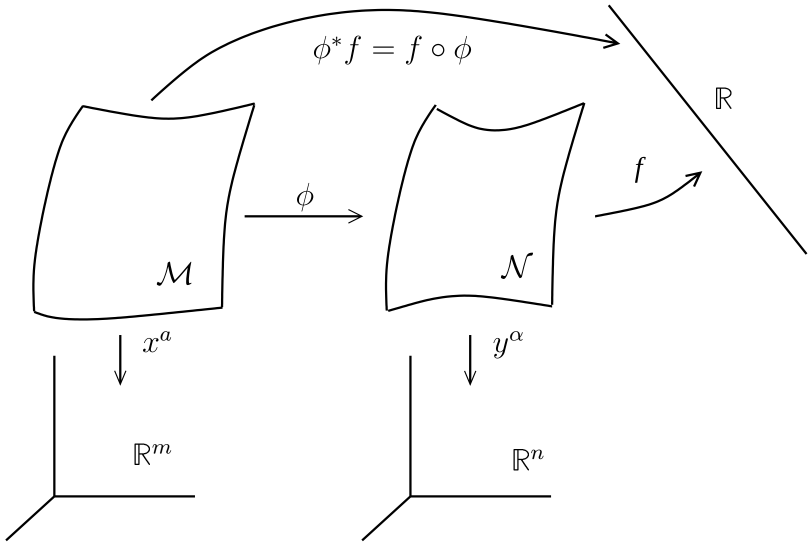


Figure C.28: The **pullback map** ϕ^* of a function f from $\mathcal{N}$ to $\mathcal{M}$ by a map $\phi : \mathcal{M} \to \mathcal{N}$ is the composition of ϕ with f.

Suppose we have a smooth map $h : \mathcal{M} \to \mathcal{N}$ as in section C.7.1. We saw that we could push-forward a vector $X_p \in T_p\mathcal{M}$ to a vector $h_* X_{h(p)} \in T_{h(p)}\mathcal{N}$ by

$$h_* X_{h(p)}(g) = X_p(g \circ h) \tag{C.-57}$$

There should be a dual map which maps co-vectors in $\mathcal{N}$ to co-vectors in $\mathcal{M}$.

$$h_p^* \omega(X_p) =< h^* \omega, X_p > = < \omega, h_* X_{h(p)} > = \omega(h_* X_{h(p)}) \tag{C.-57}$$

Let $h : \mathcal{M} \to \mathcal{N}$, $(y^1, \dots y^m)$ be local coordinates on $V \subset \mathcal{N}$ and $(x^1, \dots x^n)$ be local coordinates on $U \cap h^{-1}(V) \subset \mathcal{M}$. If

$$\omega = \sum_{b=1}^{n} \omega_b dy^b \Big|_p \tag{C.-57}$$

We have

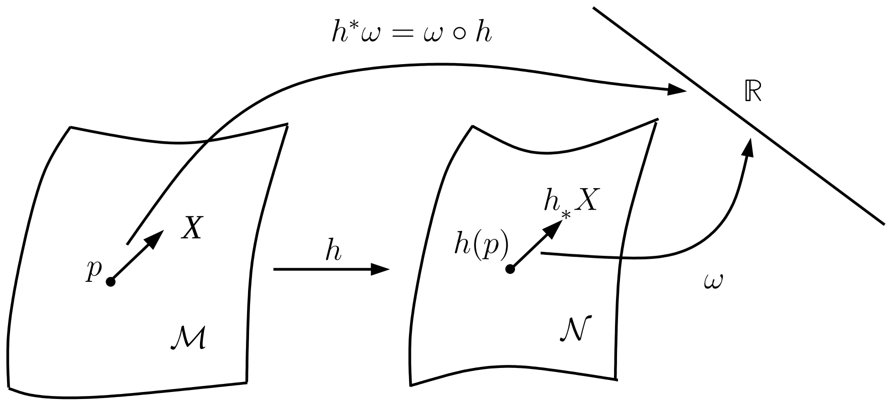


Figure C.29: pushLie. The maps the co-tangent spaces of $\mathcal{M}$ *linearly* into the co-tangent spaces of $\mathcal{N}$.

$$< \omega, h_* X_{f(p)} > \quad = \quad \sum_{b=1}^{n} \omega_b (f^* \omega X_{f(p)})^b \tag{C.-56}$$

C.8.4 More on Lie Derivative

Definition A one-parameter group of diffeomorphisms. A one-parameter family of maps $\{\phi_t\}_{t \in \mathbb{R}}$ is said to be a one parameter group of diffeommorphism if:

(i) Each $\phi_t : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism;

(ii) $\phi_0 = \mathrm{id}$;

(iii) $\phi_{s+t} = \phi_s \circ \phi_t$ for all $s, t \in \mathbb{R}$.

That is we have a group action of $\mathbb{R}$ on $\mathcal{M}$.

□

Lemma C.8.1 *Let φ_t be the one parameter group of diffeomorphisms generated by the complete vector field X on the manifold $\mathcal{M}$, and $\psi : \mathcal{M} \to \mathcal{M}$ is a diffemorphism on $\mathcal{M}$. Then $\psi \circ \varphi_t \circ \psi^{-1}$ is the one-parameter group of diffeomorphisms generated by $\psi_* X$.*

Proof: We show that the tangent to the curve $\phi_p(t) := (\psi \circ \varphi_t \circ \psi^{-1})(p)$

$$
\begin{aligned}
(\psi_* X)_p f \quad &:= \quad X_q(f \circ \psi) = \frac{d}{dt} f \circ \psi(\varphi_t(q)) \Big|_{t=0} \\
&= \quad \frac{d}{dt} f\big(\psi \circ \varphi_t \circ \psi^{-1}(\psi(q))\big) \Big|_{t=0} \\
&= \quad \frac{d}{dt} f\big(\psi \circ \varphi_t \circ \psi^{-1}(p)\big) \Big|_{t=0} \\
&= \quad \frac{d}{dt} f\big(\phi_p(t)\big) \Big|_{t=0}
\end{aligned} \tag{C.-58}
$$

□

Corollary C.8.2 *A complete tangent vector field is invariant under a diffeomorphism* $\psi : \mathcal{M} \to \mathcal{M}$ *if and only if the one-parameter group of diffeomorphisms* φ_t *generated by* X *commutes with* ψ*.*

Proof:

□

C.8.5 Isometries and Killing Vector Fields

when the metric is the same. If you move along the direction of a Killing vector field, the metric diesn't change.

A space time possess a symmetry if there exists a coordinate system such that the components of the components $g_{ab}(x)$ of the metric are independent of at least one or more of the coordinates. Then the metric has a symmetry under translations by this coordinate holding the remaining coordinates fixed.

see M. Gockeler, T. Schucker, *Differential geometry, gauge theories, and gravity*

in general these symmetries go when we go to curved space-time with fixed metric and cannot exist in general relativity where the metric becomes a dynamical variable.

Then a transformation leaving $g_{ab}(x)$ invariant is called an **isometry**.

The Lie derivative is natural to express the invariance of a tensor under a change of position. The vector ξ^a that generates the symmetry is called a Killing vector. In the original coordinates the components of the Killing vector are simply $\xi^a = \delta^a_b$. A coordinate covariant characterization of a

Killing vector is

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g'_{ab}(x') \tag{C.-58}$$

will be an isometry if

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{ab}(x') \tag{C.-58}$$

There is a coordinate system in which $g_{ab}(x)$ is the same function that $g'_{ab}(x')$, where $x'^a = x^a + \epsilon K^a$

For example

$dl^2 = dr^2 + r^2\, d\theta^2$

$g_{ab}(r, \theta) = D(1, r^2)$ and $g_{ab}(1, \theta') = D(1, r'^2)$ where $\theta' = \theta + \epsilon K_\alpha$

and give curves along which the geometrical environment is unchanged.

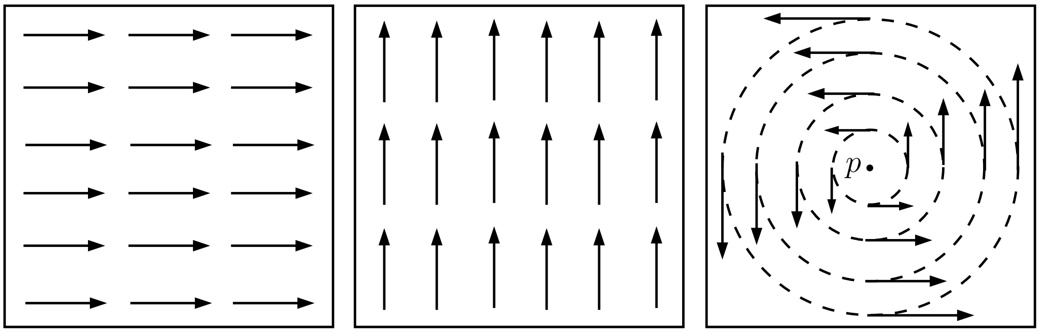


Figure C.30: The Killing vector field resulting from a congruence of curves.

Fig.(C.8.5). The first two are the "infinitesimal generators" of horizontal and vertical translations. The third is the generator of counter-clockwise rotations centered at p.

The three dimensional rotation group $O(3)$ is the isometry group for of the ordinary round sphere S^2.

$$\frac{\partial x^a}{\partial x^c}\frac{\partial x^b}{\partial x^d} \tag{C.-58}$$

$$\frac{\partial x'^a}{\partial x^b} = d^\alpha_b + \epsilon \partial_b K^a \tag{C.-58}$$

$$g'_{ab}(x') = g_{ab}(x) \text{ when } x' = xK_a. \tag{C.-58}$$

$$g'_{ab}(x^a + \epsilon K^a) = g_{ab}(x) \tag{C.-58}$$

$$g'_{\alpha\beta}(x') = (\delta^\gamma_\alpha + \partial_\alpha K^\gamma)(\delta^\delta_\beta + \partial_\beta K^\delta) g_{\gamma\delta}(x^\sigma + \epsilon K^\sigma) \tag{C.-58}$$

$$\begin{aligned} &= (\delta^\gamma_\alpha + \partial_\alpha K^\gamma)(\delta^\delta_\beta + \partial_\beta K^\delta)[g_{\gamma\delta}(x^\sigma) + \epsilon K^\sigma \partial_\sigma g_{\gamma\delta} + \dots] \\ &= g_{\alpha\beta}(x) + \epsilon[g_{\alpha\delta}\partial_\beta K^\delta + g_{\beta\delta} + K^\sigma \partial_\sigma g_{\alpha\beta}] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{C.-58}$$

$$\mathcal{L}_X = X^\sigma \partial_\sigma g_{\alpha\beta} + g_{\alpha\delta}\partial_\beta X^\delta + g_{\beta\delta}\partial_\alpha X^\delta. \tag{C.-58}$$

$$\mathcal{L}_K g_{\alpha\beta}(x) = 0 \tag{C.-58}$$

$$\mathcal{L}_K g_{\alpha\beta}(x) = \nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0, \qquad \nabla_{(\beta} K_{\alpha)} = 0 \tag{C.-58}$$

An isometry is generated by a Killing vector field K^α satisfying $\mathcal{L}_K g_{\alpha\beta}(x) = 0$

C.8.6 Conserved Quantities

$\nabla_{(a} k_{b)} = 0$

Consider a freely falling particle whose worldline has tangent vector X. Define the quantity $E = X^a k_a$, where k is a Killing vector. Then

$$\begin{aligned} X^a \nabla_a (X^c k_c) &= X^a X^b \nabla_a k_b + k_b \underbrace{X^a \nabla_a X^b}_{=0} \\ &= X^a X^b \nabla_a k_b = 0 \\ &= X^a X^b \nabla_{(a} k_{b)} = 0 \end{aligned} \tag{C.-59}$$

Thus E is conserved along the worldline of X. Given the energy-momentum tensor of a continuous distribution of matter, satisfying $\nabla_b T^{ac} = 0$. Define $J^a := T^{ac} k_c$. Then

$$
\begin{aligned}
\nabla_c(T^{bc}k_b) &= k_b \underbrace{\nabla_c T^{bc}}_{=0} + T^{bc}\nabla_c k_b \\
&= T^{bc}\nabla_c k_b \\
&= T^{bc}\nabla_{(c}k_{b)} = 0
\end{aligned}
\tag{C.-60}
$$

Thus $\nabla_a J^a = 0$, i.e., the current is conserved.

Notion of energy and angular momentum have played a key role in analyzing behaviour of physical theories. For theories of fields on a fixed, background spacetime, a locally conserved stress-energy tensor, T_{ab}, normally can be defined. If the background spaceime has a Killing field, k^a, then $J^a = T^a{}_b k^b$ is a locally conserved current. If Σ is a Cauchy surface, then $q = \int_\Sigma J^a d\Sigma_a$ defines a conserved quantity associated with k^a; if Σ is a timelike or null surface, then $\int_\Sigma J^a d\Sigma_a$ has the interpretation of the flux of this quantity through Σ.

However, in diffeomorphism covariant theories such as general relativity, there is no notion of the local stress-energy tensor of the gravitational field, so conserved quantities () cannot and their fluxes cannot be defined by the above procedures, even when Killing fields are present.

C.8.7 Adapted Coordinates

These symmetries are removed by active diffeomorphisms, symmetries help us find simple solutions to Einstein's equation but strictly it is only those properties shared by all the spacetimes in the symmetry class that have physical meaning.

However, if we are neglecting the dynamics of gravity and only consered with dynamical theories over curved spacetime, then the particulary simple spacetime in the equivalence class can be used to get your physical properties.

Spherically symmetric spacetimes

$$ds^2 = g_{00}dt^2 + 2g_{oi}dtdx^i + dr^2 + r^2\sin^2\theta d\theta^2 + r^2 d\phi^2. \tag{C.-60}$$

Axisually symmetric spacetimes

$$ds^2 = dr^2 + r^2\sin^2\theta d\theta^2 + r^2 d\phi^2. \tag{C.-60}$$

A gravitational field is said to be sationary when a reference frame exists in which all the components g are independent of the time coordinate g_{ab}. This coordinate, by the way, is usually referred to as coordinate time. Sationary spacetimes.

$$ds^2 = g_{00}dt^2 + 2g_{oi}dtdx^i + g_{ij}dx^i dx^j. \tag{C.-60}$$

Static spacetimes we when changing $dt \to -dt$ ds should remain unchanged.

$$ds^2 = g_{00}dt^2 + g_{ij}dx^i dx^j. \tag{C.-60}$$

C.8.8 Properties of Killing Fields

A very important and immediate result is the following. If $k^b k_b = 0$, then from $\nabla_a k_b = \nabla_b k_a$ we have $k^b \nabla_b k_a = -k^b \nabla_a k_b = \nabla_a (k^b k_b) = 0$, i.e., the curve to which k is tangent is a null geodesic.

$$\nabla_a \nabla_b v_c = R_{abc}{}^d v_d \tag{C.-60}$$

Proof:

$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = -R_{abc}{}^d v_d, \tag{C.-60}$$

which on using Killing's equation, gives

$$\nabla_a \nabla_b v_c + \nabla_b \nabla_c v_a = -R_{abc}{}^d v_d \tag{C.-60}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{C.-60}$$

we have that

$$\begin{aligned} 2\nabla_b \nabla_c v_a &= -(R_{abc}{}^d + R_{bca}{}^d - R_{cab}{}^d) v_d \\ &= 2R_{cab}{}^d v_d. \end{aligned} \tag{C.-60}$$

C.8.9 Diffeomorphism Gauge Group - Symmetry of GR Under Active Diffeomorphisms

It is often stated that coordinate transformations are the gauge symetries of GR. Then move onto the diffeomorphism group, however, the diff group is formed by active diffeomorphisms not coordinate transformations! The gauge symmetry referred to is GR's invariance under active diffeomorphisms!

Let us consider an infinitessimal point transformation

$$x'^a = x^a + \xi^a(x) \tag{C.-60}$$

We have already proven that the metric $g_{ab}(x)$ gets mapped to the metric $g_{ab}(x) - 2D_{(a}\xi_{b)}$ under this point transformation.

Lie algebra of vector fields ξ

A vector space V with elements $x, y, z, \dots$ and bilinear bracket $[\cdots, \cdot]$, that takes two elements of V and returns another element of V, is a Lie algerbra if the bracket is anti-symmetric and the Jacobi identity holds for all elements in V.

vector field $\xi^a(x)$ generates an infinitesimal active diffeomorphism. Has a Lie algebra

$$[\xi_{(1)}, \xi_{(2)}]f = \tag{C.-60}$$

C.9 Frame Fields

That is absolutely crucial to the loop quantum gravity programme is that GR can be put into a form that strongly resembles gauge theories in particle physics.

In appendix we introduced a natural basis for the tangent space T_P at a point P that were induced by the coordinates. We consider a set of basis vectors *not* derived from any coordinate system. Say we are given a time-like vector field v^α which defines a congruence of curves. For each of these curves, take any point P. We introduce a orthonormal frame of three unit space-like vectors.

$$e_I^\alpha = (e_1^\alpha, e_2^\alpha, e_3^\alpha) \tag{C.-60}$$

which are orthogonal to v^α and where I is a label running from 0 to 3.

We define

$$e_0^\alpha := v^\alpha \tag{C.-60}$$

orthonomality relations

$$
\begin{aligned}
&e_1^a e_{1a} = e_2^a e_{2a} = e_3^a e_{3a} = -e_0^a e_{0a} = 1 \\
&e_0^a e_{1a} = e_0^a e_{2a} = e_0^a e_{3a} = e_1^a e_{2a} = e_1^a e_{3a} = e_2^a e_{3a} = 0
\end{aligned} \tag{C.-60}
$$

The four vectors are said to form a *frame* or *tetrad* at P, and the orthonormality relations can be succinctly summarized as

$$
e_i^a e_{ja} = \eta_{ij} \tag{C.-60}
$$

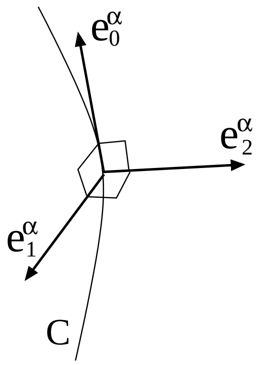


Figure C.31: Framefield or tetrad with one spatial dimension suppressed.

C.10 The Spin Connection

The triad is not a vector basis induced by a coordinate system. It turns out that the use of such frame fields brings out a different point of view on the connection and curvature, one in which GR has a strong resemblance to particle physics field theories.

Instead of a basis determined by coordinates, $\partial/\partial x^a$, we may choose any other n linearly independent vectors $\mathbf{e}_\alpha(\alpha = 1, \ldots, n)$, with components $\mathbf{e}_\alpha^{\mathbf{a}}(\alpha = 1, \ldots, n)$ with respect to the coordinate basis.

We have the freedom to choose a different basis. The metric $g_{ab}(x)$ is left invariant under local $SO(3,1)$ transformations such that

$$
e_I^a(x^a) \rightarrow e'^a_I(x^a) = O_i^J(x^a) e_J^a(x^a), \tag{C.-60}
$$

where $O^j_{\ i}(x^a)$ is a matrix in $SO(3,1)$ which depends on position in space. When "Physical quantities" are left invariant, such transformations are known as *gauge transformations*, and theories invariant under them are called *gauge theories.*

Now we have introduced these frame fields we now need to know how to compare vectors in frames at different points. Put another way; a difficulty arise when one considers partial derivatives, $\partial_a V^i$. Because the matrix $O_i^j(x^a)$ depends on spacetime, it will contribute an inhomogeneous term to the transformation of the partial derivative,

$$\begin{aligned}\partial_a V'^I(x) &= \partial_a \left(O^I{}_J(x^a)V^J(x^a)\right) \\ &= O^I j(x^a)\partial_a V^J(x^a) + V^J(x^a)\partial_a O^I{}_J(x^a)\end{aligned} \tag{C.-60}$$

The same sort of problem is encountered when considering the transformation of the partial differentiation of vector fields $\partial_a V^b(x)$. The solution there is to add the connection $\Gamma^a_{bc}(x)$ to correct for the inhomogeneous term in the transformation law, giving us the covariant derivative, $\nabla_a V_b = \partial_a V_b + \Gamma^c_{ab} V_c$. The same remedy is applied to $\partial_a V^i(x^a)$ and we introduce a connection

$$\omega_a{}^I{}_J(x) \tag{C.-60}$$

with two tetrad indicies and one spacetime index.

$$\mathcal{D}_a V^I(x) = \partial_a V^I(x) + \omega_a{}^I{}_J V^J(x) \tag{C.-60}$$

We require $\mathcal{D}_a V^i(x)$ to transform as a vector in internal space,

$$\mathcal{D}'_a V'^I(x) = O^I{}_J \mathcal{D}_a V^I(x). \tag{C.-60}$$

Therefore the connection transforms as

$$\omega'_a{}^I{}_J = O^I{}_K \omega_a{}^K{}_J - \partial_a O^I{}_J \tag{C.-60}$$

$$\mathcal{D}_a V^I = \partial_a V^I + \omega^I_{aJ} V^J \tag{C.-60}$$

$$\mathcal{D}_a V^I_b = \partial_a V^I_b + \Gamma^c_{ab} V^I_c + \omega^I_{aJ} V^J_b \tag{C.-60}$$

This covariant derivative is said to be compatible to the tetrad metric η_{IJ} if,

$$\mathcal{D}_a \eta_{IJ} = 0 \tag{C.-60}$$

This implies,

$$\partial_a \eta^{IJ} + \omega_{aK}^I \eta^{JK} + \omega_{aK}^I \eta^{KJ} = 0, \tag{C.-60}$$

This implying that the connection is antisymmetric in its tetrad indices,

$$\omega_a^{IJ} = -\omega_a^{JI}. \tag{C.-60}$$

C.10.1 The Spin Connection in terms of the Tetrads

The connection $\Gamma_{\mu\nu}^\alpha$ is uniquely determined by the requirement

$$\mathcal{D}_\mu e_\nu^I(x) = 0 \tag{C.-60}$$

that is,

$$\partial_\mu e_\nu^I + \omega_{\alpha J}^I e_b^j + \Gamma_{\alpha\beta}^\gamma e_\gamma^I = 0 \tag{C.-60}$$

It is said to be compatible to the co-triad.

The connection field $\Gamma_{\mu\nu}^\alpha$ can be calculated in much the same way as the $\Gamma_{\alpha\beta}^\gamma$ was calculated.

Consider the anti-symmetrized covariant derivative of the tetrad

$$\mathcal{D}_{[a} e_{b]}^I = \partial_{[a} e_{b]}^I + \omega_{[a}{}^{IL} e_{b]L} = 0 \tag{C.-60}$$

We can solve for ω in the same kind of way we derive the Christoffel connection. First we contract the above expression with $e_J^a e_K^b$ to obtain

$$e_J^a e_K^b \left(\partial_{[a} e_{b]}^I + \omega_{[a}{}^{IL} e_{b]L} \right) = 0. \tag{C.-60}$$

Let us define

$$\Omega_{IJK} = e_I^a e_J^b \partial_{[a} e_{b]K}.$$

This is obviously anti-symmetric in the first ttwo indices. Performing rotations of indices in (C.10.1) we get three equations,

$$
\begin{aligned}
\Omega_{JKI} + e^a_J e^b_K \omega_{[a|I|}{}^L e_{b]L} &= 0 \\
\Omega_{IJK} + e^a_I e^b_J \omega_{[a|K|}{}^L e_{b]L} &= 0 \\
\Omega_{KIJ} + e^a_K e^b_I \omega_{[a|J|}{}^L e_{b]L} &= 0
\end{aligned} \tag{C.-61}
$$

Adding the first two and subtracting the last,

$$
\begin{aligned}
&\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} \\
&+ e^a_J e^b_K \frac{1}{2}(\omega_{aI}{}^L e_{bL} - \omega_{bI}{}^L e_{aL}) \\
&+ e^a_I e^b_J \frac{1}{2}(\omega_{aK}{}^L e_{bL} - \omega_{bK}{}^L e_{aL}) \\
&- e^a_K e^b_I \frac{1}{2}(\omega_{aJ}{}^L e_{bL} - \omega_{bJ}{}^L e_{aL}) = 0
\end{aligned} \tag{C.-63}
$$

This simplifies to

$$
\begin{aligned}
&\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} \\
&+ \frac{1}{2}(e^a_J \omega_{aIK} - e^b_K \omega_{bIJ}) \\
&+ \frac{1}{2}(e^a_I \omega_{aKJ} - e^b_J \omega_{bKI}) \\
&- \frac{1}{2}(e^a_K \omega_{aJI} - e^b I \omega_{bJK}) = 0
\end{aligned} \tag{C.-65}
$$

Using $\omega_{aIJ} = -\omega_{aJI}$ the above reduces to

$$
\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} + e^a_J \omega_{aIK} = 0. \tag{C.-65}
$$

Let us swap the dummy variables J and K and replace a with b, then contract with e^K_a gives

$$
\omega_{aIJ} = e^K_a(-\Omega_{KJI} - \Omega_{IKJ} + \Omega_{JIK}). \tag{C.-65}
$$

or by using the anti-symmetry of ω_{aIJ},

$$
\omega_{aIJ} = e^K_a(\Omega_{KIJ} + \Omega_{JKI} - \Omega_{IJK}). \tag{C.-65}
$$

We now wish to use this to express ${\omega_a}^{IJ}$ in terms of the tetrads. Note

$$ {\omega_a}^{IJ} = e_a^K[{\Omega_K}^{IJ} + {\Omega^J}_K{}^I - {\Omega^{IJ}}_K] $$

and

$$ \begin{aligned} {\Omega_K}^{IJ} &= e_K^b e^{cI}\partial_{[b}e_{c]}^J \\ {\Omega^J}_K{}^I &= e^{bJ}e_K^c\partial_{[b}e_{c]}^I \\ {\Omega^{IJ}}_K &= e^{bI}e^{cJ}\partial_{[b}e_{c]K}. \end{aligned} \tag{C.-66} $$

Using these we get:

$$ \begin{aligned} {\omega_a}^{IJ} &= e_a^K\left(e_K^b e^{cI}\partial_{[b}e_{c]}^J + e^{bJ}e_K^c\partial_{[b}e_{c]}^I - e^{bI}e^{cJ}\partial_{[b}e_{c]K}\right) \\ &= e^{cI}\partial_{[a}e_{c]}^J + e^{bJ}\partial_{[b}e_{a]}^I - e^{bI}e^{cJ}e_a^K\partial_{[b}e_{c]K} \\ &= e^{bI}\partial_{[a}e_{b]}^J - e^{bJ}\partial_{[a}e_{b]}^I - e^{b[I}e^{|c|J]}e_a^K\partial_{[b}e_{c]K} \\ &= e^{b[I}\left(2\partial_{[a}e_{b]}^{J]} + e^{|c|J]}e_a^K\partial_c e_{bK}\right) \end{aligned} \tag{C.-68} $$

So finally we arrrive at

$$ {\omega_a}^{IJ} = e^{c[I}\left(2\partial_{[a}e_{c]}^{J]} + e^{|b|J]}e_a^K\partial_b e_{cK}\right) \tag{C.-68} $$

C.10.2 Curvature Associated with the Spin Connection.

Let us work out the commutator in the case of a vector λ_I.

We will need the covariant derivative of $\mathcal{D}_\beta\lambda_I$. This is a tensor with one internal space index I and one space-time index β. For covariant derivative of such an object is the standard formula is,

$$ \mathcal{D}_\alpha T_{\beta I} = \partial_\alpha T_{\beta I} + \Gamma^\rho_{\alpha\beta}T_{\rho I} + \omega^J_{\beta I}T_{\beta J}. \tag{C.-68} $$

Applying this to $\mathcal{D}_\beta\lambda_I$ gives

$$ \begin{aligned} \mathcal{D}_\alpha(\mathcal{D}_\beta\lambda_I) &= \partial_\alpha(\mathcal{D}_\beta\lambda_I) + \omega^J_{\beta I}(\mathcal{D}_\beta\lambda_J) + \Gamma^\rho_{\alpha\beta}(\mathcal{D}_\rho\lambda_I) \\ &= \partial_\alpha[\partial_\beta\lambda_I + \omega^J_{\beta I}] + \omega^K_{\alpha I}[\partial_\beta\lambda_I + \omega^J_{\beta I}] + \Gamma^\rho_{\alpha\beta}\mathcal{D}_\rho\lambda_I \end{aligned} \tag{C.-68} $$

$$
\begin{aligned}
\mathcal{D}_\alpha \mathcal{D}_\beta \lambda_I - \mathcal{D}_\beta \mathcal{D}_\alpha \lambda_I &= \Gamma^\rho_{\alpha\beta} \mathcal{D}_\rho \lambda_I - \Gamma^\rho_{\beta\alpha} \mathcal{D}_\rho \lambda_I \\
&= \partial_\alpha \partial_\beta \lambda_I - \partial_\beta \partial_\alpha \lambda_I \\
&+ \partial_\alpha (\omega^J_{\beta I} \lambda_I) - \partial_\beta (\omega^J_{\alpha I} \lambda_I) \\
&+ \omega^K_{\alpha I} \partial_\beta \lambda_K - \omega^K_{\beta I} \partial_\alpha \lambda_K \\
&+ \omega^K_{\alpha I} \omega^J_{\beta K} \lambda_J - \omega^K_{\beta I} \omega^J_{\alpha K} \lambda_J
\end{aligned} \tag{C.-71}
$$

The first line is zero as we assume partial derivatives commute. The terms in the third line are cancelled by terms in the second line. So we obtain the result

$$
\mathcal{D}_\alpha \mathcal{D}_\beta \lambda_I - \mathcal{D}_\beta \mathcal{D}_\alpha \lambda_I = R_{\alpha\beta\, I}{}^J \lambda_J \tag{C.-71}
$$

where $R_{\alpha\beta\, I}{}^J$ is defined by

$$
R_{\alpha\beta\, I}{}^J = \partial_\alpha \omega_{\beta I}{}^J - \partial_\beta \omega_{\alpha I}{}^J + \omega_{\alpha I}{}^K \omega_{\beta K}{}^J - \omega_{\beta I}{}^K \omega_{\alpha K}{}^J \tag{C.-71}
$$

which can be written in the more compact form

$$
R_{\alpha\beta}{}^{IJ} = \partial_{[\alpha} \omega_{\beta]}{}^{IJ} + \omega_{[\alpha}{}^{IK} \omega_{\beta]K}{}^J \tag{C.-71}
$$

By considering the covariant derivative of $K_\alpha = e^I_\alpha \lambda_I$ we can find the relation between the spin curvature with the

$$
\begin{aligned}
R^\alpha_{\beta\gamma\rho}(e^J_\alpha \lambda_J) = R^\alpha_{\beta\gamma\rho} K_\alpha &= \mathcal{D}_\beta \mathcal{D}_\rho K_\gamma - \mathcal{D}_\rho \mathcal{D}_\beta K_\gamma \\
&= \mathcal{D}_\beta \mathcal{D}_\rho (e^I_\gamma \lambda_I) - \mathcal{D}_\rho \mathcal{D}_\beta (e^I_\gamma \lambda_I) \\
&= e^I_\gamma [\mathcal{D}_\beta \mathcal{D}_\rho \lambda_I - \mathcal{D}_\rho \mathcal{D}_\beta \lambda_I] \\
&= e^I_\gamma R_{\beta\rho I}{}^J \lambda_J
\end{aligned} \tag{C.-73}
$$

this becomes

$$
R^\alpha_{\beta\gamma\rho} e^J_\alpha = e^I_\gamma R_{\beta\rho I}{}^J \tag{C.-73}
$$

which implies

$$
R^\alpha_{\beta\gamma\rho} = R_{\beta\rho I}{}^J e^I_\gamma e^\alpha_J \tag{C.-73}
$$

$$
R = R_{\mu\nu}{}^{IJ} e^\mu_I e^\nu_J \tag{C.-73}
$$

C.10.3 Palantini action for GR

Using $R = R_{\mu\nu}{}^{IJ} E^{\mu}_{I} E^{\nu}_{J}$ and $\sqrt{-g} = E$ we can write the Einstein Hilbert action in terms of the connection and tetrad:

$$S_{EH}[e, \omega] = \frac{1}{4\kappa} \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} E^I_\nu E^J_\nu F_{\alpha\beta}{}^{KL} \tag{C.-73}$$

where $F^{KL}_{\gamma\delta}$ is the curvature of the spin-connection.

Variation with respect to $\omega_\nu{}^{IJ}$ gives

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} D_\nu (E^I_\alpha E^J_\beta) = 0 \tag{C.-73}$$

and variation with respect to E^ν_I gives

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} E^J_\nu F_{\alpha\beta}{}^{KL} = 0 \tag{C.-73}$$

(C.10.2)

$$\epsilon_{\mu\nu\alpha\beta} = \frac{1}{4!} \epsilon_{PQRS} E^P_{[\nu} E^Q_\nu E^R_\alpha e^S_{\beta]} \tag{C.-73}$$

Hamiltonian constraints in the Palatini formalism

We can do the 3+1 split for the Palantini action (C.10.3) and obtain a Hamiltonian constraint and other constraints. While seems simpler than the of the metric variables

There are second class constraints which when solved give back the same set of constraints as obtained from the ADM framework. And so their not much improvement over the ADM formalism when it comes to quantizing it.

C.10.4 Ashtekar's New Variables

In the Palantini formalism the phase space variables are (E^a_i, Γ^i_a) (from these we get the intrinsic metric of the spacelike manifold Σ and its extrinsic curvature respectively).

We consider a canonical transformation from the phase space variables in the Palantini formalism (E^a_i, Γ^i_a) Ashtekar's variables

$$\Gamma^i_a \to \Gamma^i_a + \beta K^i_a \qquad E^a_i \to \frac{1}{\beta} E^a_i \tag{C.-73}$$

GR action written as

$$S_{EH}[E,\omega] = \frac{1}{2\kappa}\int d^4x \tilde{E}^\mu_I \tilde{E}^\nu_J F[\omega]_{\mu\nu}{}^{IJ} \tag{C.-73}$$

C.10.5 Cartan Structure Equations

Let $\{\hat{e}_i\}$ be the non-coordinate basis and $\{\hat{\theta}^i\}$ the dual basis. The vector fields satisfy

$$[\hat{e}_i, \hat{e}_j] = c_{ij}{}^k \hat{e}_k. \tag{C.-73}$$

The connection coefficients with respect to the basis $\{\hat{e}_i\}$ by

$$\nabla_i \hat{e}_j \equiv \omega^k{}_{ij} \hat{e}_k \tag{C.-73}$$

$$\omega^i{}_j := \omega^i{}_{kj} \hat{\theta}^k. \tag{C.-73}$$

The 2−forms of torsion are given by the first **Cartan's equations of structure**, which read

$$T^k = d\hat{\theta} + \omega^k{}_i \wedge \hat{\theta}^i, \tag{C.-73}$$

and the 2-forms of curvature by the second Cartan's equations of structure

$$\Omega^k{}_i = d\omega^k{}_i + \omega^k{}_j \wedge \omega^j{}_i. \tag{C.-73}$$

where $T^i \equiv \frac{1}{2} T^i{}_{jk} \hat{\theta}^j \wedge \hat{\theta}^k$ is the torsion two-form and $R^i{}_j \equiv \frac{1}{2} R^i{}_{jk\ell} \hat{\theta}^k \wedge \hat{\theta}^\ell$ the curvature two-form

Proof. Let the LHS of (C.10.5) act on the basis vectors $\hat{e}_k$ and $\hat{e}_\ell$,

$$\begin{aligned} & d\hat{\theta}^i(\hat{e}_k, \hat{e}_\ell) + [< \omega^i{}_j, \hat{e}_k >< \hat{\theta}^j, \hat{e}_\ell > - < \hat{\theta}^j, \hat{e}_k >< \omega^i{}_j, \hat{e}_\ell >] \\ = & \{\hat{e}_k[< \hat{\theta}^i, \hat{e}_\ell >] - \hat{e}_k[< \hat{\theta}^i, \hat{e}_k >] - < \hat{\theta}^i, [\hat{e}_k, \hat{e}_\ell] >\} + \{< \omega^i{}_\ell, \hat{e}_k > - < \omega^i{}_k, \hat{e}_\ell >\} \\ = & -c_{k\ell}{}^i + \omega^i_{k\ell} - \omega^i_{\ell k} = T^i_{k\ell} \end{aligned} \tag{C.-74}$$

where we have made use of (??). The RHS acting on $\hat{e}_k$ and $\hat{e}_\ell$ yields

$$\frac{1}{2}T^i{}_{jm}[<\hat{\theta}^j,\hat{e}_k><\hat{\theta}^m,\hat{e}_\ell>-<\hat{\theta}^m,\hat{e}_k><\hat{\theta}^j,\hat{e}_\ell>]=T^i{}_{k\ell}$$

which completes the proof. Equation may be proven similarly.

□

Taking the exterior derivatives of (C.10.5) and (C.10.5), we have the **Bianchi identities**

$$\begin{aligned} dT^i+\omega^i_{\ j}\wedge T^j &= R^i_{\ j}\wedge\hat{\theta}^j && \text{(C.-73)}\\ dR^i_{\ j}+\omega^i{}_k\wedge R^k_{\ j}-R^i_{\ k}\wedge\omega^k{}_j &= 0. && \text{(C.-72)} \end{aligned}$$

These are the non-coordinate basis versions of $R^a{}_{[bcd]}=0$ and $\nabla_{[e|}R^a{}_{b|cd]}=0$.

C.10.6 A Differential Geometry Translator

$$\begin{aligned} dA &= \left(\frac{\partial A_x}{\partial x}dx+\frac{\partial A_y}{\partial x}dy+\frac{\partial A_z}{\partial x}dz\right)\wedge dx\\ &= \frac{\partial A_y}{\partial x}\,dy\wedge dx+\frac{\partial A_z}{\partial x}\,dz\wedge dx && \text{(C.-72)} \end{aligned}$$

The gravitational field e

$$e^I(x)=e^I_a dx^a \qquad \text{(C.-72)}$$

The spin connection

$$\omega^I{}_J(x)=\omega^I_{aJ}(x)dx^a \qquad \text{(C.-72)}$$

$$\mathcal{D}v^I=dv^I+\omega^I{}_Jv^J \qquad \text{(C.-72)}$$

$$R^I{}_J=R^I{}_{J\,ab}dx^a\wedge dx^b \qquad \text{(C.-72)}$$

$T(X,Y)=T^a_{bc}X^bY^c$

$$T^a = \frac{1}{2} T^a{}_{bc} \omega^b \wedge \omega^c \tag{C.-72}$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{C.-72}$$

$R(X,Y)Z = R^a{}_{bcd} X^c Y^d Z^b$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \tag{C.-72}$$

Hodge-Star operation:

$$\mathcal{S}[e^I, \omega^{IJ}] = \frac{1}{4\kappa} \int_{\mathcal{M}} \epsilon_{IJKL} e^I_a e^J_b \left(R^{KL}_{\sigma\rho} - \frac{\Lambda}{6} e^K_\sigma e^L_\rho \right) dx^\mu dx^\nu dx^\sigma dx^\rho \tag{C.-72}$$

$$\mathcal{S}[e^I, \omega^{IJ}] = \frac{1}{4\kappa} \int_{\mathcal{M}} \epsilon_{IJKL} (e^I \wedge e^J \wedge R[\omega]^{KL} + \frac{\Lambda}{6} e^I \wedge e^J \wedge e^K \wedge e^L) \tag{C.-72}$$

C.11 More on Lie groups

For $g \in G$ we define the adjoint isomorphism $ad_g : G \to G$ by

$$ad_g(h) := ghg^{-1} = L_g \circ R_{g^{-1}} h, \tag{C.-72}$$

for all $h \in G$.

It is a group homomorphism as

$$\begin{aligned} ad_{[g_1,g_2]}(g_3) &= [[g_1,g_2],g_3] \\ &= [g_1,[g_2,g_3]] + [g_2,[g_3,g_1]] \\ &= ad(g_1)(ad(g_2)(g_3)) - ad(g_2)(ad(g_1)(g_3)) \\ &= [ad(g_1), ad(g_2)](g_3) \end{aligned} \tag{C.-74}$$

To see this is an isomorphism write $ad_g(h) = g'$ for any $g' \in G$, this has the solution $h = g^{-1} g' g$. This solution is unique as $ad_g(h) = ad_g(h')$ implies $h = h'$. This is called the adjoint representation of the Lie group G

The map ad_g fixes the neutral element e therefore the adjoint isomorphism ad_g on the Lie group G induces an isomorphism on $T_e(G)$

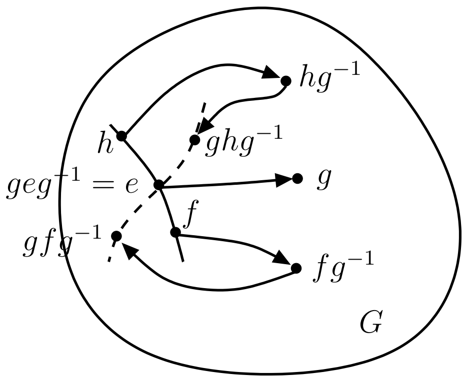


Figure C.32: A curve through e under the map $h \mapsto ghg^{-1}$, first a right action $R_{g^{-1}}$ as $h \mapsto hg^{-1}$ followed by the left action L_g as $hg^{-1} \mapsto ghg^{-1}$. The identity e is mapped to itself but points h and f near it are generally changed, so that a tangent vector at e, in $T_e(G)$, is mapped to another one in $T_e(G)$.

$$\mathcal{A}d_g := (ad_g)_* : T_e(G) \to T_e(G)$$

or

$$\mathcal{A}d_g : \mathcal{G} \to \mathcal{G}$$

on the Lie algebra $\mathcal{G}$.

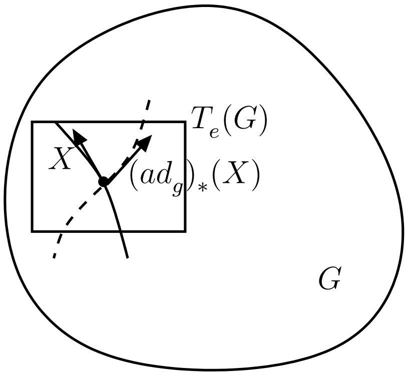


Figure C.33: A vector $X \in T_e(G)$ is mapped to another one in $\mathcal{A}d_g(X) \in T_e(G)$. Written formally as $(ad_g)_* : T_e(G) \to T_e(G)$.

This is a group homomorphism between groups

$$\begin{aligned}\mathcal{A}d(g_1g_2) &= (ad_{g_1g_2})_* = (ad_{g_1} \circ ad_{g_2})_* \\ &= (ad_{g_1})_* \circ (ad_{g_2})_* = \mathcal{A}d(g_1) \circ \mathcal{A}d(g_2)\end{aligned} \tag{C.-74}$$

Note by injectivety that $\mathcal{A}d(g)$ maps any vector $X \in T_e(G)$ to a non-zero vector for all $g \in G$. Also we have $\mathcal{A}d(g)(X+Y) = \mathcal{A}d(g)(X) + \mathcal{A}d(g)(Y)$. We thus have the map

$$\mathcal{A}d : G \to GL(n, \mathbb{R}), \tag{C.-74}$$

where $n = dim\mathcal{G}$.

This homomorphisms in turn induces a homomorphism between Lie algebras

$$Ad := \mathcal{A}d_* : \mathcal{G} \to \mathcal{GL}(n, \mathbb{R}), \quad n = dim\mathcal{G} \tag{C.-74}$$

called the adjoint representation of the Lie algebra $\mathcal{G}$.

If G is a matrix group, the adjoint representation becomes a simple matrix operation.

Definition The kernel of a group homomorphism $\varphi : G \to H$ is defined by

$$\ker \varphi := \varphi^{-1}(\{e_H\}) = \{x \in G : \varphi(x) = e_H\}.$$

Proposition C.11.1 *A group homomorphism $\varphi : G \to H$ is injective if and only if its kernel is trivial, i.e., ker $\varphi = \{e_G\}$.*

Proof:

Assume φ is injective. Since we must have $e_G \in$ ker φ, ker $\varphi = \{e_G\}$. Assume ker $\varphi = \{e_G\}$. Say φ is not injective, i.e., there is $x \neq y$ such that $\varphi(x) = \varphi(y)$. Then $\varphi(xy^{-1}) = e_H$, implying $xy^{-1} = e_G$ or $x = y$.

□

If G is a group, then by an automorphism of G we mean an isomorphism of G onto itself. The collection of such automorphisms, denoted Aut(G), is a subgroup of Sym (G).

The kernel of ad_x is the subgroup of G consisting of the elements $x \in G$ with the property that $xyx^{-1} = y$ for all $y \in G$, or, equivalently that $xy = yx$ for all $y \in G$. Thus the kernel of I equals the center $Z(G)$ of G.

C.11.1 Discrete Groups

A discrete group is a group with the discrete topology.

For example any finite group is a discrete group.

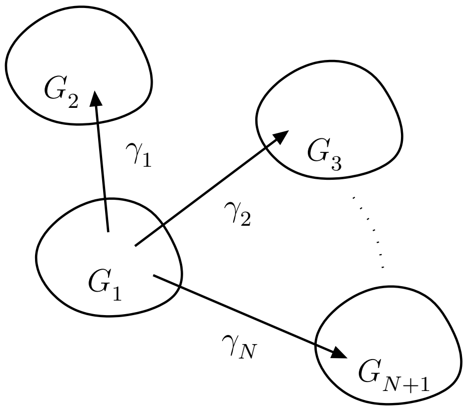


Figure C.34: If a Lie group is a direct product of the proper subgroup and some discrete subgroup then each connected component G_i is obtained from the proper subgroup G_1 by applying some discrete transformation γ_i of a discrete subgroup Γ.

C.11.2 Universal Covering Group

Two curves $g(\tau)$ and $g'(\tau)$ connecting the elements g_0 and g_1 are said to be homotomic if there exists a continuous deformation of one curve into the another, which leaves the end points g_0 and g_1 unaltered, i.e., there exists a continous function $h(\tau, s)$ of two parameters τ and s such that

$$h(0, s) = g_0, \qquad h(1, s) = g_1$$

$$h(\tau, 0) = g(\tau), \qquad h(\tau, 1) = g'(\tau).$$

A Lie group is said to be simply connected if every loop is homotopic to the null loop, i.e., every loop is contractable to one point.

A topological set not able to be partitioned into non-empty open subsets each of which has no points in common with the closure of the other. A topological space X is **connected** if $\emptyset$ and X are the only subsets of X that are both open and closed.

C.11.3 Decomposition of a Lie Group into Abelian and Non-Abelian Parts

The result of this subsection is employed in the proof of the uniqueness of Ashtekar-Lewandenski representation in LQG and the irreducibility of this representation.

Every connected compact Lie group is a quotient by a finite central subgroup of the product of a connected compact semisimple Lie group with a torus.

for a compact connected Lie group the exponential map is onto.

We prove it for the case of $SU(2)$.

Let G be any compact connected Lie group and T a maximal torus in G. We claim first that the following two statements are equivalent.

(a) the exponential map $\exp : \mathcal{L}(G) \rightarrow G$ is surjective.

(b) every element of G lies in a conjugate of T, i.e., the map $\psi : G \times T \rightarrow G$ given by $\psi(g,t) = gtg^{-1}$ is surjective.

The prove the result it is enough to establish (b) for $SU(2)$.

Proof: If (a) holds, for any $g \in G$ we have $g = \exp(\xi)$ for some $\xi \in \mathcal{L}(G)$. Therefore, g lies in the one-parameter subgroup $\{\exp(t\xi) : t \in \mathbb{R}\}$. The closure of this one-parameter subgroup is a torus in G. Therefore it is contained in a maximal torus T' in G.

...

□

Lemma C.11.2 *Let G be a connected compact Lie group. Let $\mathfrak{h}$ be an ideal in $\mathcal{L}(G)$. Then $\mathcal{L}(G)$ is the product of $\mathfrak{h}$ with the complementary ideal $\mathfrak{h}^\perp$.*

by () there exists an invariant inner product on $\mathcal{L}(G)$, i.e., $\mathcal{A}d$ is a Lie group homomorphism of G into $O(\mathcal{L}(G))$. This implies that Ad is a Lie algebra morphism of $\mathcal{L}(G)$ into the Lie algebra of $O(\mathcal{L}(G))$, i.e., all linear transformations $Ad(\xi)$, $\xi \in \mathcal{L}(G)$, are skewadjoint.

Let $\mathfrak{h}$ be an ideal in $\mathcal{L}(G)$. Then it is invariant under all $Ad(\xi)$, $\xi \in \mathcal{L}(G)$. This implies...

On the other hand, for $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}$, we have $[\xi, \eta] \in \mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$, that is, $\mathcal{L}(G)$ is the product of $\mathfrak{h}$ and $\mathfrak{h}^\perp$ as a Lie algebra.

□

Lemma C.11.3 *Every compact connected Lie group, G is isomorphic to a quotient $\tilde{G}/M$, where M is a central discrete subgroup of $\tilde{G}$, and $\tilde{G}$ is a simple product*

$$\tilde{G} = T \times P,$$

(that is, any $h \in \tilde{G}$ can be written as tp where $t \in T$ and $p \in P$), of an abelian group T and a semisimple group P.

Proof:

Recall that each Lie group possesses a Lie algebra $\mathfrak{g}$ isomorphic to the tangent vector space at the identity element of the Lie group. An ideal in a Lie algebra is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$. An ideal is said to be an invariant subalgebra.

An ideal is the Lie algebra equivalent of a closed, normal subgroup of a connected Lie group.

A connected Lie group can be defined to be simple if its Lie algebra is simple, or equivalently, if it contains no non-trivial, closed, connected normal subgroups. Under this definition, a simple connected Lie group can possess non-trivial, closed, normal subgroups, but if they exist they must be discrete.

A semisimple Lie algebra can be defined as a Lie algebra which has no non-trivial abelian ideals, but here we wish to characterise it as a Lie algebra which is the direct sum of simple Lie algebras. Semisimple Lie groups arae the direct products of simple Lie groups.

Clearly, a simple Lie algebra is semisimple.

Lemma C.11.4 *The ideal $[\mathcal{L}(G), \mathcal{L}(G)]$ in $\mathcal{L}(G)$ is a semisimple Lie algebra.*

Proof:

Let G be a compact connected Lie group.

Let $\mathfrak{h}$ be an ideal in $\mathcal{L}(G)$. Then it is invariant under all $\mathrm{ad}(\xi)$, $\xi \in \mathcal{L}(G)$. This implies that the orthogonal complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ is invariant under all $\mathrm{ad}(\xi)$, $\xi \in \mathfrak{g}$, i.e., $\mathfrak{h}^\perp$ is an ideal in $\mathfrak{g}$. It follows that $\mathcal{L}(G) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as a linear space. On the other hand, for $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^\perp$, we have $[\xi, \eta] \in \mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$, i.e., $\mathcal{L}(G)$ is the product of $\mathfrak{h}$ and $\mathfrak{h}^\perp$ as a Lie algebra.

Let $\mathfrak{a}$ be an abelean ideal in $[\mathcal{L}(G), \mathcal{L}(G)]$. Then $\mathfrak{a}^\perp$ is an ideal in $\mathcal{L}(G)$ and $\mathcal{L}(G)$ is the product of $\mathfrak{a}$ and $\mathfrak{a}^\perp$. This implies that $\mathfrak{a}$ is in the center $\mathcal{L}(Z)$ of $\mathcal{L}(G)$. It follows that $\mathfrak{a} = \{0\}$. Therefore $[\mathcal{L}(G), \mathcal{L}(G)]$ in $\mathcal{L}(G)$ is semisimple.

□

Proposition C.11.5 *Let G be connected compact Lie group. Let $C = K \cap Z_0$ and $D = \{(c, c^{-1} \in K \times Z_0 : c \in C\}$. Then $\varphi : K \times Z_0$ given by $\varphi(k, z) = kz$ induces an isomorphism of the Lie group $(K \cap Z_0)/D$ with G.*

Consider the connected compact lie group $K \times Z_0$ and the differentiable map $\varphi : K \times Z_0 \to G$ given by $\varphi(k, z) = kz$ for $k \in K$ and $z \in Z_0$. φ is a Lie group homomorphism and $L(\varphi)$ is an isomorphism of lie algebras. Therefore, φ is a covering projection.

The kernel of φ is a finite central subgroup of $K \times Z_0$.

$$\ker \varphi = \{(k, z) \in K \times Z_0 : kz = 1\} = \{(c, c^{-1}) \in K \times Z_0 : c \in K \cap Z_0\}.$$

□

Therefore, any connected compact Lie group is a quotient by a finite central subgroup of the product of a connected compact semisimple Lie group with a torus.

C.12 Group Actions on Sets

Action of a group

$$\sigma(g, x) = y \tag{C.-74}$$

$$\sigma(t, x) \tag{C.-74}$$

if the flow $\sigma(t, x)$ is periodic with period T.

We can construct a new action whose group is $U(1)$

$$\overline{\sigma}(\exp(2\pi it/T)y, x) = \sigma(t, x) \tag{C.-74}$$

and one whose group is $SO(2)$

$$\tilde{\sigma}(\left(\begin{array}{cc} \cos(2\pi t/T) & \sin(2\pi t/T) \\ -\sin(2\pi t/T) & \cos(2\pi t/T) \end{array}\right), x) = \sigma(t, x). \tag{C.-74}$$

The action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$

$$\sigma(\mathbf{M}, x) = \mathbf{M} \cdot x \tag{C.-74}$$

where $\cdot$ is the usual matrix multiplication on a vector. The action of subgroups of $GL(n,\mathbb{R})$ is defined similarly. $O(n)$ acts on $S^{(n-1)}(r)$, an $(n-1)-$sphere of radius r,

$$\sigma : O(n) \times S^{(n-1)}(r) \to S^{(n-1)}(r). \tag{C.-74}$$

Formal definition of the action of a group on a manifold:

Definition Let G be a Lie group and $\mathcal{M}$ be a manifold. The action of G on $\mathcal{M}$ is a differentional map $\sigma : G \times \mathcal{M} \to \mathcal{M}$ which satisfies the conditions

(i) $\sigma(e,p) = p$ for any $p \in \mathcal{M}$

(ii) $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$.

We as well define the **orbit** of a point x of $\mathcal{M}$ as

$$orb(x) = \{gx \mid g \in G\} \tag{C.-74}$$

i) The action of the group is *transitive* if any orbit is the whole of X.

ii) The action is *effective*, or *faithful*, if the trivial action on X, i.e.,if $\sigma(g,p) = p$ for all $p \in X$, implies $g = e$.

If the action is not effective, the set of g corresponding to the trivial action is an invariant subgroup H of G, and we can take G/H as having a faithful action.

iii) The action is *free* if the existence of an p such that $gp = p$ implies that $g = p$.

The **stabalizer** of x as

$$Stab(x) = \{g \in G \mid gx = x\} \tag{C.-74}$$

The orbits are equivalence classes - we are often interested in the quotient space.

Isotropy group

The identity element e is obviously in $H(p)$. Now, let $g_1, g_2 \in H(p)$

$$\sigma(g_1 g_2, p) = \sigma(g_1, \sigma(g_2, p)) = \sigma(g_1, p) = p$$

$g^{-1} \in H(p)$ because

$$p = \sigma(e,p) = \sigma(g^{-1}g, p) = \sigma(g^{-1}, \sigma(g,p)) = \sigma(g^{-1}, p)$$

One can consider the quotient $G/H(p)$

Example $SO(3)/SO(2)$

we have

$$SO(3)/SO(2) \cong S^2$$

Definition A group action is *effective* if the identity element is the only element that, that is, if $\sigma(g, x) = x$ for all $x \in \mathcal{M}$, then $g = e$.

C.12.1 Transitive Actions

Properties:

i) Orbits are disjoint,

ii) $\mathcal{M}$ is the union of of the orbits.

Example. Let $G = O(n)$ and $\mathcal{M} = S^n$.

Definition A group action is *transitive* if, for any $x_1, x_2 \in \mathcal{M}$, there exists a $g \in G$ such that $\sigma(g, x_1) = x_2$.

There is only one orbit!

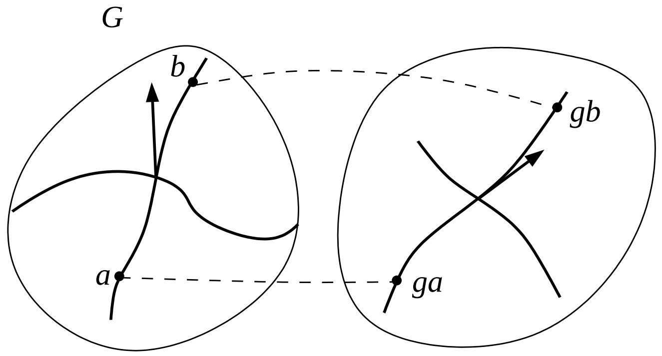


Figure C.35: leftTran. The left translation along g maps a neighbourhood of e onto one of g. There is a natural map of a vector at e to one at g.

C.12.2 Faithful Actions

The action of G on X is said to be faithful if $gx = hx$ for all $x \in X$ implies that $g = h$.

C.12.3 Free Actions

The action of G on X is said to be free if for all $g \in G$, $g \neq e$, and for all $x \in X$, $gx \neq x$.

Definition A group action is *free* if every element apart from the identity of G has no fixed points in M, that is, if there exists an element $x \in M$ such that $\sigma(g,x) = x$, then $g = e$.

C.12.4 Introduction to Gauge Invariance of the Yang-Mills Equations

$$\hat{U} = \exp\left(\frac{i}{2}\mathbf{a} \cdot \mathbf{T}\right) \tag{C.-74}$$

Minimal coupling term is

$$\begin{aligned}
\mathcal{L}_{int} &= g\sum_{i=1}^{3} \overline{\Psi}\gamma^{\mu}\frac{T^i}{2}A^i_{\mu}\Psi \\
&= g\overline{\Psi}\gamma^{\mu}\mathbf{A}_{\mu} \cdot \frac{\mathbf{T}}{2}\Psi
\end{aligned} \tag{C.-74}$$

$$\Psi \rightarrow \Psi' = \hat{U}\Psi = \exp\left(i\mathbf{a}(x) \cdot \hat{\mathbf{T}}\right)\Psi(x) \tag{C.-74}$$

$$\begin{aligned}
\partial_{\mu}\Psi \rightarrow \hat{U}\partial_{\mu}\Psi &= \hat{U}\partial_{\mu}(\hat{U}^{-1}\hat{U}\Psi) \\
&= \partial_{\mu}(\hat{U}\Psi) + \hat{U}(\partial_{\mu}\hat{U}^{-1})\hat{U}\Psi \\
&= \partial_{\mu}\Psi' + \hat{U}(\partial_{\mu}\hat{U}^{-1})\Psi' \\
&= \left[\partial_{\mu} + \hat{U}(\partial_{\mu}\hat{U}^{-1})\right]\Psi'
\end{aligned} \tag{C.-76}$$

By adding the coupling (C.-74) to the free Dirac equation the additional term $\hat{U}(\partial_{\mu}\hat{U}^{-1})$, occurring for $\partial_{\mu}\mathbf{a}$, can be absorbed by gauging the fields A^i_{μ} simultaneously. The Dirac Lagrangean is the cast into a gauge invariant form

$$\mathcal{L}(\Psi, \mathbf{A}_{\mu}) = i\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + g\overline{\Psi}\gamma^{\mu}\mathbf{A}_{\mu} \cdot \hat{\mathbf{T}}\Psi \tag{C.-76}$$

and the gauged density

$$\mathcal{L}' = i\overline{\Psi}'\gamma^\mu\partial_\mu\Psi' + g\overline{\Psi}'\gamma^\mu\mathbf{A}'_\mu\cdot\hat{\mathbf{T}}\Psi' \qquad \text{(C.-76)}$$

$$\begin{aligned}
\mathcal{L} &= i\overline{\Psi}\gamma^\mu\partial_\mu\Psi + g\overline{\Psi}\gamma^\mu\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\Psi \\
&= i\overline{\Psi}\hat{U}^{-1}\hat{U}\gamma^\mu\partial_\mu(\hat{U}^{-1}\hat{U}\Psi) + g\overline{\Psi}\hat{U}^{-1}\hat{U}\gamma^\mu\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1}\hat{U}\Psi \\
&= i\overline{\Psi}'\hat{U}\gamma^\mu\partial_\mu(\hat{U}^{-1}\Psi') + g\overline{\Psi}'\hat{U}\gamma^\mu\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1}\Psi' \\
&= i\overline{\Psi}'\gamma^\mu\hat{U}\partial_\mu(\hat{U}^{-1}\Psi') + g\overline{\Psi}'\gamma^\mu\hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1}\Psi' \\
&= i\overline{\Psi}'\gamma^\mu\partial_\mu\Psi' + i\overline{\Psi}'\gamma^\mu[\hat{U}(\partial_\mu\hat{U}^{-1})]\Psi' + g\overline{\Psi}'\gamma^\mu(\hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1})\Psi' \\
&= i\overline{\Psi}'\gamma^\mu\partial_\mu\Psi' + g\overline{\Psi}'\gamma^\mu\left[\hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1} + \frac{i}{g}\hat{U}(\partial_\mu\hat{U}^{-1})\right]\Psi' \qquad \text{(C.-80)}
\end{aligned}$$

for this to be identical to the original action we must have:

$$\mathbf{A}'_\mu\cdot\hat{\mathbf{T}} = \hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1} + \frac{i}{g}\hat{U}(\partial_\mu\hat{U}^{-1}). \qquad \text{(C.-80)}$$

We are forced to incorporate the term $\hat{U}(\partial_\mu\hat{U}^{-1})$, which is generated by gauging the kinematic energy of the field Ψ into the gauge transformation of the $\mathbf{A}_\mu$ fields. Let us look at the significance of this term in electrodynamics. In that case $\hat{U}$ is just $\hat{U} = \exp(ia(x))$. Then the gauge transformation reads

$$A'_\mu(x) = A_\mu(x) + \frac{1}{g}\partial_\mu a(x).$$

We can write the Lagrangean in the concise form

$$\begin{aligned}
\mathcal{L} &= i\overline{\Psi}(\partial_\mu - ig\mathbf{A}_\mu\cdot\hat{\mathbf{T}})\Psi \\
&\equiv i\overline{\Psi}\nabla_\mu\Psi \qquad \text{(C.-80)}
\end{aligned}$$

Here we have introduced tha covariant derivative

$$\nabla_\mu = \partial_\mu - ig\mathbf{A}_\mu\cdot\hat{\mathbf{T}} \qquad \text{(C.-80)}$$

The gauge transformation properties of $\mathbf{A}_\mu$ can be summarised as follows

$$\nabla_\mu \to \nabla'_\mu = \hat{U}\nabla_\mu\hat{U}^{-1} \qquad \text{(C.-80)}$$

The kinetic energy term of the $\mathbf{A}_\mu$ fields is missing. By analogy with electrodynamics we write

$$\mathbf{F}_{\mu\nu} = \nabla_\mu \mathbf{A}_\nu - \nabla_\nu \mathbf{A}_\mu \tag{C.-80}$$

Consider

$$\begin{aligned}
\hat{F}_{\mu\nu} &= \mathbf{F}_{\mu\nu} \cdot \hat{\mathbf{T}} = \sum_{i=1}^{3} F^i_{\mu\nu} \hat{T}^i \\
&= \nabla_\mu (\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) - \nabla_\nu (\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) \\
&= \partial_\mu (\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) + ig(\mathbf{A}_\mu \cdot \hat{\mathbf{T}})(\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) - \partial_\nu (\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) - ig(\mathbf{A}_\nu \cdot \hat{\mathbf{T}})(\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) \\
&= (\partial_\mu \mathbf{A}_\nu) \cdot \hat{\mathbf{T}} - (\partial_\nu \mathbf{A}_\nu) \cdot \hat{\mathbf{T}} - ig\big[\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \mathbf{A}_\nu \cdot \hat{\mathbf{T}}\big]
\end{aligned} \tag{C.-82}$$

From (C.12.4) we also have

$$[\nabla'_\mu, \nabla'_\nu] = \hat{U}[\nabla_\mu, \nabla_\nu]\hat{U}^{-1}$$

An explicit calculation of the commutator yields

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] &= [\partial_\mu - ig\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \partial_\nu - ig\mathbf{A}_\nu \cdot \hat{\mathbf{T}}] \\
&= [\partial_\mu, \partial_\nu] + \partial_\mu(-ig\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) - (-ig\mathbf{A}_\mu \cdot \hat{\mathbf{T}})\partial_\mu \\
&\quad + \partial_\nu(-ig\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) - (-ig\mathbf{A}_\nu \cdot \hat{\mathbf{T}})\partial_\nu \\
&\quad + (-ig)^2[\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \mathbf{A}_\nu \cdot \hat{\mathbf{T}}] \\
&= -ig\{(\partial_\mu A_\nu \cdot \hat{T}) - (\partial_\nu A_\mu \cdot \hat{T}) - ig[\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \mathbf{A}_\nu \cdot \hat{\mathbf{T}}]\}
\end{aligned} \tag{C.-85}$$

Notice that in the last line the derivatives act only on the gauges fields.

$$\mathbf{F}_{\mu\nu} = \frac{i}{g}[\nabla_\mu, \nabla_\nu] \tag{C.-85}$$

Then $\mathbf{F}_{\mu\nu}$ transforms as

$$\mathbf{F}'_{\mu\nu} = \hat{U}\mathbf{F}_{\mu\nu}\hat{U}^{-1} \tag{C.-85}$$

We obtain a gauge invariant Lagrangian by performing the trace over the internal indices

$$
\begin{aligned}
\mathcal{L}'_A &= -\frac{1}{2}Tr\{(\mathbf{F}'_{\mu\nu}\cdot\hat{\mathbf{T}})(\mathbf{F}'^{\mu\nu}\cdot\hat{\mathbf{T}})\} \\
&= -\frac{1}{2}Tr\{\hat{U}(\mathbf{F}'_{\mu\nu}\cdot\hat{\mathbf{T}})\hat{U}^{-1}\hat{U}(\mathbf{F}'^{\mu\nu}\cdot\hat{\mathbf{T}})\hat{U}^{-1}\} \\
&= -\frac{1}{2}Tr\{\hat{U}^{-1}\hat{U}(\mathbf{F}'_{\mu\nu}\cdot\hat{\mathbf{T}})(\mathbf{F}'^{\mu\nu}\cdot\hat{\mathbf{T}})\} \\
&= \mathcal{L}_A \qquad \text{(C.-87)}
\end{aligned}
$$

C.13 Principle Bundles and Connections

A principle bundle is a fibre bundle $\pi : P \to E$ with fibre F equal to the structure group G and having the property that for all U_a and U_b with $U_a \cap U_b \neq \emptyset$,

$$\varphi_{ba} : U_a \cap U_b \to Left(F) \subset Diff(F),$$

where $Left(F) = \{L_g | L_g(h) = gh, \text{ for all } h \in G, g \in G\}$. In other words, changing coordinates corresponds to multiplying the fibre on the left by some element of G.

Lemma C.13.1 *For every $G-$principle bundle, G acts naturally on P on the right.*

Given $u \in P$, we want to define ug, for each $g \in G$. Let U be a neighbourhood about $\pi(u)$ that has a trivialization. Using these coordinates, represent u as $(\pi(u), h)$ where $h \in G$. Then define ug to be a point of P that has the coordinates $(\pi(u), hg)$. It is not hard to check that this definition is independent of coordinates, and then it is clear that it is a right action.

To place gauge theory in a more general perspective, it is helpful to consider fibre bundle formulism. The idea in gauge theory is to consider group bundles, where each fibre is a copy of the internal symmetry group, and where the base space corresponds to spacetime.

think about gauge theory geometrically - to understand the gauge field as a connection on the principal bundle.

We have already encountered one fibre bundle, from general relativity: the tangent space at a point in spacetime is a fibre, with the fibre bundle

the gauge fields play the same role in gauge theory as the Christoffel symbols play in the tangent in general relativity.

A principal fibre bundle allows one to simultaneously view the physical space, $\mathcal{M}$, referred to as the base space, and the bundle space E, where the bundle space generally reflects the symmetry group of the theory by associating with each point $x \in \mathcal{M}$ a fibre in E diffeomorphic to some Lie Group $\mathcal{G}$, refined to as the gauge group or structure group.

Bundles

A principal bundle $P(\mathcal{M}, G)$, where G is a Lie group and $\mathcal{M}$ is a compact manifold.

covering $\mathcal{M}$ with topological trivial open subsets U_a and giving a set of transition functions

$$t_{ab} : U_a \cap U_b \to G. \tag{C.-87}$$

The transition functions are to satisfy three conditions:

$$\begin{aligned} t_{aa}(x) &= 1, \\ t_{ab}(x)t_{ba}(x) &= \mathbf{1}, \\ t_{ab}(x)t_{bc}(x)t_{ca}(x) &= \mathbf{1}, \end{aligned} \tag{C.-88}$$

for all points $x \in \mathcal{M}$ where the functions are all defined.

A gauge transformation is defined to be a collection of maps

$$\lambda_a : U_a \to G. \tag{C.-88}$$

acting on transition functions in the following way,

$$t_{ab}(x) \longmapsto \lambda_a^{-1}(x)t_{ab}(x)\lambda_b(x). \tag{C.-88}$$

Gauge-transformed transition functions define the same bundle. A bundle is characterized by a gauge equivalence class of transition functions satisfying (C.-88).

A connection ω is defined globally is represented by a connection one-form ω_a on each U_a, with values in the Lie algebra of G, where, on each overlap $U_a \cap U_b$,

$$\omega_a = \mathrm{Ad}_{t_{ab}^{-1}}\omega_b + t_{ab}^{-1}dt_{ab} \tag{C.-88}$$

where Ad is the adjoint representation.

$$[T_a, T_b] = C^{abc}T_c, \tag{C.-88}$$

C^{abc} forms a representation of (D.14), the matrix $\mathcal{T}^a$ with components given by

$$(\mathcal{T}^a)_{bc} = iC^{abc} \tag{C.-88}$$

This representation is called the *regular* representation or the *adjoint* representation.

C.14 Fibre Bundles

We have seen that a manifold has an associated with it a tangent space at *each* point p in the manifold. One can combine these spaces with the underlying manifold $\mathcal{M}$ to make one big manifold. Mathematically this combining (union) is expressed as,

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}. \tag{C.-88}$$

This is known as the **tangent bundle space**. In fact we can combine spaces other than the tangent space with the manifold $\mathcal{M}$ to make a bigger topological space. These other spaces need not be vector spaces as the the tangent space is. The other space is, generally, referred to as the **fibre**. The whole space is known as a **fibre bundle**.

The simplest example of a fibre bundle, which is not a vector bundle, is the cylinder formed by choosing $\mathcal{M}$ as the real line R^1 and A as the circle S^1; one constructs the cylinder as a product between these two spaces, (actually, one should really say: *one constructs the cylinder as a product over the base manifold* R^1). This **product bundle**, also referred to as a **trivial bundle**, is denoted $(\mathcal{M} \times A, \mathcal{M}, \pi)$.The first entry refers to the fibre bundle, and the second refers to the base manifold. The third is a $\pi : \mathcal{M} \times A \rightarrow \mathcal{M}$. The projection map $\pi : E \rightarrow \mathcal{M}$ associates each fibre with a point in the base manifold. The inverse of the projection map π^{-1} associates a copy of the internal symmetry group (a fibre) with each point in the base manifold.

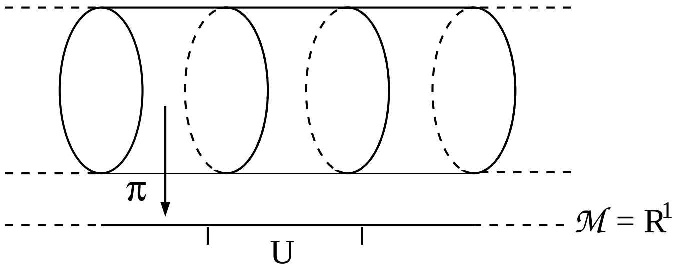


Figure C.36: Fibre bundle, $TS^1 = S^1 \times R$. The base manifold $\mathcal{M}$ (the real line R^1). The circle is the fibre. The fibre bundle consists of a manifold and a projection map π. $\pi^{-1}(U)$ is the local product space.

A fibre bundle is *locally a product bundle*. A fibre bundle which is not a product bundle known as a ***non-trivial bundle***. We give an example which shows that if a manifold is locally the direct product of two other manifolds, it is nevertheless not, in general, a product manifold. Möbius strip

The tangent bundle $T\mathcal{M}$ is locally of the form $U \times \mathcal{M}$. A ***section*** of this tangent bundle is simply a vector field $\tilde{\mathbf{w}}$ on $\mathcal{M}$. In the coordinate patch $(U, u_1, \ldots, u^n)$, $\tilde{\mathbf{w}}$ is given by its component functions $w^1_U(u), \ldots, w^n_U(u)$ with respect to the coordinate basis $\partial/\partial u$, but

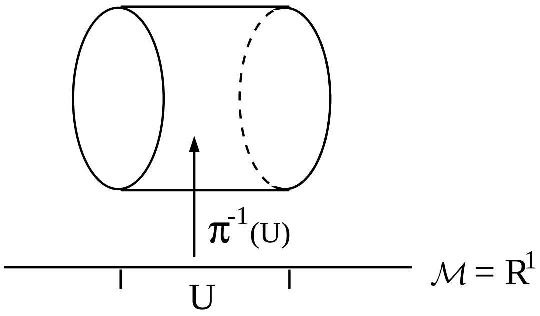


Figure C.37: The inverse map $\pi^{-1}(U)$ is the local product space.

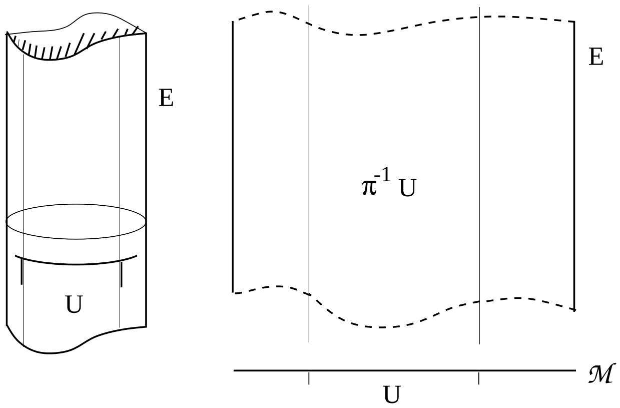


Figure C.38: tangent bundle, $TS^1 = S^1 \times R$. The base manifold $\mathcal{M}$ (the circle S^1) consists of a manifold and a projection map π. $\pi^{-1}(U)$ is the local product space.

of course these functions are defined on U only, not on all of $\mathcal{M}$. In another patch V, the same field is described by another set of component functions $w_V^1(u), \ldots, w_V^n(u)$. At a point p in the overlap $U \cap V$ these two sets of vector component functions transform as

$$w_V^a(p) = [c_{UV}(p)]_b^a w_U^b(p) \tag{C.-88}$$

where $c_{UV} = \partial v / \partial u$ the Jacobian matrix.

cotangent fibre bundle

$$T^*\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p^*\mathcal{M}. \tag{C.-88}$$

Phase space is the cotangent fibre bundle.

$$w_V^a(p) = [c_{UV}(p)^T]_b^a w_U^b(p) \tag{C.-88}$$

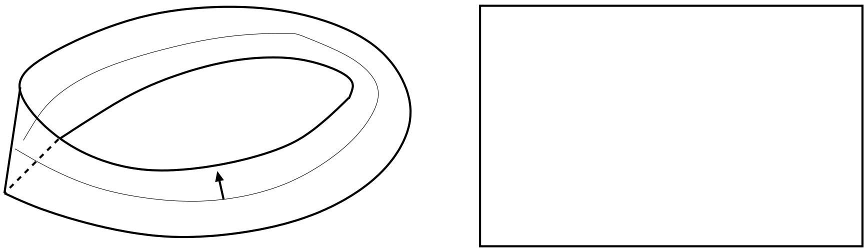


Figure C.39: Möbius.

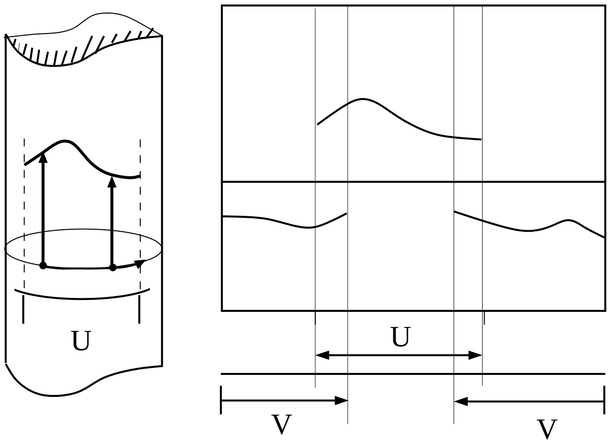


Figure C.40: tangent bundle, $TS^1 = S^1 \times R$. The curved line is a section.

Some examples:

$$\begin{aligned} \pi &\approx R^n \\ S^{n-1} &\subset \pi^{-1}(p) \end{aligned} \tag{C.-88}$$

$T_0\mathcal{M}$ is atypical, since it is a *subbundle* of the vector bundle $T\mathcal{M}$.

Let $M(n \times n)$ be the set of all $n \times n$ real matrices. We associate to the matrix $\underline{\underline{x}}$ the point in the n^2-dimensional Euclidean space whose coordinates are $x_{11}, x_{12}, \ldots x_{nn}$. The topological of the parameter space is R^n. The ***general linear group*** $Gl(n, R)$ is the group of all real $n \times n$ matrices $x = (x_{ij})$ with determinate $\det x \neq 0$.

$$X^i \mathbf{e_i} = x_{11}\mathbf{e_{11}} + x_{12}\mathbf{e_{12}} + x_{21}\mathbf{e_{21}} + x_{22}\mathbf{e_{22}} + \cdots + x_{nn}\mathbf{e_{nn}}. \tag{C.-88}$$

It is clear from $(xy)_{ij} = \sum_k x_{ik} y_{kj}$ that the product matrix has coordinates that are smooth functions of the coordinates of $\underline{\underline{x}}$ and $\underline{\underline{y}}$

a vector tangent to G is itself a matrix

Definition The ***left (right) translation*** is $G \mapsto G$, $l_a(g) = ag$, and the ***right translation***

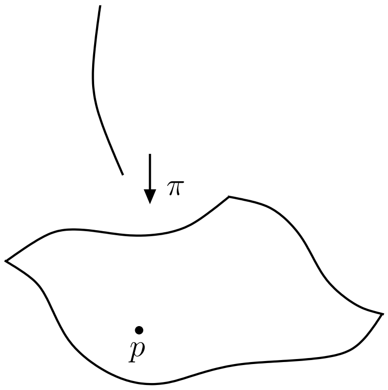


Figure C.41: Travelling up fibre.

$$\varphi_U : U \times GL(R,n) \rightarrow \pi^{-1}(U) \tag{C.-88}$$

a local trivialization, such for any $x \in U$ and $(X_i^k) \in GL(R,n)$,

$$\varphi_U(x,(X_i^k)) = (x; e_1, \ldots, e_n) \tag{C.-88}$$

$$y^a = y^a(x^1, \ldots, x^n), \;\; i = 1, \ldots, n \tag{C.-88}$$

$$\varphi_V : V \times GL(R,n) \rightarrow \pi^{-1}(V) \tag{C.-88}$$

$$\mathbf{e_i} = X_i^k \frac{\partial}{\partial x^k} = Y_i^k \frac{\partial}{\partial y^k} \tag{C.-88}$$

or

$$Y_i^k = X_i^k \left(\frac{\partial y^k}{\partial x^j} \right)_x \tag{C.-88}$$

by () the right translation of X is given by the Jacobian matrix $\left(\frac{\partial y^k}{\partial x^j} \right)_x$

one can identify a family of transition functions

$$c_{UV} : U \cap V \to GL(R, n) \tag{C.-88}$$

given by

$$c_{UV}(x) = \varphi_V^{-1} \circ \varphi_U \tag{C.-88}$$

In fact $\varphi_V^{-1} \circ \varphi_U$ is precisely the right translation of X

Details

$$L_{g_1} \circ L_{g_2} = L_{g_1 g_2} \tag{C.-88}$$

or

$$\begin{aligned} L_{g_1} \circ L_{g_2} &= L_{g_1}(x; (g_1 e)_1, \ldots, (g_1 e)_n) \\ &= (x; (g_1(g_2 e))_1, \ldots, (g_1(g_2 e))_n) \\ &= (x; (g_1 g_2 e)_1, \ldots, (g_1 g_2 e)_n) \end{aligned} \tag{C.-89}$$

The left translation along g maps a neighborhood of e onto one of g. There is a natural map of a vector at e to one on g. a vector tangent to G is itself a matrix

The one-parameter subgroup generated by any matrix A is the integral curve through e of the left-invariant vector field whose tangent at e is A.

As matrices

$$g(t+s) = g(t)g(s), \;\; \Leftrightarrow \;\; g_{ij}(t+s) = \sum_k g_{ik}(t) g_{kj}(s) \tag{C.-89}$$

$$g_A(t + \Delta t) = g_A(t) g_A(\Delta t) \tag{C.-89}$$

$$\frac{dg_A}{dt} = g_A(t) A \tag{C.-89}$$

Differentiate both sides with respect to s and put $s = 0$

$$g'(t) = g(t) g'(0). \tag{C.-89}$$

Since $g'(0)$ is a constant matrix, the solution to this is

$$g(t) = g(0)\exp(tg'(0)) \tag{C.-89}$$

$$1 + tg + \frac{1}{2!}t^2 g^2 + \ldots \tag{C.-89}$$

is the most general form for a 1-parameter subgroup of a matrix group G.

$$\Phi(x, y) = \exp_p(x\mathbf{e} + y\mathbf{f}) \tag{C.-89}$$

$$\frac{\partial}{\partial x}\Phi(x, y)\bigg|_{(x,y)} = \frac{\partial}{\partial x}\exp_p(x\mathbf{e})\bigg|_{(x,y)} = \mathbf{e}. \tag{C.-89}$$

Theorem Let X, Y be a pair of left (right) invariant vector fields. Then $[X, Y]$ is left (right) invariant.

Proof:

$$\begin{aligned}
[X', Y']_i &= X'_j \partial'^j Y'_i - Y'_j \partial'^i X'_i \\
&= X_j \partial^j Y'_i - Y_j \partial^j X'_i \\
&= X_j \partial^j \left(\frac{\partial y_i}{\partial x_k} Y_k\right) - Y_j \partial^j \left(\frac{\partial y_i}{\partial x_k} Y_k\right) \\
&= \frac{\partial y_i}{\partial x_k}(X_j \partial^j Y_k - Y_j \partial^j X_k) + \frac{\partial^2 y_i}{\partial x_j \partial x_k}(X_j Y_k - Y_j X_k).
\end{aligned} \tag{C.-91}$$

C.14.1 The Structure Group of a Bundle

In a vector bundle each $c_{UV} \in Gl(n)$. We have seen that for a Riemannian manifold $\mathcal{M}$ of dimension n, we may choose $c_{UV}(x) \in O(n)$ by using orthonormal frames. In a general bundle it may be possible to choose the $c_{UV}(x)$ such that they all lie in a specific Lie group G

$$c_{UV} : U \cap V \to G \tag{C.-91}$$

We then say that G is the ***structure group*** of the bundle.

$$c_{UV} = \begin{pmatrix} \cos\alpha(x) & -\sin\alpha(x) \\ \sin\alpha(x) & \cos\alpha(x) \end{pmatrix} \in SO(2). \tag{C.-91}$$

The orthonormal frames have allowed us to ***reduce*** the structure group from $Gl(2,R)$ to $SO(2)$.

C.14.2 Frame Bundle

Frame bundles are fundamental because from this we can construct the tangent bundle, and by a similar construction the cotangent bundle and all the tensor bundles.

If $\{v_1,\ldots,v_n\}$ is a local field of linear frames on a neighbourhood U in $\mathcal{M}$, and $\{e_1,\ldots,e_n\}$ is a frame at $x \in U$, then $\lambda_a^b V_{bx}$, where the number λ_a^b are the entries of a non-singular matrix λ. Thus relative to the local field, each linear frame determines an element of $GL(n,\mathbb{R})$; and each element of $GL(n,\mathbb{R})$ determines a linear frame. We therefore take $GL(n,\mathbb{R})$ as the typical fibre bundle.

The frame bundle is clearly not a vector bundle. Its typical fibre has instead the structure of a group.

C.14.3 The Idea of a Principal Bundle

If we have a connection in a principle bundle $P \to \mathcal{M}$ then a choice of a horizontal subspace at u is equivalent to a choice of projection

$$proj_u : T_uP \to V_u.$$

Then the vertical space (i.e. a fiber of the vertical bundle) may be viewed as the tangent space to the fibre, G. In turn, this an be viewed as g, the Lie algebra of G, so we really have

$$proJ_u : T_uP \to g.$$

Therefore a connection is equivalent to having a g valued 1-form on P which is invariant under the right action of G.

Let G be a Lie group and $\mathcal{M}$ a smooth manifold. A ***principle G bundle*** is over $\mathcal{M}$ is a manifold which locally looks like $\mathcal{M} \times G$.

We have already found the transition functions for this fibre bundle. We consider the intersection of two coordinate patches, and a point p

The transition functions are given by the Jacobian matrix. They are elements of the group $GL(n, \mathbf{R})$, the general linear transformations on an n dimensional real vector space.

The bundles $\mathcal{T}(\mathcal{M})$ and $\mathcal{T}^*(\mathcal{M})$ are vector bundles. a tangent bundle and a co-tangent bundle. Phase space is an example of a co-tangent bundle.

A principal bundle is a bundle whose fibres are the transition functions themselves - points in the fibre are elements of the structure group. We have just considered a principal fibre bundle, that of a frame space.

Definition The right action of G on $\pi^{-1}(U)$ is defined by $\phi_i^{-1}(ua) = (p, g_i a)$, that is, (fig C.14.3)

$$ua := \phi_i(p, g_i a) \tag{C.-91}$$

for any $a \in G$ and $u \in \pi^{-1}(p)$.

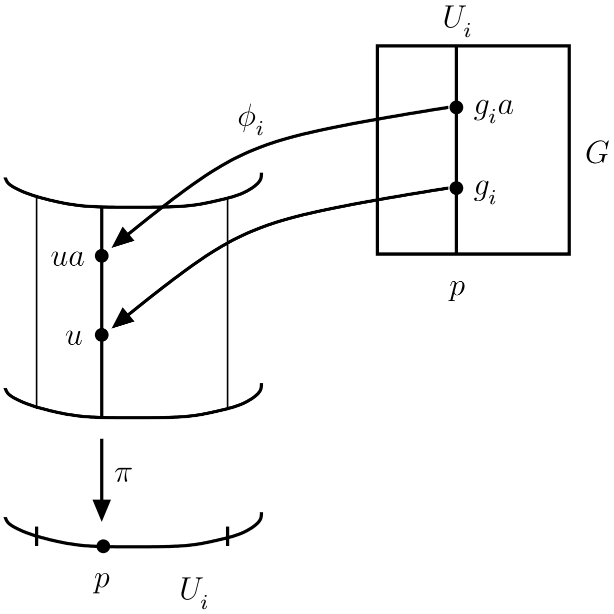


Figure C.42: Defintion of the right action of G on the principal fibre bundle P.

□

Definition Principal fibre bundle.

i) A differentiable manifold E called the total space;

ii) A differentiable manifold $\mathcal{M}$ called the base space;

iii) A surjection $\pi : E \to \mathcal{M}$ called the projection. The inverse image $\pi^{-1}(x) \equiv G_x \simeq G$ called the fibre at x;

iv) A Lie group G called the structure group, which acts on the fibre G on the left;

v) An open covering $\{U_I\}$ of $\mathcal{M}$ with a diffeomorphism $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ such that $\pi\phi_i(x,g) = x$. The map is called the local gauge or local trivialisation since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times G$;

vi) If we write $\phi_i(x,g) = \phi_{i,x}(g)$, the map $\phi_{i,x} : G \to G_x$ is a diffeommorphism. On $U_i \cap U_j \neq \emptyset$, we require that $t_{ij}(x) \equiv \phi_{i,x}^{-1}\phi_{j,x} : G \to G$ be an element of the structure group G. The ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ such that $\phi_j(x,g) = \phi_i(x, t_{ij}(x)g)$. $\{t_{ij}\}$ are the transition functions.

□

C.14.4 Principal Bundle

We are now ready to state the precise definition of a principal bundle.

We will say that a fibre bundle

$$\{P, \mathcal{M}, \pi, F, G\} \tag{C.-90}$$

is a **principal bundle** *if the fibre F is the same as the group G* and if the transition functions $c_{UV}(x)$ act on $F = G$ by *left translations.*

and

C.14.5 Action of the structure Group on a Principal Bundle

In practical situations transition functions are the gauge transformations required for pasting local charts together.

C.14.6 Connections on Principal Bundles

When we make comparison of objects in two different spaces is made by a prescribed mapping, and the mappings that connect the various spaces are called connections.

The first definition of a connection has a clear geometric meaning,

1. a $G-$invariant horizontal distribution $H_uP \subset T_uP$.

An equivalent, less geometric definition of a connection consistent with the first definition is:

2. a one-form $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying $\omega(s(X)) = X$.

We come to the final definition of the connection on a principal bundle, expressed in the form of $\mathcal{G}-$valued one-forms on the base manifold $\mathcal{M}$, instead of one-forms on the total space P. This definition is most suitable in physics applications.

3. a family of one-forms $A_i \in \Omega^1(P; \mathfrak{g})$ satisfying equation (for simplicity here in the case of matrix groups G):

$$A_j = t_{ij} A_i t_{ij}^{-1} + t_{ij}^{-1} dt_{ij}.$$

Definition (1): A connection on a principal bundle $\pi : P \to \mathcal{M}$ with group G is a smooth assignment to each $u \in P$ of an $n-$dimensional subspace $\mathcal{H}(p) \subset T_u P$ of the tangent space to P ($n = dim\ \mathcal{M}$) such that

$$T_u P = \mathcal{H}(p) \oplus \mathcal{V}(p), \tag{C.-90}$$

where $\mathcal{V}(p)$ is the subspace

$$\mathcal{V}(p) := \{X \in T_u P : \pi_*(X) = 0\}. \tag{C.-90}$$

The subspace field $\mathcal{H}$ is also required to be invariant under the left action L_g ($g \in G$):

$$(L_g)_* \mathcal{H}(p) = \mathcal{H}(gp), \tag{C.-90}$$

for all $g \in G$. $\mathcal{V}(p)$ and $\mathcal{H}(p)$ are called the vertical and horizontal subspace of $T_u P$, respectively.

$\square$

Definition (2): Suppose $\mathcal{G}$ is the Lie algebra of a Lie group G, which is the structure group of a principal fibre bundle $\pi : P \to \mathcal{M}$. Let $A \in \mathcal{G}$, and $A^\sharp$, called a fundamental field, be the vector field on P defined by

$$A^\sharp(p) := \frac{d}{dt} \exp(At)p \Big|_{t=0}. \tag{C.-90}$$

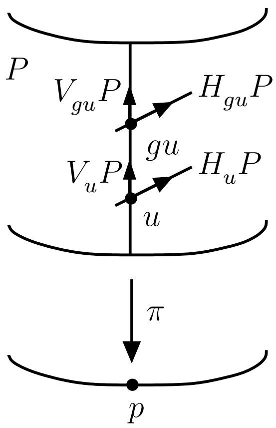


Figure C.43: The horizontal subspace $H_{gu}P$, defining by the connection in definition (1), is obtained from H_uP by the left action.

A connection on the principal bundle $\pi : P \to \mathcal{M}$ is a $\mathcal{G}-$valued one for satisfying the following two properties:

$$(i) \qquad \omega(A^\sharp(p)) = A \tag{C.-89}$$
$$(ii) \qquad (L_g)^*(\omega) = \mathcal{A}d(g)(\omega). \tag{C.-88}$$

□

Definition (3): A connection on a principal bundle is an assignment to each local trivialisation $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ (a choice of gauge in physics terms) a Lie algebra one-form ω_i on U_i which satisfies the following rule between different local trivialisations,

$$\omega_j(X_p) = (R_{t_{ij}^{-1}(p)})_*((t_{ij})_*(X_p)) + \mathcal{A}d(t_{ij}(p))(\omega_i(X_p)), \tag{C.-88}$$

for all $X_p \in T_p\mathcal{M}$ and $p \in U \cap V$.

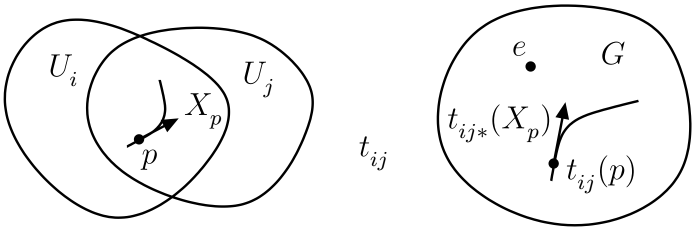


Figure C.44:

$\square$

Since a non-trivial principal bundle does not admit a global section, the pull-back $A_i = s_i^*\omega$ exists locally but not necessarily globally.

Lemma C.14.1 *The two definitions (1) and (2) of a connection on a principal fibre bundle are equivalent.*

Proof:

(2) $\Rightarrow$ (1):

Suppose we are given a $\mathcal{G}-$valued connection one-form ω, as in definition (2). Consider the field of subspaces defined by

$$\mathcal{H}(u) = \{X \in T_uP : \omega_u(\mathbb{X}) = 0\}.$$

By (), $\omega(\mathbb{Y})A \in \mathcal{G}$ for any $\mathbb{Y} \in \mathcal{V}(p)$. Hence $\mathcal{V}(p) \cap \mathcal{H}(p) \neq \emptyset$ and $T_pP = \mathcal{V}(p) \oplus \mathcal{H}(p)$. We must prove that

$$L_{g*}\mathcal{H}(u) = \mathcal{H}(gu).$$

Fix a point $u \in P$ and define $\mathcal{H}(u)$ as above. Take $\mathbb{X} \in \mathcal{H}(u)$ and construct $L_{g*}\mathbb{X} \in T_{gu}$. We find

$$\omega_{gu}(L_{g*}\mathbb{X}) = L_g^*\omega_u(\mathbb{X}) = g\omega_u(\mathbb{X})g^{-1} = 0$$

since $\omega(\mathbb{X}) = 0$. Therefore, $L_{g*}\mathbb{X} \in \mathcal{H}(gu)$. Note that L_{g*} is an invertible linear map. Hence any vector $\mathbb{Y} \in \mathcal{H}(gu)$ is expressed as $\mathbb{Y} = L_{g*}\mathbb{X}$ for some $\mathbb{X} \in \mathcal{H}(u)$. Thus $L_{g*}\mathcal{H}(u) = \mathcal{H}(gu)$.

(1) $\Rightarrow$ (2):

We define a $\mathcal{G}-$valued one-form ω by

$$\omega(A^\sharp + X) = A \tag{C.-88}$$

Note $0 \in \mathcal{H}(u)$ and setting $\mathbb{X} = 0$, we have $\omega(A^\sharp) = A$, and as the connection is linear this implies $\omega(\mathbb{X}) = 0$ for all $\mathbb{X} \in \mathcal{H}(u)$.

To complete the proof we must establish property (ii) (C.-88). Any $\mathbb{X} \in T_uP$ can be written $\mathbb{X} = a\mathbb{H} + b\mathbb{V}$, where $H \in \mathcal{H}(u)$ and $V \in \mathcal{V}(u)$. This allows us to need only verify

(C.-88) separetly for arbitrary horizontal and vertical vectors. Suppose $\mathbb{H} \in \mathcal{H}(u)$. Then $\omega(\mathbb{H}) = 0$. By (C.14.6), $\omega((L_g)_*\mathbb{H}) = 0$ and (C.-88) is verified for $\mathbb{H} \in \mathcal{H}(u)$. Consider $\mathbb{V} = A^\sharp(u) \in \mathcal{V}(u)$, for some $A \in \mathcal{G}$. We compute

$$
\begin{aligned}
\omega_{gp}((L_g)_* A^\sharp(u)) &= \omega_{gp}\left((L_g)_* \frac{d}{dt}(\exp(At)u)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(g\exp(At)u)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(g\exp(At)g^{-1}\cdot gu)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(ad_g(\exp(At))\ gu)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(\exp(\mathcal{A}d(g)At)\ gu)\Big|_{t=0}\right) \\
&= \omega_{gu}(\ (\mathcal{A}d(g)A)^\sharp\ gu) \qquad \text{(C.-92)}
\end{aligned}
$$

As $\omega(A^\sharp) = A$

$$\omega_{gu}(\ (\mathcal{A}d(g)A)^\sharp\ gu) = \mathcal{A}d(g)A$$

and then

$$\mathcal{A}d(g)A = (\mathcal{A}d(g))(\omega(A^\sharp(u)))$$

therefore we have

$$\omega_{gp}((L_g)_* A^\sharp(u)) = (\mathcal{A}d(g))(\omega(A^\sharp(u))),$$

that is, we have verified (C.-88) for vertical vectors $A^\sharp(u)$ as well.

□

In the next section provide the proof of the equivalence between of definitions (2) and (3).

C.14.7 Gauge Fields

We now make contact with the more familiar notion of gauge fields as used in physics, which live on $\mathcal{M}$ instead of P.

Theorem C.14.2 *Given a $\mathfrak{g}-$valued one-form A_i on U_i and a local section $s_i : U_i \to \pi^{-1}(U_i)$, there exists a connection one-form ω such that $A_i = s_i^* \omega$ on U_i*

Proof:

(2) $\Rightarrow$ (3):

.....

Let ω be a $\mathcal{G}-$valued connection one-form as given in definition (2), and $\phi_i : \pi^{-1}(U) \to U \times G$ be a local trivialisation. Associated with ϕ_i there is a local section given by $s_i(p) = \phi_i^{-1}(p, e)$. We define a $\mathcal{G}-$valued one-form A_i on U by

$$A_i = s_i^* \omega,$$

then we show that it is the connection one-form in the sense of definition (3). To do this we have to show that the gauge transformation

$$\omega_i \to \omega_j = s_j^* \omega$$

is given by (C.14.6).

....

Recall that we have local sections $s_i : U_i \to \pi^{-1}(U_i)$ associated canonically to the trivialisation of the bundle. g_i is the canonical local trivialisation defined by $\phi_i(u) = (p, g_i)$ for $u = s_i(p) g_i$. Let us define a $\mathfrak{g}-$valued one-form ω_i on P by

$$\omega_i = g_i^{-1} \pi^* A_i g_i + g_i^{-1} d_P g_i \tag{C.-92}$$

where d_P is the exterior derivative on P.

We first show that $s_i^* \omega_i = A_i$. For $X \in T_p M$. Note

$$s_i^* \omega(X) = \omega_i(s_{i*} X)$$

As $g_i = e$ at s_i

$$\begin{aligned} s_i^* \omega(X) &= \pi^* A_i(s_{i*} X) + d_P g_i(s_{i*} X) \\ &= A_i(\pi_{i*} s_{i*} X) + d_P g_i(s_{i*} X) \end{aligned} \tag{C.-92}$$

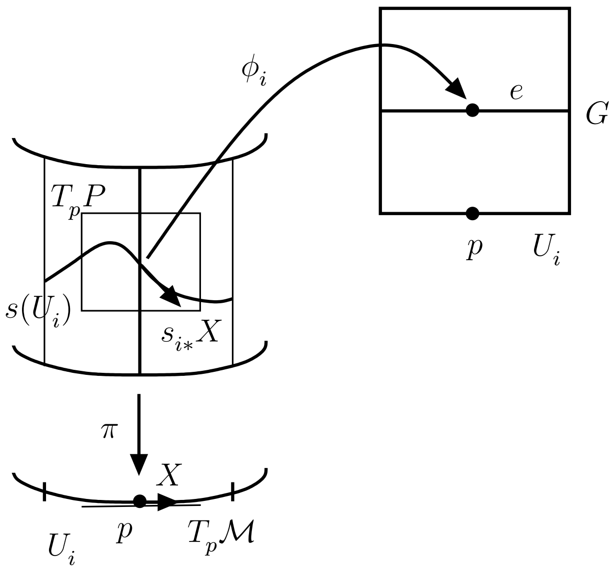


Figure C.45:

We have

$$d_P g_i(s_{i*}X) = \frac{d}{dt} g_i(s_i(t))\Big|_{t=0} = 0$$

as $g \equiv e$ along $s_i(t)$. Thus we have obtained $s_i^*\omega_i(X) = A_i(X)$.

Next we show that ω_i satisfies the axioms of a connection one-for given in definition...

We consider the subspace of T_uP tangent to the fibres. Let $X = A \in V_uP$, $A \in \mathfrak{g}$. It follows from $\pi_*X = 0$. Now we have from $g(u\exp(tA)) = g(u)\exp(tA)$

$$\begin{aligned}
\omega_i(A^\sharp) &= g_i^{-1}(d_P g_i)(A^\sharp) \\
&= g_i(u)^{-1}\frac{dg(u\exp(tA))}{dt}\Big|_{t=0} \\
&= g_i(u)^{-1}g_i(u)\frac{dg\exp(tA)}{dt}\Big|_{t=0} = A.
\end{aligned}$$

Take $X \in T_PP$ and $h \in G$. We have

$$\begin{aligned}
R_h^*\omega_i(X) &= \omega_i(R_{h*}X) \\
&= g_i^{-1}(uh)A_i(\pi_{i*}R_{i*}X)g_i(uh) + g_i^{-1}(uh)(d_P g_i(uh))(R_{h*}X)
\end{aligned}$$

Since $g_i(uh) = g_i(u)h$ and $\pi_{i*}R_{i*}X = \pi_{i*}X$ the first term above can be written

$$h^{-1}g_i^{-1}(u)A_i(\pi_{i*}X)g_i(u)h$$

For the second term we work out

$$\begin{aligned}
g_i^{-1}(uh)(d_P g_i(uh))(R_{h*}X) &= g_i^{-1}(uh)\frac{d}{dt}g_i(\gamma(t)h)\Big|_{t=0} \\
&= h^{-1}g_i^{-1}(u)\frac{d}{dt}g_i(\gamma(t))h \\
&= h^{-1}g_i^{-1}(u)(d_P g_i(u))(X)h.
\end{aligned}$$

Here $\gamma(t)$ is a curve through $u = \gamma(0)$, whose tangent vector at u is X. Therefore

$$R_h^*\omega_i(X) = h^{-1}\omega_i(X)h.$$

Hence the $\mathfrak{g}-$valued one-form ω_i defined by () satisfies $A_i = s_i^*\omega$ and the axioms of a connection one-form.

Global one-form on P?

Suppose we are given local one-forms A_i, A_j, etc., on $\mathcal{M}$ that satify the gauge transformation given in definition (3). First we will construct local one-forms ω_U, $\omega_V, \dots,$ on P from the local one-forms A_U,$A_V, \dots,$ on $\mathcal{M}$. We then need to show that $\omega_U = \omega_V$ on $\pi^{-1}(U \cap V)$, in order for the local one-forms ω_U, ω_V, etc., to collectively define a global connection one form ω as in definition (2).

.....

(3) $\Rightarrow$ (2):

We first define the map $\tilde{\omega}_i : T_uP \to \mathcal{G}$ by

$$\tilde{\omega}_i((s_i)_* X_p + A^\sharp) = A_i(X_p) + A. \tag{C.-100}$$

A local $\mathcal{G}-$valued one-form on all of $\pi^{-1}(U_i)$ can be constructed from this map via

$$\omega_i(\mathbb{X}_{gu}) = (\mathcal{A}d(g))\,\tilde{\omega}_i\,((L_{g^{-1}})_*\mathbb{X}_{gu}) \tag{C.-100}$$

for $\mathbb{X}_{gu} \in T_{gu}P$. This one-form reduces to the one form $\tilde{\omega}_i$ on T_uP for $g = e$. The one-form ω_i satifies condition (ii) of definition 2 on the restricted bundle $\pi^{-1}(U)$

$$
\begin{aligned}
\omega_{igu}((L_g)_*\mathbb{X}_u) &= \mathcal{A}d(g))\,\tilde{\omega}_{iu}\,((L_{g^{-1}})_*((L_g)_*\mathbb{X}_u)) \\
&= \mathcal{A}d(g))\,\tilde{\omega}_{iu}\,(\mathbb{X}_u) \\
&= \mathcal{A}d(g))\,\omega_{iu}\,(\mathbb{X}_u),
\end{aligned}
$$

and thus is a connection in the sense of definition (2) on this restricted bundle.

Suppose s_j is another local trivialisation, where $U_i \cap U_j \neq \emptyset$. We can similarly define a local connect ω_j on $s_{U_j}(U_i \cap U_j)$, as then they would agree on all of $\pi^{-1}(U \cap V)$ by virtue of (C.14.7). That is, we wish to show that

$$
\tilde{\omega}_i((s_j)_* X_p + A^\sharp) = \tilde{\omega}_j((s_j)_* X_p + A^\sharp)
$$

Now since $\omega_i(A^\sharp) = A = \omega_j(A^\sharp)$, it is sufficient to check that $\tilde{\omega}_i((s_j)_* X_p) = \tilde{\omega}_j((s_j)_* X_p)$, for $x \in U \cap V$ and $X_p \in T_p\mathcal{M}$.

suppose $\gamma(t)$ is a curve in $\mathcal{M}$ with $p = \gamma(0)$ and $X_p = \gamma'(0)$. We compute $(s_j)_*(X_p)$ by

$$
\begin{aligned}
(s_j)_*(X_p) &= \frac{d}{dt} s_j(\gamma(t))\Big|_{t=0} \\
&= \frac{d}{dt}\big(t_{ij}(\gamma(t)) s_j(\gamma(t))\big)\Big|_{t=0} \\
&= \frac{d}{dt}\big(t_{ij}(p) s_j(\gamma(t))\big)\Big|_{t=0} + \frac{d}{dt}\big(t_{ij}(\gamma(t)) s_j(p)\big)\Big|_{t=0} \\
&= (L_{t_{ij}(p)})_*(s_i)_*(X_p) + \frac{d}{dt}\big(t_{ij}(\gamma(t)) t_{ij}^{-1}(p) s_j(p)\big)\Big|_{t=0} \\
&= (L_{t_{ij}(p)})_*(s_i)_*(X_p) + \big((R_{t_{ij}^{-1}(p)})_*(t_{ij})_*(X_p)\big)^\sharp_{s_j(p)}.
\end{aligned}
$$

Using this and by using the gauge transformation formula we then have

$$
\begin{aligned}
\tilde{\omega}_i((s_j)_* X_p) &= \tilde{\omega}_i\Big(\big((R_{t_{ij}^{-1}(p)})_*(t_{ij})_*(X_p)\big)^\sharp_{s_j(p)} + (L_{t_{ij}(p)})_*(s_i)_*(X_p)\Big) \\
&= (R_{t_{ij}^{-1}(p)})_*(t_{ij})_*(X_p) + \mathcal{A}d(t_{ij})\,(A_i(X_p)) \\
&= \tilde{\omega}_i((s_j)_* X_p) = A_j(X_p) \qquad \text{(C.-109)}
\end{aligned}
$$

By () $\tilde{\omega}_j((s_j)_* X_p) = A_j(X_p)$ the proof is complete.

□

If the principal bundle is non-trivial, it is not generally possible to describe the connection ω on P in terms of a single Yang-Mills $A_\mu(x)$ field on $\mathcal{M}$. Instead, one must cover $\mathcal{M}$ with local trivalising charts, and then local Yang-Mills fields associated with any pair of overlapping charts U_i, U_j will be related on $U_i \cap U_j$ by

$$A_\mu^{(j)}(x) = \Omega(x)^{-1} A_\mu^{(i)}(x)\Omega(x) + \Omega(x)^{-1}\partial_\mu \Omega(x)$$

with the corresponding local gauge function $\Omega_{ij}(x)$ satifying the $s_i(x) = s_j(x)\Omega_{ij}(x)$. Note that these functions $\Omega_{ij} : U_i \cap U_j \to G$ are precisely the bundle transition functions.

Let $P(\mathcal{M}, G)$ be a principal bundle over $\mathcal{M}$ and U a chart of $\mathcal{M}$. Take two local sections s_1 and s_2 over U such that $s_2(p) = s_1(p)g(p)$. The corresponding local forms A_1 and A_2 are related as

$$A_2 = g^{-1}A_1 g + g^{-1}dg. \qquad \text{(C.-109)}$$

In components, this reads

$$A_{2\mu}(p) = g^{-1}(p)A_{1\mu}(p)g(p) + g^{-1}(p)\partial_\mu g(p). \qquad \text{(C.-109)}$$

which is simply the gauge transformation ().

Example Electrodynamics:

Let P be a U(1) bundle over $\mathcal{M}$. Take overlapping charts U_i and U_j. Let A_i (A_j) be a local connection form on U_i (U_j). The transition function $t_{ij} : U_i \cap U_j \to U(1)$ is given by

$$t_{ij}(p) = \exp(i\theta(p)) \qquad \theta(p) \in \mathbb{R}. \qquad \text{(C.-109)}$$

A_i and A_j are related by

$$\begin{aligned} A_j(p) &= t_{ij}(p)^{-1}A_i(p)t_{ij}(p) + t_{ij}(p)^{-1}dt_{ij}(p) \\ &= A_i(p) + id\theta(p) \qquad \text{(C.-109)} \end{aligned}$$

In components, we have

$$A_{j\mu} = A_i\mu + i\partial_\mu\theta. \qquad \text{(C.-109)}$$

□

Example General relativity:

The important example is a connection in the principal $GL(n,\mathbb{R})-$bundle $\mathbf{B}(\mathcal{M})$ of frames on a $n-$dimensional manifold $\mathcal{M}$. Any local coordinate chart (U,φ) on $\mathcal{M}$ provides a local section $s : U \to \mathbf{B}(\mathcal{M})$ by associating with $x \in U \subset \mathcal{M}$, the local frame $(\partial_1, \partial_2, \ldots, \partial_n)_x$. If ω is a connection one-form on $\mathbf{B}(\mathcal{M})$, let $\Gamma := s^*\omega$ denote the associated $\mathcal{L}(GL(n,\mathbb{R}))-$valued one-form on U, and consider the relation between Γ and the local one-form Γ' associated with another coordinate chart (U', φ') such that $U \cap U' \neq \emptyset$.

The local section of $\mathbf{B}(\mathcal{M})$ associated with s' is $(\partial/\partial x'^1, \ldots, \partial/\partial x'^m)$, and the transition function $J : U \cap U' \to GL(n,\mathbb{R})$ is just the Jacobian of the coordinatre transformations:

$$(\partial_{\mu'})_x = (\partial_\nu)_x J^\nu{}_\mu(x)$$

where

$$J^\nu{}_\mu(x) := \partial x^\nu / \partial x'^\mu.$$

Then,

$$\begin{aligned}
\Gamma'_\mu(x) &= (s^*\omega)_x(\partial_{\mu'}) \\
&= J^\alpha_\mu(x)(s'^*\omega)_x(\partial_\alpha) \\
&= J^\alpha_\mu(x)\big(J^{-1}(x)\Gamma_\alpha(x)J(x) + J^{-1}(x)\partial_\alpha J(x)\big)
\end{aligned} \tag{C.-110}$$

where we have used the result () for a connection in a bundle whose srtucture group is a matrix group. The second term reads

$$\begin{aligned}
J^\alpha_\mu(x)(J^{-1}(x)\partial_\alpha J(x))^\epsilon{}_\delta &= \frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial x'^\epsilon}{\partial x^\lambda}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\lambda}{\partial x'^\delta}\right)\right) \\
&= \frac{\partial x'^\epsilon}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\delta}.
\end{aligned}$$

If $G_\lambda{}^\rho$ is some basis for the Lie algebra $M(n,\mathbb{R})$ (the set of all $n \times n$ real matrices) of $GL(n,\mathbb{R})$ then we can write the matrix-valued one-form Γ_μ as

$$(\Gamma_\mu)^\epsilon{}_\delta = \Gamma^\lambda_{\mu\rho}(G_\lambda{}^\rho)^\epsilon{}_\delta.$$

The first term in () is then

$$
\begin{aligned}
J^{\alpha}{}_{\mu}(x)(J^{-1}(x)\Gamma_{\alpha}(x)J(x))^{\epsilon}{}_{\delta} &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\left(\frac{\partial x'^{\epsilon}}{\partial x^{\beta}}(\Gamma_{\alpha})^{\beta}{}_{\lambda}(x)\frac{\partial x^{\lambda}}{\partial x'^{\delta}}\right) \\
&= \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\left(\frac{\partial x'^{\epsilon}}{\partial x^{\beta}}(G_{\gamma}{}^{\rho})^{\beta}{}_{\lambda}\frac{\partial x^{\lambda}}{\partial x'^{\delta}}\right)\Gamma^{\gamma}_{\alpha\rho}(x).
\end{aligned}
$$

In particular, if we pick the natural basis set $(G_{\gamma}{}^{\rho})^{\beta}{}_{\lambda} = \delta^{\beta}_{\gamma}\delta^{\rho}_{\lambda}$ then () becomes the well known transformation law for the components $\Gamma^{\epsilon}_{\mu\delta}$ of an affine connection on $\mathcal{M}$:

$$
\Gamma'^{\epsilon}_{\mu\delta}(x) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\rho}}{\partial x'^{\delta}}\frac{\partial x'^{\epsilon}}{\partial x^{\gamma}}\Gamma^{\gamma}_{\alpha\rho}(x) + \frac{\partial x'^{\epsilon}}{\partial x^{\lambda}}\frac{\partial^2 x^{\lambda}}{\partial x'^{\mu}\partial x'^{\delta}}.
$$

□

C.14.8 Parallel Transport in a Principal Bundle

Parallel transport was defined as transport without change.

What is parallel transport of an element of a principal bundle along a curve in $\mathcal{M}$?

We use the notion of a horizontal lift.

Horizontal vector fields are fields whose flow lines move from one fibre into another.

Definition Let P be a principal fibre bundle and let $\gamma[0,1] \to \mathcal{M}$ be a curve in $\mathcal{M}$. A curve $\tilde{\gamma}[0,1] \to P$ is said to a horizontal lift of γ if $\pi\tilde{\gamma} = \gamma$ and the tangent vector to $\tilde{\gamma}(t)$ always belongs to $\mathcal{H}_{\tilde{\gamma}(t)}P$

□

C.14.9 Curvature on a Principal Bundle

Suppose a connection is given on a principal bundle $\pi : P \to \mathcal{M}$, with group G. Then $X \in T_u P$ can be written uniquely as the sum of a vertical and a horizontal vector

$$
D\varphi \equiv (d\varphi)^H
$$

where

$$(d\varphi)^H(\mathbb{X}_1,\ldots,\mathbb{X}_{i+1}) \equiv d\varphi(\mathbb{H}_1,\ldots,\mathbb{H}_{i+1})$$

and $\mathbb{H}_1,\ldots,\mathbb{H}_{i+1}$ are the horizontal vectors of $\mathbb{X}_1,\ldots,\mathbb{X}_{i+1} \in T_uP$.

Lemma C.14.3 *If $A, B \in \mathcal{G}$, the map $\sharp$ preserves the Lie algebra structure:*

$$[A^\sharp, B^\sharp] = [A,B]^\sharp \tag{C.-114}$$

Proof:

$$[A^\sharp, B^\sharp]_p = \lim_{t\to 0}\frac{1}{t}\{B^\sharp - (L_{g(t)})_* B^\sharp\}$$

$$\begin{aligned}[A^\sharp, B^\sharp]_p &= \lim_{t=0}\{(\sharp)_* B - (\sharp)_* \mathcal{A}d(g(t))B\} \\ &= (\sharp)_*\left(\lim_{t=0}\frac{1}{t}\{B - \mathcal{A}d(g(t))B\}\right)\end{aligned} \tag{C.-114}$$

$$\mathcal{A}d(g(t))B = (L_{g(t)})_*(R_{g^{-1}(t)})_* B.$$

by $(R_{g^{-1}(t)})_* B = B$,

$$\mathcal{A}d(g(t))B = (L_{g(t)})_* B$$

$$[A^\sharp, B^\sharp]_p = (\sharp_p)_*\left(\lim_{t\to 0}\frac{1}{t}\{B - (L_{g(t)})_* B\}\right) = (\sharp_p)_*([A,B]) = [A,B]^\sharp(p),$$

□

C.14.10 Extension and Reduction of Principal Bundles

Given any bundle E we can construct a principal bundle $P(E)$, by replacing the fibres in E with the transition functions, while keeping the transition functions the same.

Let $H \subset G$ be a closed subgroup. P is reduced to a principal H bundle Q if:

(i) $Q \subset P$ is a submanifold,

(ii) $qh \in Q$ for all $q \in Q$, $h \in H$, such that

(1) $\pi(Q) = M$,

(2) H acts transitively in each fibre $Q_x = \pi^{-1}(x) \cap Q$, (remember $\pi^{-1}(x) \subset P$); by acts transitively we mean given any two points $p, q \in Q_x$ there exists at least one $h \in H$ such that $hp = q$.

$$\{v \in H(P)| < \omega|v >= 0\} \tag{C.-114}$$

$$< dx^\nu|\frac{\partial}{\partial x^\mu} >= \delta_{\mu\nu}, \quad < dg_{kj}|\frac{\partial}{\partial g_{lm}} >= \delta_{kl}\delta_{jm}, \quad < dx^\nu|\frac{\partial}{\partial g_{lm}} >=< dg_{kj}|\frac{\partial}{\partial x^\mu} >= 0. \tag{C.-114}$$

$$\begin{aligned} < \omega_{ij}|X_\mu > &= < i(g^{-1})_{ik}dg_{kj} + (g^{-1}A^a_\nu T^a g)_{ij}dx^\nu|\frac{\partial}{\partial x^\mu} + iB_{\mu lm}\frac{\partial}{\partial g_{lm}} > \\ &= (ig^{-1})_{ik} < dg_{kj}|\frac{\partial}{\partial g_{lm}} > iB_{\mu lm} + (g^{-1}A^a_\nu T^a g)_\nu < dx^\nu|\frac{\partial}{\partial x^\mu} > \\ &= (g^{-1}A^a_\mu T^a g - g^{-1}B_\mu)_{ij} \end{aligned} \tag{C.-115}$$

Hence

$$B_{\mu ij} = \left(A^b_\mu T^b g\right)_{ij}. \tag{C.-115}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} + i\left(A^b_\mu T^b g\right)_{ij}\frac{\partial}{\partial g_{ij}} \tag{C.-115}$$

$$D_\mu(gWg^{-1}) = \frac{\partial}{\partial x^\mu}\tilde{W} + iA^b_\mu T^b gWg^{-1} - igWg^{-1}A^b_\mu T^b = [\partial_\mu + A^a_\mu T^a, \tilde{W}], \tag{C.-115}$$

$$ig^{-1}\mathbf{d}g + g^{-1}\mathbf{A}g = ig^{-1}h^{-1}\mathbf{d}(hg) + g^{-1}h^{-1}\mathbf{A}'hg. \tag{C.-115}$$

This gives

$$\mathbf{A}' = -i\mathbf{d}hh^{-1} + h\mathbf{A}h^{-1} \tag{C.-115}$$

$$[D_\mu, D_\nu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu) g\frac{\partial}{\partial g} - A^a_\mu A^b_\nu \left(T^a g \frac{\partial}{\partial g} T^b g \frac{\partial}{\partial g} - T^b g \frac{\partial}{\partial g} T^a g \frac{\partial}{\partial g} \right). \qquad \text{(C.-115)}$$

$$[D_\mu, D_\nu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]) g\frac{\partial}{\partial g} = iF_{\mu\nu} g\frac{\partial}{\partial g} \qquad \text{(C.-115)}$$

Thus we see that the curvature of the principal bundle is associated with the field strength tensor.

C.14.11 The Complex Line Bundle

$$z = a + ib \quad \leftrightarrow \quad a\mathbf{e_1} + b\mathbf{e_2}, \text{ or in component form } \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{(C.-115)}$$

We multiply a complex number $z = a + ib$ by i, $-b + ia$

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad J^2 = -I \qquad \text{(C.-115)}$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{(C.-115)}$$

$$iz = -b + ia \quad \leftrightarrow \quad \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{(C.-114)}$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y) \qquad \text{(C.-114)}$$

Of course, this can be expressed entirely in real terms

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \circ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix} \qquad \text{(C.-114)}$$

C.15 Summary of Differential Geometry

- Coordinate independent formalism.
- Anti-symmetric objects are important - unifying notations.

C.16 Some Algebraic Geomerty

C.16.1 Homology

boundaries $\subset$ cycles .

We are interested in things that do not have boundaries, but are not themselves boundaries of anything. In other words, we are interested in chains that are elements of $Z_n(K)$ but not in $B_n(K)$.

We define the $n-th$ Homology group, denoted $H_n(K)$, as

The following theorem simplifies calculations of Homology groups.

$$H_n(K) \equiv Z_n(K)/B_n(K). \tag{C.-114}$$

Lemma C.16.1 *Let $f : G_1 \to G_2$ be a homomorphism.*

(i) $ker(f) = \{x : x \in G_1, f(x) = 0\}$ is a subgroup of G_1

(ii) $im(f) = \{x : x \in f(G_1) \subset G_2\}$ is a subgroup of G_2.

Proof: (i) Let $x, y \in ker(f)$. Then $x+y \in ker(f)$ since $f(x+y) = f(x)+f(y) = 0+0 = 0$. Now $0 \in ker(f)$ as $f(0) = f(0+0) = f(0) + f(0)$. We also have $-x \in ker(f)$ since $f(0) = f(x-x) = f(x) + f(-x) = 0$.

(ii) Let $y_1 = f(x_1), y_2 = f(x_2) \in im(f)$ where $x_1, x_2 \in G_1$. Since f is a homomorphism we have $y_1 + y_2 = f(x_1) + f(x_2) = f(x_1 + x_2) \in im(f)$. Clearly $0 \in im(f)$ since $f(0) = 0$. If $y = f(x)$, $-y \in im(f)$ since $0 = f(x-x) = f(x) + f(-x)$ implies $f(-x) = -y$.

□

Note that $G_1/ker(f)$ is a quotient group. In fact we have the following.

Theorem C.16.2 Fundamental theorem of homomorphisms. *Let $f : G_1 \to G_2$ be a homomorphism. Then*

$$G_1/ker(f) \simeq im(f) \tag{C.-114}$$

where $im(f)$ is the image of f in G_2.

Proof: Both sides are groups according to lemma C.16.1. Define a map $\varphi : G_1/ker(f) \to im(f)$ by

$$\varphi([x]) = f(x)$$

This map is well defined since for $x' \in [x]$, there exists $h \in ker(f)$ such that $x' = x + h$ and

Now we show that φ is an isomorphism. Firstly φ is a homomorphism,

$$\begin{aligned}\varphi([x]+[y]) &= \varphi([x+y]) \\ &= f(x+y) \\ &= f(x)+f(y) = \varphi([x]) + \varphi([y]).\end{aligned} \tag{C.-115}$$

Next we show that it is one-to-one. If $\varphi([x]) = \varphi([y])$, then $f(x) = f(y)$ or $f(x) - f(y) = f(x-y) = 0$. This shows that $x - y \in ker(f)$ and $[x] = [y]$. Finally, we show φ is onto. If $y \in im(f)$, there exists $x \in G_1$ such that $f(x) = y = \varphi([x])$.

□

If we can express $B_n(K)$ as the kernel of some map f, then

$$H_n(K) = Z_n(K)/B_n(K) = Z_n(K)/\ker(f) \simeq im(f) \tag{C.-115}$$

which is generally much easier to calculate.

Examples

Let $K = \{p_0, p_1, (p_0p_1)\}$. We have

$$\begin{aligned}C_0(K) &= \{ip_0 + jp_1 : i, j \in \mathbb{Z}\} \\ C_1(K) &= \{k(p_0p_1) : k \in \mathbb{Z}\}.\end{aligned}$$

Since (p_0p_1) is not the boundary of any simplex in K, $B_1(K) = 0$, and

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(K).$$

If $z = m(p_0p_1) \in Z_1(K)$, it satifies

$$\partial_1 z = m\partial_1(p_0p_1) = m\{p_1 - p_0\} = mp_1 - mp_0 = 0.$$

Thus we must have $m = 0$ and so $Z_1(K) = 0$, hence

$$H_1(K) = \{0\} \tag{C.-117}$$

Theorem C.16.3 *Let K be a connected simplicial complex. Then*

$$H_0(K) = \mathbb{Z}. \tag{C.-117}$$

Proof: Since K is connected, for any pair of 0-simplexies p_i and p_j, there exists a sequence of 1-simplexies $(p_ip_k), (p_kp_l), \ldots, (p_mp_j)$ such that $\partial_1((p_ip_k)+(p_kp_l)+\cdots+(p_mp_j)) = p_i - p_j$. It follows

$$p_i = p_j + \partial_1(p_{ij}).$$

where p_{ij} is a 1-simplex.

Suppose

$$z = \sum_i^{I_0} n_i p_i$$

where I_0 is the number of 0-simplexies in K. Then

$$\begin{aligned} z &= \sum_i^{I_0} n_i p_i = \sum_i^{I_0} n_i p_1 + \sum_i^{I_0} n_i \partial_1(p_{1j}) \\ &= \sum_i^{I_0} n_i p_1 + \partial_1[\sum_i^{I_0} n_i p_{1j}] \end{aligned} \tag{C.-117}$$

It is clear that $z \in B_0(K)$ if $\sum n_i = 0$.

Let $\sigma_j = (p_{j,1}p_{j,2})$ $(1 \leq j \leq I_1)$ be 1-simplexies in K, I_1 being the number of

$$\begin{aligned} B_0(K) &= im\partial_1 \\ &= \{\partial_1[n_1\sigma_1 + \cdots + n_{I_1}\sigma_{I_1}] : n_1, \ldots, n_{I_1} \in \mathbb{Z}\} \\ &= \{n_1(p_{1,2} - p_{1,1}) + \cdots + n_{I_1}(p_{I_1,2} - p_{I_1,1}) : n_1, \ldots, n_{I_1} \in \mathbb{Z}\} \end{aligned} \tag{C.-118}$$

$$z = \sum_i n_i p_i$$

We have proved for a connected complex K that $z = \sum n_i p_i \in B_0(K)$ if and only if $\sum n_i = 0$.

Define a surjective homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(n_1 p_1 + \cdots + n_{I_0} p_{I_0}) = \sum_{i=1}^{I_0} n_i.$$

We then have $ker(f) = f^{-1}(0) = B_0(K)$. From theorem , it follows that $H_0(K) = Z_0(K)/B_0(K) = Z_0(K)/ker(f) = \mathbb{Z}$.

□

If also

$$H_1(K) = \{0\} \tag{C.-118}$$

then it is a simply connected complex K.

Meaning of the $nth-$homology group

$$H_0(K) = \mathbb{Z} \tag{C.-118}$$

for any connected complex K.

C.16.2 De'Rham Cohomology

Two gauge potentials are gauge equivalent if they differ by a gauge potential

$$B_\mu(x) = A_\mu(x) + \partial_\mu \phi(x) \tag{C.-118}$$

we write

$$A_\mu(x) \sim B_\mu(x) = A_\mu(x) + \partial_\mu \phi(x) \tag{C.-118}$$

We denote a gauge equivalence class $[A_\mu(x)]$ where $A_\mu(x)$ is a representative of the equivalence class. Say the gauge potentials $A_\mu(x)$ and $A'_\mu(x)$ are not related to each by a gauge transformation then their sum

$$A''_\mu \equiv A_\mu + A'_\mu \tag{C.-118}$$

A''_μ cannot be related to either A_μ or A'_μ by a gauge transformation and hence belongs to another distinct equavalence class, $[A''_\mu]$. $[A''_\mu] = [A_\mu] + [A'_\mu]$. With this definition class addition, it is not hard to see that these classes form an addative abelian group with the identity $[\partial_\mu \phi(x)]$. We need to establish this we need to show that that the addittion operation is independent representative choosen.

Also can have $aA_\mu + bA'_\mu$ where a and b are real numbers.

$$\epsilon^{\mu\nu}\partial_\nu\partial_\mu\phi(x) \equiv 0 \tag{C.-118}$$

In differential geometry notation Eq (C.16.2) is written

$$A \sim B = A + d\phi \tag{C.-118}$$

$$H^p(\mathcal{M}) = \frac{Z^p(\mathcal{M})}{B^p(\mathcal{M})} \tag{C.-118}$$

Two closed forms (elements of $Z^p(\mathcal{M})$) define the same cohomology class (elements of $H^p(\mathcal{M})$) if they differ by an exact form (an element of $B^p(\mathcal{M})$).

Consider all $r-$forms λ that satisfy $d\lambda = 0$. Such forms are called closed. Certainly forms that satisfy $\lambda = d\Phi$ are themselves closed bacause of the niloptence of d. Such forms are called exact. not all closed forms are exact. We will denote $Z^r(\mathcal{M})$ refers to all closed forms and $B^r(\mathcal{M})$ refers to all exact $r-$forms.

we consider all closed forms modded out by exact forms. In other words two forms are said to be equivalent if

$$\lambda_1 = \lambda_2 + d\Phi \quad \rightarrow \quad [\lambda_1] = [\lambda_2] \tag{C.-118}$$

for any Φ. two closed forms are called cohomology. Elements of the cohomology are the closed forms modded out by all exact forms. This we write as

$$H^r(\mathcal{M}, \mathbb{R}) = Z^r(\mathcal{M})/B^r(\mathcal{M}), \tag{C.-118}$$

Definition (i) An n-form is called a cocycle if it is closed, that is, $d\omega = 0$.

(ii) An n-form is called a coboundary if it is exact, that is, if there is an $(n-1)$-form σ such that $\omega = d\sigma$.

(iii) The vector spaces of closed (exact) n-forms are denoted $Z^n(\mathcal{M})$ and $B^n(\mathcal{M})$ respectively.

(iv) The de Rham cohomology group (under addition) is given by

$$H^n(\mathcal{M}) := Z^n(\mathcal{M})/B^n(\mathcal{M}).$$

$\square$

Examples

(i) We first take the example $\mathcal{M} = \mathbb{R}$. First let us find $H^0(\mathbb{R})$. The set $B^0(\mathbb{R})$ has no meaning since there are no (-1)-form. We define $\Omega^{-1}(\mathcal{M})$ to be empty, hence $H^0(\mathbb{R}) = Z^0(\mathbb{R})$. The set $Z^0(\mathbb{R})$ is the set of all 0-forms f that are closed, i.e. $df = 0$. The only way to have $df = \frac{df}{dx}dx = 0$ is if f is constant and so the set of 0-forms are isomorphic to $\mathbb{R}$. So

$$H^0(\mathbb{R}) = Z^0(\mathbb{R})/B^0(\mathbb{R}) \simeq \mathbb{R}/\{0\} = \mathbb{R}.$$

In 1-dimension a 1-form is $f(x)dx$, if we take the differential operator we get

$$d(f(x)dx) = \frac{\partial f}{\partial x} dx \wedge dx = 0.$$

So in 1-dimension, any 1-form on $\mathbb{R}$ is closed. Furthermore, we can take any 1-form fdx and integrate it:

$$F = \int f dx$$

so that

$$f = dF.$$

Therefore all 1-forms are exact. So, all one forms are closed and all one forms are exact therefore the quotient is the identity:

$$H^1(\mathbb{R}) = Z^1(\mathbb{R})/B^1(\mathbb{R}) = \{0\}.$$

Obviously there can be no 2-forms, 3-forms, ... etc in one dimension so that $H^n(\mathbb{R}) = 0$ for $n \geq 2$.

Summarising:

$$\begin{aligned} H^0(S^1) &= \mathbb{R} \\ H^1(S^1) &= \{0\} \\ H^n(S^1) &= 0, \quad n \geq 2. \end{aligned} \qquad \text{(C.-119)}$$

(ii) $\mathcal{M} = S^1$. Let $S^1 = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. Again the only way to have $df = \frac{df}{d\theta}d\theta = 0$ is if f is constant and so the set of 0-forms are isomorphic to $\mathbb{R}$. Therefore

$$H^0(S^1) = \mathbb{R}.$$

This is always the case when $\mathcal{M}$ is connected. Next we compute $H^1(S^1)$. Let $\omega = f(\theta)d\theta$ be a 1-form. Is it possible to, as in the previous example, to write any closed form as dF? If $\omega = dF$, then F must be given by

$$F(\theta) = \int_0^\theta f(\theta')d\theta'.$$

Any such F must be uniquely defined on S^1. For this to be the case F must satisfy the periodicity: $F(\theta) = F(\theta + 2\pi)$. $F(2\pi) = F(0) = \int_0^0 f(\theta')d\theta' = 0$. Namely F must satisfy

$$F(2\pi) = \int_0^{2\pi} f(\theta')d\theta' = 0.$$

Conversely, let ω is a closed form whos integration is zero. Define

$$g(\theta) = c + \int_0^{2\pi} \omega$$

We have

$$\begin{aligned} g(\theta + 2\pi) &= c + \int_0^{\theta+2\pi} \omega \\ &= c + \int_0^{\theta} \omega + \int_\theta^{\theta+2\pi} \omega \\ &= g(\theta) + \int_0^{2\pi} \omega = g(\theta). \end{aligned} \tag{C.-120}$$

Therefore g is well defined with $\omega = dg$. We find that a one form is exact if and only if its integration is zero. As with the computation of homology groups, we can employ the fundamental theorem of homomorphisms to simplify the calculation. Let us define the map $\lambda : \Omega^1(S^1) \to \mathbb{R}$

$$\lambda : \omega = f d\theta \mapsto \int_0^{2\pi} f(\theta')d\theta'$$

We then have $\ker(\lambda) = \lambda^{-1}(0) = B^1(S^1)$. By theorem (C.16.2),

$$H^1(S^1) = \Omega^1(S^1)/ker(\lambda) \simeq im(\lambda) = \mathbb{R}.$$

Just as in the case of $\mathcal{M} = \mathbb{R}$ there can be no 2-forms, 3-forms, ... etc in one dimension so that $H^n(S^1) = 0$ for $n \geq 2$. Summarising:

$$\begin{aligned} H^0(S^1) &= \mathbb{R} \\ H^1(S^1) &= \mathbb{R} \\ H^n(S^1) &= 0, \quad n \geq 2. \end{aligned} \tag{C.-121}$$

(iii) $\mathcal{M} = S^2$.

$$\begin{aligned} H^0(S^2) &= \mathbb{R} \\ H^1(S^2) &= \{0\} \\ H^2(S^2) &= \mathbb{R} \\ H^n(S^2) &= \{0\}, \quad n \geq 3 \end{aligned} \tag{C.-123}$$

(iv) $\mathcal{M} = S^n$.

$$
\begin{aligned}
H^0(S^1) &= \mathbb{R} \\
H^n(S^1) &= \mathbb{R} \\
H^j(S^1) &= \{0\}, \quad j \neq 0, n
\end{aligned}
\tag{C.-124}
$$

□

Meaning of $nth-$cohomology group

The Duality of Homology and Cohomology

As the name suggests, the cohomology group is a dual space of the homology group.

To do with how Stokes' theorem can be used to study those properties of a manifold which determine the relation between closed and exact forms.

Stokes' theorem establishes a precise duality between $H^n(\mathcal{M})$ and $H_n(\mathcal{M})$ via

$$
(\omega, \mathcal{M}) := \int_{\mathcal{M}} \omega \tag{C.-124}
$$

$$
\begin{aligned}
\int_{\mathcal{M}} \omega &= (d\omega, \mathcal{M}) \\
\int_{\partial\mathcal{M}} \omega &= (\omega, \partial\mathcal{M})
\end{aligned}
\tag{C.-124}
$$

$$
(d\omega, \mathcal{M}) = (\omega, \partial\mathcal{M}) \tag{C.-124}
$$

C.16.3 Cohomology and Homology

One can determine if two principals are equivalent or not with the use of something called *characteristic classes.*

second chern class

Roughly speaking, the cohomology $H^p(M, \mathbb{R})$ counts the number of noncontractable p-dimensional surfaces in M.

[385])

$$b^p = \dim H^p(M) = \dim H_p(M). \tag{C.-124}$$

$$\Omega(c_p, \omega_q) = \int_{c_p} \omega_q. \tag{C.-124}$$

Ω period matrix.

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i. \tag{C.-124}$$

Definition The set U is smoothly contractible to p_o if there exists a smooth map $F : U \times I \to U$ such that

$$F(x,0) = x, \quad F(x,1) = p_0 \ \text{ for } \ x \in U.$$

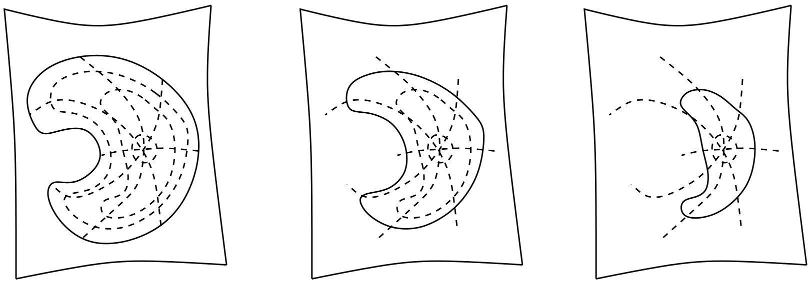


Figure C.46: .

$\square$

Theorem C.16.4 (Poincare's lemma). *If a coordinate neighbourhood U of a manifold M is contractible to a point $p_0 \in M$, any closed r-form on U is also exact.*

Proof: We may choose coordinates such that $x(p_0) = 0$ and use $F(x,t) := tx$. We claim that

$$\sigma(p) := \int_0^1 dt \frac{t^{n-1}}{(n-1)!} \, x^\nu \omega_{\nu\mu_2\ldots\mu_n}(tx(p)) \, dx^{\mu_2}(p) \wedge \cdots \wedge dx^{\mu_n}(p) \tag{C.-124}$$

satifies $d\sigma = \omega$. We compute

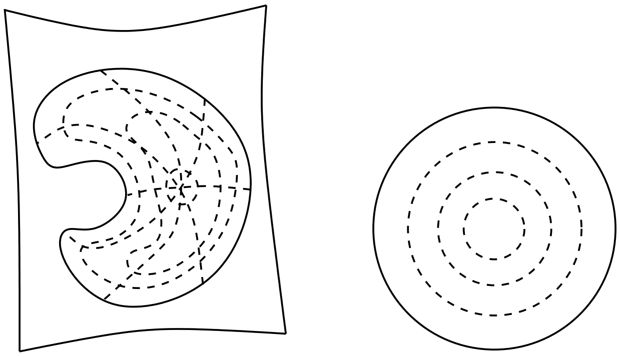


Figure C.47: .

$$
\begin{aligned}
d\sigma(p) &= \int_0^1 dt \frac{t^{n-1}}{(n-1)!}\left[\omega_{\mu_1\ldots\mu_n}(tx(p)) + t^n x^\nu(p)(\partial_{\mu_1}\omega_{\nu\mu_2\ldots\mu_n})(tx(p))\right] \\
&\quad \times dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p). \qquad \text{(C.-124)}
\end{aligned}
$$

As the r-form ω is closed we have $\partial_{[\nu}\omega_{\mu_1\ldots\mu_n]} = 0$, from which we see that

$$
\begin{aligned}
0 &= (\partial_{[\nu}\omega_{\mu_1\ldots\mu_n]})dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \frac{1}{n!}[\partial_\nu\omega_{\mu_1\ldots\mu_n} - \partial_{\mu_1}\omega_{\nu\mu_2\ldots\mu_n} + \partial_{\mu_2}\omega_{\nu\mu_1\ldots\mu_n} - \partial_{\mu_3}\omega_{\nu\mu_1\mu_2\ldots\mu_n} + \ldots] \\
&\quad \times dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \frac{1}{n!}[\partial_\nu\omega_{\mu_1\ldots\mu_n} - n\partial_{\mu_1}\omega_{\nu\ldots\mu_n}]dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p). \qquad \text{(C.-126)}
\end{aligned}
$$

Using this, (C.-124) simplifies to

$$
\begin{aligned}
d\sigma(p) &= \int_0^1 dt\frac{1}{n!}\left[nt^{n-1}\omega_{\mu_1\ldots\mu_n}(tx(p)) + t^n x^\nu(p)(\partial_\nu\omega_{\mu_1\ldots\mu_n})(tx(p))\right] \\
&\quad \times dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \left[\int_0^1 dt\frac{1}{n!}\frac{d}{dt}[t^n\omega_{\mu_1\ldots\mu_n}(tx(p))]\right] dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \left[\frac{1}{n!}[t^n\omega_{\mu_1\ldots\mu_n}(tx(p))]\right]_0^1 dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \omega(p). \qquad \text{(C.-129)}
\end{aligned}
$$

□

The theroem shows that every closed form is locally exact. The de Rham cohomology obstructs the global exactness of closed forms and giving us global information about M. For instance, if M is not simply connected (not all points are contractable to a point - homology is non-trivial) then the cohomology will be non-trivial.

C.16.4 Homotopy

two spaces *homeomorphic* need invariant so that spaces that are homeomorphic have the same result

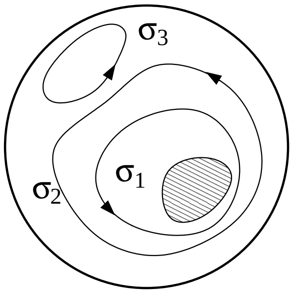


Figure C.48: σ_1 and σ_2 are homotopic to each other but not to σ_3.

$$(a \cdot b)(t) := \begin{cases} a(t), & 0 \leq t \leq \|a\| \\ b(t - \|a\|) & \|a\| \leq t \leq \|a\| + \|b\| \end{cases} \tag{C.-129}$$

The product of two paths is not defined if the terminal point of the first is the same as the intial point of the other.

For any path α, we denote by α^{-1} the *inverse path* formed by transversing α in the opposite direction.

Classes of Paths and Loops

Homotopy1.fig

Figure C.49: Homotopy1A.

Proposition C.16.5 *The relation $\sim$ is*

*(i) $a \sim a$ (*reflexive*)*

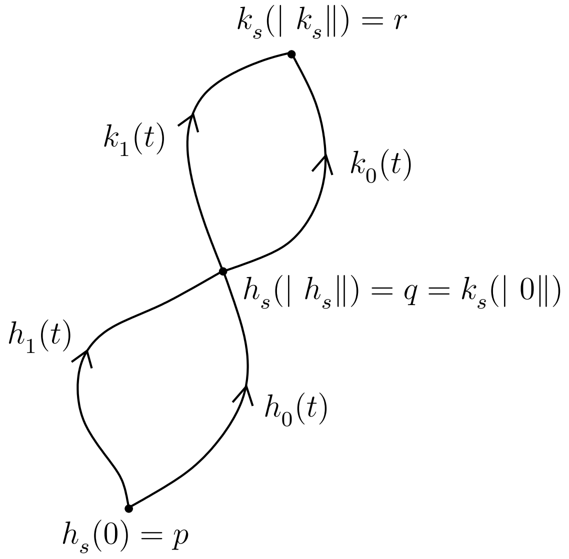


Figure C.50: Homotopy2. The product of two equivalence classes of paths $[a]$ and $[b]$.

*(ii) $a \sim b$ implies $b \sim a$ (*symmetric*)*

*(iii) $a \sim b$ and $b \sim c$ imply $a \sim c$ (*transitive*).*

$$j_s(t) = \begin{cases} h_{2s}(t), & 0 \le s \le \frac{1}{2}, \\ k_{2s-1}(t), & \frac{1}{2} \le s \le 1, \end{cases} \tag{C.-129}$$

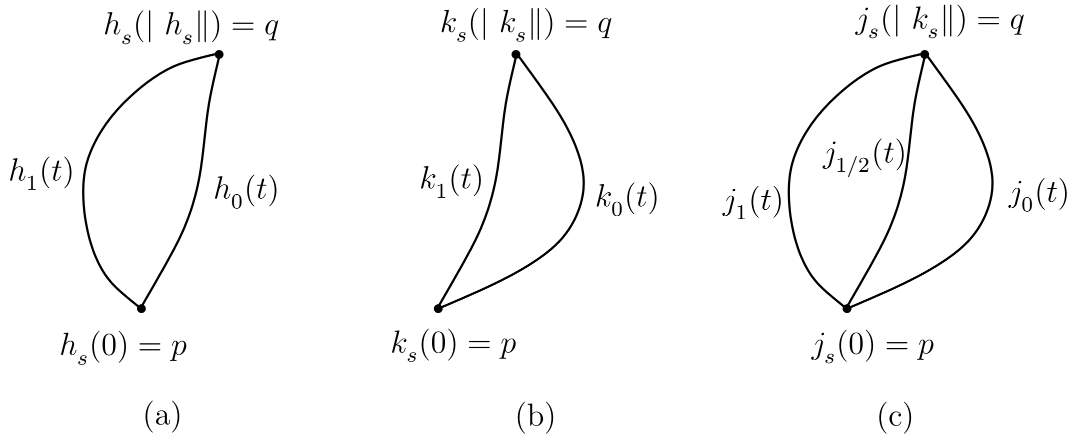


Figure C.51: transitivity. (a) $0 \le s \le 1$. (b) $0 \le s \le 1/2$. (c) $1/2 \le s \le 1$.

paths a and b are equivalent if and only if one can be continuously deformed into the other in X. $a \sim b$. We denote the equivalence class containing the path a by $[a]$.

$$[a] = [b] \;\; if\ and\ only\ if \;\; a \sim b \tag{C.-129}$$

The collection of all equivalence classes of paths in a topological space X is called the *fundamental groupoid*, denoted $\Gamma(X)$.

The fundamental Homotopy group

Proposition C.16.6 *For any paths a, a', b, b' in X, if $a \cdot b$ is defined and $a \sim a'$ and $b \sim b'$, then $a' \cdot b'$ is defined and $a \cdot b \sim a' \cdot b'$.*

$$[a] \cdot [b] = [a \cdot b]. \tag{C.-129}$$

Proposition C.16.7 *For any path a in X, there exist identity paths e_1 and e_2 such that $a \cdot a^{-1} \sim e_1$ and $a^{-1} \cdot a \sim e_2$.*

$$\begin{aligned} h_0 &= e_1 \\ h_1(t) &= \begin{cases} a(t), & 0 \le t \le \|a\| \\ a(2\|a\| - t), & \|a\| \le t \le 2\|a\| \end{cases} \\ &= \begin{cases} a(t), & 0 \le t \le \|a\| \\ a^{-1}(t - \|a\|), & \|a\| \le t \le 2\|a\| \end{cases} \\ &= (a \cdot a^{-1})(t), \end{aligned} \tag{C.-131}$$

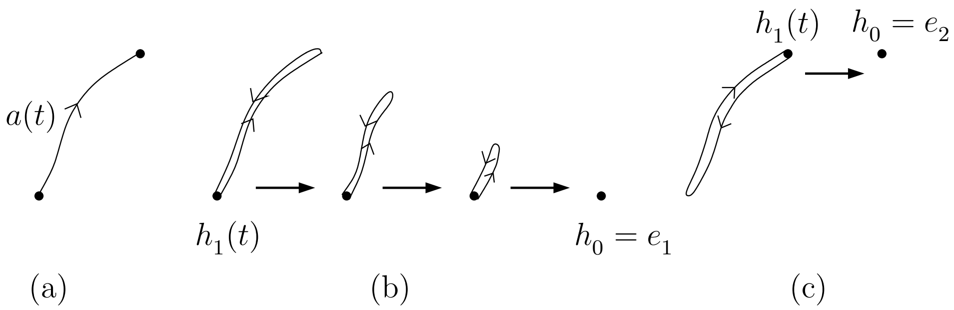


Figure C.52: (a) $a(t)$. (b) For decreasing s, $a \cdot a^{-1} \sim e_1$. (c) $a^{-1} \cdot a \sim e_2$.

Proposition C.16.8 *The set $\pi(X, p)$, together with the multiplication defined, is a group.*

This is the *fundamental group of X relative to the basepoint p.*

Sets and a binary operation. The minimal requirement

Type	identity	inverse	associative	closed	all products defined
group	yes	yes	yes	yes	yes
groupoid	no	no	no	yes	no
semi-group	no	no	yes	yes	
monoid	yes	no	yes	yes	

Obviously a group qualifies as a semi-group, a monoid and groupoid. A monoid is a semi-group with identity.

C.16.5 Homeomorphisms

So that all surfaces that can be obtained from rectangle and gluing together are homeomorphic.

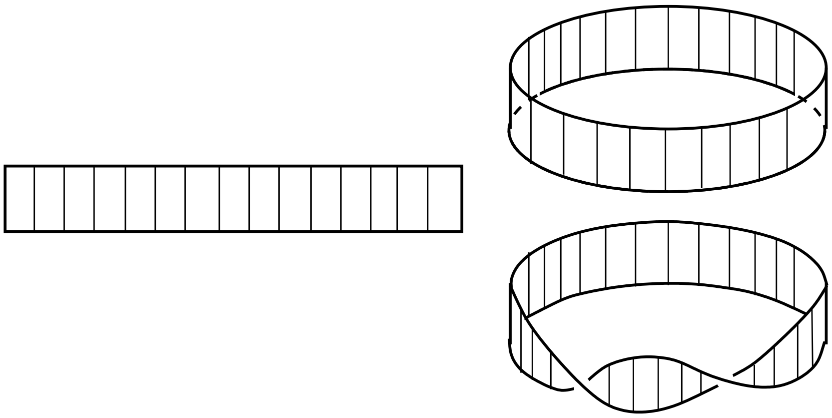


Figure C.53: HomeoFigDef.

Two genometric objects are homeomorphic if there exists a continuous one-to-one mapping $f : X \to Y$ such that the inverse mapping $f^{-1}Y :\to X$ is also continuous.

Twist homeomorphisms

canonical homology basis $\{a_1, a_2, b_1, b_2\}$

C.17 Summary

- Tensor Calculus
- Group theory

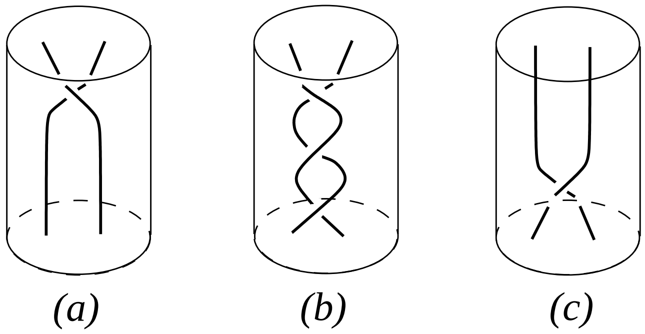


Figure C.54: homeoClylind.

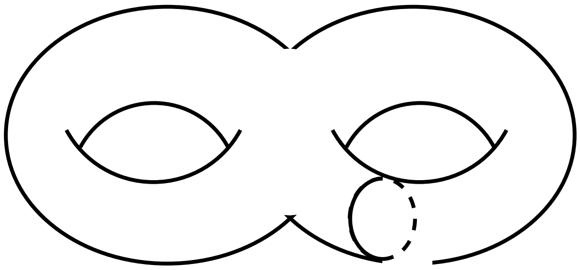


Figure C.55: homDehnfig1.

C.18 Biblioliographical notes

In this chapter I have relied on the following references:

Dana P. Williams notes on the Spectral Theorem for bounded normal operators

Introduction to topology and modern analysis G.F Simons.

Manifolds at Dr Neil Lambert Department of Mathematics King's College

C.19 Worked Exercises and Details

Commutativity of exterior derivative and $\mathcal{L}_\omega$.

Use the fact that $X^a_{;\,b}$ transforms as a tensor to deduce the transformation law of the connection given by (C.6.1).

Proof. $\Gamma^a_{cb} X^c = X^a_{;\,b} - X^a_{,b}$ we get

$$\Gamma'^a_{cb} X'^c = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\,\Gamma^d_{fe} X^f - \frac{\partial x'^b}{\partial x^e}\frac{\partial^2 x^a}{\partial x'^f \partial x'^b} X^f. \tag{C.-131}$$

Expressing X'^a in terms of X^f and since X^f is an arbitrary vector field, we have

$$\Gamma'^a_{cb}\frac{\partial x'^c}{\partial x^f} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\,\Gamma^d_{fe} - \frac{\partial x'^b}{\partial x^e}\frac{\partial^2 x^a}{\partial x'^f \partial x'^b}. \qquad \text{(C.-131)}$$

Multiplying through by $\partial x^c/\partial_g$ and rearranging the indices, we arrive at

$$\Gamma'^a_{cb} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^e}{\partial x'^c}\,\Gamma^d_{fe} - \frac{\partial x'^b}{\partial x^e}\frac{\partial x^e}{\partial x'^c}\frac{\partial^2 x^a}{\partial x'^f \partial x'^b}. \qquad \text{(C.-131)}$$

Commutativity of exterior derivative and $\mathcal{L}_\omega$.

C.19.1 Dynamical and Non-Dynamical Symmetries

Dynamical symmetries constrain the solutions of the equations of motion. *Non-Dynamical symmetries* are redundantcies of the mathematical formulation of the theory - these are often referred to as *Gauge symmetries*.

Example:

(a) Dynamical symmetry: Time translation invariance.

The Lagrangian does not explicitly depend on time

$$\mathcal{L} = \mathcal{L}(q_n, \dot{q}_n). \qquad \text{(C.-131)}$$

From this it follows that

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q_n}\dot{q}_n + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\ddot{q}_n \qquad \text{(C.-131)}$$

Now, at this point we make use of *the Euler-Lagrange equations of motion*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right) - \frac{\partial \mathcal{L}}{\partial q_n} = 0 \qquad \text{(C.-131)}$$

one finds

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \dot{q}_n \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right) + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\ddot{q}_n \\
&= \frac{d}{dt}\left(\dot{q}_n \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right) \qquad \text{(C.-131)}
\end{aligned}
$$

or

$$
\frac{d}{dt}\left(\dot{q}_n \frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \mathcal{L}\right) \qquad \text{(C.-131)}
$$

That is, the Hamiltonian is conserved in time

$$
\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L}. \qquad \text{(C.-131)}
$$

This reflects the conservation of energy E for isolated systems. Note that, as we needed to use the equations of motion this condition only holds for trajectories that extremalize the action.

(b) Non-dynamical symmetry: reparameterization invariance.

$$
\begin{aligned}
S[q(t+\epsilon)] - S[q(t)] &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t}\epsilon + \frac{\partial \mathcal{L}}{\partial q^i}\dot{q}^i\epsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\frac{d}{dt}(\dot{q}^i\epsilon)\right] dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t}\epsilon + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)\frac{d\epsilon}{dt}\right] dt \\
&= [\mathcal{L}\epsilon]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L}\right)\frac{d\epsilon}{dt}\, dt \qquad \text{(C.-132)}
\end{aligned}
$$

Parameterization-invariance means that the integral must vanish for arbitrary $d\epsilon/dt$, so that we have

$$
\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L} = 0. \qquad \text{(C.-132)}
$$

Even if the action is not extremal for some trajectory, it is still invariant under reparameterization of that trajectory.

Example: Diff (S^1) - the Virasoro algebra

As will be explained in chapter N.-19, in closed string theory there is an invariance under active diffeomorphisms acting on a circle, the corresponding group is denoted Diff (S^1). If $\theta \in (0, 2\pi]$ is a coordinate on S^1,

$$\tau(\theta) \rightarrow \tau'(\theta) = \tau(\theta) + f(\tau(\theta)) \tag{C.-132}$$

where f is periodic, i.e., $f(\tau + 2\pi) = f(\tau)$. There is a complete Fourier series expansion

$$f(\tau(\theta)) = \sum_{n=0}^{\infty} A_n \cos n\tau(\theta) + B_n \sin n\tau(\theta)$$

For an infinitesmal form transformation

$$\tau(\theta) \rightarrow \tau'(\theta) = \tau(\theta) + V(\theta)\frac{\partial}{\partial\theta}(\tau(\theta)) \tag{C.-132}$$

where $V(\theta)$ is the vector generating the infinitesimal diffeomorphism. As $V(\theta) = V(\theta+2\pi)$ we can expand $V(\theta)$,

$$\begin{aligned} V(\theta)\frac{\partial}{\partial\theta} &= (\sum_{n=0}^{\infty} b_n \cos n\theta + c_n \sin n\theta)\frac{\partial}{\partial\theta} \\ &= \sum_{n=0}^{\infty} [\frac{(b_n - ic_n)}{2} e^{in\theta} + \frac{(b_n + ic_n)}{2} e^{-in\theta}]\frac{\partial}{\partial\theta} \\ &= \sum_{n=-\infty}^{\infty} a_n i e^{in\theta}\frac{\partial}{\partial\theta} \end{aligned} \tag{C.-133}$$

where $a_n = (c_n - ib_n)/2$ and $a_{-n} = a_n^*$ for $n > 0$.

$$\tau'(\theta) = \tau(\theta) + \sum_{n\in\mathbb{Z}} a_n i e^{in\theta}\theta^\alpha \partial_\alpha \tau \tag{C.-133}$$

Therefore one has a basis indexed by $n \in \mathbb{Z}$

$$D_n = ie^{in\theta}\frac{\partial}{\partial\theta} \tag{C.-133}$$

The Lie algebra of Diff (S^1) is

$$
\begin{aligned}
[D_n, D_m]f &= [ie^{in\theta}\theta^\alpha\partial_\alpha ie^{im\theta}\theta^\beta\partial_\beta - ie^{im\theta}\theta^\alpha\partial_\alpha ie^{in\theta}\theta^\beta\partial_\beta]f \\
&= e^{i(n+m)\theta}\big[(in - im)\theta^\alpha\partial_\alpha f + i(\theta^\alpha\partial_\alpha\theta^\beta\partial_\beta - \theta^\beta\partial_\beta\theta^\alpha\partial_\alpha)f\big] \\
&= (n-m)D_{m+n}f \qquad \text{(C.-134)}
\end{aligned}
$$

or

$$
[D_n, D_m] = (n-m)D_{m+n}. \qquad \text{(C.-134)}
$$

Appendix D

Yang-Mills and Gauge Theory

think about gauge theory geometrically - to understand the gauge field as a connection on the principal bundle.

We have already encountered one fibre bundle, from general relativity: the tangent space at a point in spacetime is a fibre, with the fibre bundle

D.1 Summary

The origin of such functions could be diffrent. For example, it could be a awve function $\psi(x)$ from quantum mechanics. Recall that its absolute value $|\psi(x)|$ can be interpreted as the distribution density for the probability of finding the particle at the point x.

we may think of values of ψ as vectors in a 2-plane $\mathbb{R}^2$ which can be intepreted as the internal space of the particle. No particular direction in this plane has special meaning.

A particle is moving from point to point and carrying its internal spaqce with it. This has a geometric structure - a fibre bundle. The disjoint union of the internal spaces forms a fibre bundle $\pi : S \to \mathcal{M}$. Its fibre over a point $x \in \mathcal{M}$ is the internal space S_x. It could be a vector space (then we speak about a vector bundle) or a group (this leads to a principal bundle). For example, we can interpret the phase angle θ as an example of the group $U(1)$.

There is no common internal space in general, so one cannot identify the all internal spcaes with the spaces with the same space F. Or the fibre bundle is not trival, in general. However, we allow to identify the fibres along paths in $\mathcal{M}$. Thus, if $x(t)$ depends on t, the internal state $\overline{x}(t) \in S_{x(t)}$ describes a path in S lying over the original path. This leads to the notion of paralle transport in the fibre, or equivalently, a connection. In general, there is no reasonto expect that diferent paths from x to y lead to the same parallel transport of the internal state. They could differ by application of a symmetry group acting on the fibres (the structure grouop of fibre bundle). Physically, this is viewed as a phase shift. It is produced by the external field. Quantatively, the phase shift is described by the "curvative" of the connection.

The phase shift is determined by the commutator of the coovariant derivative ∇_μ:

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Here the group G is a Lie group, and the commutator takes its value in the Lie algebra **g** of G.

In general our state bundle $S \to \mathcal{M}$ is only locally trivial. This means that, for any point $x \in \mathcal{M}$, one can find a coordinate system in its neighbourhod U such that $\pi^{-1}(U_i)$ can be identified with $U \times F$, and the connection is given as above. If V is another neighbourhood, thwn we asssume that the new identification $\pi^{-1}(V) = V \times F$ differs over $U < capV$ from the old one by the "gauge transformation" $g(x) \in G$ so that $(x, s) \in U \times F$ corresponds to $(x, g(x)s)$ in $V \times F$. We shall see that the connection changes by the formula

$$A_\mu \to g^{-1} A_\mu + g^{-1} \partial_\mu g.$$

Note that we may take $V = U$ and change the trivialization by a function $g : U \to G$. The set of such functions forms a group (infinite-dminsional). It is called the gauge group.

D.2 Maxwell Equations and Gauge Theory

D.2.1 Gauge Symmetry

From Global to local symmetries. We have a single, free non-relativistic particle described by a wavefuncion $\psi(x)$. Multiplying the wavefunction by a complex number with unit yields a wavefunction that is physically equivalent to the original.

Consider a *local* transformation, that is , a transformation that depends on $\vec{r}$ and t,

$$\psi'(\vec{r}, t) = e^{i\alpha(\vec{r},t)} \psi(\vec{r}, t) \tag{D.0}$$

where $\alpha(\vec{r}, t)$ a real-valued function. The Schrödinger equation

for although they have the same probability distribution for position, they will have different momentum probability distribution

$$\partial_\mu \psi' = e^{i\alpha} [\partial_\mu \psi + i(\partial_\mu \alpha) \psi] \tag{D.0}$$

and the term in $\partial_\mu \alpha$. We want a the gauge transformation to be a symmetry of the theory, we have to modify

At the end of this chapter we will place these ideas in the more general mathematical context of what are known as fibre bundles.

The demand for gauge invariance does not give us the equation of motion for the electromagnetic field - it only gives us the interaction between the charged particle and the gauge potential. The simplest non-trivial gauge invariant Lagraigian

$$\mathcal{L}(A) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{D.0}$$

$$\begin{aligned} \mathcal{L}(A) \to \mathcal{L}(A') &= \partial_\mu (A_\nu(x) + \partial_\nu \alpha(x)) - \partial_\nu (A_\mu(x) + \partial_\mu \alpha(x)) \\ &= \mathcal{L}(A) + \partial_\mu \partial_\nu \alpha(x) - \partial_\nu \partial_\mu \alpha(x) \\ &= \mathcal{L}(A). \end{aligned} \tag{D.-1}$$

In the presence of no charges the system is completely described by the. A point in phase space represents the state of a particular system at time t. As time goes on, the state of the system changes according to Maxwell's equations. The entire history, past, present and future, of the system comes to be represented by a certain trajectory in phase space. When a system has gauge symmetries there is more than one trajectory in phase space that represents a physically equivalent system. The collection of such trajectories forms a (hyper-)surface in phase space. If the gauge symmetry is . A particular trajectory is called a gauge slice. Two distinct gauge slices are related to each other through a gauge transformation; there is a trajectory along which the gauge transformation drags the first gauge slice onto the other - this called a *gauge orbit*.

In the case of spacetime in GR, an action is invariant under general coordinate transformations could be built from tensors by contracting all their indices, so that the Jacobian (and inverse Jacobian) transformation matrices cancel.

D.2.2 Simple example

consider the integral

$$Z = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{(x-y)^2} \tag{D.-1}$$

$$\begin{aligned} x &\to x + a \\ y &\to y + a \end{aligned} \tag{D.-1}$$

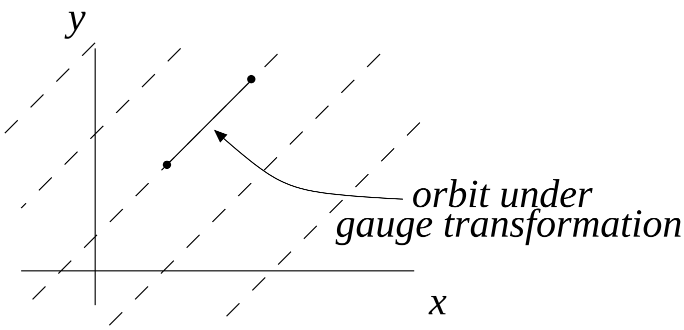


Figure D.1: simplegauge. The motion of a "configuration," (x, y), in the configuration space under the "gauge" transformation defined in (D.-1). The path is called a gauge orbit. In the simple example here, the gauge orbits are lines of constant $x - y$.

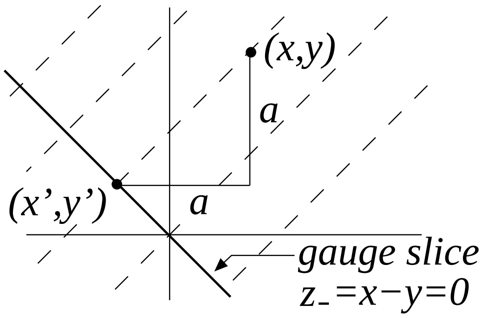


Figure D.2: simplegauge2. The gauge choice $x + y = 0$ defines a "gauge slice" through the configuration space. (x', y') is a configuration on the slice, that is, it satisfies the gauge condition. (x, y) is a gauge equivalent configuration, since both (x, y) and (x', y') reside on the same gauge orbit. a is the gauge transformation that takes us from the slice (x, y).

the gauge orbits are lines of constant $x - y$. The "action", $(x - y)^2$ is a gauge invariant.

$$\overset{-1}{\det}\left(\frac{\partial f}{\partial x}\right)\bigg|_{f=a} = \int dx\delta(f(x) - a) \tag{D.-1}$$

The degeneracy is caused by the fact that we integrate over a redundant set of integration variables which results in an infinite volume factor. This situation occurs because of the way we formulate the theory as based on the principle of a local gauge invariance. The complete physical content is contained by one contribution out of each equivalence class. One selects to one such member by imposing a condition called a *gauge fixing condition.*

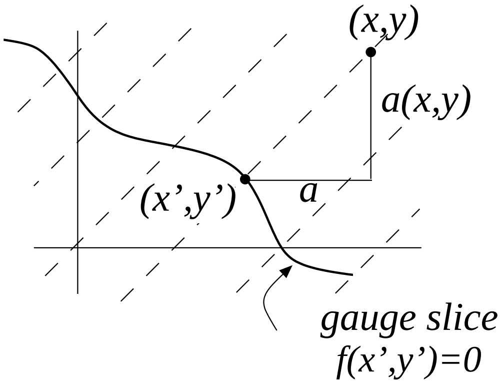


Figure D.3: simplegauge3. Illustration of a general choice of gauge, $f(x,y) = 0$. The desired change of coordinates is from (x,y) to (s,y), s being a variable that runs along the slice, and a doing the gauge transformation that runs from the slice (x,y).

D.2.3 Gauge Constraints

In the case of constrained coherent state path integral, we will end up integrating over the gauge orbits in effect averaging over all possible gauge orbits. Because of this, we will not have to fix a gauge and will not encounter the Gribov problem.(quant-ph/9611026)

$$\boxed{\partial_a E^a(x) \approx 0} \tag{D.0}$$

which is the Gauss's law in the absence of charge.

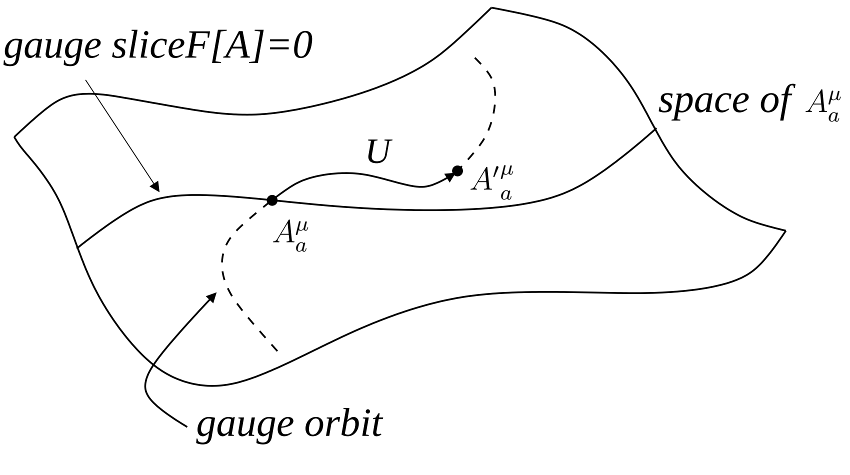


Figure D.4: orbitgenerator. A gauge choice is made. Configurations A'^μ_a that satisfy the gauge condition, $F[A] = 0$, fall on the gauge slice. Two configurations that are gauge equivalent lie on the same gauge orbit. U is the gauge transformation relating A'^μ_a to A^μ_a.

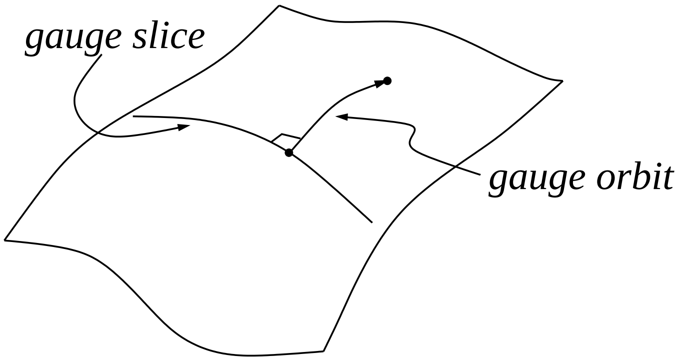


Figure D.5: gaugeslice.

D.2.4 Gauge Fixing in Electrodynamics

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{D.0}$$

and

$$\vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{D.0}$$

where

$$\vec{E} = (F^{10}, F^{20}, F^{30}), \quad \vec{B} = (-F^{23}, -F^{31}, -F^{12}).$$

Equation (D.2.4) together with

$$\Box A^\alpha - \partial^\alpha (\partial_\beta A^\beta) = j^\alpha. \tag{D.0}$$

are equivalent to Maxwell's equations.

We have residual gauge invariance. Given any A^α, choosing ϕ obeying

$$\Box \phi + \partial_\alpha A^\alpha = 0 \tag{D.0}$$

$A'^\alpha = A^\alpha + \partial^\alpha \phi$ will lead to $\partial_\alpha A'^\alpha = 0$

$$\Box A'^\alpha - \partial^\alpha (\partial_\beta A'^\beta) = \Box A^\alpha - \partial^\alpha (\partial_\beta A^\beta) + \Box \phi + \partial_\alpha A^\alpha = j^\alpha$$

and (D.2.4) is unchanged.

Substitution of $A_\mu \to A'_\mu = A_\mu + \partial_\mu \phi$ and doesn't change the form of the equations of motion and give rise to the same electromagnetic field.

It is possible to use the gauge invariance to require some particular condition of the field A_μ. For instance, we can perform a gauge transformation in such a way that the transformed field satisfies

$$\partial_\mu A^\mu = 0. \tag{D.0}$$

This is called the Lorentz gauge giving the wave equation for the potential

$$\Box A^\alpha = j^\alpha. \tag{D.0}$$

Or we can choose a gauge such that

$$A_0 = 0. \tag{D.0}$$

This is called the temporal gauge, or

$$\vec{\nabla} \cdot \vec{A} = 0. \tag{D.0}$$

which is called the Coulomb gauge. Each time we fix the constraint by restricting the gauge field A_μ, we need to check that there exists a choice of ϕ such that the gauge condition is possible.

The Wave Equation

Coulomb gauge

With the choice of the Coulomb gauge, time component of Maxwell equation (N.-19) becomes

$$-\nabla^2 A^0 = e^2 j^0. \tag{D.0}$$

We see that A^0 is not a dynamical variable, but rather is determined by the charge density a the same time,

$$A^0(\vec{x}, t) = \int d^3x' \frac{1}{4\pi|\vec{x} - \vec{x}'|} e^2 j^0. \tag{D.0}$$

Writing the electric field

the longitudinal piece

$$
\begin{aligned}
F^{ok(L)} &= -\partial^k A^0 \\
&= \int d^3x' \frac{x^k - x'^k}{4\pi|\vec{x} - \vec{x}'|^{3/2}} e^2 j^0 .
\end{aligned}
\tag{D.0}
$$

Lorentz Gauge

• From the wave equation $k \cdot k = 0$ hence $k^2 = \omega^2$, i.e. $E^2 = p^2c^2$ (massless photons).

• From rom Lorenz gauge condition $\epsilon \cdot \mathbf{k} = 0$ implies $\epsilon^0 = \epsilon \cdot \mathbf{k}/\omega$.

• Polarization 4-vector $\epsilon'^\mu = \epsilon^\mu + ak^\mu$ is equivalent to ϵ^μ for any constant a. Hence we can always choose $\epsilon^0 = 0$. Then Lorenz condition becomes the transverstity condition: $\epsilon \cdot \mathbf{k} = 0$.

• For k along z-axis -we can express ' in terms of plane polarization states ,

$$
\epsilon^\mu_x = (0, 1, 0, 0), \qquad \epsilon^\mu_y = (0, 0, 1, 0) \tag{D.0}
$$

or circular polarization states $\epsilon^\mu_{R,L} = (0, 1, \pm i, 0)/\sqrt{2}$. Note only two polarization states for real photons.

Temporal Gauge

D.2.5 Gauge Invariance Noether

$$
\int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - T^\mu_\lambda X^\lambda_\nu \right] \delta\omega^\nu d\sigma_\mu = 0
$$

since $\delta\omega^\nu$ is arbitrary,

$$
\int_{\partial R} J^\mu_\nu d\sigma_\mu = 0 \tag{D.0}
$$

where

$$
J^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - T^\mu_\lambda X^\lambda_\nu . \tag{D.0}
$$

by Gauss's theorem that $\int_R \partial_\mu J^\mu_\nu d^4 = 0$, and since R is arbitrary,

$$\partial_\mu J^\mu_\nu = 0. \tag{D.0}$$

We therefore have a conserved current J^μ_ν because of gauge invariance of the action under the transformations (N.-19). This gives rise to a conserved charge Q_ν defined by

$$Q_\nu = \int_\sigma J^\mu_\nu d\sigma_\mu \tag{D.0}$$

where the integral is taken over a spacelike hypersurface σ_μ.

$$\int_V \partial_0 J^0_\nu d^3x + \int_V \partial_i J^i_\nu d^3x = 0. \tag{D.0}$$

The second term is transformed into a surface integral by Gauss's theorem. By taking the surface far enough away we see that this term vanishes. So we have

$$\frac{d}{dt}\int J^0_\nu d^3x = \frac{dQ_\nu}{dt} = 0. \tag{D.0}$$

D.3 Canonical Formulation of Yang-Mills

$$\mathcal{L} = \frac{1}{4}F^I_{\mu\nu}F^{I\mu\nu} - \frac{1}{2}F^{I\mu\nu}(\partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g\epsilon^{IJK}A^J_\mu A^K_\mu) \tag{D.0}$$

Palatini method can be applied to Yang-Mills theory. A^I_μ and $F^I_{\mu\nu}$ are taken to be independent fields; they are totally unrelated. However, by eliminating the $F^I_{\mu\nu}$ field by its equations of motion, we can show the equivalence with the standard Yang-Mills action,

$$\mathcal{L}[A] = -\frac{1}{4}F^I_{\mu\nu}F^{I\mu\nu} \tag{D.0}$$

We show this in the following. We take the variation of Eq.(D.3) with with respect to $F^I_{\mu\nu}$ as well as respect to A^I_a. The two equations of motion yield:

$$\frac{\delta S}{\delta F^I_{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial F^i_{\mu\nu}} = \frac{\partial}{\partial F^i_{\mu\nu}}\frac{1}{4}\left[\eta^{ce}\eta^{df}F^i_{cd}F^i_{ef} + \eta^{ce}\eta^{df}(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)F^i_{fe}\right] \tag{D.1}$$

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g\epsilon^{IJK}A^J_\mu A^K_\mu \tag{D.2}$$
$$0 = \partial_\mu F^{I\mu\nu} + g\epsilon^{IJK}A^{I\mu}F^I_{\mu\nu}K \tag{D.3}$$

inserting Eq.(D.3) into Eq.(D.3) we recover the standard action of Eq.(D.3).

First, we eliminate F^i_{ab} in terms of A^i_a fields, keeping F^i_{0a} an independent field. Written in these variables, we find that the Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4}F^i_{ab}F^I_{iab} + \frac{1}{2}(F^i_{0a})^2 - F^{a,0i}(\partial_i A^a_0 - \partial_0 A^a_i + gf^{abc}A^b_0 A^c_i) \tag{D.3}$$

Let us define,

$$\begin{aligned} E^a_i &:= F^a_{0i} && \text{(D.4)} \\ B^a_i &:= \frac{1}{2}\epsilon_{ijk}F^{iab} && \text{(D.5)} \end{aligned}$$

In terms of these fields, we now have,

$$E^a_i \dot{A}^a_i - \mathcal{H} \tag{D.5}$$

where,

$$\boxed{\mathcal{H} = \frac{1}{2}[(E^a_i)^2 + (B^a_i)^2] + A^i_0(\partial_a E^a_i + g\epsilon_{ijk}A^k_a E^{aj}) \qquad \text{(D.6)}}$$

$$\mathcal{D}_a E^a_i = \partial_a E^a_i + g\epsilon_{ijk}A^k_a E^{aj} \tag{D.6}$$

$$\begin{aligned} F_{ab} &= \partial_a A_b - \partial_b A_a - i[A_a, A_b] \\ &= \partial_a A^i_b T_i - \partial_b A^j_a T_j - i[A^i_a T_i, A^j_b T_j] \\ &= (\partial_a A^i_b - \partial_b A^i_a)T_i - iA^i_a A^j_b [T_i, T_j] \\ &= (\partial_a A^i_b - \partial_b A^i_a) \\ &= (\partial_a A^i_b - \partial_b A^i_a + A^j_a A^k_b f_{ijk})T_i. && \text{(D.3)} \end{aligned}$$

$$\boxed{F_{ab} = (\partial_a A^i_b - \partial_b A^i_a + A^j_a A^k_b f_{ijk})T_i \qquad \text{(D.4)}}$$

D.3.1 The Geometry of Gauge Invariance - Gauge Field as a Connection

How do you compare Ψ on one fibre with Ψ another? we can choose coordinates for one fibre that are different than others, that is, apply a rotation of one fibre without rotating others.

the gauge fields play the same role in gauge theory as the Christoffel symbols play in general relativity.

In electrodynamics

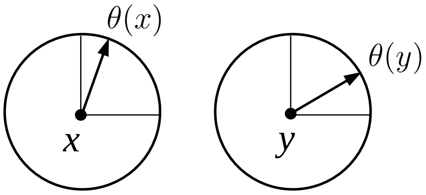


Figure D.6: gaugeElectric. The wavefunction at point x is $\psi(x) = |\psi(x)|e^{i\theta(x)}$ and the wavefunction at a nearby point y is $\psi(y) = |\psi(y)|e^{i\theta(y)}$.

For the sake of simplicity we consider wavefunction combined as a vector

$$\Psi = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \vdots \\ \Psi_N(x) \end{pmatrix} \tag{D.4}$$

We assume a local symmetry such that

$$\Psi'(x) = \exp(-ig\theta(x))\Psi(x) \tag{D.4}$$

is physically equivalent to $\Psi(x)$.

The situation is somewhat analogous to an arbitrary contravariant vector and its Lorentz transformation

$$V^\mu(x), \quad V'^\mu(x) = \Lambda^\mu{}_\nu(x)V^\nu(x) \tag{D.4}$$

The local Lorentz transformation $\Lambda^\mu{}_\nu(x)$ corresponds to the gauge transformation $\exp(-ig\theta(x))$.

$$\frac{V^\mu(x+\delta x) - V^\mu(x)}{\delta x} \tag{D.4}$$

yields additional terms owing to the dependence of the metric tensor $g_{\mu\nu}(x)$ on the position.

In a completely analogous manner we can write down () for the wavefunction $\Psi(x)$:

$$\partial_\mu \Psi(x) \to (\partial_\mu + \tilde{\Gamma}_\mu)\Psi(x), \tag{D.4}$$

$$D_\mu \Psi(x) = (\partial_\mu - igA_\mu)\Psi(x) \tag{D.4}$$

$\Psi(x) \to e^{-ig\theta(x)}\Psi(x)$

Introduce a mathematical notion we consider and electric field in a spacial slice in Minkowskian spacetime. The structure is $\mathcal{M}_{Mink} \times R^3$ They can taken together is a single entity. It is an example of what mathematicians call a ***vector bundle***.

Let G be a Lie group and $\mathcal{M}$ a smooth manifold.

A ***principle G bundle*** is over $\mathcal{M}$ is a manifold which locally looks like $\mathcal{M} \times G$.

The connection tells us how points on one fibre, that is the value of Ψ at one point, are mapped into points on another fibre. If we rotate one fibre (or simply change the coordinate basis we are using to describe it), the rotated points should still map to the same points on the neighbouring fibre

The geometry of the $U(1)$ fibre bundle of electromagnetism is the relationships between phases of wavefunctions at different points in spacetime is determined by the gauge potential $A_\mu(x)$, to be compared with the case of GR the spacetime geometry is determined by the connection $\Gamma^c_{ab}(x)$.

In the case of spacetime in GR, an action is invariant under general coordinate transformations could be built from tensors by contracting all their indices, so that the Jacobian (and inverse Jacobian) transformation matrices cancel.

D.4 Summary of Yang Mills

A Lie group is a group object G in the category of smooth manifolds. For any $g \in G$ we denote by L_g (resp. R_g) the left (resp. right) transaltion action of the group G on itself.

The differnetials of the maps R_g, L_g transform a vecter field η to the vector field

$$(L_g)_*(\eta), (R_g)_*(\eta),$$

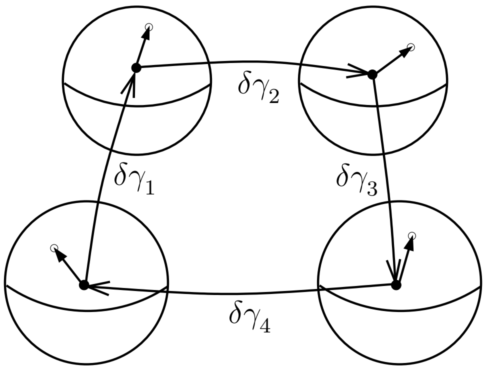


Figure D.7: ConnCurvat. $F_{ab} \approx$

respectively. The Lie algerbra of G is the vector space $\mathbf{g}$ of vector fields which are invariant with respect to left translations. Its Lie algerba structure is the bracket operation of vector fields.

For any $\eta \in \mathbf{g}$, the vector

$$(dL_{g^{-1}})_g(\eta_g)$$

belongs to the vector space

$$T(G)_{\mathbf{1}}$$

and $\tilde{v}$ be the corresponding vectotr field. We have

$$\tilde{v}_g = (dL_g)_{\mathbf{1}}(v),$$

so that

$$(dR_{g^{-1}})_g(\tilde{v}) = dc(g)_1(v)$$

where $c(g) : G \to G$ is the conjugacy action

$$h \to g \cdot h \cdot g^{-1}.$$

D.5 Propagation Kernel Yang-Mills

Comparison Between Gauge Yang-Mills Theories and GR

GR is invariant under diffeomorphisms, namely under the pull back $q \rightarrow g^{\xi} = \xi_{*} g$ of the gravitational field by a map $\xi : \mathcal{M} \rightarrow \mathcal{M}$ from the spacetime to itself. An active diffeormorphism can be thought of as simultaneously dragging the metric and matter fields over the manifold, or keeping the fields "where" they are and mapping the points of the manifold to other points of the manifold, i.e., $\xi : \mathcal{M} \rightarrow \mathcal{M}$. Should not be confused with the freedom of choosing coordinates on $\mathcal{M}$: once coordinates are fixed, one still gets physically identical fields of g and g^{ξ}.

$$g_{ab}(x) = \frac{\partial \xi^{c}(x)}{\partial x^{a}} \frac{\partial \xi^{d}(x)}{\partial x^{b}} g^{\xi}_{cd}(\xi(x)) \tag{D.4}$$

or

$$g^{\xi}_{ab}(x) = \frac{\partial \phi^{c}(x)}{\partial x^{a}} \frac{\partial \phi^{d}(x)}{\partial x^{b}} g_{cd}(\phi(x)) \tag{D.4}$$

This is the change in the metric under an infinitesimal active diffeomorphism along the vector field ξ^{a}.

In electromagnetism the invariance in A_{μ} comes about because the field strength $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is left unchanged by the gauge transformations $A_{\mu} \rightarrow A'_{\mu} = A_{\mu} + \partial_{\mu}\phi$.

Gauge transformation	Active diffeomorphism
Gauge group	Group of all active diffeomorphism
Gauge potential, A_a	Connection coefficient, Γ^{c}_{ab}
Field strength, F_{ab}	Curvature tensor, R^{a}_{bcd}
Bianchi identity:	Bianchi identity:
$\mathcal{D}_a F_{bc} = 0$	$\mathcal{D}_a R^{b}_{cde} = 0$

Faddeev-Popov

$$A^{\Omega}_{\mu} = A_{\mu} + \partial_{\mu}\Omega \tag{D.4}$$

$$A^{\Omega}_{\mu} = \Omega A_{\mu} \Omega^{-1} + i\Omega(\partial_{\mu})\Omega^{-1} \tag{D.4}$$

it has the invariance property:

$$\mathcal{D}\Omega = \mathcal{D}(\Omega'\Omega) \tag{D.4}$$

$$\begin{aligned}
\Delta_{FP}^{-1}(A_\mu^{\Omega'}) &= \int \mathcal{D}\Omega' \; \delta(F(A_\mu^{\Omega'\Omega})) \\
&= \int \mathcal{D}[\Omega'\Omega] \; \delta(F(A_\mu^{\Omega'\Omega})) \\
&= \int \mathcal{D}\Omega'' \; \delta(F(A_\mu^{\Omega''})) \\
&= \Delta_{FP}^{-1}(A_\mu)
\end{aligned} \tag{D.2}$$

$$\Delta_{FP}(A_\mu) = \Delta_{FP}(A_\mu^{\Omega}) \tag{D.2}$$

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}\Omega \delta(F(A_\mu^{\Omega})) \tag{D.2}$$

inserting and then making gauge transformation

$$\int \mathcal{D}A_\mu \; \left(\Delta_{FP}(A_\mu) \int \mathcal{D}\Omega \; \delta(F(A_\mu^{\Omega})) \right) e^{i\int d^4x \mathcal{L}[A]} \tag{D.2}$$

and then make the replacement followed by gauge transformation on the entire functional integral, so that $A_\mu \to A_\mu^{-\Omega}$

$$\int \mathcal{D}\Omega \; \int \mathcal{D}A_\mu \; \Delta_{FP}(A_\mu) \; \delta(F(A_\mu)) e^{i\int d^4x \mathcal{L}[A]} \tag{D.2}$$

Temporal gauge:

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}\Omega \; \delta(A_0^{\Omega}) \tag{D.2}$$

The YM gauge transformation (Eq. D.5) does not mix different components of the 4-vector A_μ. The integral (Eq. D.5) reads explicitly

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}\Omega \; \delta(\Omega A_0 \Omega^{-1} + i\Omega(\partial_0)\Omega^{-1}), \tag{D.2}$$

hence, Δ_{FP} only depends on the component A_0, i.e., $\Delta_{FP}(A_\mu) = \Delta_{FP}(A_0)$; but this is, in turn, fixed to be zero by the $\delta-$function appearing in the integral:

$$
\begin{aligned}
1 &= \int \mathcal{D}\Omega \; \Delta_{FP}(A_0)\delta(A_0^\Omega) \\
&= \int \mathcal{D}\Omega \; \Delta_{FP}(A_0^\Omega)\delta(A_0^\Omega) \\
&= \int \mathcal{D}\Omega \; \Delta_{FP}(A_0^\Omega = 0)\delta(A_0^\Omega) \\
&= \Delta_{FP}(A_0 = 0) \int \mathcal{D}\Omega \; \delta(A_0^\Omega) \qquad \text{(D.0)}
\end{aligned}
$$

where we used the gauge invariance of Δ_{FP} to write the second line. It turns out that Δ_{FP} is just a constant.

$$
W[A_i', A_i'', T] = \Delta_{FP} \int_{A_i'}^{A_i''} \mathcal{D}A_\mu \int \mathcal{D}\Omega \; \delta(A_0^\Omega) e^{iS[A_\mu]} \qquad \text{(D.0)}
$$

changing the order of the two integrations, changing variables $A_\mu \to A_\mu^\Omega$ and integrating out A_0, we obtain

$$
W[A_i', A_i'', T] = \Delta_{FP} \int \mathcal{D}\Omega \int_{A_i'^{\Omega(0)}}^{A_i''^{\Omega(T)}} \mathcal{D}A_i e^{S[A_i;A_0=0]} \qquad \text{(D.0)}
$$

Bulk integration on $\Omega(x,t), 0 > t > T$, factors out leaving

$$
W[A_i', A_i'', T] = \Delta_{FP} \int \mathcal{D}\Omega(0) \; \mathcal{D}\Omega(T) \int_{A_i'^{\Omega(0)}}^{A_i''^{\Omega(T)}} \mathcal{D}A_i e^{iS[A_i;A_0=0]} \qquad \text{(D.0)}
$$

That $\mathcal{D}A_i$ and $S[A_i; A_0 = 0]$ are invariant under a time-independent gauge transformation will mean that one of the integrals $\Lambda(\vec{x}, 0)$ and $\Lambda(\vec{x}, T)$ is redundant. $A_\mu \to A_\mu^\Lambda$, replacing $S[A_i; A_0 = 0]$ with $S[A_i^\Lambda; A_0 = 0]$ and $\mathcal{D}[A_i]$ with $\mathcal{D}[A_i^\Lambda]$

$$
\begin{aligned}
\int_{A_i'^{\Omega(0)}}^{A_i''^{\Omega(T)}} \mathcal{D}A_i \; e^{S[A_i;A_0=0]} &= \int_{A_i'^{\Lambda(0)\Omega^{-1}(T)}}^{A_i''} \mathcal{D}(A_i^{\Omega(T)}) \; e^{iS[A_i^{\Omega(T)};A_0=0]} \\
&= \int_{A_i'^{\Lambda(0)\Omega^{-1}(T)}}^{A_i''} \mathcal{D}A_i e^{S[A_i;A_0=0]} \qquad \text{(D.0)}
\end{aligned}
$$

$\mathcal{D}\Omega(0) \rightarrow \mathcal{D}(\Omega(0)\Omega^{-1}(T)) = \mathcal{D}\lambda$

$$W[A'_i, A''_i, T] = \left(\int \mathcal{D}\Omega(T)\right) \Delta_{FP} \int \mathcal{D}\lambda \int_{A'^{\lambda}_i}^{A''_i} \mathcal{D}A_i \; e^{iS[A_i;A_0=0]} \tag{D.0}$$

we are able to drop the second integral, we therefore have

$$W[A'_i, A''_i, T] = \int \mathcal{D}\lambda \tilde{W}[A'^{\lambda}_i, A''_i, T] \tag{D.0}$$

where

$$\tilde{W}[A'_i, A''_i, T] = \Delta_{FP} \int \mathcal{D}A_i \; e^{iS[A_i;A_0=0]} \tag{D.0}$$

it is equal to the integration over all gauge factors λ, and hence independent of the gauge.

$$\Psi_{t+T}[A_i] = \mathcal{D}A'_i \int W[A_i, A'_i, T]\Psi_t[A'_i]. \tag{D.0}$$

$\Psi_t[A_i] = \Psi_t[A^{\lambda}_i]$

$$W[A_i, A'_i, T] = \sum_n e^{-iE_nT}\overline{\Psi_n[A'_i]}\Psi_n[A_i]. \tag{D.0}$$

$$\int_{-\infty}^{+\infty} dT \; W[A_i, A'_i = 0, T] = const. \; \Psi_0[A_i]. \tag{D.0}$$

$$W[A'_i, A''_i, T] = \mathcal{N}(T) \exp\left\{ \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} p \frac{(|A'^T|^2 + |A''^T|^2)\cos T - 2A'^T \cdot A''^T}{\sin T} \right\}. \tag{D.0}$$

$$D_{ij} = \delta_{ij} - \frac{p_i p_j}{p^2}. \tag{D.0}$$

$$W[A_i, A'_i, T] = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_1 \dots \epsilon_n} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_n}{(2\pi)^3} e^{-i\sum_{\alpha=1}^{n} E_\alpha T} \; \overline{\psi_{p_1\epsilon_1,\dots,p_n\epsilon_n}[A_i]} \; \psi_{p_1\epsilon_1,\dots,p_n\epsilon_n}[A'_i] \tag{D.0}$$

Using $\Psi_0[A_i] = \lim_{T\rightarrow\infty} W[A_i, A'_i = 0, T]$ the (non-normalized) vacuum state can be read from the zero'th order of (N.-19)

D.5.1 n Uncoupled Harmonic Oscillators

The Hamiltonian of a general one-dimensional harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}k(\hat{x} - x_0)^2$$

if we measure the energy H we observe that the outcome is quantized. We can interpret as the number of quanta in x. we could call these quanta particles (Fock states - more on that later - also what Rovelli terms *global* particles.)

$$W[x, x', T] =< x'|e^{-iHT}|x > \tag{D.0}$$

$$W[x, x', T] = \mathcal{N}(T) \exp \frac{i}{2}\left(\frac{(x^2 + x'^2)\cos T - 2xx'}{\sin T}\right) \tag{D.0}$$

$$W[x_1, x_2, T] = \sum_n e^{-iE_nT}\, \overline{\psi_n(x_2)}\psi_n(x_1) \tag{D.0}$$

we expand (N.-19) in a power series in e^{-iT} by writing the sines and cosines in terms of exponentials and expanding.

$$\begin{aligned} W[x_1, x_2, T] &= \mathcal{N}(T) \exp \frac{i}{4}\left(\frac{(x^2 + x'^2)\cos T - 2xx'}{\sin T}\right) \\ &= \mathcal{N}(T) \exp -\frac{1}{4}\left((x^2 + x'^2)\frac{(1 + e^{-2iT})}{1 - e^{-2iT}} - \frac{2xx'e^{-iT}}{1 - e^{-2iT}}\right) \end{aligned} \tag{D.0}$$

$$\frac{(1 + e^{-2iT})}{1 - e^{-2iT}} = (1 + e^{-2iT})(1 + e^{-2iT} + e^{-4iT} + \dots) = 1 + 2e^{-2iT} + 2e^{-4iT} + \dots$$

$\mathcal{N}(T)$ part is

$$\mathcal{N}(T) = \left(\frac{\omega}{2\pi i \sin T}\right)^{1/2} = \left(\frac{\omega}{2\pi}\right)^{1/2} e^{-T/2}(1-e^{-2iT})^{-1/2} = \left(\frac{\omega}{2\pi}\right)^{1/2} e^{-T/2}\left(1+\frac{1}{2}e^{-2iT}+\dots\right).$$

Putting it together

$$\begin{aligned} W[x_1, x_2, T] &= \left(\frac{\omega}{\pi}\right)^{1/2} e^{-iT/2} \exp\left(-\frac{\omega}{2}(x^2 + x'^2)\right)\left(1 + \frac{1}{2}e^{-2iT} + \dots\right) \\ & (1 - \omega(x^2 + x'^2)e^{-2iT} + \dots)(1 + 2\omega x x' e^{-iT} + \dots). \end{aligned} \tag{D.0}$$

the leading term in the $iT \to \infty$ limit,

$$\lim_{iT\to\infty} W[x_1, x_2, T] = \lim_{iT\to\infty} e^{-i\omega T/2} \left(\frac{\omega}{\pi}\right)^{1/2} \exp\left(-\frac{\omega}{2}(x^2 + x'^2)\right)$$

Ground state is

$$\Psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp(-y^2/2) \tag{D.0}$$

and first three excited states are

$$\begin{aligned} \Psi_1 &= \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2} y \exp(-y^2/2) \\ \Psi_2 &= \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}}(2y^2 - 1) \exp(-y^2/2) \\ \Psi_3 &= \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{3}}(2y^3 - 3y) \exp(-y^2/2) \end{aligned} \tag{D.-1}$$

where $\alpha = m\omega/\hbar$ and $y = \sqrt{\alpha}x$

the wavefunctions $\psi_n(x)$ can be identified as a Hermite polynomial times $\exp\left(-\frac{1}{2}m\omega x^2\right)$, as is well known.

Now, if we have n uncoupled harmonic oscillators:

$$\begin{aligned} & \exp\left(\sum_{m=1}^{n} m \frac{H(m)\cos mT - H(m)}{\sin mT}\right) \\ &= \\ &= \left[1 + \frac{1}{2}\sum_m mH(m)e^{-imT} + \dots\right] \end{aligned} \tag{D.-2}$$

there are first excited states

$$
\begin{aligned}
\Psi_{1,0,0,\ldots,0} &= \left(\frac{\alpha}{\pi}\right)^{n/4} \sqrt{2} y_1 \exp(-\sum_{m=1}^{n} y_m^2/2) \\
\Psi_{0,1,0,\ldots,0} &= \left(\frac{\alpha}{\pi}\right)^{n/4} \sqrt{2} y_2 \exp(-\sum_{m=1}^{n} y_m^2/2) \\
& \ldots \\
\Psi_{0,0,0,\ldots,1} &= \left(\frac{\alpha}{\pi}\right)^{n/4} \sqrt{2} y_n \exp(-\sum_{m=1}^{n} y_m^2/2) \qquad \text{(D.-6)}
\end{aligned}
$$

And higher excited states, for example

$$
\Psi_{1,0,1,\ldots,0} = \left(\frac{\alpha}{\pi}\right)^{n/4} (\sqrt{2} y_1)(\sqrt{2} y_3) \exp(-\sum_{m=1}^{n} y_m^2/2)
$$

$$
\Psi_{1,2,0,\ldots,0} = \left(\frac{\alpha}{\pi}\right)^{n/4} (\sqrt{2} y_1) \left(\frac{1}{\sqrt{2}}(2y_2^2 - 1)\right) \exp(-\sum_{m=1}^{n} y_m^2/2)
$$

and

$$
\Psi_{3,2,0,\ldots,0} = \left(\frac{\alpha}{\pi}\right)^{n/4} \left(\frac{1}{\sqrt{3}}(2y_1^3 - 3y_1)\right) \left(\frac{1}{\sqrt{2}}(2y_2^2 - 1)\right) \exp(-\sum_{m=1}^{n} y_m^2/2)
$$

multi-particle propagator

$$
\begin{aligned}
W[T] &= \sum_{m} < x'|e^{-iE_m T}|x > + \frac{1}{2!} \sum_{m_1,m_2} < x_1', x_2'|e^{-i(E_{m_1}+E_{m_2})T}|x_1, x_2 > + \\
&+ \frac{1}{2!} \sum_{m_1,m_2,m_3} < x_1', x_2', x_3'|e^{-i(E_{m_1}+E_{m_2}+E_{m_3})T}|x_1, x_2, x_3 > + \ldots \qquad \text{(D.-6)}
\end{aligned}
$$

$$
W[T] = \sum_{n=0}^{\infty} \frac{1}{n!} e^{-i\sum_{m=1}^{n} E_m T}\, \overline{\psi(x_1, x_2, \ldots, x_n)}\, \psi(x_1, x_2, \ldots, x_n) \qquad \text{(D.-6)}
$$

$n-$**photon states**

(D.5) lends itself to a particle interpretation. The ground state is the absence of particles:

$$\Psi_0[A^T] = \mathcal{N} \exp\left(-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_p \, A^{T\prime}A^{T\prime\prime}\right) \tag{D.-6}$$

A wave function representing a state with one photon of momentum $\vec{p}_1$ is obtained from (D.5.1) by replacing the ground state wave function with the mode $\vec{p}_1$ with the harmonic oscillator wave function for the first excited state,

$$\Psi_1[A^T] = \sqrt{2}A^{T\prime}(p_1)\ \Psi_0[A^T]. \tag{D.-6}$$

A wave function with two photons of momentum $\vec{p}_1$, then we replace (D.5.1) the mode $\vec{p}_1$ with the second excited wave function,

$$\Psi_2[A^T] = \frac{1}{\sqrt{2}}(A^{T\prime}(p_1)\cdot A^{T\prime}(p_1) - 1)\ \Psi_0[A^T]. \tag{D.-6}$$

A wave function with two photons of momentum $\vec{p}_1$ and momentum $\vec{p}_2$, then we replace (D.5.1) the modes $\vec{p}_1$ and $\vec{p}_2$ with the product of two first excited wave function,

$$\Psi_{11}[A^T] = 2(A^{T\prime}(p_1))(A^{T\prime}(p_2))\ \Psi_0[A^T], \tag{D.-6}$$

and so on.

D.6 Propagation Kernel in General Relativity

$$W[g'_{ij}, g''_{ij}, T] = \int_{g'_{ij}}^{g''_{ij}} \mathcal{D}g_{ab}\ e^{iS[g_{ab}]} \tag{D.-6}$$

The action

We start with the covariant Lorentzian action for a metric g and generic matter fields ϕ:

$$S[g_{ab}] = \int_0^T \int_{\Sigma_t} d^3x\ \sqrt{-g}g^{ab}R_{ab} + \int_{\Sigma_0\cup\Sigma_T} d^3x\ \mathcal{K} \equiv \int_0^T \int_{\Sigma_t} d^3x\ \mathcal{L} \tag{D.-6}$$

where R_{ab} is the Ricci tensor and g is the determinant of the metric, and $\mathcal{K}$ is the trace of the extrinsic curvature of the boundary. In the path-integral formulation, the surface term, sometimes called the *Gibbons-Hawking* term, is needed in order to have only first-order time derivatives in the action, so that the convolution property of the propagator kernel

$$W[g'''_{ij}, g'_{ij}] = \int_{\Sigma_{t_2}} \mathcal{D}g''_{ij}\, W[g'''_{ij}, g''_{ij}] W[g''_{ij}, g'_{ij}]$$

is guaranteed [270].

The surface term is also required so that the action yields the correct equations of motion subject only to the condition that the induced three metric on the boundary is held fixed.

The invariance under active diffeomorphisms of GR makes the integral (D.6) infinite. This situation is analogous to the YM case, and can be cured with a gauge-fixing.

D.6.1 Gauge-Fixing

For the action (D.6) to be covariant, ξ must not change the boundaries of the spacetime region considered, that is

$$\xi^0(0, \vec{x}) = 0, \qquad \xi^0(T, \vec{x}) = T. \tag{D.-6}$$

The GR analogue of the temporal gauge

$$g_{00} = -1, \qquad g_{0i} = 0. \tag{D.-6}$$

was introduced in section **??**.

As for YM, this is not a complete gauge fixing, but we expect that additional gauge fixing is not required in the path integral. In the linearized case we shall explicitly see that the remaining part of the gauge is taken care by the integration over the gauge parameters. Thus, we gauge fix the path integral by inserting in (N.-19) the FP identity

$$1 = \Delta_{\mathrm{FP}} \int \mathcal{D}\xi\ \delta(g^{\xi}_{00} + 1)\delta(g^{\xi}_{01}), \tag{D.-6}$$

where $\mathcal{D}\xi$ is a formal measure over the active diffeomorphisms.

$$W[g'_{ij}, g''_{ij}, T] = \int \mathcal{D}\xi^i_{fin} \mathcal{D}\xi^0_{fin} \mathcal{D}\xi^0_{ini} \int_{g'_{ij}{}^{\xi^0}}^{g''_{ij}{}^{\xi^\mu}} \mathcal{D}g_{ij} \Delta_{\mathrm{FP}} \exp\left\{ i \int d^3x \int_{\xi^0(\vec{x},0)}^{\xi^0(\vec{x},T)} dt \mathcal{L}[g_{ij}] \right\} \tag{D.-6}$$

D.6.2 The Disappearance of Time

The scalar constraint can then be written in the form

$$\left(g^{ik}g^{jl} - \frac{1}{2}g^{ij}g^{kl}\right)\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{kl}}{\partial t} - g_{00}\det g_{ij}\ R[g_{ij}] = 0. \tag{D.-6}$$

$$T(\vec{x}) = \int_0^1 \sqrt{g_{00}(\vec{x},t)}dt = \int_0^1 \sqrt{\frac{(g^{ij}g^{kl} - \frac{1}{2}g^{ij}g^{kl})\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{kl}}{\partial t}}{\det g_{ij}\ R[g_{ij}]}} \tag{D.-6}$$

between initial and final surface, along the $\vec{x} = const.$ lines, which in this gauge are the geodesics normal to the initial surface.

D.6.3 Linearized Theory

$$g_{ab}(x) = \eta_{ab} + h_{ab}(x) \tag{D.-6}$$

in the temporal gauge the action reads

$$\begin{aligned}\int d^3x \int_0^T dt\ \mathcal{L} = \int d^3x \int_0^T dt\ \Big\{ & \partial_i\partial_j h^{ij} - \nabla^2 h + \\ +\frac{1}{4}\Big[\partial_a h^i{}_i \partial^a h^j{}_j - \partial^a h^{ij}\partial_a h_{ij} + \partial^k h^{ij}\partial_i h_{jk} + & \partial^k h^{ij}\partial_j h_{ik} - \partial^j h_{ij}\partial^i h - \partial_i h^{ij}\partial_j h^k{}_k\Big]\Big\}.\end{aligned} \tag{D.-7}$$

Writing

$$\begin{aligned} H^{TT}(\vec{p}) &= \overline{h^{TT\prime\prime}_{ij}(\vec{p})}h^{TT\prime\prime}_{ij}(\vec{p}) + \overline{h^{TT\prime}_{ij}(\vec{p})}h^{TT\prime}_{ij}(\vec{p}) \\ \tilde{H}^{TT}(\vec{p}) &= \overline{h^{TT\prime\prime}_{ij}(\vec{p})}h^{TT\prime}_{ij}(\vec{p}) + \overline{h^{TT\prime}_{ij}(\vec{p})}h^{TT\prime\prime}_{ij}(\vec{p}) \end{aligned} \tag{D.-7}$$

$$W[A_i', A_i'', T] = \mathcal{N}(T)\ \delta(h_0')\delta(h_0'')\exp\left\{\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}p\ \frac{H^{TT}(\vec{p})\cos T - \tilde{H}^{TT}(\vec{p})}{\sin T}\right\}. \tag{D.-7}$$

where the function $\mathcal{N}(T)$ is a normalization factor. This is the field propagation kernel of linearized GR.

D.6.4 Ground-State and Graviton States

$$W[h'_{ij}, h''_{ij}, T] = \mathcal{N}(T)\ \delta(h'_o)\delta(h''_0)e^{-\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}p\ H^{TT}(\vec{p})}\left[1 + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}p\ H^{TT}(\vec{p}) + \ldots\right]. \tag{D.-7}$$

Using $\Psi_0[h_{ij}] = \lim_{T\to\infty} W[h_{ij}, h'_{ij} = 0, T]$ the (non-normalized) vacuum state can be read from the zero'th order of (D.6.4)

$$\overline{\Psi_0[h''_{ij}]}\Psi_0[h'_{ij}] = \delta(h'_0)\delta(h''_0)\ \exp\left\{-\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}p\ H^{TT}(\vec{p})\right\}, \tag{D.-7}$$

and therefore

$$\Psi_0[h_{ij}] = \delta(h_0)\ \exp\left\{-\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}p\ h^{TT}_{ij}(\vec{p})h^{TT}_{ij}(-\vec{p})\right\}. \tag{D.-7}$$

The graviton states can be obtained from the analog of (N.-19).

$$W[h_{ij}, h'_{ij}, T] = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\epsilon_1\ldots\epsilon_n}\int \frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_n}{(2\pi)^3}\ e^{-i\sum_{\alpha=1}^{n}E_\alpha T}\ \overline{\psi_{p_1\epsilon_1,\ldots,p_n\epsilon_n}[h_{ij}]}\ \psi_{p_1\epsilon_1,\ldots,p_n\epsilon_n}[h'_{ij}] \tag{D.-7}$$

This expression can be matched with (D.6.4) to extract the n-graviton $n-$states. The (non-normalized) wave functional of the one-graviton state with momentum p and polarization ϵ, for example, reads

$$\Psi_{p,\epsilon}[h_{ij}] = \delta(h_0)\ \sqrt{p}\epsilon^{ij}\ h^{TT}_{ij}(\vec{p})\ \Psi_0[h_{ij}]. \tag{D.-7}$$

D.6.5 Newton Potential from the Propagation Kernel

$$E_0 = -\frac{1}{32\pi}\int d^3x\int d^3y\frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}, \qquad m = \int \rho(\vec{x})\ d^3x. \tag{D.-7}$$

D.7 Holonomies

$$\begin{aligned}\Delta\phi &= \frac{e}{\hbar c}\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S}\\ &= \frac{e\Phi}{\hbar c}\end{aligned} \tag{D.-7}$$

$\Phi = \int \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux through the surface

$$A \rightarrow A' = A - A \tag{D.-7}$$

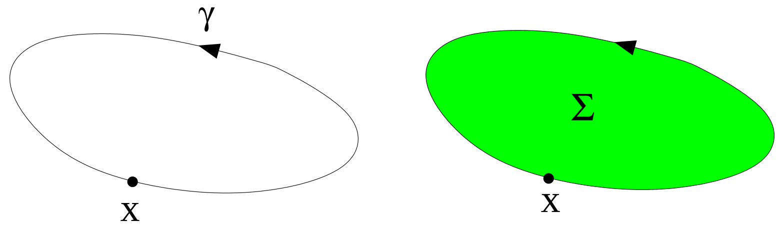


Figure D.8: The Wilson loop integral is taken around a closed loop. It can be expressed in terms of the flux integral of the field strength over a surface bounded by the loop γ.

$$\boxed{{A'}_a = UA_aU^{-1} + iU(\partial_a)U^{-1}.} \tag{D.-6}$$

whereas the holonomy transforms homogeneously

$$\boxed{H(\gamma; A') = e^{-i\Lambda(P_2)}H(\gamma; A)e^{-i\Lambda(P_1)}.} \tag{D.-5}$$

Suppose that we consider the displacement

$$P_1 = x, \quad P_2 = x + dx. \tag{D.-5}$$

$$H(\gamma; A) = 1 + iA \cdot dx. \tag{D.-5}$$

$$\prod_{l=1}^{N}(1 - i\Delta x_a(l)A^a(x(l))). \tag{D.-5}$$

The combined effect of many infinitesimal parallel transports leads to the integral

$$\boxed{H(\gamma, A) = \mathcal{P}\exp\left\{\int_\gamma A_a(\gamma(s))\frac{d\gamma^a}{ds}ds\right\}} \tag{D.-4}$$

Hence, the Wilson loop is the power-series expansion of the exponential, with matrices in each term ordered so that higher values of s stand to the left. This prescription is called *path-ordering* and is denoted by the symbol $\mathcal{P}$. This expression is similar to the time-ordered exponential for the interaction-picture perturbation expansion. If we consider U to be a continuous function of the parameter s, rather than fixing $s = 1$ at the end point we can write the analogous differential equation,

$$\boxed{\frac{d}{ds}H(x(s),y)=\left(g\frac{dx^a}{ds}A^i_a(x(s))T^i\right)H(x(s),y). \qquad \text{(D.-3)}}$$

$$\boxed{U(x)(1-idx^aA_a)U^{-1}(x+dx)=1-idx^aA'_a. \qquad \text{(D.-2)}}$$

$$\begin{aligned} U(x)(1-idx^aA_a)U^{-1}(x+dx) &= U(x)U^{-1}(x)+dx^aU(x)\partial_aU^{-1}(x)-idx^a[U(x)A_aU^{-1}(x)]. \\ &= 1-idx^a[iU(\partial_a)U^{-1}+UA_aU^{-1}] \\ &= 1-idx^aA'_a. \end{aligned} \tag{D.-3}$$

where we substituted (D.-6) in the final step.

$$\begin{aligned} U(P_1)H(\gamma,A)U(P_N) &= U(P_1)\,[1-idx^aA_a(P_2)]\,U(P_2)U^{-1}(P_2)\ldots \\ &\quad U(P_{N-2})^{-1}\,U(P_{N-2})[1-idx^aA_a(P_{N-1})]U^{-1}(P_{N-1}) \\ &\quad U(P_{N-1})[1-idx^aA_a(P_N)]U^{-1}(P_N) \\ &= H(\gamma,A') \end{aligned} \tag{D.-5}$$

$$\boxed{U(P_1)H(\gamma,A)U^{-1}(P_N)=H(\gamma,A') \qquad \text{(D.-4)}}$$

We now consider the case where γ is a closed loop i.e. $P_N=P_1=P$

$$U(P)H(\gamma,A)U^{-1}(P)=H(\gamma,A') \tag{D.-4}$$

taking the trace of the above equation and use the cyclic property of matrices that $\mathrm{Tr}AB=\mathrm{Tr}BA$,

$$\boxed{\begin{aligned} \mathrm{Tr}H(\gamma,A') &= \mathrm{Tr}\{U(P)H(\gamma,A)U^{-1}(P)\} \\ &= \mathrm{Tr}\{U^{-1}(P)U(P)H(\gamma,A)\}=\mathrm{Tr}H(\gamma,A). \qquad \text{(D.-3)} \end{aligned}}$$

We find that the trace of the holonomy around a closed loop is gauge invariant.

$$\mathrm{Tr}(\phi^i\tau_i\phi'^j\tau_j)=\phi^i\phi'^j\mathrm{Tr}(\tau_i\tau_j)=\phi^i\phi'^j\frac{1}{2}\mathrm{Tr}(\tau_i\tau_j+\tau_j\tau_i)=\phi^i\phi'^j\delta_{ij}= \tag{D.-3}$$

Geometric interpretation in internal space.

$$H(\gamma,A)=\phi^a\tau_a=\begin{pmatrix}\phi^0+\phi^3 & \phi^1-i\phi^2 \\ \phi^1+i\phi^2 & \phi^0-\phi^3\end{pmatrix} \tag{D.-3}$$

The trace and determinant of

$$\mathrm{Tr} H(\gamma, A) = 2\phi^0, \quad \det H(\gamma, A) = \phi^0 - (\phi^1)^2 - (\phi^2)^2 - (\phi^3)^2 \tag{D.-3}$$

The two above relations tell one that the similarity transformation in (D.-3) conserves the dot product $\phi^i\phi^i$. This suggests that the similarity transformation effects a rotation of ϕ^i. In fact

$$\exp\left(-i\frac{\mathbf{a}^i}{2}\tau_i\right)\phi^i\tau_i \exp\left(i\frac{\mathbf{a}^i}{2}\tau_i\right) = \phi'^i\tau_i \tag{D.-3}$$

this effects a rotation on ϕ^i by an angle of $|\mathbf{a}|/2$ around the axis whose unit vector is $\mathbf{a}$.

$$A_a = A_a(x) = A^I_a(x)T_I \tag{D.-3}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b] \tag{D.-3}$$

where $[A_a, A_b] = A_a A_b - A_b A_a$. Consider an infinitesimal circuit

$$= \exp\{iF_{ab}dx^a dx^b\} \tag{D.-3}$$

D.8 Topological Field Theories

Topological quantum field theories have revealed the existence of deep connections between 3-dimensional topology, complex analysis, and algebra, particularly the algebra of quantum groups.

One of the most interesting topological quantum field theory is Chern-Simons theory. This a field theory in 3-dimensions, and the reason it's called "topological" is that you don't need any metric or other geometrical structure on your 3d spacetime manifold for this theory to make sense. Thus it admits all diffeomorphisms as symmetries.

D.8.1 $U(1)$ Chern-Simons theory

$$\boxed{S = \int_\Sigma d^3x \epsilon^{\mu\nu\gamma} A_\mu \partial_\nu A_\gamma \qquad \text{(D.-2)}}$$

$$\begin{aligned}
\epsilon^{\mu\nu\gamma} A_\mu(x) B_\nu(x) C_\gamma(x) &= \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\tau}\frac{\partial x'^\gamma}{\partial x^\rho}\, \epsilon^{\sigma\tau\gamma}\, A'_\mu(x')B'_\nu(x')C'_\gamma(x') \\
&= \det\left(\frac{\partial x'}{\partial x}\right) \epsilon^{\sigma\tau\gamma}[A'_\mu B'_\nu C'_\gamma](x') \qquad \text{(D.-2)}
\end{aligned}$$

On the other hand $d^3x' = d^3x \det(\partial x'/\partial x)$.

$$\boxed{d^3x A_\mu(x) B_\nu(x) C_\gamma(x) = d^3x' A'_\mu(x') B'_\nu(x') C'_\gamma(x') \qquad \text{(D.-1)}}$$

which is invariant without the benefit of $\sqrt{-g}$.

Let's take a look at this again but this time in the language of exterior calculus:

$$S = \int_\Sigma A \wedge dA \qquad \text{(D.-1)}$$

where $dA = A_\gamma\, dx^\gamma$

$$dA = d(A_\gamma\, dx^\gamma) = \frac{\partial A_\gamma}{\partial x^\nu}\, dx^\nu \wedge dx^\gamma \qquad \text{(D.-1)}$$

$$\begin{aligned}
A \wedge dA &= (A_\mu\, dx^\mu) \wedge \frac{\partial A_\nu}{\partial x^\mu}\, dx^\nu \wedge dx^\gamma \\
&= A_\mu \frac{\partial A_\nu}{\partial x^\mu}\, dx^\mu \wedge dx^\nu \wedge dx^\gamma \qquad \text{(D.-1)}
\end{aligned}$$

D.8.2 Canonical Structure

We recall the Hamiltonian formulation of Maxwell theory. In the gauge $A_0 = 0$ the spacial components of the gauge field $\vec{A}$ are canonically conjugate to the electric field components $\vec{E}$, and Gauss's law $\vec{\nabla} \cdot \vec{E} = \rho$ appears as a constraint, for which the non-dynamical field A_0 is a Lagrange multiplier.

Now let us consider the canonical structure of the Maxwell Chern-Simons theory with Lagrangian,

$$\mathcal{L}_{MCS} = \frac{1}{2e^2} \tag{D.-1}$$

In the A^0 gauge we identify the A_i as "coordinate" fields, with corresponding "momentum" fields

$$\Gamma^i \equiv \frac{1}{e^2}\dot{A}_i + \frac{\kappa}{2}\epsilon^{ij}A_j \tag{D.-1}$$

Pure Chern-Simons Canonical Theory

$$\boxed{\mathcal{L}_{CS} = \frac{\kappa}{2}\epsilon^{ij}\dot{A}_i A_j + \kappa A_0 B} \tag{D.0}$$

Once again, A_0 is a Lagrange multiplier field, imposing the Gauss law: $B = 0$. The Lagrangian is first order in time derivatives, so it is already in the Legendre transformed form $\mathcal{L} = p\dot{(x)} - \mathcal{H}$, with $\mathcal{H} = 0$.The only dynamics would be inherited from coupling matter fields.

At first sight, pure Chern-Simmons theory looks rather boring, because the source-free classical equations of motion (N.-19) reduce to $F_{\mu\nu} = 0$, the solutions of which are just flat connections (this is saying that in Chern-Simons theory the analog of the magnetic field vanishes). This is in contrast to source free Maxwell theory

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{\det g}}\frac{\partial S_{CS}}{\partial q_{\mu\nu}} \tag{D.0}$$

The components of the gauge field are canonically conjugate to each other,

$$\{A_i(x), A_j(y)\} = \frac{i}{\kappa}\epsilon_{ij}\delta(\vec{x} - \vec{y}) \tag{D.0}$$

$$\prod^{i} = \frac{\kappa}{2}\epsilon^{ij}A_j \tag{D.0}$$

The canonical commutation relations arise because of the constraints, noting that these are second class constraints so we must use Dirac brackets to find the canonical relations between A_i and A_J.

U(1)

$$\tilde{A}_i = \oint_{C_i} \tilde{A}_j(x)dx^j \tag{D.0}$$

of any connection i.e. that can be reduced to a constant one by a homotopically trivial gauge transformation ??? . A homotopically non-trivial gauge transformation

$$g(x) = e^{2\pi i(m_1 x_1 + m_2 x_2)} \tag{D.0}$$

Eq (connection transform) show that $\tilde{A}_1$ and $\tilde{A}_2$ remain constant under this transformation, but their values are shifted:

$$\tilde{A}_i \to \tilde{A}_i + 2\pi m_i. \tag{D.0}$$

The phase space $S_! \times S_1$ has volume $(2\pi)^2$. The WKB approximation tells us that the dimension of the Hilbert space is approximately

$$\dim \mathcal{H}_{T^2}^{U(1)} \approx \frac{(2\pi)^2}{2\pi\hbar} \equiv 2k \tag{D.0}$$

in the limit of large k.

D.8.3 Chern-Simons with Magnetic Flux Lines

The charge density

$$\rho(\vec{x}, t) = e \sum_{n=1}^{N} \delta(\vec{x} - \vec{x}_n(t)) \tag{D.0}$$

describes N such particles, with the n^{th} particle following the trajectory $\vec{x}_n(t)$. The corresponding current density is

$$\vec{j}(\vec{x}, t) = e \sum_{n=1}^{N} \dot{\vec{x}}_n(t)\delta(\vec{x} - \vec{x}_n(t)). \tag{D.0}$$

$$B(\vec{x}, t) = \frac{1}{\kappa} e \sum_{n=1}^{N} \delta(\vec{x} - \vec{x}_n(t)) \tag{D.0}$$

which follows each point particle through out its motion.

If each particle has mass m, the total action is

$$\boxed{S = \frac{m}{2}\sum_{n=1}^{N}\int dt\dot{\vec{x}}^2 + \frac{\kappa}{2}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \int d^3xA_\mu J^\mu. \qquad \text{(D.1)}}$$

$$\boxed{A^i(x_n) = \frac{e}{2\pi\kappa}\sum_{m\neq n}\epsilon^{ij}\frac{(x_n^j - x_m^j)}{|\vec{x}_n - \vec{x}_m|^2} \qquad \text{(D.2)}}$$

D.8.4 $SU(2)$ Chern-Simons theory

$$\begin{aligned}
\delta\frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(A\wedge\, dA + \frac{2}{3}A\wedge A\wedge A) = \\
&= \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(\delta A\wedge\, dA + A\wedge\delta\, dA + 2A\wedge A\wedge\delta A) \\
&= \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(-\, dA\wedge\delta A + A\wedge\delta\, dA + 2A\wedge A\wedge\delta A) \\
&= \frac{\mathcal{A}}{4\pi}\int_H 2\mathrm{tr}[(A\wedge A -\, dA)\wedge\delta A] \\
&= \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(F\wedge\delta A)
\end{aligned} \tag{D.-2}$$

$$\int_H \mathrm{tr}A\wedge\delta\, dA \tag{D.-2}$$

$$\mathcal{CS} = \tag{D.-2}$$

$$A = A_\gamma^i T_i dx^\gamma \tag{D.-2}$$

$$dA = \partial_\nu A_\gamma^i T_i dx^\nu\wedge dx^\gamma \tag{D.-2}$$

$$A\wedge\, dA = A_i^\mu\partial_\nu A_\gamma^j T_i T_j dx^\mu\wedge dx^\nu\wedge dx^\gamma \tag{D.-2}$$

$$A\wedge\, dA=\epsilon^{\mu\nu\gamma}A^i_\mu\partial_\nu A^j_\gamma T_iT_j dx^1\wedge dx^2\wedge dx^3 \tag{D.-2}$$

Similarly $A\wedge A\wedge A=\epsilon^{\mu\nu\gamma}A^i_\mu A^j_\nu A^k_\gamma T_iT_jT_k dx^1\wedge dx^2\wedge dx^3$

$$\mathcal{CS}=\epsilon^{\mu\nu\gamma}\left(A^\mu_i\partial_\nu A^j_\gamma T_iT_j-\frac{2}{3}A^\mu_i A^j_\nu A^k_\gamma T_iT_jT_k\right)\,dv \tag{D.-2}$$

$$dv=dx^1\wedge dx^2\wedge dx^3 \tag{D.-2}$$

$$\frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(A\wedge\, dA+\frac{2}{3}A\wedge A\wedge A) \tag{D.-2}$$

$$\int_{\mathcal{M}}\mathrm{tr}(\mathcal{CS}^g)=\int_{\mathcal{M}}\mathrm{tr}(\mathcal{CS})+8\pi^2 n(g) \tag{D.-2}$$

where $n(g)$ is the degree of the mapping

by exponentiating the integral we obtain a gauge invariant quantity

$$\exp\left(\frac{ik}{4\pi}\mathrm{tr}\mathcal{CS}\right). \tag{D.-2}$$

$$K_\mu=\frac{1}{6}\epsilon^{\mu\nu\gamma\delta}\mathrm{tr}(\Omega^{-1}\partial_\nu\Omega)(\Omega^{-1}\partial_\gamma\Omega)(\Omega^{-1}\partial_\delta\Omega) \tag{D.-2}$$

we parameterize the invariant $SU(2)$ group measure dU (which we introduced in appendix A) as follows:

$$dU=\rho(\sigma_1,\sigma_2,\sigma_3)\,d\sigma_1\,d\sigma_2\,d\sigma_3 \tag{D.-2}$$

where U is an element of $SU(2)$ parameterized by some coordinates σ_i. Let U_0 be a fixed element of $SU(2)$ and $U'=U_0U$. If $\{\sigma\}$ are the coordinates that parameterize U and $\{\sigma\}$ the coordinates that parameterize U', then the group measure obeys:

$$dU=\,dU'=\rho(\sigma'_1,\sigma'_2,\sigma'_3)\,d\sigma'_1\,d\sigma'_2\,d\sigma'_3 \tag{D.-2}$$

that is, the group measure obeys $dU=\,dU_0U)$ for fixed U_0 - the rearrangement theorem for continuous groups.

The invariant measure is given by

$$\boxed{\rho(\sigma_1, \sigma_2, \sigma_3) = \epsilon^{ijk}\mathrm{tr}\left(U^{-1}\frac{\partial U}{\partial \sigma_i}U^{-1}\frac{\partial U}{\partial \sigma_j}U^{-1}\frac{\partial U}{\partial \sigma_k}\right) \qquad \text{(D.-1)}}$$

With this expression one can check explicitly that:

$$\rho(\sigma_i) = \rho({\sigma'}_i)\det\left(\frac{\partial \sigma'}{\partial \sigma}\right) \tag{D.-1}$$

$$\begin{aligned} n &= \frac{1}{4\pi^2}\int d^4x\partial_\mu K^\mu \\ &= \frac{1}{24\pi^2}\oint_{S^3} d^3\sigma\epsilon^{\mu\nu\gamma\delta}\vec{n}_\mu \mathrm{tr}(\Omega^{-1}\partial_\nu\Omega)(\Omega^{-1}\partial_\gamma\Omega)(\Omega^{-1}\partial_\delta\Omega) \\ &= \frac{1}{24\pi^2}\int_G dU. \end{aligned} \tag{D.-2}$$

Let $\phi(\theta)$ map one circle S^1 ($0 \le \theta \le 2\pi$) onto another circle S^1 such that it satisfies the boundary condition $\phi(0) = \phi(2\pi) + N2\pi$ where N can be any negative or positive integer i.e. the set $\mathbf{Z}$.

for example $\phi(\theta) = 2N\pi\theta$

$$\mathcal{Q} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\frac{d\phi(\theta)}{d\theta} = \frac{1}{2\pi}[\phi(2\pi) - \phi(0)] \tag{D.-2}$$

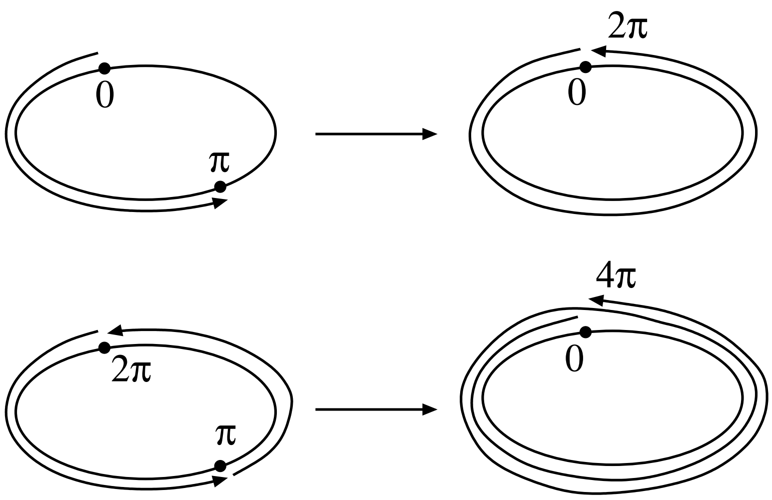


Figure D.9: $\phi(\theta) = 2\theta$ as an example of a mapping with winding number = 2.

It does not change if we smoothly deform the function $\phi(\theta)$ while keeping the boundary conditions the same. $\mathcal{Q}$ is known as the "winding number" and is a topological invariant.

Two functions are said to be in the same equivalence or "homotopy class" if they have the same N.

$$\pi_1(S^1) = \mathbb{Z} \tag{D.-2}$$

Symplectic Structure algebra

Note that in terms of the Poisson bracket $\{\ ,\ \}$ defined by ${}^G\Omega_{CS}$, we have

$$\{A_a^i(x), A_b^j(y)\} = (undretilde)\eta_{ab}k^{ij}\delta(x,y), \tag{D.-2}$$

where $(undertilde)\eta_{ab}$ and k^{ij} denote the inverse of $\tilde{\eta}^{ab}$ and k_{ij}. This result follows from the fact that for any $f : \Gamma_{CS} \to R$, the Hamilton vector field X_f is given by

$$X_f = \int_\Sigma (undretilde)\eta_{ab}k^{ij}\,\frac{\delta f}{\delta A_b^j}\frac{\delta}{\delta A_a^i}.$$

Hence the Poisson bracket of any two functions f, g is

$$\{f, g\} = \int_\Sigma (undretilde)\eta_{ab}k^{ij}\,\frac{\delta f}{\delta A_b^j}\frac{\delta g}{\delta A_a^i}$$

Constraint algebra

we construct a constraint function associated with

$$k_{ij}\tilde{\eta}^{ab}F_{ab}^i = 0.$$

given test field v^i (which takes values in the Lie algebra $\mathcal{G}$), we define

$$G(v) := \frac{1}{2}\int_\Sigma v^i k_{ij}\tilde{\eta}^{ab}F_{ab}^i \tag{D.-2}$$

$$\frac{\delta G(v)}{\delta A_a^i} = k_{ij}\tilde{\eta}^{ab}\mathcal{D}_b v^i \tag{D.-2}$$

where $\mathcal{D}_a$ is any torsion free (compatible) extension of the generalized derivative operator associated with A_a^i, (so that $\mathcal{D}_b v^j = \partial_b v^j + C_n m^j A_a^n v^m$). From this it follows that the Hamiltonian vector field $X_{G(v)}$ is given by

$$X_{G(v)} = \int_\Sigma -(\mathcal{D}_a v^i)\frac{\delta}{\delta A^i_a} \tag{D.-2}$$

so that

$$A^i_a \mapsto A^i_a - \epsilon \mathcal{D}_a v^i + \mathcal{O}(\epsilon^2) \tag{D.-2}$$

under the 1-parameter family of diffeomorphisms on the Chern-Simon's phase space ${}^G\Gamma_{CS}$ associated with $X_{G(v)}$. This is the usual gauge transformation of the connection $A^i_a(x)$ that we find in Yang-mills theory, (also compare with section **??**). Thus, $G(v)$ can be appropriately called a Gauss constraint function. (Relation to interpretation of the isolated quantum constraint in Black hole entropy calculation???)

D.8.5 The Quantum Hall Effect

$$\begin{aligned} S_{bulk} &= \int_{\mathcal{M}} d^3x \left[-\frac{t}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\sigma_H}{2}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda \right], \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{D.-1}$$

while our metric is $(-1,+1,+1)_{diag}$

under a gauge transformation the surface term

$$-\frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x \epsilon^{\mu\nu}(\partial_\mu \alpha)A_\nu. \tag{D.-2}$$

$$S_{tot} = S_{bulk} + \frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\epsilon^{\mu\nu}(\partial_\mu\phi)A_\nu - \frac{\sigma_H}{4}\int_{\partial\mathcal{M}} d^2x\epsilon^{\mu\nu} D_\mu\phi D^\mu\phi, \tag{D.-1}$$

$$D_\mu\phi = \partial_\mu\phi - A_\mu. \tag{D.0}$$

The field transforms as

$$\phi \to \phi + \alpha \tag{D.0}$$

so that

$$D\phi := d\phi - A \to D\phi. \tag{D.0}$$

As we will now demonstrate, it is the second term in which restores gauge invariance.

$$\begin{aligned}\frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\epsilon^{\mu\nu}\partial_\mu\phi A_\nu &\rightarrow \frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\epsilon^{\mu\nu}(\partial_\mu\phi+\partial_\mu\alpha)(A_\nu+\partial_\nu\alpha)\\ &= \frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\left[\epsilon^{\mu\nu}(\partial_\mu\phi)A_\nu+\epsilon^{\mu\nu}(\partial_\mu\alpha)A_\nu+\epsilon^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\alpha)\right]\end{aligned} \quad \text{(D.-1)}$$

D.9 How to Quantize 2+1 Gravity and Solve it Exactly *a la* Witten

Witten [388] was able to show that the 2+1 theory of gravity simplifies considerably when expressed in Palantini form. in fact, Witten demonstrated the the 2+1 Palantini Tetrad theory was equivalent to Chern-Simons thoery based on the inhomogeneous Lie group $ISO(2,1)$ and thus could be explicitly canonically quantized.

$$S_{EH}(e)=\frac{1}{2}\int\tilde{\eta}^{abc}\epsilon_{IJK}\, e_a^I\, R_{bc}^{\ \ JK} \quad \text{(D.-1)}$$

$$S_{EH}(e,\omega)=\frac{1}{2}\int\tilde{\eta}^{abc}\epsilon_{IJK}\, e_a^I\, F_{bc}^{\ \ JK} \quad \text{(D.-1)}$$

where

$$F_{abI}^{\ \ \ \ J}=2\partial_{[a}\omega_{b]I}^{\ \ \ J}+[\omega_a,\omega_b]_I^{\ J}$$

Since the 2+1 Palantini action is a functional of both a co-triad and a connection 1-form, we will obtain two Euler-Lagrange equations of motion.

$$e^a=e^a_\mu dx^\mu, \quad \omega^a=\frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu, \quad \text{(D.-1)}$$

The first order action takes the form

$$T^a_{\mu\nu} = \partial_\mu e_\nu^{\ a}-\partial_\nu e_\mu^{\ a}+\epsilon^a_{\ bc}(\omega_\mu^{\ b}e_\nu^{\ c}+\omega_\nu^{\ b}e_\mu^{\ c})=0 \quad \text{(D.0)}$$

$$R^a_{\mu\nu} = \partial_\mu\omega_\nu^{\ a}-\partial_\nu\omega_\mu^{\ a}+\epsilon_{abc}\omega_\mu^{\ b}\omega_\nu^{\ c}=0 \quad \text{(D.1)}$$

The first of these implies that the connection is torsion-free, and, if e is invertible, that ω has the standard expression in terms of the triad. Given such a spin connection, (D.1) is then equivalent to the standard Einstein field equations.

In a vacuum in three-dimensions the vanishing of the Ricci tensor implies the Reinmann tensor vanishes

$$R_{ab} = 0 \implies R^a{}_{bcd} = 0 \tag{D.1}$$

so the solution of the equations of motion is that curvature vanishes.

$$S_{CS} = \frac{k}{4\pi} tr \int (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{D.1}$$

This condition, which is imposed *a priori* in Einstein's formulation of general relativity, is seen to be part of the equations of motion.

The local trivialiaation of the gauge field gives rise to a parametrization of the phase space in terms of the holonomies along a set of generators of the fundamental group $\pi_1(S_g)$ and ...

Using the explicit algebra basis, the commutation relations

$$[T_a, T_b] = C_{ab}{}^c T_c.$$

and the inner product $< T_a, T_b >$, we can obtain a more explicit expression for the Cher-Simons action. First we consider the term $A \wedge A \wedge A$

$$\begin{aligned}
\epsilon^{ijk} A_i{}^a A_j{}^b A_k{}^c \, T_a T_b T_c &= \frac{1}{2}\epsilon^{ijk}(A_i{}^a A_j{}^b A_k{}^c - A_j{}^a A_i{}^b A_k{}^c) \, T_a T_b T_c \\
&= \frac{1}{2}\epsilon^{ijk}(T_a T_b T_c - T_b T_a T_c) A_i{}^a A_j{}^b A_k{}^c \\
&= \frac{1}{2}\epsilon^{ijk} \, [T_a, T_b] T_c \, A_i{}^a A_j{}^b A_k{}^c
\end{aligned} \tag{D.0}$$

with the notation $tr(\cdot,\cdot) =< \cdot, \cdot >$ we write

$$\begin{aligned}
S_{CS} &= k \int_{\mathcal{M}} d^3x \Big\{ < T_a, T_b > \epsilon^{ijk} A_i{}^a (\partial_j A_k{}^b - \partial_k A_j{}^b) + \\
&\quad + \frac{2}{3} < \frac{1}{2}[T_a, T_b], T_c > \epsilon^{ijk} A_i{}^a A_j{}^b A_k{}^d \Big\},
\end{aligned} \tag{D.0}$$

which on substituting the commutation relations becomes

$$
\begin{aligned}
S_{CS} &= < T_c, T_d > k \int_{\mathcal{M}} d^3x \epsilon^{ijk} \left(A_i{}^c (\partial_j A_k{}^d - \partial_k A_j{}^d) + \frac{1}{3} C_{ab}{}^c A_i{}^a A_j{}^b A_k{}^d \right) \\
&= k \int_{\mathcal{M}} d^3x \epsilon^{ijk} k_{cd} \left(A_i{}^c (\partial_j A_k{}^d - \partial_k A_j{}^d) + \frac{1}{3} C_{ab}{}^c A_i{}^a A_j{}^b A_k{}^d \right)
\end{aligned} \tag{D.0}
$$

It is important to note that Chern-Simons theory is not defined for arbitrary Lie groups - we need the additional structureprovided by the invariant , nondegenerate bilinear form k_{ab}.

$$
< T_a, T_b >= k_{ab} \tag{D.0}
$$

For the Lie group $SU(2)$ we have

$$
k_{ab} = \frac{\delta_{ab}}{2} \tag{D.0}
$$

??

semi-direct products

N is the group of all translations in three-space and H is the group of all rotations about some fixed origin. $N \otimes_S H$ is the group of all, the group generated by all translations and rotations.

N is the group of all translations in spacetime and H is the group of all homogeneous Lorentz transformations. $N \otimes_S H$ is called the Poincarë group. the symmetry group of spacetime special relativity.

??

$$
\mathbb{R}^3 \otimes_S SO(3). \tag{D.0}
$$

We introduce translation generators P_a, $a = 1, 2, 3$, which satisfy

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{ab}{}^c J_c, \\
[J_a, P_b] &= \epsilon_{ab}{}^c P_c, \\
[P_a, P_b] &= 0
\end{aligned} \tag{D.-1}
$$

The group has a nondegenerate invariant bilinear form, a form that commutes with all algebra elements. The expression $J^a P_a$ satisfies these conditions as we can check:

$$
\begin{aligned}
[J^a P_a, J_b] &= J^a[P_a, J_b] + [J^a, J_b]P_a = \epsilon_{ab}{}^c(J^a P_c + \eta^{ad} J_c P_d) = 0 \\
[J^a P_a, P_b] &= J^a[P_a, P_b] + [J^a, P_b]P_a = \epsilon_{ab}{}^c \eta^{ad} P_c P_d = 0
\end{aligned}
\tag{D.-1}
$$

$$
Tr(T_a T_b) = \sum_{\alpha\beta} C_{a\alpha\beta} C_{b\alpha\beta} = k_{ab} \tag{D.-1}
$$

Lie algebra $su(2)$ and can be represented by the matrices

$$
J_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad J_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{D.-1}
$$

in (D.0)

$$
\begin{array}{ll}
T_1 = J_1 & T_4 = P_1 \\
T_2 = J_2 & T_5 = P_2 \\
T_3 = J_3 & T_6 = P_3
\end{array}
\tag{D.-2}
$$

$[T_a, T_b] = C_{ab}{}^c T_c$ and $\eta = (-1, 1, 1)$

$$
\begin{aligned}
[T_1, T_2] &= C_{12}{}^a T_a = C_{12}{}^3 T_3 = \epsilon_{123} \eta^{33} T_3, \\
[T_1, T_3] &= C_{13}{}^a T_a = \epsilon_{12}{}^3 T_2 = C_{13}{}^2 T_2, \\
[T_2, T_3] &= C_{23}{}^a T_a = \epsilon_{23}{}^1 T_1 = C_{23}{}^1 T_1 \\
\\
[T_1, T_1] &= C_{11}{}^a T_a = 0, \\
[T_2, T_2] &= C_{11}{}^a T_a = 0, \quad etc... \\
\\
[T_1, T_4] &= [J_1, P_1] = C_{14}{}^a T_a = 0 \\
[T_1, T_5] &= [J_1, P_2] = C_{15}{}^a T_a = \epsilon_{12}{}^3 T_6 \\
[T_1, T_6] &= [J_1, P_3] = C_{16}{}^a T_a = \epsilon_{13}{}^2 T_5 \\
\\
[T_4, T_5] &= [P_1, P_2] = C_{45}{}^a T_a = 0, \quad etc...
\end{aligned}
\tag{D.-12}
$$

$$
(C_{ab}{}^1) = \begin{pmatrix} 0 & 0 & 0 & 0 & & \\ 0 & 0 & -1 & & & \\ 0 & 1 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \quad (C_{ab}{}^2) = \begin{pmatrix} 0 & -1 & 0 & 0 & & \\ 0 & 0 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \tag{D.-12}
$$

$$
(C_{ab}{}^3) = \begin{pmatrix} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \quad (C_{ab}{}^4) = \begin{pmatrix} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \tag{D.-12}
$$

An element of the semi-direct product $\mathbb{R}^3 \otimes_S SO(3)$ is

$$
\exp\left\{\alpha_c (C_{ab})^c\right\}
$$

$$
\begin{aligned}
k_{11} &= Tr(T_1 T_1) = \sum_{\alpha,\beta=1}^{6} C_{1\alpha\beta} C_{1\alpha\beta} = \\
&=
\end{aligned} \tag{D.-12}
$$

The inner product

$$
< \cdot\,,\ \cdot >= k_{ab} \cdot\ \cdot \tag{D.-12}
$$

The inner product can be identified:

$$
\begin{aligned}
< J_a, P_b > &= \eta_{ab}, \\
< J_a, J_b > &= 0, \\
< P_a, P_b > &= 0
\end{aligned} \tag{D.-13}
$$

the Cartan connection can be written as

$$
A_\mu(x) = \sum_{a'=1}^{6} A_\mu^{a'}(x) T_{a'} = \sum_{a=1}^{3} (\omega_{a\mu}(x) J^a + e_{a\mu}(x) P^a) \tag{D.-13}
$$

It is a connection on a principle bundle $SO(3)$ bundle, but takes values in $iso(3)$.

the terms $< \frac{1}{2}[T_a, T_b], T_c >$ are

$$\begin{aligned} < [J_a, P_b], J_c > &= \epsilon_{ab}{}^d < P_d, J_c >= \epsilon_{ab}{}^d \eta_{cd} = -\epsilon_{abc} \\ < [J_a, J_b], P_c > &= \epsilon_{ab}{}^d < J_d, P_c >= \epsilon_{abc} \\ \\ < [P_a, P_b], P_c > &= 0, \quad < [P_a, P_b], J_c >= 0 \\ < [J_a, P_b], P_c > &= \epsilon_{ab}{}^d < P_d, P_c >= 0 \\ < [J_a, J_b], J_c > &= \epsilon_{ab}{}^d < J_d, J_c >= 0 \end{aligned} \tag{D.-17}$$

Substituting these into (D.0) we get

$$\begin{aligned} S_{CS} &= \frac{k}{2}\int d^3x\epsilon^{\mu\nu\rho}\Big\{ < J_a, P_b > \omega_\mu{}^a\partial_\nu e_\rho{}^b + < P_a, J_b > e_\mu{}^a\partial_\nu\omega_\rho{}^b + \\ &\quad \frac{2}{3}(\frac{1}{2} < [P_a, J_b], J_c > e^a_\mu\omega_\nu{}^b\omega_\rho{}^c \\ &\quad +\frac{1}{2} < [J_a, P_b], J_c > \omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c + \frac{1}{2} < [J_a, J_b], P_c > \omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c)\Big\} \\ &= \frac{k}{2}\int d^3x\epsilon^{\mu\nu\rho}\Big(\omega_\mu{}^a\partial_\nu e_{a\rho} + e^a_\mu\partial_\nu\omega_\rho{}^a + \epsilon_{abc}e^a_\mu\omega_\nu{}^b\omega_\rho{}^c\Big) \\ &= \frac{k}{2}\int \epsilon^{\mu\nu\rho}\Big(e_{\mu a}(\partial_\nu\omega_\rho{}^a - \partial_\rho\omega_\nu{}^a + \epsilon^a{}_{bc}\omega_\nu{}^b\omega_\rho{}^c)\Big) + \text{ (surface term)} \\ &= \int d^3x\epsilon^{\mu\nu\rho}e_{\rho a}R^a{}_{\mu\nu} \end{aligned} \tag{D.-21}$$

D.9.1 Gauge transformations

gauge transformation generated by $\epsilon = \rho^a P_a + \tau^a J_a$, where ρ^a and τ^a are infinitesimally small.

$$\begin{aligned} \delta A_\mu &= -D_\mu\epsilon = -\partial_\mu\epsilon - [A_\mu, \epsilon] \\ &= -(\partial_\mu\rho^a + \epsilon^a{}_{bc}\omega_\mu{}^b\rho^c + \epsilon^a{}_{bc}e_\mu{}^b\tau^c)P_a - (\partial_\mu\tau^a + \epsilon^a{}_{bc}\omega_\mu{}^b\tau^c)J_a \end{aligned} \tag{D.-21}$$

so that

$$\delta\omega_\mu{}^a = -(\partial_\mu\tau^a + \epsilon^a{}_{bc}\omega_\mu{}^b\tau^c) \tag{D.-20}$$
$$\delta e_\mu{}^a = -(\partial_\mu\rho^a + \epsilon^a{}_{bc}\omega_\mu{}^b\rho^c + \epsilon^a{}_{bc}e_\mu{}^b\tau^c) \tag{D.-19}$$

Now a diffeomorphism generated by a vector field $-v^\mu$

$$\begin{aligned}
\delta_{Diff}\omega_\mu{}^a &= -v^\nu(\partial_\nu\omega_\mu{}^a - \partial_\mu\omega_\nu{}^a) - \partial_\mu(v^\nu\omega_\nu{}^a) && \text{(D.-18)} \\
\delta_{Diff}e_\mu{}^a\delta &= -v^\nu(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a) - \partial_\mu(v^\nu e_\nu{}^a) && \text{(D.-17)}
\end{aligned}$$

If we take

$$\rho^a = v^\mu e_\mu{}^a$$

and compare with (D.-17) with (D.-19)

$$\delta_{Diff}e_\mu{}^a - \delta e_\mu{}^a = -v^\nu(D_\nu e_\mu{}^a - D_\mu e_\nu{}^a) + \epsilon^{abc}v^\nu\omega_{\nu b}e_{\mu c}. \tag{D.-17}$$

The expression

$$D_\nu e_\mu{}^a - D_\mu e_\nu{}^a$$

is just the torsion, which should vanish as a consequence of the equations of motion. The remaining term $\epsilon^{abc}v^\nu\omega_{\nu b}e_{\mu c}$ corresponds to a local Lorentz transformation with parameter $\tau^a = v^\mu\omega_{\mu a}$.

Similarly, the difference between the transformations of $\omega_{\mu a}$ is

$$\delta_{Diff}\omega_\mu{}^a - \delta\omega_\mu{}^a = -v^\nu(D_\nu\omega_\mu{}^a - D_\mu\omega_\nu{}^a) + \epsilon^{abc}v^\nu\omega_{\nu b}\omega_{\mu c}. \tag{D.-17}$$

Here we have chosen again $\tau^a = v^\mu\omega_{\mu a}$. Part of the difference is given by the curvature $R^a = D_\nu\omega_\mu{}^a - D_\mu\omega_\nu{}^a$, which upon imposing the equations of motion vanish es. The remaining term is again a local Lorentz transformation.

So we see that the Lorentz transformations and diffeomorphisms agree with the gauge transformations of $ISO(2,1)$ as long as we impose the equations of motion following from the Chern-Simons action. The space of metrics solving Einstein's equation up to infinitesimal active diffeomorphisms is therefor isomorphic to the space of flat Chern-Simons gauge fields modulo infinitesimal Chern-Simons gauge transformations.

Inhomogeneous Lie groups

$\mathcal{L}_G \oplus \mathcal{L}_G^*$

The Lie bracket

$$[v, w] := C^I{}_{JK} v^J w^K$$

and the co-adjoint bracket

$$\{v, \beta\} := C^K{}_{JI} v^J \beta_K$$

We can define a bracket on $\mathcal{L}_G \oplus \mathcal{L}_G^*$

$$[(\alpha, v), (\beta, w)]^i := (-\{w, \alpha\} + \{w, \alpha\}, [v, w])^i, \tag{D.-17}$$

$$\{[v, w], \alpha\}_I = -\{v, \{w, \alpha\}\}_I + \{w, \{v, \alpha\}\}_I$$

use this to show that (D.9.1) satisfies the Jacobi identity.

$$\{C^L{}_{JK} v^J w^K, \alpha\}_I = -\{v$$

the inhomogeneous Lie group IG is obtained by exponentiating the Lie algebra $\mathcal{L}_{IG}$.

D.9.2 Cosmological Constant

$$S_{CS} = \frac{k}{2} \int \epsilon^{\mu\nu\rho} \Big(e_{\mu a} (\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a) + \epsilon_{abc} e_\mu{}^a (\omega_\nu{}^b \omega_\rho{}^c - \frac{\lambda}{3} e_\nu{}^b e_\rho{}^c) \Big) \tag{D.-17}$$

$-\lambda$ is the cosmological constant. The curvature tensor is here

$$R^a{}_{\mu\nu} = \partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + \epsilon_{abc} \omega_\mu{}^b \omega_\nu{}^c - \frac{\lambda}{3} e_\nu{}^b e_\rho{}^c \tag{D.-17}$$

The torsion $T_{\mu\nu}{}^a$ is the same. If we don't change the inner product on the algebra, this term can come from the term

$$\frac{k}{2} \int d^3x \epsilon^{\mu\nu\rho} \frac{2}{3} \big(\frac{1}{2} < [P_a, P_b], P_c > e_\mu{}^a e_\nu{}^b e_\rho{}^c \big) \tag{D.-16}$$

if we take $[P_a, P_b]$ equal to $-\lambda \epsilon_{ab}{}^c J_c$ instead of zero. This changes the gauge group:

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{ab}{}^c J_c, \\
[J_a, P_b] &= \epsilon_{ab}{}^c P_c, \\
[P_a, P_b] &= -\lambda \epsilon_{ab}{}^c J_c
\end{aligned}
\qquad \text{(D.-17)}
$$

D.10 Witten and Link Polynomials

A knot is a non-intersecting smooth closed curve in a three-manifold (see fig.(D.10)). It is said to be oriented if there is a direction...

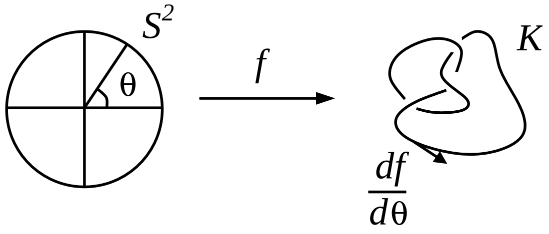


Figure D.10: knotDef.

With a given knot we associate a *knot diagram* obtained by projecting the knot on to a plane.

D.10.1 Knot and Link Polynomials

Two shadows of the same knot are related by a sequence of Reidermeister moves.

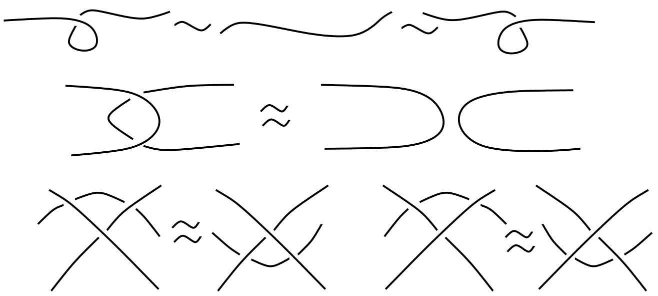


Figure D.11: Reidemeister moves.

The bracket polynomial

Try to form a invriant under Reidermeister moves - then if two shadow representations of the same knot will share this invariant.

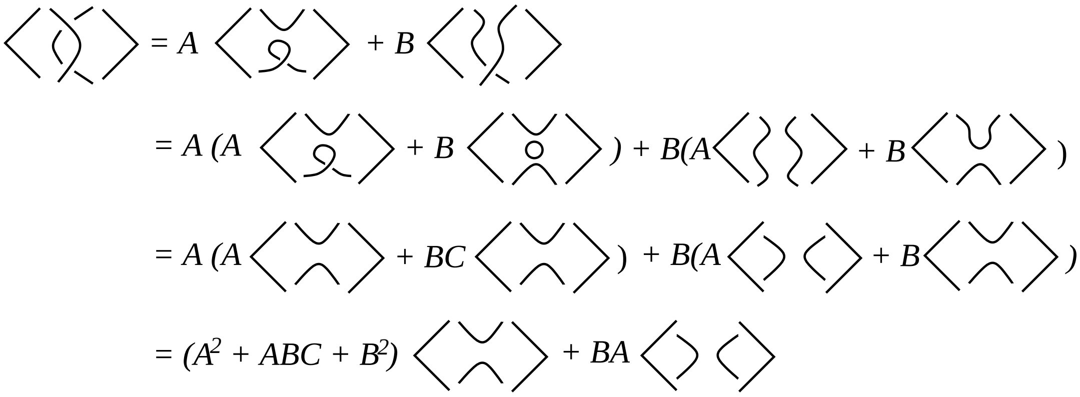


Figure D.12: Bracketpoly1.

$$AB = 1 \quad \Rightarrow B = A^{-1} \qquad A^2 + ABC + B^2 = 0 \quad \Rightarrow C = -A^2 - A^{-2} \tag{D.-17}$$

Then if two shadows have different polynomials they are topologically inequivalent.

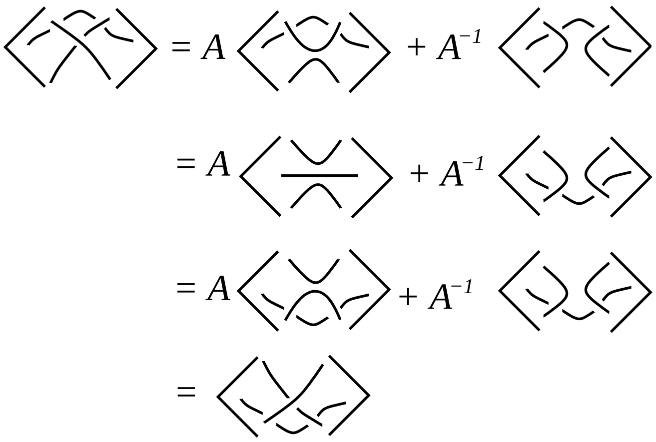


Figure D.13: Bracketpoly2.

Where we used move II a couple of times. So move III follows from move RII!

So not an invariant of ambeint isotopy. But we can fix it

The bracket polynomial is an invariant of the regular isotopy - the framing is not irrelevant.

$$\mathcal{L}_K(A) = (-A^3)^{-w(K)} < K > \tag{D.-17}$$

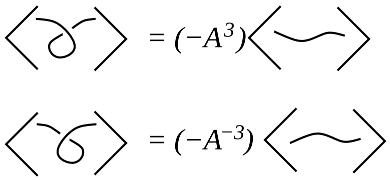


Figure D.14: Bracketpoly4.

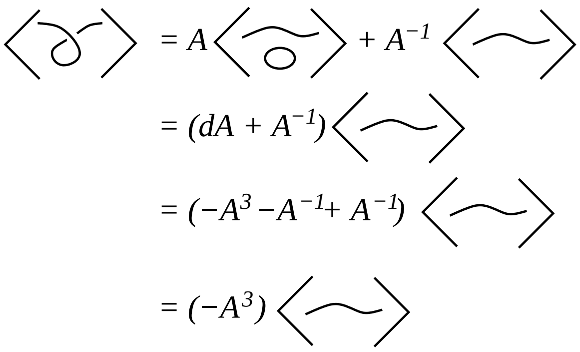


Figure D.15: Brackpoly4Pr. Proof of bracket of curl

The Jones polynomial

$$V_K(t) = \mathcal{L}_K(t^{-1/4}) \tag{D.-17}$$

The Jones polynomial is an invariant of oriented links - the framing is irrelevant.

$$A(-A^3) <> (-A^3)^{-(w(K)+1)} - A^{-1}(A^3)^{-1} <> (-A^3)^{-(w(K)-1)} = (A^2 - A^{-1}) <> (-A^3)^{-w(K)} \tag{D.-17}$$

$$\begin{aligned} A(-A^3)\mathcal{L} + A^{-1}(A^3)^{-1}\mathcal{L} &= (A^2 - A^{-1})\mathcal{L} \\ -A^4\mathcal{L} + A^{-4}\mathcal{L} &= (A^2 - A^{-2})\mathcal{L} \end{aligned} \tag{D.-17}$$

Letting $A = t^{-1/4}$, we finally arrive at

$$t^{-1}\mathcal{L}X - t\mathcal{L}X = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\mathcal{L}X \qquad \mathcal{L}\,O = 1 \tag{D.-17}$$

HOMFLY Polynomial

The first discovery was a direct generalization of the original Jones polynomial to an invariant $P_K(a;z)$ in two variables a and z such that

$$aP_{K_+} - a^{-1}P_{K_-} = zP_{K_0}. \tag{D.-17}$$

PK is called the HOMFLY polynomial after the different people that discovered it and proved its properties. They are Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, (Przytcki and Trawzyk) in the order of the acronym.

D.10.2 Link Invariants from Chern-Simons Field Theory

$$W(\gamma) = Tr\ \mathcal{P} \oint ds\ \dot{\gamma}^a(s)A_a(\gamma(s)) \tag{D.-17}$$

Smooth deformations in the ambient space do not modify the topological properties of the link.

$$< W(\gamma) > = \int \mathcal{D}[A]\ e^{CS}\ W(A,\gamma) \tag{D.-17}$$

Now this expectation value should be invariant under diffeomorphisms, i.e. it should be a knot invariant, parameterized by the coupling constant k.

D.10.3 Witten On the Jones Polynomial

$$W_\gamma[A] = 1 + \int \mathcal{L}_{CS} ds + \int_{CS} ds \int_{CS} \mathcal{L}_{CS} dt + \dots \tag{D.-17}$$

Figure D.16: Chern-Simons regularization.

Perturbation theory

Smooth deformations in the ambient space do not modify the topological properties of the link defined by $\{C, C_f\}$. An ambient isotopy invariant of the framed knot C.

If we expand this out, remembering the expressions for X and g we get,

$$< W(A,\gamma) >^{(1)} = \oint ds \oint dt \dot{\gamma}^a(s)\dot{\gamma}^b(t)\epsilon_{abc}\frac{\gamma^c(s)-\gamma^c(t)}{|\gamma^c(s)-\gamma^c(t)|^3} = \mathrm{GSL}\gamma) \tag{D.-17}$$

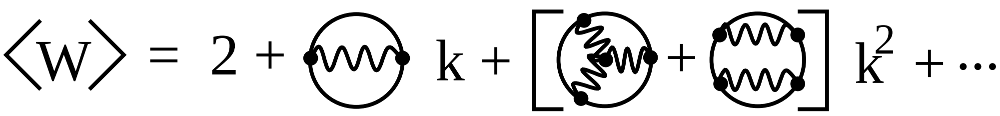


Figure D.17: DiagExpanW. Chern-Simons.

$$\text{Kauffmann bracket}(\gamma)[k] = e^{kGSL(\gamma)}\text{Jones polynomial}(\gamma)[k] \tag{D.-17}$$

so we see that all the framing dependence can be concentrated in the "phase factor" $\exp(kGSL(\gamma))$.

This translates into a recursion relation for the expectation values of Wilson loops given by

$$\alpha\mathcal{Z}(\mathcal{L}) + \beta\mathcal{Z}(\mathcal{L}_1) + \alpha\mathcal{Z}(\mathcal{L}_2) = 0. \tag{D.-17}$$

Diagrammatically this is

$$\begin{aligned}
\alpha &= -\exp\left(\frac{2\pi i}{N(N+k)}\right) \\
\beta &= -\exp\left(\frac{i\pi(2-N-N^2)}{N(N+k)}\right) + \exp\left(\frac{i\pi(2+N-N^2)}{N(N+k)}\right) \\
\gamma &= \exp\left(\frac{2\pi i(1-N^2)}{N(N+k)}\right)
\end{aligned} \tag{D.-19}$$

The generalisation of the Jones polynomial to the gauge group $G = SU(N)$ is the HOMFLY polynomial. In the case where $G = SU(2)$ we get the Jones polynomial.

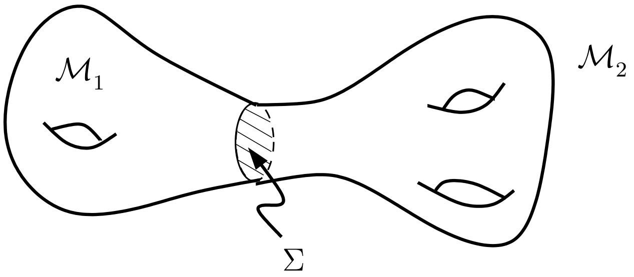


Figure D.18: Heegard. Heegard spilting of a three manifold $\mathcal{M}$ into two three manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ with common boundary Σ.

Witten' paper

The boundary of $\mathcal{M}_2$ is also Σ, but with the opposite orientation , so its Hilbert space is the dual space $\mathcal{H}(\Sigma)^*$. The homeomorphism $f : \Sigma \rightarrow$ is represented by an operator acting on $\mathcal{H}(\Sigma)$

$$U_f : \mathcal{H}(\Sigma) \rightarrow \mathcal{H}(\Sigma). \tag{D.-19}$$

and the partition function is

$$\mathcal{Z}(\mathcal{M}) = < \Psi_{\mathcal{M}_2} | U_f | \Psi_{\mathcal{M}_1} > . \tag{D.-19}$$

If we know the wavefunctions and the operators associated to homeomorphisms we can compute the partition function.

For a conformal field theory in two dimensions

$$< \phi(z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n) >= \sum_{ab} C^{ab} \mathcal{F}_a(z_1, \ldots, z_n) \overline{\mathcal{F}}_b(\overline{z}_1, \ldots, \overline{z}_n) \tag{D.-19}$$

$\mathcal{F}_a(z_1, \ldots, z_n)$ are called the conformal blocks or chiral blocks.

The physical Hilbert space of a three-dimensional topological Chern-Simons theory can be interpreted as the space of conformal blocks of the corresponding Wess-Zumino-Witten model of two dimensions.

The space of conformal blocks of a WZW model on Σ with gauge group G and level k.

The connection between Chern-Simons theory and Wess-Zumino-Witten conformal field theory, the functional integrals of these three-balls correspond to states in the space of four point correlator conformal blocks of the WZW conformal theory.

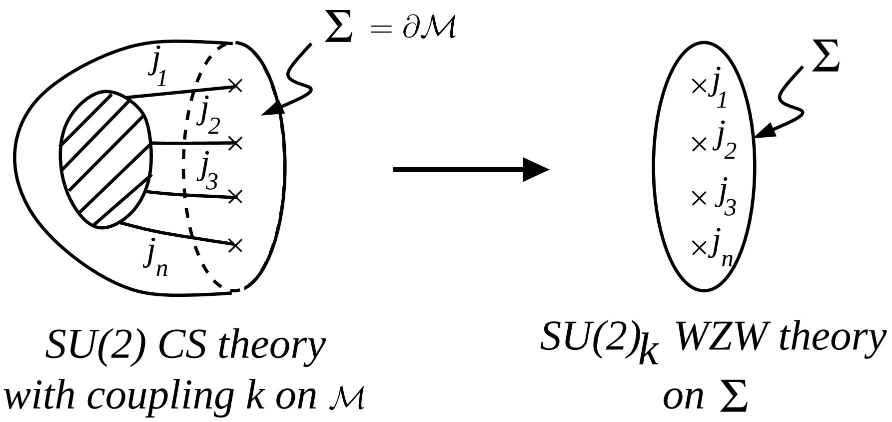


Figure D.19: ChernStoWZW. (a) $SU(2)$ CS theory with coupling k with Wilson lines carrying representations $j_1, j_2, \ldots, j_n$, ending at n points in the boundary Σ. (b) $SU(2)_k$ WZW theory on Σ with n punctures carrying primary fields in representations $j_1, j_2, \ldots, j_n$.

The dimension of the space of conformal blocks depends on a simply connected complex Lie grouop and an integer called the level.

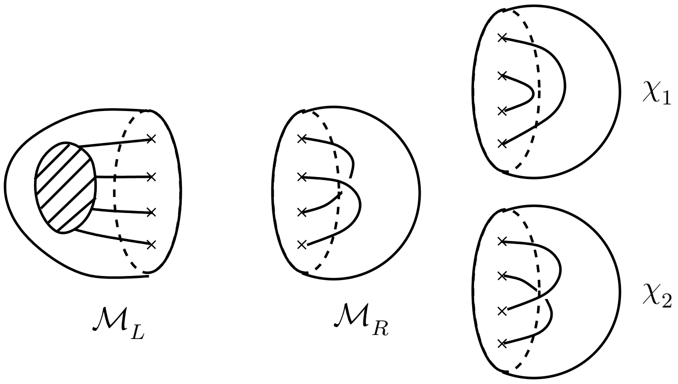


Figure D.20: CuttingMan.

D.10.4 Dehn Surgery

There is a procedure, called Dehn surgery, by which all closed (compact and without boundary), orientable, connected three-dimensional manifolds can be constructed []. One begins by drawing a knot or link (a link is a set of knotted loops all tangled up) on a given manifold. One then cuts out a tubular neighbourhood around each of the knotted lines a glues them back in differently.

We can see the number of times a knot wraps meridionally around the torus and the

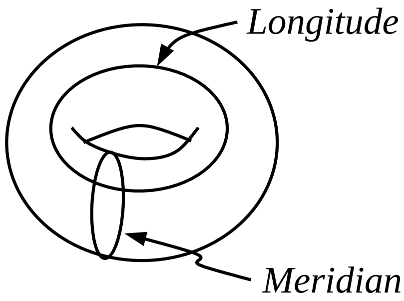


Figure D.21: TorusLongMer. Ameridian and longitude on a torus.

number of times it wraps longitudinally by counting the number of times the knot crosses a meridian and the number of times it crosses a longitude curve respectively.

A three-sphere S^3 can be constructed by gluing together two solid tori.

“The Knot Book, An Elementary Introduction to the Mathematical Theory of Knots”, Colin C. Adams.

“Intuitive Topology”, V. V. Prasolov.

gluing the two boundaries

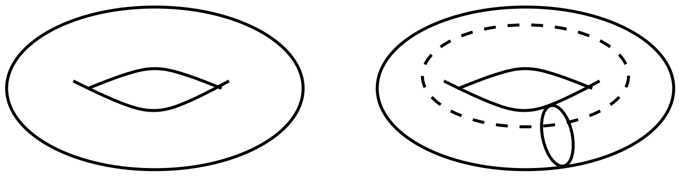


Figure D.22: gluingTori. tori .

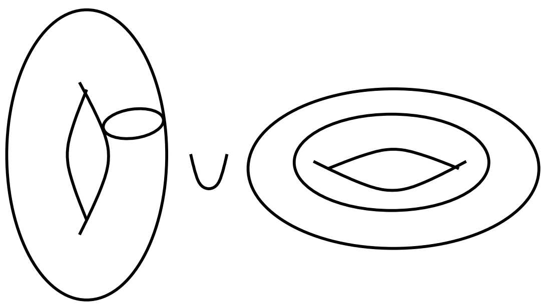


Figure D.23: gluingTori2. Gluing together two tori to get a 3-sphere S^3.

the complement of a knot

that two Dehn surgery descriptions of a three-manifold they must be related by Kirby moves. these are like the Reidemeister. instead of going from one projection of a knot onto another through a sequence of Reidermeister moves, Kirby moves take us from one Dehn surgery description of three-manifold to another through a sequence of Dehn descriptions of the same three-manifold.

Kerby calculus forms a basis for new invariants distinguishing three-manifolds

D.10.5 Chern-Simons Theory on a Torus

$SL(2,\mathbb{Z})$ transformation that maps the meridian M' of T' onto $nM + mL$, where M and L are a meridian and a longitude on T and m, n are respectively prime.

D.10.6 Chern-Simons Quantum Field Theory and Dehn Surgery

The Jones Polynomial Vaughan F.R. Jones Department of Mathematics, University of California at Berkeley, Berkeley CA 94720, U.S.A. 18 August 2005

The expectation value $< W(L) >$ can be inetrpreted as a scalar product between two states. Given a surface which separates $\mathbb{R}^3$ into two parts P_1 and P_2, the computation of the fucntional intergral can be divided into two parts. The integral is performed in P_1 and P_2 with given boundary conditions on the surface for the fields . These can be thought of as the “bra” and “ket” state vectors. The scalar product between these two states gives the result of the whole functional integral in $\mathbb{R}^3$.

D.10.7 Three-Manifold Invariants

Witten showed how the quantum field theory context defined many invariants of three dimensional manifolds and how these invariants could be calculated (by a surgery description) through invariants of knots and links in the three dimensional sphere.

Framed unoriented links in S^3 modulo equivalence under Kirby moves $\leftrightarrow$ Closed (compact without boundary), orientable, connected three-manifolds manifolds modulo homeomorphisms.

Combination of framed link invariants which do not change under Kirby moves = Invariants of associated three-manifold.

Quantum gravity vacuum and invariants of embedded spin networks

arXiv:gr-qc/0301047 v2 24 Jun 2003 *Quantum gravity vacuum and invariants of embedded spin networks* A. Mikovic

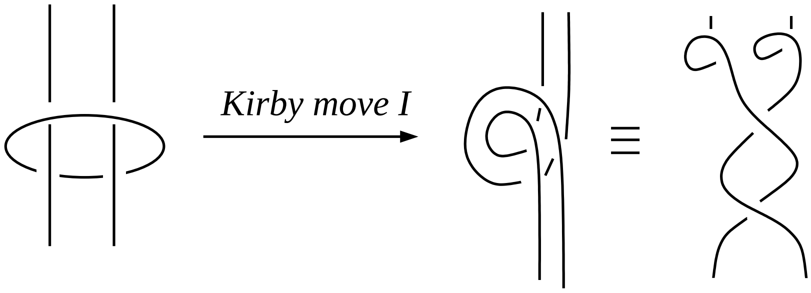


Figure D.24: KirbyMIfigs2.

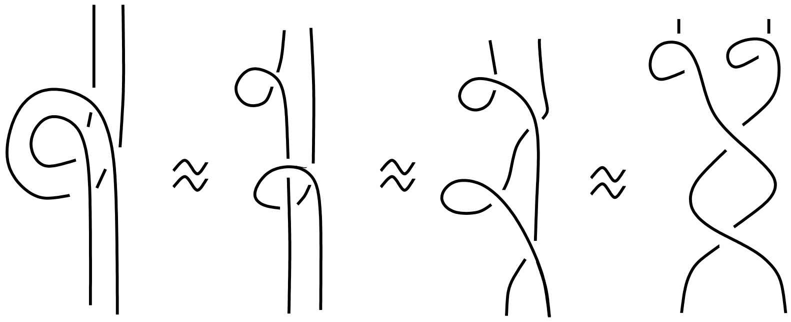


Figure D.25: KirbyMIfigs3.

Abstract:

We show that the path integral for the three-dimensional $SU(2)$ BF theory with a Wilson loop or a spin network function inserted can be understood as the Rovelli-Smolin loop transform of a wavefunction in the Ashtekar connection representation, where the wavefunction satisfies the constraints of quantum general relativity with zero cosmological constant. This wavefunction is given as a product of the delta functions of the $SU(2)$ field strength and therefore it can be naturally associated to a flat connection spacetime. The loop transform can be defined rigorously via the quantum $SU(2)$ group, as a spin foam state sum model, so that one obtains invariants of spin networks embedded in a three-manifold. These invariants define a flat connection vacuum state in the q-deformed spin network basis. We then propose a modification of this construction in order to obtain a vacuum state corresponding to the flat metric spacetime.

D.11 BF-Theory

Topological field theory:

$$S^{BF} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_i. \tag{D.-19}$$

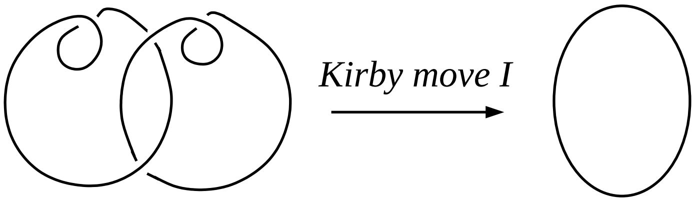


Figure D.26: KirbyMIfigs1.

No local degrees of freedom:

$$F^i = -\Lambda B^i, \qquad \mathcal{D} \wedge B^i = 0 \tag{D.-19}$$

We add a quadratic constraint

$$B^{(i} \wedge B^{j)} = \frac{1}{3}\delta^{ij} B^k \wedge B_k \tag{D.-19}$$

The result is general relativity:

$$S_{Plebanski} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_I - \frac{1}{2}\phi_{ij} B^i \wedge B^j \tag{D.-19}$$

Quantization is determined by the quantization of the topological field theory. As the constraints are non-derivative the gravitational filed has the same commutation relations as the topological theory.

Consider the action

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2) \tag{D.-19}$$

which has two constrained degrees of freedom (q_1, q_2), which we assume to live on a circle, with conjugate momentum (p_1, p_2). This theory is completely constrained because both q_1 and q_2 must be zero. There are no degrees of freedom. Now let us impose another constraint with corresponding Lagrange multiplier ξ. The action principle becomes

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2 + \xi(\lambda_1 - \lambda_2)) \tag{D.-19}$$

The two original Lagrange multipliers are constrained and we now have one degree of freedom: λ_1 has to be equal to λ_2 and thus only $q_1 + q_2$ has to be zero whereas the difference is free,

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1(q_1 + q_2)). \tag{D.-19}$$

This mimics the transition from BF-theory to gravity where additional constraints (the simplicity constraints) reduce the freedom the original Lagrange multipliers of the BF-theory and thereby introduce local degrees of freedom.

FIBRE BUNDLES USE TO BE HERE!!!!

D.12 Characteristic Classes

The information about the topology of the $P(\mathcal{M}, G)$, which is encoded in the transition functions, can be extracted from integrals of polynomials in the curvature, called characteristic classes, of an arbitrary connection, which represent topological invariants.

$$\det(t\mathbf{I} + a_c T^c) \tag{D.-19}$$

$$\det(t\mathbf{I} + a_c T^c) = \sum_{i=0}^{m} t^i P_{m-i}(a_c) \tag{D.-19}$$

$$\det(t\mathbf{I} + a_c T^c) = \det(t\mathbf{I} + g a_c T^c g^{-1}) = \det(g(t\mathbf{I} + a_c T^c g^{-1}) = \det(t\mathbf{I} + a_c T^c) \tag{D.-19}$$

$$a_i(P) = \frac{(-1)^i}{(2\pi)^i} \sum_{j_1 < j_2 < \cdots < j_i \leq n} \prod_j \frac{1}{i_j! j^{i_j}} (TrF^j)^{i_j} \tag{D.-19}$$

second *Chern class*

$$c_2(P) = \frac{1}{4\pi^2} \left(\frac{1}{2}(TrF) \wedge (TrF) - \frac{1}{2} Tr(F \wedge F) \right) \tag{D.-19}$$

For $SU(n)$

$$c_2(P) = -\frac{1}{8\pi^2} Tr(F \wedge F) \tag{D.-19}$$

to do with the calculation of black hole entropy.

D.13 Biblioliographical notes

In this chapter I have relied on the following references:

D.14 Worked Exercises

$$\frac{\partial}{\partial g_{ij}} g_{mn} = \delta_{im}\delta_{jn} \tag{D.-19}$$

This implies

$$\frac{\partial}{\partial g_{ij}} (g^{-1})_{mn} = -g^{-1}_{im} g^{-1}_{jn} \tag{D.-19}$$

$$0 = \frac{\partial}{\partial g_{ij}} g g^{-1} = \left(\frac{\partial}{\partial g_{ij}} g\right) g^{-1} + g \frac{\partial}{\partial g_{ij}} g^{-1} \tag{D.-19}$$

$$\mathbf{d}(ig^{-1}\mathbf{d}g + g^{-1}\mathbf{A}g) = i\mathbf{d}g^{-1} \wedge \mathbf{d}g + (\mathbf{d}g^{-1} \wedge \mathbf{A}g + g^{-1}\mathbf{d}\mathbf{A}g - g^{-1}\mathbf{A} \wedge \mathbf{d}g) \tag{D.-19}$$

where we used $d^2 g = 0$ and that, in the last term, in passing through $\mathbf{A}$ we picked up a minus sign.

Proofs

Questions

1.

where Ad is the adjoint representation.

$$[T_a, T_b] = C^{abc} T_c, \tag{D.-19}$$

C^{abc} forms a representation of (D.14), the matrix $\mathcal{T}^a$ with components given by

$$(\mathcal{T}^a)_{bc} = iC^{abc} \tag{D.-19}$$

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] \equiv 0. \tag{D.-19}$$

Proof

1. Forms

$$[\mathcal{T}^a, \mathcal{T}^b] = C^{abc}\mathcal{T}^c, \tag{D.-19}$$

Which can easily be shown to be so from the Jacobi identity for C_{abc} and its antisymmetry in the first two indices - $C_{abc} = -C_{bac}$,

$$C_{abd}C_{dce} + C_{bcd}C_{dae} + C_{cad}C_{dbe} \equiv 0 \tag{D.-19}$$

$$C_{abd}C_{dce} = C_{ade}C_{bcd} - C_{bde}C_{acd} \tag{D.-19}$$

$$C^{abd}(\mathcal{T}^d)_{ce} = (\mathcal{T}^a)_{de}(\mathcal{T}^b)_{cd} - (\mathcal{T}^b)_{de}(\mathcal{T}^a)_{cd} \tag{D.-19}$$

chern-Simons.

$$\mathcal{L}_{\mathcal{CS}} = \kappa\epsilon^{ijk}\mathrm{tr}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right) \tag{D.-19}$$

$$A_i \to A_i^g \equiv g^{-1}A_ig + g^{-1}\partial_i g \tag{D.-19}$$

We prove:

$$\mathcal{L}_{\mathcal{CS}} \to \mathcal{L}_{\mathcal{CS}} - \kappa\epsilon^{ijk}\partial_i\mathrm{tr}\left(\partial_j gg^{-1}A_k\right) - \frac{\kappa}{3}\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i gg^{-1}\partial_j gg^{-1}\partial_k g\right) \tag{D.-18}$$

Taking a derivative of the identity $I = g^{-1}g$ we get an identity involving derivatives:

$$\partial_i(I) = 0 = \partial_i(g^{-1}g) = \partial_i g^{-1}g + g^{-1}\partial_i g \tag{D.-18}$$

which means that

$$\boxed{\partial_i g^{-1}g = -g^{-1}\partial_i g \quad \text{or} \quad \partial_i g^{-1} = -g^{-1}\partial_i gg^{-1}.} \tag{D.-17}$$

we will use this to replace derivatives of the inverse matrix g^{-1} in favour of $-g^{-1}\partial_i gg^{-1}$ which involves the derivative of g. We will also make use of the cyclic symmetry of $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

$$\boxed{\begin{aligned}\mathcal{L}^g_{\mathcal{CS}} &= \kappa\epsilon^{ijk}\mathrm{tr}\left([g^{-1}A_ig+g^{-1}\partial_ig]\partial_j[g^{-1}A_kg+g^{-1}\partial_kg]\right.\\ &+ \left.\frac{2}{3}[g^{-1}A_ig+g^{-1}\partial_ig][g^{-1}A_jg+g^{-1}\partial_jg][g^{-1}A_kg+g^{-1}\partial_kg]\right) \qquad \text{(D.-16)}\end{aligned}}$$

Multiplying out the second term of (D.-16) is made easier if we split evaluate the terms with three, two, one and zero A's separately.

3 A**'s**:

$$\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_iA_jA_kg\right) = \frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{A_iA_jA_k\right\} \tag{D.-16}$$

0 A**'s**:

$$\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_igg^{-1}\partial_jgg^{-1}\partial_jg\right) \tag{D.-16}$$

2 A**'s**:

$$\begin{aligned}&\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{g^{-1}A_iA_j\partial_kg+g^{-1}A_i\partial_jgg^{-1}A_kg+g^{-1}\partial_igg^{-1}A_jA_kg\right\}\\ &\quad=\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{A_iA_j\partial_kgg^{-1}+A_kA_i\partial_jgg^{-1}+A_jA_k\partial_igg^{-1}\right\}\\ &\quad=2\kappa\epsilon^{ijk}\mathrm{tr}(A_iA_j\partial_kgg^{-1})\end{aligned} \tag{D.-17}$$

we can $i\to j$, $j\to k$, $k\to i$ etc and not change the value of this term. We could have written down the last line of (D.-17) straight away from noting the from the permutative symmetry of the second term of (D.-16)

1 A:

$$\begin{aligned}&\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{g^{-1}A_i\partial_jgg^{-1}\partial_kg+g^{-1}\partial_jgg^{-1}A_jg+g^{-1}\partial_igg^{-1}\partial_jgg^{-1}A_kg\right\}\\ &\quad=2\kappa\epsilon^{ijk}\mathrm{tr}\left\{\partial_igg^{-1}\partial_jgg^{-1}A_k\right\}\\ &\quad=2\kappa\epsilon^{ijk}\mathrm{tr}\left\{\partial_jg\partial_ig^{-1}A_k\right\}\end{aligned} \tag{D.-18}$$

From the first term of (D.-16) for convenience of comparison:

2 A**'s**:

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i g\partial_j(g^{-1}A_k g)\right)\\
&\quad=\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i g\partial_j g^{-1}A_k g+g^{-1}A_i\partial_j A_k g+g^{-1}A_i A_k\partial_j g\right)\\
&\quad=\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i(-g^{-1}\partial_j g g^{-1})A_k g+g^{-1}A_i A_k\partial_j g+A_i\partial_j A_k)\right)\\
&\quad=\kappa\epsilon^{ijk}\mathrm{tr}\left(-2A_i\partial_j g g^{-1}A_k+A_i\partial_j A_k\right) \qquad \text{(D.-20)}
\end{aligned}
$$

0 A**'s**:

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g\partial_j(g^{-1}\partial_k g)\right)\\
&\quad=\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g\partial_j g^{-1}\partial_k g+g^{-1}\partial_i g g^{-1}\partial_j\partial_k g\right)\\
&\quad=\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g\partial_j g^{-1}\partial_k g\right)\\
&\quad=-\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g\right) \qquad \text{(D.-22)}
\end{aligned}
$$

because $\epsilon^{ijk}\partial_j\partial_k g=0$.

1 A:

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i g\partial_j(g^{-1}\partial_k g)+g^{-1}\partial_i g\partial_j(g^{-1}A_k g)\right)\\
&\quad=\kappa\epsilon^{ijk}\mathrm{tr}\left(g\partial_j g^{-1}\partial_k g g^{-1}A_i+\right.\\
&\qquad\left.+g^{-1}\partial_i g\partial_j g^{-1}A_k g+g^{-1}\partial_i g g^{-1}\partial_j A_k g+g^{-1}\partial_i g g^{-1}A_k\partial_j g\right) \qquad \text{(D.-23)}
\end{aligned}
$$

Let us consider the first term in the last line of (D.-23)

$$
g\partial_j g^{-1}\partial_k g g^{-1}A_i=(g\partial_j g^{-1}g)(g^{-1}\partial_k g g^{-1})A_i=\partial_j g\partial_k g^{-1}A_i \qquad \text{(D.-23)}
$$

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(\partial_j g\partial_k g^{-1}A_i+\partial_i g\partial_j g^{-1}A_k+\partial_i g g^{-1}\partial_j A_k-\partial_i g^{-1}A_k\partial_j g\right)\\
&\quad=-\kappa\epsilon^{ijk}\mathrm{tr}\left(3\partial_j g\partial_i g^{-1}A_k+\partial_j g g^{-1}\partial_i A_k\right) \qquad \text{(D.-23)}
\end{aligned}
$$

Adding (D.14) and the second term of (D.-20) we have the $\mathcal{L}_{\mathcal{CS}}$ term in (D.-16)

Adding (D.-22) and (D.14)

$$
-\frac{1}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g\right) \qquad \text{(D.-23)}
$$

Adding (D.-23) and (D.-18)

$$-\kappa\epsilon^{ijk}\mathrm{tr}\left(\partial_i g\partial_j g^{-1}A_k + \partial_i g g^{-1}\partial_j A_k\right) = -\kappa\epsilon^{ijk}\partial_j\mathrm{tr}\left(\partial_i g g^{-1}A_k\right) \qquad \text{(D.-23)}$$

And we are done.

Jone's polynomial (satellite formulae)

(1)

$T^a_{(\rho_1)} + T^a_{(\rho_2)} := T^a_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}$

Answers:

$T^a_{(\rho_1)}T^b_{(\rho_1)} = T^c_{(\rho_1)}$

$$\begin{aligned}
& \left(T^a_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}\right)\left(T^b_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^b_{(\rho_2)}\right) \\
= & \left(T^a_{(\rho_1)}T^b_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}T^b_{(\rho_2)}\right) + T^a_{(\rho_1)} \otimes T^b_{(\rho_2)} + T^b_{(\rho_1)} \otimes T^a_{(\rho_2)} \\
= & \left(T^c_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)}\right) + T^a_{(\rho_1)} \otimes T^b_{(\rho_2)} + T^b_{(\rho_1)} \otimes T^a_{(\rho_2)} \qquad \text{(D.-24)}
\end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{(D.-24)}$$

As the last two terms are symmetric in the indices a and b,

$$\begin{aligned}
& \left[T^a_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}\ ,\ T^b_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^b_{(\rho_2)}\right] \\
= & \left[T^a_{(\rho_1)}, T^b_{(\rho_1)}\right] \otimes \mathbf{1} + \mathbf{1} \otimes \left[T^a_{(\rho_2)}, T^b_{(\rho_2)}\right] \\
= & C^{ab}{}_c\left(T^c_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)}\right) \qquad \text{(D.-25)}
\end{aligned}$$

These are generators of a *reducible* representation of G. That is, there exists a unitary matrix U such that

$$U(T^a_{(\rho_1)} + T^a_{(\rho_2)})U^\dagger = \begin{pmatrix} T^a_{\rho(t_1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & T^a_{\rho(t_n)} \end{pmatrix} \qquad \text{(D.-25)}$$

where $\rho(t_r)$ are irreducible representations. We have

$$Tr_{\rho_1}\, Tr_{\rho_2} = Tr_{\rho_1 \otimes \rho_2}$$

$$W' = W_{\rho_1 \otimes \rho_2}(U_1)$$

$$\begin{aligned}
&\left(T^c_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)} + T^{(a}_{(\rho_1)} \otimes T^{b)}_{(\rho_2)}\right)\left(T^d_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^d_{(\rho_2)}\right) \\
&T^c_{(\rho_1)}T^d_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)}T^d_{(\rho_2)} + T^{(c}_{(\rho_1)} \otimes T^{d)}_{(\rho_2)} + \\
&T^a_{(\rho_1)}T^d_{(\rho_1)} \otimes T^b_{(\rho_2)} + T^b_{(\rho_1)}T^d_{(\rho_1)} \otimes T^a_{(\rho_2)} + T^a_{(\rho_1)} \otimes T^b_{(\rho_2)}T^d_{(\rho_2)} + T^b_{(\rho_1)} \otimes T^a_{(\rho_2)}T^d_{(\rho_2)}
\end{aligned} \tag{D.-27}$$

Appendix E

Covariant Classical and Quantum Mechanins

E.1 Conventional Mechanics

E.1.1 Action Principle for Several Dependent variables

A dynamic system with m degrees of freedom describes the evolution in time t of m Lagrangian variables q^i, where $i = 1, \ldots, m$. We denote the space in which the variables q^i take value is a $m-$dimensional configuration space $\mathcal{C}_0$.

The dynamics of the system is determined by the Lagrangian $\mathcal{L}$ which is a function of several dependent variables $q^1(t), \ldots, q^m(t)$ and $\dot{q}^1(t), \ldots, \dot{q}^m(t)$ all of which depend on t, the independent variable.

Given two times t_1 and t_2 and two points q_1^i and q_2^i in $\mathcal{C}_0$, physical motions are such that the action

$$S[q] = \int_{t_1}^{t_2} dt \, \mathcal{L}\left(q^i(t), \frac{dq^(t)}{dt}\right) \tag{E.0}$$

is an extremum in the space of motions $q^i(t)$ such that $q^i(t_1) = q_1^i$ and $q^i(t_2) = q_2^i$. Physical motions satisfy the Euler-Lagrange equations, which we derive now.

We compare neighbouring paths by writing

$$q^i(t; \alpha) = q^i(t; 0) + \alpha \eta^i(t), \qquad i = 1, 2, \ldots, m, \tag{E.0}$$

where the η_i are independent, but subject them to the condition

$$\eta^i(t_1) = \eta^i(t_2) = 0, \qquad i = 1, 2, \ldots, m. \tag{E.0}$$

Differentiating with respect to α and setting $\alpha = 0$ we obtain

$$\int_{t_1}^{t_2} \sum_i \left[\frac{\partial \mathcal{L}}{\partial q^i} \eta^i(t) + \frac{\partial \mathcal{L}}{\partial q_t^i} \frac{d\eta^i(t)}{dt} \right] dt = 0. \tag{E.0}$$

Integrating by parts

$$\int_{t_1}^{t_2} \sum_i \left[\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial q_t^i} \right) \right] \eta^i(t) dt + \sum_i \left[\frac{\partial \mathcal{L}}{\partial q_t^i} \eta^i(t) \right]_{t_1}^{t_2} = 0. \tag{E.0}$$

Inserting (E.1.1) into this we have

$$\int_{t_1}^{t_2} \sum_i \left[\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial q_t^i} \right) \right] \eta^i(t) dt = 0. \tag{E.0}$$

Since the η^i are arbitrary and independent of each other, each of the terms vanishes independently

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial q_t^i} \right) = 0, \qquad i = 1, 2, \ldots, m. \tag{E.0}$$

These are the Euler-Lagrange equations. A dynamical system is therefore specified by the couple $(\mathcal{C}_0, \mathcal{L})$.

E.1.2 The Hamilton Equations

The Hamiltonian from a Lagrangian

The momenta are calculated by differentiating the Lagrangian via

$$p_i(q^i, \dot{q}^i, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}. \tag{E.0}$$

Inverting the function $p_i(q^i, \dot{q}^i)$ yields the function $\dot{q}^i(q^i, p_i)$. Defing the Hamiltonian by

$$\begin{aligned} H_0(q^i, p_i) &= \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}(q^i, p_i) - \mathcal{L}(q^i, \dot{q}(q^i, p_i)) \\ &= p_i \dot{q}(q^i, p_i) - \mathcal{L}(q^i, \dot{q}(q^i, p_i)). \end{aligned} \tag{E.0}$$

We derive Hamilton's equations by considering the total derivative of the Lagrangian

$$d\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q^i} dq^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} d\dot{q}^i \right) + \frac{\partial \mathcal{L}}{\partial t} dt. \tag{E.0}$$

We substitute (E.1.2) into the total differential of the Lagrangian

$$d\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q^i} dq^i + p_i d\dot{q}^i \right) + \frac{\partial \mathcal{L}}{\partial t} dt. \tag{E.0}$$

This can then be written as

$$d\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q^i} dq^i + d(p_i \dot{q}^i) - \dot{q}^i dp_i \right) + \frac{\partial \mathcal{L}}{\partial t} dt, \tag{E.0}$$

which after rearangment becomes

$$d\left(\sum_i p_i \dot{q}^i - \mathcal{L} \right) = \sum_i \left(-\frac{\partial \mathcal{L}}{\partial q^i} dq^i + \dot{q}^i dp_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt. \tag{E.0}$$

The term on the LHS is the Hamiltonian, therefore

$$dH_0 = \sum_i \left(-\frac{\partial \mathcal{L}}{\partial q^i} dq^i + \dot{q}^i dp_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt. \tag{E.0}$$

We now consider the total derivative of the Hamiltonian

$$dH_0 = \sum_i \left(\frac{\partial H_0}{\partial q^i} dq^i + \frac{\partial H_0}{\partial p_i} dp_i \right) + \frac{\partial H_0}{\partial t} dt. \tag{E.0}$$

Comparing the terms of (E.1.2) and (E.1.2) give Hamilton's equations are then

$$
\begin{aligned}
\frac{dq^i(t)}{dt} &= \frac{\partial H_0(q^i,p_i)}{\partial p_i} \\
\frac{dp_i(t)}{dt} &= -\frac{\partial H_0(q^i,p_i)}{\partial q^i}
\end{aligned} \tag{E.0}
$$

and

$$
\frac{\partial H_0}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{E.0}
$$

The $2m-$dimensional space coordinatised by the coordinates q^i and the momenta p_i is called the "conventional" phase space Γ_0.

Legendre transformations

A function $y = f(x)$ is convex if

$$
\frac{d^2 f}{dx^2} > 0 \tag{E.0}
$$

The functional relationship specified by $f(x)$ can be represented equally well as a set of tangent lines specified by their slope and intercept values. The tangent line at $x = x_0$ intersects the vertical axis at $(0, -f^*)$ and f^* is the value of the Legendre transform $f^*(p)$ where $p = \dot{f}(x_0)$.

Hamilton's equations from a variational principle

Hamilon's canonical equations can be obtained from a variational principle where we regard $q^i(t)$ and $p_i(t)$ are uncorrelated and independently adjustable functions. One abandons the formula $p_i = \partial\mathcal{L}(q^i, \dot{q}^i, t)/\partial\dot{q}^i$. Now, variations will be over paths in the (q^i, p_i) phase space, which have $2m$ dimensions.

$$
\mathcal{L}(q^i, \dot{q}^i, p_i, t) \equiv \sum_i \dot{q}^i p_i - H_0(q^i, p_i, t) \tag{E.0}
$$

(E.1.2) is an extremum in the space of motions in phase space (q^i, p_i) such that $q^i(t_1) = q^i_1$ and $q^i(t_2) = q^i_2$. Physical motions satisfy the Hamilton equations equations, which we derive now.

We compare neighbouring paths by writing

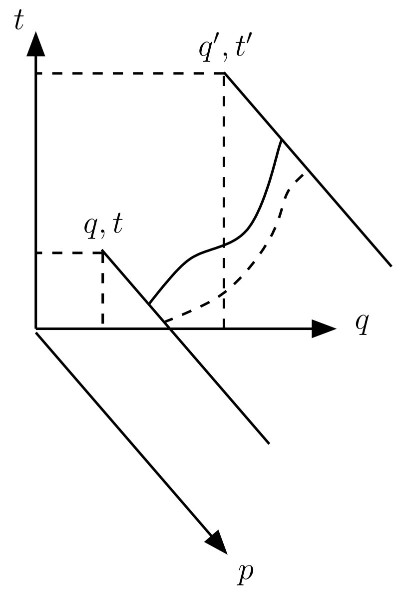


Figure E.1: We consider variations in the path that extremises (E.1.2) subject to the conditions that at initial time t the position is q and at final time t' the position is q'.

$$q^i(t;\alpha) = q^i(t;0) + \alpha\eta_q^i(t) \quad \text{and} \quad p_i(t;\alpha) = p_i(t;0) + \alpha\eta_{pi}(t), \quad i = 1, 2, \ldots, m, \tag{E.0}$$

where the η_q^i and η_{pi} are independent, but subject them to the condition

$$\eta_q^i(t_1) = \eta_q^i(t_2) = 0, \qquad i = 1, 2, \ldots, m. \tag{E.0}$$

Differentiating with respect to α and setting $\alpha = 0$ we obtain

$$\int_{t_1}^{t_2} \sum_i \left[\frac{\partial \mathcal{L}}{\partial q^i} \eta_q^i(t) + \frac{\partial \mathcal{L}}{\partial q_t^i} \frac{d\eta_q^i(t)}{dt} + \frac{\partial \mathcal{L}}{\partial p_i} \eta_{pi}(t) \right] dt = 0. \tag{E.0}$$

Upon substituting the RHS of (E.1.2) we have

$$\int_{t_1}^{t_2} \sum_i \left[-\frac{\partial H_0}{\partial q^i}\, \eta_q^i(t) + p_i \frac{d\eta_q^i(t)}{dt} + \left(\dot{q}^i - \frac{\partial H_0}{\partial p_i} \right) \eta_{pi}(t) \right] dt = 0. \tag{E.0}$$

Integrating by parts gives

$$\int_{t_1}^{t_2}\sum_i\left(-\frac{\partial H_0}{\partial q^i}-\frac{dp_i}{dt}\right)\eta_q^i(t)dt+\sum_i\Big[p_i\eta_q^i(t)\Big]_{t_1}^{t_2}+\int_{t_1}^{t_2}\sum_i\left(\dot{q}^i-\frac{\partial H_0}{\partial p_i}\right)\eta_{pi}(t)dt=0. \tag{E.0}$$

Substituting in (E.1.2) gives

$$\int_{t_1}^{t_2}\sum_i\left(-\frac{\partial H_0}{\partial q^i}-\frac{dp_i}{dt}\right)\eta_q^i(t)dt+\int_{t_1}^{t_2}\sum_i\left(\frac{dq^i}{dt}-\frac{\partial H_0}{\partial p_i}\right)\eta_{pi}(t)dt=0. \tag{E.0}$$

Since the η_q^i and η_{pi} are arbitrary and independent of each other, each of the terms vanishes independently,

$$\frac{dp_i}{dt}=-\frac{\partial H_0}{\partial q^i} \tag{E.0}$$

and

$$\frac{dq^i}{dt}=\frac{\partial H_0}{\partial p_i}. \tag{E.0}$$

Recovery of Second order formulism from Hamilton's equations

We now employ Hamilton's equations,

$$\frac{dq^i}{dt}=\frac{\partial H_0}{\partial p_i}\qquad\text{and}\qquad\frac{dp_i}{dt}=-\frac{\partial H_0}{\partial q^i}$$

and take $\mathcal{L}(q^i,\dot{q}^i,p_i,t)=\sum_i\dot{q}^ip_i-H_0(q^i,p_i,t)$ to recover $p_i=\partial\mathcal{L}(q^i,\dot{q}^i,t)/\partial\dot{q}^i$ together with the Euler-Lagrange equations for $\mathcal{L}=\mathcal{L}(q^i,\dot{q}^i,t)$. We write:

$$\begin{aligned}d\mathcal{L}&=\sum_i\frac{\partial\mathcal{L}}{\partial q^i}dq^i+\sum_i\frac{\partial\mathcal{L}}{\partial\dot{q}^i}d\dot{q}^i+\sum_i\frac{\partial\mathcal{L}}{\partial p_i}dp^i+\frac{\partial\mathcal{L}}{\partial t}\\&=\sum_i p_id\dot{q}^i+\sum_i\dot{q}^idp_i-\sum_i\frac{\partial H_0}{\partial q^i}dq^i-\sum_i\frac{\partial H_0}{\partial p^i}dp^i-\frac{\partial H_0}{\partial t}.\end{aligned} \tag{E.0}$$

Comparing terms, and employing Hamilton's equations, we obtain:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial p_i} &= \dot{q}^i - \frac{\partial H_0}{\partial p_i} = 0 \\
\frac{\partial \mathcal{L}}{\partial \dot{q}^i} &= p_i \\
\frac{\partial \mathcal{L}}{\partial q^i} &= -\frac{\partial H_0}{\partial q^i} = -\dot{p}_i
\end{aligned} \tag{E.-1}
$$

The first equation implies $\mathcal{L} = \mathcal{L}(q^i, \dot{q}^i, t)$. From the second equation we recover the formula $p_i = \partial\mathcal{L}(q^i, \dot{q}^i, t)/\partial\dot{q}^i$. From the second and third equations we obtain:

$$
\frac{\partial \mathcal{L}}{\partial q^i}(q^i, \dot{q}^i, t) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}(q^i, \dot{q}^i, t)\right)
$$

which is the Euler-Larange equation for $\mathcal{L}(q^i, \dot{q}^i, t)$.

E.1.3 Sympletic Geometry

Unified coordinates on $T^*\mathcal{C}_0$

Let

$$
\begin{aligned}
\xi^\mu &= (q^1, q^2 \dots q^m, p_1, p_2 \dots p_m) \quad \text{for} \quad (\mu = 1, \dots, 2m) \quad \text{and} \\
\partial_\mu &= (\partial_{q^1}, \partial_{q^2}, \dots, \partial_{q^m}, \partial_{p_1}, \partial_{p_2}, \dots, \partial_{p_m}).
\end{aligned} \tag{E.-1}
$$

Then

$$
\frac{d\xi^\mu}{dt} = \sum_{\nu=1}^{2m} \Omega^{\mu\nu}\partial_\nu \mathcal{H} \tag{E.-1}
$$

$$
\{A, B\} = \sum_{\nu=1}^{2m} \partial^\mu A \Omega_{\mu\nu} \partial^\nu B := \partial_\mu A \partial^\mu B \tag{E.-1}
$$

It's sort of like a scalar product. In Minkoskian spacetime the scalar product is

$$
\begin{aligned}
a^\mu b_\mu &= a^\mu \eta_{\mu\nu} b^\nu \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned} \tag{E.-1}
$$

In Hamilton mecahanics the matrix

$$
\begin{pmatrix} 0_m & \mathbb{1}_m \\ -\mathbb{1}_m & 0_m \end{pmatrix} \tag{E.-1}
$$

plays the role η. Lorentzian tranformations leave scalar product $a^\mu b_\mu$ invariant. The transformations that leave (E.1.3) invariant will be the subject of the next section and are called *canonical transformations*.

Just as space and time can be treated on an equal footing in special relativity by using Minkowskian spacetime notation, the q and p variables can be treated on an equal footing by using the ξ notation introduced above.

Example 4-dimensiona phase space:

Let us check (E.-1), (E.1.3) and (E.1.3) for $m = 2$:

$$
\frac{d\xi^\mu}{dt} = \sum_{\nu=1}^{4} \Omega^{\mu\nu} \partial_\nu \mathcal{H}
$$

reads

$$
\begin{pmatrix} \frac{d\xi^1}{dt} \\ \frac{d\xi^2}{dt} \\ \frac{d\xi^3}{dt} \\ \frac{d\xi^4}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_1 \mathcal{H} \\ \partial_2 \mathcal{H} \\ \partial_3 \mathcal{H} \\ \partial_4 \mathcal{H} \end{pmatrix}, \tag{E.-1}
$$

this gives the Hamilton equations:

$$
\begin{aligned}
\frac{d\xi^1}{dt} &= \partial_3 \mathcal{H} \quad \Longrightarrow \quad \frac{dq^1}{dt} = \frac{\partial H_0}{\partial p_1} \\
\frac{d\xi^2}{dt} &= \partial_4 \mathcal{H} \quad \Longrightarrow \quad \frac{dq^2}{dt} = \frac{\partial H_0}{\partial p_2} \\
\frac{d\xi^3}{dt} &= -\partial_1 \mathcal{H} \quad \Longrightarrow \quad \frac{dp_1}{dt} = -\frac{\partial H_0}{\partial q^1} \\
\frac{d\xi^4}{dt} &= -\partial_2 \mathcal{H} \quad \Longrightarrow \quad \frac{dp_2}{dt} = -\frac{\partial H_0}{\partial q^2}
\end{aligned} \tag{E.-3}
$$

Let us check (E.1.3). We have

$$
\{A, B\} = \sum_{\mu,\nu=1}^{4} \partial^\mu A \Omega_{\mu\nu} \partial^\nu B \tag{E.-3}
$$

which reads

$$
\begin{aligned}
\sum_{\mu,\nu=1}^{4} \partial^\mu A \Omega_{\mu\nu} \partial^\nu B &= (\partial_1 A, \partial_2 A, \partial_3 A, \partial_4 A) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_1 B \\ \partial_2 B \\ \partial_3 B \\ \partial_4 B \end{pmatrix} \\
&= \Big(\partial_{q^1} A, \partial_{q^2} A, \partial_{p_1} A, \partial_{p_2} A\Big) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{q^1} B \\ \partial_{q^2} B \\ \partial_{p_1} B \\ \partial_{p_2} B \end{pmatrix} \\
&= \Big(\partial_{q^1} A, \partial_{q^2} A, \partial_{p_1} A, \partial_{p_2} A\Big) \begin{pmatrix} \partial_{p_1 1} B \\ \partial_{p_2} B \\ -\partial_{q^1} B \\ -\partial_{q^2} B \end{pmatrix} \\
&= \left(\frac{\partial A}{\partial q^1} \frac{\partial B}{\partial p_1} - \frac{\partial A}{\partial p_1} \frac{\partial B}{\partial q^1} \right) + \left(\frac{\partial A}{\partial q^2} \frac{\partial B}{\partial p_2} - \frac{\partial A}{\partial p_2} \frac{\partial B}{\partial q^2} \right) \\
&= \{A, B\}.
\end{aligned} \tag{E.-6}
$$

□

Now, let us define for an aribrary function on phase space, f,

$$
X_0 f = v_i(q^i, p_i) \frac{\partial f}{\partial q^i} + f_i(q^i, p_i) \frac{\partial f}{\partial p_i} \tag{E.-6}
$$

where

$$v_i(q^i, p_i) = \frac{\partial H_0(q^i, p_i)}{\partial p_i} \qquad f_i(q^i, p_i) = -\frac{\partial H_0(q^i, p_i)}{\partial q^i}. \tag{E.-6}$$

Note

$$\begin{aligned} X_0 q^j &= v_i(q^i, p_i)\frac{\partial q^j}{\partial q^i} + f_i(q^i, p_i)\frac{\partial q^j}{\partial p_i} \\ &= \frac{\partial H_0(q^i, p_i)}{\partial p_i}\delta_{ij} \\ &= \frac{\partial H_0(q^i, p_i)}{\partial p_j} \\ &= \frac{dq^j}{dt} \end{aligned} \tag{E.-8}$$

and similarly

$$\begin{aligned} X_0 p_j &= v_i(q^i, p_i)\frac{\partial p_j}{\partial q^i} + f_i(q^i, p_i)\frac{\partial p_j}{\partial p_i} \\ &= -\frac{\partial H_0(q^i, p_i)}{\partial q^j} \\ &= \frac{dp_j}{dt}. \end{aligned} \tag{E.-9}$$

Time evolution is a flow $(q^i(t), p_i(t))$ in this space; we see that the vector field on Γ_0 tangent to this flow is X_0. If $f = f(q^j, p_j)$ then

$$\begin{aligned} X_0 f &= v_i(q^i, p_i)\frac{\partial f(q^j, p_j)}{\partial q^i} + f_i(q^i, p_i)\frac{\partial f(q^j, p_j)}{\partial p_i} \\ &= \frac{dq^j}{dt}\frac{\partial f(q^j, p_j)}{\partial q^j} + \frac{dp_j}{dt}\frac{\partial f(q^j, p_j)}{\partial p_j} \\ &= \frac{df(q^j, p_j)}{dt}. \end{aligned} \tag{E.-10}$$

E.1.4 Canonical Transformations

We can perform a change of variables on phase space

$$\begin{aligned} Q^i &= Q^i(q^i, p_i, t) \\ P^i &= P^i(q^i, p_i, t) \end{aligned} \tag{E.-10}$$

such that

$$\dot{Q}^i = \frac{\partial \mathcal{K}}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial \mathcal{K}}{\partial Q^i} \tag{E.-10}$$

where $\mathcal{K}$ is the new Hamiltonian, that is it obeys Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left(\sum_i P_i Q^i - \mathcal{K}(Q^i, P_i, t) \right) dt = 0. \tag{E.-10}$$

It is easily seen that two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ that differ by a toal time derivative yield exactly the same equations of motion,

$$S'(q, \dot{q}) = \int_{t_1}^{t_2} dt\, \mathcal{L} = \int_{t_1}^{t_2} dt \left(\mathcal{L} + \frac{dF}{dt} \right) \tag{E.-10}$$

$$\sum_i p_i \dot{q}_i - \mathcal{H} = \sum_i P_i \dot{Q}_i - \mathcal{K} + \frac{dF}{dt} \tag{E.-10}$$

Which variables does F depend on?

$4m$ variables $\{q, p, Q, P\}$ $2m$ indices

$$\text{Type 1. } F_1(q^i, Q^i, t)\ (q^i, Q^i) \text{ are independent.} \tag{E.-9}$$
$$\text{Type 2. } F_2(q^i, P_i, t)\ (q^i, P_i) \text{ are independent.} \tag{E.-8}$$
$$\text{Type 3. } F_3(p_i, Q^i, t)\ (p_i, Q^i) \text{ are independent.} \tag{E.-7}$$
$$\text{Type 4. } F_4(p_i, P_i, t)\ (p, P) \text{ are independent.} \tag{E.-6}$$

Canonical transformations: Type 1

Suppose we considre a generating function of type 1:

$$F = F_1(q^i, Q^i, t). \tag{E.-6}$$

Equation (E.1.4) is then

$$\sum_i p_i \dot{q}_i - \mathcal{H} = \sum_i P_i \dot{Q}_i - \mathcal{K} + \frac{dF_1}{dt}(q, Q). \tag{E.-6}$$

We have

$$\frac{dF_1}{dt}(q, Q) = \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \tag{E.-6}$$

Substituting this into (E.1.4) gives

$$\sum_i p_i \dot{q}_i - \mathcal{H} = \sum_i P_i \dot{Q}_i - \mathcal{K} + \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \tag{E.-6}$$

So only two terms depend on $\dot{q}_i$ (or $\mathcal{H}$, $\mathcal{K}$) and F_1 do not depend on $\dot{q}_i$. So we may compare coefficients and obtain

$$p_i = \frac{\partial F_1}{\partial q^i} \qquad \text{for } i = 1, \ldots, m. \tag{E.-6}$$

Now compare coefficients of Q_i to obtain

$$P_i = -\frac{\partial F_1}{\partial Q^i} \qquad \text{for } i = 1, \ldots, m. \tag{E.-6}$$

We are then left with

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_1}{\partial t}. \tag{E.-6}$$

Equation (E.1.4) represents m relations defining the p_i as a function of q^i, Q^i, and t. Assuming that they can be inverted, we can then solve for the m Q^i's in terms of q^i, p_i, and t, thus yielding the first equation of the transformation equations (E.-10). We the substitue $Q^i(q^i, p_i, t)$ into (E.1.4) so that they give the m P_i's as functions of q^i, p_i, and t, thus yielding the secong equation of the transformation equations (E.-10).

Canonical transformations: Type 2

Suppose we considre a generating function of type 2.

$$\frac{dF_2}{dt}(q,P) = \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_2}{\partial P^i}\dot{P}^i + \frac{\partial F_2}{\partial t} \tag{E.-6}$$

Inserting this into the RHS of (E.1.4) would not produce terms that are not likewise. So we cannot simply compare. Instead we use a Legendre transform to change variables

$$F_2(q,P,t) = F_1(q,Q,t) + \sum_i P^i Q_i \tag{E.-6}$$

So replacing F_1 by $F_2 - \sum PQ$, which represents a Legendre transformation, we see:

$$\frac{dF_1}{dt} = \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P^i}\dot{P}^i + \frac{\partial F_2}{\partial t} - \sum_i \dot{P}^i Q_i - \sum_i P^i \dot{Q}_i \tag{E.-6}$$

and so obtain

$$\sum_i p_i \dot{q}_i - \mathcal{H} = \sum_i P_i \dot{Q}_i - \mathcal{K} + \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P^i}\dot{P}^i + \frac{\partial F_2}{\partial t} - \sum_i \dot{P}^i Q_i - \sum_i P^i \dot{Q}_i. \tag{E.-6}$$

This leads to the equations

$$p_i = \frac{\partial F_2}{\partial q^i} \qquad \text{for } i = 1, \ldots, m, \tag{E.-6}$$

$$Q^i = \frac{\partial F_2}{\partial P_i} \qquad \text{for } i = 1, \ldots, m, \tag{E.-6}$$

and

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_2}{\partial t}. \tag{E.-6}$$

Canonical transformations: Type 3 and 4

The type 3 generating function F_3 depends only on the old generalised momenta and the new generalised coordinates, in this case we use

$$F = \sum_i q^i p_i + F_3(p_i, Q^i, t). \tag{E.-6}$$

We then have the equations

$$\begin{aligned} q^i &= -\frac{\partial F_3}{\partial p_i} \\ P_i &= -\frac{\partial F_3}{\partial Q^i} \\ \mathcal{K} &= H + \frac{\partial F_3}{\partial t}. \end{aligned} \tag{E.-7}$$

The type 4 generating function F_3 depends only on the old and new momenta, in this case we use

$$F = \sum_i q^i p_i - \sum_i Q^i P_i + F_4(p_i, P_i, t). \tag{E.-7}$$

We then have the equations

$$\begin{aligned} q^i &= -\frac{\partial F_4}{\partial p_i} \\ Q^i &= -\frac{\partial F_4}{\partial P_i} \\ \mathcal{K} &= H + \frac{\partial F_4}{\partial t}. \end{aligned} \tag{E.-8}$$

E.1.5 The Hamilton-Jacobi Equation

Canonical transformations can be a way of picking phase space coordinates to simplify a problem:

$$\begin{aligned} Q &= Q(q, p, t) \\ P &= P(q, p, t) \end{aligned} \tag{E.-8}$$

where Q, P are new variables and q, p are the old ones, or inversely

$$\begin{aligned} q &= q(Q, P, t) \\ p &= p(Q, P, t) \end{aligned} \tag{E.-8}$$

Now, the solution to our problem is to express q and p in terms of the intial conditions q_0 and p_0 $(= q(t = 0), p(t = 0))$ and time t.

$$\begin{aligned} q &= q(q_0, p_0, t) \\ p &= p(q_0, p_0, t). \end{aligned} \tag{E.-8}$$

The obvious suggestion from a comparison of (E.-8) and (E.-8) is to make $Q = q_0$ and $P = p_0$, i.e. the new variables equal the intial conditions. So the "motion" in the new coordinates is the system remaining staionary at a point (q_0, p_0).

Thus Hamilton's equations in the new variables are:

$$\frac{\partial \mathcal{K}}{\partial P} = \dot{Q} = \dot{q_0} = 0 \tag{E.-7}$$
$$-\frac{\partial \mathcal{K}}{\partial Q} = \dot{P} = \dot{p_0} = 0 \tag{E.-6}$$

This implies that $\mathcal{K}$ is equal to an arbitary functon of t - we will *choose* the simplest case:

$$\mathcal{K} = 0 \tag{E.-6}$$

If $\mathcal{K} = 0$, then the original Hamiltonian must obey :

$$\mathcal{K} = 0 = H_0(q, p, t) + \frac{\partial F}{\partial t} \tag{E.-6}$$

where F is the generating function of the transformation. We choose a canonical transformation as Type 1:

$$F_1(q, Q, t) \tag{E.-6}$$

(though it is often choosen to be of type 2). Recall for Type 1 we have

$$
\begin{aligned}
\frac{\partial F_1}{\partial q} &= p, &\text{(E.-5)}\\
\frac{\partial F_1}{\partial Q} &= -P &\text{(E.-4)}
\end{aligned}
$$

and conventually (and for reasons we will see in the next section) we denote F_1 by S. To write the Hamiltonian in (E.1.5) in terms of the same variables, one may use (E.-5). Then (E.1.5) becomes, with substitution of (E.-5), what is called the Hamilton-Jacobi equation

$$
\frac{\partial S}{\partial t} + H_0\left(q, \frac{\partial S}{\partial q}\right) = 0. \tag{E.-4}
$$

The Hamilton-Jacobi equation can be obtained from the classical limit of the Schrodinger equation.

E.1.6 The Hamilton Function

Let us first give the simplest example of the Hamilton function, the one with the least number of variables. The Hamilton function $S(q, t, q', t')$ is a function of four variables, q, t, q', t', defined as the action of a physical motion that starts at q at time t and ends at q' at time t'.

Let us denote $q_{qt,q't'}(\tilde{t})$ as a physical motion that starts at (q, t) and ends at (q', t'). That is, a function of time that solves the equantions of motion and such

$$
\begin{aligned}
q_{qt,q't'}(t) &= q,\\
q_{qt,q't'}(t') &= q'. &\text{(E.-4)}
\end{aligned}
$$

Example : Free particle

The action is

$$
S[q] = \int_t^{t'} d\tilde{t}\, \frac{1}{2} m\dot{q}^2(\tilde{t}). \tag{E.-4}
$$

Physical motions are straight motions

$$
v \equiv \dot{q} = \frac{q' - q}{t' - t}, \tag{E.-4}
$$

$$
\begin{aligned}
S(q,t,q',t') &= \int_t^{t'} d\tilde{t}\,\frac{1}{2}m\left(\frac{q'-q}{t'-t}\right)^2 \\
&= \frac{m(q'-q)^2}{2(t'-t)}
\end{aligned}
\qquad \text{(E.-4)}
$$

where we have used that the integrand is not a function of $\tilde{t}$.

□

As a worked exercise, at the end of the appendix we calculate that the Hamilton function of a harmonic oscillator is

$$
S(q,t,q',t') = m\omega\frac{(q^2+q'^2)cos\omega(t'-t) - 2qq'}{2\sin\omega(t'-t)} \qquad \text{(E.-4)}
$$

for motion that starts at (q,t) and ends at (q',t').

Definition : The Hamilton Function

Consider two points (t_1, q_1^i) and (t_2, q_2^i) in $\mathcal{C}$. The function on $\mathcal{G} = \mathcal{C}\times\mathcal{C}$

$$
S(t_1, q_1^i, t_2, q_2^i) = \int_{t_1}^{t_2} dt\;\mathcal{L}\left(q^i(t), \dot{q}^i(t)\right), \qquad \text{(E.-4)}
$$

where $q^i(t)$ is the physical motion from $q_1^i(t_1)$ to $q_2^i(t_2)$.

□

Properties of the Hamilton Function

Recall that the momentum is

$$
p(q,\dot{q}) = \frac{\partial\mathcal{L}(q,\dot{q})}{\partial\dot{q}} \qquad \text{(E.-4)}
$$

We will prove:

$$
\frac{\partial S(q,t,q',t')}{\partial q} = -p(q,t,q',t'), \qquad \frac{\partial S(q,t,q',t')}{\partial q'} = p'(q,t,q',t') \qquad \text{(E.-4)}
$$

where $p(q,t,q',t') = p(q,\dot{q}(q,t,q',t'))$ and $p'(q,t,q',t') = p'(q',\dot{q}'(q,t,q',t'))$ are the initial and final momenta respectively, expressed as functions of initial and final positions and times, via physical motions, determined by the initial and final data.

We also have

$$\frac{\partial S(q,t,q',t')}{\partial t} = E(q,t,q',t'), \qquad \frac{\partial S(q,t,q',t')}{\partial t'} = -E'(q,t,q',t') \tag{E.-4}$$

Proof:

We first prove the second equation of (E.1.6) which refers to the final momemtum. We vary the final point q' while keeping the time fixed (see fig E.1). There will be a variation along the way, $\delta q(t)$, connecting the original physical motion to the new physical motion with different final position. The change in the Hamilton function δS

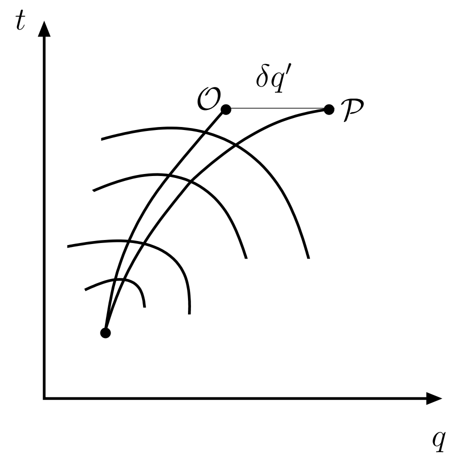


Figure E.2: Variation in the Hamilton function.

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1} \mathcal{L}(q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t), t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t),\dot{q}(t),t)dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial\mathcal{L}}{\partial q}\delta q + \frac{\partial\mathcal{L}}{\partial\dot{q}}\delta\dot{q}\right) dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\delta q dt + \left[\frac{\partial\mathcal{L}}{\partial\dot{q}}\delta q(t)\right]_{t_0}^{t_1} \\
&= \frac{\partial\mathcal{L}}{\partial\dot{q}'}\delta q'
\end{aligned} \tag{E.-6}$$

where we have put the quantity inside the parentheses of the integral to zero by the Euler-Lagrange equation. So

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{q}'}\delta q'$$

and hence

$$\frac{\partial S}{\partial q'} = p'. \tag{E.-6}$$

We first prove the second equation of (E.1.6) which refers to the final energy.

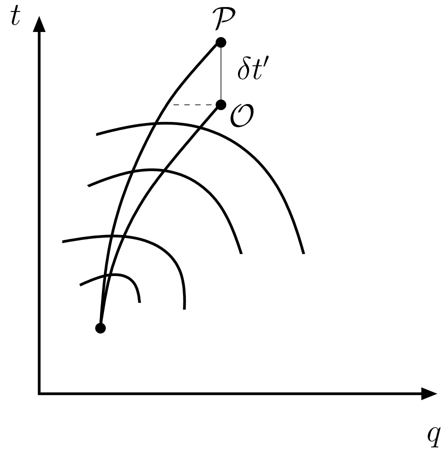


Figure E.3: Variation in the Hamilton function.

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1+\delta t} \mathcal{L}(q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t), t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t), \dot{q}(t), t)dt \\
&= \mathcal{L}\delta t' + \int_{t_0}^{t_1} \mathcal{L}(q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t), t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t), \dot{q}(t), t)dt \\
&= \mathcal{L}\delta t' + \frac{\partial \mathcal{L}}{\partial \dot{q}'}\delta q' \qquad \text{(E.-7)}
\end{aligned}$$

where

$$\delta q' = -\dot{q}'\delta t'$$

(see fig E.2). So

$$\delta S = \left(\mathcal{L} - \dot{q}'\frac{\partial \mathcal{L}}{\partial \dot{q}'}\right)\delta t'$$

and hence

$$\frac{\partial S}{\partial t'} = -H_0 = -E. \tag{E.-7}$$

In general we have

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{q}'}\delta q' - \left(\dot{q}'\frac{\partial \mathcal{L}}{\partial \dot{q}'} - \mathcal{L}\right)\delta t' \tag{E.-7}$$

We have proved

$$\frac{\partial S}{\partial q'} = \frac{\partial \mathcal{L}(q', \dot{q}', t)}{\partial \dot{q}'} = p' \tag{E.-7}$$

$$\frac{\partial S}{\partial t'} = \mathcal{L} - \dot{q}'\frac{\partial \mathcal{L}}{\partial \dot{q}'} = -H_0. \tag{E.-7}$$

The Hamilton Function solves the Hamilton-Jacobi equation

Solve the first equation for $\dot{q}$ so that we have

$$\dot{q}' = \dot{q}'(q', \frac{\partial S}{\partial q'}, t')$$

and substitute the value of $\dot{q}$ into the second giving

$$\frac{\partial S}{\partial t'} + H_0\left(q', \frac{\partial S}{\partial q'}, t'\right) = 0, \tag{E.-7}$$

that is, the Hamilton function solves the Hamiltonian-Jacobi equation. The Hamilton-Jacobi equation is also solved in the initial variables (q, t) in the sense that

$$-\frac{\partial S}{\partial t} + H_0\left(q, -\frac{\partial S}{\partial q}, t\right) = 0 \tag{E.-7}$$

where we have a minus signs in front of the partial derivatives.

If we know the Hamilton function, we have solved the equations of motion because we obtain the general solution of the equations of motion in the form $q'(q, p, t, t')$ by simply inverting the function

$$p(q,t,q',t') = -\frac{\partial S(q,t,q',t')}{\partial q} \tag{E.-7}$$

with respect to q'. The resulting function $q'(q,p,t,t')$ is the general solution of the equations of motion where the initial coordinate and momentum are q, p at time t.

Example : **Free particle**

The ("conventional") Hamiltonian is

$$H_0(q,p) = \frac{1}{2m}p^2. \tag{E.-7}$$

From the Hamilton function (E.-4)

$$S(q,t,q',t') = \frac{m(q'-q)^2}{2(t'-t)} \tag{E.-7}$$

We have

$$\begin{aligned} \frac{\partial S}{\partial q'} &= m\frac{q'-q}{t'-t} = mv = p', \\ \frac{\partial S}{\partial t'} &= -\frac{m(q'-q)^2}{2(t'-t)^2} = -E' \end{aligned} \tag{E.-7}$$

So that

$$\frac{\partial S}{\partial t'} + \frac{1}{2m}\left(\frac{\partial S}{\partial q'}\right)^2 = 0 \tag{E.-7}$$

and therefore the Hamilton-Jacobi equation is solved.

Consider the derivative

$$\frac{\partial S}{\partial q} = -m\frac{q'-q}{t'-t} = -mv = -p. \tag{E.-7}$$

Inverting this gives the general solution

$$q' = q + \frac{p}{m}(t'-t). \tag{E.-7}$$

$\square$

A second example.

Example : **Harmonic oscillator**

The ("conventional") Hamiltonian is

$$H_0(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2. \tag{E.-7}$$

The Hamilton function is (E.1.6)

$$S(q,t,q',t') = m\omega \frac{(q^2+q'^2)cos\omega(t'-t) - 2qq'}{2\sin\omega(t'-t)} \tag{E.-7}$$

for motion that starts at (q,t) and ends at (q',t').

We have

$$\begin{aligned} \frac{\partial S}{\partial q'} &= m\omega \frac{q'\cos\omega(t'-t) - q}{\sin\omega(t'-t)} = mv = p', \\ \frac{\partial S}{\partial t'} &= -m\omega^2 \frac{(q^2+q'^2)}{2\sin^2\omega(t'-t)} + m\omega^2 \frac{qq'\cos\omega(t'-t)}{\sin^2\omega(t'-t)} = -E' \end{aligned} \tag{E.-7}$$

So that

$$\begin{aligned} &\frac{\partial S}{\partial t'} + \frac{1}{2m}\left(\frac{\partial S}{\partial q'}\right)^2 + \frac{1}{2}m\omega^2 q'^2 = \\ &= -m\omega^2 \frac{(q^2+q'^2)}{2\sin^2\omega(t'-t)} + m\omega^2 \frac{qq'\cos\omega(t'-t)}{\sin^2\omega(t'-t)} \\ &\quad + m\omega^2 \frac{q^2+q'^2\cos^2\omega(t'-t) - 2qq'\cos\omega(t'-t)}{2\sin^2\omega(t'-t)} + \frac{1}{2}m\omega^2 q'^2 \\ &= 0. \end{aligned} \tag{E.-9}$$

and therefore the Hamilton-Jacobi equation is solved.

Consider the derivative

$$\frac{\partial S}{\partial q} = m\omega \frac{q\cos\omega(t'-t) - q'}{\sin\omega(t'-t)} = -p. \tag{E.-9}$$

Inverting this gives the general solution

$$q' = q\cos\omega(t'-t) + \frac{p}{m\omega}\sin\omega(t'-t). \tag{E.-9}$$

□

E.1.7 Presympletic Formulation

A very elegant formulation of mechanics, and a crucial step in the direction of the generally covariant formulism, is provided by the presympletic formulism.

Define the covariant configuration space

$$\mathcal{C} = \mathbb{R} \times \mathcal{C}_0 \tag{E.-9}$$

coordinatised by the $m+1$ variables (t, q^i).

The graph of the function $(q^i(t), p_i(t))$ is an unparametrised curve $\tilde{\gamma}$ in the $(2m+1)-$dimensional space $\Sigma = \mathbb{R}\times\Gamma_0$, with coordinates $(t, q^i(t), p_i(t))$; it is formed by all the points $(t, q^i(t), p_i(t))$ in this space. The vector field

$$X = \frac{\partial}{\partial t} + v_i(q^i, p_i)\frac{\partial f}{\partial q^i} + f_i(q^i, p_i)\frac{\partial f}{\partial p_i} \tag{E.-9}$$

E.2 Generally Covariant Mechanics

The purpose of this appendix is to express dynamics in the language more wildly used, to give visual examples as well as to provide workings out. More details of ideas and physical considerations should be sought from Rovelli's book.

$$\partial_\nu = (\partial_{q^1}, \partial_{q^2} \ldots \partial_{q^m};, \partial_{p_1}, \partial_{p_2} \ldots, \partial_{p_m}) \tag{E.-9}$$

$$X^1 = \frac{\partial\mathcal{H}}{\partial q}, \quad X^2 = \frac{\partial\mathcal{H}}{\partial p} \tag{E.-9}$$

$$X_0 g = \frac{\partial \mathcal{H}}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial \mathcal{H}}{\partial q}\frac{\partial g}{\partial p} \tag{E.-9}$$

$$\omega_{\mu\nu} = \frac{1}{2}\left(\frac{\partial \theta_\mu}{\partial \xi^\nu} - \frac{\partial \theta_\nu}{\partial \xi^\mu}\right) \tag{E.-9}$$

$$\Omega_{\mu\nu}\frac{d\xi^\nu}{ds} = -\partial_\mu f \tag{E.-9}$$

$$(d\theta_0)(X_f) = -df \tag{E.-9}$$

$$\Omega_{\mu\nu}\frac{d\xi^\nu}{dt} = -\partial_\mu \mathcal{H} \tag{E.-9}$$

$$d\theta_0 = dx^\alpha \frac{\partial \theta_0}{\partial x^\alpha} \quad d\mathcal{H}_0 = dx^\alpha \frac{\partial \theta_0}{\partial x^\alpha} \tag{E.-9}$$

$$(d\theta_0)(X) = -d\mathcal{H} \tag{E.-9}$$

Presympletic

Extended phase space

$$\mathcal{C} = \mathbb{R} \times \mathcal{C}_0 \tag{E.-9}$$

A simple harmonic oscillator can be viewed as a system with two partial observables, q and t A motion of the system defines a relation between q and t. A given motion is characterised by two constants $A \in [0, \infty]$ and $\phi \in [0, 2\pi]$, and is given by the equation

$$f(q, t) = q - A\sin(\omega t + \phi) = 0. \tag{E.-9}$$

E.2.1 Hamiltonian Mechanics

Definition : Variational principle. A curve γ connecting the events q_1^a and q_2^a is a physical motion if $\tilde{\gamma}$ extremises the action

$$S[\tilde{\gamma}] = \int_{\tilde{\gamma}} p_a dq^a \tag{E.-9}$$

in the class of curves $\tilde{\gamma}$ satisfying

$$H(q^a, p_a) = 0 \tag{E.-9}$$

whose restriction γ to $\mathcal{C}$ connects q_1^a and q_2^a.

$\square$

E.2.2 Relativistic Hamilton-Jacobi Equation

$$H\left(q^a, \frac{\partial S(q^a)}{\partial q^a}\right) = 0 \tag{E.-9}$$

defined on the extended configuration space $\mathcal{C}$.

E.2.3 Double Timeless Pendulum - Classical Theory

Introduction to the Double Timeless Pendulum

The Double Timeless Pendulum is a mechanical model with two partial observables, a and b. A given motion is characterised by two constants $A \in [0, \sqrt{2E}]$ and $\phi \in [0, 2\pi]$, and is given by the equation

$$f(a,b) = \left(\frac{a}{A}\right)^2 + \left(\frac{b}{B}\right)^2 - 2\frac{a}{A}\frac{b}{B}\cos\phi = \sin^2\phi. \tag{E.-9}$$

Whose dynamics is defined by the relativistic Hamiltonian

$$H(a,b,p_a,p_b) = -\frac{1}{2}(p_a^2 + p_b^2 + a^2 + b^2 - 2E) = 0. \tag{E.-9}$$

The Hamilton-Jacobi equation

The Hamilton-Jacobi equation is

$$\left(\frac{\partial S(a,b)}{\partial a}\right)^2 + \left(\frac{\partial S(a,b)}{\partial b}\right)^2 + a^2 + b^2 - 2E = 0. \tag{E.-9}$$

We solve this using method of separation of variables. We put:

$$S(a,b) = g(a) + h(b)$$

and substitute this into (E.2.3),

$$\left(\frac{\partial g(a)}{\partial a}\right)^2 + a^2 = -\left(\frac{\partial h(b)}{\partial b}\right)^2 - b^2 + 2E. \qquad \text{(E.-9)}$$

Each side is equal to a constant, therefore we can write

$$\begin{aligned} \left(\frac{dg(a)}{da}\right)^2 + a^2 &= A^2 \\ -\left(\frac{dh(b)}{db}\right)^2 - b^2 + 2E &= A^2. \end{aligned} \qquad \text{(E.-9)}$$

The first equation becomes,

$$\frac{dg(a)}{da} = \sqrt{A^2 - a^2} \qquad \text{(E.-9)}$$

and so

$$g(a) = \int^a \sqrt{A^2 - \tilde{a}^2} d\tilde{a} \qquad \text{(E.-9)}$$

In order to arrive at the evaluation of this integral, we first write:

$$\begin{aligned} I &= \int^a \sqrt{A^2 - \tilde{a}^2} d\tilde{a} \\ &= a\sqrt{A^2 - a^2} - \int^a \tilde{a}\frac{d}{d\tilde{a}}\left(\sqrt{A^2 - \tilde{a}^2}\right) d\tilde{a} \\ &= a\sqrt{A^2 - a^2} + \int^a \frac{\tilde{a}^2}{\sqrt{A^2 - \tilde{a}^2}} d\tilde{a} \\ &= a\sqrt{A^2 - a^2} - \int^a \frac{A^2 - \tilde{a}^2}{\sqrt{A^2 - \tilde{a}^2}} d\tilde{a} + A^2 \int^a \frac{d\tilde{a}}{\sqrt{A^2 - \tilde{a}^2}} \\ &= a\sqrt{A^2 - a^2} - I + A^2 \int^a \frac{d\tilde{a}}{\sqrt{A^2 - \tilde{a}^2}} \end{aligned} \qquad \text{(E.-12)}$$

where we used integration by parts in the first step. From this we have,

$$I = \frac{a}{2}\sqrt{A^2 - a^2} + \frac{A^2}{2}\int^a \frac{d\tilde{a}}{\sqrt{A^2 - \tilde{a}^2}}. \tag{E.-12}$$

To solve the integral on the RHS we use the substitution $\tilde{a} = A\sin u$,

$$\begin{aligned} \int^a \frac{d\tilde{a}}{\sqrt{A^2 - \tilde{a}^2}} &= \int^{\arcsin(a/A)} \frac{A\cos u du}{\sqrt{A^2 - A^2\sin^2 u}} \\ &= \int^{\arcsin(a/A)} du \\ &= \arcsin\left(\frac{a}{A}\right) \\ &= \arctan\left(\frac{a}{\sqrt{A^2 - a^2}}\right). \end{aligned} \tag{E.-14}$$

Therefore

$$g(a) = \frac{a}{2}\sqrt{A^2 - a^2} + \frac{A^2}{2}\arctan\left(\frac{a}{\sqrt{A^2 - a^2}}\right). \tag{E.-14}$$

The second equation, (E.-9), can be written

$$\left(\frac{dh(b)}{db}\right)^2 = 2E - A^2 - b^2 \tag{E.-14}$$

We immediately see that, in analogy to (E.2.3), that we have

$$h(b) = \frac{b}{2}\sqrt{2E - A^2 - b^2} + \frac{2E - A^2}{2}\arctan\left(\frac{b}{\sqrt{2E - A^2 - b^2}}\right). \tag{E.-14}$$

So finally we have the one parameter family of solutions

$$\begin{aligned} S(a, b, A) &= \frac{a}{2}\sqrt{A^2 - a^2} + \frac{A^2}{2}\arctan\left(\frac{a}{\sqrt{A^2 - a^2}}\right) \\ &\quad + \frac{b}{2}\sqrt{2E - A^2 - b^2} + \frac{2E - A^2}{2}\arctan\left(\frac{b}{\sqrt{2E - A^2 - b^2}}\right). \end{aligned} \tag{E.-14}$$

Derivation of the Evolution equation

The evolution equation can be obtained from

$$\frac{\partial S(a,b,A)}{\partial A} - p_A = 0. \tag{E.-14}$$

We define $\phi = p_A/A$, then we have

$$\frac{\partial S(a,b,A)}{\partial A} - A\phi = 0. \tag{E.-14}$$

Calculating the derivative,

$$\begin{aligned}
\frac{\partial S(a,b,A)}{\partial A} &= \frac{\partial}{\partial A}(g(a,A) + h(b,A)) \\
&= \frac{\partial}{\partial A}\int^a \sqrt{A^2 - \tilde{a}^2}d\tilde{a} + \frac{\partial}{\partial A}\int^b \sqrt{2E - A^2 - \tilde{b}^2}d\tilde{b} \\
&= A\int^a \frac{d\tilde{a}}{\sqrt{A^2 - \tilde{a}^2}} - A\int^b \frac{d\tilde{b}}{\sqrt{2E - A^2 - \tilde{b}^2}} \\
&= A \arcsin\left(\frac{a}{A}\right) - A\arcsin\left(\frac{b}{\sqrt{2E - A^2}}\right).
\end{aligned} \tag{E.-16}$$

We then have

$$\arcsin\left(\frac{a}{A}\right) - \arcsin\left(\frac{b}{B}\right) = \phi. \tag{E.-16}$$

where $B = \sqrt{2E - A^2}$. Using $\sin(x-y) = \sin x \cos y - \cos x \sin y$ on (E.2.3) gives

$$\frac{a}{A}\cos\left[\arcsin\left(\frac{b}{B}\right)\right] - \frac{b}{B}\cos\left[\arcsin\left(\frac{a}{A}\right)\right] = \sin\phi \tag{E.-16}$$

Then squaring both sides gives

$$\begin{aligned}
&\left(\frac{a}{A}\right)^2\cos^2\left[\arcsin\left(\frac{b}{B}\right)\right] + \left(\frac{b}{B}\right)^2\cos^2\left[\arcsin\left(\frac{a}{A}\right)\right] \\
&-2\frac{a}{A}\frac{b}{B}\cos\left[\arcsin\left(\frac{a}{A}\right)\right]\cos\left[\arcsin\left(\frac{b}{B}\right)\right] = \sin^2\phi
\end{aligned} \tag{E.-16}$$

Using $\cos^2 x = 1 - \sin^2 x$ this becomes

$$\left(\frac{a}{A}\right)^2 \left[1 - \left(\frac{b}{B}\right)^2\right] + \left(\frac{b}{B}\right)^2 \left[1 - \left(\frac{a}{A}\right)^2\right]$$
$$-2\frac{a}{A}\frac{b}{B}\cos\left[\arcsin\left(\frac{a}{A}\right)\right]\cos\left[\arcsin\left(\frac{b}{B}\right)\right] = \sin^2\phi. \qquad \text{(E.-16)}$$

Using $\cos(x-y) = \cos x \cos y + \sin x \sin y$ on (E.2.3) we have

$$\cos\left[\arcsin\left(\frac{a}{A}\right)\right]\cos\left[\arcsin\left(\frac{b}{B}\right)\right] + \frac{a}{A}\frac{b}{B} = \cos\phi. \qquad \text{(E.-16)}$$

Substituting this into (E.-16) gives

$$\left(\frac{a}{A}\right)^2 \left[1 - \left(\frac{b}{B}\right)^2\right] + \left(\frac{b}{B}\right)^2 \left[1 - \left(\frac{a}{A}\right)^2\right] - 2\frac{a}{A}\frac{b}{B}\left[\cos\phi - \frac{a}{A}\frac{b}{B}\right] = \sin^2\phi. \qquad \text{(E.-16)}$$

which easily simplifies to

$$\left(\frac{a}{A}\right)^2 + \left(\frac{b}{B}\right)^2 - 2\frac{a}{A}\frac{b}{B}\cos\phi = \sin^2\phi. \qquad \text{(E.-16)}$$

which is the evolution equation (E.2.3).

The Hamilton function

$$\begin{aligned} a' &= a'(\tau') = A\sin(\tau' + \phi) \\ b' &= b'(\tau') = B\sin(\tau') \end{aligned} \qquad \text{(E.-16)}$$

$$\begin{aligned} a &= a(\tau' + \tau) = A\sin(\tau' + \tau + \phi) \\ b &= b(\tau' + \tau) = B\sin(\tau' + \tau) \end{aligned} \qquad \text{(E.-16)}$$

We now derive the formula

$$A^2 = \frac{a^2 + a^{'2} - 2aa' \cos\tau}{\sin^2\tau} \tag{E.-16}$$

and

$$E = \frac{(a^2 + a^{'2} + b^2 + b^{'2}) - 2(aa' + bb') \cos\tau}{\sin^2\tau}. \tag{E.-16}$$

We first prove (E.2.3)

$$\begin{aligned}
&a^2 + a^{'2} - 2aa' \cos\tau = \\
&= A^2\{\sin^2(\tau' + \phi + \tau) + \sin^2(\tau' + \phi) - 2\sin(\tau' + \phi + \tau)\sin(\tau' + \phi)\cos\tau\} \\
&= A^2\{[\sin(\tau' + \phi)\cos\tau + \cos(\tau' + \phi)\sin\tau]^2 + \sin^2(\tau' + \phi) \\
&\qquad -2[\sin(\tau' + \phi)\cos\tau + \cos(\tau' + \phi)\sin\tau]\sin(\tau' + \phi)\cos\tau\} \\
&= A^2\{\sin^2(\tau' + \phi)\cos^2\tau + \cos^2(\tau' + \phi)\sin^2\tau + 2\sin(\tau' + \phi)\cos\tau\cos(\tau' + \phi)\sin\tau \\
&\qquad + \sin^2(\tau' + \phi)[\cos^2\tau + \sin^2\tau] \\
&\qquad -2\sin^2(\tau' + \phi)\cos^2\tau - 2\cos(\tau' + \phi)\sin\tau\sin(\tau' + \phi)\cos\tau\} \\
&\qquad = A^2\sin^2\tau
\end{aligned} \tag{E.-22}$$

giving (E.2.3). Similarly

$$B^2 = \frac{b^2 + b^{'2} - 2aa' \cos\tau}{\sin^2\tau} \tag{E.-22}$$

Therefore

$$2E = A^2 + B^2 = \frac{a^2 + a^{'2} + b^2 + b^{'2} - 2(aa' + bb')\cos\tau}{\sin^2\tau} \tag{E.-22}$$

giving (E.2.3)? Rearanging gives

$$\cos^2\tau - \frac{(aa' + bb')}{E}\cos\tau + \frac{a^2 + a^{'2} + b^2 + b^{'2} - 2E}{2E} = 0. \tag{E.-22}$$

So

$$\cos\tau = \frac{aa' + bb' + \pm\sqrt{(aa' + bb')^2 + 2E(2E - a^2 - a'^2 - b^2 - b'^2)}}{2E}. \tag{E.-22}$$

and

$$\tau(a,b,a',b') = \arccos \frac{aa' + bb' + \pm\sqrt{(aa'+bb')^2 + 2E(2E - a^2 - a'^2 - b^2 - b'^2)}}{2E}. \tag{E.-22}$$

E.2.4 Nonrelativistic Systems as a Special Case

E.3 Conventional Quantum Mechanics

E.3.1 Transition Amplitudes

Let $\hat{q}$ be a set of operators that commute, are complete in the sense of Dirac (form a maximally commuting set) and whose corresponding classical variables coordinatise the configuration space. Consider the basis that diagonalises these operators $\hat{q}|q>= q|q>$. The transition amplitude is then given by

$$W(q,t,q',t') = < q'|e^{-\frac{i}{\hbar}H_0(t'-t)}|q> \tag{E.-22}$$

That is, the transition amplitude is given by the matrix elements of the evolution operator

$$U(t) = e^{-\frac{i}{\hbar}H_0 t} \tag{E.-22}$$

in the basis $|q>$.

Notice that the transition amplitude is a function of the same variables as the Hamilton function. The transition amplitude gives the dynamics of quantum theory.

Example : Free particle For a free particle

$$W(q,t,q',t') = < q'|e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}(t'-t)}|q> \tag{E.-22}$$

Inserting the resolution of identity $\mathbf{1} = \int dp|p>< p|$,

$$\begin{aligned} W(q,t,q',t') &= \int dp \int dp' \; < q'|p'>< p'|e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}(t'-t)}|p>< p|q> \\ &= \frac{1}{2\pi\hbar}\int dp \; e^{\frac{i}{\hbar}p(q-q') - \frac{i}{\hbar}\frac{p^2}{2m}(t'-t)} \end{aligned} \tag{E.-22}$$

where we have used $< p|q >= \frac{1}{\sqrt{2\pi\hbar}} e^{ipq/\hbar}$. Evaluating the Gaussian integral we get

$$W(q,t,q',t') = A \exp\left\{ \frac{i}{\hbar} \frac{m(q'-q)^2}{2(t'-t)} \right\} \quad \text{(E.-22)}$$

were the amplitude is $A = \sqrt{\frac{m}{2\pi\hbar i(t'-t)}}$. Recall that $\frac{m(q'-q)^2}{2(t'-t)}$ was the Hamilton function of this system. Therefore

$$W(q,t,q',t') \propto e^{\frac{i}{\hbar} S(q,t,q',t')}. \quad \text{(E.-22)}$$

$\square$

E.3.2 The Feynman Path Integral

The above result is general via the construction given by Feynman. Start from

$$W(q,t,q',t') = < q'|U(t'-t)|q > \quad \text{(E.-22)}$$

and using the fact that the evolution operator defines a group

$$U(t'-t) = U(t'-t'')U(t''-t) \quad \text{(E.-22)}$$

to write it as a product of short-time evolution operators

$$W(q,t,q',t') = < q'|U(\epsilon) \ldots U(\epsilon)|q > \quad \text{(E.-22)}$$

where

$$\epsilon = \frac{t'-t}{N} \quad \text{(E.-22)}$$

At each step we insert the resolution of the identity

$$\mathbb{1} = \int dq_n |q_n >< q_n|. \quad \text{(E.-22)}$$

The transition amplitude is then expressed by a multiple integral

$$W(q,t,q',t') = \int dq_1 \dots dq_{N-1} \prod_{n=1}^{N} < q_n|U(\epsilon)|q_{n-1} > \qquad \text{(E.-22)}$$

where we take $q = q_0$ and $q' = q_N$. This expression is true for any N. We take the limit where $N \to \infty$:

$$W(q,t,q',t') = \lim_{N\to\infty} \int dq_1 \dots dq_{N-1} \prod_{n=1}^{N} < q_n|U(\epsilon)|q_{n-1} > . \qquad \text{(E.-22)}$$

Now, consider the particular case where the Hamiltonian is of the form

$$H_0 = \frac{p^2}{2m} + V(q). \qquad \text{(E.-22)}$$

The "infitesimal" evolution operator is then

$$U(\epsilon) = e^{-\frac{i}{\hbar}\left(\frac{p^2}{2m}+V(q)\right)\epsilon}. \qquad \text{(E.-22)}$$

For small ϵ we can make the replacement

$$U(\epsilon) \sim e^{-\frac{i}{\hbar}\frac{p^2}{2m}\epsilon} e^{-\frac{i}{\hbar}V(q)\epsilon}, \qquad \text{(E.-22)}$$

(see subsection on the Trotter product formula below). In the $|q>$ basis the second exponential gives just a number. The first was computed above in (E.-22 - E.3.1). Together they yeild

$$\begin{aligned} < q_n|e^{-\frac{i}{\hbar}\frac{p^2}{2m}\epsilon} e^{-\frac{i}{\hbar}V(q)\epsilon}|q_{n-1} > &= < q_n|e^{-\frac{i}{\hbar}\frac{p^2}{2m}\epsilon}|q_{n-1} > e^{-\frac{i}{\hbar}V(q_n)\epsilon} \\ &= \sqrt{\frac{m}{2\pi\hbar i\epsilon}} e^{\frac{i}{\hbar}\left(\frac{m(q_n-q_{n-1})^2}{2(t_n-t_{n-1})^2}+V(q_m)\right)\epsilon} \end{aligned} \qquad \text{(E.-22)}$$

The exponent in the last expression is a discretisation of the classical action. The transition amplitude can therefore be written as a multiple integral of the discretisation of the action in the limit that the discretisation is replaced by the continuous limit, namely $\epsilon \to 0$.

We then have for the tranistion amplitude

$$
\begin{aligned}
W(q,t,q',t') &= \lim_{N\to\infty}\int dq_1\ldots dq_{N-1}\prod_{n=1}^{N} < q_n|U(\epsilon)|q_{n-1} > \\
&= \lim_{N\to\infty}\left(\frac{m}{2\pi\hbar i\epsilon}\right)^{N/2}\int dq_1\ldots dq_{N-1}e^{\frac{i}{\hbar}\sum_{n=1}^{N-1}\left(\frac{m(q_n-q_{n-1})^2}{2\epsilon^2}+V(q_m)\right)\epsilon} \\
&= \lim_{N\to\infty}\mathcal{N}\int dq_1\ldots dq_{N-1}e^{\frac{i}{\hbar}S_N(q_n)}. \qquad \text{(E.-23)}
\end{aligned}
$$

We now come to the reason why this object is called a "path integral" and can be seen as a "sum over histories". Imagine that the points $q, q_1, \ldots, q_{N-1}, q'$ are connected by lines. The sum in the exponential of (E.-23) interpreted as a Riemann sum of a certain integral along the path - the action S.

The argument of the exponential in (E.-23) is $iS/\hbar$ evaluated along the broken path connecting $q, q_1, \ldots, q_{N-1}, q'$. The integrals over the quantities $q_1, \ldots, q_{N-1}$ can be interpreted as summing over all possible broken paths connecting q and q'.

The notation for the path integral is

$$
W(q,t,q',t') = \int \mathcal{D}[q(t)]e^{\frac{i}{\hbar}S[q]}. \qquad \text{(E.-23)}
$$

In the absence of a satifactory hamiltonian operator we can take an expression like (E.-23) as a tentative ansatz for *defining* the quantum theory.

Trotter product formula

We give an outline of the argument. Putting $\lambda := i(t'-t)/\hbar$ we can write

$$
W(q,t,q',t') = < q'|e^{-\lambda(T+V)/N}\ldots e^{-\lambda(T+V)/N}|q > \qquad \text{(E.-23)}
$$

It can be shown that

$$
e^{-\lambda(T+V)/N} = e^{-\lambda T/N}e^{-\lambda V/N} + \mathcal{O}\left(\frac{\lambda^2}{N^2}\right) \qquad \text{(E.-23)}
$$

(worked exercie). In the limit $N\to\infty$. We wish to establish that we can replace the term

$$
\left[e^{-\lambda(T+V)/N}\right]^N \qquad \text{(E.-23)}
$$

with the term

$$\left[e^{-\lambda T/N}e^{-\lambda V/N}\right]^N \tag{E.-23}$$

We express the difference between (E.3.2) and (E.3.2) as

$$\begin{aligned}
&\left(e^{-\lambda T/N}e^{-\lambda V/N}\right)^N - \left(e^{-\lambda(T+V)/N}\right)^N \\
&\qquad = \left[e^{-\lambda T/N}e^{-\lambda V/N} - e^{-\lambda(T+V)/N}\right]\left(e^{-\lambda(T+V)/N}\right)^{N-1} \\
&\qquad = +e^{-\lambda T/N}e^{-\lambda V/N}\left[e^{-\lambda T/N}e^{-\lambda V/N} - e^{-\lambda(T+V)/N}\right]e^{-\lambda(T+V)^{(N-2)/N}} \\
&\qquad + \cdots + (e^{-\lambda T/N}e^{-\lambda V/N})^{N-1}\left[e^{-\lambda T/N}e^{-\lambda V/N} - e^{-\lambda(T+V)/N}\right]
\end{aligned} \tag{E.-25}$$

This is an identity. It contains N terms, each of which has the factor $\exp(-\lambda T/N)\exp(-\lambda V/N) - \exp(-\lambda(T+V)/N)$, which by (E.3.2) is of order $1/N^2$. This justifies the replacement of (E.3.2) by the expression

$$W(q,t,q',t') = \lim_{N\to\infty} < q'|(e^{-\lambda T/N}e^{-\lambda V/N})^N|q > . \tag{E.-25}$$

E.3.3 General Properties of Transition Amplitudes

The transition amplitude gives the wavefunction $\psi(q,t)$ given the intial wavefunction $\psi(q',t')$,

$$\psi(q',t') = \int dq\ \psi(q,t)W(q,t,q',t'). \tag{E.-25}$$

The transition amplitude $W(q,t,q',t')$ is then (as a function of q') the wavefunction at time t' for a state that at time t was a delta function concentrated at q. Therefore it satisfies the Schrodinger equation in the variables (q',t') (and the conjugate equation in the variables (q,t)).

$$-i\hbar\frac{\partial}{\partial t'}W(q,t,q',t') + H_0\left(q', -i\hbar\frac{\partial}{\partial q'}\right)W(q,t,q',t') = 0. \tag{E.-24}$$

Example : Free particle For a free particle

First we have

$$
\begin{aligned}
-i\hbar\frac{\partial}{\partial t'}W(q,t,q',t') &= -i\hbar\frac{\partial}{\partial t'}\left(\sqrt{\frac{m}{2\pi\hbar i(t'-t)}}\exp\left\{\frac{i}{\hbar}\frac{m(q'-q)^2}{2(t'-t)}\right\}\right) \\
&= -i\hbar\left(-\frac{1}{2(t'-t)}-\frac{i}{\hbar}\frac{m(q'-q)^2}{2(t'-t)^2}\right)W(q,t,q',t'). \quad \text{(E.-24)}
\end{aligned}
$$

Then

$$
\begin{aligned}
& -H_0\left(q',-i\hbar\frac{\partial}{\partial q'}\right)W(q,t,q',t') \\
& = -A\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial q'}\right)^2\exp\left\{\frac{i}{\hbar}\frac{m(q'-q)^2}{2(t'-t)}\right\} \\
& = \hbar^2 A\frac{1}{2m}\frac{\partial}{\partial q'}\left(\frac{i}{\hbar}\frac{m(q'-q)}{(t'-t)}\exp\left\{\frac{i}{\hbar}\frac{m(q'-q)^2}{2(t'-t)}\right\}\right) \\
= & \ \hbar^2\frac{1}{2m}\left(\frac{i}{\hbar}\frac{m}{(t'-t)}+\left(\frac{i}{\hbar}\right)^2\frac{m^2(q'-q)^2}{(t'-t)^2}\right)W(q,t,q',t') \qquad \text{(E.-26)}
\end{aligned}
$$

and hence Schrodinger equation is solved. It is easy to see that $W(q,t,q',t')$ satisfies the conjugate Schrodinger equation in the variables (q,t),

$$
+i\hbar\frac{\partial}{\partial t}W(q,t,q',t')+H_0^*\left(q,-i\hbar\frac{\partial}{\partial q}\right)W(q,t,q',t')=0. \qquad \text{(E.-26)}
$$

□

Example : The SHO. Normalisation factor for the SHO transition amplitude:

Use the fact that

$$
W(q,t,q',t')=A(t'-t)e^{\frac{i}{\hbar}S(q,t,q',t')} \qquad \text{(E.-26)}
$$

satisfies the Schödinger equation in the variables t',q' and that $S(q,t,q',t')$ satifies the Hamilton-Jacobi equation in the variables t',q' to determine the normalisation factor $A(t'-t)$.

Solution.

The Schödinger equation in the variables t',q' is

$$-i\hbar\frac{\partial}{\partial t'}W(q,t,q',t') + \left(\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial q'}\right)^2 + \frac{1}{2}m\omega^2 q'^2\right)W(q,t,q',t') = 0. \qquad \text{(E.-26)}$$

or

$$\begin{aligned} &-i\hbar e^{\frac{i}{\hbar}S(q,t,q',t')}\frac{\partial}{\partial t'}A(t'-t) - i\hbar A(t'-t)\frac{\partial}{\partial t'}e^{\frac{i}{\hbar}S(q,t,q',t')} \\ &\qquad + \left(-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial q'}\right)^2 + \frac{1}{2}m\omega^2 q'^2\right)W(q,t,q',t') = 0. \end{aligned} \qquad \text{(E.-26)}$$

This becomes

$$\begin{aligned} &i\hbar e^{\frac{i}{\hbar}S(q,t,q',t')}\frac{\partial}{\partial t'}A(t'-t) - W(q,t,q',t')\frac{\partial}{\partial t'}S(q,t,q',t') \\ &\quad = A(t'-t)\left(-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial q'}\right)^2 + \frac{1}{2}m\omega^2 q'^2\right)e^{\frac{i}{\hbar}S(q,t,q',t')} \\ &\quad = W(q,t,q',t')\left(-\frac{\hbar^2}{2m}\left(\frac{i}{\hbar}\frac{\partial^2 S}{\partial q'^2} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial q'}\right)^2\right) + \frac{1}{2}m\omega^2 q'^2\right) \end{aligned} \qquad \text{(E.-27)}$$

This is greatly simplified by substitution of the Hamilton-Jacobi equation,

$$\frac{\partial S}{\partial t'} + \frac{1}{2m}\left(\frac{\partial S}{\partial q'}\right)^2 + \frac{1}{2}m\omega^2 q'^2 = 0 \qquad \text{(E.-27)}$$

to obtain

$$\frac{\partial}{\partial t'}A(t'-t) = -\frac{1}{2m}A(t'-t)\frac{\partial^2 S}{\partial q'^2}. \qquad \text{(E.-27)}$$

Now inserting the explicit expression for the Hamilton function of a harmonic oscillator, (E.1.6), which I reproduce here

$$S(q,t,q',t') = m\omega\frac{(q^2+q'^2)\cos\omega(t'-t) - 2qq'}{2\sin\omega(t'-t)} \qquad \text{(E.-27)}$$

for motion that starts at (q,t) and ends at (q',t'). Equation (E.3.3) becomes

$$\frac{\partial}{\partial t'}A(t'-t) = -\frac{1}{2}\omega A(t'-t)\cot\omega(t'-t) \tag{E.-27}$$

or

$$\frac{\partial}{\partial t'}\ln A(t'-t) = -\frac{1}{2}\frac{\partial}{\partial t'}\left(\ln\sin\omega(t'-t)\right) \tag{E.-27}$$

so that

$$\ln A(t'-t) = \ln C - \frac{1}{2}\left(\ln\sin\omega(t'-t)\right) \tag{E.-27}$$

or

$$\begin{aligned} A(t'-t) &= C\exp\left(-\frac{1}{2}\left(\ln\sin\omega(t'-t)\right)\right) \\ &= C\sqrt{\frac{1}{\sin\omega(t'-t)}}. \end{aligned} \tag{E.-27}$$

The overall constant of integration may be obtained from the free particle limit, i.e. $\omega \to 0$, of the transition amplitude which we know is

$$\sqrt{\frac{m}{2\pi\hbar i(t'-t)}}\exp\left\{\frac{i}{\hbar}\frac{m(q'-q)^2}{2(t'-t)}\right\}.$$

Therefore for the SHO we have

$$W(q,t,q',t') = \sqrt{\frac{m\omega}{2\pi\hbar i\sin\omega(t'-t)}}\exp\left\{\frac{i}{\hbar}m\omega\frac{(q^2+{q'}^2)\cos\omega(t'-t)-2qq'}{2\sin\omega(t'-t)}\right\} \tag{E.-27}$$

□

E.4 Generally Covariant Quantum Mechanics

We have

$$< \alpha', t'| = < \alpha| e^{-iH_0 t'} \quad \text{and} \quad |\alpha, t> = e^{iH_0 t}|\alpha> \tag{E.-27}$$

The propagator is defined by

$$\begin{aligned}
W(\alpha, t, \alpha', t') &= < \alpha', t'|\alpha, t > \\
&= < \alpha'|e^{-iH_0(t'-t)}|\alpha > \\
&= \sum_{mn} < \alpha'|m >< m|e^{-iH_0(t'-t)}|n >< n|\alpha > \\
&= \sum_{n} H_n(\alpha') e^{-iE_n(t'-t)} \overline{H_n(\alpha)},
\end{aligned} \tag{E.-29}$$

where $H_n(\alpha)$ is the eigenfunction of H_0 with eigenvalue E_n.

E.4.1 Boundary Formalism

The probability amplitude of measuring a state ψ_t at t if the state ψ_0 was measured at time $t = 0$ is

$$A = < \psi_t|e^{-iHt}|\psi_0 > \tag{E.-29}$$

Fix a time t and consider the non-relativistic boundary space

$$\mathcal{K}_t = \mathcal{H}_t^* \otimes \mathcal{H}_0 = L_2[\mathbb{R}^2, d\alpha d\alpha'] \tag{E.-29}$$

where the notation $\mathcal{H}^*$ indicates the dual of the Hilbert space $\mathcal{H}$. The space $\mathcal{K}_t$ can be called a kinematic Hilbert space.

Dynamical vacuum

The linear functional W_t defined by

$$W_t\,(\psi_t \otimes \psi) := < \psi_t|e^{-iHt}|\psi_0 > \tag{E.-29}$$

is well defined on $\mathcal{K}_t$. This functional captures the entire dynamics of the system. A linear functional on a Hilbert space defines a state. We denote this state $|0_t >$ defined by W_t

$$W_t(\psi) = < 0_t|\psi >_{\mathcal{K}_t} \tag{E.-29}$$

and call it the "dynamical vacuum" state in boundary state space $\mathcal{K}_t$.

The states

$$|\alpha, \alpha' > = < \alpha'|_t \otimes |\alpha >_0 \tag{E.-29}$$

represent a basis of the system for α at time $t = 0$ to α' at time t.

Recall

$$W(\alpha, t, \alpha', t') = < \alpha'|e^{-iH(t'-t)}|\alpha > . \tag{E.-29}$$

$$\begin{aligned} < 0_t|\alpha, \alpha' > &= W_t\,(< \alpha'| \otimes |\alpha >) \\ &= < \alpha|e^{-iHt}|\alpha' > \\ &= W(\alpha, t, \alpha', 0) \end{aligned} \tag{E.-30}$$

"Minkowski" vacuum

Denote $|0_M >$ the lowest eigenstate of H_0 in $\mathcal{H}_0$,

$$< \alpha|0_M > = H_0(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha^2} \tag{E.-30}$$

Consider the analytic continuation in imaginary time of the propagator

$$W(\alpha, -it, \alpha', 0) \to_{t\to\infty} H_0(\alpha) e^{-E_0 t} \overline{H_0(\alpha')}. \tag{E.-30}$$

This can be written

$$\lim_{t\to\infty} e^{E_0 t}|0_{-it} >= \ |0_M > \otimes < 0_M|. \tag{E.-30}$$

This expression relates the dynamical vacuum $|0_t >$ and the Minkowski $|0_M >$. This equation can be used to find the quantum states corresponding to Minkowski spacetime from the spinfoam formulation of quantum gravity.

E.5 Conditional Probabilities

$$\mathcal{P}(a\ when\ b) = \frac{< s|\mathbb{P}\ \pi_a \pi_b \mathbb{P}|s >}{< s|\mathbb{P}\pi_b \mathbb{P}|s >} \tag{E.-30}$$

E.5.1 Dolby

$$\mathcal{U}(t_{n+1}) = \mathcal{U}(t_n)\psi(t_n)$$

In order to illustrate the diffculty with this definition of probability, consider the two-state system introduced in Section 2.2, but let us imagine, for simplicity, that time is discrete. That is, the states are $\psi_S(t_n), S =\uparrow, \downarrow$ where $\psi_S(t) =< S, t_n|\psi >$, with integer n.

In the conventional formalism one focus on probabilities of the form

$$P(\uparrow \text{ when } t_n),$$

where the event $(\uparrow, t_n)$ is considered as one element of the set of equal time alternatives $S_{t_n} = \{(\uparrow, t_n)(\downarrow, t_n)\}$.

But the general formalism does not privilege the time variable and therefore allows us to consider also probabilities of the form

$$\mathcal{P}(t_n \text{ when } \uparrow),$$

where the event $(\uparrow, t_n)$ is considered as one element of the set of alternatives

$$S = \{\ldots, (\uparrow, t_{n-1}), (\uparrow, t_n), (\uparrow, t_{n+1}), \ldots\}.$$

$$\psi_\uparrow(t_n) = \begin{cases} 1, & \text{if } n = 1, 2 \\ 0, & \text{otherwise} \end{cases} \tag{E.-30}$$

i.e.

$$\mathbb{P}\psi_\uparrow(t_n) = \psi_\uparrow(t_n)$$

then

$$\begin{aligned}\mathcal{P}(t_n \text{ when } \uparrow) &= \frac{< s|\pi_{t_n}\pi_{\uparrow}|s >}{< s|\pi_{\uparrow}|s >} \\ &= \sum_{s'} \frac{< s|\pi_{t_n}|s' >< s'|\pi_{\uparrow}|s >}{|\psi_s|^2} \\ &= \sum_{s'} \frac{< s|\pi_{t_n}|s' >< s'|\pi_{\uparrow}|s >}{|\psi_s|^2} \end{aligned} \tag{E.-31}$$

where

$$\mathcal{U}(t_0) = \mathcal{U}(t_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{U}(t_2) = \mathcal{U}(t_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \tag{E.-31}$$

E.5.2 Hamilton Function of GR

$$S_{GR}[g] = \int_{\mathcal{R}} d^n x \sqrt{\det g}\; R + \int_{\Sigma} \sqrt{\det q}\; k \tag{E.-31}$$

$$S[q] = \int_{\Sigma} \sqrt{\det q}\; k \tag{E.-31}$$

E.6 Schrödinger Representation of Field Theory

The action for the free scalar field theory is

$$\int d^4x\; \mathcal{L}(x) = \frac{1}{2} \int d^4x\; \left(\partial^{\mu}\varphi\partial_{\mu}\varphi - m^2\varphi^2\right). \tag{E.-31}$$

The conjugate field momentum is then

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_t \varphi} = \dot{\varphi}(x) \tag{E.-31}$$

The Hamiltonian density is the given in terms of the Lagrangian density via

$$\mathcal{H}_0(x) = \pi(x)\dot{\varphi}(x) - \mathcal{L}(x) \tag{E.-31}$$

and so the Hamiltonian is

$$H_0 = \frac{1}{2}\int d^3x \left(\pi^2 + |\nabla\varphi|^2 + m^2\varphi^2\right). \tag{E.-31}$$

We go to a coordinate Scrödinger representation and work with a basis for Fock space where the operator $\varphi(\vec{x})$, now time independent, is diagonal. Let $|\phi>$ be an eigenstate of φ with eigenvalue ϕ.

$$\varphi(\vec{x})|\phi>= \phi(\vec{x})|\phi> . \tag{E.-31}$$

note $\varphi(\vec{x})$ is an operator, while $\phi(\vec{x})$ is just an ordinary scalar function.

$$\left[\frac{\delta}{\delta\phi(\vec{x})}, \phi(\vec{y})\right] = \delta(\vec{x}-\vec{y}). \tag{E.-31}$$

therefore

$$\pi(\vec{x}) = -i\frac{\delta}{\delta\phi(\vec{x})} \tag{E.-31}$$

in terms of the coordinate basis

$$< \phi'|\pi(\vec{x})|\phi >= -i\frac{\delta}{\delta\phi(\vec{x})}\delta(\phi-\phi'). \tag{E.-31}$$

turn the Hamiltonian operator into a functional differential operator,

$$H_0 = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\vec{x})} + |\nabla\phi(\vec{x})|^2 + m^2\phi^2(\vec{x})\right), \tag{E.-31}$$

(where $|\nabla\phi| = \partial_i\phi\partial^i\phi$) and the Schödinger

$$i\frac{\partial}{\partial t}|\Psi >= H|\Psi >, \tag{E.-31}$$

turns into a functional differential equation,

$$i\frac{\partial}{\partial t}\Psi[\phi,t] = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\vec{x})} + |\nabla\phi|^2 + m^2\phi^2\right)\Psi[\phi,t]. \tag{E.-31}$$

whose solutions, the eigenfunctionals of the hamiltonian functional differential operator, represent possible states of the system. For time-independent hamiltonians it is possible to separate the variables

$$\Psi[\phi,t] = e^{-iEt}\Psi[\phi], \tag{E.-31}$$

obtaining a functional eigenvalue problem for the time-independent Schödinger equation

$$\frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\vec{x})} + |\nabla\phi|^2 + m^2\phi^2\right)\Psi[\phi] = E\Psi[\phi]. \tag{E.-31}$$

E.6.1 Ground State and Excited States

Creation and anihilation operators can be written

$$\begin{aligned} a^\dagger(\vec{k}) &= \int d^3x e^{-i\vec{k}\cdot\vec{x}}\left(\omega_k\phi(\vec{x}) - \frac{\delta}{\delta\phi(\vec{x})}\right) \\ a(\vec{k}) &= \int d^3x e^{i\vec{k}\cdot\vec{x}}\left(\omega_k\phi(\vec{x}) + \frac{\delta}{\delta\phi(\vec{x})}\right). \end{aligned} \tag{E.-31}$$

The Minkowski vacuum state is determined by $a\left(\vec{k}\right)|0_M>= 0$. In the functional representation, this state reads

$$\Psi_{0_M}[\phi] \equiv <\phi|0_M> . \tag{E.-31}$$

and is determined by

$$a\left(\vec{k}\right)\Psi_{0_M}[\phi] = \frac{\hbar}{\sqrt{2\omega}}\frac{\delta}{\delta\phi(\vec{k})}\Psi_{0_M}[\phi] + \sqrt{\frac{\omega}{2}}\phi(\vec{k})\Psi_{0_M}[\phi] = 0. \tag{E.-31}$$

The solution of this equation gives the functional form of the vacuum state

$$\Psi_{0_M}[\phi] = Ne^{-\frac{1}{2\hbar}\int d^3k\, \omega(\vec{k})\phi(\vec{k})\phi(\vec{k})}. \tag{E.-31}$$

The one-particle state with momentum $\vec{k}$ is created by $a^\dagger(\vec{k})$:

$$\Psi_{\vec{k}}[\phi] \equiv <\phi|\vec{k}>= a^\dagger(\vec{k})\Psi_{0_M}[\phi] = \sqrt{2\omega}\phi(\vec{k})\Psi_{0_M}[\phi]. \tag{E.-31}$$

It has energy $\hbar\omega(\vec{k})$. Therefore, the time-dependent state

$$\Psi_{\vec{k}}[t,\phi] = \sqrt{2\omega}e^{-i\omega(\vec{k})t}\phi(\vec{k})\Psi_{0_M}[\phi] \tag{E.-31}$$

is a solution of the Wheeler-DeWitt equation

$$\left(i\hbar\frac{\partial}{\partial t} - H_0\right)\Psi = 0. \tag{E.-31}$$

A generic one-particle state with wave function $f(\vec{k})$ is defined by

$$|f>\equiv \int \frac{d^3k}{\sqrt{2\omega}} f(\vec{k})|\vec{k}>, \tag{E.-31}$$

and its functional representation is therefore

$$\Psi_f[\phi] \equiv <\phi|f>= \int d^3k f(\vec{k})\phi(\vec{k})\Psi_0[\phi] \tag{E.-31}$$

or

$$\Psi_f[\phi] = \phi[f]\Psi_0[\phi], \tag{E.-31}$$

where

$$\phi[f] = \int d^3k f(\vec{k})\phi(\vec{k}). \tag{E.-31}$$

The corresponding solution to the Wheeler-DeWitt equation is

$$\Psi_f[t,\phi] = \int d^3k f(\vec{k})e^{-i\omega(\vec{k})t}\phi(\vec{k})\Psi_0[\phi] \tag{E.-31}$$

as follows from

$$\left(i\hbar\frac{\partial}{\partial t} - H_0\right)\Psi_f[t,\phi] = \int \frac{d^3k}{\sqrt{2\omega}} f(\vec{k})\left(i\hbar\frac{\partial}{\partial t} - H_0\right)\sqrt{2\omega}e^{-i\omega(\vec{k})t}\phi(\vec{k})\Psi_0[\phi]$$
$$= 0. \qquad \text{(E.-31)}$$

or, in Fourier transform,

$$\Psi_f[t,\phi] = \int d^3x F(t,\vec{x})\phi(\vec{x})\Psi_0[\phi] \qquad \text{(E.-31)}$$

where

$$F(x) = F(t,\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i(\vec{k}\cdot\vec{x}-\omega(\vec{k})t)} f(\vec{k}) \qquad \text{(E.-31)}$$

The $n-$(Fock)-particle state $|k_1,\ldots,k_n>$ can be obtained be repeated application of the creation operator $a^\dagger(\vec{k})$. They have energy $\hbar(\omega_1 + \cdot + \omega_n)$ where $\omega_i(\vec{k_i})$:

$$H_0|k_1,\ldots,k_n >= \hbar(\omega_1 + \cdot + \omega_n)|k_1,\ldots,k_n > . \qquad \text{(E.-31)}$$

The general solution of the Wheeler-DeWitt equation is therefore

$$\Psi[t,\phi] = \sum_n \int \frac{d^3k_1\ldots d^3k_n}{\sqrt{2\omega_1\ldots 2\omega_n}} f(\vec{k}_1\ldots\vec{k}_n)e^{-i(\omega_1+\cdots+\omega_n)t}$$
$$\times a^\dagger(k_1)\ldots a^\dagger(k_n)\Psi_0[\phi]. \qquad \text{(E.-31)}$$

Scalar product

E.7 Transition Amplitude on an Infinite Strip

We calculate the transition amplitude for the free real massive scalar field ϕ, both in the Minkowskian space and in Euclidean space between $t_1 = 0$ and $t_2 = T$.

E.7.1 Minkowskian case

$$W_M[\varphi_1, 0, \varphi_2, T] = \int_{\varphi_1,0,\varphi_2,T} \mathcal{D}\phi \exp\left(\frac{i}{2}\int_0^T d^4x \left(\partial_\mu\phi(x)\partial^\mu\phi(x) - m^2\phi^2\right)\right) \qquad \text{(E.-31)}$$

where

$$\int_0^T d^4x = \int_0^T \int_{-\infty}^{\infty} d^3x. \qquad \text{(E.-31)}$$

The classical equation is

$$\left(\Box_x + m^2\right)\phi(x) = 0. \qquad \text{(E.-31)}$$

We must now solve this equation in the infinite strip bounded by the two hyperplanes $t = 0$ and $t = T$, with boundary conditions

$$\begin{aligned} \phi(\vec{x}, 0) &= \varphi_1(\vec{x}) \\ \phi(\vec{x}, T) &= \varphi_2(\vec{x}). \end{aligned} \qquad \text{(E.-31)}$$

We can solve this problem by considring the Fourier transform $\tilde{\phi}(\vec{k}, t)$ of the field $\phi(\vec{x}, t)$. From equation (E.7.1) we have

$$\begin{aligned} \left(\Box_x + m^2\right)\phi(x) &= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\left(\Box_x + m^2\right) e^{-i\vec{k}\cdot\vec{x}}\tilde{\phi}(\vec{k}, t) \\ &= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}}\left(\frac{\partial^2}{\partial t^2} + \vec{k}^2 + m^2\right)\tilde{\phi}(\vec{k}, t) = 0. \end{aligned} \qquad \text{(E.-31)}$$

This implies

$$\left(\frac{\partial^2}{\partial t^2} + \vec{k}^2 + m^2\right)\tilde{\phi}(\vec{k}, t) = 0. \qquad \text{(E.-31)}$$

Solving this gives:

$$\tilde{\phi}(\vec{k}, t) = \frac{\tilde{\phi}(\vec{k}, T)\sin\omega_k t - \tilde{\phi}(\vec{k}, 0)\sin\omega_k(t - T)}{\sin\omega_k T}. \qquad \text{(E.-31)}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and

$$\begin{aligned} \tilde{\phi}(\vec{k}, 0) &= \int_{-\infty}^{\infty} d^3 y e^{i\vec{k}\cdot\vec{y}} \varphi_1(\vec{y}) \\ \tilde{\phi}(\vec{k}, T) &= \int_{-\infty}^{\infty} d^3 y e^{i\vec{k}\cdot\vec{y}} \varphi_2(\vec{y}). \end{aligned} \tag{E.-31}$$

Putting it altogether we have

$$\overline{\phi}(x) = \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 y e^{-\vec{k}\cdot(\vec{x}-\vec{y})} \frac{\varphi_2(\vec{y}) \sin \omega_k t - \varphi_1(\vec{y}) \sin \omega_k (t-T)}{\sin \omega_k T}. \tag{E.-31}$$

Now the functional integral can be solved by substituting $\phi(x) = \overline{\phi}(x) + \eta(x)$ where $\eta(x)$ is a fluctuation:

$$\begin{aligned} W_M[\varphi_1, 0, \varphi_2, T] &= \int_{\varphi_1, 0, \varphi_2, T} \mathcal{D}\phi \exp\left(\frac{i}{2}\int_0^T d^4 x \left(\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi^2\right)\right) = \\ &= \int_{\varphi_1, 0, \varphi_2, T} \mathcal{D}\phi \exp\left(\frac{i}{2}\int_0^T d^4 x \left(\left(\partial_\mu \overline{\phi}(x) + \partial_\mu \eta(x)\right)^2 - m^2 \left(\overline{\phi}(x) + \eta(x)\right)^2\right)\right) \end{aligned} \tag{E.-32}$$

by using (E.7.1) and the boundary conditions for $\eta(x)$, namely $\eta(\vec{x}, t = 0) = 0$ and $\eta(\vec{x}, t = T) = 0$. These imply for the integral in the exponetial:

$$\begin{aligned} &\int_0^T d^4 x \left(\left(\partial_\mu \overline{\phi}(x) + \partial_\mu \eta(x)\right)\left(\partial^\mu \overline{\phi}(x) + \partial^\mu \eta(x)\right) - m^2 \left(\overline{\phi}(x) + \eta(x)\right)^2\right) = \\ &= \int_0^T d^4 x \left(\partial_\mu \overline{\phi}(x) \partial^\mu \overline{\phi}(x) - m^2 \overline{\phi}^2\right) + 2\left(\partial_\mu \overline{\phi}(x) \partial^\mu \eta(x) - m^2 \overline{\phi}(x)\eta(x)\right) + \\ &\quad + \left(\partial_\mu \eta(x) \partial^\mu \eta(x) - m^2 \eta\right) \\ &= \int_0^T d^4 x \left(\partial_\mu \overline{\phi}(x) \partial^\mu \overline{\phi}(x) - m^2 \overline{\phi}^2\right) - 2\left(\partial^\mu \partial_\mu \overline{\phi}(x) + m^2 \overline{\phi}(x)\right)\eta(x) + \\ &\quad + \left(\partial_\mu \eta(x) \partial^\mu \eta(x) - m^2 \eta\right) \\ &= \int_0^T d^4 x \left(\partial_\mu \overline{\phi}(x) \partial^\mu \overline{\phi}(x) - m^2 \overline{\phi}^2\right) + \left(\partial_\mu \eta(x) \partial^\mu \eta(x) - m^2 \eta^2\right) \end{aligned} \tag{E.-36}$$

so that (E.-32) becomes

$$W_M[\varphi_1, 0, \varphi_2, T] = \exp\left(\frac{i}{2}\int_0^T d^4x \left(\partial_\mu\overline{\phi}(x)\partial^\mu\overline{\phi}(x) - m^2\overline{\phi}^2\right)\right)$$
$$\int_{0,t=0,0,t=T} \mathcal{D}\eta \exp\left(\frac{i}{2}\int_0^T d^4x \left(\partial_\mu\eta(x)\partial^\mu\eta(x) - m^2\eta^2\right)\right) \tag{E.-37}$$

Note the first term is $e^{iS_{cl}/2}$ where S_{cl} is the action evaluated for the classical field configuration $\overline{\phi}(x)$ - i.e. the Hamilton functional.

We can make a simplification:

$$\begin{aligned}
&\int_0^T d^4x \left(\partial_\mu\overline{\phi}(x)\partial^\mu\overline{\phi}(x) - m^2\overline{\phi}^2\right) \\
&= \int_0^T dx^0 \int_{-\infty}^{\infty} d^3x \left(\partial_\mu[\overline{\phi}(x)\partial^\mu\overline{\phi}(x)] - \overline{\phi}(x)\partial_\mu\partial^\mu\overline{\phi}(x) - m^2\overline{\phi}^2\right) \\
&= \int_0^T dx^0 \int_{-\infty}^{\infty} d^3x \left(\partial_{x^0}[\overline{\phi}(x)\partial_{x^0}\overline{\phi}(x)] - \partial_{x^i}[\overline{\phi}(x)\partial_{x^i}\overline{\phi}(x)]\right) \\
&= \int_{-\infty}^{\infty} d^3x\overline{\phi}(x)\partial_{x^0}\overline{\phi}(x)\Big|_0^T - \int_0^T dx_0\overline{\phi}(x)\partial_{x^i}\overline{\phi}(x)\Big|_{-\infty}^{\infty} \\
&= \int_{-\infty}^{\infty} d^3x\, \overline{\phi}(x)\partial_{x^0}\overline{\phi}(x)\Big|_0^T
\end{aligned} \tag{E.-40}$$

where we have made use of the classical equation (E.7.1) again and that the field $\overline{\phi}(x)$ falls off to zero at spatial infinity. Then inserting (E.7.1) we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} d^3x\, \overline{\phi}(x)\partial_{x^0}\overline{\phi}(x)\Big|_0^T = \\
&= \int_{-\infty}^{\infty} d^3x \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} d^3y e^{-\vec{k}\cdot(\vec{x}-\vec{y})} \frac{\varphi_2(\vec{y})\sin\omega_k t - \varphi_1(\vec{y})\sin\omega_k(t-T)}{\sin\omega_k T} \\
&\times \int_{-\infty}^{\infty} \frac{d^3k'}{(2\pi)^3} \int_{-\infty}^{\infty} d^3z e^{-\vec{k}'\cdot(\vec{x}-\vec{z})}\omega_{k'} \frac{\varphi_2(\vec{z})\cos\omega_{k'}t - \varphi_1(\vec{z})\cos\omega_{k'}(t-T)}{\sin\omega_{k'}T}\Big|_0^T \\
&= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^3k'}{(2\pi)^3} \int_{-\infty}^{\infty} d^3x e^{-i\vec{x}\cdot(\vec{k}+\vec{k}')} \int_{-\infty}^{\infty} d^3y \int_{-\infty}^{\infty} d^3z e^{i\vec{k}\cdot\vec{y}} e^{i\vec{k}'\cdot\vec{z}}\omega_{k'} \\
&\left(\varphi_2(\vec{y})(\varphi_2(\vec{z})\cot\omega_{k'}T - \frac{\varphi_1(\vec{z})}{\sin\omega_{k'}T}) - \varphi_1(\vec{y})(\frac{\varphi_2(\vec{z})}{\sin\omega_{k'}T} - \varphi_1(\vec{z})\cot\omega_{k'}T)\right) \\
&= \int_{-\infty}^{\infty} d^3y \int_{-\infty}^{\infty} d^3z e^{i\vec{k}\cdot\vec{y}} \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{y}-\vec{z})}\omega_k \times \\
&\times \left(-\frac{\varphi_1(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{z})\varphi_2(\vec{y})}{\sin\omega_k T} + \cot\omega_k T(\varphi_2(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{y})\varphi_1(\vec{z}))\right)
\end{aligned} \tag{E.-45}$$

Now the second term in (E.-37) becomes

$$\int_{0,t=0,0,t=T} \mathcal{D}\eta \exp\left(-\frac{i}{2}\int_0^T d^4x \left(\eta(x)\left(\Box_x + m^2\right)\eta(x)\right)\right) = \left(\det(-\Box - m^2)\right)^{-\frac{1}{2}}. \tag{E.-45}$$

Substituting this and (E.-45) into (E.-37) we obtain for the transition amplitude

$$\begin{aligned} W_M[\varphi_1, 0, \varphi_2, T] &= \frac{1}{\sqrt{\det(-\Box - m^2)}} \exp\left(\frac{i}{2}\int_{-\infty}^{\infty} d^3y \int_{-\infty}^{\infty} d^3z \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{y}-\vec{z})}\omega_k \right. \\ &\left. \left(-\frac{\varphi_1(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{z})\varphi_2(\vec{y})}{\sin\omega_k T} + \cot\omega_k T(\varphi_2(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{y})\varphi_1(\vec{z}))\right)\right). \end{aligned} \tag{E.-46}$$

The inifinite factor $(\det(-\Box - m^2))^{-\frac{1}{2}}$ will be dealt with in a later subsection.

E.7.2 Euclidean case

We now explicitly calculate the transition amplitude for a free massive scalar field in euclidean space, using a slightly different method that will be used again in the calculation of the generalised Tomonaga-Schwinger equation.

$$W_E[\varphi_1, 0, \varphi_2, T] = \int_{\varphi_1,0,\varphi_2,T} \mathcal{D}\phi \exp\left(-\frac{1}{2}\int_0^T d^4x \left(\partial_\mu\phi(x)\partial^\mu\phi(x) + m^2\phi^2\right)\right) \tag{E.-46}$$

classical equation

$$\left(\Box_x - m^2\right)\phi(x) = 0. \tag{E.-46}$$

We now solve this equation using the Green function technique. The Green's function satisfies the equation

$$\left(\Box_x - m^2\right)G(x,y) = -\delta^4(x-y). \tag{E.-46}$$

This can be rewritten as

$$\int_0^T d^4x \left(G(x,y)\left(\Box_x - m^2\right)\overline{\phi}(x) - \overline{\phi}(x)\left(\Box_x - m^2\right)G(x,y)\right) = \\ = \int_0^T d^4x\delta^4(x-y)\overline{\phi}(x) \tag{E.-46}$$

that is

$$\begin{aligned}
\overline{\phi}(y) &= \int_0^T d^4x \left(G(x,y)\Box_x\overline{\phi}(x) - \overline{\phi}(x)\Box_x G(x,y)\right) \\
&= \int_0^T dx^0 \int_{-\infty}^{\infty} d^3x \left(G(x,y)\partial^2_{x^i}\overline{\phi}(x) - \overline{\phi}(x)\partial^2_{x^i}G(x,y) + \right. \\
&\qquad \left. + G(x,y)\partial^2_{x^0}\overline{\phi}(x) - \overline{\phi}(x)\partial^2_{x^0}G(x,y)\right) \\
&= \int_0^T dx^0 \int_{-\infty}^{\infty} d^3x \left[\partial_{x^i}\left(G(x,y)\partial_{x^i}\overline{\phi}(x) - \overline{\phi}(x)\partial_{x^i}G(x,y)\right) + \right. \\
&\qquad \left. + \partial_{x^0}\left(G(x,y)\partial_{x^0}\overline{\phi}(x) - \overline{\phi}(x)\partial_{x^0}G(x,y)\right)\right] \\
&= \int_0^T dx^0 \int_{-\infty}^{\infty} d^3x \left[\partial_{x^i}\left(G(x,y)\overleftrightarrow{\partial}_{x^i}\overline{\phi}(x)\right) + \partial_{x^0}\left(G(x,y)\overleftrightarrow{\partial}_{x^0}\overline{\phi}(x)\right)\right]
\end{aligned} \tag{E.-51}$$

where $G\overleftrightarrow{\partial}\phi = G\partial\phi - \phi\partial G$. Supposing that $G(x,y)$ and $\phi(x)$ go to zero fast enough at spatial infinity the first term of the sum is zero, and so

$$\begin{aligned}
\overline{\phi}(y) &= \int_{-\infty}^{\infty} d^3x \int_0^T dx^0\partial_{x^0}\left(G(x,y)\overleftrightarrow{\partial}_{x^0}\overline{\phi}(x)\right) \\
&= \int_{-\infty}^{\infty} d^3x \left[G(x,y)\overleftrightarrow{\partial}_{x^0}\overline{\phi}(x)\right]_{x^0=0}^{x^0=T} \\
&= \int_{-\infty}^{\infty} d^3x \left[G(\vec{x},T,y)\,\partial_{x^0}\overline{\phi}(x)\big|_{x^0=T} - \partial_{x^0}G(x,y)\big|_{x^0=T}\,\varphi_2(\vec{x}) + \right. \\
&\qquad \left. - G(\vec{x},0,y)\,\partial_{x^0}\overline{\phi}(x)\big|_{x^0=0} + \partial_{x^0}G(x,y)\big|_{x^0=0}\,\varphi_1(\vec{x})\right]
\end{aligned} \tag{E.-53}$$

To reproduce the boundary conditions $\phi(\vec{x},0) = \varphi_1(\vec{x})$ and $\phi(\vec{x},T) = \varphi_2(\vec{x})$ the Green function G must be zero for $x_0 = 0$ and $x_0 = T$. Then

$$\overline{\phi}(y) = \int_{-\infty}^{\infty} d^3x \left(\partial_{x^0}G(\vec{x},0,y)\,\varphi_1(\vec{x}) - \partial_{x^0}G(\vec{x},T,y)\,\varphi_2(\vec{x})\right). \tag{E.-53}$$

We solve (E.7.2) with a "spatial" Fourier transform:

$$\begin{aligned} G(x,y) &= \int \frac{d^3k}{(2\pi)^3}\tilde{G}(x_0,y_0)e^{-i(\vec{x}-\vec{y})} \\ \delta^4(x-y) &= \int \frac{d^3k}{(2\pi)^3}\delta(x_0-y_0)e^{-i(\vec{x}-\vec{y})} \end{aligned} \tag{E.-53}$$

imposing the conditions

$$\tilde{G}(0,y_0) = \tilde{G}(t,y_0) = 0. \tag{E.-53}$$

Employing (E.-53) in (E.7.2) we have:

$$\int \frac{d^3k}{(2\pi)^3}\left(\Box_x - m^2\right)\tilde{G}(x_0,y_0)e^{-i(\vec{x}-\vec{y})} = -\int \frac{d^k}{(2\pi)^3}\delta(x_0-y_0)e^{-i(\vec{x}-\vec{y})}. \tag{E.-53}$$

Which gives

$$\left(\partial^2_{x0} - (\vec{k}^2+m^2)\right)\tilde{G}(x_0,y_0) = -\delta(x_0-y_0). \tag{E.-53}$$

The general solution is of the form

$$\tilde{G}(x_0,y_0) = \tilde{G}_p(x_0,y_0) + \left[A(\vec{k},y_0)f_1(\vec{k},x_0) + B(\vec{k},y_0)f_2(\vec{k},x_0)\right] \tag{E.-53}$$

where $\tilde{G}_p(x_0,y_0)$ is a solution (i.e., any solution) of the inhomogeneous equation and $f_1(\vec{k},x_0)$, $f_2(\vec{k},x_0)$ are two linearly independent solutions of the homogeneous equation. The homogeneous solutions are helpful in obtaing the correct boundary conditions.

A solution of the inhomogeneous equation is

$$\frac{1}{2\omega_k}e^{-\omega_k|x_0-y_0|}. \tag{E.-53}$$

We check this,

$$\begin{aligned} \partial^2_{x0}\left[\frac{1}{2\omega_k}e^{-\omega_k|x_0-y_0|}\right] &= \partial_{x0}\left[-\omega_k\left(2\Theta(x_0-y_0)-1\right)\frac{1}{2\omega_k}e^{-\omega_k|x_0-y_0|}\right] \\ &= \left(-\delta(x_0-y_0) + \omega_k^2\frac{1}{2\omega_k}e^{-\omega_k|x_0-y_0|}\right), \end{aligned}$$

which rearranged gives

$$\left(\partial_{x_0}^2 - (\vec{k}^2 + m^2)\right) \frac{1}{2\omega_k} e^{-\omega_k |x_0 - y_0|} = -\delta(x_0 - y_0). \qquad \text{(E.-55)}$$

Two linearly independent homogeneous solutions are obviously $e^{\omega_k x_0}$ and $e^{-\omega_k x_0}$. Therefore

$$\tilde{G}(x_0, y_0) = \frac{1}{2\omega_k} e^{-\omega_k |x_0 - y_0|} + A(\vec{k}, y_0) e^{\omega_k x_0} + B(\vec{k}, y_0) e^{-\omega_k x_0}. \qquad \text{(E.-55)}$$

We now impose the boundary conditions (E.7.2)

$$\begin{aligned} \tilde{G}(0, y_0) &= \frac{1}{2\omega_k} e^{-\omega_k y_0} + A(\vec{k}, y_0) + B(\vec{k}, y_0) = 0 \\ \tilde{G}(T, y_0) &= \frac{1}{2\omega_k} e^{-\omega_k (T - y_0)} + A(\vec{k}, y_0) e^{\omega_k T} + B(\vec{k}, y_0) e^{-\omega_k T} = 0 \end{aligned} \qquad \text{(E.-56)}$$

which are solved by appropriate choices for $A(\vec{k}, y_0)$ and $B(\vec{k}, y_0)$. From these equations we obtain

$$\tilde{G}(T, y_0) - \tilde{G}(0, y_0) e^{-\omega_k T} = \frac{1}{2\omega_k} e^{-\omega_k (T - y_0)} - \frac{1}{2\omega_k} e^{-\omega_k y_0} e^{-\omega_k T} + A(\vec{k}, y_0)(e^{\omega_k T} - e^{-\omega_k T}) = 0 \qquad \text{(E.-56)}$$

and

$$\tilde{G}(0, y_0) e^{\omega_k T} - \tilde{G}(T, y_0) = \frac{1}{2\omega_k} e^{-\omega_k y_0} e^{\omega_k T} - \frac{1}{2\omega_k} e^{-\omega_k (T - y_0)} + B(\vec{k}, y_0)(e^{\omega_k T} - e^{-\omega_k T}) = 0 \qquad \text{(E.-56)}$$

so that

$$A(\vec{k}, y_0) = \frac{e^{-\omega_k y_0} e^{-\omega_k T} - e^{-\omega_k (T - y_0)}}{2\omega_k (e^{\omega_k T} - e^{-\omega_k T})} \qquad \text{(E.-56)}$$

$$B(\vec{k}, y_0) = \frac{e^{-\omega_k (T - y_0)} - e^{-\omega_k y_0} e^{\omega_k T}}{2\omega_k (e^{\omega_k T} - e^{-\omega_k T})} \qquad \text{(E.-56)}$$

Substituting these into (E.7.2), and expressing $G(x,y)$ as the "spatial" Fourier transformation, we finally have for the Green function $G(x,y)$:

$$
\begin{aligned}
G(x,y) &= \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \left(\frac{1}{2\omega_k} e^{-\omega_k|x_0-y_0|} + \right. \\
&\left. + \frac{e^{-\omega_k y_0} e^{-\omega_k T} - e^{-\omega_k (T-y_0)}}{2\omega_k \left(e^{\omega_k T} - e^{-\omega_k T}\right)} e^{\omega_k x_0} + \frac{e^{-\omega_k (T-y_0)} - e^{-\omega_k y_0} e^{\omega_k T}}{2\omega_k \left(e^{\omega_k T} - e^{-\omega_k T}\right)} e^{-\omega_k x_0} \right). \qquad \text{(E.-57)}
\end{aligned}
$$

We now wish to substitute this into (E.7.2). First we calculate

$$
\begin{aligned}
\partial_{x^0} \tilde{G}(x_0, y_0) &= -\frac{1}{2}(2\Theta(x_0 - y_0) - 1) e^{-\omega_k|x_0-y_0|} + \\
&+ e^{-\omega_k y_0} \frac{e^{\omega_k T} e^{-\omega_k x_0} + e^{-\omega_k T} e^{\omega_k x_0}}{2\left(e^{\omega_k T} - e^{-\omega_k T}\right)} \\
&- e^{\omega_k y_0} \frac{e^{\omega_k x_0} + e^{-\omega_k x_0}}{2\left(e^{\omega_k T} - e^{-\omega_k T}\right)} e^{-\omega_k T} \qquad \text{(E.-58)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\partial_{x^0} \tilde{G}(0, y_0) &= \frac{1}{2} e^{-\omega_k y_0} + \frac{1}{2} e^{-\omega_k y_0} \frac{\cosh \omega_k T}{\sinh \omega_k T} - \frac{1}{2} e^{\omega_k y_0} \frac{e^{-\omega_k T}}{\sinh \omega_k T} \\
&= -\frac{\sinh \omega_k (y_0 - T)}{\sinh \omega_k T} \qquad \text{(E.-58)}
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{x^0} \tilde{G}(T, y_0) &= -\frac{1}{2} e^{-\omega_k (T-y_0)} + \frac{1}{2} e^{-\omega_k y_0} \frac{1}{\sinh \omega_k T} - \frac{1}{2} e^{-\omega_k (T-y_0)} \frac{\cosh \omega_k T}{\sinh \omega_k T} \\
&= +\frac{1}{2} e^{-\omega_k y_0} \frac{1}{\sinh \omega_k T} - \frac{1}{2} e^{-\omega_k (T-y_0)} \frac{1}{\sinh \omega_k T} e^{\omega_k T} \\
&= -\frac{\sinh \omega_k y_0}{\sinh \omega_k T} \qquad \text{(E.-59)}
\end{aligned}
$$

substituting this into (E.7.2), the formula that we repeat here

$$\overline{\phi}(y) = \int_{-\infty}^{\infty} d^3x \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \left(\partial_{x^0}\tilde{G}(0,y_0)\,\varphi_1(\vec{x}) - \partial_{x^0}\tilde{G}(T,y_0)\,\varphi_2(\vec{x})\right) \qquad \text{(E.-59)}$$

we have

$$\overline{\phi}(x) = \int_{-\infty}^{\infty} d^3y \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{\sinh\omega_k t\ \varphi_2(\vec{y}) - \sinh\omega_k(t-T)\varphi_1(\vec{y})}{\sinh\omega_k T}. \qquad \text{(E.-59)}$$

The functional integral can be soved exactly as in the minkowskian case, to obtain

$$\begin{aligned} W_E[\varphi_1, 0, \varphi_2, T] \;\; = \;\; & \frac{1}{\sqrt{\det(-\Box - m^2)}} \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} d^3y \int_{-\infty}^{\infty} d^3z \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{y}-\vec{z})}\omega_k \right. \\ & \left.\left(-\frac{\varphi_1(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{z})\varphi_2(\vec{y})}{\sinh\omega_k T} + \coth\omega_k T(\varphi_2(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{y})\varphi_1(\vec{z}))\right)\right). \end{aligned} \qquad \text{(E.-60)}$$

E.7.3 Normalisation Factor

Minkowskian case

To find the correct normaqlisation factor for the transition amplitude, we sue that it satify the functional Schrödinger equation

$$i\frac{\partial}{\partial T} W_M[\varphi_1, 0, \varphi_2, 0, T] = \frac{1}{2}\left(-\frac{\delta^2}{\delta\varphi_2^2(\vec{x})} + |\nabla\varphi_2(\vec{x})|^2 + m^2\varphi_2^2(\vec{x})\right) W_M[\varphi_1, 0, \varphi_2, 0, T] \qquad \text{(E.-60)}$$

By writing the transition amplitude as $W_M[\varphi_1, 0, \varphi_2, 0, T] = M(T)\exp(iS[\varphi_1, \varphi_2])$, (E.7.3) reads as

$$\begin{aligned} i\exp(iS[\varphi_1,\varphi_2])\frac{\partial}{\partial T}M(T) - W_M[\varphi_1, 0, \varphi_2, T]\frac{\partial}{\partial T}S[\varphi_1,\varphi_2] = W_M[\varphi_1, 0, \varphi_2, 0, T]\,\cdot \\ \cdot\frac{1}{2}\int d^3x \left(\left(-i\frac{\delta^2 S[\varphi_1,\varphi_2]}{\delta\varphi_2^2(\vec{x})} + \left(\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_2(\vec{x})}\right)^2\right) + \left(|\nabla\varphi_2(\vec{x})|^2 + m^2\varphi_2^2(\vec{x})\right)\right) \end{aligned} \qquad \text{(E.-61)}$$

This is greatly simplified by substitution of the Hamilton-Jacobi equation, that is, the classical action calculated on the boundary conditions, and as such obeys

$$
\begin{aligned}
0 &= \frac{\partial S}{\partial T} + H_0\left(\varphi_2, \frac{\delta S}{\delta \varphi_2}\right) \\
&= \frac{\partial S}{\partial T} + \frac{1}{2}\int d^3x \left(\left(\frac{\delta S}{\delta \varphi_2}\right)^2 + |\nabla\varphi_2(\vec{x})|^2 + m^2\varphi_2^2(\vec{x})\right), \qquad \text{(E.-61)}
\end{aligned}
$$

to obtain

$$
i\exp\left(iS[\varphi_1,\varphi_2]\right)\frac{\partial}{\partial T}M(T) = -\frac{i}{2}W_M[\varphi_1, 0, \varphi_2, 0, T]\int d^3x \frac{\delta^2 S[\varphi_1,\varphi_2]}{\delta\varphi_2^2(\vec{x})}. \qquad \text{(E.-61)}
$$

From the exponent of (E.-46),

$$
\begin{aligned}
S[\varphi_1,\varphi_2] &= \frac{1}{2}\int_{-\infty}^{\infty} d^3y \int_{-\infty}^{\infty} d^3z \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\cdot(\vec{y}-\vec{z})}\omega_k \\
&\quad \left(-\frac{\varphi_1(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{z})\varphi_2(\vec{y})}{\sin\omega_k T} + \cot\omega_k T(\varphi_2(\vec{y})\varphi_2(\vec{z}) + \varphi_1(\vec{y})\varphi_1(\vec{z}))\right)
\end{aligned} \qquad \text{(E.-62)}
$$

we obtain

$$
\begin{aligned}
\int d^3x \frac{\delta^2 S[\varphi_1,\varphi_2]}{\delta\varphi_2^2(\vec{x})} &= \int d^3x \frac{1}{2}\int_{-\infty}^{\infty} d^3y \int_{-\infty}^{\infty} d^3z \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\cdot(\vec{y}-\vec{z})} \\
&\quad \cot\omega_k T(2\delta(\vec{x}-\vec{y})\delta(\vec{x}-\vec{z})) \\
&= V\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\omega_k\cot\omega_k T. \qquad \text{(E.-63)}
\end{aligned}
$$

where V is a volume. Substituting this into (E.7.3) we get

$$
\frac{\partial}{\partial T}M(T) = -\frac{1}{2}M(T)V\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\omega_k\cot\omega_k T \qquad \text{(E.-63)}
$$

or

$$
\frac{\partial}{\partial T}\ln M(T) = -\frac{1}{2}V\frac{\partial}{\partial T}\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\omega_k\ln\sin\omega_k T \qquad \text{(E.-63)}
$$

so that

$$M(T) = C \exp\left(-\frac{1}{2}V\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\omega_k \ln\sin\omega_k T\right). \tag{E.-63}$$

E.7.4 Relation to the Vacuum State

Minkowski vacuum

$$\Psi_0[\phi] = \exp\left(-\frac{1}{2}\int d^3y\int d^3z\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{y}-\vec{z})}\sqrt{\vec{p}^2+m^2}\psi(\vec{z})\psi(\vec{y})\right) \tag{E.-63}$$

This functional corresponds to the vacuum state defined as the state with the lowest energy. In the following we call the Minkowski vacuum the vacuum state defined this way, to distinguish it from another vacuum state that will be defined shortly.

Choose a basis $|n>$ of eigenstates of H_0 with eigenvalues E_n, and consider the operator

$$W(T) = \sum_n e^{-TE_n}|n><n|. \tag{E.-63}$$

In the large T limit, this becomes the projector on the vacuum

$$\lim_{T\to\infty} W(T) = |0_M><0_M|. \tag{E.-63}$$

Nonperturbative vacuum

We define a kinematic Hilbert space $\mathcal{K}_{\Sigma_T}$, as the tensor product

$$\mathcal{K}_{\Sigma_T} := \mathcal{H}^*_{t=T}\otimes\mathcal{H}_{t=0} \tag{E.-63}$$

where the notation $\mathcal{H}^*$ indicates the dual of the Hilbert space $\mathcal{H}$; which of course is canonically isomorphic to $\mathcal{H}$. We denote a field on Σ_T by $\phi = (\varphi_1, \varphi_2)$. The field basis of the Fock space induces in $\mathcal{K}_{\Sigma_T}$ the basis

$$|\phi>=|\varphi_1,\varphi_2> = <\varphi_2|_{t=T}\otimes|\varphi_1>_{t=0}, \tag{E.-63}$$

which in the language of wave functionals translates as

$$\Psi[\phi] = \Psi[\varphi_1, \varphi_2] = < \varphi_1, \varphi_2 | \Psi > . \qquad \text{(E.-63)}$$

In the kinematic Hilbert space $\mathcal{K}_{\Sigma_T}$ the transition amplitude $W[\varphi_1, 0; \varphi_2, T]$ defines a preferred (bra) state

$$< 0_{\mathcal{K}_{\Sigma_T}} | \Psi > = W[\varphi_1, 0; \varphi_2, T] \qquad \text{(E.-63)}$$

is this Hilbert space. The state is referred to as the *nonperturbative vacuum*, or covariant vacuum. This state expresses the dynamics from $t = 0$ to $t = T$. As a state in $\mathcal{K}_{\Sigma_T}$, which is the tensor product of two Hilbert spaces, it defines a linear mapping between the two spaces $\mathcal{H}_{t=0}$ and $\mathcal{H}_{t=T}$. The linear mapping is precisely the (imaginary time) evolution e^{-TH}. We have by construction

$$< 0_{\mathcal{K}_{\Sigma_T}} | \left(< \psi_{out} | \otimes | \psi_{in} > \right) = < \psi_{out} | e^{-TH} | \psi_{in} > . \qquad \text{(E.-63)}$$

Or

$$< 0_{\mathcal{K}_{\Sigma_T}} | \psi_{in} > = e^{-TH} | \psi_{in} > . \qquad \text{(E.-63)}$$

Notice that the bra/ket mismatch is apparent only, as the three states live in different Hilbert spaces.

Equation (E.7.4) shows that in the limit $T \to \infty$ we have the projector on the vacuum

$$\lim_{T \to \infty} < 0_{\mathcal{K}_{\Sigma_T}} | \left(< \psi_{out} | \otimes | \psi_{in} > \right) = < \psi_{out} | 0_M > \; < 0_M | \psi_{in} > . \qquad \text{(E.-63)}$$

We can therefore write the relation the two notions of vacuum that we have defined as

$$\lim_{T \to \infty} | 0_{\mathcal{K}_{\Sigma_T}} > = | 0_M > \otimes < 0_M | . \qquad \text{(E.-63)}$$

E.8 General Boundary Formulation

E.9 Generalised Schödinger Equation in Euclidean Field Theory

E.9.1 Surfaces and Surface Derivatives

Consider a finite region $\mathcal{R}$ in the euclidean $4d$ space $\mathbb{R}^4$. we use cartesian coordinates $x, y, z, \ldots$ on $\mathbb{R}^4$ where $x = (x^a)$, $a = 1, 2, 3, 4$. Let $\Sigma = \partial\mathcal{R}_\Sigma$ be a compact 3d surface that bounds a finite region $\mathcal{R}_\Sigma$. We denote $s, t, u \ldots$ coordinates on Σ, where $s = (s^q)$, $q = 1, 2, 3$.

A line element is given by

$$d\tau^2 = q_{qr}(s) ds^q ds^r = g_{ab} \left(\frac{\partial x^a}{\partial s^q} ds^q \right) \left(\frac{\partial x^b}{\partial s^q} ds^r \right) \tag{E.-63}$$

the induced metric is then

$$q_{ab}(s) = \frac{\partial x^a(s)}{\partial s^q} \frac{\partial x_a(s)}{\partial s^r} \tag{E.-63}$$

The surface gradient is defined as

$$\nabla^q := \frac{\partial}{\partial s^q} \tag{E.-63}$$

The normal one-form of the surface is

$$\tilde{n}_a(s) = \epsilon_{abcd} \frac{\partial x^b(s)}{\partial s^1} \frac{\partial x^c(s)}{\partial s^2} \frac{\partial x^d(s)}{\partial s^3}. \tag{E.-63}$$

We orient the coordinate system s so that $\tilde{n}_a$ is outward directed. Its norm is easily seen to be the determinate of the induced metric q_{qr}, (see append on measurement of area),

$$\tilde{n}_a \tilde{n}^a = \det q. \tag{E.-63}$$

Proof.

$$
\begin{aligned}
\tilde{n}^a\tilde{n}_a &= \epsilon^{abcd}\epsilon_{ab'c'd'}\frac{\partial x_b(s)}{\partial s^1}\frac{\partial x_c(s)}{\partial s^2}\frac{\partial x_d(s)}{\partial s^3}\frac{\partial x^{b'}(s)}{\partial s^1}\frac{\partial x^{c'}(s)}{\partial s^2}\frac{\partial x^{d'}(s)}{\partial s^3} \\
&= \delta^{bcd}_{b'c'd'}\frac{\partial x_b(s)}{\partial s^1}\frac{\partial x_c(s)}{\partial s^2}\frac{\partial x_d(s)}{\partial s^3}\frac{\partial x^{b'}(s)}{\partial s^1}\frac{\partial x^{c'}(s)}{\partial s^2}\frac{\partial x^{d'}(s)}{\partial s^3} \\
&= \det\begin{pmatrix} \delta^b_{b'} & \delta^c_{b'} & \delta^d_{b'} \\ \delta^b_{c'} & \delta^c_{c'} & \delta^d_{c'} \\ \delta^b_{d'} & \delta^c_{d'} & \delta^d_{d'} \end{pmatrix}\frac{\partial x_b(s)}{\partial s^1}\frac{\partial x_c(s)}{\partial s^2}\frac{\partial x_d(s)}{\partial s^3}\frac{\partial x^{b'}(s)}{\partial s^1}\frac{\partial x^{c'}(s)}{\partial s^2}\frac{\partial x^{d'}(s)}{\partial s^3} \\
&= \det\begin{pmatrix} \frac{\partial x_b(s)}{\partial s^1}\frac{\partial x^b(s)}{\partial s^1} & \frac{\partial x_b(s)}{\partial s^1}\frac{\partial x^b(s)}{\partial s^2} & \frac{\partial x_b(s)}{\partial s^1}\frac{\partial x^b(s)}{\partial s^3} \\ \frac{\partial x_b(s)}{\partial s^2}\frac{\partial x^b(s)}{\partial s^1} & \frac{\partial x_b(s)}{\partial s^2}\frac{\partial x^b(s)}{\partial s^2} & \frac{\partial x_b(s)}{\partial s^2}\frac{\partial x^b(s)}{\partial s^3} \\ \frac{\partial x_b(s)}{\partial s^3}\frac{\partial x^b(s)}{\partial s^1} & \frac{\partial x_b(s)}{\partial s^3}\frac{\partial x^b(s)}{\partial s^2} & \frac{\partial x_b(s)}{\partial s^3}\frac{\partial x^b(s)}{\partial s^3} \end{pmatrix} \\
&= \det\begin{pmatrix} q_{11}(s) & q_{12}(s) & q_{13}(s) \\ q_{21}(s) & q_{22}(s) & q_{23}(s) \\ q_{31}(s) & q_{32}(s) & q_{33}(s) \end{pmatrix} \\
&= \det q(s) \qquad \text{(E.-67)}
\end{aligned}
$$

where we have used (E.9.1).

In the following we use the normalised normal

$$
n_a \equiv (\det q)^{-\frac{1}{2}}\,\tilde{n}_a \qquad \text{(E.-67)}
$$

and the induced volume element on Σ

$$
d\Sigma(s) \equiv (\det q)^{\frac{1}{2}}\;d^3 s. \qquad \text{(E.-67)}
$$

Given a functional $F[\Sigma]$ that depends on the surface, we define the functional deriviative with respect to the surface as the normal projection of the functional derivative with respect to the embedding that defines the surface

$$
\frac{\delta}{\delta\Sigma(s)} \equiv n^a(s)\frac{\delta}{\delta x^a(s)} \qquad \text{(E.-67)}
$$

Here the functional derivative on the right hand side is defined in terms of the volume element $d\Sigma$.

$$
\int d\Sigma(s)N(s)\frac{\delta F[\Sigma]}{\delta\Sigma(s)} = \int d\Sigma N(s)\frac{\delta F[\Sigma]}{\delta x^a(s)}n^a(s) = \lim_{\epsilon\to 0}\frac{F[\Sigma_{\epsilon N}]-F[\Sigma]}{\epsilon} \qquad \text{(E.-67)}
$$

where $\Sigma_{\epsilon N}$ is the deformed surface defined by

$$x^a(s) + \epsilon N(s) n^a(s). \tag{E.-67}$$

$$F_f[\Sigma] := \int_{\mathcal{R}} d^4x f(x) \tag{E.-67}$$

Then

$$\frac{\delta F_f[\Sigma]}{\delta \Sigma(s)} = f(x(s)). \tag{E.-67}$$

That is, the variation of the bulk integral under normal variation of the surface is the integrand in the variation point.

E.10 Relation to the Vacuum State

corresponds to the state with the lowest energy.

E.11 Generalised Tomonaga-Swinger Equation

It was demonstrated that $W[\varphi, \Sigma]$ satisfies a local functional equation governing the variation of $W[\varphi, \Sigma]$ under arbitrary local deformations of Σ, namely

$$\frac{\delta W[\varphi, \Sigma]}{\delta \Sigma(s)} = H_0 \left(\varphi(s), \nabla\varphi(s), \frac{\delta}{\delta\varphi(s)} \right) W[\varphi, \Sigma] \tag{E.-67}$$

Equation (E.11) was derived on the basis of a lattice regularisation of the functional integral

$$W[\varphi, \Sigma] = \int_{\phi|_\Sigma = \varphi} \mathcal{D}\phi e^{-S[\phi]} \tag{E.-67}$$

defining $W[\varphi, \Sigma]$, and under certain hypotheses on the existence of the continuum limit. In this section, working in context of the free euclidean theory, we show that this equation can be derived from the functional integral definition of $W[\varphi, \Sigma]$ directly in the continuum setting, using a formula of Hadamard which expresses the variation of a Green function under a variation of the boundary (V. Volterra, *Theory of functionals and of integral and integro-differential equations*, Dover (1959)).

E.11.1 Hadamard

$$\int_V d^4x \partial^\mu X_\mu = \int_\Sigma n^\mu X_\mu \tag{E.-67}$$

$$\begin{aligned} &(-\Delta_x + m^2) G_\Sigma(x,y) = \delta^{(4)}(x-y), \quad , G_\Sigma(x(s),y) = 0, \\ &(-\Delta_x + m^2)\Phi(x) = 0, \quad , x \in V, \ \Phi(x(s)) = \varphi(s) \end{aligned} \tag{E.-67}$$

$$\begin{aligned} \Phi(y) &= \int d^4x \delta^{(4)}(x-y)\Phi(x) \\ &= \int d^4x \Phi(x)(-\Delta_x + m^2) G_\Sigma(x,y) \\ &= \int d^4x \Phi(x)(-\Delta_x + m^2) G_\Sigma(x,y) - G_\Sigma(x,y)(-\Delta_x + m^2)\Phi(x) \\ &= -\int d^4x \left(\Phi(x)\partial^\mu \partial_\mu G_\Sigma(x,y) - G_\Sigma(x,y)\partial^\mu\partial_\mu \Phi(x)\right) \\ &= -\int d^4x \partial_\mu \left(\Phi(x)\partial_\mu G_\Sigma(x,y) - G_\Sigma(x,y)\partial_\mu \Phi(x)\right) \\ &= -\int d\Sigma(s) n^\mu (\Phi(x)\partial_\mu G_\Sigma(x,y) - 0) \end{aligned} \tag{E.-71}$$

$$\Phi(y) = -\int d\Sigma(s) n^\mu \varphi(s) n^\mu(s) \frac{\partial G_\Sigma(x(s),y)}{\partial x^\mu} \tag{E.-71}$$

E.12 Local and Global Particles

E.12.1 Quick Reminder: the harmonic Oscillator

The SHO's dynamics is governed by the Hamiltonian

$$H_0 = \frac{1}{2m}(p^2 + m^2\omega^2 q^2). \tag{E.-71}$$

The state space of the quantum theory is $\mathcal{H} = L_2[\mathbb{R}, dq]$ formed by the functions $\psi(q)$.

The time-independent Schödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(q)}{dq^2} + \frac{m}{2}\omega^2 q^2 \psi(q) = E\psi(q). \tag{E.-71}$$

We introduce the variable

$$q = \sqrt{\frac{\hbar}{m\omega}}\zeta \tag{E.-71}$$

the Schödinger equation is then

$$\frac{\hbar\omega}{2}\left(\frac{d^2}{d\zeta^2} + \zeta^2\right)\psi(\zeta) = E\psi(\zeta). \tag{E.-71}$$

If one defines

$$a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\zeta} + \zeta\right). \tag{E.-71}$$

as the creation operator and

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{d\zeta} + \zeta\right). \tag{E.-71}$$

as the annihilator operator, the Schödinger equation for a harmonic oscillator reduces to

$$\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\psi(\zeta) = E\psi(\zeta). \tag{E.-71}$$

Note

$$\begin{aligned}
[a, a^\dagger] &= \frac{1}{2}[\left(\frac{d}{d\zeta} + \zeta\right), \left(-\frac{d}{d\zeta} + \zeta\right)] \\
&= \frac{1}{2}([\frac{d}{d\zeta}, \zeta] + [\zeta, -\frac{d}{d\zeta}]) \\
&= 1.
\end{aligned} \tag{E.-72}$$

In general

$$|n >= \frac{(a^\dagger)^n}{\sqrt{n!}} \tag{E.-72}$$

The solutions of the corresponding time-independent Schödinger equation are

$$\psi_n(q) = < q|n >= \left(\frac{1}{\pi\epsilon^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{q}{\epsilon}\right) e^{-q^2/2\epsilon^2} \tag{E.-72}$$

where

$$\epsilon = \sqrt{\frac{\hbar}{m\omega}} \tag{E.-72}$$

and $H_n(\zeta)$ are the Hermite polynomials, the first few being:

$$\begin{aligned} H_0(\zeta) &= 1 \\ H_1(\zeta) &= 2\zeta \\ H_2(\zeta) &= 4\zeta^2 - 2. \end{aligned} \tag{E.-73}$$

the first few wavefunctions being:

$$\begin{aligned} \psi_0(q) &= \left(\frac{1}{\pi\epsilon^2}\right)^{\frac{1}{4}} e^{-q^2/2\epsilon^2} \\ \psi_1(q) &= \left(\frac{1}{\pi\epsilon^2}\right)^{\frac{1}{4}} \sqrt{2}\left(\frac{q}{\epsilon}\right) e^{-q^2/2\epsilon^2} \\ \psi_2(q) &= \left(\frac{1}{\pi\epsilon^2}\right)^{\frac{1}{4}} \left(\frac{1}{2}\right)^{\frac{3}{2}} \left(4\left(\frac{q}{\epsilon}\right)^2 - 2\right) e^{-q^2/2\epsilon^2}. \end{aligned} \tag{E.-74}$$

The eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots \tag{E.-74}$$

The eigenmodes of H_0, $|n>$ form an orthonormal basis that diagonalises H_0. In the $q-$representation an arbitrary wavefunction can be written:

$$\begin{aligned} \psi(q) &= < q|\sum_{n=0}^{\infty} a_n |n> \\ &= \sum_{n=0}^{\infty} a_n < q|n> \\ &= \sum_{n=0}^{\infty} a_n \psi_n(q), \end{aligned} \tag{E.-75}$$

where

$$a_n = \int_{-\infty}^{\infty} dq \psi_n^*(q)\psi(q). \tag{E.-75}$$

So that

$$\begin{aligned}
\psi(q) &= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dq' \psi_n^*(q')\psi(q')\psi_n(q) \\
&= \int_{-\infty}^{\infty} dq' \left(\sum_{n=0}^{\infty} \psi_n^*(q')\psi_n(q) \right) \psi(q')
\end{aligned} \tag{E.-75}$$

implying the completeness relation

$$\delta(q - q') = \sum_{n=0}^{\infty} \psi_n^*(q')\psi_n(q). \tag{E.-75}$$

E.12.2 Two Oscillators

To begin with, we consider two weakly coupled harmonic oscillators q_1, q_2 with unit mass and with the same angular frequency ω; the dynamics is governed by the Hamiltonian

$$H_0 = H_1 + H_2 + V = \frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) + \lambda q_1 q_2. \tag{E.-75}$$

where p_1, p_2 are the momenta conjugate to q_1, q_2 and, say, $\lambda \ll \omega^2$. The state space of the system is $\mathcal{H} = L_2[\mathbb{R}^2, dq_1 dq_2]$ formed by the functions $\psi(q_1, q_2)$. We can define an orthonormal basis in this Hilbert space by diagonalising a complete set of commuting self-adjoint operators. We choose the set formed by the operators H_1 and H_2. Denote E_1 and E_2 the eigenvalues of the operators H_1 and H_2 respectively, and $|n_1, n_2>_{loc}$ their common eigenstates. The reason for the suffix "loc" will be clear in a moment. The integers n_1 and n_2 are the quantum numbers of E_1 and E_2 and we can interpret them as the number of quanta in the first and second oscillator respectively. More precisely, if we measure H_1 of the first oscillator we observe that the result of the measurement outcome is quantised: $E_1 = \hbar\omega(n_1 + 1/2)$ and n_1 can be interpreted as the number of quanta in q_1. It is suggestive to call these quanta "particles". Call $N_{12} = n_1 + n_2$ the total particle number. Introducing a Fock-like notation, we can write the state with no particles also as

$$|0>_{loc} = |0,0>_{loc}; \tag{E.-75}$$

the two one-particle states with particles localised on each oscillator as

$$\begin{aligned} |1>_{loc} &= |1,0>_{loc}, && \text{(E.-74)} \\ |2>_{loc} &= |0,1>_{loc}, && \text{(E.-73)} \end{aligned}$$

where $|1>_{loc}$ represents a particle on the first oscillator and the state $|2>_{loc}$ represents a particle on the second oscillator. Notice that, according to standard Fock-space terminology, any linear combination of one-particle states

$$|\psi>_{loc} = c_1|1>_{loc} + c_2|2>_{loc} \tag{E.-73}$$

is also called a one-particle state.

Let us introduce normal coordinates:

$$q_a = \frac{q_1 + q_2}{\sqrt{2}}, \qquad q_b = \frac{q_1 - q_2}{\sqrt{2}} \tag{E.-73}$$

We are then able to factorise the hamiltonian as

$$H_0 = H_a + H_b = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2). \tag{E.-73}$$

where

$$\omega_1^2 = \omega^2 + \lambda, \qquad \omega_2^2 = \omega^2 - \lambda. \tag{E.-73}$$

Let E_a (E_b) be the eigenvalues of H_a (H_b), and denote $|n_a, n_b>$ the common eigenstates of $H_a + H_b$. The number n_a (n_b) is the number of quanta (or "particles") in the mode a (b). Call $N_{ab} = n_a + n_b$ the total number of these particles in the system. For instance the no-particle state is

$$|0> = |0,0>; \tag{E.-73}$$

the two one-particle states with particles localised on each mode are

$$
\begin{aligned}
|a> &= |1,0>, \\
|b> &= |0,1> .
\end{aligned}
\tag{E.-73}
$$

A generic one-particle state is a state of the form

$$
|\psi>= c_a|a> + c_b|b> . \tag{E.-73}
$$

What is the relation between the one-particle states $|\psi>_{loc}$ defined by (E.12.2) and the one-particle states $|\psi>$ defined in (E.12.2)?

Denote itin

$$
|1>= \frac{1}{\sqrt{2}}|a> + \frac{1}{\sqrt{2}}|b> . \tag{E.-73}
$$

Is this state equal to $|1>_{loc}$? No. If λ is small the two states differ slightly, but they do differ. Both states are, in some sense, "one-particle states" and in both states the "particle" is on the first oscillator. However, they are distininct states.

We illustrate the difference in two ways.

Explicit comparison

First, we can simply write both of them explicitly in the coordinate basis.

$$
\begin{aligned}
<q_1, q_2|1>_{loc} &= <q_1, q_2|1,0>_{loc} \\
&= \psi_1(q_1)\psi_0(q_2) \\
&= \left(\frac{1}{\pi\epsilon^2}\right)^{\frac{1}{4}} 2\left(\frac{q_1}{\epsilon}\right)\frac{1}{\sqrt{2}}e^{-q_1^2/2\epsilon^2} \times \left(\frac{1}{\pi\epsilon^2}\right)^{\frac{1}{4}} e^{-q_2^2/2\epsilon^2} \\
&= \sqrt{\frac{2\omega^2}{\pi}}q_1 e^{-\frac{\omega}{2}(q_1^2+q_2^2)}
\end{aligned}
\tag{E.-75}
$$

while

$$
\begin{aligned}
< q_1, q_2|1 > &= \frac{< q_1, q_2|a > + < q_1, q_2|b >}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}(< q_1, q_2|1, 0 > + < q_1, q_2|0, 1 >) \\
&= \frac{1}{\sqrt{2}}\psi_1^{(\omega_a)}\left(\frac{q_1+q_2}{\sqrt{2}}\right)\psi_0^{(\omega_b)}\left(\frac{q_1-q_2}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\psi_0^{(\omega_a)}\left(\frac{q_1+q_2}{\sqrt{2}}\right)\psi_1^{(\omega_b)}\left(\frac{q_1-q_2}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\cdot\left(\frac{\omega_a}{\pi}\right)^{\frac{1}{4}} 2\left(\sqrt{\omega_a}\cdot\frac{q_1+q_2}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}e^{-\frac{\omega_a}{4}(q_1+q_2)^2}\times\left(\frac{\omega_b}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\omega_b}{4}(q_1-q_2)^2} \\
&\quad +\frac{1}{\sqrt{2}}\cdot\left(\frac{\omega_a}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\omega_a}{4}(q_1+q_2)^2}\times\left(\frac{\omega_b}{\pi}\right)^{\frac{1}{4}} 2\left(\sqrt{\omega_b}\cdot\frac{q_1-q_2}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}e^{-\frac{\omega_b}{4}(q_1-q_2)^2} \\
&= \sqrt{\frac{2}{\pi}}(\omega_a\omega_b)^{\frac{1}{4}}\left(\frac{\sqrt{\omega_a}+\sqrt{\omega_b}}{2}q_1 + \frac{\sqrt{\omega_a}-\sqrt{\omega_b}}{2}q_2\right)e^{-\frac{1}{2}\left(\frac{\omega_a+\omega_b}{2}(q_1^2+q_2^2)+(\omega_a-\omega_b)q_1q_2\right)}
\end{aligned}
\tag{E.-80}
$$

If λ is small, $\omega_a \sim \omega_b \sim \omega$ and the two states are similar. In fact, we can compute that their scalar product is

$$ {}_{loc}\!<1|1>= 1 - \mathcal{O}(\lambda^2). \tag{E.-80}$$

(See worked exercises).

Comparison via perturbation theory in λ

Second, we can compare them using perturbation theory in λ. This is instructive because we will be able to do the same in the context of field theory. Let us take $H_0 = H_1 + H_2$ as the unperturbed hamiltonian. The two states $|1>_{loc}$ and $|2>_{loc}$ span a degenerate eigenspace of H_0.

Recall the theroem regarding simultaneous eigfunctions of operators that commute; if an operator $\hat{Q}$ commutes with V, i.e. $[V, \hat{Q}] = 0$, then they have simultaneous eigenfunctions. The interaction term $\lambda q_1 q_2$ has the "symmetry" under the exchange $q_1 \leftrightarrow q_2$. Let us call the operator $\hat{S}$ where

$$\hat{S}\psi(q_1, q_2) = \psi(q_2, q_1) \tag{E.-80}$$

Then

$$\begin{aligned}\hat{S}V(f(q_1,q_2)) &= \hat{S}(Vf(q_1,q_2)) \\ &= \hat{S}(\lambda q_1 q_2 f(q_1,q_2)) \\ &= \lambda q_2 q_1 f(q_2,q_1) \\ &= V\hat{S}f(q_1,q_2) \end{aligned} \quad \text{(E.-82)}$$

where $f(q_1, q_2)$ is a test function. This implies that $[\hat{S}, V] = 0$. Obviously $\hat{S}^2 = \hat{\mathbb{1}}$, and therefore if $\hat{S}\psi = \beta\psi$, then

$$\begin{aligned}\hat{S}^2\psi(q_1,q_2) &= \beta^2\psi(q_1,q_2) \\ &= \psi(q_1,q_2)\end{aligned} \quad \text{(E.-82)}$$

and so $\beta = \pm 1$. If $\hat{S}\psi_1(q_1, q_2) = \psi_2(q_1, q_2)$ then

$$\hat{S}\frac{\psi_1(q_1,q_2) \pm \psi_2(q_1,q_2)}{\sqrt{2}} = \pm\frac{(\psi_1(q_1,q_2) \pm \psi_2(q_1,q_2)}{\sqrt{2}}. \quad \text{(E.-82)}$$

Clearly V is diagonalised in the above degenerate eigensubspace by the two states

$$\begin{aligned}|a>_0 &= \frac{|1>_{loc} + |2>_{loc}}{\sqrt{2}} \\ |b>_0 &= \frac{|1>_{loc} - |2>_{loc}}{\sqrt{2}}\end{aligned} \quad \text{(E.-82)}$$

We can compute the first order correction to these states using first order perturbation theory. It is convenient to use creation and annihilation operators. From (E.12.1), (E.12.1) and (E.12.1) we have

$$\begin{aligned}q_{1,2} &= \frac{1}{\sqrt{2\omega}}(a_{1,2} + a^\dagger_{1,2}), \\ p_{1,2} &= \frac{-i\sqrt{\omega}}{\sqrt{2}}(a_{1,2} - a^\dagger_{1,2})\end{aligned} \quad \text{(E.-82)}$$

In terms of the which the perturbation reads

$$V = \frac{\lambda}{2\omega}(a^\dagger_1 a^\dagger_2 + a_1 a_2 + a^\dagger_1 a_2 + a_1 a^\dagger_2). \quad \text{(E.-82)}$$

Notice that the term $a_1^\dagger a_2^\dagger$ takes us out of the one particle sector, giving the non-vanishing matrix elements

$$\begin{aligned}
{}_{loc}\!<2,1|V|a>_0 &= \frac{\lambda}{2\omega}{}_{loc}\!<2,1|(a_1^\dagger a_2^\dagger + a_1 a_2 + a_1^\dagger a_2 + a_1 a_2^\dagger)(\frac{|1,0>_{loc} + |0,1>_{loc}}{\sqrt{2}}) \\
&= \frac{\lambda}{2\omega} \qquad \text{(E.-82)}
\end{aligned}$$

where we have used that

$$|2,1>_{loc} = \frac{(a_1^\dagger)^2}{\sqrt{2}} a_2^\dagger |0,0>_{loc} = a_1^\dagger a_2^\dagger \frac{1}{\sqrt{2}} |1,0>_{loc}, \qquad \text{(E.-82)}$$

and similarly

$$\begin{aligned}
{}_{loc}\!<1,2|V|a>_0 &= \frac{\lambda}{2\omega} \\
{}_{loc}\!<2,1|V|b>_0 &= \frac{\lambda}{2\omega} \\
{}_{loc}\!<1,2|V|b>_0 &= -\frac{\lambda}{2\omega}. \qquad \text{(E.-83)}
\end{aligned}$$

To first order in λ, the hamiltonian eigenstates $|a>$ and $|b>$ are therefore

$$\begin{aligned}
|a> &= |a>_0 + \frac{{}_{loc}\!<2,1|V|a>_0}{E_{a_0} - E_{(2,1)_0}} |2,1>_{loc} + \frac{{}_{loc}\!<1,2|V|a>_0}{E_{a_0} - E_{(1,2)_0}} |1,2>_{loc} \\
&= |a>_0 - \frac{\lambda}{4\omega^2} |2,1>_{loc} - \frac{\lambda}{4\omega^2} |1,2>_{loc} \qquad \text{(E.-83)}
\end{aligned}$$

where we have used $E_{a_0} = (3/2)\omega + (1/2)\omega$ and $E_{(2,1)_0} = E_{(1,2)_0} = (5/2)\omega + (3/2)\omega$. Similarly,

$$\begin{aligned}
|b> &= |b>_0 + \frac{{}_{loc}\!<2,1|V|b>_0}{E_{a_0} - E_{(2,1)_0}} |2,1>_{loc} + \frac{{}_{loc}\!<1,2|V|b>_0}{E_{a_0} - E_{(1,2)_0}} |1,2>_{loc} \\
&= |b>_0 - \frac{\lambda}{4\omega^2} |2,1>_{loc} + \frac{\lambda}{4\omega^2} |1,2>_{loc}. \qquad \text{(E.-83)}
\end{aligned}$$

And therefore, to first order in λ

$$|1>=|1>_{loc} - \frac{\lambda}{\sqrt{8}\omega^2}\,|2,1>_{loc}\,. \tag{E.-83}$$

Thus, the two states (E.-74) and (E.12.2) are both "one-particle states" in which the particle is concentrated on the oscillator q_1, but they are distinct states. They represent two distinct kinds of one-quantum states, or two distinct kinds of quanta. We call $|1>_{loc}$ a *local* particle state, and $|1>$ a *global* particle state. They represent the simplest example of the distinction between the two classes of states.

More in general, we call "global patricle states" the eigenstates of the "global" number operator

$$N_{ab}|n_a, n_b>=(n_a+n_b)|n_a, n_b>, \tag{E.-83}$$

and we call "local particle states" the eigenstates of the "local" number operator

$$N_1|n_1, n_2>_{loc}=n_1|n_1, n_2>_{loc}\,. \tag{E.-83}$$

Let us illustrate the different properties that these states have. The state $|1>_{loc}$ is an eigenstate of H_1, which is an observable that depends just on q_1 and its momentum, namely just on the variable associated to the first oscillator. If we want to measure how many local particles are in the first oscillator, namely to measure n_1, we can make a measurement that involves solely variables of the q_1 oscillator. In this sense $|1>_{loc}$ is "local".

The state $|1>$, on the other hand, describes a single particle "on the first oscillator", but is not an eigenstate of observables that depend on variables of the sole first oscillator. This can be seen from the fact that it is a state in which the two oscillators are (weakly) correlated. The source of these correlations can be traced to the vacuum state: local particles are excitation over the local vacuum (E.12.2) which has no correlations:

$$< q_1, q_2|0>_{loc}=\sqrt{\frac{\omega}{\pi}}e^{-\frac{\omega}{2}q_1^2}e^{-\frac{\omega}{2}q_2^2}=\psi_0(q_1)\psi_0(q_2) \tag{E.-83}$$

while global particles are excitations over the global vacuum (E.12.2)

$$< q_1, q_2|0>=\frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}}e^{-\frac{1}{2}\frac{\omega_a+\omega_b}{2}q_1^2}e^{-\frac{1}{2}\frac{\omega_a+\omega_b}{2}q_2^2}e^{-\frac{1}{2}(\omega_a-\omega_b)q_1q_2}=\psi_0^{(\omega_a)}\left(\frac{q_1+q_2}{\sqrt{2}}\right)\psi_0^{(\omega_b)}\left(\frac{q_1-q_2}{\sqrt{2}}\right) \tag{E.-83}$$

which does not factorise (i.e. cannot be put in the form $\psi(q_1)\phi(q_2)$), and therefore represents vacuum correlations between the two oscillators.

What is the physical relevance of the state $|1>$?

Notice that $|1>_{loc}$ is not an energy eigenstate, because of the interaction term V, but $|1>$ isn't an energy eigenstate either, because $|1,0>$ and $|0,1>$ have different energies:

$$\begin{aligned} H|1> &= H\left(\frac{1}{\sqrt{2}}|1,0> + \frac{1}{\sqrt{2}}|0,1>\right) \\ &= (H_a + H_b)\left(\frac{1}{\sqrt{2}}|1,0> + \frac{1}{\sqrt{2}}|0,1>\right) \\ &= 2\omega_a\frac{1}{\sqrt{2}}|1,0> + 2\omega_b\frac{1}{\sqrt{2}}|0,1> \\ &\not\propto |1> . \end{aligned} \tag{E.-85}$$

Its defining property is just the fact of being a linear combination of one-quantum excitations of the normal modes of the system. What then is the physical relevance of the state $|1>$? It is the following: the one-particle Fock states of QFT are precisely states of the same kind as $|1>$. To see this, consider the following: one-particle Fock states of QFT are precisely states of the same kind as $|1>$. To see this, consider a Fock particle localised in a region R. This state can be described by means of a function $f(x)$ with compact support in R, as

$$|f> = \int dk \tilde{f}(k)|k> \tag{E.-85}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$ and the states $|k>$ are the one-particle Fock states with momentum k. They are energy eigenstates (with different energies) and they are single-particle excitations of the normal modes of the system. Therefore they are the analog of the states $|1,0>$ and $|0,1>$of the two-oscillator model. The linear combination (E.12.2) is the analog of the linear combination (E.12.2), which picks the one-particle global state maximally concentrated in the region chosen (the oscillator q_1 in the model, the region R in the QFT). Thus, Fock particles are global particles. No measurement in a finite region R can count these particles, because Fock particles are not eigenstates of local field operators, precisely in the same sense in which $|1>$ is not an eigenstate of an observable localised on the q_1 oscillator. If we make a measurement with an apparatus located in a region R, we can count the number of particles the apparatus detect. However, these particles are not global particles. They are local particles, that can be described by appropriate QFT states which are close, but not identical, to $n-$ particle Fock states, like $|1>_{loc}$ is close, but not identical to $|1>$. Later on we will discuss local particle states, analogous to the $|n_1, n_2>_{loc}$ states, in the context of QFT.

Probability of detector measuring the first oscillator measuring a particle on the first oscillator

Suppose now the state of the system is $|0>$ and we measure whether a particle is on the first oscillator by measuring the energy E_1. The probability of not seeing any particle is not determined by the sole scalar product (E.12.2), because we are in fact tracing over n_2. Rather it is given by

$$\mathcal{P} = \left| \sum_{n_2} {}_{loc} < 0, n_2|0 > \right|^2 =< 0|P_{0loc}|0 > \qquad \text{(E.-85)}$$

where

$$P_{0loc} = \sum_{n_2} |0, n_2 >_{loc\ loc}<0, n_2| \qquad \text{(E.-85)}$$

is the projection on the lowest eigenspace of H_1. A strightforward calculation gives

$$\mathcal{P} =< 0|P_{0loc}|0 >= 1 - \frac{1}{16}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4) \qquad \text{(E.-85)}$$

(see worked exercise).

E.12.3 Chain of Oscillators

As an intermediate step before moving on to field theory, we consider a chain of harmonic oscillators. This system allows one to emphasise several important points regarding the relation between local and global particle states.

We study a system of n harmonic oscillators $\mathbf{q} = (q_i)$, $i = 1, \ldots, n$ with the same frequencies $\omega = 1$ and coupled by a constant λ. Each oscillator is coupled with its two neighbouring (except for the first and last oscillator that have only one coupling)

$$H = \frac{1}{2}\left(|\mathbf{p}|^2 + |\mathbf{q}|^2\right) + \lambda \sum_{i=1}^{n-1} q^i q^{i+1} \qquad \text{(E.-85)}$$

where $|\mathbf{q}|^2 = \sum_i (q^i)^2$. Notice that we are not considering a ring but an open chain of oscillators. Diagonalising the hamiltonian of the system we obtain the normal frequencies

$$\omega_a = \sqrt{1 + 2\lambda \cos\theta_a}, \qquad \text{where} \quad \theta_a = \frac{a\pi}{n+1}, \qquad \text{and} \quad a = 1, \ldots, n. \qquad \text{(E.-85)}$$

The normal modes $\mathbf{Q} = (Q_a)$, $a = 1, \ldots, n$ are given by $\mathbf{Q} = U^{(n)}\mathbf{q}$, where $U^{(n)}$ is the orthogonal $n \times n$ matrix

$$U^{(n)}_{ai} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{ai\pi}{n+1}\right). \tag{E.-85}$$

The vacuum state is

$$< \mathbf{q}|0 >= \prod_{a=1}^{n} \left(\frac{\omega_a}{\pi}\right)^{1/4} e^{-\frac{1}{2}q^i D^{(n)}_{ij} q^j}, \tag{E.-85}$$

where

$$D^{(n)}_{ij} = \sum_a U^{(n)}_{ai} \omega_a U^{(n)}_{aj}. \tag{E.-85}$$

(See worked exercises).

A basis that diagonalises H is given by the states $|\mathbf{n} >= |n_1, \ldots, n_n >$ with n_a quanta in the $a-$th mode. The number operator is

$$N|\mathbf{n} >= \left(\sum_{a=1}^{n} n_a\right) |\mathbf{n} > . \tag{E.-85}$$

Denote $|a >$ the one particle state $|0, \ldots, 1, \ldots, 0 >$ in which the vacuum state except for the $a-$th mode which is in its first excitation. The state

$$|i >= \sum_{a=1}^{n} U^{-1}_{ia} |a > \tag{E.-85}$$

is the one particle state maximally concentrated on the $i-$th oscillator. It is the analog of the global one particale states (E.12.2) and (). This is the global one-particle state, with the particle on the $i-$th oscillator.

Partitioning the chain

Now, consider a partition of the chain in two regions R_1 and R_2. Let the region R_1 be formed by the first n_1 oscillators, and the region R_2 be formed by the remaining n_2 oscillators, with $n_1 + n_2 = n$. We write $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2)$, where $\mathbf{q}_1$ (respectively $\mathbf{q}_2$) is a vector with n_1 (n_2) components. We regard the first region of oscillators as a generalisation of the oscillator q_1 in the previous section, and the second region as the analog of the oscillator q_2 . The total Hilbert space of the system factorises as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We can rewrite the hamiltonian (E.12.3) in the form

$$\begin{aligned} H &= H_1 + H_2 + V \\ &= \left(\frac{1}{2}\left(|\mathbf{p}_1|^2 + |\mathbf{q}_1|^2\right) + \lambda \sum_{i=1}^{n_1-1} q_1^i q_1^{i+1}\right) + \left(\frac{1}{2}\left(|\mathbf{p}_2|^2 + |\mathbf{q}_2|^2\right) + \lambda \sum_{i=1}^{n_2-1} q_2^i q_2^{i+1}\right) + \lambda q_1^{n_1} q_2^1. \end{aligned} \tag{E.-86}$$

E.12.4 Convergence Between Local and Global States

E.13 Worked Exercises and Details

Hamilton Function

Proposition: Hamilton function for a harmonic oscillator

Prove that the Hamilton function of a harmonic oscillator is

$$S(q,t,q',t') = m\omega \frac{(q^2 + q'^2)cos\omega(t-t') - 2qq'}{2\sin\omega(t'-t)}$$

for motion that starts at (q,t) and ends at (q',t').

Solution

First we need the solution of

$$\ddot{q} = -\frac{k}{m}q(\tilde{t}) = -\omega^2 q(\tilde{t})$$

with boundary conditions $q = q(\tilde{t} = t)$ and $q' = q(\tilde{t} = t')$. It is well know the general solution is of the form $A\cos(\omega\tilde{t}) + B\sin(\omega\tilde{t})$. The solution with correct boundary conditions is easily seen to be

$$q_{qtq't'}(\tilde{t}) = \frac{q\sin(\omega t') - q'\sin(\omega t)}{\sin\omega(t'-t)}\cos(\omega\tilde{t}) + \frac{q'\cos(\omega t) - q\cos(\omega t')}{\sin\omega(t'-t)}\sin(\omega\tilde{t}) \tag{E.-86}$$

This simplifies

$$
\begin{aligned}
q_{qtq't'}(\tilde{t}) &= \frac{q[\sin(\omega t')\cos(\omega\tilde{t}) - \cos(\omega t')\sin(\omega\tilde{t})] + q'[\sin(\omega\tilde{t})\cos(\omega t) - \cos(\omega\tilde{t})\sin(\omega t)]}{\sin\omega(t'-t)} \\
&= \frac{q\sin\omega(t'-\tilde{t}) + q'\sin\omega(\tilde{t}-t)}{\sin\omega(t'-t)}.
\end{aligned}
\tag{E.-87}
$$

The velocity is then

$$
\dot{q}_{qtq't'}(\tilde{t}) = \frac{-q\cos\omega(t'-\tilde{t}) + q'\cos\omega(\tilde{t}-t)}{\sin\omega(t'-t)}. \tag{E.-87}
$$

The Hamilton function is

$$
\begin{aligned}
S(q,t,q',t') &= \int_t^{t'} d\tilde{t}\left(\frac{1}{2}m\dot{q}^2_{qtq't'} - \frac{1}{2}kq^2_{qtq't'}\right) \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\int_t^{t'} d\tilde{t}\,\big[\{-q\cos\omega(t'-\tilde{t}) + q'\cos\omega(\tilde{t}-t)\}^2 \\
&\qquad -\{q\sin\omega(t'-\tilde{t}) + q'\sin\omega(\tilde{t}-t)\}^2\big] \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\int_t^{t'} d\tilde{t}\Big[q^2(\cos^2\omega(t'-\tilde{t}) - \sin^2\omega(t'-\tilde{t})) \\
&\qquad +{q'}^2(\cos^2\omega(\tilde{t}-t) - \sin^2\omega(\tilde{t}-t)) \\
&\qquad -2qq'\big(\cos\omega(t'-\tilde{t})\cos\omega(\tilde{t}-t) + \sin\omega(t'-\tilde{t})\sin\omega(\tilde{t}-t)\big)\Big] \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\int_t^{t'} d\tilde{t}\Big[q^2\cos 2\omega(t'-\tilde{t}) + {q'}^2\cos 2\omega(\tilde{t}-t) - 2qq'\cos\omega(2\tilde{t}-t-t')\Big] \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\Big(q^2\frac{1}{2\omega}\Big[-\sin 2\omega(t'-\tilde{t})\Big]_t^{t'} + {q'}^2\frac{1}{2\omega}\Big[\sin 2\omega(\tilde{t}-t)\Big]_t^{t'} \\
&\quad -2qq'\frac{1}{2\omega}\Big[\sin\omega(2\tilde{t}-t-t')\Big]_t^{t'}\Big) \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\left((q^2+{q'}^2)\frac{1}{2\omega}\sin 2\omega(t'-t) - 2qq'\frac{1}{2\omega}(2\sin\omega(t'-t))\right) \\
&= \frac{m\omega^2}{2\sin^2\omega(t'-t)}\left((q^2+{q'}^2)\frac{1}{2\omega}2\sin\omega(t-t')\cos\omega(t-t') - 2qq'\frac{1}{2\omega}(2\sin\omega(t'-t))\right) \\
&= m\omega\frac{(q^2+{q'}^2)cos\omega(t-t') - 2qq'}{2\sin\omega(t'-t)}.
\end{aligned}
\tag{E.-97}
$$

Expansion in λ.

Idenity: Expansion in λ.

Prove that

$$e^{-\lambda(T+V)/N} = e^{-\lambda T/N} e^{-\lambda V/N} + \mathcal{O}\left(\frac{\lambda^2}{N^2}\right) \tag{E.-97}$$

where T and V are operators and that the coefficent of λ^2/N^2 is given by the commutator $\frac{1}{2}[V, T]$.

Solution

We define the operator-valued function of the parameter λ

$$F(\lambda) := e^{\lambda T/N} e^{-\lambda(T+V)/N} e^{\lambda V/N} \tag{E.-97}$$

It can be expanded as a Taylor series about $\lambda = 0$

$$F(\lambda) := \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{d^n F}{d\lambda^n}\right)_{\lambda=0} \lambda^n \tag{E.-97}$$

where we have used $F(0) = \mathbb{1}$. Taking the first derivative gives

$$\begin{aligned}
\frac{d}{d\lambda}F(\lambda) &= \frac{d}{d\lambda}\left\{e^{\lambda T/N} e^{-\lambda(T+V)/N} e^{\lambda V/N}\right\} \\
&= \frac{1}{N}\Big(e^{\lambda T/N} T e^{-\lambda(T+V)/N} e^{\lambda V/N} - e^{\lambda T/N}(T+V)e^{-\lambda(T+V)/N} e^{\lambda V/N} \\
&\quad + e^{\lambda T/N} e^{-\lambda(T+V)/N} V e^{\lambda V/N}\Big) \\
&= \frac{1}{N} e^{\lambda T/N}[e^{-\lambda(T+V)/N}, V] e^{\lambda V/N}
\end{aligned} \tag{E.-99}$$

Putting $\lambda = 0$ we have:

$$\frac{d}{d\lambda}F(\lambda)\Big|_{\lambda=0} = 0 \tag{E.-99}$$

as $[\mathbb{1}, V] = 0$. Now consider the second derivative of $F(\lambda)$:

$$
\begin{aligned}
\frac{d^2}{d\lambda^2}F(\lambda) &= \frac{1}{N}\frac{d}{d\lambda}\left\{e^{\lambda T/N}[e^{-\lambda(T+V)/N}, V]e^{\lambda V/N}\right\} \\
&= \frac{1}{N^2}(e^{\lambda T/N}T[e^{-\lambda(T+V)/N}, V]e^{\lambda V/N} \\
&\quad + e^{\lambda T/N}[e^{-\lambda(T+V)/N}, V]Ve^{\lambda V/N} \\
&\quad + e^{\lambda T/N}[-e^{-\lambda(T+V)/N}(T+V)V + V(T+V)e^{-\lambda(T+V)/N}]e^{\lambda V/N}). \qquad \text{(E.-101)}
\end{aligned}
$$

Putting $\lambda = 0$ and dividing by 2! gives the coefficient of the third term in the Taylor expansion,

$$
\frac{1}{2!}\frac{d^2}{d\lambda^2}F(\lambda)\Big|_{\lambda=0} = \frac{1}{N^2}\frac{1}{2}[V, T]. \qquad \text{(E.-101)}
$$

Therefore we have:

$$
e^{\lambda T/N}e^{-\lambda(T+V)/N}e^{\lambda V/N} = \mathbb{1} + \frac{\lambda^2}{N^2}\frac{1}{2}[V, T] + \mathcal{O}\left(\frac{\lambda^3}{N^3}\right). \qquad \text{(E.-101)}
$$

Multiplying by the left with $e^{-\lambda T/N}$ and by the right by $e^{-\lambda V/N}$ gives:

$$
\begin{aligned}
e^{-\lambda(T+V)/N} &= e^{-\lambda T/N}e^{-\lambda V/N} + \frac{\lambda^2}{N^2}e^{-\lambda T/N}\left(\frac{1}{2}[V, T]\right)e^{-\lambda V/N} + \mathcal{O}\left(\frac{\lambda^3}{N^3}\right) \\
&= e^{-\lambda T/N}e^{-\lambda V/N} + \frac{\lambda^2}{N^2}\left(\mathbb{1} - \frac{\lambda T}{N} + \dots\right)\frac{1}{2}[V, T]\left(\mathbb{1} - \frac{\lambda V}{N} + \dots\right) + \mathcal{O}\left(\frac{\lambda^3}{N^3}\right) \\
&= e^{-\lambda T/N}e^{-\lambda V/N} + \frac{\lambda^2}{N^2}\frac{1}{2}[V, T] + \mathcal{O}\left(\frac{\lambda^3}{N^3}\right) \qquad \text{(E.-102)}
\end{aligned}
$$

which is the desired result.

Spacetime Smeared States

Proposition: Spacetime Smeared States

Functional Integral

Proposition: Functional Integral

$$S_T^E[\phi] = \frac{1}{2}\int_0^T dt \int d^3x \left[(\partial_\mu \phi_{\varphi,\Sigma})^2 + m^2\phi_{\varphi,\Sigma}^2\right] \tag{E.-102}$$

The gausian integral

$$W[\varphi_1, \varphi_2, T] = \int_{\phi|_{t=0}=\varphi_2}^{\phi|_{t=T}=\varphi_1} \mathcal{D}\phi\; e^{S_T^E[\phi]} \tag{E.-102}$$

can be solved by finding the extremal value of the exponent, that is, by solving the classical equation with boundary conditions

(a) Find the classical solution corresponding to Eq.(E.13).

(b) Do the integral to Eq.(E.13).

$$W[\varphi_1, \varphi_2, T] = \mathcal{N}\exp\left\{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega\left(\frac{|\tilde{\varphi}_1|^2 + |\tilde{\varphi}_2|^2}{\tanh(\omega T)} - \frac{2\tilde{\varphi}_1\overline{\tilde{\varphi}_2}}{\sinh(\omega T)}\right)\right\} \tag{E.-102}$$

where $\omega = \sqrt{k^2+m^2}$

$$\tilde{\varphi}(k) = \int d^3x e^{ikx}\varphi(x) \tag{E.-102}$$

Solutions

(a)

$$\left.\frac{\delta S^E}{\delta\overline{\phi}}\right|_{\overline{\phi}=\phi_{cl}} = 0 \tag{E.-102}$$

$$\begin{aligned}\frac{\delta S^E}{\delta\overline{\phi}} &= \frac{1}{2}\frac{\delta}{\delta\overline{\phi}}\int_0^T dt\left(\int d^3x\left[(\partial_t\phi)^2 - (\partial_i\phi)^2 + m^2\phi^2\right]\right) \\ &= \int d^3x\left[-\partial_t^2\phi + \partial_i^2\phi + m\phi\right] = 0\end{aligned} \tag{E.-102}$$

$$\begin{aligned}\phi_{cl}(x,t) &= \int \frac{d^3k d^3y}{(2\pi)^3} e^{-i\vec{k}(\vec{x}-\vec{y})}\ \frac{\varphi_2(\vec{y})\sinh(\omega t) - \varphi_1(\vec{y})\sinh(\omega(t-T))}{\sinh(\omega T)} \\ &= \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}}\ \frac{\tilde{\varphi}_2(\vec{k})\sinh(\omega t) - \tilde{\varphi}_1(\vec{k})\sinh(\omega(t-T))}{\sinh(\omega T)}\end{aligned} \tag{E.-102}$$

$$\tilde{\phi}_{cl}(k,t) = \frac{\tilde{\varphi}_2(\vec{k})\sinh(\omega t) - \tilde{\varphi}_1(\vec{k})\sinh(\omega(t-T))}{\sinh(\omega T)} \qquad \text{(E.-102)}$$

(b)

$$S^E[\phi,T] = S^E[\overline{\phi},T] + \frac{\delta S^E}{\delta\overline{\phi}}\bigg|_{\overline{\phi}=\phi_{cl}}(\overline{\phi}-\phi_{cl}) + \frac{\delta^2 S^E}{\delta\phi^2}\bigg|_{\overline{\phi}=\phi_{cl}}(\overline{\phi}-\phi_{cl})^2 \qquad \text{(E.-102)}$$

Taylor expansion

$$\phi_{cl} = \varphi_2\frac{\sinh(\omega(T-t))}{\sinh(\omega T)} + \varphi_1\frac{\sinh(\omega t)}{\sinh(\omega T)} \qquad \text{(E.-102)}$$

so that the first term in the Taylor expansion is

$$S^E[\phi_{cl}] = \frac{1}{2}\frac{\omega}{\sinh(\omega T)}((\varphi_2^2+\varphi_1^2)\cosh(\omega T) - 2\varphi_1\varphi_2) \qquad \text{(E.-102)}$$

$$W[\varphi_1,\varphi_2,T] = \mathcal{N}\exp\left(-\frac{1}{2}\frac{\omega}{\sinh(\omega T)}((\varphi_2^2+\varphi_1^2)\cosh(\omega T) - 2\varphi_1\varphi_2)\right) \qquad \text{(E.-102)}$$

where

$$\mathcal{N} = \int \mathcal{D}\overline{\phi}\exp() \qquad \text{(E.-102)}$$

Lorentzian case

Elementary geometry of an equilateral tetrahedron

Elementary geometry of an equilateral tetrahedron

$$\sin\frac{\theta_a}{2} = \frac{b}{\sqrt{4c^2-a^2}}, \qquad \sin\frac{\theta_b}{2} = \frac{a}{\sqrt{4c^2-b^2}}, \qquad \sin\frac{\theta_c}{2} = \frac{ab}{\sqrt{(4c^2-a^2)(4c^2-b^2)}} \qquad \text{(E.-102)}$$

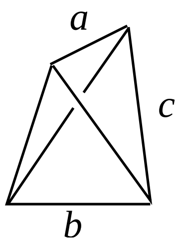


Figure E.4: equiltetra.

Factorising chain of two oscillators

Show that

$$H_0 = \frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) + \lambda q_1 q_2 \tag{E.-102}$$

factorises to

$$H_0 = \frac{1}{2}(p_a^2 + \omega_a^2 q_a^2) + \frac{1}{2}(p_b^2 + \omega_b^2 q_b^2) \tag{E.-102}$$

via the normal coordinates

$$q_a = \frac{q_1 + q_2}{\sqrt{2}}, \qquad q_b = \frac{q_1 - q_2}{\sqrt{2}}. \tag{E.-102}$$

Proof:

First

$$\begin{aligned}
\frac{1}{2}(\omega_a^2 q_a^2 + \omega_b^2 q_b^2) &= \frac{1}{2}\left\{\omega_a^2 \left(\frac{q_1+q_2}{\sqrt{2}}\right)^2 + \omega_b^2 \left(\frac{q_1-q_2}{\sqrt{2}}\right)^2\right\} \\
&= \frac{1}{2}\left\{\frac{\omega_a^2+\omega_b^2}{2}(q_1^2+q_2^2) + (\omega_a^2-\omega_b^2) q_1 q_2\right\} \\
&\equiv \frac{1}{2}\omega^2(q_1^2+q_2^2) + \lambda q_1 q_2
\end{aligned} \tag{E.-103}$$

Implying

$$\omega^2 = \frac{\omega_a^2+\omega_b^2}{2}, \qquad \lambda = \frac{1}{2}(\omega_a^2 - \omega_b^2) \tag{E.-103}$$

We find

$$\omega_a^2 = \omega^2 + \lambda, \qquad \omega_b^2 = \omega^2 - \lambda. \tag{E.-103}$$

Now we turn to the momenta operators

$$\begin{aligned} p_1 &= -i\hbar \frac{\partial}{\partial q_1} \\ &= -i\hbar \left(\frac{\partial q_a}{\partial q_1} \frac{\partial}{\partial q_a} + \frac{\partial q_b}{\partial q_1} \frac{\partial}{\partial q_b} \right) \\ &= -i\hbar \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial q_a} + \frac{\partial}{\partial q_b} \right) \end{aligned} \tag{E.-104}$$

similarly

$$p_2 = -i\hbar \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial q_a} - \frac{\partial}{\partial q_b} \right). \tag{E.-104}$$

Then

$$\begin{aligned} p_1^2 + p_2^2 &= -\hbar^2 \frac{1}{2} \left\{ \left(\frac{\partial}{\partial q_a} + \frac{\partial}{\partial q_b} \right)^2 + \left(\frac{\partial}{\partial q_a} - \frac{\partial}{\partial q_b} \right)^2 \right\} \\ &= -\hbar^2 \frac{\partial^2}{\partial q_a^2} + -\hbar^2 \frac{\partial^2}{\partial q_b^2} \\ &= p_1^2 + p_2^2. \end{aligned} \tag{E.-105}$$

Comparison of local and global states for two oscillators.

Take

$$\omega_a^2 = \omega^2 + \lambda, \qquad \omega_b^2 = \omega^2 - \lambda \tag{E.-105}$$

and attempt an expansion of

$$< q_1, q_2 | 0 >= \sqrt{\frac{2}{\pi}} (\omega_a \omega_b)^{\frac{1}{4}} \left(\frac{\sqrt{\omega_a} + \sqrt{\omega_b}}{2} q_1 + \frac{\sqrt{\omega_a} - \sqrt{\omega_b}}{2} q_2 \right) e^{-\frac{1}{2}\left(\frac{\omega_a + \omega_b}{2} (q_1^2 + q_2^2) + (\omega_a - \omega_b) q_1 q_2 \right)} \tag{E.-105}$$

in λ. How does it compare to

$$\sqrt{\frac{2\omega^2}{\pi}} q_1 e^{-\frac{\omega}{2}(q_1^2+q_2^2)}?. \tag{E.-105}$$

Proof:

Note:

$$\begin{aligned}
(\omega_a \omega_b)^{\frac{1}{4}} &= (\omega^4 - \lambda^2)^{\frac{1}{8}} \\
&= \sqrt{\omega}\left(1 - \frac{1}{8}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4)\right)
\end{aligned} \tag{E.-105}$$

$$\begin{aligned}
\frac{\sqrt{\omega_a}+\sqrt{\omega_b}}{2} &= \frac{1}{2}\left(\omega^2+\lambda\right)^{\frac{1}{4}} + \frac{1}{2}\left(\omega^2-\lambda\right)^{\frac{1}{4}} \\
&= \frac{\sqrt{\omega}}{2}\left(1 + \frac{1}{4}\frac{\lambda}{\omega^2} - \frac{3}{4^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^3)\right) + \frac{\sqrt{\omega}}{2}\left(1 - \frac{1}{4}\frac{\lambda}{\omega^2} - \frac{3}{4^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^3)\right) \\
&= \sqrt{\omega}\left(1 - \frac{3}{4^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4)\right) \\
&= \sqrt{\omega}\left(1 - \frac{3}{32}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4)\right)
\end{aligned} \tag{E.-107}$$

$$\frac{\sqrt{\omega_a}-\sqrt{\omega_b}}{2} = \sqrt{\omega}\left(\frac{1}{4}\frac{\lambda}{\omega^2} + \mathcal{O}(\lambda^3)\right) \tag{E.-106}$$

$$\begin{aligned}
\frac{\omega_a+\omega_b}{2} &= \frac{1}{2}(\omega^2+\lambda)^{\frac{1}{2}} + \frac{1}{2}(\omega^2-\lambda)^{\frac{1}{2}} \\
&= \frac{\omega}{2}\left(1 + \frac{1}{2}\frac{\lambda}{\omega^2} - \frac{1}{2^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^3)\right) + \frac{\omega}{2}\left(1 - \frac{1}{2}\frac{\lambda}{\omega^2} - \frac{1}{2^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^3)\right) \\
&= \omega\left(1 - \frac{1}{2^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4)\right) \\
&= \omega\left(1 - \frac{1}{8}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4)\right)
\end{aligned} \tag{E.-108}$$

$$\begin{aligned}
\omega_a - \omega_b &= (\omega^2+\lambda)^{\frac{1}{2}} - (\omega^2-\lambda)^{\frac{1}{2}} \\
&= \omega\left(1 + \frac{1}{2}\frac{\lambda}{\omega^2} - \frac{1}{2^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^3)\right) - \omega\left(1 - \frac{1}{2}\frac{\lambda}{\omega^2} - \frac{1}{2^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^3)\right) \\
&= \omega\left(\frac{1}{2}\frac{\lambda}{\omega^2} + \mathcal{O}(\lambda^3)\right)
\end{aligned} \tag{E.-109}$$

Inserting these into (E.13) we see

$$
\begin{aligned}
< q_1, q_2|0 > \;&=\; \sqrt{\frac{2}{\pi}} \times \sqrt{\omega}\left(1-\frac{1}{8}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right) \times \left\{\sqrt{\omega}\left(1-\frac{3}{4^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^3)\right) q_1 + \right. \\
&\left. \sqrt{\omega}\left(\frac{1}{4}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right) q_2\right\} \times \exp\left\{-\frac{1}{2}\omega\left(1-\frac{1}{2^2}\frac{1}{2!}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^3)\right)(q_1^2+q_2^2)\right\} \\
&\times \exp\left\{-\frac{1}{2}\omega\left(\frac{1}{2}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right) q_1 q_2\right\}
\end{aligned} \tag{E.-111}
$$

and with further simplification

$$
\begin{aligned}
< q_1, q_2|0 > \;&=\; \sqrt{\frac{2\omega^2}{\pi}}\left\{\left(1-\frac{3}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^3)\right) q_1 + \left(\frac{1}{4}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right) q_2\right\} \\
&\times\; \exp\left\{-\frac{1}{2}\omega\left(1-\frac{1}{8}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^3)\right)(q_1^2+q_2^2)\right\} \exp\left\{-\frac{1}{2}\omega\left(\frac{1}{2}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right) q_1 q_2\right\}
\end{aligned} \tag{E.-112}
$$

Comparison of local and global states for two oscillators: scalar product.

Calculate the scalar product

$$
{}_{loc}{<}0|0> \tag{E.-112}
$$

as an expansion in λ.

Proof:

$$
\begin{aligned}
{}_{loc}{<}1|1> \;&=\; \int dq_1 dq_2 \;{}_{loc}< 1|q_1, q_2 >< q_1, q_2|1 > \\
&=\; \int dq_1 dq_2 \sqrt{\frac{2\omega^2}{\pi}} q_1 e^{-\frac{\omega}{2}(q_1^2+q_2^2)} \times \\
&\times\; \sqrt{\frac{2}{\pi}}(\omega_a\omega_b)^{\frac{1}{4}}\left(\frac{\sqrt{\omega_a}+\sqrt{\omega_b}}{2} q_1 + \frac{\sqrt{\omega_a}-\sqrt{\omega_b}}{2} q_2\right) e^{-\frac{1}{2}\left(\frac{\omega_a+\omega_b}{2}(q_1^2+q_2^2)+(\omega_a-\omega_b)q_1q_2\right)} \\
&=\; \frac{2}{\pi}\omega(\omega_a\omega_b)^{\frac{1}{4}} \int dq_1 dq_2 \left(\frac{\sqrt{\omega_a}+\sqrt{\omega_b}}{2} q_1^2 + \frac{\sqrt{\omega_a}-\sqrt{\omega_b}}{2} q_1 q_2\right) \\
&\times e^{-\frac{1}{2}\left(\frac{\omega_a+\omega_b+2\omega}{2}(q_1^2+q_2^2)+(\omega_a-\omega_b)q_1q_2\right)}.
\end{aligned} \tag{E.-115}
$$

We need to evaluate integrals of the form:

$$I_1 = \int dq_1 dq_2 \; q_1^2 e^{-\frac{1}{2}\left(\alpha(q_1^2+q_2^2)+2\beta q_1 q_2\right)} \tag{E.-115}$$

and

$$I_2 = \int dq_1 dq_2 \; q_1 q_2 e^{-\frac{1}{2}\left(\alpha(q_1^2+q_2^2)+2\beta q_1 q_2\right)}. \tag{E.-115}$$

Note that (E.13) is equal to

$$\frac{1}{2}\int dq_1 dq_2 \; (q_1^2+q_2^2) e^{-\frac{1}{2}\left(\alpha(q_1^2+q_2^2)+2\beta q_1 q_2\right)}. \tag{E.-115}$$

If we define

$$I_0(\alpha,\beta) = \int dq_1 dq_2 \; e^{-\frac{1}{2}\left(\alpha(q_1^2+q_2^2)+2\beta q_1 q_2\right)} \tag{E.-115}$$

then

$$I_1 = -\frac{\partial}{\partial\alpha} I_0(\alpha,\beta), \qquad I_2 = -\frac{\partial}{\partial\beta} I_0(\alpha,\beta). \tag{E.-115}$$

It is easy to evaluate $I_0(\alpha,\beta)$:

$$\begin{aligned}
I_0(\alpha,\beta) &= \int dq_1 dq_2 \; e^{-\frac{1}{2}\left(\alpha(q_1^2+q_2^2)+2\beta q_1 q_2\right)} \\
&= \int dq_1 dq_2 \; e^{-\frac{\alpha}{2}\left((q_1^2+q_2^2)+2(\beta/\alpha) q_1 q_2\right)} \\
&= \int dq_1 e^{-\frac{\alpha}{2}q_1^2} \int dq_2 \; e^{-\frac{\alpha}{2}\left(q_2^2+2(\beta/\alpha) q_1 q_2\right)} \\
&= \int dq_1 e^{-\frac{\alpha}{2}q_1^2} \int dq_2 \; e^{-\frac{\alpha}{2}\left((q_2+q_1(\beta/\alpha))^2-q_1^2(\beta/\alpha)^2\right)} \\
&= \int dq_1 e^{-\frac{1}{2}\left(\frac{\alpha^2-\beta^2}{\alpha}\right)q_1^2} \int dq_2 \; e^{-\frac{\alpha}{2}q_2^2} \\
&= \sqrt{\frac{2\pi\alpha}{\alpha^2-\beta^2}} \cdot \sqrt{\frac{2\pi}{\alpha}} \\
&= \frac{2\pi}{\sqrt{\alpha^2-\beta^2}}.
\end{aligned} \tag{E.-120}$$

Inserting this into (E.13) we have

$$I_1 = \frac{2\pi\alpha}{(\alpha^2-\beta^2)^{3/2}}, \qquad I_2 = -\frac{2\pi\beta}{(\alpha^2-\beta^2)^{3/2}}. \tag{E.-120}$$

As $\alpha = \frac{\omega_a+\omega_b+2\omega}{2}$ and $\beta = \frac{\omega_a-\omega_b}{2}$,

$$\begin{aligned}
\alpha^2-\beta^2 &= \frac{(\omega_a+\omega_b+2\omega)^2-(\omega_a-\omega_b)^2}{4} \\
&= \frac{(\omega_a+\omega_b)^2-(\omega_a-\omega_b)^2+4\omega(\omega_a+\omega_b)+4\omega^2}{4} \\
&= \omega^2+\omega(\omega_a+\omega_b)+\omega_a\omega_b
\end{aligned} \tag{E.-121}$$

and

$$\begin{aligned}
I_1 &= \frac{\pi(2\omega+\omega_a+\omega_b)}{(\omega^2+\omega(\omega_a+\omega_b)+\omega_a\omega_b)^{3/2}}, \\
I_2 &= -\frac{\pi(\omega_a-\omega_b)}{(\omega^2+\omega(\omega_a+\omega_b)+\omega_a\omega_b)^{3/2}}.
\end{aligned} \tag{E.-121}$$

Then

$$_{loc}<1|1> = \frac{1}{\pi}\omega(\omega_a\omega_b)^{\frac{1}{4}}\big((\sqrt{\omega_a}+\sqrt{\omega_b})I_1+(\sqrt{\omega_a}-\sqrt{\omega_b})I_2\big) \tag{E.-121}$$

Let us expand

$$\begin{aligned}
\omega^2+\omega(\omega_a+\omega_b)+\omega_a\omega_b &= \omega^2+\omega\big((\omega^2+\lambda)^{1/2}+(\omega^2-\lambda)^{1/2}\big)+(\omega^4-\lambda^2)^{1/2} \\
&= \omega^2\left[1+\left(1+\frac{\lambda}{\omega^2}\right)^{1/2}+\left(1-\frac{\lambda}{\omega^2}\right)^{1/2}+\left(1-\frac{\lambda^2}{\omega^4}\right)^{1/2}\right] \\
&= (2\omega)^2\left[1-\frac{3}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right]
\end{aligned} \tag{E.-122}$$

and thus

$$\begin{aligned}
\big(\omega^2+\omega(\omega_a+\omega_b)+\omega_a\omega_b\big)^{-3/2} &= (2\omega)^{-3}\left[1-\frac{3}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right]^{-3/2} \\
&= (2\omega)^{-3}\left[1+\frac{9}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right].
\end{aligned} \tag{E.-122}$$

As well

$$2\omega+\omega_a+\omega_b = 4\omega\left[1-\frac{1}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right] \tag{E.-122}$$

and

$$\omega_a-\omega_b = \omega\left(\frac{1}{2}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right). \tag{E.-122}$$

So that

$$\begin{aligned} I_1 &= \pi\times 4\omega\left[1-\frac{1}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right]\times(2\omega)^{-3}\left[1+\frac{9}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right] \\ &= \pi\frac{1}{2\omega^2}\left[1+\frac{7}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right] \end{aligned} \tag{E.-122}$$

and

$$\begin{aligned} I_2 &= -\pi\times\omega\left[\frac{1}{2}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right]\times(2\omega)^{-3}\left[1+\frac{9}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right] \\ &= -\pi\frac{1}{\omega^2}\left[\frac{1}{16}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right] \end{aligned} \tag{E.-122}$$

Putting it all together

$$\begin{aligned} {}_{loc}{<}1|1> &= \frac{1}{\pi}\omega(\omega_a\omega_b)^{\frac{1}{4}}\left\{(\sqrt{\omega_a}+\sqrt{\omega_b})I_1+(\sqrt{\omega_a}-\sqrt{\omega_b})I_2\right\} \\ &= \frac{1}{\pi}\omega\times\sqrt{\omega}\left(1-\frac{1}{8}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)\times \\ &\quad \left\{2\sqrt{\omega}\left(1-\frac{3}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)\times\pi\frac{1}{2\omega^2}\left[1+\frac{7}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right]-\right. \\ &\quad \left.\sqrt{\omega}\left(\frac{1}{2}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right)\times\pi\frac{1}{\omega^2}\left[\frac{1}{16}\frac{\lambda}{\omega^2}+\mathcal{O}(\lambda^3)\right]\right\} \\ &= \left(1-\frac{1}{8}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)\times\left\{\left(1+\frac{1}{8}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)-\left(\frac{1}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)\right\} \\ &= 1-\frac{1}{32}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4) \end{aligned} \tag{E.-126}$$

Probability of seeing a local particle on first oscillator.

Suppose the state of the system is the global ground state $|0\rangle$. Calculate probability of detecting a particle on the first oscillator (we trace over the local particle states on the second oscillator). The result should be

$$\mathcal{P} =< 0|P_{0loc}|0 >= 1 - \frac{1}{16}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^3). \tag{E.-126}$$

Proof:

$$\begin{aligned}
\mathcal{P} &= < 0| \left(\sum_{n_2=0}^{\infty} |0, n_2 >_{loc\ loc}<0, n_2| \right) |0 > \\
&= \int dq_1 dq_2 \int dq_1' dq_2' \sum_{n_2=0}^{\infty} < 0|\mathbf{q} >< q_1, q_2|0, n_2 >_{loc\ loc}<0, n_2|q_1', q_2' >< \mathbf{q}'|0 > \\
&= \int dq_1 dq_2 \int dq_1' dq_2'\ \psi_0^*(q_1, q_2)\psi_0^{loc}(q_1) \left(\sum_{n_2=0}^{\infty} \psi_{n_2}^{loc}(q_2)\psi_{n_2}^{loc*}(q_2') \right) \psi_0^{loc*}(q_1')\psi_0(q_1', q_2') \\
&= \int dq_1 dq_2\ \psi_0^*(q_1, q_2)\psi_0^{loc}(q_1) \int dq_1' dq_2' \left(\delta(q_2 - q_2')\right) \psi_0^{loc*}(q_1')\psi_0(q_1', q_2') \\
&= \int dq_1 dq_2\ \psi_0^*(q_1, q_2)\psi_0^{loc}(q_1) \int dq_1'\ \psi_0^{loc*}(q_1')\psi_0(q_1', q_2)
\end{aligned} \tag{E.-129}$$

We evaluate

$$\mathcal{P} = \int dq_1 dq_2 \; \psi_0^*(q_1, q_2)\psi_0^{loc}(q_1) \int dq_1' \; \psi_0^{loc*}(q_1')\psi_0(q_1', q_2). \tag{E.-129}$$

Let us perform the q_1' integration.

$$\begin{aligned}
\int dq_1' \psi_0^{loc*}(q_1')\psi_0(q_1', q_2) &= \int dq_1' \psi_0^{loc*}(q_1')\psi_0^{(\omega_a)}\left(\frac{q_1' + q_2}{\sqrt{2}}\right)\psi_0^{(\omega_b)}\left(\frac{q_1' - q_2}{\sqrt{2}}\right) \\
&= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\frac{\omega_a+\omega_b}{2}q_2^2} \int dq_1' e^{-\frac{1}{2}\frac{\omega_a+\omega_b+2\omega}{2}{q_1'}^2} e^{-\frac{1}{2}(\omega_a-\omega_b)q_1'q_2} \\
&= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\frac{\omega_a+\omega_b}{2}q_2^2} \int dq_1' e^{-\frac{1}{2}(\alpha {q_1'}^2 - 2\beta q_1' q_2)} \\
&= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\frac{\omega_a+\omega_b}{2}q_2^2} \int dq_1' e^{-\frac{\alpha}{2}({q_1'}^2 - 2(\beta/\alpha) q_1' q_2)} \\
&= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\frac{\omega_a+\omega_b}{2}q_2^2} \int dq_1' e^{-\frac{\alpha}{2}\left(({q_1'}^2 - (\beta/\alpha) q_2)^2) - (\beta/\alpha)^2 q_2^2\right)} \\
&= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{\omega_a+\omega_b}{2} - \frac{\beta^2}{\alpha}\right)q_2^2} \int dq_1' e^{-\frac{\alpha}{2}{q_1'}^2} \\
&= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{\omega_a+\omega_b}{2} - \frac{\beta^2}{\alpha}\right)q_2^2} \sqrt{\frac{2\pi}{\alpha}}
\end{aligned} \tag{E.-134}$$

where

$$\alpha = \frac{\omega_a + \omega_b + 2\omega}{2} \quad \text{and} \quad \beta = \frac{\omega_a - \omega_b}{2} \tag{E.-134}$$

The q_1 integration is identical

$$\begin{aligned}
\int dq_1 \psi_0^*(q_1, q_2)\psi_0^{loc}(q_1) &= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{\omega_a+\omega_b}{2} - \frac{\beta^2}{\alpha}\right)q_2^2} \int dq_1 e^{-\frac{\alpha}{2}q_1^2} \\
&= \left(\frac{\omega}{\pi}\right)^{1/4} \frac{(\omega_a\omega_b)^{1/4}}{\sqrt{\pi}} \sqrt{\frac{2\pi}{\alpha}} e^{-\frac{1}{2}\left(\frac{\omega_a+\omega_b}{2} - \frac{\beta^2}{\alpha}\right)q_2^2}
\end{aligned} \tag{E.-134}$$

Substituting these results into the expression for $\mathcal{P}$ gives

$$
\begin{aligned}
\mathcal{P} &= \left(\sqrt{\frac{\omega}{\pi}}\right)\left(\sqrt{\frac{\omega_a}{\pi}}\sqrt{\frac{\omega_b}{\pi}}\right)\frac{2\pi}{\alpha}\int dq_2 e^{-\frac{1}{2}\left(\omega_a+\omega_b-\frac{2\beta^2}{\alpha}\right)q_2^2} \\
&= \left(\sqrt{\frac{\omega}{\pi}}\right)\left(\sqrt{\frac{\omega_a}{\pi}}\sqrt{\frac{\omega_b}{\pi}}\right)\frac{2\pi}{\alpha}\int dq_2 e^{-\frac{1}{2}\frac{2(\alpha^2-\beta^2-\omega\alpha)}{\alpha}q_2^2} \\
&= \left(\sqrt{\frac{\omega}{\pi}}\right)\left(\sqrt{\frac{\omega_a}{\pi}}\sqrt{\frac{\omega_b}{\pi}}\right)\frac{2\pi}{\alpha}\sqrt{\frac{2\pi\alpha}{2(\alpha^2-\beta^2-\omega\alpha)}} \\
&= \sqrt{\omega}\sqrt{\omega_a\omega_b}\frac{2}{\alpha}\sqrt{\frac{2\alpha}{2(\alpha^2-\beta^2-\omega\alpha)}} \qquad \text{(E.-136)}
\end{aligned}
$$

We use

$$
\omega_a = \sqrt{\omega^2+\lambda}, \qquad \omega_b = \sqrt{\omega^2-\lambda} \qquad \text{(E.-136)}
$$

to find the expansion in λ. First

$$
\begin{aligned}
\sqrt{\omega_a\omega_b} &= (\omega^4-\lambda^2)^{1/4} \\
&= \omega\left(1-\frac{1}{4}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right). \qquad \text{(E.-136)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{2}{\alpha} &= 4(2\omega+\omega_a+\omega_b)^{-1} \\
&= 4(\omega)^{-1}\left(2+\sqrt{1+\frac{\lambda}{\omega^2}}+\sqrt{1-\frac{\lambda}{\omega^2}}\right)^{-1} \\
&= 4(\omega)^{-1}\left(2+\left(1+\frac{1}{2}\frac{\lambda}{\omega^2}-\frac{1}{8}\frac{\lambda^2}{\omega^4}+\ldots\right)+\left(1-\frac{1}{2}\frac{\lambda}{\omega^2}-\frac{1}{8}\frac{\lambda^2}{\omega^4}+\ldots\right)\right)^{-1} \\
&= \frac{1}{\omega}\left(1-\frac{1}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)^{-1} \\
&= \frac{1}{\omega}\left(1+\frac{1}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right). \qquad \text{(E.-139)}
\end{aligned}
$$

We have

$$
\begin{aligned}
2(\alpha^2-\beta^2-\omega\alpha) &= 2\frac{(\omega_a+\omega_b+2\omega)^2-(\omega_a-\omega_b)^2}{4} - 2\frac{\omega(\omega_a+\omega_b)+2\omega^2}{2} \\
&= \frac{(\omega_a+\omega_b)^2-(\omega_a-\omega_b)^2+4\omega(\omega_a+\omega_b)+4\omega^2}{2} - \omega(\omega_a+\omega_b) - 2\omega^2 \\
&= 2(\omega^2+\omega(\omega_a+\omega_b)+\omega_a\omega_b) - \omega(\omega_a+\omega_b) - 2\omega^2 \\
&= \omega(\omega_a+\omega_b) + 2\omega_a\omega_b \\
&= \omega^2\left(\sqrt{1+\frac{\lambda}{\omega^2}} + \sqrt{1-\frac{\lambda}{\omega^2}} + 2\sqrt{1-\frac{\lambda^2}{\omega^4}}\right) \\
&= 4\omega^2\left(1-\frac{1}{16}\frac{\lambda^2}{\omega^4} - \frac{1}{4}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4)\right) \\
&= 4\omega^2\left(1-\frac{5}{16}\frac{\lambda^2}{\omega^4} + \mathcal{O}(\lambda^4)\right) \qquad \text{(E.-144)}
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{2\alpha}{2(\alpha^2-\beta^2-\omega\alpha)} &= \frac{(\omega_a+\omega_b+2\omega)}{2(\alpha^2-\beta^2-\omega\alpha)} \\
&= \frac{4\times\omega\left(1-\frac{1}{16}\frac{\lambda^2}{\omega^4}+\dots\right)}{4\omega^2\left(1-\frac{5}{16}\frac{\lambda^2}{\omega^4}+\dots\right)} \\
&= \frac{1}{\omega}\left(1+\frac{1}{4}\frac{\lambda^2}{\omega^4}+\dots\right). \qquad \text{(E.-145)}
\end{aligned}
$$

Putting it together we obtain

$$
\begin{aligned}
\mathcal{P} &= \sqrt{\omega}\sqrt{\omega_a\omega_b}\frac{2}{\alpha}\sqrt{\frac{2\alpha}{2(\alpha^2-\beta^2-\omega\alpha)}} \\
&= \sqrt{\omega}\omega\left(1-\frac{1}{4}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)\times\frac{1}{\omega}\left(1+\frac{1}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right)\times\frac{1}{\sqrt{\omega}}\left(1+\frac{1}{8}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4)\right) \\
&= 1-\frac{1}{16}\frac{\lambda^2}{\omega^4}+\mathcal{O}(\lambda^4). \qquad \text{(E.-146)}
\end{aligned}
$$

Factorising chain of four oscillators

Show that

$$H_0 = \frac{1}{2}\sum_{i=1}^{4}(p_i^2 + q_i^2) + \lambda \sum_{i=1}^{3} q_i q_{i+1} \tag{E.-146}$$

factorises to

$$H_0 = \frac{1}{2}\sum_{a=1}^{4}(p_a^2 + \omega_a^2 Q_a^2). \tag{E.-146}$$

Hint: Consider an eigenvector problem. Use the trig-identity:

$$2\cos x \sin y = \sin(y-x) + \sin(y+x). \tag{E.-146}$$

Proof:

We wish to equate

$$\begin{aligned}
&\frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2) + \lambda q_1 q_2 + \lambda q_2 q_3 + \lambda q_3 q_4 = \\
&= \frac{1}{2}(q_1, q_2, q_3, q_4)\begin{pmatrix} 1 & \lambda & 0 & 0 \\ \lambda & 1 & \lambda & 0 \\ 0 & \lambda & 1 & \lambda \\ 0 & 0 & \lambda & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \frac{1}{2}(Q_1, Q_2, Q_3, Q_4)\begin{pmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 \\ 0 & 0 & 0 & \omega_4^2 \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}
\end{aligned} \tag{E.-147}$$

where the $\mathbf{Q} = (Q_a)$, $a = 1, \ldots, 4$.

We look at the eigenvector problem:

$$\begin{pmatrix} 1 & \lambda & 0 & 0 \\ \lambda & 1 & \lambda & 0 \\ 0 & \lambda & 1 & \lambda \\ 0 & 0 & \lambda & 1 \end{pmatrix}\begin{pmatrix} e_1^{(a)} \\ e_2^{(a)} \\ e_3^{(a)} \\ e_4^{(a)} \end{pmatrix} = \beta^{(a)} \begin{pmatrix} e_1^{(a)} \\ e_2^{(a)} \\ e_3^{(a)} \\ e_4^{(a)} \end{pmatrix} \tag{E.-147}$$

From the trig-identity:

$$2\cos x \sin y = \sin(y-x) + \sin(y+x) \tag{E.-147}$$

we have:

$$\begin{aligned}
\lambda \sin\left(\frac{0 \times a\pi}{5}\right) + \sin\left(\frac{a\pi}{5}\right) + \lambda \sin\left(\frac{2a\pi}{5}\right) &= \left(1 + 2\lambda\cos\left(\frac{a\pi}{5}\right)\right)\sin\left(\frac{a\pi}{5}\right) \\
\lambda \sin\left(\frac{a\pi}{5}\right) + \sin\left(\frac{2a\pi}{5}\right) + \lambda \sin\left(\frac{3a\pi}{5}\right) &= \left(1 + 2\lambda\cos\left(\frac{a\pi}{5}\right)\right)\sin\left(\frac{2a\pi}{5}\right) \\
\lambda \sin\left(\frac{2a\pi}{5}\right) + \sin\left(\frac{3a\pi}{5}\right) + \lambda \sin\left(\frac{4a\pi}{5}\right) &= \left(1 + 2\lambda\cos\left(\frac{a\pi}{5}\right)\right)\sin\left(\frac{3a\pi}{5}\right) \\
\lambda \sin\left(\frac{3a\pi}{5}\right) + \sin\left(\frac{4a\pi}{5}\right) + \lambda \sin\left(\frac{5a\pi}{5}\right) &= \left(1 + 2\lambda\cos\left(\frac{a\pi}{5}\right)\right)\sin\left(\frac{4a\pi}{5}\right)
\end{aligned} \tag{E.-149}$$

where $a = 1, 2, 3, 4$. From which we see the eigenvector problem, (E.13), is solved for

$$e_i^{(a)} = C_{(4)} \sin\left(\frac{ia\pi}{4+1}\right) \tag{E.-149}$$

with eigenvalue:

$$\beta^{(a)} = 1 + 2\lambda\cos\left(\frac{a\pi}{4+1}\right). \tag{E.-149}$$

We define a scalar product by $\mathbf{e}^{(b)T}\mathbf{e}^{(a)} = \sum_i e_i^{(b)} e_i^{(a)}$. Obviously, $\mathbf{e}^{(a)T}\mathbf{e}^{(a)} > 0$. It is easy to prove that, because the eigenvalues are distinct, that the eigenvectors $\mathbf{e}^{(a)}$ are orthogonal; denote the matrix in (E.13) by M. First we have

$$\mathbf{e}^{(b)T} M \mathbf{e}^{(a)} = \beta^{(a)} \mathbf{e}^{(b)T}\mathbf{e}^{(a)}. \tag{E.-149}$$

Since the matrix M is symmetric the LHS can also be written as

$$\begin{aligned}
(\mathbf{e}^{(b)T} M \mathbf{e}^{(a)})^{TT} &= (\mathbf{e}^{(a)T} M^T \mathbf{e}^{(b)})^T \\
&= (\mathbf{e}^{(a)T} M \mathbf{e}^{(b)})^T \\
&= (\beta^{(b)} \mathbf{e}^{(a)T}\mathbf{e}^{(b)})^T \\
&= \beta^{(b)} \mathbf{e}^{(b)T}\mathbf{e}^{(a)}.
\end{aligned} \tag{E.-151}$$

The difference between the last two equations is

$$0 = (\beta^{(a)} - \beta^{(b)})\mathbf{e}^{(b)T}\mathbf{e}^{(a)}. \tag{E.-151}$$

Thus,

$$\mathbf{e}^{(b)T}\mathbf{e}^{(a)} = 0 \quad \text{if} \quad a \neq b. \tag{E.-151}$$

which is what we wished to establish.

We need to normalise $e_i^{(a)}$:

$$\begin{aligned}
1 &= \sum_{j=1}^{4} e_j^{(a)} e_j^{(a)} \\
&= C_{(4)}^2 \sum_{j=1}^{4} \sin^2\left(\frac{\pi a j}{4+1}\right) \\
&= C_{(4)}^2 \frac{1}{2} \sum_{j=1}^{4} \left(1 - \cos\left(\frac{2\pi a j}{4+1}\right)\right) \\
&= C_{(4)}^2 \left(\frac{4}{2} - \frac{1}{4}\sum_{j=1}^{4} \exp\left(i\frac{2\pi a j}{4+1}\right) - \frac{1}{4}\sum_{j=1}^{4} \exp\left(-i\frac{2\pi a j}{4+1}\right)\right) \\
&= C_{(4)}^2 \left(\frac{4}{2} - \frac{1}{4}\frac{\exp\left(i\frac{2\pi a}{4+1}\right) - \exp\left(i\frac{2\pi a(4+1)}{4+1}\right)}{1 - \exp\left(i\frac{2\pi a}{4+1}\right)} - \frac{1}{4}\frac{\exp\left(-i\frac{2\pi a}{4+1}\right) - \exp\left(-i\frac{2\pi a(4+1)}{4+1}\right)}{1 - \exp\left(-i\frac{2\pi a}{4+1}\right)}\right) \\
&= C_{(4)}^2 \left(\frac{4}{2} - \frac{1}{4}\frac{\exp\left(i\frac{2\pi a}{4+1}\right) - 1}{1 - \exp\left(i\frac{2\pi a}{4+1}\right)} - \frac{1}{4}\frac{\exp\left(-i\frac{2\pi a}{4+1}\right) - 1}{1 - \exp\left(-i\frac{2\pi a}{4+1}\right)}\right) \\
&= C_{(4)}^2 \frac{4+1}{2}.
\end{aligned} \tag{E.-156}$$

Therefore we have

$$e_i^{(a)} = \sqrt{\frac{2}{4+1}} \sin\left(\frac{ia\pi}{4+1}\right). \tag{E.-156}$$

Construction of the orthogonal matrix:

$$\begin{aligned}
(U_{ia}^{(4)T}) &:= \begin{pmatrix} e_1^{(1)} & e_1^{(2)} & e_1^{(3)} & e_1^{(4)} \\ e_2^{(1)} & e_2^{(2)} & e_2^{(3)} & e_2^{(4)} \\ e_3^{(1)} & e_3^{(2)} & e_3^{(3)} & e_3^{(4)} \\ e_4^{(1)} & e_4^{(2)} & e_4^{(3)} & e_4^{(4)} \end{pmatrix} \\
&= (e_i^{(a)}).
\end{aligned} \tag{E.-156}$$

We have $(U^{(4)}_{ai})(U^{(4)T}_{ib}) = \mathbf{e}^{(a)T}\mathbf{e}^{(b)} = \delta_{ab}$. We then have

$$\begin{aligned}
&\begin{pmatrix} e_1^{(1)} & e_2^{(1)} & e_3^{(1)} & e_4^{(1)} \\ e_1^{(2)} & e_2^{(2)} & e_3^{(2)} & e_4^{(2)} \\ e_1^{(3)} & e_2^{(3)} & e_3^{(3)} & e_4^{(3)} \\ e_1^{(4)} & e_2^{(4)} & e_3^{(4)} & e_4^{(4)} \end{pmatrix} \begin{pmatrix} 1 & \lambda & 0 & 0 \\ \lambda & 1 & \lambda & 0 \\ 0 & \lambda & 1 & \lambda \\ 0 & 0 & \lambda & 1 \end{pmatrix} \begin{pmatrix} e_1^{(1)} & e_1^{(2)} & e_1^{(3)} & e_1^{(4)} \\ e_2^{(1)} & e_2^{(2)} & e_2^{(3)} & e_2^{(4)} \\ e_3^{(1)} & e_3^{(2)} & e_3^{(3)} & e_3^{(4)} \\ e_4^{(1)} & e_4^{(2)} & e_4^{(3)} & e_4^{(4)} \end{pmatrix} \\
= &\begin{pmatrix} e_1^{(1)} & e_2^{(1)} & e_3^{(1)} & e_4^{(1)} \\ e_1^{(2)} & e_2^{(2)} & e_3^{(2)} & e_4^{(2)} \\ e_1^{(3)} & e_2^{(3)} & e_3^{(3)} & e_4^{(3)} \\ e_1^{(4)} & e_2^{(4)} & e_3^{(4)} & e_4^{(4)} \end{pmatrix} \begin{pmatrix} \beta^{(1)}e_1^{(1)} & \beta^{(2)}e_1^{(2)} & \beta^{(3)}e_1^{(3)} & \beta^{(4)}e_1^{(4)} \\ \beta^{(1)}e_2^{(1)} & \beta^{(2)}e_2^{(2)} & \beta^{(3)}e_2^{(3)} & \beta^{(4)}e_2^{(4)} \\ \beta^{(1)}e_3^{(1)} & \beta^{(2)}e_3^{(2)} & \beta^{(3)}e_3^{(3)} & \beta^{(4)}e_3^{(4)} \\ \beta^{(1)}e_4^{(1)} & \beta^{(2)}e_4^{(2)} & \beta^{(3)}e_4^{(3)} & \beta^{(4)}e_4^{(4)} \end{pmatrix} \\
= &\begin{pmatrix} \beta^{(1)} & 0 & 0 & 0 \\ 0 & \beta^{(2)} & 0 & 0 \\ 0 & 0 & \beta^{(3)} & 0 \\ 0 & 0 & 0 & \beta^{(4)} \end{pmatrix}.
\end{aligned} \tag{E.-157}$$

Write

$$(D_{ab}) = \begin{pmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 \\ 0 & 0 & 0 & \omega_4^2 \end{pmatrix} = \begin{pmatrix} \beta^{(1)} & 0 & 0 & 0 \\ 0 & \beta^{(2)} & 0 & 0 \\ 0 & 0 & \beta^{(3)} & 0 \\ 0 & 0 & 0 & \beta^{(4)} \end{pmatrix}. \tag{E.-157}$$

We have

$$\begin{aligned} q_i M_{ij} q_j &= Q_a(U^{(4)}_{ai})M_{ij}(U^{(4)}_{jb})Q_b \\ &= Q_a D_{ab} Q_b \end{aligned} \tag{E.-157}$$

giving the original equation (E.-147), implying $q_i = (U^{(4)}_{ib})Q_b$ or $(U^{(4)})_{ai}q_i = (U^{(4)}_{ai})(U^{(4)}_{ib})Q_b = \delta_{ab}Q_b = Q_a$.

Thus the normal coordinates are given by

$$Q_a = U^{(4)}_{ai} q_i, \tag{E.-157}$$

with the normal frequencies

$$\omega_a = \sqrt{1 + 2\lambda\cos\theta_a}, \quad \text{where} \quad \theta_a = \frac{a\pi}{4+1}, \quad \text{and} \quad a = 1, \ldots, 4. \tag{E.-157}$$

We check the momenta remain in a "diagonalised" form: First

$$
\begin{aligned}
p_i &= -i\hbar\frac{\partial}{\partial q^i} \\
&= -i\hbar\sum_a \frac{\partial Q_a}{\partial q^i}\frac{\partial}{\partial Q_a} \\
&= -i\hbar\sum_a \frac{\partial(\sum_j U_{aj}^{(4)} q^j)}{\partial q^i}\frac{\partial}{\partial Q_a} \\
&= -i\hbar\sum_a U_{ai}^{(4)}\frac{\partial}{\partial Q_a} \qquad \text{(E.-159)}
\end{aligned}
$$

then

$$
\begin{aligned}
\sum_{i=1}^{4}(p_i)^2 &= -\hbar^2\sum_{i=1}^{n}\left(\sum_a U_{ai}^{(4)}\frac{\partial}{\partial Q_a}\right)^2 \\
&= -\hbar^2\sum_{a,b}\left(\sum_{i=1}^{n} U_{ai}^{(4)}U_{bi}^{(4)}\right)\frac{\partial}{\partial Q_a}\frac{\partial}{\partial Q_b} \\
&\equiv \sum_{a=1}^{4}(P_a)^2. \qquad \text{(E.-160)}
\end{aligned}
$$

Chain of oscillators

(a) Diagonilisation of the hamiltonian.

The hamiltonian

$$H = \frac{1}{2}(|\mathbf{p}|^2 + |\mathbf{q}|^2) + \lambda \sum_{i=1}^{n-1} q^i q^{i+1} \tag{E.-160}$$

where $|\mathbf{q}|^2 = \sum_{i=1}^{n}(q^i)^2$ is diagonalised in the coordinates $\mathbf{Q} = U^{(n)}\mathbf{q}$, where $U^{(n)}$ is an orthogonal $n \times n$ matrix

$$U^{(n)}_{ai} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{ai\pi}{n+1}\right) \tag{E.-160}$$

with normal frequencies

$$\omega_a = \sqrt{1 + 2\lambda\cos\theta_a}, \qquad \text{where} \quad \theta_a = \frac{a\pi}{n+1}, \quad \text{and} \quad a = 1, 2, \ldots, n. \tag{E.-160}$$

(b) The vacuum state.

The vacuum state is

$$< \mathbf{q}|0 >= \prod_{a=1}^{n} \left(\frac{\omega_a}{\pi}\right)^{1/4} e^{-\frac{1}{2}q^i D^{(n)}_{ij} q^j} \tag{E.-160}$$

where

$$D^{(n)}_{ij} = \sum_a U^{(n)}_{ai}\omega_a U^{(n)}_{aj}. \tag{E.-160}$$

Proof:

Part (a)

We require

$$
\begin{aligned}
\sum_{a=1}^{n}\omega_a^2(Q_a)^2 &= \sum_{a=1}^{n}\omega_a^2\left(\sum_{i=1}^{n}U_{ai}^{(n)}q^i\right)^2 \\
&= \sum_{i,j=1}^{n}\left(\sum_{a=1}^{n}\omega_a^2U_{ai}^{(n)}U_{aj}^{(n)}\right)q^iq^j \\
&\equiv \sum_{i=1}^{n}(q^i)^2+\lambda\sum_{i=1}^{n-1}q^iq^{i+1}
\end{aligned} \tag{E.-161}
$$

or

$$
(q_1,q_2,q_3,q_4,\cdots,q_{n-1},q_n)\begin{pmatrix}
1 & \lambda & 0 & 0 & \cdots & 0 & 0\\
\lambda & 1 & \lambda & 0 & \cdots & 0 & 0\\
0 & \lambda & 1 & \lambda & \cdots & 0 & 0\\
0 & 0 & \lambda & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 1 & \lambda\\
0 & 0 & 0 & 0 & \cdots & \lambda & 1
\end{pmatrix}\begin{pmatrix} q_1\\ q_2\\ q_3\\ q_4\\ \vdots\\ q_{n-1}\\ q_n\end{pmatrix}
$$
$$
=(Q_1,Q_2,Q_3,Q_4,\cdots,Q_{n-1},Q_n)\begin{pmatrix}
\omega_1^2 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & \omega_2^2 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \omega_3^2 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \omega_4^2 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & \omega_{n-1}^2 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & \omega_n^2
\end{pmatrix}\begin{pmatrix} Q_1\\ Q_2\\ Q_3\\ Q_4\\ \vdots\\ Q_{n-1}\\ Q_n\end{pmatrix} \tag{E.-162}
$$

where $Q_a = U_{ai}^{(n)}q^i$. We are led to the eigenvector problem:

$$
\begin{pmatrix}
1 & \lambda & 0 & 0 & \cdots & 0 & 0\\
\lambda & 1 & \lambda & 0 & \cdots & 0 & 0\\
0 & \lambda & 1 & \lambda & \cdots & 0 & 0\\
0 & 0 & \lambda & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 1 & \lambda\\
0 & 0 & 0 & 0 & \cdots & \lambda & 1
\end{pmatrix}\begin{pmatrix} e_1^a\\ e_2^a\\ e_3^a\\ e_4^a\\ \vdots\\ e_{n-1}^a\\ e_n^a\end{pmatrix} = \beta^a\begin{pmatrix} e_1^a\\ e_2^a\\ e_3^a\\ e_4^a\\ \vdots\\ e_{n-1}^a\\ e_n^a\end{pmatrix} \tag{E.-162}
$$

From the trig-identity:

$$
2\cos x\sin y = \sin(y-x)+\sin(y+x) \tag{E.-162}
$$

we have:

$$\begin{aligned}
\lambda \sin\left(\frac{0\times a\pi}{n+1}\right) + \sin\left(\frac{a\pi}{n+1}\right) + \lambda \sin\left(\frac{2a\pi}{n+1}\right) &= \left(1+2\lambda\cos\left(\frac{a\pi}{n+1}\right)\right)\sin\left(\frac{a\pi}{n+1}\right)\\
\lambda \sin\left(\frac{a\pi}{n+1}\right) + \sin\left(\frac{2a\pi}{n+1}\right) + \lambda \sin\left(\frac{3a\pi}{n+1}\right) &= \left(1+2\lambda\cos\left(\frac{a\pi}{n+1}\right)\right)\sin\left(\frac{2a\pi}{n+1}\right)\\
\lambda \sin\left(\frac{2a\pi}{n+1}\right) + \sin\left(\frac{3a\pi}{n+1}\right) + \lambda \sin\left(\frac{4a\pi}{n+1}\right) &= \left(1+2\lambda\cos\left(\frac{a\pi}{n+1}\right)\right)\sin\left(\frac{3a\pi}{n+1}\right)\\
\lambda \sin\left(\frac{3a\pi}{n+1}\right) + \sin\left(\frac{4a\pi}{n+1}\right) + \lambda \sin\left(\frac{5a\pi}{n+1}\right) &= \left(1+2\lambda\cos\left(\frac{a\pi}{n+1}\right)\right)\sin\left(\frac{4a\pi}{n+1}\right)\\
\vdots &= \vdots\\
\lambda \sin\left(\frac{(n-1)a\pi}{n+1}\right) + \sin\left(\frac{na\pi}{n+1}\right) + \lambda \sin\left(\frac{(n+1)a\pi}{n+1}\right) &= \left(1+2\lambda\cos\left(\frac{a\pi}{n+1}\right)\right)\sin\left(\frac{na\pi}{n+1}\right)
\end{aligned} \tag{E.-167}$$

where $a = 1, 2, \ldots, n$. From which we see the eigenvectors are

$$e_i^{(a)} = C_{(n)} \sin\left(\frac{ia\pi}{n+1}\right) \tag{E.-167}$$

where the eigenvalues are

$$\beta^{(a)} = 1 + 2\lambda\cos\left(\frac{a\pi}{n+1}\right). \tag{E.-167}$$

We define a scalar product as $e^{(b)T}e^{(a)} = \sum_i e_i^{(b)} e_i^{(a)}$. It is easy to prove that, because the eigenvalues are distinct, that the eigenvectors $\mathbf{e}^{(a)}$ are orthogonal; denote the matrix on the LHS in (E.-162) by M. First we have

$$\mathbf{e}^{(b)T} M \mathbf{e}^{(a)} = \beta^{(a)} \mathbf{e}^{(b)T} \mathbf{e}^{(a)}. \tag{E.-167}$$

Since the matrix M is symmetric the LHS can also be written as

$$\begin{aligned}
(\mathbf{e}^{(b)T} M \mathbf{e}^{(a)})^{TT} &= (\mathbf{e}^{(a)T} M^T \mathbf{e}^{(b)})^T\\
&= (\mathbf{e}^{(a)T} M \mathbf{e}^{(b)})^T\\
&= (\beta^{(b)} \mathbf{e}^{(a)T} \mathbf{e}^{(b)})^T\\
&= \beta^{(b)} \mathbf{e}^{(b)T} \mathbf{e}^{(a)}.
\end{aligned} \tag{E.-169}$$

The difference between the last two equations is

$$0 = (\beta^{(a)} - \beta^{(b)})\mathbf{e}^{(b)T}\mathbf{e}^{(a)}. \qquad \text{(E.-169)}$$

Thus,

$$\mathbf{e}^{(b)T}\mathbf{e}^{(a)} = 0 \qquad \text{for} \quad a \neq b. \qquad \text{(E.-169)}$$

We need the $e_i^{(a)}$ to be normalised:

$$\begin{aligned}
1 &= \sum_{j=1}^{n} e_j^{(a)} e_j^{(a)} \\
&= C_{(n)}^2 \sum_{j=1}^{n} \sin^2\left(\frac{\pi a j}{n+1}\right) \\
&= C_{(n)}^2 \frac{1}{2} \sum_{j=1}^{n} \left(1 - \cos\left(\frac{2\pi a j}{n+1}\right)\right) \\
&= C_{(n)}^2 \left(\frac{n}{2} - \frac{1}{4}\sum_{j=1}^{n} \exp\left(i\frac{2\pi a j}{n+1}\right) - \frac{1}{4}\sum_{j=1}^{n} \exp\left(-i\frac{2\pi a j}{n+1}\right)\right) \\
&= C_{(n)}^2 \left(\frac{n}{2} - \frac{1}{4}\frac{\exp\left(i\frac{2\pi a}{n+1}\right) - \exp\left(i\frac{2\pi a(n+1)}{n+1}\right)}{1 - \exp\left(i\frac{2\pi a}{n+1}\right)} - \frac{1}{4}\frac{\exp\left(-i\frac{2\pi a}{n+1}\right) - \exp\left(-i\frac{2\pi a(n+1)}{n+1}\right)}{1 - \exp\left(-i\frac{2\pi a}{n+1}\right)}\right) \\
&= C_{(n)}^2 \left(\frac{n}{2} - \frac{1}{4}\frac{\exp\left(i\frac{2\pi a}{n+1}\right) - 1}{1 - \exp\left(i\frac{2\pi a}{n+1}\right)} - \frac{1}{4}\frac{\exp\left(-i\frac{2\pi a}{n+1}\right) - 1}{1 - \exp\left(-i\frac{2\pi a}{n+1}\right)}\right) \\
&= C_{(n)}^2 \frac{n+1}{2}. \qquad \text{(E.-174)}
\end{aligned}$$

Therefore we have

$$e_i^{(a)} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{i a \pi}{n+1}\right). \qquad \text{(E.-174)}$$

Construction of the orthogonal matrix:

$$
\begin{aligned}
(U^{(n)}_{ia}) \quad &:= \quad \begin{pmatrix}
e^{(1)}_1 & e^{(2)}_1 & e^{(3)}_1 & e^{(4)}_1 & \cdots & e^{(n-1)}_1 & e^{(n)}_1 \\
e^{(1)}_2 & e^{(2)}_2 & e^{(3)}_2 & e^{(4)}_2 & \cdots & e^{(n-1)}_2 & e^{(n)}_2 \\
e^{(1)}_3 & e^{(2)}_3 & e^{(3)}_3 & e^{(4)}_3 & \cdots & e^{(n-1)}_3 & e^{(n)}_3 \\
e^{(1)}_4 & e^{(2)}_4 & e^{(3)}_4 & e^{(4)}_4 & \cdots & e^{(n-1)}_4 & e^{(n)}_4 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e^{(1)}_{n-1} & e^{(2)}_{n-1} & e^{(3)}_{n-1} & e^{(4)}_{n-1} & \cdots & e^{(n-1)}_{n-1} & e^{(n)}_{n-1} \\
e^{(1)}_n & e^{(2)}_n & e^{(3)}_n & e^{(4)}_n & \cdots & e^{(n-1)}_n & e^{(n)}_n
\end{pmatrix} \\
&= \quad (e^{(a)}_i). \qquad \text{(E.-174)}
\end{aligned}
$$

From (E.13) and $\mathbf{e}^{(a)T}\mathbf{e}^{(a)} = 1$ for all a, we have

$$
\sum_i U^{(n)}_{ai}(U^{(n)}_{ia}) = \delta_{ab}. \qquad \text{(E.-174)}
$$

Then we have

$$
\begin{aligned}
q_i M_{ij} q_j \quad &= \quad Q_a(U^{(n)}_{ai})M_{ij}(U^{(n)}_{jb})Q_b \\
&= \quad Q_a D_{ab} Q_b \qquad \text{(E.-174)}
\end{aligned}
$$

Now turn to the momenta. We have

$$
\begin{aligned}
p_i \quad &= \quad -i\hbar \frac{\partial}{\partial q^i} \\
&= \quad -i\hbar \sum_a \frac{\partial Q_a}{\partial q^i} \frac{\partial}{\partial Q_a} \\
&= \quad -i\hbar \sum_a \frac{\partial(\sum_j U^{(n)}_{aj} q^j)}{\partial q^i} \frac{\partial}{\partial Q_a} \\
&= \quad -i\hbar \sum_a U^{(n)}_{ai} \frac{\partial}{\partial Q_a} \qquad \text{(E.-176)}
\end{aligned}
$$

which then means for the momentum squared

$$
\begin{aligned}
\sum_{i=1}^n (p_i)^2 \quad &= \quad -\hbar^2 \sum_{i=1}^n \left(\sum_a U^{(n)}_{ai} \frac{\partial}{\partial Q_a} \right)^2 \\
&= \quad -\hbar^2 \sum_{a,b} \left(\sum_{i=1}^n U^{(n)}_{ai} U^{(n)}_{bi} \right) \frac{\partial}{\partial Q_a} \frac{\partial}{\partial Q_b} \\
&\equiv \quad \sum_{a=1}^n (P_a)^2 \qquad \text{(E.-177)}
\end{aligned}
$$

by (E.13).

(b) The vacuum state

$$\psi_0(Q) = \prod_{a=1}^{n} \left(\frac{\omega_a}{\pi}\right)^{1/4} e^{-\frac{1}{2}\sum_a \omega_a Q_a^2} \qquad \text{(E.-177)}$$

but

$$\begin{aligned} \sum_a \omega_a Q_a^2 &= \sum_a \omega_a \left(U_{ai}^{(n)} q^i\right)^2 \\ &= \sum_a q^i \left(U_{ai}^{(n)} \omega_a U_{aj}^{(n)}\right) q^j. \end{aligned} \qquad \text{(E.-177)}$$

Gaussian integral with source

If the matrix A is symmetric and strictly positive, prove that

$$\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x + J^T x\right) = \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2} \exp\left(\frac{1}{2}J^T A^{-1} J\right) \qquad \text{(E.-177)}$$

where $x^T A x = \sum_{i,j=1}^{n} x_i A_{ij} x_j$ and $J^T x = \sum_{i=1}^{n} J_i x_i$

Proof:

Make a change of variables

$$x = x' + A^{-1}J \qquad \text{(E.-177)}$$

then

$$-\frac{1}{2}x^T A x + J^T x = -\frac{1}{2}x'^T A x' + \frac{1}{2}J^T A^{-1} J \qquad \text{(E.-177)}$$

and

$$\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x + J^T x\right) = \int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x\right) \exp\left(\frac{1}{2}J^T A^{-1} J\right). \qquad \text{(E.-177)}$$

We need to evaluate the integral

$$\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x\right). \qquad \text{(E.-177)}$$

Let O be an orthogonal transformation ($OO^T = \mathbb{1}$) diagonalising A:

$$A = O^T D O, \qquad D = \begin{pmatrix} d_1 & & & \\ & d_1 & & 0 \\ 0 & & \ddots & \\ & & & d_n \end{pmatrix}, \qquad d_i > 0 \quad \text{for all } i. \qquad \text{(E.-177)}$$

Make the following change of variables with unit Jacobian:

$$x' = Ox \qquad (\det O = 1), \qquad \text{(E.-177)}$$

then

$$\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x\right) = \int \prod_{i=1}^{n} dx'_i \exp\left(-\frac{1}{2}x'^T D x'\right). \tag{E.-177}$$

The integral is the product of n independent gaussian integrals, and is given by

$$(2\pi)^{2/n} \prod_{i=1}^{n} d_i = \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2}. \tag{E.-177}$$

Thus

$$\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x + J^T x\right) = \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2} \exp\left(\frac{1}{2}J^T A^{-1} J\right). \tag{E.-177}$$

Note that this implies:

$$\int \prod_{i=1}^{n} dx_i \exp\left(-x^T M x + J^T x\right) = \left(\det\left(\frac{M}{\pi}\right)\right)^{-1/2} \exp\left(\frac{1}{4}J^T M^{-1} J\right). \tag{E.-177}$$

Projection for a chain of oscillators split into two regions.

Prove

$$\mathcal{P} =< 0|P_{0loc}|0 >= \left(\prod_{a=1}^{n}\sqrt{\frac{\omega_a}{\pi}}\prod_{a=1}^{n_1}\sqrt{\frac{\tilde{\omega}_a}{\pi}}\right)\left(\det\left(\frac{A}{2\pi}\right)\right)^{-1}\left(\det\left(\frac{B}{2\pi}\right)\right)^{-1/2} \qquad \text{(E.-177)}$$

where A and B are defined by

$$\begin{aligned} A_{ij} &:= D^{(n_1)}_{ij} + D^{(n)}_{ij}, \qquad i,j = 1,\dots,n_1; \\ B_{kl} &:= 2D^{(n)}_{kl} - 2\left(D^{(n)}A^{-1}D^{(n)}\right)_{kl}, \qquad k,l = n_1+1,\dots,n. \end{aligned} \qquad \text{(E.-177)}$$

Proof:

Denote by $|n_{\{I\}} >= |n^{(2)}_1,\dots,n^{(2)}_n >$ the eigenmodes of H_2 from (E.-86), then in analogy to (E.-129)

$$\begin{aligned} < 0|P_{0loc}|0 > &= \int d\mathbf{q} \;< 0|\mathbf{q} > \sum_{\{I\}}\big(< \mathbf{q}_1| \otimes < \mathbf{q}_2|\big)\big(|0 >_1 \otimes |n_{\{I\}} >\big) \\ &\quad \int d\mathbf{q}'\big({}_1{<}0| \otimes < n_{\{I\}}|\big)\big(|\mathbf{q}'_1 > \otimes |\mathbf{q}'_2 >\big) < \mathbf{q}'|0 > \\ &= \int d\mathbf{q}\; \psi^*_0(\mathbf{q}) < \mathbf{q}_1|0 >_1 \int d\mathbf{q}' \\ &\quad {}_1{<}0|\mathbf{q}'_1 > \Big(\sum_{\{I\}} < \mathbf{q}_2|n_{\{I\}} >< n_{\{I\}}|\mathbf{q}'_2 >\Big) < \mathbf{q}'|0 > \\ &= \int d\mathbf{q}\; \psi^*_0(\mathbf{q}) < \mathbf{q}_1|0 >_1 \int d\mathbf{q}'_1 d\mathbf{q}'_2 \;{}_1{<}0|\mathbf{q}'_1 > \Big(\delta(\mathbf{q}_2 - \mathbf{q}'_2)\Big) < \mathbf{q}'_1, \mathbf{q}'_2|0 > \\ &= \int d\mathbf{q}_1 d\mathbf{q}_2\; \psi^*_0(\mathbf{q}) < \mathbf{q}_1|0 >_1 \int d\mathbf{q}'_1 \;{}_1{<}0|\mathbf{q}'_1 >< \mathbf{q}'_1, \mathbf{q}_2|0 > \\ &= \int d\mathbf{q}_2 \left(\int d\mathbf{q}_1 < \mathbf{q}_1|0 >_1 \psi^*_0(\mathbf{q}_1;\mathbf{q}_2)\right)\left(\int d\mathbf{q}'_1 \;{}_1{<}0|\mathbf{q}'_1 >< \mathbf{q}'_1, \mathbf{q}_2|0 >\right) \end{aligned}$$

(E.-183)

The global vacuum wavefunction, given by (E.13), is

$$\langle \mathbf{q}|0\rangle = \prod_{a=1}^{n}\left(\frac{\omega_a}{\pi}\right) e^{-\frac{1}{2}q^T D^{(n)} q}. \qquad \text{(E.-183)}$$

The local vacuum wavefunction is obviously

$$\langle \mathbf{q}|0\rangle_1 = \prod_{a=1}^{n_1}\left(\frac{\tilde{\omega}_a}{\pi}\right) e^{-\frac{1}{2}q^T D^{(n_1)} q} \tag{E.-183}$$

where $\tilde{\omega}_a$ is the eigenfrequencies of H_1. Inserting these wavefunctions,

$$\begin{aligned} < 0|P_{0loc}|0 > \quad = \quad & \int d\mathbf{q}_2 \int d\mathbf{q}_1 \left(\prod_{a=1}^{n}\left(\frac{\omega_a}{\pi}\right)^{1/4}\right) e^{-\frac{1}{2}q^T D^{(n)} q} \left(\prod_{a=1}^{n_1}\left(\frac{\tilde{\omega}_a}{\pi}\right)^{1/4}\right) e^{-\frac{1}{2}q_1^T D^{(n_1)} q_1} \\ & \int d\mathbf{q}_1' \left(\prod_{a=1}^{n_1}\left(\frac{\tilde{\omega}_a}{\pi}\right)^{1/4}\right) e^{-\frac{1}{2}\sum_{i,j=1}^{n_1} q_1'^i D_{ij}^{(n_1)} q_1'^j} \\ & \left(\prod_{a=1}^{n}\left(\frac{\omega_a}{\pi}\right)^{1/4}\right) e^{-\frac{1}{2}\sum_{i,j=1}^{n_1} q_1'^i D_{ij}^{(n)} q_1'^j - \sum_{i=n_1+1}^{n}\sum_{j=1}^{n_1} q_2^i D_{ij}^{(n)} q_1'^j - \frac{1}{2}\sum_{i,j=n_1+1}^{n} q_2^i D_{ij}^{(n)} q_2^j} \end{aligned} \tag{E.-185}$$

Let us perform the $\mathbf{q}_1'$ integration using the result (E.13):

$$\begin{aligned} & \int d\mathbf{q}_1'\ {}_1<0|\mathbf{q}_1' >< \mathbf{q}_1', \mathbf{q}_2|0 >= \\ & = e^{-\frac{1}{2}\sum_{k,l=n_1+1}^{n} q_2^k D_{kl}^{(n)} q_2^l} \int d\mathbf{q}_1' e^{-\frac{1}{2}\sum_{i,j=1}^{n_1} q_1'^i \left(D_{ij}^{(n_1)} + D_{ij}^{(n)}\right) q_1'^j + \sum_{j=1}^{n_1}\left(-\sum_{i=n_1+1}^{n} q_2^i D_{ij}^{(n)}\right) q_1'^j} \\ & = e^{-\frac{1}{2}\sum_{k,l=n_1+1}^{n} q_2^k D_{kl}^{(n)} q_2^l} \int d\mathbf{q}_1' e^{-\frac{1}{2}\sum_{i,j=1}^{n_1} q_1'^i A_{ij} q_1'^j + \sum_{i=1}^{n_1} J^i q_1'^i} \\ = \quad & \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{k,l=n_1+1}^{n} q_2^k D_{kl}^{(n)} q_2^l\right) \exp\frac{1}{2}J^T A^{-1} J \\ = \quad & \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{k,l=n_1+1}^{n} q_2^k D_{kl}^{(n)} q_2^l\right) \exp\left(\frac{1}{2}\sum_{i,j=1}^{n_1}\Big(\sum_{k=n_1+1}^{n} q_2^k D_{ki}^{(n)}\Big) A_{ij}^{-1}\Big(\sum_{l=n_1+1}^{n} D_{jl}^{(n)} q_2^l\Big)\right) \\ = \quad & \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{k,l=n_1+1}^{n} q_2^k \left(D_{kl}^{(n)} - \sum_{i,j=1}^{n_1} D_{ki}^{(n)} A_{ij}^{-1} D_{jl}^{(n)}\right) q_2^l\right) \end{aligned} \tag{E.-189}$$

Inserting this into (E.-185) gives

$$\begin{aligned} < 0|P_{0loc}|0 > \quad = \quad & \left(\prod_{a=1}^{n}\sqrt{\frac{\omega_a}{\pi}}\prod_{a=1}^{n_1}\sqrt{\frac{\tilde{\omega}_a}{\pi}}\right)\int d\mathbf{q}_2 \int d\mathbf{q}_1\ e^{-\frac{1}{2}q^T D^{(n)} q} e^{-\frac{1}{2}q_1^T D^{(n_1)} q_1} \\ & \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2} e^{-\frac{1}{2}q_2^T\left[D^{(n)} - \left(D^{(n)} A^{-1} D^{(n)}\right)\right] q_2}. \end{aligned} \tag{E.-189}$$

Performing the $\mathbf{q}_1$ integration is identical to the previous integration and so gives

$$
\begin{aligned}
& \int d\mathbf{q}_1 \; \psi_0^*(\mathbf{q}) < \mathbf{q}_1 | 0 >_1 = \\
& = e^{-\frac{1}{2}\sum_{k,l=n_1+1}^{n} q_2^k D_{kl}^{(n)} q_2^l} \int d\mathbf{q}_1 e^{-\frac{1}{2}\sum_{i,j=1}^{n_1} q_1^i \left(D_{ij}^{(n_1)} + D_{ij}^{(n)}\right) q_1^j + \sum_{j=1}^{n_1}\left(-\sum_{i=n_1+1}^{n} q_2^i D_{ij}^{(n)}\right) q_1^j} \\
= \; & \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{k,l=n_1+1}^{n} q_2^k \left(D_{kl}^{(n)} - \sum_{i,j=1}^{n_1} D_{ki}^{(n)} A_{ij}^{-1} D_{jl}^{(n)}\right) q_2^l\right). \qquad \text{(E.-190)}
\end{aligned}
$$

Inserting this into (E.-189) we obtain

$$
\begin{aligned}
< 0 | P_{0loc} | 0 > \; &= \; \left(\prod_{a=1}^{n} \sqrt{\frac{\omega_a}{\pi}} \prod_{a=1}^{n_1} \sqrt{\frac{\tilde{\omega}_a}{\pi}}\right) \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1} \int d\mathbf{q}_2 \; e^{-\frac{1}{2} q_2^T \left[2D^{(n)} - 2\left(D^{(n)} A^{-1} D^{(n)}\right)\right] q_2} \\
&= \; \left(\prod_{a=1}^{n} \sqrt{\frac{\omega_a}{\pi}} \prod_{a=1}^{n_1} \sqrt{\frac{\tilde{\omega}_a}{\pi}}\right) \left(\det\left(\frac{A}{2\pi}\right)\right)^{-1} \left(\det\left(\frac{B}{2\pi}\right)\right)^{-1/2} \qquad \text{(E.-190)}
\end{aligned}
$$

where we have used the result (E.13) again but, this time, with $J = 0$.

$$\boxed{}$$

Repeat the previous for the case $n_1 = 1$ and $n = 2$ for $\omega = 1$.

Proof:

First

$$(U^{(2)}_{ai}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{E.-190}$$

so that

$$(D^{(2)}_{ij}) = (\sum_a U^{(2)}_{ai}\omega_a U^{(2)}_{aj}) = \frac{1}{2}\begin{pmatrix} \omega_1+\omega_2 & \omega_1-\omega_2 \\ \omega_1-\omega_2 & \omega_1+\omega_2 \end{pmatrix}, \tag{E.-190}$$

where

$$\omega_1^2 = 1+\lambda, \qquad \omega_2^2 = 1-\lambda. \tag{E.-190}$$

Whereas for the $D^{(n_1)}$ there is just the one component

$$(D^{(n_1)}_{11}) = 1. \tag{E.-190}$$

Then

$$\begin{aligned} < 0|P_{0loc}|0 > &= \int d\mathbf{q}_1 d\mathbf{q}_2\ \psi_0^*(\mathbf{q}) < \mathbf{q}_1|0 >_1 \int d\mathbf{q}'_1\ {}_1{<}0|\mathbf{q}'_1 >< \mathbf{q}'_1, \mathbf{q}_2|0 > \\ &= \int d\mathbf{q}_1 d\mathbf{q}_2\ \left(\frac{\omega_1}{\pi}\right)^{1/4}\left(\frac{\omega_2}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}(q_1,q_2)\frac{1}{2}\begin{pmatrix} \omega_1+\omega_2 & \omega_1-\omega_2 \\ \omega_1-\omega_2 & \omega_1+\omega_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right) \\ &\left(\frac{1}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}q_1^2\right)\int d\mathbf{q}'_1\ \left(\frac{1}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}q_1^{'2}\right) \\ &\quad \left(\frac{\omega_1}{\pi}\right)^{1/4}\left(\frac{1}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}(q'_1,q_2)\frac{1}{2}\begin{pmatrix} \omega_1+\omega_2 & \omega_1-\omega_2 \\ \omega_1-\omega_2 & \omega_1+\omega_2 \end{pmatrix}\begin{pmatrix} q'_1 \\ q_2 \end{pmatrix}\right) \end{aligned} \tag{E.-192}$$

Let us perform the q'_1 integration

$$\begin{aligned}
\int d\mathbf{q}_1' e^{-\frac{1}{2}q_1'\left(D_{11}^{(n_1)}+D_{11}^{(n)}\right)q_1'+\left(-q_2 D_{21}^{(n)}\right)q_1'} &= \int d\mathbf{q}_1' e^{-\frac{1}{2}q_1' A_{11} q_1'+\left(-q_2 D_{21}^{(n)}\right)q_1'} \\
&= \int d\mathbf{q}_1' e^{-\frac{1}{2}A_{11}\left(q_1'^2-2q_2 D_{21}^{(n)} q_1'/A_{11}\right)} \\
&= \int d\mathbf{q}_1' e^{-\frac{1}{2}A_{11}q_1'^2+\frac{1}{2}q_2\left(D_{21}^{(n)}\frac{1}{A_{11}}D_{21}^{(n)}\right)q_2} \\
&= \left(\frac{A_{11}}{2\pi}\right)^{-1/2} e^{\frac{1}{2}q_2\left(D_{21}^{(n)}\frac{1}{A_{11}}D_{12}^{(n)}\right)q_2} \qquad \text{(E.-194)}
\end{aligned}$$

where we have defined

$$A_{11} = D_{11}^{(n_1)} + D_{11}^{(n)}. \qquad \text{(E.-194)}$$

Inserting this into (E.-192) we obtain

$$\begin{aligned}
< 0|P_{0loc}|0 > &= \left(\sqrt{\frac{\omega_1}{\pi}}\sqrt{\frac{\omega_2}{\pi}}\sqrt{\frac{1}{\pi}}\right)\int d\mathbf{q}_1 d\mathbf{q}_2\ \exp\left(-\frac{1}{2}(q_1,q_2)\frac{1}{2}\begin{pmatrix}\omega_1+\omega_2 & \omega_1-\omega_2\\ \omega_1-\omega_2 & \omega_1+\omega_2\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right) \\
&\quad \exp\left(-\frac{1}{2}q_1^2\right)\left(\frac{A_{11}}{2\pi}\right)^{-1/2}\exp-\left(\frac{1}{2}q_2\left(D_{22}^{(2)}-D_{21}^{(2)}\frac{1}{A_{11}}D_{12}^{(2)}\right)q_2\right) \\
&= \left(\sqrt{\frac{\omega_1}{\pi}}\sqrt{\frac{\omega_2}{\pi}}\sqrt{\frac{1}{\pi}}\right)\int d\mathbf{q}_2 e^{-\frac{1}{2}q_2 D_{22}^{(2)}q_2}\int d\mathbf{q}_1 e^{-\frac{1}{2}q_1\left(D_{11}^{(n_1)}+D_{11}^{(n)}\right)q_1+\left(-q_2 D_{21}^{(n)}\right)q_1} \\
&\quad \left(\frac{A_{11}}{2\pi}\right)^{-1/2}\exp-\left(\frac{1}{2}q_2\left(D_{22}^{(2)}-D_{21}^{(2)}\frac{1}{A_{11}}D_{12}^{(2)}\right)q_2\right) \\
&= \left(\sqrt{\frac{\omega_1}{\pi}}\sqrt{\frac{\omega_2}{\pi}}\sqrt{\frac{1}{\pi}}\right)\left(\frac{A_{11}}{2\pi}\right)^{-1}\int d\mathbf{q}_2 \exp-\left(\frac{1}{2}q_2\left(2D_{22}^{(2)}-2D_{21}^{(2)}\frac{1}{A_{11}}D_{12}^{(2)}\right)q_2\right) \\
&= \left(\sqrt{\frac{\omega_1}{\pi}}\sqrt{\frac{\omega_2}{\pi}}\sqrt{\frac{1}{\pi}}\right)\left(\frac{A_{11}}{2\pi}\right)^{-1}\int d\mathbf{q}_2 \exp-\left(\frac{1}{2}q_2 B_{22} q_2\right) \\
&= \left(\sqrt{\frac{\omega_1}{\pi}}\sqrt{\frac{\omega_2}{\pi}}\sqrt{\frac{1}{\pi}}\right)\left(\frac{A_{11}}{2\pi}\right)^{-1}\left(\frac{B_{22}}{2\pi}\right)^{-1/2} \\
&= \sqrt{\omega_1\omega_2}\left(\frac{A_{11}}{2}\right)^{-1}\left(\frac{B_{22}}{2}\right)^{-1/2} \qquad \text{(E.-200)}
\end{aligned}$$

where we have defined

$$B_{22} = 2D_{22}^{(2)} - 2D_{21}^{(2)}\frac{1}{A_{11}}D_{12}^{(2)}. \qquad \text{(E.-200)}$$

We have

$$
\begin{aligned}
\sqrt{\omega_1\omega_2} &= \left(1-\frac{1}{4}\lambda^2+\mathcal{O}(\lambda^4)\right) \\
&= \left(1-\frac{n_1+n-2}{4}\lambda^2+\mathcal{O}(\lambda^4)\right). \qquad \text{(E.-200)}
\end{aligned}
$$

Now

$$
\begin{aligned}
\left(\frac{A_{11}}{2}\right)^{-1} &= 4(\omega_1+\omega_2+2)^{-1} \\
&= 4\left(2+(1+\lambda)^{1/2}+(1-\lambda)^{1/2}\right)^{-1} \\
&= \left(1-\frac{1}{16}\lambda^2+\mathcal{O}(\lambda^4)\right)^{-1} \\
&= 1+\frac{1}{16}\lambda^2+\mathcal{O}(\lambda^4) \\
&= 1+\frac{8n_1-7}{16}\lambda^2+\mathcal{O}(\lambda^4) \qquad \text{(E.-203)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\frac{B_{22}}{2}\right)^{-1/2} &= \left[D_{22}^{(2)}-D_{21}^{(2)}\frac{1}{A_{11}}D_{12}^{(2)}\right]^{-1/2} \\
&= \left[\frac{1}{2}(\omega_1+\omega_2)-\frac{1}{2}(\omega_1-\omega_2)\left(\frac{1}{A_{11}}\right)\frac{1}{2}(\omega_1-\omega_2)\right]^{-1/2} \\
&= \left[1-\frac{1}{8}\lambda^2+\mathcal{O}(\lambda^4)-\frac{1}{4}(\lambda+\mathcal{O}(\lambda^3))\left(\frac{1}{2}+\mathcal{O}(\lambda^2)\right)(\lambda+\mathcal{O}(\lambda^3))\right]^{-1/2} \\
&= \left[1-\frac{1}{4}\lambda^2+\mathcal{O}(\lambda^4)\right]^{-1/2} \\
&= 1+\frac{1}{8}\lambda^2+\mathcal{O}(\lambda^4) \\
&= 1+\left(\frac{n_2}{4}-\frac{1}{8}\right)\lambda^2+\mathcal{O}(\lambda^4) \qquad \text{(E.-207)}
\end{aligned}
$$

Putting it all together, we obtain

$$
\begin{aligned}
<0|P_{0loc}|0> &= \left(1-\frac{1}{4}\lambda^2+\mathcal{O}(\lambda^4)\right)\left(1+\frac{1}{16}\lambda^2+\mathcal{O}(\lambda^4)\right)\left(1+\frac{1}{8}\lambda^2+\mathcal{O}(\lambda^4)\right) \\
&= 1-\frac{\lambda^2}{16}+\mathcal{O}(\lambda^4). \qquad \text{(E.-207)}
\end{aligned}
$$

Expansions in λ

Given

$$\begin{aligned}
\tilde{\omega}_a &= \sqrt{1+2\lambda\cos\theta_{n_1}}, \quad \text{where } \tilde{\theta}_a = \frac{a\pi}{n_1+1},\ a=1,\ldots,n_1, \\
A_{ij} &:= D^{(n_1)}_{ij} + D^{(n)}_{ij}, \qquad i,j=1,\ldots,n_1; \\
B_{kl} &:= 2D^{(n)}_{kl} - 2\left(D^{(n)}A^{-1}D^{(n)}\right)_{kl}, \qquad k,l=n_1+1,\ldots,n
\end{aligned} \tag{E.-208}$$

establish the following expansions in λ:

$$\prod_{a=1}^{n_1}\sqrt{\tilde{\omega}_a} \approx 1-\lambda^2\frac{n_1-1}{4} \tag{E.-207}$$

$$\left(\det\left(\frac{A}{2}\right)\right)^{-1} \approx 1+\lambda^2\left(\frac{n_1}{2}-\frac{7}{16}\right) \tag{E.-206}$$

$$\left(\det\left(\frac{B}{2}\right)\right)^{-1/2} \approx 1+\lambda^2\left(\frac{n_2}{4}-\frac{1}{8}\right) \tag{E.-205}$$

Hint:

$$\begin{aligned}
\det\left(\mathbb{1}+h\right) &= \exp\left\{\ln\det\left(\mathbb{1}+h\right)\right\} \\
&= \exp\left\{\operatorname{tr}\ln\left(\mathbb{1}+h\right)\right\} \\
&= \exp\left\{\operatorname{tr}\left(h-\frac{h^2}{2}+\mathcal{O}(h^3)\right)\right\} \\
&= \exp\left\{\left(\operatorname{tr}h-\frac{1}{2}\operatorname{tr}h^2+\mathcal{O}(h^3)\right)\right\}.
\end{aligned} \tag{E.-207}$$

Proof:

Part (a) Expression (E.-207).

$$
\begin{aligned}
\prod_{a=1}^{n_1}\sqrt{\tilde{\omega}_a} &= \prod_{a=1}^{n_1}\left(1+2\lambda\cos\left(\frac{a\pi}{n_1+1}\right)\right)^{1/4} \\
&= \prod_{a=1}^{n_1}\left(1+\frac{\lambda}{2}\cos\left(\frac{a\pi}{n_1+1}\right)-\frac{3}{8}\lambda^2\cos^2\left(\frac{a\pi}{n_1+1}\right)+\mathcal{O}(\lambda^3)\right) \\
&= 1+\frac{\lambda}{2}\sum_{a=1}^{n_1}\cos\left(\frac{a\pi}{n_1+1}\right)+\frac{\lambda^2}{4}\sum_{a<b=1}^{n_1}\cos\left(\frac{a\pi}{n_1+1}\right)\cos\left(\frac{b\pi}{n_1+1}\right) \\
&\quad -\frac{3}{8}\lambda^2\sum_{a=1}^{n_1}\cos^2\left(\frac{a\pi}{n_1+1}\right)+\mathcal{O}(\lambda^3)
\end{aligned}
\tag{E.-209}
$$

Let us take each term at a time

$$
\begin{aligned}
\frac{\lambda}{2}\sum_{a=1}^{n_1}\cos\left(\frac{a\pi}{n_1+1}\right) &= \frac{\lambda}{4}\left(\sum_{a=1}^{n_1}\exp\left(i\frac{a\pi}{n_1+1}\right)+\sum_{a=1}^{n_1}\exp\left(-i\frac{a\pi}{n_1+1}\right)\right) \\
&= \frac{\lambda}{4}\left(\frac{\exp\left(i\frac{\pi}{n_1+1}\right)+1}{1-\exp\left(i\frac{\pi}{n_1+1}\right)}+\frac{1+\exp\left(i\frac{\pi}{n_1+1}\right)}{\exp\left(i\frac{\pi}{n_1+1}\right)-1}\right) \\
&= 0.
\end{aligned}
\tag{E.-210}
$$

A result that should have been obvious. Next

$$
\begin{aligned}
&\frac{\lambda^2}{4}\sum_{a<b=1}^{n_1}\cos\left(\frac{a\pi}{n_1+1}\right)\cos\left(\frac{b\pi}{n_1+1}\right) = \\
&= \frac{1}{2}\left\{\frac{\lambda^2}{4}\sum_{a,b=1}^{n_1}\cos\left(\frac{a\pi}{n_1+1}\right)\cos\left(\frac{b\pi}{n_1+1}\right)-\frac{\lambda^2}{4}\sum_{a=1}^{n_1}\cos^2\left(\frac{a\pi}{n_1+1}\right)\right\} \\
&= \frac{\lambda^2}{8}\sum_{a=1}^{n_1}\cos\left(\frac{a\pi}{n_1+1}\right)\sum_{b=1}^{n_1}\cos\left(\frac{b\pi}{n_1+1}\right)-\frac{\lambda^2}{8}\sum_{a=1}^{n_1}\cos^2\left(\frac{a\pi}{n_1+1}\right) \\
&= -\frac{\lambda^2}{8}\sum_{a=1}^{n_1}\cos^2\left(\frac{a\pi}{n_1+1}\right) \\
&= -\frac{\lambda^2}{16}\sum_{a=1}^{n_1}\left\{1+\cos\left(\frac{a2\pi}{n_1+1}\right)\right\} \\
&= -\frac{\lambda^2}{16}(n_1-1)
\end{aligned}
\tag{E.-214}
$$

where we used (E.-210), and a result from (E.-174) in the last line. The last term in (E.-209) is then

$$-\frac{3}{8}\lambda^2\sum_{a=1}^{n_1}\cos^2\left(\frac{a\pi}{n_1+1}\right) = -\frac{3}{16}\lambda^2(n_1-1). \tag{E.-214}$$

Therefore, putting these results together, we have

$$\prod_{a=1}^{n_1}\sqrt{\tilde{\omega}_a} \approx 1-\lambda^2\frac{n_1-1}{4}. \tag{E.-214}$$

Part (b) Expression (E.-206).

$$\begin{aligned}
D_{ij}^{(n_1)} &= \sum_{a=1}^{n_1}\sqrt{\frac{2}{n_1+2}}\sin\left(\frac{ai\pi}{n_1+1}\right)\sqrt{1+2\lambda\cos\left(\frac{a\pi}{n_1+1}\right)}\sqrt{\frac{2}{n_1+2}}\sin\left(\frac{aj\pi}{n_1+1}\right)\\
&= \frac{2}{n_1+2}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right)\\
&\quad +\frac{2}{n_1+2}\lambda\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\cos\left(\frac{a\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right)\\
&\quad -\frac{2}{n_1+2}\frac{1}{2}\lambda^2\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\cos^2\left(\frac{a\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right)+\mathcal{O}(\lambda^3). \qquad \text{(E.-216)}
\end{aligned}$$

First

$$\begin{aligned}
\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right) &= \delta_{ij}\quad \text{for } i,j=1,\ldots,n_1; \qquad \text{(E.-215)}\\
\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right) &= \delta_{ij}\quad \text{for } i,j=1,\ldots,n. \qquad \text{(E.-214)}
\end{aligned}$$

which easily follows from (E.13), (E.13) and (E.-174) (where here we have swapped around the roles of i,j and a,b, which we can obviously do). So $D^{(n_1)} = \mathbb{1}_{n_1}+\mathcal{O}(\lambda)$. Also, $D^{(n)} = \mathbb{1}+\mathcal{O}(\lambda)$. We can write

$$D^{(n_1)} = \mathbb{1}_{n_1}+\lambda M^{(n_1)}-\frac{1}{2}\lambda^2\tilde{M}^{(n_1)}+\mathcal{O}(\lambda^3), \tag{E.-213}$$

$$D^{(n)} = \mathbb{1}_{n_1}+\lambda M^{(n)}-\frac{1}{2}\lambda^2\tilde{M}^{(n)}+\mathcal{O}(\lambda^3) \tag{E.-212}$$

(here in part (b) the $D^{(n)}$ is understood as a matrix whose indices $i,j=1,\ldots,n_1$) where

$$M_{ij}^{(n_1)} = \frac{2}{n_1+1}\sum_{a=1}^{n_1} \sin\left(\frac{ai\pi}{n_1+1}\right)\cos\left(\frac{a\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right), \tag{E.-211}$$

$$\tilde{M}_{ij}^{(n_1)} = \frac{2}{n_1+1}\sum_{a=1}^{n_1} \sin\left(\frac{ai\pi}{n_1+1}\right)\cos^2\left(\frac{a\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right), \tag{E.-210}$$

$$M_{ij}^{(n)} = \frac{2}{n+1}\sum_{a=1}^{n} \sin\left(\frac{ai\pi}{n+1}\right)\cos\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right), \tag{E.-209}$$

$$\tilde{M}_{ij}^{(n)} = \frac{2}{n+1}\sum_{a=1}^{n} \sin\left(\frac{ai\pi}{n+1}\right)\cos^2\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right). \tag{E.-208}$$

From the definition of the matrix A, we have

$$\begin{aligned} \frac{A}{2} &= \frac{1}{2}(D^{(n_1)} + D^{(n)}) \\ &= \mathbb{1}_{n_1} + \frac{1}{2}\lambda\left(M^{(n_1)} + M^{(n)}\right) - \frac{1}{4}\lambda^2\left(\tilde{M}^{(n_1)} + \tilde{M}^{(n)}\right) + \mathcal{O}(\lambda^3) \\ &=: (\mathbb{1}_{n_1} + h). \end{aligned} \tag{E.-209}$$

A similar calculation to (E.-207) gives

$$\begin{aligned} \left(\det\left(\frac{A}{2}\right)\right)^{-1} &= (\det(\mathbb{1}_{n_1} + h))^{-1} \\ &= \exp\left\{-\ln\det\left((\mathbb{1}_{n_1} + h)\right)\right\} \\ &= \exp\left\{-\mathrm{tr}_{n_1}\ln\left(\mathbb{1}_{n_1} + h\right)\right\} \\ &= \exp\left\{-\mathrm{tr}_{n_1}\left(h - \frac{h^2}{2} + \mathcal{O}(h^3)\right)\right\} \\ &= \exp\left\{-\mathrm{tr}_{n_1}h + \frac{1}{2}\mathrm{tr}_{n_1}h^2 + \mathcal{O}(h^3)\right\} \\ &= \left(1 - \mathrm{tr}_{n_1}h + \frac{1}{2}\mathrm{tr}_{n_1}h^2 + \frac{1}{2!}(\mathrm{tr}_{n_1}h)^2 + \mathcal{O}(h^3)\right). \end{aligned} \tag{E.-214}$$

The matrix h defined in (E.-209) is

$$h = \frac{1}{2}\lambda\left(M^{(n_1)} + M^{(n)}\right) - \frac{1}{4}\lambda^2\left(\tilde{M}^{(n_1)} + \tilde{M}^{(n)}\right) + \mathcal{O}(\lambda^3). \tag{E.-213}$$

Inserting this expression for h into (E.-214) we have

$$\left(\det\left(\frac{A}{2}\right)\right)^{-1} = \left(1 - \mathrm{tr}_{n_1}h + \frac{1}{2}\mathrm{tr}_{n_1}h^2 + \frac{1}{2}(\mathrm{tr}_{n_1}h)^2 + \mathcal{O}(h^3)\right) =$$
$$= 1 - \frac{1}{2}\lambda\left(\mathrm{tr}_{n_1}M^{(n_1)} + \mathrm{tr}_{n_1}M^{(n)}\right) + \frac{1}{4}\lambda^2\left(\mathrm{tr}_{n_1}\tilde{M}^{(n_1)} + \mathrm{tr}_{n_1}\tilde{M}^{(n)}\right)$$
$$+\frac{1}{8}\lambda^2\mathrm{tr}_{n_1}\left(M^{(n_1)} + M^{(n)}\right)^2 + \frac{1}{8}\lambda^2\left(\mathrm{tr}_{n_1}M^{(n_1)} + \mathrm{tr}_{n_1}M^{(n)}\right)^2 + \mathcal{O}(\lambda^3). \quad \text{(E.-214)}$$

In the following (E.-214) will be extensively employed.

First, from

$$\begin{aligned} M_{ij}^{(n_1)} &= \frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\cos\left(\frac{a\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right) \\ &= \frac{1}{2}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\left\{\sin\left(\frac{a(j-1)\pi}{n_1+1}\right) + \sin\left(\frac{a(j+1)\pi}{n_1+1}\right)\right\} \\ &= \frac{1}{2}(\delta_{i,j-1} + \delta_{i,j+1}) \end{aligned} \quad \text{(E.-215)}$$

(where $\delta_{i,0} = \delta_{i,n_1+1} = 0$) we obviously have

$$\begin{aligned} \mathrm{tr}_{n_1}M^{(n_1)} &= \frac{1}{2}\sum_{i=1}^{n_1}(\delta_{i,i-1} + \delta_{i,i+1}) \\ &= 0. \end{aligned} \quad \text{(E.-215)}$$

Consider

$$\begin{aligned} M_{ij}^{(n)} &= \frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\cos\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right) \\ &= \frac{1}{2}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\left(\sin\left(\frac{a(j-1)\pi}{n+1}\right) + \sin\left(\frac{a(j+1)\pi}{n+1}\right)\right) \\ &= \frac{1}{2}(\delta_{i,j-1} + \delta_{i,j+1}) \end{aligned} \quad \text{(E.-216)}$$

(where $\delta_{i,0} = 0$) then

$$\begin{aligned} \mathrm{tr}_{n_1}M^{(n)} &= \frac{1}{2}\sum_{i=1}^{n_1}(\delta_{i,i-1} + \delta_{i,i+1}) \\ &= 0. \end{aligned} \quad \text{(E.-216)}$$

Substituting these results into (E.-216) we see that $\mathrm{tr}_{n_1} M^{(n)} = 0$. The expression (E.-214) then simplifies to

$$\left(\det\left(\frac{A}{2}\right)\right)^{-1} = 1 + \frac{1}{4}\lambda^2\left(\mathrm{tr}_{n_1}\tilde{M}^{(n_1)} + \mathrm{tr}_{n_1}\tilde{M}^{(n)}\right) + \frac{1}{8}\lambda^2 \mathrm{tr}_{n_1}\left(M^{(n_1)} + M^{(n)}\right)^2 + \mathcal{O}(\lambda^3). \quad \text{(E.-216)}$$

Now we turn to the matrix (E.-210). We will consider the special case first where $n_1 = 1$. Then $\mathrm{tr}_{n_1}\tilde{M}^{(n_1)}$ is given by

$$\begin{aligned}
\mathrm{tr}_{n_1}\tilde{M}^{(n_1)} &= \sum_{i=1}^{1} \frac{2}{1+1} \sin\left(\frac{i\pi}{1+1}\right)\cos^2\left(\frac{\pi}{1+1}\right)\sin\left(\frac{i\pi}{1+1}\right) \\
&= \frac{2}{1+1}\sin\left(\frac{\pi}{1+1}\right)\cos^2\left(\frac{\pi}{1+1}\right)\sin\left(\frac{\pi}{1+1}\right) \\
&= 0.
\end{aligned} \quad \text{(E.-217)}$$

Now assume $n_1 \neq 1$. We calculate

$$\begin{aligned}
\tilde{M}^{(n_1)}_{ij} &= \frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\cos^2\left(\frac{a\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right) \\
&= \frac{1}{2}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\left\{1+\cos\left(\frac{2a\pi}{n_1+1}\right)\right\}\sin\left(\frac{aj\pi}{n_1+1}\right) \\
&= \frac{1}{2}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{aj\pi}{n_1+1}\right) \\
&\quad + \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\left\{\sin\left(\frac{a(j-2)\pi}{n_1+1}\right)+\sin\left(\frac{a(j+2)\pi}{n_1+1}\right)\right\} \\
&= \frac{1}{2}\delta_{i,j} + \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{a(j-2)\pi}{n_1+1}\right) \\
&\quad + \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{a(j+2)\pi}{n_1+1}\right)
\end{aligned} \quad \text{(E.-222)}$$

We now look at the remaining summations. Consider the first summation. For $1 \leq j-2$ equations (E.-215) apply. For $j = 2$ this summation obviously vanishes. For $j = 1$ we have:

$$\frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{-a\pi}{n_1+1}\right) = -\frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{a\pi}{n_1+1}\right) = -\frac{1}{4}\delta_{i,1} \qquad \text{(E.-222)}$$

by (E.-215) - this will contribute $-\frac{1}{4}\delta_{i,1}\delta_{j,1}$.

We now consider the second summation. For $j+2 \leq n_1$ equations (E.-215) apply. For $j = n_1 - 1$ the summation obviously vanishes. The case $j = n_1$ needs special consideration. First consider the case where $i = n_1$, then

$$\begin{aligned} & \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{an_1\pi}{n_1+1}\right)\sin\left(\frac{a(n_1+2)\pi}{n_1+1}\right) \\ = & \frac{1}{4}\frac{1}{2}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\left\{\cos\left(\frac{a2\pi}{n_1+1}\right)-\cos\left(\frac{a2(n_1+1)\pi}{n_1+1}\right)\right\} \\ = & \frac{1}{4}\frac{1}{2}\frac{2}{n_1+1}\left\{-1-\sum_{a=1}^{n_1}\cos(a2\pi)\right\} \\ = & -\frac{1}{4}\delta_{i,n_1}. \end{aligned} \qquad \text{(E.-224)}$$

This will contribute $-\frac{1}{4}\delta_{i,n_1}\delta_{j,n_1}$. Now consider the case where $i < n_1$. We write

$$\begin{aligned} & \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{ai\pi}{n_1+1}\right)\sin\left(\frac{a(n_1+2)\pi}{n_1+1}\right) \\ = & \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\left\{\cos\left(\frac{a(n_1-i+2)\pi}{n_1+1}\right)-\cos\left(\frac{a(n_1+i+2)\pi}{n_1+1}\right)\right\}. \end{aligned} \qquad \text{(E.-224)}$$

We now prove that for $1 < i < n_1$

$$\sum_{a=1}^{n_1}\cos\left(\frac{a(n_1-i+2)\pi}{n_1+1}\right) = \begin{cases} 0 & n_1 - i \text{ even} \\ 1 & n_1 - i \text{ odd} \end{cases} \qquad \text{(E.-224)}$$

We have

$$
\begin{aligned}
\sum_{a=1}^{n_1} \cos\left(\frac{a(n_1-i+2)\pi}{n_1+1}\right) &= \frac{1}{2}\sum_{a=1}^{n_1}\left\{\exp i\frac{a(n_1-i+2)\pi}{n_1+1} + \exp -i\left(\frac{a(n_1-i+2)\pi}{n_1+1}\right)\right\} \\
&= \frac{1}{2}\sum_{a=1}^{n_1}\left\{\frac{1-\exp i(n_1-i+2)\pi}{1-\exp i\left(\frac{(n_1-i+2)\pi}{n_1+1}\right)} + \frac{1-\exp -i(n_1-i+2)\pi}{1-\exp -i\left(\frac{(n_1-i+2)\pi}{n_1+1}\right)}\right\}
\end{aligned} \tag{E.-225}
$$

This is obviously zero for $n_1 - i$ even. Now say $n_1 - i$ is odd then

$$
\begin{aligned}
\sum_{a=1}^{n_1} \cos\left(\frac{a(n_1-i+2)\pi}{n_1+1}\right) &= \frac{1}{2}\sum_{a=1}^{n_1}\left\{\frac{2}{1-\exp i\left(\frac{(n_1-i+2)\pi}{n_1+1}\right)} + \frac{2}{1-\exp -i\left(\frac{(n_1-i+2)\pi}{n_1+1}\right)}\right\} \\
&= \sum_{a=1}^{n_1}\left\{\frac{1}{1-\exp i\left(\frac{(n_1-i+2)\pi}{n_1+1}\right)} + \frac{\exp i\left(\frac{(n_1-i+2)\pi}{n_1+1}\right)}{\exp i\left(\frac{(n_1-i+2)\pi}{n_1+1}\right)-1}\right\} \\
&= 1.
\end{aligned} \tag{E.-226}
$$

Similar calculations give that for $1 \leq i < n_1$

$$
\sum_{a=1}^{n_1} \cos\left(\frac{a(n_1+i+2)\pi}{n_1+1}\right) = \begin{cases} 0 & n_1 - i \text{ even} \\ 1 & n_1 - i \text{ odd.} \end{cases} \tag{E.-226}
$$

Substituting (E.13) and (E.13) into (E.-224) we have for $1 < i < n_1$ the summation (E.-224) vanishes.

Lastly we calculate the summation (E.-224) for $i = 1$.

$$
\begin{aligned}
& \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\sin\left(\frac{a\pi}{n_1+1}\right)\sin\left(\frac{a(n_1+2)\pi}{n_1+1}\right) \\
=\ & \frac{1}{4}\frac{2}{n_1+1}\sum_{a=1}^{n_1}\left\{\cos(a\pi) - \cos\left(\frac{a(n_1+3)\pi}{n_1+1}\right)\right\} \\
=\ & \frac{1}{4}\frac{2}{n_1+1}\begin{cases} -1-0 & n_1-1 \text{ even} \\ 0-1 & n_1-1 \text{ odd} \end{cases}. \\
=\ & -\frac{1}{2(n_1+1)}
\end{aligned} \tag{E.-228}
$$

This will contribute $-\frac{1}{2(n_1+1)}\delta_{i,1}\delta_{j,n_1}$. Putting it together we obtain

$$\tilde{M}^{(n_1)}_{ij} = \frac{1}{2}\delta_{i,j} + \frac{1}{4}(\delta_{i,j-2}+\delta_{i,j+2}) - \frac{1}{4}\delta_{i,1}\delta_{j,1} - \frac{1}{4}\delta_{i,n_1}\delta_{j,n_1} - \frac{1}{2(n_1+1)}\delta_{i,1}\delta_{j,n_1} \tag{E.-228}$$

(where $\delta_{i,-1}=\delta_{i,0}=\delta_{i,n_1+1}=\delta_{i,n_1+2}=0$) so that

$$\begin{aligned} \mathrm{tr}_{n_1}\tilde{M}^{(n_1)} &= \sum_{i=1}^{n_1}(\frac{1}{2}\delta_{i,i} - \frac{1}{4}\delta_{i,1} - \frac{1}{4}\delta_{i,n_1}) \\ &= \frac{1}{2}(n_1-1), \end{aligned} \tag{E.-228}$$

which combined with the result (E.-217) establishes this result holds in general.

Now we turn to the matrix (E.-208).

$$\begin{aligned} \tilde{M}^{(n)}_{ij} &= \frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\cos^2\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right) \\ &= \frac{1}{2}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\left\{1+\cos\left(\frac{2a\pi}{n+1}\right)\right\}\sin\left(\frac{aj\pi}{n+1}\right) \\ &= \frac{1}{2}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right) \\ &\quad +\frac{1}{4}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\left\{\sin\left(\frac{a(j-2)\pi}{n+1}\right)+\sin\left(\frac{a(j+2)\pi}{n+1}\right)\right\} \\ &= \frac{1}{2}\delta_{i,j} + \frac{1}{4}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\sin\left(\frac{a(j-2)\pi}{n+1}\right) + \\ &\qquad +\frac{1}{4}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\sin\left(\frac{a(j+2)\pi}{n+1}\right) \end{aligned} \tag{E.-232}$$

We now look at the remaining summations. Consider the first summation. For $1 \leq j-2$ equations (E.-214) apply. For $j=2$ this summation obviously vanishes. For $j=1$ we have:

$$\begin{aligned} \frac{1}{4}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\sin\left(\frac{-a\pi}{n+1}\right) &= -\frac{1}{4}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ai\pi}{n+1}\right)\sin\left(\frac{a\pi}{n+1}\right) \\ &= -\frac{1}{4}\delta_{i,1} \end{aligned} \tag{E.-232}$$

by (E.-214) - this will contribute $-\frac{1}{4}\delta_{i,1}\delta_{j,1}$.

We now consider the second summation. For $j+2 \leq n$ equations (E.-214) apply. For $j=n-1$ the summation obviously vanishes. We do not need to consider the case $j=n$ at this point as we are considering $\tilde{M}^{(n)}_{ij}$ for $i,j = 1,\ldots,n_1$ at the moment and $n_1 < n$. We can already calculate the trace $\mathrm{tr}_{n_1}\tilde{M}^{(n)}$. We have

$$\tilde{M}^{(n)}_{ij} = \frac{1}{2}\delta_{i,j} + \frac{1}{4}(\delta_{i,j-2} + \delta_{i,j+2}) - \frac{1}{4}\delta_{i,1}\delta_{j,1} \quad \text{for } i,j = 1,\ldots,n_1 \tag{E.-232}$$

(where $\delta_{i,-1} = \delta_{i,0} = \delta_{i,n+1} = 0$) so that

$$\begin{aligned}\mathrm{tr}_{n_1}\tilde{M}^{(n)} &= \sum_{i=1}^{n_1}(\frac{1}{2}\delta_{i,i} - \frac{1}{4}\delta_{i,1}) \\ &= \frac{1}{4}(2n_1 - 1).\end{aligned} \tag{E.-232}$$

We will need the general expression for $\tilde{M}^{(n)}$ later on. The case $j = n$ works out in a similar way to how $\tilde{M}^{(n_1)}$ worked out. We obtain in general that

$$\tilde{M}^{(n)}_{ij} = \frac{1}{2}\delta_{i,j} + \frac{1}{4}(\delta_{i,j-2} + \delta_{i,j+2}) - \frac{1}{4}\delta_{i,1}\delta_{j,1} - \frac{1}{4}\delta_{i,n}\delta_{j,n} - \frac{1}{2(n+1)}\delta_{i,1}\delta_{j,n} \tag{E.-232}$$

(where $\delta_{i,-1} = \delta_{i,0} = \delta_{i,n+1} = \delta_{i,n+2} = 0$).

Next, we easily see

$$\begin{aligned}((M^{(n_1)})^2)_{ij} &= \sum_{k=1}^{n_1} M^{(n_1)}_{ik} M^{(n_1)}_{kj} \\ &= \left(\frac{2}{n_1+1}\right)^2 \sum_{a,b=1}^{n_1} \sin\left(\frac{ai\pi}{n_1+1}\right) \cos\left(\frac{a\pi}{n_1+1}\right) \sum_{k=1}^{n_1} \sin\left(\frac{ak\pi}{n_1+1}\right) \sin\left(\frac{bk\pi}{n_1+1}\right) \times \\ &\quad \times \cos\left(\frac{b\pi}{n_1+1}\right) \sin\left(\frac{bj\pi}{n_1+1}\right) \\ &= \frac{2}{n_1+1} \sum_{a=1}^{n_1} \sin\left(\frac{ai\pi}{n_1+1}\right) \cos^2\left(\frac{a\pi}{n_1+1}\right) \sin\left(\frac{aj\pi}{n_1+1}\right) \\ &= (\tilde{M}^{(n_1)})_{ij}\end{aligned} \tag{E.-235}$$

and thus

$$\mathrm{tr}_{n_1}((M^{(n_1)})^2) = \mathrm{tr}_{n_1}\tilde{M}^{(n_1)} = \frac{1}{2}\,(n_1 - 1)\,. \tag{E.-235}$$

Next

$$
\begin{aligned}
((M^{(n)})^2)_{ij} &= \sum_{k=1}^{n_1} M_{ik}^{(n)} M_{kj}^{(n)} \\
&= \sum_{k=1}^{n_1} \frac{2}{n+1} \sum_{a=1}^{n} \sin\left(\frac{ai\pi}{n+1}\right) \cos\left(\frac{a\pi}{n+1}\right) \sin\left(\frac{ak\pi}{n+1}\right) \times \\
&\quad \times \frac{2}{n+1} \sum_{b=1}^{n} \sin\left(\frac{bk\pi}{n+1}\right) \cos\left(\frac{b\pi}{n+1}\right) \sin\left(\frac{bj\pi}{n+1}\right) \\
&= \frac{1}{4} \sum_{k=1}^{n_1} \frac{2}{n+1} \sum_{a=1}^{n} \left\{ \sin\left(\frac{a(i-1)\pi}{n+1}\right) + \sin\left(\frac{a(i+1)\pi}{n+1}\right) \right\} \sin\left(\frac{ak\pi}{n+1}\right) \times \\
&\quad \times \frac{2}{n+1} \sum_{b=1}^{n} \sin\left(\frac{bk\pi}{n+1}\right) \left\{ \sin\left(\frac{b(j-1)\pi}{n+1}\right) + \sin\left(\frac{b(j+1)\pi}{n+1}\right) \right\} \\
&= \frac{1}{4} \sum_{k=1}^{n_1} (\delta_{i-1,k} + \delta_{i+1,k})(\delta_{k,j-1} + \delta_{k,j+1}) \\
&= \frac{1}{4} (\delta_{i-1,j-1} + \delta_{i-1,j+1} + \delta_{i+1,j-1} + \delta_{i+1,j+1}) \qquad \text{(E.-240)}
\end{aligned}
$$

so

$$
\begin{aligned}
\mathrm{tr}_{n_1}((M^{(n)})^2) &= \frac{1}{4} \sum_{i=1}^{n_1} (\delta_{i-1,i-1} + \delta_{i-1,i+1} + \delta_{i+1,i-1} + \delta_{i+1,i+1}) \\
&= \frac{1}{2}(n_1 - 1). \qquad \text{(E.-240)}
\end{aligned}
$$

Next

$$
\begin{aligned}
(M^{(n_1)}M^{(n)})_{ij} &= \sum_{k=1}^{n_1} M_{ik}^{(n_1)} M_{kj}^{(n)} \\
&= \sum_{k=1}^{n_1} \frac{2}{n_1+1} \sum_{a=1}^{n_1} \sin\left(\frac{ai\pi}{n_1+1}\right) \cos\left(\frac{a\pi}{n_1+1}\right) \sin\left(\frac{ak\pi}{n_1+1}\right) \times \\
&\quad \times \frac{2}{n+1} \sum_{b=1}^{n} \sin\left(\frac{bk\pi}{n+1}\right) \cos\left(\frac{b\pi}{n+1}\right) \sin\left(\frac{bj\pi}{n+1}\right) \\
&= \frac{1}{4} \sum_{k=1}^{n_1} \frac{2}{n_1+1} \sum_{a=1}^{n_1} \left\{ \sin\left(\frac{a(i-1)\pi}{n_1+1}\right) + \sin\left(\frac{a(i+1)\pi}{n_1+1}\right) \right\} \sin\left(\frac{ak\pi}{n_1+1}\right) \times \\
&\quad \times \frac{2}{n+1} \sum_{b=1}^{n} \sin\left(\frac{bk\pi}{n+1}\right) \left\{ \sin\left(\frac{b(j-1)\pi}{n+1}\right) + \sin\left(\frac{b(j+1)\pi}{n+1}\right) \right\} \\
&= \frac{1}{4} \sum_{k=1}^{n_1} (\delta_{i-1,k} + \delta_{i+1,k})(\delta_{k,j-1} + \delta_{k,j+1}) \\
&= \frac{1}{4} (\delta_{i-1,j-1} + \delta_{i-1,j+1} + \delta_{i+1,j-1} + \delta_{i+1,j+1}) \qquad \text{(E.-245)}
\end{aligned}
$$

so

$$
\begin{aligned}
2\mathrm{tr}_{n_1}(M^{(n_1)})(M^{(n)}) &= 2 \sum_{i=1}^{n_1} \sum_{k=1}^{n_1} M_{ik}^{(n_1)} M_{ki}^{(n)} \\
&= \frac{1}{2} \sum_{i=1}^{n_1} (\delta_{i-1,i-1} + \delta_{i-1,i+1} + \delta_{i+1,i-1} + \delta_{i+1,i+1}) \\
&= (n_1 - 1). \qquad \text{(E.-246)}
\end{aligned}
$$

The expression (E.-214) then gives

$$
\begin{aligned}
\left(\det\left(\frac{A}{2}\right)\right)^{-1} &= 1 + \frac{1}{4}\lambda^2 \left(\mathrm{tr}_{n_1} \tilde{M}^{(n_1)} + \mathrm{tr}_{n_1} \tilde{M}^{(n)} \right) \\
&\quad + \frac{1}{8}\lambda^2 \left(\mathrm{tr}_{n_1}(M^{(n_1)})^2 + 2\mathrm{tr}_{n_1}(M^{(n)}M^{(n_1)}) + \mathrm{tr}_{n_1}(M^{(n)})^2 \right) + \mathcal{O}(\lambda^3) \\
&= 1 + \frac{1}{4}\lambda^2 \left(\frac{1}{2}(n_1-1) + \frac{1}{4}(2n_1-1) \right) \\
&\quad + \frac{1}{8}\lambda^2 \left(\frac{1}{2}(n_1-1) + (n_1-1) + \frac{1}{2}(n_1-1) \right) + \mathcal{O}(\lambda^3) \\
&= 1 + \lambda^2 \left(\frac{n_1}{2} - \frac{7}{16} \right) + \mathcal{O}(\lambda^3). \qquad \text{(E.-249)}
\end{aligned}
$$

Part (c): Expression (E.-205).

Now

$$D^{(n)} = \mathbb{1} + \lambda M^{(n)} - \frac{1}{2}\lambda^2 \tilde{M}^{(n)} + \mathcal{O}(\lambda^3) \tag{E.-249}$$

where

$$\begin{aligned}
M_{ij}^{(n)} &= \frac{2}{n+1}\sum_{a=1}^{n} \sin\left(\frac{ai\pi}{n+1}\right)\cos\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right), && \text{(E.-248)}\\
\tilde{M}_{ij}^{(n)} &= \frac{2}{n+1}\sum_{a=1}^{n} \sin\left(\frac{ai\pi}{n+1}\right)\cos^2\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{aj\pi}{n+1}\right), && \text{(E.-247)}
\end{aligned}$$

and we have

$$A^{-1} = \frac{1}{2}\mathbb{1}_{n_1} + \mathcal{O}(\lambda). \tag{E.-247}$$

Note

$$\begin{aligned}
\sum_{i,j=1}^{n_1} \delta_{ki}\delta_{ij}\delta_{jl} &= 0 \quad \text{for } k, l = n_1+1, \ldots, n,\\
\sum_{i,j=1}^{n_1} M_{ki}^{(n)}\delta_{ij}\delta_{jl} &= 0 \quad \text{for } k, l = n_1+1, \ldots, n,\\
\sum_{i,j=1}^{n_1} \delta_{ki}\delta_{ij}M_{jl}^{(n)} &= 0 \quad \text{for } k, l = n_1+1, \ldots, n,\\
\sum_{i,j=1}^{n_1} P_{ki}L_{ij}\delta_{jl} &= 0 \quad \text{for } k, l = n_1+1, \ldots, n,\\
\sum_{i,j=1}^{n_1} \delta_{ki}L_{ij}P_{jk} &= 0 \quad \text{for } k, l = n_1+1, \ldots, n, && \text{(E.-250)}
\end{aligned}$$

for arbitrary matrix L, and $P = M^{(n)}$ or $P = \tilde{M}^{(n)}$. As such

$$
\begin{aligned}
\frac{B_{kl}}{2} &= D^{(n)}_{kl} - \left(D^{(n)}A^{-1}D^{(n)}\right)_{kl} \\
&= \delta_{kl} + \lambda M^{(n)}_{kl} - \frac{1}{2}\lambda^2\tilde{M}^{(n)}_{kl} - \lambda^2\sum_{i,j=1}^{n_1} M^{(n)}_{ki}\left(\frac{1}{2}\delta_{ij}\right)M^{(n)}_{jl} + \mathcal{O}(\lambda^3) \\
&= \delta_{kl} + \lambda M^{(n)}_{kl} - \frac{1}{2}\lambda^2\left(\sum_{i,j=1}^{n_1} M^{(n)}_{ki}\delta_{ij}M^{(n)}_{jl} + \tilde{M}^{(n)}_{kl}\right) + \mathcal{O}(\lambda^3) \\
&= \delta_{kl} + b_{kl}. \qquad \text{(E.-252)}
\end{aligned}
$$

A similar calculation to (E.-207) gives

$$
\begin{aligned}
\left(\det\left(\frac{B}{2}\right)\right)^{-1/2} &= \left(\det(\mathbb{1}_{n_2} + b)\right)^{-1/2} \\
&= \exp\left\{-(1/2)\ln\det\left((\mathbb{1}_{n_2} + b)\right)\right\} \\
&= \exp\left\{-(1/2)\mathrm{tr}_{n_2}\ln\left(\mathbb{1}_{n_2} + b\right)\right\} \\
&= \exp\left\{-(1/2)\mathrm{tr}_{n_2}\left(b - \frac{b^2}{2} + \mathcal{O}(b^3)\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\mathrm{tr}_{n_2}b + \frac{1}{4}\mathrm{tr}_{n_2}b^2 + \mathcal{O}(b^3)\right\} \\
&= 1 - \frac{1}{2}\mathrm{tr}_{n_2}b + \frac{1}{4}\mathrm{tr}_{n_2}b^2 + \frac{1}{8}(\mathrm{tr}_{n_2}b)^2 + \mathcal{O}(b^3).
\end{aligned}
\qquad \text{(E.-257)}
$$

where

$$
b = \lambda M^{(n)} - \frac{1}{2}\lambda^2\left(M^{(n)}\mathbb{1}_{n_1}M^{(n)} + \tilde{M}^{(n)}\right) + \mathcal{O}(\lambda^3). \qquad \text{(E.-257)}
$$

Substituting this into (E.-252) gives

$$
\begin{aligned}
\left(\det\left(\frac{B}{2}\right)\right)^{-1/2} &= 1 - \frac{1}{2}\lambda\mathrm{tr}_{n_2}M^{(n)} + \frac{1}{4}\lambda^2\left(\mathrm{tr}_{n_2}(M^{(n)}\mathbb{1}_{n_1}M^{(n)}) + \mathrm{tr}_{n_2}\tilde{M}^{(n)}\right) \\
&\quad + \frac{1}{4}\lambda^2\mathrm{tr}_{n_2}(M^{(n)})^2 + \frac{1}{8}\lambda^2(\mathrm{tr}_{n_2}M^{(n)})^2 + \mathcal{O}(\lambda^3). \qquad \text{(E.-257)}
\end{aligned}
$$

Note

$$
\begin{aligned}
M^{(n)}_{kl} &= \frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\cos\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{al\pi}{n+1}\right) \\
&= \frac{1}{2}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\left(\sin\left(\frac{a(l-1)\pi}{n+1}\right)+\sin\left(\frac{a(l+1)\pi}{n+1}\right)\right) \\
&= \frac{1}{2}(\delta_{k,l-1}+\delta_{k,l+1}) \qquad \text{(E.-258)}
\end{aligned}
$$

(where $\delta_{k,n+1}=0$) so that

$$
\mathrm{tr}_{n_2}M^{(n)} = \sum_{k=n_1+1}^{n}\frac{1}{2}(\delta_{k,k-1}+\delta_{k,k+1}) = 0. \qquad \text{(E.-258)}
$$

Next

$$
\begin{aligned}
\sum_{i,j=1}^{n_1} M^{(n)}_{ki}\delta_{ij}M^{(n)}_{jl} &= \sum_{i,j=1}^{n_1}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\cos\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{ai\pi}{n+1}\right) \\
&\quad \delta_{ij}\frac{2}{n+1}\sum_{b=1}^{n}\sin\left(\frac{bj\pi}{n+1}\right)\cos\left(\frac{b\pi}{n+1}\right)\sin\left(\frac{bl\pi}{n+1}\right) \\
&= \frac{1}{4}\sum_{i=1}^{n_1}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\left\{\sin\left(\frac{a(i-1)\pi}{n+1}\right)+\sin\left(\frac{a(i+1)\pi}{n+1}\right)\right\}\times \\
&\quad \frac{2}{n+1}\sum_{b=1}^{n}\left\{\sin\left(\frac{b(i-1)\pi}{n+1}\right)+\sin\left(\frac{b(i+1)\pi}{n+1}\right)\right\}\sin\left(\frac{bl\pi}{n+1}\right) \\
&= \frac{1}{4}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\sin\left(\frac{a(n_1+1)\pi}{n+1}\right)\times \\
&\qquad \times\frac{2}{n+1}\sum_{b=1}^{n}\sin\left(\frac{b(n_1+1)\pi}{n+1}\right)\sin\left(\frac{bl\pi}{n+1}\right) \\
&= \frac{1}{4}(\delta_{k,n_1+1}\times\delta_{n_1+1,l}) \qquad \text{(E.-263)}
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathrm{tr}_{n_2}(M^{(n)}\mathbb{1}_{n_1}M^{(n)}) &= \frac{1}{4}\sum_{k=n_1+1}^{n}(\delta_{k,n_1+1}\times\delta_{n_1+1,k}) \\
&= \frac{1}{4}. \qquad \text{(E.-263)}
\end{aligned}
$$

Next, for $k,l=n_1+1,\dots,n$, we have

$$
\begin{aligned}
\tilde{M}^{(n)}_{kl} &= \frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\cos^2\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{al\pi}{n+1}\right) \\
&= \frac{1}{2}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\left\{1+\cos\left(\frac{2a\pi}{n+1}\right)\right\}\sin\left(\frac{al\pi}{n+1}\right) \\
&= \frac{1}{2}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\sin\left(\frac{al\pi}{n+1}\right) \\
&\quad +\frac{1}{4}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\left\{\sin\left(\frac{a(l-2)\pi}{n+1}\right)+\sin\left(\frac{a(l+2)\pi}{n+1}\right)\right\} \\
&= \frac{1}{2}\delta_{k,l}+\frac{1}{4}(\delta_{k,l-2}+\delta_{k,l+2})-\frac{1}{4}\delta_{k,n}\delta_{l,n} \qquad \text{(E.-266)}
\end{aligned}
$$

(here $\delta_{k,-1}=\delta_{k,0}=\delta_{k,n+1}=\delta_{k,n+2}=0$) where we have used (E.13), so that

$$
\begin{aligned}
\mathrm{tr}_{n_2}\tilde{M}^{(n)} &= \frac{1}{2}\sum_{k=n_1+1}^{n}\delta_{k,k}+\frac{1}{4}\sum_{k=n_1+1}^{n}(\delta_{k,k-2}+\delta_{k,k+2})-\frac{1}{4} \\
&= \frac{1}{2}n_2-\frac{1}{4}. \qquad \text{(E.-266)}
\end{aligned}
$$

Next

$$
\begin{aligned}
((M^{(n)})^2)_{kl} &= \sum_{p=n_1+1}^{n} M^{(n)}_{kp}M^{(n)}_{pl} \\
&= \sum_{p=n_1+1}^{n}\frac{2}{n+1}\sum_{a=1}^{n}\sin\left(\frac{ak\pi}{n+1}\right)\cos\left(\frac{a\pi}{n+1}\right)\sin\left(\frac{ap\pi}{n+1}\right)\times \\
&\quad \times\frac{2}{n+1}\sum_{b=1}^{n}\sin\left(\frac{bp\pi}{n+1}\right)\cos\left(\frac{b\pi}{n+1}\right)\sin\left(\frac{bl\pi}{n+1}\right) \\
&= \frac{1}{4}\sum_{p=n_1+1}^{n}\frac{2}{n+1}\sum_{a=1}^{n}\left\{\sin\left(\frac{a(k-1)\pi}{n+1}\right)+\sin\left(\frac{a(k+1)\pi}{n+1}\right)\right\}\sin\left(\frac{ap\pi}{n+1}\right)\times \\
&\quad \times\frac{2}{n+1}\sum_{b=1}^{n}\sin\left(\frac{bp\pi}{n+1}\right)\left\{\sin\left(\frac{b(l-1)\pi}{n+1}\right)+\sin\left(\frac{b(l+1)\pi}{n+1}\right)\right\} \\
&= \frac{1}{4}\sum_{p=n_1+1}^{n}(\delta_{k-1,p}+\delta_{k+1,p})(\delta_{p,l-1}+\delta_{p,l+1}) \\
&= \frac{1}{4}(\delta_{k-1,l-1}+\delta_{k-1,l+1}+\delta_{k+1,l-1}+\delta_{k+1,l+1}) \qquad \text{(E.-271)}
\end{aligned}
$$

(here $\delta_{q,r}=0$ for $q,r<n_1+1$ and $q,r=0$ for $q,r=n+1$) so that

$$\begin{aligned}\mathrm{tr}_{n_2}(M^{(n)})^2 &= \frac{1}{4}\sum_{k=n_1+1}^{n}(\delta_{k-1,k-1}+\delta_{k-1,k+1}+\delta_{k+1,k-1}+\delta_{k+1,k+1}) \\ &= \frac{1}{2}(n_2-1). \qquad \text{(E.-271)}\end{aligned}$$

We have

$$\begin{aligned}\left(\det\left(\frac{B}{2}\right)\right)^{-1/2} &= 1+\frac{1}{4}\lambda^2\left(\mathrm{tr}_{n_2}(M^{(n)}\mathbb{1}_{n_1}M^{(n)})+\mathrm{tr}_{n_2}\tilde{M}^{(n)}\right) \\ &\quad +\frac{1}{4}\lambda^2\mathrm{tr}_{n_2}(M^{(n)})^2+\mathcal{O}(\lambda^3) \\ &= 1+\frac{1}{4}\lambda^2\left(\frac{1}{4}+(\frac{1}{2}n_2-\frac{1}{4})\right)+\frac{1}{4}\lambda^2\left(\frac{1}{2}(n_2-1)\right) \\ &= 1+\lambda^2\left(\frac{n_2}{4}-\frac{1}{8}\right)+\mathcal{O}(\lambda^3). \qquad \text{(E.-273)}\end{aligned}$$

Repeat the previous for the case $n_1=1$ and $n=2$ for $\omega=1$.

Proof:

First

$$(D^{(2)}_{ij})=(\sum_a U^{(2)}_{ai}\omega_a U^{(2)}_{aj})=\frac{1}{2}\begin{pmatrix}\omega_1+\omega_2 & \omega_1-\omega_2\\ \omega_1-\omega_2 & \omega_1+\omega_2\end{pmatrix}, \qquad \text{(E.-273)}$$

where

$$\omega_1^2=1+\lambda, \qquad \omega_2^2=1-\lambda. \qquad \text{(E.-273)}$$

Whereas for the $D^{(n_1)}$ there is just the one component

$$(D^{(n_1)}_{11})=1. \qquad \text{(E.-273)}$$

$$M^{(1)}=0, \qquad \tilde{M}^{(1)}=0. \qquad \text{(E.-273)}$$

We expand $D^{(2)}$ in λ

$$D^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{8}\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(\lambda^3). \tag{E.-273}$$

We can then read off

$$M^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{M}^{(2)} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.-273}$$

We have, for $i, j = 1$ part of the matrices,

$$\begin{aligned}
\mathrm{tr}_{n_1} M^{(1)} &= 0, \\
\mathrm{tr}_{n_1} \tilde{M}^{(1)} &= 0, \\
\mathrm{tr}_{n_1} (M^{(1)})^2 &= 0, \\
\mathrm{tr}_{n_1} (M^{(2)} M^{(1)}) &= 0, \\
\mathrm{tr}_{n_1} M^{(2)} &= 0, \\
\mathrm{tr}_{n_1} \tilde{M}^{(2)} &= \frac{1}{4} \\
\mathrm{tr}_{n_1} (M^{(2)})^2 &= 0
\end{aligned} \tag{E.-278}$$

and have, for $k, l = 2$ part of the matrices,

$$\begin{aligned}
\mathrm{tr}_{n_2} M^{(2)} &= 0, \\
\mathrm{tr}_{n_2} \tilde{M}^{(2)} &= \frac{1}{4}, \\
\mathrm{tr}_{n_2} (M^{(2)} \mathbb{1}_{n_1} M^{(2)}) &= M^{(2)}_{21} M^{(2)}_{12} = \frac{1}{4}, \\
\mathrm{tr}_{n_2} (M^{(2)})^2 &= 0.
\end{aligned} \tag{E.-280}$$

So that

$$\begin{aligned}
\left(\det\left(\frac{A}{2}\right)\right)^{-1} &= 1 + \frac{1}{4}\lambda^2 \left(\mathrm{tr}_{n_1} \tilde{M}^{(1)} + \mathrm{tr}_{n_1} \tilde{M}^{(2)}\right) \\
&\quad + \frac{1}{8}\lambda^2 \left(\mathrm{tr}_{n_1} (M^{(1)})^2 + 2\mathrm{tr}_{n_1} (M^{(2)} M^{(1)}) + \mathrm{tr}_{n_1} (M^{(2)})^2\right) + \mathcal{O}(\lambda^3) \\
&= 1 + \frac{1}{4}\lambda^2 \left(0 + \frac{1}{4}\right) + \frac{1}{8}\lambda^2 (0 + 0 + 0) + \mathcal{O}(\lambda^3) \\
&= 1 + \frac{1}{16}\lambda^2 + \mathcal{O}(\lambda^3) \\
&= 1 + \frac{1}{4}\lambda^2 \left(\frac{1}{2}(n_1 - 1) + \frac{1}{4}(2n_1 - 1)\right) \\
&\quad + \frac{1}{8}\lambda^2 \left(\frac{1}{2}(n_1 - 1) + (n_1 - 1) + \frac{1}{2}(n_1 - 1)\right) + \mathcal{O}(\lambda^3).
\end{aligned} \tag{E.-284}$$

And

$$\begin{aligned}\left(\det\left(\frac{B}{2}\right)\right)^{-1/2} &= 1+\frac{1}{4}\lambda^2\left(\mathrm{tr}_{n_2}(M^{(2)}\mathbb{1}_{n_1}M^{(2)})+\mathrm{tr}_{n_2}\tilde{M}^{(2)}\right)\\ &\quad+\frac{1}{4}\lambda^2\mathrm{tr}_{n_2}(M^{(2)})^2+\mathcal{O}(\lambda^3)\\ &= 1+\frac{1}{4}\lambda^2\left(\frac{1}{4}+\frac{1}{4}\right)+\frac{1}{4}\lambda^2\times 0+\mathcal{O}(\lambda^3)\\ &= 1+\frac{1}{8}\lambda^2+\mathcal{O}(\lambda^3)\\ &= 1+\frac{1}{4}\lambda^2\left(\frac{1}{4}+(\frac{1}{2}n_2-\frac{1}{4})\right)+\frac{1}{4}\lambda^2\left(\frac{1}{2}(n_2-1)\right)+\mathcal{O}(\lambda^3). \quad \text{(E.-287)}\end{aligned}$$

Expansions in λ of correlation function.

Prove that

$$\left(D^{(n_1)}\right)^{-1}_{ij}\approx\left(1+\frac{3}{4}\lambda^2\right)\delta_{i,j}-\frac{1}{2}\lambda\delta_{i,j\pm1}+\frac{3}{8}\lambda^2\delta_{i,j\pm2}+\ldots \quad \text{(E.-287)}$$

Proof:

If $D^{(n_1)}=(\mathbb{1}_{n_1}+b)$ then

$$\begin{aligned}\left(D^{(n_1)}\right)^{-1} &= (\mathbb{1}_{n_1}+b)^{-1}\\ &= \mathbb{1}_{n_1}-b+b^2-b^3+\mathcal{O}(b^4). \quad \text{(E.-287)}\end{aligned}$$

Recall

$$D^{(n_1)}=\mathbb{1}_{n_1}+\lambda M^{(n_1)}-\frac{1}{2}\lambda^2\tilde{M}^{(n_1)}+\mathcal{O}(\lambda^3). \quad \text{(E.-287)}$$

Substituting this into (E.-287)

$$
\begin{aligned}
\left(D^{(n_1)}\right)^{-1}_{ij} &= (\mathbb{1}_{n_1}+b)^{-1}_{ij} \\
&= \delta_{i,j} - \left(\lambda M^{(n_1)}_{ij} - \frac{1}{2}\lambda^2 \tilde{M}^{(n_1)}_{ij}\right) + \lambda^2 \left(M^{(n_1)}\right)^2_{ij} + \mathcal{O}(\lambda^3) \\
&= \delta_{i,j} - \lambda\frac{1}{2}(\delta_{i,j-1}+\delta_{i,j-1}) + \frac{1}{2}\lambda^2\left(\frac{1}{2}\delta_{i,j}+\frac{1}{4}(\delta_{i,j-2}+\delta_{i,j+2})-\frac{1}{2}\delta_{i,1}\right) \\
&\quad +\lambda^2\left(\frac{1}{2}\delta_{i,j}+\frac{1}{4}(\delta_{i,j-2}+\delta_{i,j+2})-\frac{1}{2}\delta_{i,1}\right)+\mathcal{O}(\lambda^3) \\
&= \left(1+\frac{3}{4}\lambda^2\right)\delta_{i,j} - \frac{1}{2}\lambda(\delta_{i,j-1}+\delta_{i,j-1}) + \frac{3}{8}\lambda^2(\delta_{i,j-2}+\delta_{i,j+2}) - \frac{3}{4}\lambda^2\delta_{i,1}+\mathcal{O}(\lambda^3)
\end{aligned}
\tag{E.-291}
$$

where we have used (E.-215), (E.13) and (E.-235).

Open chain of oscillators

Given the hamiltonian for an open chain of oscillators

$$
H = \sum_{i=1}^{N}\frac{1}{2}p_i^2 + \frac{1}{2}\sum_{i=1}^{N-1}\omega^2(q_i-q_{i+1})^2 \tag{E.-291}
$$

find normal coordinates which diagonalise it.

Proof:

$N=2$

$$
\begin{aligned}
H &= \sum_{i=1}^{2}\frac{1}{2}p_i^2 + \frac{1}{2}\omega^2(q_1-q_2)^2 \\
&= \sum_{i=1}^{2}\frac{1}{2}p_i^2 + \frac{1}{2}\omega^2(q_1^2-2q_1q_2-q_2^2).
\end{aligned}
\tag{E.-291}
$$

We require

$$
\frac{1}{2}(q_1,q_2)\begin{pmatrix}\omega^2 & -\omega^2\\ -\omega^2 & \omega^2\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = \frac{1}{2}(Q_1,Q_2)\begin{pmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\end{pmatrix} \tag{E.-291}
$$

We obviously have the eigenvectors

$$e^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad e^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{E.-291}$$

with eigenvalues

$$\beta^{(1)} = 0, \qquad \beta^{(1)} = 2\omega^2. \tag{E.-291}$$

We obviously have $e^{(1)T}e^{(2)} = e^{(2)T}e^{(1)} = 0$ and $e^{(1)T}e^{(1)} = e^{(2)T}e^{(2)} = 1$.

Construct the matrix

$$(U_{ia}) = (e_i^{(a)}) = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \tag{E.-291}$$

We obviously have

$$U_{ai}U_{ib} = \sum_i e_i^{(a)}e_i^{(b)} = \delta_{ab} \tag{E.-291}$$

$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\omega^2 & -\omega^2\\ -\omega^2 & \omega^2\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\beta(1) & 0\\ 0 & \beta(2)\end{pmatrix} \tag{E.-291}$$

Denote

$$(M_{ij}) = \begin{pmatrix}\omega^2 & -\omega^2\\ -\omega^2 & \omega^2\end{pmatrix} \qquad (D_{ab}) = \begin{pmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{pmatrix} \tag{E.-291}$$

(E.13)

$$\begin{aligned} q_iM_{ij}q_j &= Q_aU_{ai}M_{ij}U_{jb}Q_b \\ &= Q_aD_{ab}Q_b \end{aligned} \tag{E.-291}$$

$N = 3$

$$\begin{aligned} H &= \sum_{i=1}^{3}\frac{1}{2}p_i^2 + \frac{1}{2}\omega^2(q_1-q_2)^2 + \frac{1}{2}\omega^2(q_2-q_3)^2 \\ &= \sum_{i=1}^{3}\frac{1}{2}p_i^2 + \frac{1}{2}\omega^2(q_1^2 - 2q_1q_2 + 2q_2^2 - 2q_2q_3 + q_3^2). \end{aligned} \tag{E.-291}$$

$$\frac{1}{2}(q_1,q_2,q_3)\begin{pmatrix}\omega^2 & -\omega^2 & 0\\ -\omega^2 & 2\omega^2 & -\omega^2\\ 0 & -\omega^2 & \omega^2\end{pmatrix}\begin{pmatrix}q_1\\ q_2\\ q_3\end{pmatrix} = \frac{1}{2}(Q_1,Q_2,Q_3)\begin{pmatrix}\omega_1^2 & 0 & 0\\ 0 & \omega_2^2 & 0\\ 0 & 0 & \omega_3^2\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\\ Q_3\end{pmatrix} \qquad \text{(E.-291)}$$

We obviously have the eigenvector

$$e^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} \qquad \text{(E.-291)}$$

with eigenvalue 0. We consider the secular determinant

$$\det\begin{pmatrix}\omega^2-\beta & -\omega^2 & 0\\ -\omega^2 & 2\omega^2-\beta & -\omega^2\\ 0 & -\omega^2 & \omega^2-\beta\end{pmatrix} = 0. \qquad \text{(E.-291)}$$

This determinant obvious has the eignvalue $\beta = \omega^2$. Direct evaluation of the secular equation gives the cubic equation

$$\beta(\omega^2-\beta)(3\omega^2-\beta) = 0. \qquad \text{(E.-291)}$$

with the eigenvalues

$$\beta^{(1)} = 0, \qquad \beta^{(1)} = \omega^2, \qquad \beta^{(1)} = 3\omega^2. \qquad \text{(E.-291)}$$

The other eigenvectors are

$$e^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}, \qquad e^{(3)} = \frac{1}{2}\begin{pmatrix}1\\ -2\\ 1\end{pmatrix}. \qquad \text{(E.-291)}$$

E.13.1 Lattice Scalar Field

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{N}\dot{q}_i^2 - \frac{1}{2}\sum_{i=1}^{N-1}\omega^2(q_i-q_{i+1})^2 \qquad \text{(E.-291)}$$

The Euler-Lagrange equations of motion are

$$\begin{aligned} \ddot{q}_1 + \omega^2(q_1 - q_2) &= 0, \\ \ddot{q}_i + \omega^2(q_{i-1} + 2q_i - q_{i+1}) &= 0 \qquad 2 \le i \le N-1 \\ \ddot{q}_N + \omega^2(q_N - q_{N-1}) &= 0 \end{aligned} \tag{E.-291}$$

Instead trying to diagonalise Λ analytically, we look for solutions of the normal mode type, but with a wavelike structure

$$q_n(t) = Ce^{i(kna - \omega t)} \tag{E.-292}$$

Such a solution, if it exists, looks like a wave of amplitude C propagating down a chain of masses. This means that at time $t + \Delta t$ there is an index $n + \Delta n$ whose mass is doing what the mass with index n was doing at time t. To find Δn we simply solve the equation

$$kna - \omega t = k(n + \Delta n)a - \omega(t + \Delta t)$$

or

$$k\Delta na = \omega \Delta t.$$

With such solutions the the disturbance is moving toward the right with rate $\Delta n/\Delta t = \omega/ka$. Since an measures the distance along the chain, the disturbance, propagates at speed

$$c = a\Delta n/\Delta t = \omega/k.$$

To find such solutions, insert (E.13.1) into (E.-290)

$$-\omega_n^2 q_i = \omega^2[e^{-ika} - 2 + e^{ika}].$$

This equation gives a dispertion relation between ω and k:

$$\omega_n^2 = 4\omega^2 \sin^2 \frac{ka}{2}. \tag{E.-292}$$

Finite chain

In any finite chain the solutions depend on the boundary conditions imposed at the end-point masses. The boundary condition we impose here is 'fixed ends':

$$q_1 = q_N = 0. \tag{E.-292}$$

Since the masses at q_1 and q_N do not move, the dynamical system now consists of the $N-2$ masses with indices running from $i = 2$ to $i = N-1$.

The end-point conditioncan be satisfied by combining pairs of running-waves of () so as to interfere destructively and cancel out at the end points, forming standing waves.

Appendix F

Generally Covariant Thermodynamics and Statistical Mechanics

F.1 Gravitational Statistical Mechanics

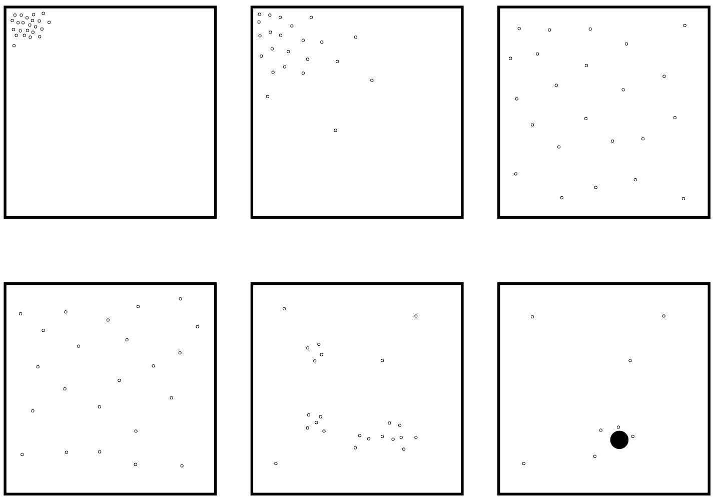


Figure F.1: For a gas, increasing entropy tends to make the distribution more uniform. For a system of graviating bodies the reverse is true. High entropy is achieved by gravitational clumping and highest of all, by collapse to a black hole.

It would seem at first that the primordial gas was in a high entropy, that is, a disordered state. But thus turns out not to be true. This completely ignores gravity.

Entropy overall increases even though certain constituents have become more ordered.

No object in the universe contains more entropy that a black hole.

for some reason the early universe was filled with a hot, uniform gaseous mixture of hydrogen and helium.

How is it that the universe began in such a highly ordered configuration? Such an extemely low state of entropy?

Consider a "high temperature" early-universe regime. This is usually described in terms of fluctuations around a background metric; is there a genuinely general covariant description of this physics? And what is temperature in this context, if we do not fix a background metric?

Appendix G

Spin Networks

Mostly follows “A Spin Network Primer” [56]

G.1 Diagrammatic Mathematics

Diagrammatic algebra designed too handle the combinatorics of irreducible representations, all the familiar results of representation theory have diagrammatical form.

G.1.1 Line, Bend ad Loop

Consider the tensor

$$(\delta^A_B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which can be represented diagramatically as in fig G.1

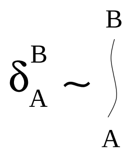


Figure G.1: Diagramatically representation of δ^A_B

Consider the two antisymmetric tensors

$$(\epsilon_{AB}) = (\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad \text{(G.0)}$$

We associate a curve with a matrix with two upper (lower) indices. The first trial for ϵ_{AB} we look at is in fig (G.2) and for ϵ^{AB} fig (G.3)

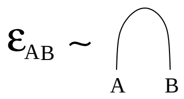


Figure G.2: epdown

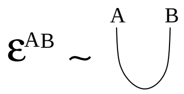


Figure G.3: epup

This fits well with the diagramatitics of $\delta_A^C\epsilon_{CB} = \epsilon_{AB}$. We soon find trouble with this choose however. Firstly:

$$\delta_A^C\epsilon_{CD}\epsilon^{DE}\delta_E^B = -\delta_A^B$$

and straightening a line yields a minus sign:

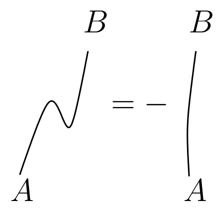


Figure G.4: firstprob

Secondaly, as a conseqence of

$$\epsilon_{AD}\epsilon_{BC}\epsilon^{CD} = -\epsilon_{AB},$$

topological difficulties can be fixed by modifying the definition

$$\epsilon_{AB} \to \tilde{\epsilon}_{AB} = i\epsilon_{AB}.$$

As the indices take two values, we have the identity

$$\epsilon_{[EB}\epsilon_{C]F} = 0 \tag{G.0}$$

which reduces to

$$\epsilon_{EB}\epsilon_{CF} + \epsilon_{BC}\epsilon_{EF} + \epsilon_{CE}\epsilon_{BF} = 0 \tag{G.0}$$

Contracting this with ϵ^{EA} and ϵ^{FD}, then using $\epsilon_{EB}\epsilon^{EA} = \delta^A_B$, $\epsilon_{CF}\epsilon^{FD} = -\delta^D_C$ and $\epsilon_{EF}\epsilon^{EA}\epsilon^{FD} = \epsilon^{AD}$ etc we obtain the so-called binor identity:

$$\epsilon_{AC}\epsilon^{BD} = \delta^B_A\delta^D_C - \delta^D_A\delta^B_C \tag{G.0}$$

Using the definitions of the $\tilde{\epsilon}$ matrices, the binor identity becomes

$$\tilde{\epsilon}_{AC}\tilde{\epsilon}^{BD} - \delta^D_A\delta^B_C + \delta^B_A\delta^D_C = 0. \tag{G.0}$$

Then introducing the rule that we assign a minus sign to each crossing, equation (G.1.1) can be diagrammatically, represented as in fig G.5.

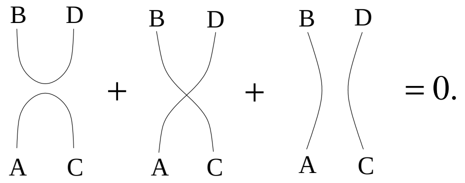


Figure G.5: The binor indentity $\tilde{\epsilon}_{AC}\tilde{\epsilon}^{BD} - \delta^D_A\delta^B_C + \delta^B_A\delta^D_C = 0$

For more than

$\delta^C_A\epsilon_{CD} = \epsilon_{AB}$.

$$\delta^C_A\epsilon_{CD}\epsilon^{DE}\delta^B_E = \epsilon_{AD}\epsilon^{DB} = -\delta^B_A, \tag{G.0}$$

$$\epsilon_{AD}\epsilon_{BC}\epsilon^{CD} = -\epsilon_{AB} \tag{G.0}$$

$$\epsilon_{AB} \to \tilde{\epsilon}_{AB} = i\epsilon_{AB} \tag{G.0}$$

$$\delta^D_A \tag{G.0}$$

Using these rules, we can show that these strands behave as would thin strings in the plane; one can arbitrary deform a graphical expression without changing its meaning.

In translating a diagram into tensor notation, we use

1. assign a minus sign to each
2. assign a minus sign to each crossing

G.1.2 Symmetrizing Products of Delta Functions

Define the $D^{A'B'}_{(A\,B)}$ as the symmetric product of two delta functions:

$$D^{A'B'}_{(A\,B)} := \frac{1}{2!}\left(\delta^{A'}_{A}\delta^{B'}_{B} + \delta^{A'}_{B}\delta^{B'}_{A}\right) \tag{G.0}$$

$D^{A'B'}_{(A\,B)}$ are projectors i.e.

$$D^{A'B'}_{(C\,D)}D^{C\,D}_{(A\,B)} = \frac{1}{2!}\left(\delta^{A'}_{C}\delta^{B'}_{D} + \delta^{A'}_{D}\delta^{B'}_{C}\right)D^{C\,D}_{(A\,B)} = D^{(A'B')}_{(A\,B)} = D^{A'B'}_{(A\,B)} \tag{G.0}$$

More generally $D^{A'B'...D'}_{(A\,B\,...D)}$, the symmetric product of n delta functions, is a projector:

$$\begin{aligned} D^{A'B'...D'}_{(E\,F\,...H)}D^{EF...H}_{(A\,B\,...D)} &= \frac{1}{n!}\left(\delta^{A'}_{E}\delta^{B'}_{F}\ldots\delta^{H'}_{H} + \ldots\right)D^{EF...H}_{(A\,B\,...D)} \\ &= D^{(EF...H)}_{(A\,B\,...D)} = D^{EF...H}_{(A\,B\,...D)}. \end{aligned} \tag{G.0}$$

This general result can be represented diagramatically as in fig G.1.2.

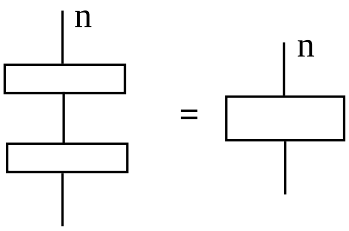


Figure G.6: Projector. The symmetric product of n delta functions, is a projector

G.1.3 Jones-Wenzl Projectors

Starting from the binor identity

$$-\tilde{\epsilon}^{A'B'}\tilde{\epsilon}_{AB} = \delta_A^{A'}\delta_B^{B'} - \delta_B^{A'}\delta_A^{B'}, \tag{G.0}$$

a simple rearrangement gives

$$\frac{1}{2}(\delta_A^{A'}\delta_B^{B'} + \delta_A^{B'}\delta_B^{A'}) = \delta_A^{A'}\delta_B^{B'} + \frac{1}{2}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'B'} \tag{G.0}$$

Written in tha stanard form (see in a momemt)

$$\delta_{(A}^{A'}\delta_{B)}^{B'} = \delta_A^{A'}\delta_B^{B'} - \mu_1\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'B'} \tag{G.0}$$

where $\mu_1 = -1/2$. Which is diagramatically represented in fig G.1.3.

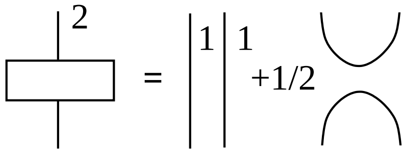


Figure G.7: Diagrammatical representation of equation (G.1.3) with $\mu_1 = -1/2$.

Jones-Wenzl Projectors for $n = 3$

We can rearange the symmetric product of the three deltas as follows

$$\begin{aligned} 3\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} &= \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \delta_B^{A'}\delta_A^{(B'}\delta_C^{C')} + \delta_C^{A'}\delta_{(B}^{B'}\delta_{A)}^{C'} \\ &= 3\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C')}\right) - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \delta_C^{A'}\delta_B^{(B'}\delta_A^{C')}\right) \end{aligned} \tag{G.-1}$$

This rearangment (G.-1) can be represented diagramatically as in fig (G.1.3)

Multiply (G.1.3) by $\delta_C^{C'}$ and symmetrize over the upper indices B' and C' to get

$$\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C')} = -\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C')}, \tag{G.-1}$$

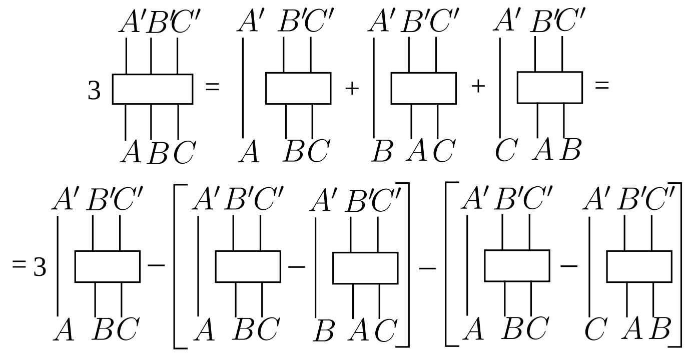


Figure G.8: graphmath7. Diagramatical representation of (G.-1).

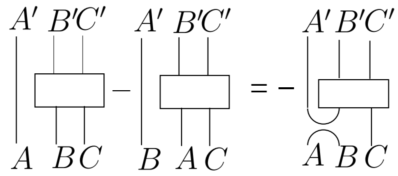


Figure G.9: graphmath9. Diagramatical representation of (G.1.3)

which is expressed diagramatically as fig.(G.1.3)

Using this in fig.(graphmath7) we obtain

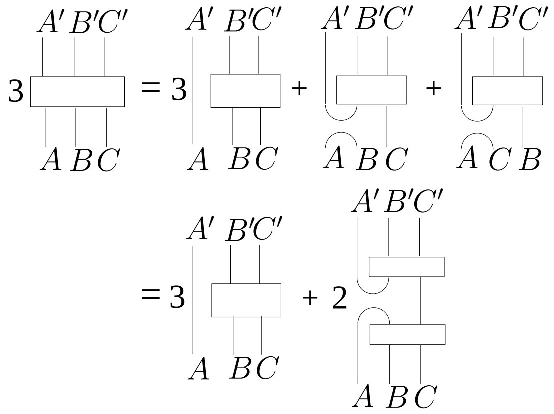


Figure G.10: graphmath8.

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \frac{1}{3}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C')} + \frac{1}{3}\tilde{\epsilon}_{AC}\tilde{\epsilon}^{A'(C'}\delta_B^{B')} \tag{G.-1}$$

we obtain

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \frac{2}{3}\tilde{\epsilon}_{A(B}\delta_{C)}^{(C'}\tilde{\epsilon}^{B')A'} \tag{G.-1}$$

Or

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \mu_1\tilde{\epsilon}_{A(B}\delta_{C)}^{(C'}\tilde{\epsilon}^{B')A'} \tag{G.-1}$$

where

$$\mu_1 = -2/3. \tag{G.-1}$$

This is represented in fig (Diagrelfor3)

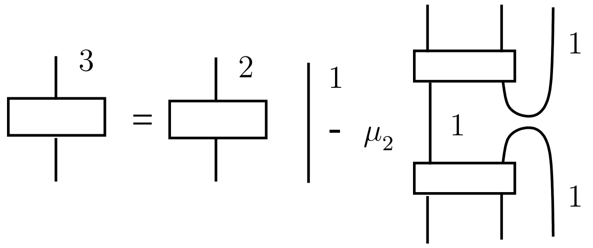


Figure G.11: Diagrelfor3. Compact diagrammatical representation of (G.1.3)

Jones-Wenzl Projectors for Arbitrary n

We now consider the symmetric product of n δ's. We have:

$$\begin{aligned} n\delta_A^{(A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_F^{F')} &= \delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \dots + \delta_F^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_A^{F')} \\ &= n\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')}\right) - \dots \\ &\quad \dots - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_F^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_A^{F')}\right) \end{aligned} \tag{G.-4}$$

This is represented by diagram (graphmath11)

$$\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')} = -\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} \tag{G.-4}$$

This is represented by diagram graphmath12

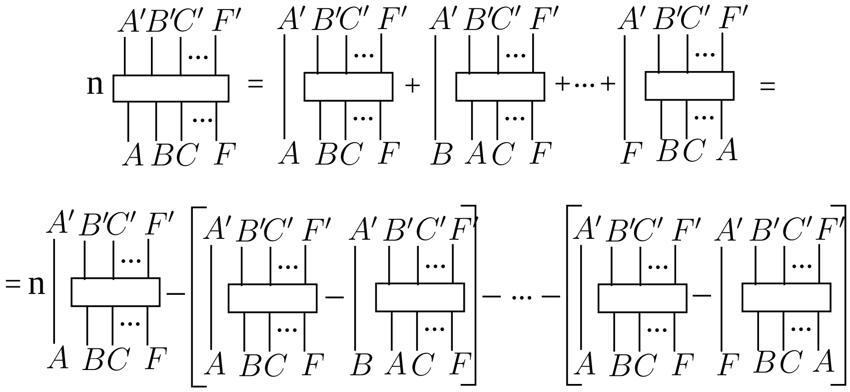


Figure G.12: graphmath11. Representing equation (G.-4).

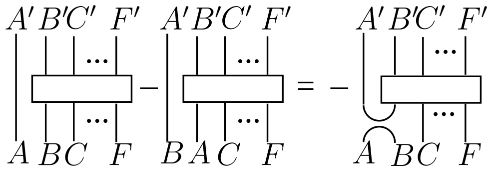


Figure G.13: graphmath12. (G.1.3)

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_F^{F')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \frac{1}{n}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} + \dots + \frac{1}{n}\tilde{\epsilon}_{AF}\tilde{\epsilon}^{A'(F'}\delta_C^{C'}\dots\delta_B^{B')} \tag{G.-4}$$

$$\underbrace{\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} + \tilde{\epsilon}_{AC}\tilde{\epsilon}^{A'(C'}\delta_B^{B'}\dots\delta_F^{F')}\dots + \tilde{\epsilon}_{AF}\tilde{\epsilon}^{A'(F'}\delta_C^{C'}\dots\delta_B^{B')}}_{n-1 \text{ terms}} = +(n-1)\tilde{\epsilon}_{A(B}\delta_C^{C'}\dots\delta_{F)}^{(F'}\tilde{\epsilon}^{B')A'}. \tag{G.-4}$$

we obtain

$$\delta_{(A}^{A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_{F)}^{F'} = \delta_A^{A'}\delta_{(B}^{B'}\delta_C^{C'}\dots\delta_{F)}^{F'} + (n-1)\epsilon_{A(B}\delta_C^{C'}\dots\delta_{F)}^{(F'}\epsilon^{B')A'} \tag{G.-4}$$

G.1.4 Contractions of Symmetrised Lines

We perform a contraction the symmetrised lines as given by fig (G.1.4), and we denote the resulting value Δ_n. For example $\Delta_1 = -2$:

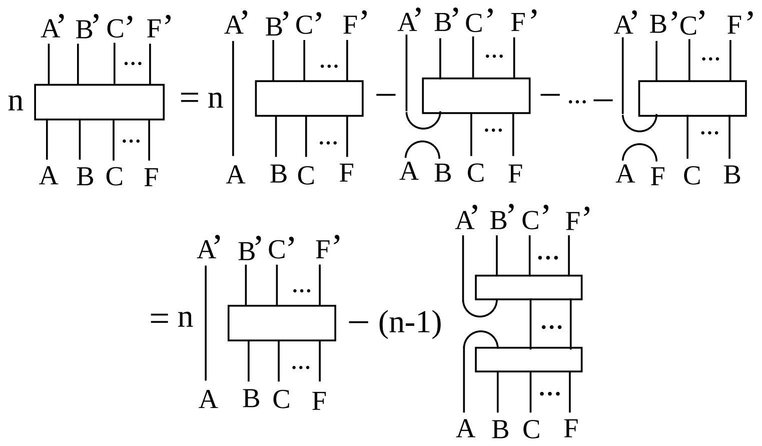


Figure G.14: graphmath13.

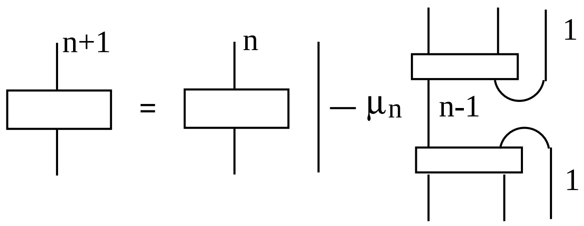


Figure G.15: This one.

$$\Delta_1 = \delta_A^{A'}\left(\tilde{\epsilon}_{A'C}\delta_D^C\tilde{\epsilon}^{DA}\right) = -\delta_A^{A'}\delta_{A'}^A = -2.$$

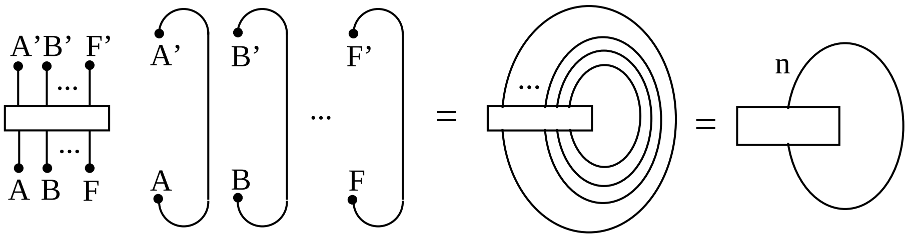


Figure G.16: graphmath2. The contraction of the symmetric product of deltas, the resulting values denoted Δ_n.

As an example, we explicitly work out the value of Δ_2 using the graphical method as shown in fig (graphmath2A). We find that the result is $\Delta_2 = 3$.

In order to find the value of Δ_n for $n > 2$ we derive recursive relations.

a relation between $\Delta_{n+2}, \Delta_{n+1}$ and Δ_n.

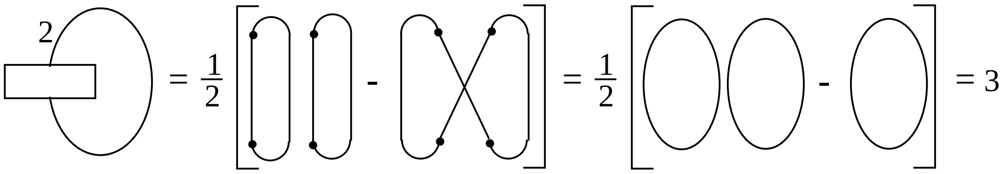


Figure G.17: graphmath2A. Calculation of Δ_2.

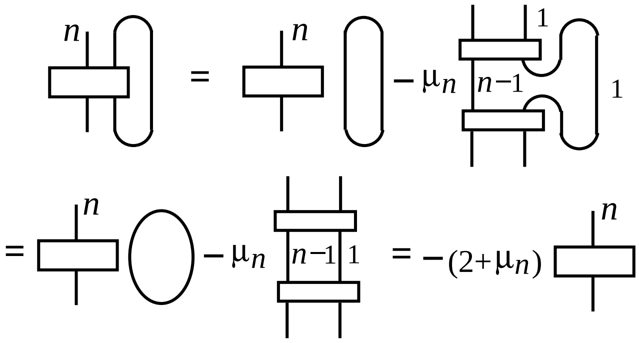


Figure G.18: graphmath3A .

$$\Delta_{n+1} = -(2+\mu_n)\Delta_n$$

$$\mu_n = -\frac{\Delta_{n+1}}{\Delta_n} - 2$$

$$\begin{aligned}
\Delta_n &= -2\Delta_{n-1} + \frac{n-1}{n}\Delta_{n-1} \\
&= \Delta_{n-1}[\frac{n-1}{n} - 2] \\
&= -\Delta_{n-1}[\frac{n+1}{n}]
\end{aligned} \tag{G.-5}$$

Employing this recursive relation we obtain,

$$
\begin{aligned}
\Delta_n &= -\Delta_{n-1}[\frac{n+1}{n}] \\
&= (-1)^2\Delta_{n-2}[\frac{n}{n-1}][\frac{n+1}{n}] \\
&= (-1)^2\Delta_{n-2}[\frac{n+1}{n-1}] \\
&= (-1)^3\Delta_{n-3}[\frac{n+1}{n-2}] \\
&\vdots \\
&= (-1)^{n-1}\frac{(n+1)}{2}\Delta_1 \\
&= (-1)^n(n+1) \qquad \text{(G.-10)}
\end{aligned}
$$

where we have used $\Delta_1 = -2$.

So that

$$\Delta_n = (-1)^n(n+1). \qquad \text{(G.-10)}$$

The one contraction of an $(n+1)-$symmetrised prodct is proportional to an $n-$symmetrised product, as shown in fig graphmath3

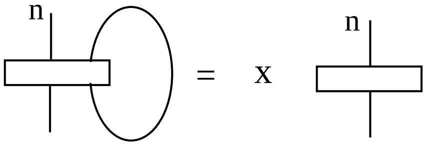


Figure G.19: graphmath3 .

By definition of Δ_n, we see that x is given by Δ_{n+1}/Δ_n. (see fig graphmath1)

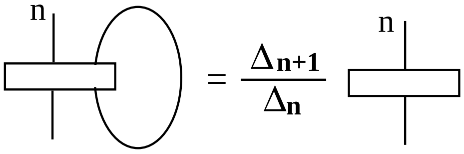


Figure G.20: graphmath1.

Now, if

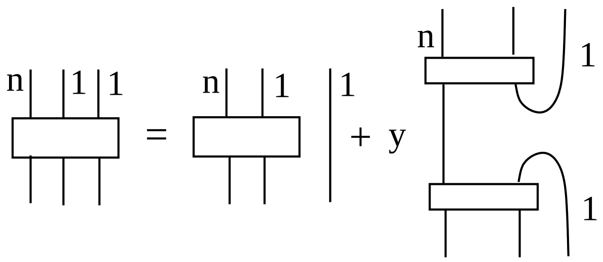


Figure G.21: P math4.

it follows that

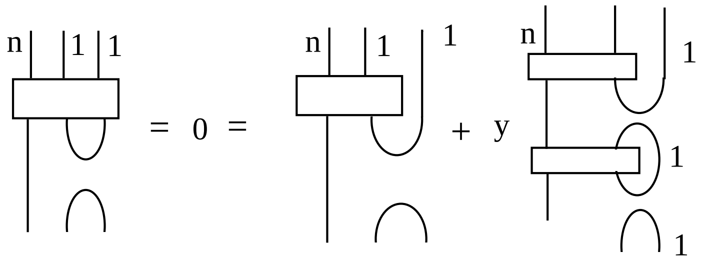


Figure G.22: graphmath5.

Hence

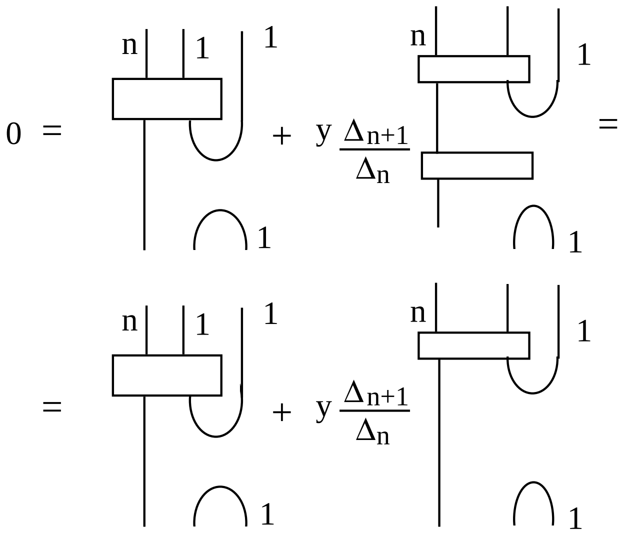


Figure G.23: graphmath6.

Therefore $y = \Delta_n/\Delta_{n+1}$ and the recursion takes the form

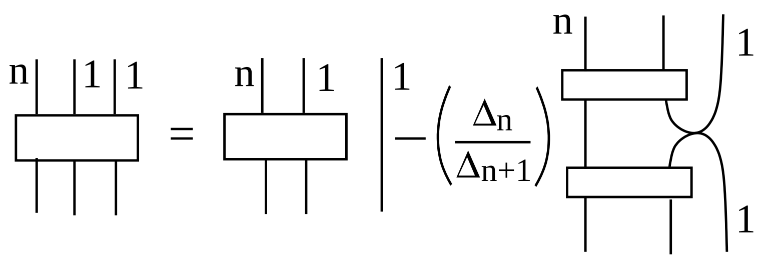


Figure G.24: P math4.

It follows that

$$\Delta_{n+2} = -2\Delta_{n+1} - \Delta_n \tag{G.-10}$$

with $\Delta_1 = -2$ and $\Delta_2 = 3$. This obviously has a unique solution which is

$$\Delta_n = (-1)^n(n+1), \tag{G.-10}$$

as is easily checked:

$$\begin{aligned} -2\Delta_{n+2} - \Delta_n &= -2(-1)^{n+1}(n+2) - (-1)^n(n+1) \\ &= (-1)^{n+2}[2(n+2) - (n+1)] \\ &= (-1)^{n+2}(n+3), \end{aligned} \tag{G.-11}$$

$$\Delta_1 = (-1)^1(1+1)$$

$$\Delta_2 = (-1)^2(2+1).$$

$$\begin{aligned} \Delta_{n+1} + \Delta_n &= (-1)(\Delta_n + \Delta_{n-1}) \\ &= (-1)^2(\Delta_{n-1} + \Delta_{n-2}) \\ &\dots \\ &= (-1)^{n-1}(\Delta_2 + \Delta_1) \\ &= (-1)^{n-1}(3-2) = (-1)^{n-1} \end{aligned} \tag{G.-14}$$

each containing a turn back.

G.1.5 3-Vertices

We define a 3-vertex as in fig...

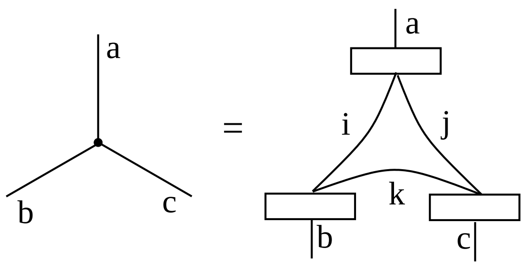


Figure G.25: P.

$$\begin{aligned} i &= \frac{a+b-c}{2} \\ j &= \frac{a+c-b}{2} \\ k &= \frac{b+c-a}{2}. \end{aligned} \tag{G.-15}$$

We consider the "bubble" diagram.

Lemma G.1.1 *The network is zero if $a \neq b$. If $a = b$, then*

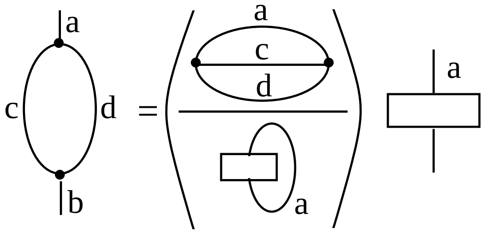


Figure G.26: P.

Proof:

Assume that $a > b$.

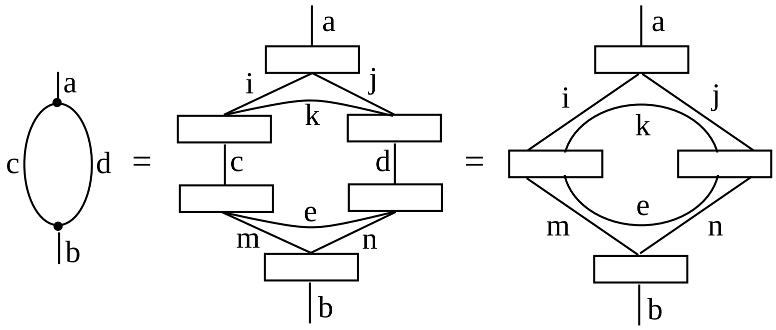


Figure G.27: graphmath15.

where

$$i = \frac{a+c-d}{2} \qquad l = \frac{c+d-b}{2}$$
$$j = \frac{a+d-c}{2} \qquad m = \frac{c+b-d}{2}$$
$$k = \frac{c+d-a}{2} \qquad n = \frac{b+d-c}{2}. \tag{G.-16}$$

Rewritting, we find $e = (a-b)/2$

Consider expanding each of the two middle projectors into their sum of products of δ's. It followes that each term will contain a turn-back with respect to the $a-$projector above and give zero.

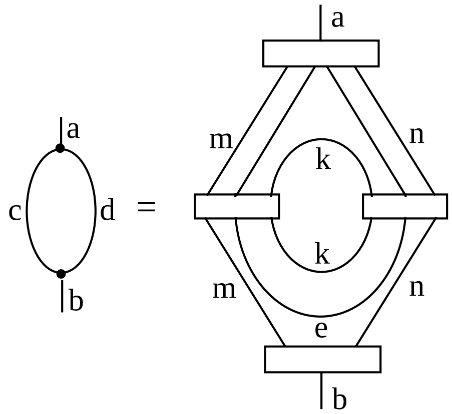


Figure G.28: graphmath16.

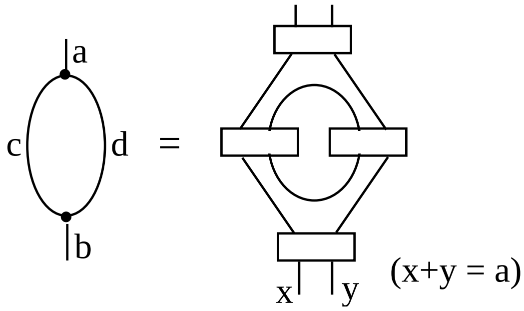


Figure G.29: graphmath17.

Now assume that $a = b$. Consider

Hence

and

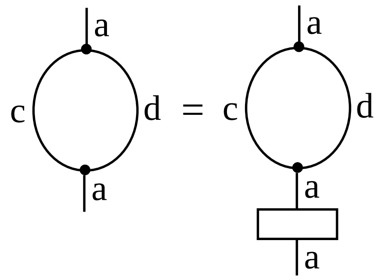


Figure G.30: graphmath18.

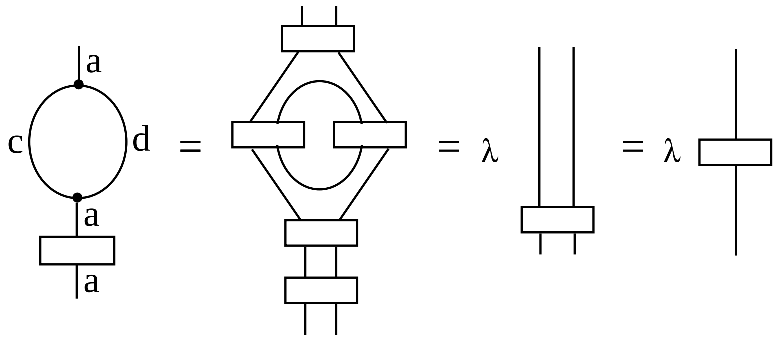


Figure G.31: graphmath18A.

Consider expanding each of the two middle projectors into their sum of products of δ's. Only straight-ahead terms survive the extra projector at the bottom. Thus

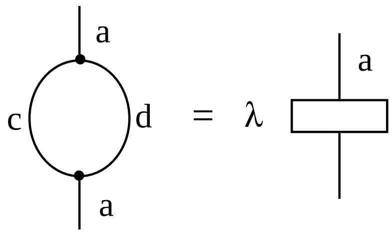


Figure G.32: graphmath19.

Hence

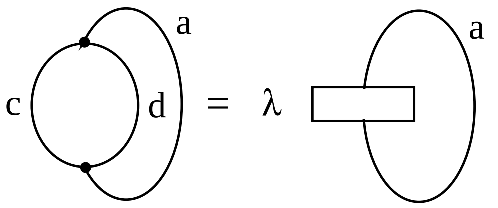


Figure G.33: graphmath20.

□

$$N_{abc}\begin{pmatrix} a & b & c \\ m_a & m_b & m_c \end{pmatrix} \tag{G.-16}$$

$$N_{abc} = \left[\frac{(a+b-c)!(b+c-a)!(c+a-b)!}{2^2(a+b+c+2)!}\right]^{1/2} \tag{G.-16}$$

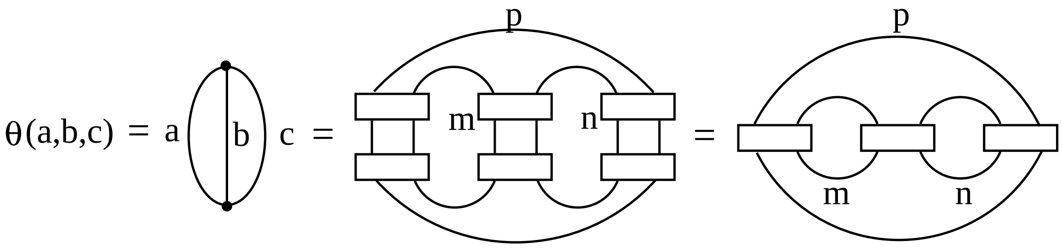


Figure G.34: graphmath21.

where

$$\begin{aligned} m &= \frac{a+b-c}{2}, \\ n &= \frac{b+c-a}{2}, \\ p &= \frac{a+c-b}{2}. \end{aligned} \tag{G.-17}$$

In the case $p = 0$, we get

$$Net(m, n, 0) \tag{G.-17}$$

$$\Delta_{m+n}. \tag{G.-17}$$

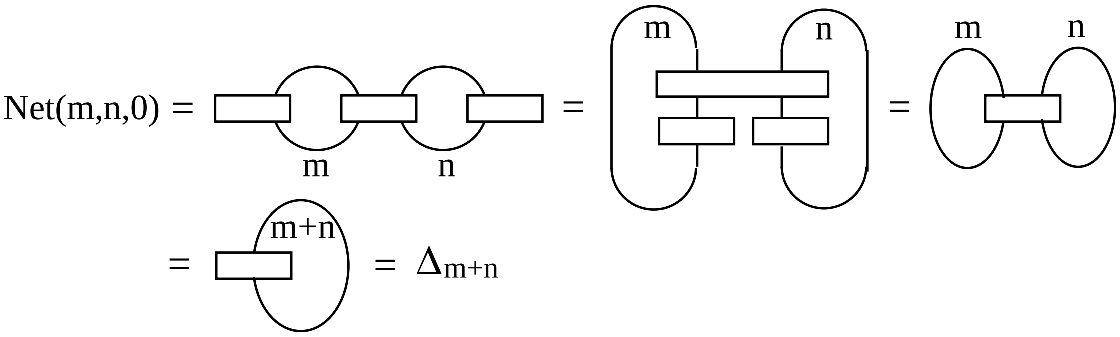


Figure G.35: graphmath22.

$Net(n-1,1,1)$ we will need to get the eigenvalue of the area operator. $Net(m,n,1) =$ is easy to deal with.

$$Net(m,n,1) = -(2 + \mu_m + \mu_n)\Delta_{m+n}. \tag{G.-17}$$

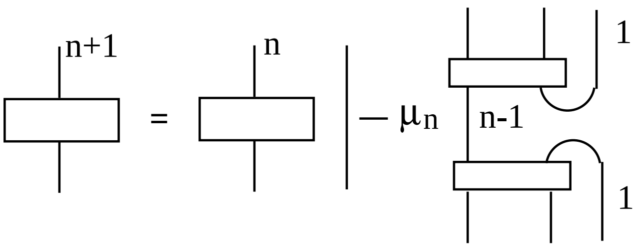


Figure G.36: This one.

Applying this to $Net(m,n,1)$ as shown below.

We see that the last term is equivalent to (G.1.5) and so is zero.

The first network is $-2Net(m,n,0)$, where $Net(m,n,0)$ has already been calculated in the previous Lemma as Δ_{m+n}. The second and third nets are each equivalent to $Net(m,n,0)$. The forth network vanishes. Thus

$$Net(m,n,1) = -(2\Delta_{m+n} + \mu_m\Delta_{m+n} + \mu_n\Delta_{m+n}).$$

Definition Let $Net(m,n,p_e,p_i)$, for $p_e + p_i = p - 1 \geq 1$

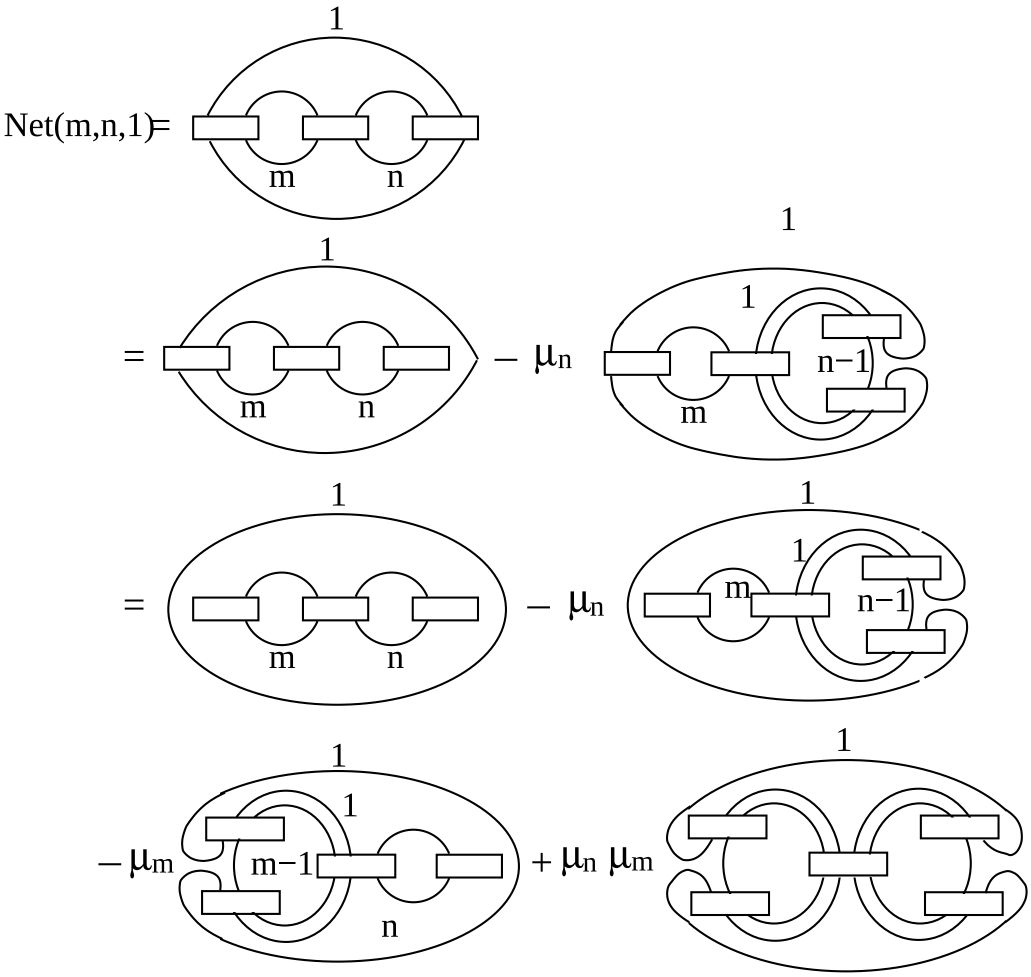


Figure G.37: graphmath23.

Similarly,

$$
\begin{aligned}
Net(m,n,p) \quad &= \quad (-2-\mu_{m+p-1}-\mu_{n+p-1})Net(m,n,p-1) \\
&\quad + \mu_{m+p-1}\mu_{n+p-1}Net(m,n,1,p-2) \qquad \text{(G.-17)}
\end{aligned}
$$

Recursion relation for Net

(a) $Net(m,n,p-1,0) = (-2-\mu_m-\mu_n)Net(m,n,p-1)$

(b) $Net(m,n,p_e,p_i) = (-2-\mu_{m+p_i}-\mu_{n+p_i})Net(m,n,p-1) + \mu_{m+p_i}\mu_{n+p_i}Net(m,n,p_e+1,p_i-1)$

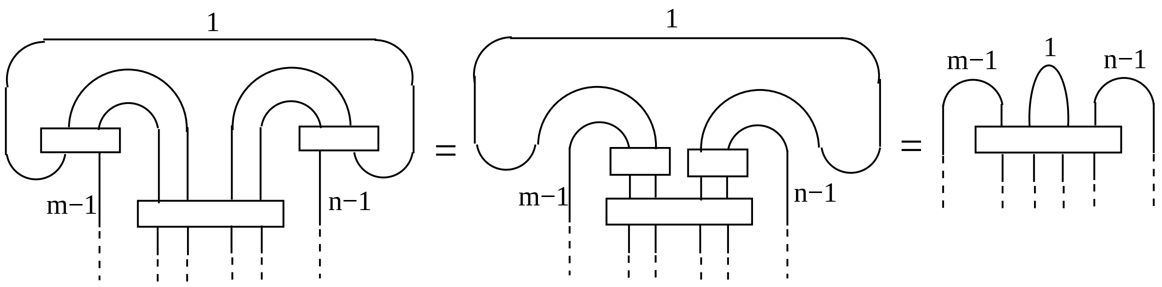


Figure G.38: graphmath25.

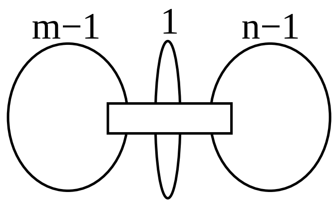


Figure G.39: graphmath24.

Starting with (G.-17) and using (b) over again $p-2$ times, and then finally using (a) we can obtain a realtion:

$$Net(m,n,p) = \rho(m,n,p)Net(m,n,p-1). \tag{G.-17}$$

To simplify the analysis, we introduce the following. Since $\mu_{m+j} = \Delta_{m-1+j}/\Delta_{m+j}$,

$$-2 - \mu_{m+j} - \mu_{n+j} = \frac{-2\Delta_{m+j}\Delta_{n+j} - \Delta_{m-1+j}\Delta_{n+j} - \Delta_{m+j}\Delta_{n-1+j}}{\Delta_{m+j}\Delta_{n+j}}$$

Write

$$\alpha_j = -2\Delta_{m+j}\Delta_{n+j} - \Delta_{m-1+j}\Delta_{n+j} - \Delta_{m+j}\Delta_{n-1+j}$$

and

$$\beta_j = \Delta_{m+j}\Delta_{n+j}$$

First (G.-17) becomes

$$Net(m,n,p) = \frac{\alpha_{p-1}}{\beta_{p-1}}Net(m,n,p-1) + \frac{\beta_{p-2}}{\beta_{p-1}}Net(m,n,1,p-2) \tag{G.-17}$$

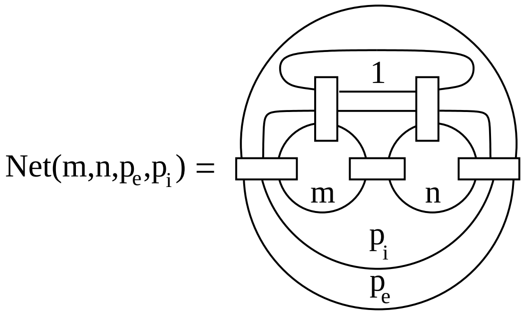


Figure G.40: graphmath26.

then we would use (b) with $p_e = 1$ and $p_i = p - 2$,

$$Net(m, n, 1, p-2) = \frac{\alpha_{p-2}}{\beta_{p-2}} Net(m, n, p-1) + \frac{\beta_{p-3}}{\beta_{p-2}} Net(m, n, 2, p-3) \tag{G.-17}$$

and next we would use (b) with $p_e = 2$ and $p_i = p - 3$,

$$Net(m, n, 2, p-3) = \frac{\alpha_{p-3}}{\beta_{p-3}} Net(m, n, p-1) + \frac{\beta_{p-4}}{\beta_{p-3}} Net(m, n, 3, p-4) \tag{G.-17}$$

and so on until

$$\begin{aligned} Net(m, n, p-2, 1) &= \frac{\alpha_1}{\beta_1} Net(m, n, p-1) + \frac{\beta_0}{\beta_1} Net(m, n, p-1, 0) \\ &= \frac{\alpha_1}{\beta_1} Net(m, n, p-1) + \frac{\beta_0}{\beta_1} \frac{\alpha_0}{\beta_0} Net(m, n, p-1) \end{aligned} \tag{G.-18}$$

where in the last line we used (a). Putting it together,

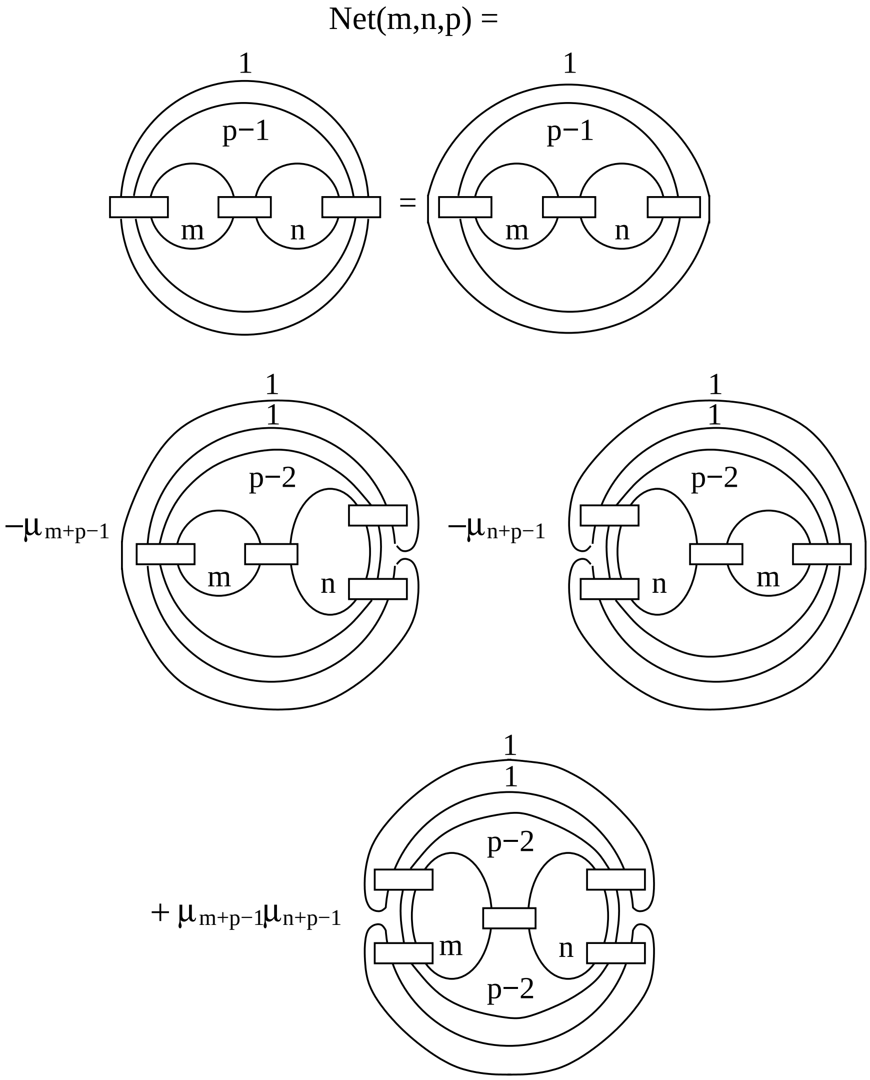


Figure G.41: Netmnp.

$$
\begin{aligned}
Net(m,n,p) &= \frac{\alpha_{p-1}}{\beta_{p-1}}Net(m,n,p-1)+\frac{\beta_{p-2}}{\beta_{p-1}}Net(m,n,1,p-2)\\
&= (\frac{\alpha_{p-1}+\alpha_{p-2}}{\beta_{p-1}})Net(m,n,p-1)+\frac{\beta_{p-3}}{\beta_{p-1}}Net(m,n,2,p-3)\\
&= (\frac{\alpha_{p-1}+\alpha_{p-2}+\alpha_{p-3}}{\beta_{p-1}})Net(m,n,p-1)+\frac{\beta_{p-4}}{\beta_{p-1}}Net(m,n,2,p-4)\\
&= \cdots\\
&= \frac{1}{\beta_{p-1}}\left(\sum_{j=0}^{p-1}\alpha_j\right)Net(m,n,p-1) \qquad \text{(G.-21)}
\end{aligned}
$$

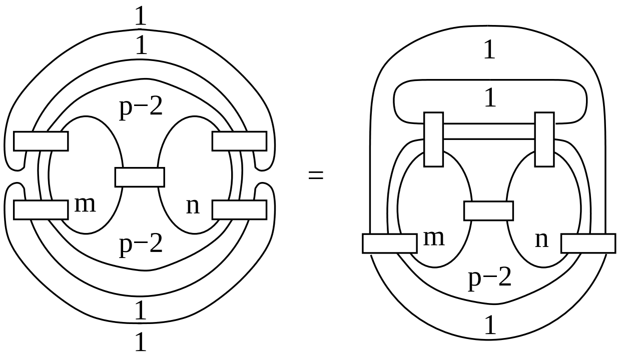


Figure G.42: Netmn1p-2. Equivalence of last network in fig (G.1.5) with $Net(m,n,1,p-2)$.

Therefore

$$\rho(m,n,p) = \frac{1}{\beta_{p-1}} \sum_{j=0}^{p-1} \alpha_j. \tag{G.-21}$$

Or

$$\begin{aligned}
\rho(m,n,p) &= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \big(-2\Delta_{m+j}\Delta_{n+j} - \Delta_{m+j-1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \Big((-2\Delta_{m+j} - \Delta_{m+j-1})\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\Big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \Big(\Delta_{m+j+1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\Big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \Big(\Delta_{m+1}\Delta_{n} - \Delta_{m}\Delta_{n-1} \\
&\qquad \Delta_{m+2}\Delta_{n+1} - \Delta_{m+1}\Delta_{n} \\
&\qquad \Delta_{m+3}\Delta_{n+2} - \Delta_{m+2}\Delta_{n+1} \\
&\qquad \cdots \\
&\qquad \Delta_{m+j+1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1} \\
&\qquad \cdots \\
&\qquad \Delta_{m+p-1}\Delta_{n+p-2} - \Delta_{m+p-2}\Delta_{n+p-3} \\
&\qquad \Delta_{m+p}\Delta_{n+p-1} - \Delta_{m+p-1}\Delta_{n+p-2}\Big) \\
&= \frac{\Delta_{m+p}\Delta_{n+p-1} - \Delta_{m}\Delta_{n-1}}{\Delta_{m+p-1}\Delta_{n+p-1}}
\end{aligned} \tag{G.-31}$$

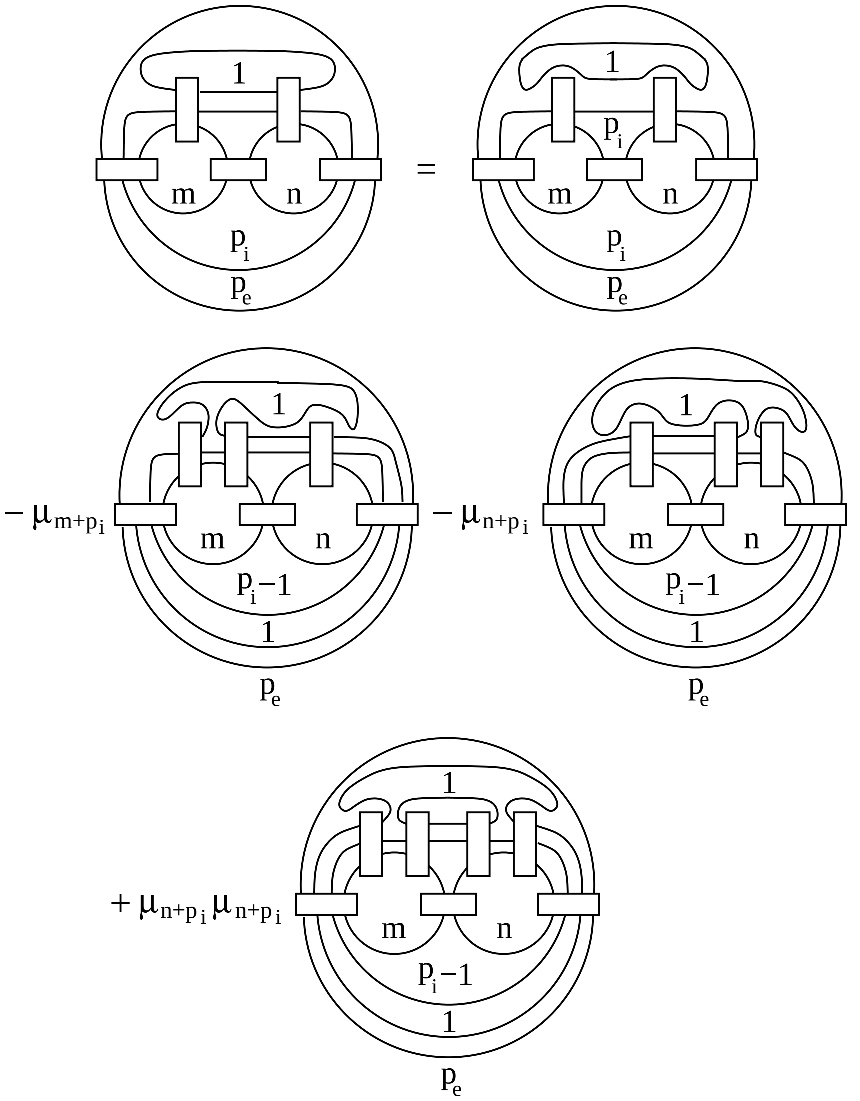


Figure G.43: Netmnpepi.

where we used $\Delta_{k+2} = -2\Delta_{k+1} - \Delta_k$. We can simplify futher,

$$
\begin{aligned}
\Delta_{m+p}\Delta_{n+p-1} - \Delta_m\Delta_{n-1} &= (-1)^{m+p}(m+p+1)(-1)^{n+p-1}(n+p) - (-1)^m(m+1)(-1)^{n-1}n \\
&= (-1)^{m+n+2p-1}[(m+p+1)(n+p) - (m+1)n] \\
&= (-1)^{m+n+2p-1}[np + (m+p+1)p] \\
&= (-1)^{m+n+p}(m+n+p+1)(-1)^{p-1}p \\
&= \Delta_{m+n+p}\Delta_{p-1} \qquad \text{(G.-34)}
\end{aligned}
$$

Therefore

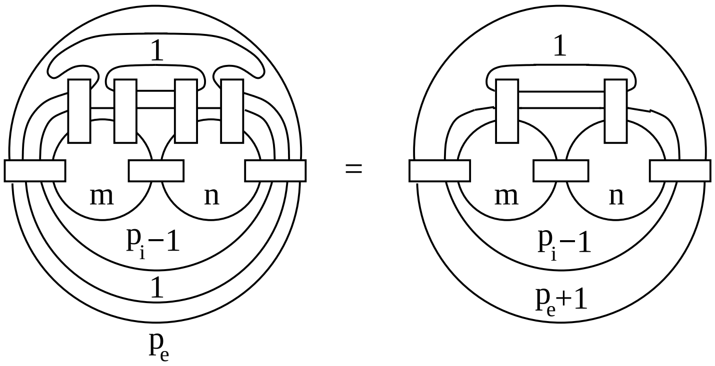


Figure G.44: Netmnpe+1pi-1.

$$\rho(m,n,p) = \frac{\Delta_{m+n+p}\Delta_{p-1}}{\Delta_{m+p-1}\Delta_{n+p-1}} \tag{G.-34}$$

Denote

$$\Delta_n! := \Delta_n \Delta_{n-1} \Delta_{n-2} \cdots \Delta_1.$$

$$\begin{aligned}
\theta(a,b,c) &= \rho(m,n,p) Net(m,n,p-1) \\
&= \left(\prod_{j=1}^{p} \rho(m,n,j)\right) Net(m,n,0) \\
&= \left(\prod_{j=1}^{p} \rho(m,n,j)\right) \Delta_{m+n}
\end{aligned} \tag{G.-35}$$

Hence, by (G.1.5)

$$\begin{aligned}
\theta(a,b,c) &= \prod_{j=1}^{p} \left[\frac{\Delta_{m+n+j}\Delta_{j-1}}{\Delta_{m+j-1}\Delta_{n+j-1}}\right] \Delta_{m+n} \\
&= \frac{\left(\Delta_{m+n+p}\Delta_{m+n+p-1}\cdots\Delta_{m+n}\right)\Delta_{p-1}!}{\left(\Delta_{m+p-1}\Delta_{m+p-2}\cdots\Delta_{m}\right)\left(\Delta_{n+p-1}\Delta_{n+p-2}\cdots\Delta_{n}\right)} \\
&= \frac{\Delta_{m+n+p}!\Delta_{n-1}!\Delta_{m-1}!\Delta_{p-1}!}{\Delta_{m+p-1}!\Delta_{n+p-1}!\Delta_{m+n-1}!}
\end{aligned} \tag{G.-36}$$

The minus signs in the factorial

$$\begin{aligned}
\Delta_{m+n+p}! &= (-1)^{m+n+p}(m+n+p+1)(-1)^{m+n+p-1}(m+n+p)\cdots(-1)^{1}2! \\
&= (-1)^{(m+n+p)+(m+n+p-1)+\cdots+1}(m+n+p+1)! \\
&= (-1)^{(m+n+p)(m+n+p+1)/2}(m+n+p+1)! \qquad \text{(G.-37)}
\end{aligned}$$

So that we get

$$\begin{aligned}
\Delta_{m+n+p}! &= (-1)^{(m+n+p)(m+n+p+1)/2}(m+n+p+1)! \\
\Delta_{m+n-1}! &= (-1)^{(m+n-1)(m+n)/2}(m+n)! \\
\Delta_{m-1}! &= (-1)^{(m-1)m/2}m! \qquad \text{(G.-38)}
\end{aligned}$$

Collecting the exponents of (-1) in (G.-36) is

$$\begin{aligned}
&\frac{1}{2}\big[(m+n+p)(m+n+p+1)+(n-1)n+(m-1)m+(p-1)p+ \\
&+(m+p-1)(m+p)+(n+p-1)(n+p)+(m+n-1)(m+n)\big] \\
&=\frac{1}{2}\big[(m+n+p)^2+n^2+m^2+p^2 \\
&+(m+p)^2-(m+p)+(n+p)^2-(n+p)+(m+n)^2-(m+n)\big] \\
&=\frac{1}{2}\big[(m^2+n^2+p^2+2mn+2mp+2np)+n^2+m^2+p^2+ \\
&+2m^2+2n^2+2p^2+2mp+2np+2mn-2(m+n+p)\big] \\
&=2m^2+2n^2+2p^2+2mn+2np+2pm-m-n-p\equiv m+n+p\ (mod\ 2). \qquad \text{(G.-44)}
\end{aligned}$$

Therefore,

$$\theta(a,b,c)=\frac{(-1)^{m+n+p}(m+n+p+1)!m!n!p!}{(m+n)!(n+p)!(m+p)!} \qquad \text{(G.-43)}$$

where

$$\begin{aligned}
m &= \frac{a+b-c}{2}, \\
n &= \frac{b+c-a}{2}, \\
p &= \frac{a+c-b}{2}. \qquad \text{(G.-44)}
\end{aligned}$$

$$
\begin{aligned}
m+p &= 2a \\
m+n &= 2b \\
n+p &= 2c \\
m+n+p &= 2a+2b+2c. \qquad \text{(G.-46)}
\end{aligned}
$$

TET

Recoupling formula

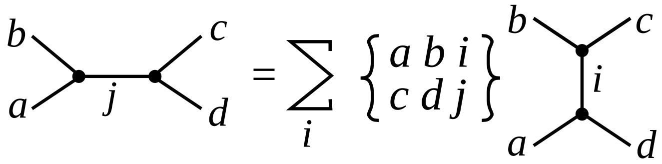


Figure G.45: recouplefig. The recoupling equation

The tetrahedron network.

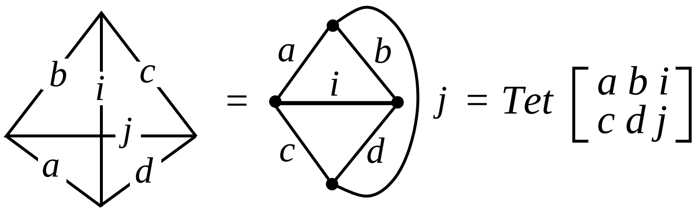


Figure G.46: TetDef.

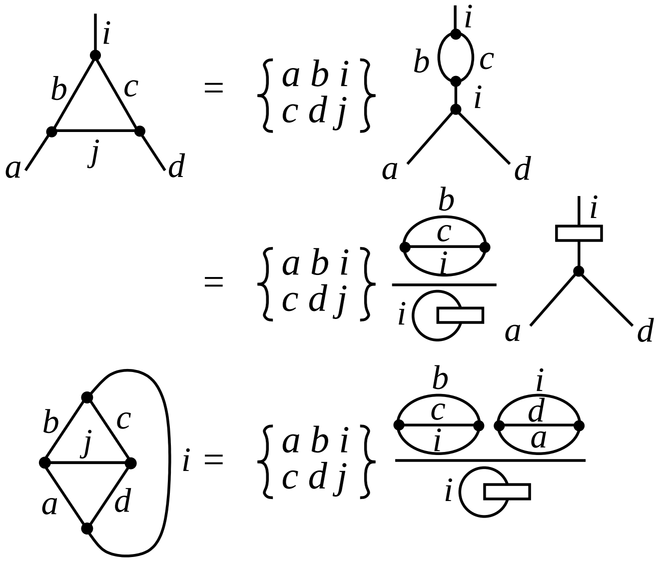


Figure G.47: 6jandTET.

The tetrahedron formula for recoupling theoery.

The evaluation of the tetrahedron network.

$$\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} = \frac{Tet \left[\begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right] \Delta_i}{\theta(a,d,i)\theta(b,c,j)} \tag{G.-45}$$

$$Tet \left[\begin{array}{ccc} A & B & E \\ C & D & F \end{array} \right] = \frac{\mathcal{I}}{\mathcal{E}} \sum_{m \le S \le M} \frac{(-1)^S (S+1)!}{\prod_i (S-a_i)! \prod_j (b_j - S)!} \tag{G.-44}$$

where

$$
\begin{aligned}
a_1 &= \frac{A+D+E}{2}, & b_1 &= \frac{B+D+E+F}{2} \\
a_2 &= \frac{B+C+E}{2}, & b_2 &= \frac{A+C+E+F}{2} \\
a_3 &= \frac{A+B+F}{2}, & b_3 &= \frac{A+B+C+D}{2} \\
a_4 &= \frac{C+D+F}{2}, & & \\
m &= \max\{a_i\}, & M &= \min\{b_j\} \\
\mathcal{E} &= A!B!C!D!E!F!, & \mathcal{I} &= \prod_{ij}(b_j - a_i)!. \qquad \text{(G.-48)}
\end{aligned}
$$

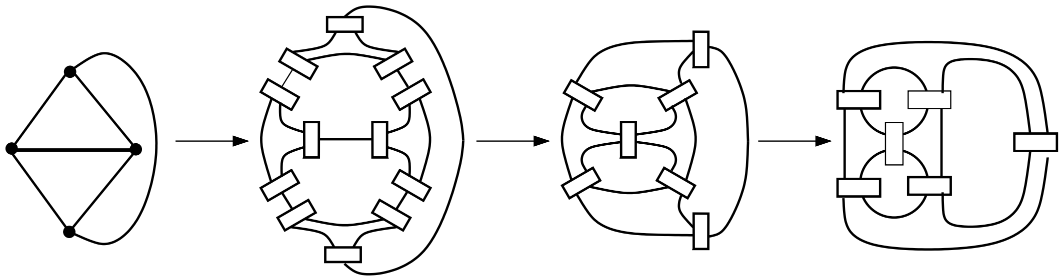


Figure G.48: Tetfig2.

The $6j-$sybols have a number of properties including the orthogonal identity

$$
\sum_l \left\{ \begin{array}{ccc} a & b & l \\ c & d & j \end{array} \right\} \left\{ \begin{array}{ccc} d & a & i \\ b & c & l \end{array} \right\} = \delta_i^j \qquad \text{(G.-48)}
$$

and the Biedenharn-Elliot or Pentagon identity

$$
\sum_l \left\{ \begin{array}{ccc} d & i & l \\ e & m & c \end{array} \right\} \left\{ \begin{array}{ccc} a & b & f \\ e & l & i \end{array} \right\} \left\{ \begin{array}{ccc} a & f & k \\ d & d & l \end{array} \right\} = \left\{ \begin{array}{ccc} a & b & k \\ c & d & i \end{array} \right\} \left\{ \begin{array}{ccc} k & b & f \\ e & m & c \end{array} \right\} \qquad \text{(G.-48)}
$$

The reduction formula

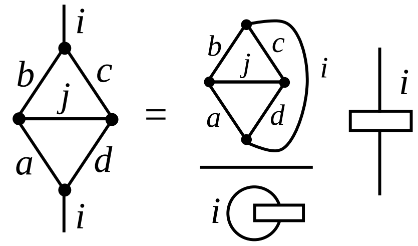


Figure G.49: reductfigs.

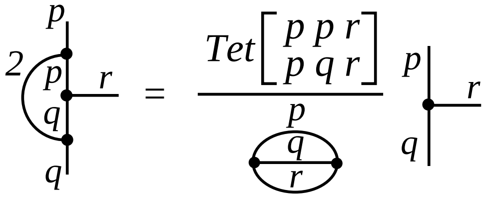


Figure G.50: reductfigs2.

Change of basis for 4-valent spin networks.

(1)

Answers:

Rotate the network on the RHS by clockwise and apply the recoupling identity again.

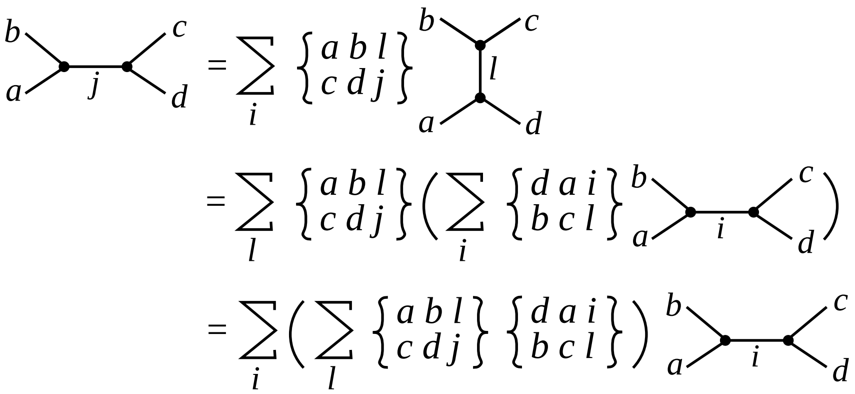


Figure G.51: recoupfig2. Proof of the orthogonality identity.

G.1.6 Angular Momentum Representation

The Pauli matrices are:

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{G.-48}$$

The three matrices

$$\tau^{\mathbf{i}} = -\frac{i}{2}\hat{\sigma}^i \tag{G.-48}$$

are the generators of $SU(2)$ in its fundamental representation.

$$\tau^{\mathbf{i}}\tau^{\mathbf{j}} - \tau^{\mathbf{j}}\tau^{\mathbf{i}} = \epsilon_{ijk}\tau^{\mathbf{k}} \tag{G.-48}$$

Higher-order representations are generated from

$$\tau^{\mathbf{i}}_{(\mathbf{j})} = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes 1 \tag{G.-48}$$

as these can be shown to satisfy

$$\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})} = \epsilon_{ijk}\tau^{\mathbf{k}}_{(\mathbf{j})} \tag{G.-48}$$

$$\left| \frac{1}{2}\, \frac{1}{2} \right\rangle = u^A \quad \text{and} \quad d\left| \frac{1}{2}\, -\frac{1}{2} \right\rangle = d^A, \tag{G.-48}$$

which diagrammatically represented

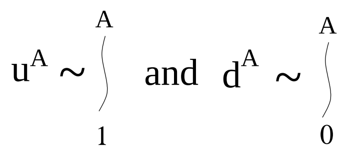


Figure G.52: The "u" stands for up and corresponds to index value $A = 1$. Likewise the "d" for down and corresponds to index value $A = 0$.

The "u" stands for up and corresponds to index value $A = 1$. Likewise the "d" for down and corresponds to index value $A = 0$. The inner product is given by linking upper and lower indices, for instance

For higher representations

$$| \, j \; m\rangle := | \, r \; s\rangle = N_{rs} u^{(A} u^B \dots u^C \;\; d^D d^E \dots d^{F)} \tag{G.-48}$$

in which

$$N_{rs} \left(\frac{1}{r!s!(r+s)!} \right)^{1/2} , \quad j = \frac{r+s}{2}, \quad \text{and} \quad m = \frac{r-s}{2} \tag{G.-48}$$

The parentheses in (G.1.6) around the indices indicate symmetrization, e.g. $u^{(A}u^{B)} = u^A u^B + u^B u^A$. The normalization ensures that the states are orthonormal in the usual inner product.

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{G.-48}$$

with

$$\hat{S}_i = \frac{\hbar}{2} \hat{\sigma}^i \tag{G.-48}$$

for $i = 1, 2, 3$. One has

$$\frac{\hat{\sigma}^3}{2} \left| \, \frac{1}{2} \; \frac{1}{2} \right\rangle = \frac{1}{2} \left| \, \frac{1}{2} \; \frac{1}{2} \right\rangle, \tag{G.-48}$$

$$(\tau_{AB}) := (\sigma_A) \otimes (\rho_B) \tag{G.-48}$$

$$(\tau_{AB}) = (\eta_A) \otimes (\rho_B) \neq (\rho_B) \otimes (\eta_A) = (\tau_{BA}) \tag{G.-48}$$

$$(\hat{\sigma}_{AC} \otimes \hat{\sigma}_{BD})((\eta_C) \otimes (\rho_D)) = (\hat{\sigma}_{AC} \eta_C) \otimes (\hat{\sigma}_{BD} \rho_D) \tag{G.-48}$$

The antisymmetric tensor is invariant under the action of $SU(2)$

$$U^A{}_C U^B{}_D \epsilon^{CD} = \epsilon^{AB} \tag{G.-48}$$

(analogous to how the 3×3 tensor (δ_{ab} is invariant under the action of $SO(3)$ i.e. $O^c{}_a O^d{}_b \delta_{cd} = \delta_{ab}$, which preserves the scalar product between two vectors under rotation).

Contracting this equation with ϵ_{AB} we obtain the condition that the determinate of U is one

$$\det U = \frac{1}{2}\epsilon_{AC}\epsilon^{BD}U^{A}_{\ B}U^{C}_{\ D} = 1 \tag{G.-48}$$

since

$$\epsilon_{AB}\epsilon^{AB} = 2. \tag{G.-48}$$

The inverse of an $SU(2)$ matrix can be written as

$$(U^{-1})^{A}_{\ B} = -\epsilon_{BD}U^{D}_{\ C}\epsilon^{CA}. \tag{G.-48}$$

Recall the anti-symmetric tensors (G.1.1) are used for raising or lowering but must be careful about the down-left-up-right rule:

$$\begin{aligned} \eta^{A} &= \epsilon^{AB}\eta_{B} \\ \zeta_{A} &= \zeta^{B}\epsilon_{BA}. \end{aligned} \tag{G.-48}$$

We have the identity

$$\frac{1}{2}\sum_{i=1}^{3}\hat{\sigma}_{i\ A}^{\ B}\hat{\sigma}_{i\ C}^{\ D} = \frac{1}{2}(\epsilon_{AC}\epsilon^{BD} - \delta_{A}^{D}\delta_{C}^{B}) \tag{G.-48}$$

We check this by direct calculation, in reference to (G.1.6). There are 16 possibilities in total.

$A = C$, $B \neq D$:

$$\begin{aligned} \delta^{ij}\sigma_{i\ 0}^{\ 0}\sigma_{j\ 1}^{\ 0} &= 0 \\ \delta^{ij}\sigma_{i\ 1}^{\ 0}\sigma_{j\ 0}^{\ 0} &= 0 \\ \delta^{ij}\sigma_{i\ 0}^{\ 1}\sigma_{j\ 1}^{\ 1} &= 0 \\ \delta^{ij}\sigma_{i\ 1}^{\ 1}\sigma_{j\ 0}^{\ 1} &= 0 \end{aligned} \tag{G.-50}$$

$B = D$, $A \neq C$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^1{}_0 &= 0\\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^0{}_0 &= 0\\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^1{}_1 &= 0\\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^0{}_1 &= 0
\end{aligned}
\tag{G.-52}
$$

$A = C$ and $B = D$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^0{}_0 &= 1\\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^0{}_1 &= 0\\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^1{}_0 &= 0\\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^1{}_1 &= 1
\end{aligned}
\tag{G.-54}
$$

$A \neq C$ and $B \neq D$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^1{}_1 &= -1\\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^1{}_0 &= 2\\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^0{}_1 &= 2\\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^0{}_0 &= -1
\end{aligned}
\tag{G.-56}
$$

All 16 possible answers have been shown to be in accordance with (G.1.6).

The identity (G.1.6) is expressed diagrammatically as in fig G.53

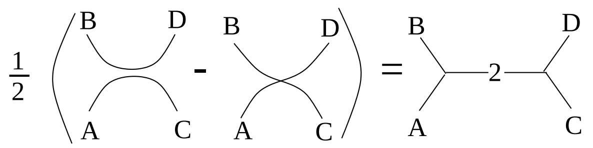
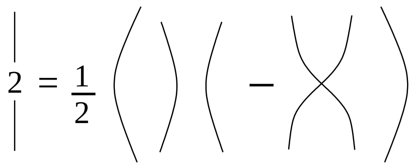


Figure G.53: The identity: $\frac{1}{2}(\epsilon_{AC}\epsilon^{BD} - \delta_A^D\delta_C^B) = \frac{1}{2}\sum_{i=1}^3 \hat{\sigma}_i{}^B_A\hat{\sigma}_i{}^D_C$.

Edges may be further joined into networks by making use of internal trivalent vertices

G.1.7 Recoupling Theory: Combinatorics of Angular Momentum

The rules of addition of angular momentum are known as recoupling theory



G.1.8 Loop states and the spin network states

$$\psi_\alpha(A) = -\mathrm{Tr}U_\alpha(A) \tag{G.-56}$$

Proof: The orthogonality relations for loop-network states on a given graph γ follow from basic group integration theory. By the Peter-Weyl theorem we have

$$< S, S >= \tag{G.-55}$$

where we have used that the non-equivalent irreducible - as well as our choice of equivalent - representations of a compact group are orthogonal, that is, $\pi(1)$ is a projector.

We can thus follow the contraction along the graph, obtaining a sequence of the edges $(e_1, e_2, \ldots)$. Since the graph is finite, the sequence must close on itself.

The completeness of these states for $L_2(\mathcal{A}/\mathcal{G}, d\mu_{0,\gamma})$ follows also from the Peter-Weyl theorem together with a gauge-invariance argument:

hep-th/9601105 19th Jan 1996

$$|\Psi_\alpha|^2 = \int dU |\mathrm{Tr}U|^2 = 1 \tag{G.-55}$$

$$\Psi_{[\alpha]}[A] = \Psi_{\alpha_1}[A] \ldots \Psi_{\alpha_n}[A] = \mathrm{Tr}H(A,\alpha_1) \ldots \mathrm{Tr}H(A,\alpha_n) \tag{G.-55}$$

The function

$$\Psi(\alpha) = \int d\mu_0[A] \mathrm{Tr}\mathcal{P}e^{\oint_\alpha A}\Psi[A] \tag{G.-55}$$

G.1.9 Summary of binary calculus and recoupling theory

The dashed circle is a magnification of the dot in the diagram on the left. Such dashed lines indicate spin network structures at a point. The internal labels i, j, k are positive integers determined by the external labels a, b, c via

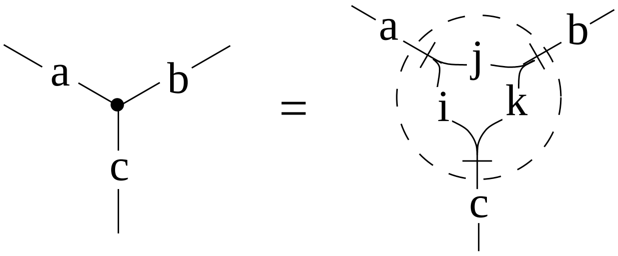
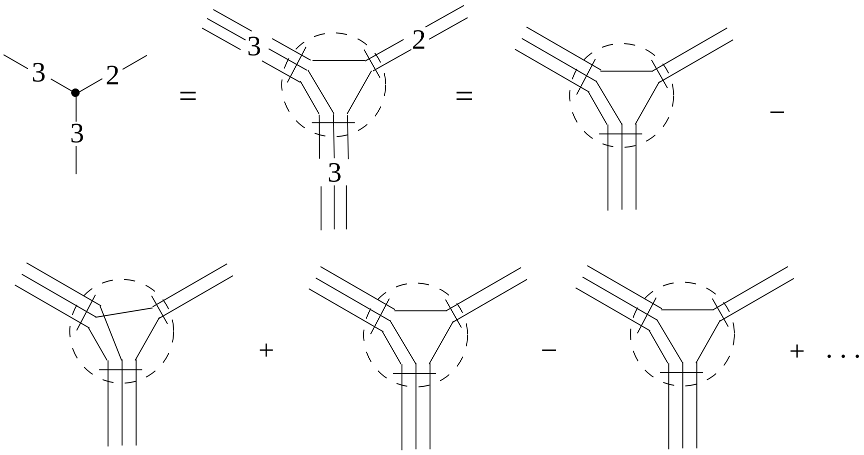


Figure G.54: Elementary recoupling.

$$a = i + j, \quad b = j + k, \quad c = i + k \tag{G.-55}$$

or

$$i = (a + c - b)/2, \quad j = (a + b - c)/2, \quad k = (b + c - a)/2. \tag{G.-55}$$



as can be seen by solving the simultaneous equations (G.1.9), or by drawing the strands through the vertex (see fig()). As in quantum mechanics the external labels satisfy the triangular inequalities

$$a + b \geq c, \quad b + c \geq a, \quad a + c \geq b \tag{G.-55}$$

and the sum $a + b + c$ is an even integer (see Eq.(G.1.9)).

• Jones-Wenzl projectors.

• q-deformed binary calculus.

G.1.10 The Spectrum of the Area Operator

$$\mathcal{A}[S] = \int_S d^2\sigma \sqrt{g^S} = \int_S d^2\sigma \sqrt{n_a n_b \tilde{E}^a_i \tilde{E}^b_j} \tag{G.-55}$$

We need to turn express $\mathcal{A}[S]$ in terms of loop variables, which can be accomplished through a limiting procedure.

$$U(\gamma, A) = \mathcal{P}\exp\left\{\int_\gamma A_a(\gamma(s))\frac{d\gamma^a}{ds}ds\right\} \tag{G.-55}$$

$$\frac{\delta}{\delta A^i_a(x)}U(\gamma, A) = \int_\gamma ds\; \dot{x}^a(s)\; \delta^3(\gamma(s), x)\; U(\gamma(0,s), A))\; \tau^i\; U(\gamma(s,1), A)). \tag{G.-55}$$

$$\tilde{E}(S) = \sqrt{\tilde{E}(S)^i \tilde{E}(S)^i}. \tag{G.-55}$$

Acting on a state $\mathbf{\Psi_s}$, that intersects Σ only once, it gives

$$\begin{aligned} E^2(\Sigma)\mathbf{\Psi_s}(A) &= -\mathbf{\Psi_{s-\gamma}}{}^{lm}(A)\left(j(U[\gamma(0,s),A)]\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)}j(U[\gamma(s,\gamma),A))\right)_{lm} \\ &= j(j+1)\mathbf{\Psi_s}(A) \end{aligned} \tag{G.-55}$$

Since one has for the Casimir operator

$$\tau^{(j)}_i \tau^{(j)}_i = -j(j+1)\mathbf{1} \tag{G.-55}$$

Thus,

$$E(\Sigma)\mathbf{\Psi_s}(A) = \sqrt{j(j+1)}\mathbf{\Psi_s}(A) \tag{G.-55}$$

$$\hat{A}_S|s> \frac{G}{4c^3}\sum_i \hat{O}_i^{1/2}|s> \tag{G.-55}$$

$$\begin{aligned} \hat{O}_e|s> &= -\hat{J}^2|s> \\ &= -\hbar^2\frac{n^2}{2} \end{aligned} \tag{G.-55}$$

$$\theta(n,n,1) = Net(n-1,1,1) \;=\; (d-\mu_{n-1}-\mu_1)\Delta_n$$
$$= (-2+\frac{n}{n+1}\frac{1}{2})(-1)^n(n+1) \qquad \text{(G.-55)}$$

$$\frac{\theta(n,n,1)}{\Delta_n} = (-2+\frac{n}{n+1}\frac{1}{2}) = \left(-\frac{n+2}{2n}\right) \qquad \text{(G.-55)}$$

Now the diagram may be reduced using the recoupling identities. The bubble may be extracted with ()

$$\begin{aligned}
\hat{O}_e|s> &= -\hbar^2\frac{n^2}{2}|(s-e)> \\
&= -\hbar^2\frac{n^2}{2}\frac{\theta(n,n,2)}{\Delta_n}|(s-e)> \\
&= -\hbar^2\frac{n^2}{2}\left(-\frac{n+2}{2n}\right)|s> \\
&= -\hbar^2\frac{n(n+1)}{4}|s>
\end{aligned} \qquad \text{(G.-57)}$$

Details J

$$j(U[\gamma(s,\gamma)])^{A'B'\ldots D'}_{AB\ldots D} = U[\gamma(s,\gamma)]^{(A'}_{A}U[\gamma(s,\gamma)]^{B'}_{B}\ldots U[\gamma(s,\gamma)]^{C')}_{C} \qquad \text{(G.-57)}$$

$$\delta^D_{A'}\delta^E_{B'}\ldots\delta^F_{C'}\; j(U[\gamma(s,\gamma)])^{A'B'\ldots D'}_{AB\ldots C} = \delta^D_{(A'}\delta^E_{B'}\ldots\delta^F_{C')}\; j(U[\gamma(s,\gamma)])^{A'B'\ldots D'}_{AB\ldots D} \qquad \text{(G.-57)}$$

$$\begin{aligned}
j(U[\gamma(s,\gamma)])^{DE\ldots F}_{AB\ldots D} &= \delta^D_{(0}\delta^E_{0}\ldots\delta^F_{0)}\; j(U[\gamma(s,\gamma)])^{00\ldots0}_{AB\ldots C} + \delta^D_{(1}\delta^E_{0}\ldots\delta^F_{0)}\; j(U[\gamma(s,\gamma)])^{10\ldots0}_{AB\ldots C} + \ldots \\
&+ \delta^D_{(1}\delta^E_{1}\ldots\delta^F_{1)}\; j(U[\gamma(s,\gamma)])^{11\ldots1}_{AB\ldots C}
\end{aligned} \qquad \text{(G.-57)}$$

or

$$\begin{aligned}
j(U[\gamma(s,\gamma)])^{DE\ldots F}_{AB\ldots D} &= \omega^{DE\ldots F}(i=0)\; j(U[\gamma(s,\gamma)])^{00\ldots0}_{AB\ldots C} + \omega^{DE\ldots F}(i=1)\; j(U[\gamma(s,\gamma)])^{10\ldots0}_{AB\ldots C} + \ldots \\
&+ \omega^{DE\ldots F}(i=2j)\; j(U[\gamma(s,\gamma)])^{11\ldots1}_{AB\ldots C}
\end{aligned} \qquad \text{(G.-57)}$$

$$\left(\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)}\right)^{AB...C}_{DE...F} \omega^{DE...F}(i=0,1,\ldots,2j) = j(j+1)\omega^{AB...C}(i=0,1,\ldots,2j) \quad \text{(G.-57)}$$

Hence

$$\left(\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)}\right)^{AB...C}_{DE...F} j(U[\gamma(s,\gamma)])^{DE...F}_{A'B'...C'} = j(j+1)\; j(U[\gamma(s,\gamma)])^{AB...C}_{A'B'...C'} \quad \text{(G.-57)}$$

or

$$\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)}\; j(U[\gamma(s,\gamma)]) = j(j+1)\; j(U[\gamma(s,\gamma)]) \quad \text{(G.-57)}$$

Quotations to do with important conceptual points in Rovelli's paper which I have attempted to cover in this last section.

gr-qc/9806079 Loop quantum gravity and quanta of space: a primer

$$\theta(a,b,c) = \frac{(-1)^{m+n+p}(m+n+p+1)m!n!p!}{a!b!c!} \quad \text{(G.-57)}$$

where $m=(a+b-c)/2$, $m=(b+c-a)/2$, $p=(c+a-b)/2$,

G.1.11 The Spectrum of the Volume Operator

$$V[\sigma]_\epsilon = \int_\sigma \sqrt{g} = \int \sqrt{\epsilon_{abc}\epsilon_{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k} \quad \text{(G.-57)}$$

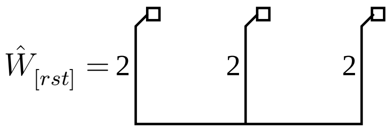


Figure G.55: grasp3edge.

Since the "comb" basis spans the space of all intertwiners, we can write the action of the W operator as sending the original vertex to a superposition of other vertices in the same basis:

$$\sum_{k_2,\ldots,k_{n-2}} \hat{W}_{[rst]i_2,\ldots,i_{n-1}}^{\quad k_2,\ldots,k_{n-1}}$$

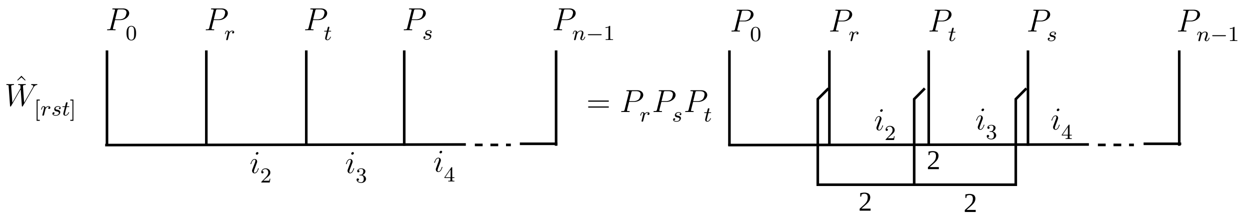


Figure G.56: grasp3edge2.

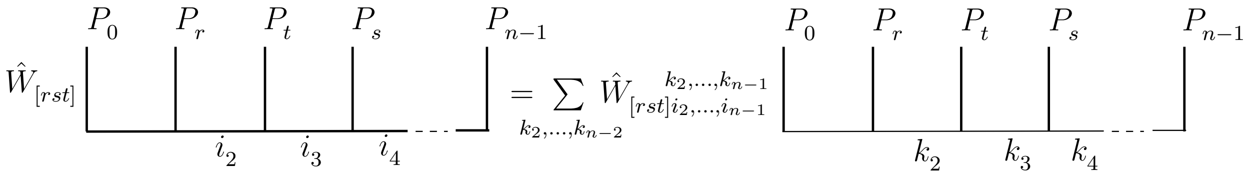


Figure G.57: grasp3edge3.

$$W^{(n)\ k_2\cdots k_{n-2}}_{[rst]\ i_2\cdots i_{n-2}} = -P_r P_s P_t \left\{ \begin{array}{ccc} k_2 & P_t & k_3 \\ i_2 & P_t & i_3 \\ 2 & 2 & 2 \end{array} \right\} (\lambda^{i_2}_{k_2})^2 \delta^{k_4}_{i_4} \cdots \delta^{k_{n-2}}_{i_{n-2}}$$
$$\times \frac{Tet \left[\begin{array}{ccc} P_r & P_r & P_0 \\ k_2 & i_2 & 2 \end{array} \right] Tet \left[\begin{array}{ccc} P_s & P_s & k_4 \\ k_3 & i_3 & 2 \end{array} \right] \Delta_{k_2} \Delta_{k_3}}{\theta(k_2, i_2, 2)\theta(k_3, i_3, 2)\theta(P_0, P_r, k_2)\theta(k_2, k_3, P_t)\theta(k_3, k_4, P_s)} \quad \text{(G.-57)}$$

Using (G.-45), this can be expressed in terms of the Kauffman-Lins 6-j symbols

$$W^{(n)\ k_2\cdots k_{n-2}}_{[rst]\ i_2\cdots i_{n-2}} = -P_r P_s P_t \left\{ \begin{array}{ccc} k_2 & P_t & k_3 \\ i_2 & P_t & i_3 \\ 2 & 2 & 2 \end{array} \right\} (\lambda^{i_2}_{k_2})^2 \delta^{k_4}_{i_4} \cdots \delta^{k_{n-2}}_{i_{n-2}}$$
$$\frac{\left\{ \begin{array}{ccc} P_r & P_r & P_0 \\ k_2 & i_2 & 2 \end{array} \right\} \left\{ \begin{array}{ccc} P_s & P_s & k_4 \\ k_3 & i_3 & 2 \end{array} \right\}}{\theta(P_0, P_r, k_2)} \quad \text{(G.-57)}$$

In both these formulas we have used the 9-j symbol, which is given by the spin network fig.(G.58)

The eigenvalues of the volume operator are then proportional to the sum of the absolute values of the W-eigenvalues:

$$\hat{V} = \sqrt{\sum_{0 \le r < s < t \le n-1} |\frac{i}{16} \hat{W}_{[rst]}|} \quad \text{(G.-57)}$$

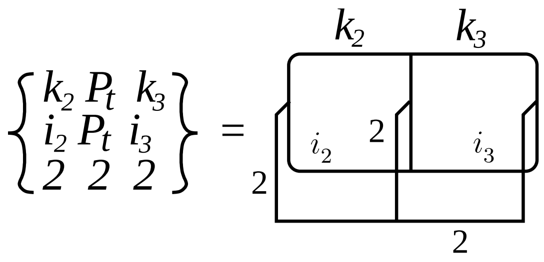


Figure G.58: 9jsymbolFig.

$$W_{[012]}{}^{k}_{i} = \frac{P_0 P_1 P_2 \begin{Bmatrix} P_0 & P_1 & k \\ P_0 & P_1 & i \\ 2 & 2 & 2 \end{Bmatrix} \Delta_j}{\theta(P_0, P_1, j)\theta(P_2, P_3, j)\theta(i, j, 2)} \tag{G.-57}$$

G.1.12 Reidermeister Moves

Remarkably, a knot in three dimensional space can be continuously deformed into another knot, if and only if, the planar projection of the knots can be transformed into each other via a sequence of four moves called the "Reidermeister moves" [76].

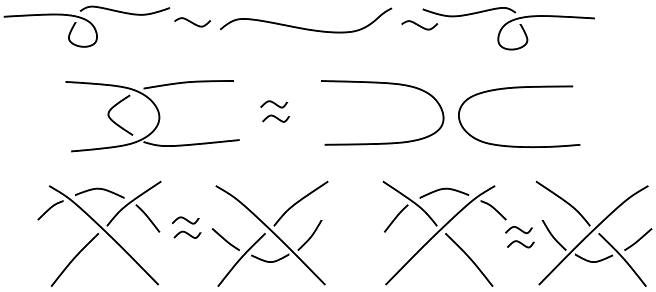


Figure G.59: Reidemeister moves.

G.1.13 Kauffman Bracket

Quantum $SU(2)$ group.

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{G.-57}$$

$$U \tilde{\epsilon} U^T = \tilde{\epsilon} \tag{G.-57}$$

$$
\begin{aligned}
&ba = qab \quad dc = qcd \\
&ca = qac \quad db = qbd \\
&bc - cb = 0 \\
&ad - da = -(q - q^{-1})bc \\
&ad - q^{-1}bc = 1
\end{aligned}
\tag{G.-60}
$$

where $q = aA^2$. complex non-commuting components a, b, c, d.

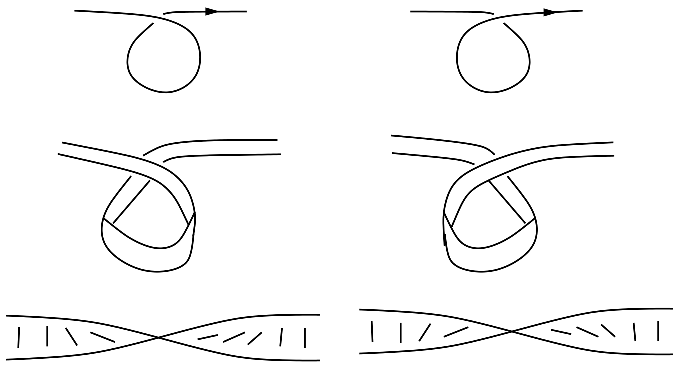


Figure G.60: Twist and q-deformed $su(2)$ - $su_q(2)$ (or quantum group of $su(2)$).

G.1.14 The Braid Group B_N

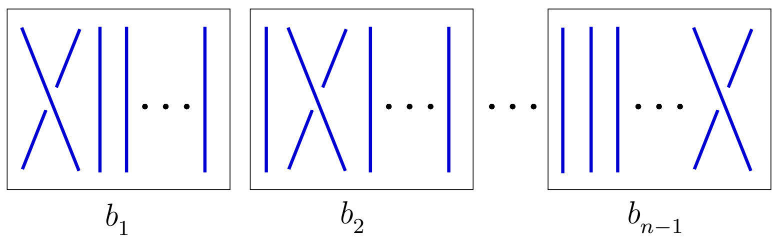


Figure G.61: braidgroup. The generators of the braid group.

The n-stranded braid group

$$
\begin{cases}
b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & \text{for } 1 \leq i < n \\
b_i b_j = b_j b_i & \text{for } |i - j| \geq 2
\end{cases}
\tag{G.-60}
$$

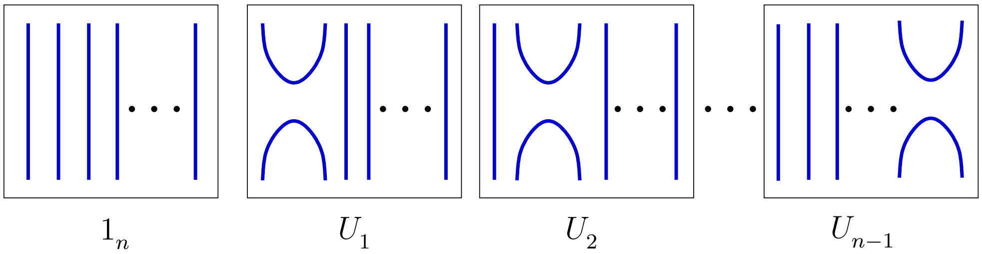


Figure G.62: Tempalgebra. The generators of the Temperley-algebra.

G.1.15 Temperley-Lieb Algebra

d is the value assigned to a closed loop

$$\begin{cases} U_i^2 = dU_i \\ U_i U_{i\pm 1} U_i = U_i \\ U_i U_j = U_J U_i \qquad \text{for } \ |i-j| \geq 1 \end{cases} \tag{G.-60}$$

Proposition G.1.2 *If $g_n \in T_n$ denotes the image of ... in the Temperley-Lieb algebra T_n, then*

(i) $g_n^2 = g_n$

(ii) $g_n U_i = U_i g_n = 0$ for all $i = 1, 2, \ldots, n-1$.

The canonical construction

$$g_n = \frac{1}{\{n\}!} \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)} \hat{\sigma} \tag{G.-60}$$

$$\{n\}! := \sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)} \tag{G.-60}$$

$$\{n\}! := \prod_{k=1}^{n} \frac{1 - A^{-4k}}{1 - A^{-4}} \tag{G.-60}$$

$$\{2\}! = 1 + A^{-4} = \frac{1 - (A^{-4})^2}{1 - A^{-4}} \tag{G.-60}$$

From fig.(G.65). From the we have first row is $1 + A^{-4k} + (A^{-4k})^2$, while the second row is the same thing but multiplied by A^{-4k}. This is obviously because, while in the first row the blue lines are not crossing, in the second they are crossing once. Hence,

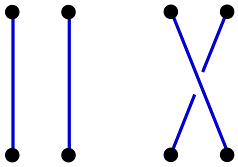


Figure G.63: 2tangEx.

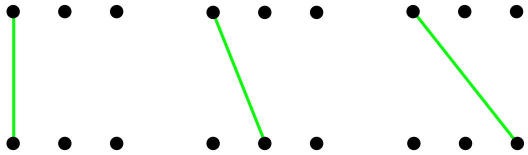


Figure G.64: posBraids.

$$\begin{aligned}\{3\}! &= \big(1 + A^{-4k} + (A^{-4k})^2\big) + A^{-4}\big(1 + A^{-4k} + (A^{-4k})^2\big) \\ &= \underbrace{(1 + A^{-4})}_{\text{blue line factor}} \underbrace{\big(1 + A^{-4k} + (A^{-4k})^2\big)}_{\text{green line factor}} \\ &= \{2\}! \cdot \frac{1 - (A^{-4})^3}{1 - A^{-4}} \qquad \text{(G.-61)}\end{aligned}$$

Similar reasoning applies to $\{4\}!$,

$$\begin{aligned}\{4\}! &= \{3\}! \cdot \big(1 + A^{-4k} + (A^{-4k})^2 + (A^{-4k})^3\big) \\ &= \{3\}! \cdot \frac{1 - (A^{-4})^4}{1 - A^{-4}} \qquad \text{(G.-61)}\end{aligned}$$

and so on,

$$\begin{aligned}\{r+1\}! &= \{r\}! \cdot \big(1 + A^{-4k} + (A^{-4k})^2 + \cdots + (A^{-4k})^r\big) \\ &= \{r\}! \cdot \frac{1 - (A^{-4})^{r+1}}{1 - A^{-4}} \qquad \text{(G.-61)}\end{aligned}$$

$\square$

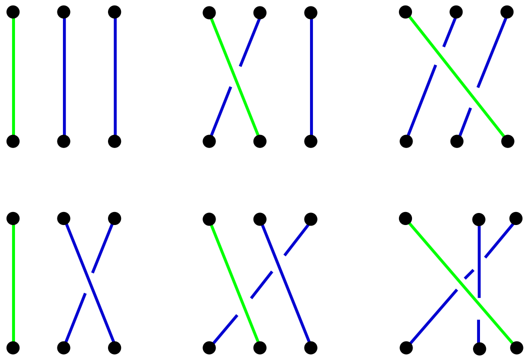


Figure G.65: canconconstr.

Proof of properties

Given any $i \in \{1, 2, \ldots, n-1\}$ choose the set braids W that do not end in σ_i or σ_i^{-1}, as is easy to see from the canonical construction, the remaining braids in $\{n\}!g_n$ are given by the set $W' = \{w\sigma_i \mid w \in W\}$ (note that the choice of the set W ensures the minimality of the braids in W'). So that we have

$$\{n\}!g_n = \sum_{w \in W} (A^{-3})^{t(w)} w + (A^{-3})^{t(w)+1} w\sigma_i, \tag{G.-61}$$

we have given examples in fig.(G.66). Since $w\sigma_i U_i = (-A^{-3}) w U_i$ in T_n, it follows that $g_n U_i = 0$ for $i = 1, 2, \ldots, n-1$.

$\sum_{w \in W} (A^{-3})^{t(w)} w =$

$\sum_{w \in W} (A^{-3})^{t(w)} w\sigma_1 =$

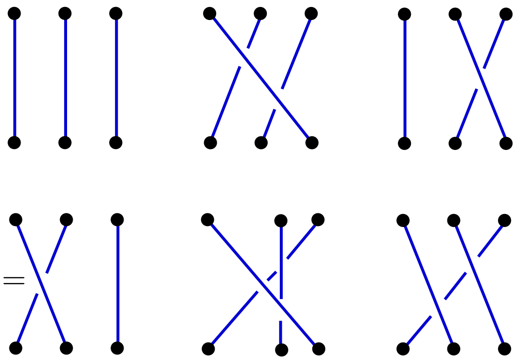
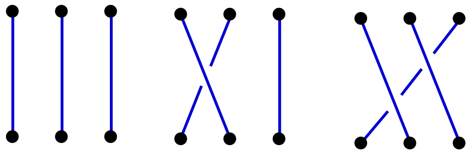


Figure G.66: ProofpropQ1.

it is claimed that the coefficient of 1_n is $\{n\}!$

$$\tilde{g}_n := \sum_{\sigma} (A^{-3})^{t(\sigma)} \hat{\sigma}$$

$\sum_{w \in W} (A^{-3})^{t(w)} w =$ 

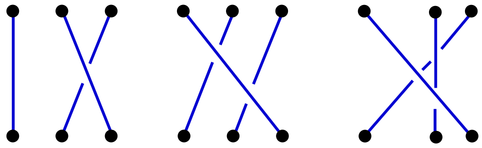
$\sum_{w \in W} (A^{-3})^{t(w)} w \sigma_2 =$ 

Figure G.67: ProofpropQ2.

$$\hat{\sigma} = (A^{-1})^{t(\sigma)} 1_n + \dots$$

$$\hat{g}_n := \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)} \left[(A^{-1})^{t(\sigma)} 1_n + \dots \right] + \dots = \{n\}! 1_n + \dots$$

Hence the coefficient of 1_n in $\tilde{g}_n$ is the sum $\sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)} = \{n\}!$.

$\square$

G.1.16 The twist move

$$\lambda_c^{ab} = (-1)^{(a+b-c)/2} A^{[a(a+2)+(b(b+2)+(c(c+2)]/2} \quad \text{(G.-61)}$$

G.1.17 q-deformed Recoupling Theory

We will modify our spin network technology slightly to make there be only finitely many vector spaces "j".

q-deformed graphs are ribbon (framed) graphs with braiding. Thus, any undeformed spin network has to be supplemented with information about twists and crossings before evaluation.

[PDF] SPIN NETWORKS AND THE BRACKET POLYNOMIAL File Format: PDF/Adobe Acrobat - View as HTML LC Jeffrey, Chern-Simons-Witten invariants of lens spaces and torus bundles, and the ... E. Witten, Quantum field theory and the Jones polynomial, Comm. ...

We will modify the binor identity.

We would like to pick A and B so that the Reidemeister moves are preserved.

Note: the first Reidemeister move didn't hold before!

We'll call it d. So we want:

$$A^2 + A^{-2} + d = 0 \tag{G.-61}$$

This tells us $d = -(A^2 + A^{-2})$.

We want identity:

Which implies $A = B$. $d = -2$

Applications:

(i) Theory with non-zero cosmological constant seem to require the use of q-deformed spin networks.

(ii) q-spin networks are manifold invariants making them usefull in mathematical investigations in topology.

Smolin and Markoppoulou have used this to define abstract states of quantum gravity which encode the topology in the quantum state itself []. Smolin has developed a very tentative formulation of M-theory, arguing that topology change that is needed for mirror symmetry in string theory []. Smolin has also shown that perturbations in the q-deformed theory look very much like propagating strings [].

(iii)

(v) Infrared regularization in .spin foam state sum models.

G.1.18 Jones Polynomial

$B = A^{-1}$ we also need $A^2 + C + A^{-1} = 0$ so

$$C = -A^2 - A^{-2} \tag{G.-61}$$

G.2 Spatialy Diffeomorphism Invariant Space

$$f^a(\alpha(t(s))) = \beta^a(s) \tag{G.-61}$$

G.3 General Madlestam Identities

Mandelstam identiies of the first kind are a simple consequence of the cyclic property of matrices, ($Tr(\mathbb{A}\mathbb{B}) = Tr(\mathbb{B}\mathbb{A})$ for matrices $\mathbb{A}$ and $\mathbb{B}$),

$$W(\gamma_1 \circ \gamma_2) = W(\gamma_2 \circ \gamma_1). \tag{G.-61}$$

This holds for any gauge group of any dimension.

There are various identities of the second kind. Here is an example. First note that the product of $N+1$ δ's of dimension$N$ and anti-symmetrised indices is identically zero,

$$\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_{N+1}}_{B_{N+1}]} = 0. \tag{G.-61}$$

Then contract this with

$$H(\gamma_1)_{A_1}{}^{B_1}\dots H(\gamma_{N+1})_{A_{N+1}}{}^{B_{N+1}}. \tag{G.-61}$$

The result is an identically vanishing sum of products of the traces of products of holonomies.

For example for $N = 1$

$$\begin{aligned} 0 &\equiv 2\sum_{A_1,B_1,A_2,B_2=1}^{1} \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2]} H(\gamma_1)_{A_1}{}^{B_1} H(\gamma_2)_{A_2}{}^{B_2} \\ &= \sum_{A_1,B_1,A_2,B_2=1}^{1} (\delta^{A_1}_{B_1}\delta^{A_2}_{B_2} - \delta^{A_1}_{B_2}\delta^{A_2}_{B_1}) H(\gamma_1)_{A_1}{}^{B_1} H(\gamma_2)_{A_2}{}^{B_2} \\ &= W(\gamma_1)W(\gamma_2) - W(\gamma_1 \circ \gamma_2). \end{aligned} \tag{G.-62}$$

There is a compact way of writing this identity for an arbitrary order in terms of the quantities

$$M_K := \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_K]} H(\gamma_1)_{A_1}{}^{B_1}\dots H(\gamma_K)_{A_K}{}^{B_K}. \tag{G.-62}$$

We show below that M_K satifies the following recursive relation

$$\begin{aligned}(K+1)M_{K+1}(\gamma_1,\dots,\gamma_{K+1}) &= W(\gamma_{K+1})M_K(\gamma_1,\gamma_2,\dots,\gamma_K) - \cdots \\ &\cdots - M_K(\gamma_1\circ\gamma_{K+1},\gamma_2,\dots,\gamma_K) \\ &\cdots - M_K(\gamma_1\dots\gamma_i\circ\gamma_{K+1},\dots,\gamma_K) \\ &\cdots - M_K(\gamma_1,\gamma_2,\dots,\gamma_K\circ\gamma_{K+1}) \qquad \text{(G.-64)}\end{aligned}$$

with

$$M_1(\gamma) = W(\gamma). \qquad \text{(G.-64)}$$

In terms of the Ms, the identity for an $N\times N$ matrix group can be written as

$$M_{N+1}(\gamma_1,\dots,\gamma_{N+1}) = 0. \qquad \text{(G.-64)}$$

Let us now derive the recursive relation. For $K=2$ this comes from the identity

$$2\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2]} = 2\frac{1}{2}(\delta^{A_1}_{B_1}\delta^{A_2}_{B_2} - \delta^{A_1}_{B_2}\delta^{A_2}_{B_1}) \qquad \text{(G.-64)}$$

As can be seen by contracting this with $H(\gamma_1)_{A_1}{}^{B_1}H(\gamma_1)_{A_2}{}^{B_2}$ resulting in

$$2M_2(\gamma_1,\gamma_2) = W(\gamma_1)M_1(\gamma_2) - M_1(\gamma_1\circ\gamma_2). \qquad \text{(G.-64)}$$

For $K=3$ we have

$$\begin{aligned}3\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\delta^{A_3}_{B_3]} &= 3\frac{1}{3!}(\delta^{A_1}_{B_1}\delta^{A_2}_{B_2}\delta^{A_3}_{B_3} - \delta^{A_1}_{B_3}\delta^{A_2}_{B_2}\delta^{A_3}_{B_1} - \delta^{A_1}_{B_1}\delta^{A_2}_{B_3}\delta^{A_3}_{B_2} \\ &\quad - \delta^{A_1}_{B_2}\delta^{A_2}_{B_1}\delta^{A_3}_{B_3} + \delta^{A_1}_{B_2}\delta^{A_3}_{B_3}\delta^{A_3}_{B_1} + \delta^{A_1}_{B_3}\delta^{A_2}_{B_1}\delta^{A_2}_{B_2} \\ &= \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2]}\delta^{A_3}_{B_3} - \delta^{A_1}_{[B_3}\delta^{A_2}_{B_2]}\delta^{A_3}_{B_1} - \delta^{A_1}_{[B_1}\delta^{A_2}_{B_3]}\delta^{A_3}_{B_2}) \qquad \text{(G.-65)}\end{aligned}$$

Contracting this with $H(\gamma_1)_{A_1}{}^{B_1}H(\gamma_2)_{A_2}{}^{B_2}H(\gamma_3)_{A_3}{}^{B_3}$ gives

$$3M_3(\gamma_1,\gamma_2,\gamma_3) = W(\gamma_3)M_2(\gamma_1,\gamma_2) - M_2(\gamma_1\circ\gamma_3,\gamma_2) - M_2(\gamma_1,\gamma_2\circ\gamma_3) \qquad \text{(G.-65)}$$

For arbitary K we have

$$
\begin{aligned}
(K+1)\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_{K+1}}_{B_{K+1}]} &= \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_K]}\delta^{A_{K+1}}_{B_{K+1}} \\
&- \delta^{A_1}_{[B_{k+1}}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_K]}\delta^{A_{K+1}}_{B_1} \\
&- \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_i}_{B_{k+1}}\cdots\delta^{A_K}_{B_K]}\delta^{A_{K+1}}_{B_i} \\
&- \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_{K+1}]}\delta^{A_{K+1}}_{B_K} \qquad \text{(G.-67)}
\end{aligned}
$$

To see this note that the LHS is made up of a total of $(K+1)!$ distinct terms where we have a plus sign when we have an even permutation of $(B_1, B_2, \ldots, B_{K+1})$ and a minus sign when we have an odd permutation of $(B_1, B_2, \ldots, B_{K+1})$. The RHS comprises of $K+1$ collections of terms where each of these comprises of $K!$ terms, and hence there are $(K+1)!$ individual terms altogether. Each of these corresponds to a distinct permutation of $(B_1, B_2, \ldots, B_{K+1})$ and appears with the correct plus sign or minus sign to agree with the LHS. Contracting this with $H(\gamma_1)_{A_1}{}^{B_1} H(\gamma_2)_{A_2}{}^{B_2} \cdots H(\gamma_{K+1})_{A_{K+1}}{}^{B_{K+1}}$ and noting

$$
\delta^{A_{K+1}}_{B_i} H(\gamma_i)_{A_i}{}^{B_i} H(\gamma_i)_{A_{K+1}}{}^{B_{K+1}} = [H(\gamma_i)H(\gamma_{K+1})]_{A_i}{}^{B_{K+1}}
$$

gives (G.-64).

An immediate consequence of the recurrence relation (G.-64), obtained indentifying the loop $N+1$ with i (the identity loop), is

$$
(N+1)M_{N+1}(\gamma_1, \ldots, \gamma_N, i) = (W(i) - N)M_N(\gamma_1, \ldots, \gamma_N) = 0 \qquad \text{(G.-67)}
$$

from which we see that

$$
W(i) = N. \qquad \text{(G.-67)}
$$

Let us consider the identity for 2×2 matrices.

N=2

$$
\begin{aligned}
0 &\equiv 3M_3(\gamma_1, \gamma_2, \gamma_3) \\
&= W(\gamma_3)M_2(\gamma_1, \gamma_2) - M_2(\gamma_1 \circ \gamma_3, \gamma_2) - M_2(\gamma_1, \gamma_2 \circ \gamma_3) \\
&= W(\gamma_3)\frac{1}{2}[W(\gamma_2)M_1(\gamma_1) - M_1(\gamma_1 \circ \gamma_2)] - \frac{1}{2}[W(\gamma_2)M_1(\gamma_1 \circ \gamma_3) - M_1(\gamma_1 \circ \gamma_3 \circ \gamma_2)] \\
&\quad - \frac{1}{2}[W(\gamma_2 \circ \gamma_3)M_1(\gamma_1) - M_1(\gamma_1 \circ \gamma_2 \circ \gamma_3)] \qquad \text{(G.-69)}
\end{aligned}
$$

so that

$$
\begin{aligned}
W(\gamma_1)W(\gamma_2)W(\gamma_3) \;=\; & W(\gamma_1 \circ \gamma_2)W(\gamma_3) + W(\gamma_2 \circ \gamma_3)W(\gamma_1) + W(\gamma_3 \circ \gamma_1)W(\gamma_2) \\
& - W(\gamma_1 \circ \gamma_2 \circ \gamma_3) - W(\gamma_1 \circ \gamma_3 \circ \gamma_2)
\end{aligned}
\tag{G.-70}
$$

G.4 Summary

G.5 Biblioliographical notes

In this chapter I have relied on the following references: Kauffman and Lins

G.6 Worked Exercises and Details

Change of basis for 4-valent spin networks.

Appendix H

Black Hole Entropy

Compare "ordinary" thermodynamics. The quantum theory of an ideal gas allows us to specify and count states, and the resulting entropy agrees, up to typically small corrections, with the classical prediction. But this is to be expected: the correspondence principle relates the quantum states to the classical phase space, forcing an approximate agreement between the two theories. A classical black hole, on the other hand, has no hair there is no classical phase space to explain the thermodynamics. The states responsible for black hole entropymust be fundamentally quantum mechanical, and there is no obvious reason for them to have any preconceived behavior.

http://math.ucr.edu/home/baez/week148.html

This Week's Finds in Mathematical Physics (Week 148)

H.1 Review of Thermodynamics and Statistical Mechanics

Thermodynamics deals with large systems in terms of macroscopic observables alone.

The system's classical state is described by generalized coordinates $\{q^i\}$ and generalized momenta $\{p_i\}$, where the index j runs from 1 to $n =$ (the number of degrees of freedom). The evolution of q, p is governed by Hamilton's equations

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = \frac{\partial H}{\partial q^j}, \tag{H.0}$$

in statistical mechanics we shall describe the statistical properties of an ensemble of systems by a distribution function equal to the number of systems per unit volume in a 2n-dimensional phase space that is analogous to kinetic theory's 6-dimensional one.

H.2 Statistical Mechanics of Black Holes

from theomodynamics we se the temperature and entropy arises from underlying statistical mechanics. What microstaes are responsible for black hole thermodynamics?

a classical, stationary black hole is determined completely by its mass, charge, and angular momentum, with no room for additional microscopic states to account for thermal behaviour.

If black hole thermodynamics has a statistical mechanical origin then the relavent states must therefore be non-classical.

A microstate is not given by the Schwarzschild metric, but by some complicated time-dependent non-symmetric metric.

such time-dependent non-symmetric microstates of the geometry into account is essential for a statistical understanding of the thermal behavior of black holes

The interior degrees of freedom of the black hole are indistguishable to an exterior orbserver - classically because there is a causal barrier at horizon stops the interior effecting the exterior - [1], hence these degrees of freedon do not contribute to the entropy and so don't effect the energy exchange between the black hole and the exterior.

Searching for a derivation of black hole thermodynamics from properties of stationary or symmetric metrics alone is like trying to derive the thermodynamics of an ideal gas in a spherical box just from spherically symmetric motions of the molecules.

As Ashetkar notes, *the surface degrees of freedom are born quantum mechanically.*

H.2.1 Isolated Horizons: the Classical Phase Space

In the application of physical theory to model the world, one shapes the theory with boundary conditions to fit the system under study. One does exactly the same thing with gravity. Here we model a spacetime with a black hole as a system with two boundaries, one at asymptotic infinity and one at the apparent horizon fig(**??**). While it satifies these boundary conditions, we demand also that the action princle leads to Einstein's equations outside the black hole horizon, in the bulk of the spacetime. When such conditions are met we say that the action principle satisfies *functional differentiability.*

$$S'[\sigma, A] \quad = \quad -\frac{i}{8\pi}\left[\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) - \int_T \mathrm{tr}(\Sigma \wedge A)\right] \tag{H.1}$$

[1]but must show that in full quantum gravity of checking that this can be assumed

where $F = dA + A \wedge A$.

Variation with respect to A gives rise to a surface term,

$$[\delta S']_H = -\frac{i}{8\pi G}\int_H \mathrm{tr}\Sigma \wedge \delta A \tag{H.1}$$

$$\begin{aligned} S[\sigma, A] &= -\frac{i}{8\pi}\Bigg[\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) - \int_T \mathrm{tr}(\Sigma \wedge A) \\ &\quad + \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)\Bigg] \end{aligned} \tag{H.1}$$

$U(1)$ Chern-Simons surface term - necessary to ensure functional differentiability of the action.

$$\begin{aligned} \delta\frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) &= \\ &= \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(F \wedge \delta A) \\ &= -\int_H \mathrm{tr}(\Sigma \wedge \delta A) \end{aligned} \tag{H.0}$$

using $F = -\frac{2\pi}{\mathcal{A}}\Sigma$ (on each S^2). Hence, it immediatly follows that the action (H.1) has well defined variation with respect to A and gives rise only to the bulk equations of motion.

Details: $SU(2)$ Chern-Simons theory

$$S'[\sigma, A] = -\frac{i}{8\pi}\left[\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) - \int_T \mathrm{tr}(\Sigma \wedge A)\right] \tag{H.1}$$

Variation with respect to A gives rise to a surface term. Let us specify which part of the action this comes from

$$\begin{aligned} F = dA + A \wedge A &= \partial_\alpha A_\beta dx^\alpha \wedge dx^\beta + (A_\alpha dx^\alpha) \wedge (A_\beta dx^\beta) \\ &= (\partial_\alpha A_\beta + A_\alpha A_\beta) dx^\alpha \wedge dx^\beta \\ &= (\partial_\alpha A_\beta + A_\alpha A_\beta)\epsilon^{\alpha\beta} dx^1 dx^2 \end{aligned} \tag{H.0}$$

$\partial_\alpha A_\beta dx^\alpha \wedge dx^\beta$

$$\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) = \int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge [\partial_\alpha A_\beta dx^\alpha \wedge dx^\beta]) \tag{H.0}$$

$$\begin{aligned} &= -\int_{\mathcal{M}} \mathrm{tr}(\Sigma_\alpha \partial_\beta A_\gamma) dx^\alpha \wedge dx^\beta \wedge dx^\gamma \\ &= -\int_{\mathcal{M}} \mathrm{tr}(\partial_\beta \Sigma_\alpha A_\gamma)\epsilon^{\alpha\beta\gamma} dv + \int_{\mathcal{M}} \mathrm{tr}\partial_\beta(\Sigma_\alpha A_\gamma)\epsilon^{\alpha\beta\gamma} dv \end{aligned} \tag{H.0}$$

$$[\delta S']_H = -\frac{i}{8\pi G}\int_H \mathrm{tr}(\Sigma_\alpha \delta A_\beta)\epsilon^{\alpha\beta} ds \tag{H.0}$$

$$[\delta S']_H = -\frac{i}{8\pi G}\int_H \mathrm{tr}(\Sigma \wedge \delta A) \tag{H.0}$$

Details: Edge states in $U(1)$ Chern-Simons theory

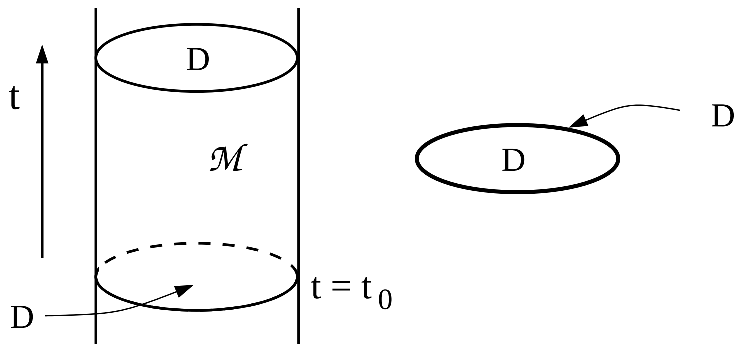


Figure H.1: disc D. $\mathcal{M} = D \times R$.

$$\begin{aligned} \mathcal{CS} &= \frac{k}{4\pi}\int_{\mathcal{M}} (A_i dx^i) \wedge (\partial_i A_k dx^j \wedge dx^k) \\ &= \frac{k}{4\pi}\int_{\mathcal{M}} dv \epsilon^{ijk} A_i \partial_j A_k \end{aligned} \tag{H.0}$$

where $dv = dx^1 \wedge dx^2 \wedge dx^3$.

$$\mathcal{G}_\Lambda = \int_D \Lambda G(A) d^2x = \int_D \Lambda \wedge dA = \int_D dv \epsilon^{ij} \Lambda \partial_i A_j \approx 0. \tag{H.0}$$

Now we require that this functional generates gauge transformations, which in turn requires that $\left(\frac{\delta \mathcal{G}_\Lambda}{\delta A}\right)$ exists. However, we note from (H.2.1) that

$$\mathcal{G}_\Lambda = \int_D \epsilon^{ij} A_i \partial_j \Lambda + \int_{\partial D} \Lambda A \tag{H.0}$$

$$\mathcal{G}_\Lambda = \int_D A \, d\Lambda + \int_{\partial D} \Lambda A, \tag{H.0}$$

and therefore differentiability requires the boundary condition

$$\Lambda|_{\partial D} = 0. \tag{H.0}$$

Hence the Gauss law reduces to

$$\mathcal{G}_\Lambda = \int_D \epsilon^{ij} A_i \partial_j \Lambda \tag{H.0}$$

$$\mathcal{G}_\Lambda = \int_D dA\Lambda, \tag{H.0}$$

with the gauge parameter Λ subject to (H.2.1). Then, due to this boundary condition,

$$\begin{aligned}
\{\mathcal{G}_\Lambda, \mathcal{G}_{\Lambda'}\} &= \left\{ \int_D dv \epsilon^{ij} \Lambda \partial_i A_j, \int_D dv' \epsilon^{i'j'} \Lambda' \partial_{i'} A_{j'} \right\} \\
&= \int_D dv \int_D dv' \epsilon^{ij} \epsilon^{i'j'} \partial_i \Lambda \partial_{i'} \Lambda' \{A_j(x), A_{j'}(x')\} \\
&= \frac{i}{\kappa} \int_D dv \epsilon^{ij} \epsilon^{i'j'} \partial_i \Lambda \partial_{i'} \Lambda' \epsilon_{jj'}
\end{aligned} \tag{H.-1}$$

$$\{A_j(x), A_{j'}(x')\} = \frac{i}{\kappa} \epsilon_{jj'} \delta(\vec{x} - \vec{x'}) \tag{H.-1}$$

$$\{\mathcal{G}_\Lambda, \mathcal{G}_{\Lambda'}\} = \frac{2\pi}{k} \int_D \Lambda \, d\Lambda' = 0. \tag{H.-1}$$

Finally we see that these observables generate a $U(1)$ Lie algebra at the edge,

$$\{\mathcal{Q}(\xi), \mathcal{Q}(\xi')\} = \frac{2\pi}{k} \int_{\partial D} \xi \, d\xi' \tag{H.-1}$$

Translated:

$$A = A_i dx^i, \quad dA = \partial_i A_j dx^i \wedge dx^j, \quad dx^i \wedge dx^j = \epsilon^{ij} dx^1 \wedge dx^2,$$
$$dx^i \wedge dx^j = \epsilon^{ijk} dx^1 \wedge dx^2 \wedge dx^3. \qquad \text{(H.-1)}$$

$$\mathcal{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} A \wedge dA \qquad \text{(H.-1)}$$

$$G(A) \equiv \epsilon^{ijk} \partial_i A_j \approx 0. \qquad \text{(H.-1)}$$

$$\mathcal{G}_\Lambda = \int_D \Lambda G(A)\, d^2x = \int_D \Lambda\, dA \approx 0. \qquad \text{(H.-1)}$$

$$\{A_j(x), A_{j'}(x')\} = ? \qquad \text{(H.-1)}$$

$$\begin{aligned} \{\mathcal{G}_\Lambda, \mathcal{G}_{\Lambda'}\} &= \\ &= \frac{2\pi}{k} \int_D dx^i \Lambda \partial_j \Lambda' \end{aligned} \qquad \text{(H.-1)}$$

$$\int_{\partial D} \Lambda d\Lambda' = \int_{\partial D} dx^i \Lambda \partial_i \Lambda' = \int_{\partial D} ds \Lambda \frac{d\Lambda'}{ds} \qquad \text{(H.-1)}$$

H.3 Quantum Geometry and Black Hole Entropy

In the classical theory, the fields in the bulk - metric, triad, connection... determine the fields on the boundary by continuity. So there would appear to by no independent surface degrees of freedom! This is correct in the classical theory but it is not the case in the quantum theory. A feature of any quantum field theory, not just quatum GR, is that quantum fields are really distributional and so can be arbriarly discontinous. So just because you know the quantum field in the bulk, it doesn't tell you what the field at the boundary is.

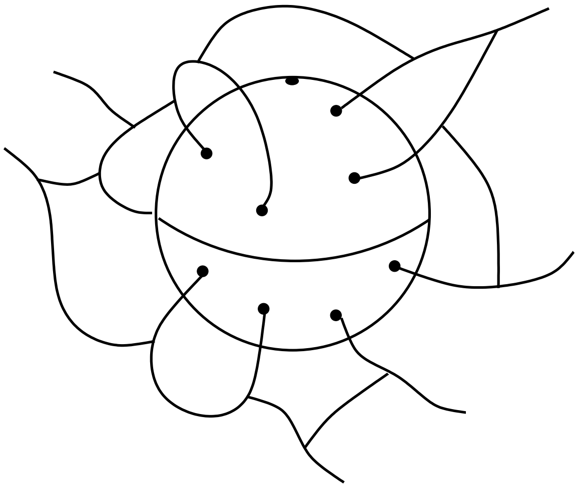


Figure H.2: sepbulksurface.

Basic Strategy:

Canonically quantize the vacuum Einstein equtions with boundary conditions describing horizon of a non-rotating black hole, using techniques of loop quantum gravity:

Seperate "bulk" and "surface" degrees of freedom and count surface states with an area $\mathcal{A}$.

Also works for gravity coupled to electromagonetism and dilaton field!

H.3.1 Classical Phase Space

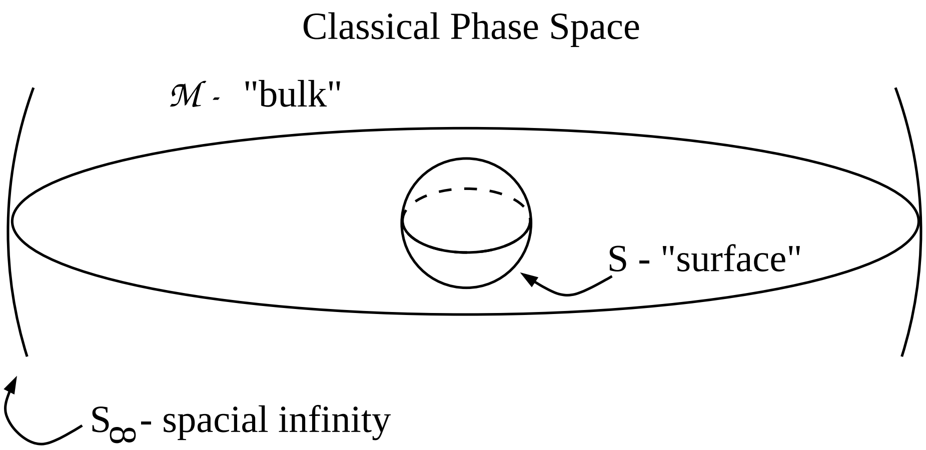


Figure H.3: classphase.

A point in the classical space consists of the fields:

$$A_a^i = \Gamma_a^i - \gamma K_a^i \tag{H.-1}$$

γ real and $\gamma \neq 0$,

$$E_{ab}^i = \frac{1}{\gamma}\Sigma_{ab}^i \tag{H.-1}$$

satisfying:

(i) Asymptotic flatness at S_∞

(ii) A_a^i reduces to $U(1)$ connection A_a on S.

(iii) $F_{ab} = -\frac{2\pi\gamma}{\mathcal{A}}$

There consists generating:

(i) $SU(2)$ gauge transformations on $\mathcal{M}$ reducing to $U(1)$ on S and identify on S_∞.

(ii) Diffeomorphisms of $\mathcal{M}$ mapping S to itself and reducing to identity at S_∞.

(iii) Time evolution with lapse goinig to 0 at S and constant at S_∞.

H.3.2 Sympletic Structure

The Lie algebra of vector fields on Γ induces a Lie algebra structure on the space of functions, given by (need to improve on this)

$$\{f, g\} := \Omega^{ab}\partial_a f \partial_b g \tag{H.-1}$$

$$\Omega(y_1, y_2) = \sum_\mu (p_{1\mu}q_{2\mu} - p_{2\mu}q_{1\mu}) \tag{H.-1}$$

$$\omega\left((\delta A, \delta E), (\delta A', \delta E')\right) = \frac{1}{8\pi}\left[\int_{\mathcal{M}} \mathrm{tr}(\delta E \wedge \delta A' - \delta E' \wedge \delta A) - \frac{\mathcal{A}}{\pi\gamma}\int_S \delta A \wedge \delta A'\right] \tag{H.-1}$$

$U(1)$ Chern-Simons surface tern! Equals

$$-\frac{k}{2\pi}\int_S \delta A \wedge \delta A' \tag{H.-1}$$

where the "level"

$$k = \frac{\mathcal{A}}{4\pi\gamma} \tag{H.-1}$$

must be an *integer* to quantize the theory!

diads on a 2-sphere. Connection compatible to these diads. This connection is a $U(1)$ connection. On sphere we get first $SO(2)$ - because of double covering we get $U(1)$ connection W. $SU(2)$ is trival on an oreinetable 3-manifold - so don't have to do patches. Cannot have a global field of diads on a 2-sphere because every vector field on a sphere has to vanish somewhere. Spin bundle on 2-sphere is not rrival. Need 2 patches and in the overlap they are related to each other by a $U(1)$ transformation. W not globally defined 1-form.

H.3.3 Quantization Strategy

We separately quantize:

BULK: (A, E) on $\mathcal{M}$ with usual symplectic structure.

SURFACE: $U(1)$ connection A on S with Chern-Simons symplectic structure. $k = \frac{\mathcal{A}}{4\pi\gamma}$ must be integer!

Obtaining Hilbert space spaces H_{bulk} and H_{surface}.

$$H_{\text{bulk}} = \lim_{P} H^{P}_{\text{bulk}} \tag{H.0}$$

$$H_{\text{surface}} = \lim_{P} H^{P}_{\text{surface}} \tag{H.1}$$

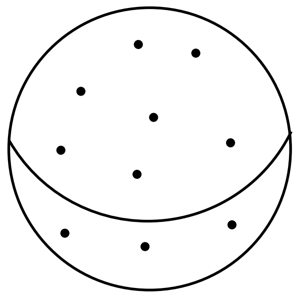


Figure H.4: Classical bourndary conditions for isolated horizons.

$$P = \{p_1, \dots, p_n\} \tag{H.1}$$

Then we set

$$H_{\text{physical}} = \lim_P \frac{H^P_{\text{bulk}} \otimes H^P_{\text{surface}}}{Gauge} \tag{H.1}$$

Note that $U(1)$ gauge transformations on S are generated by

$$F_{\alpha\beta} + \frac{2\pi\gamma}{\mathcal{A}} E_{\alpha\beta} \tag{H.1}$$

so "Gauge" includes imposing constraint!

H.3.4 Bulk States

$$\mathcal{H}_{bulk} = L^2(\{\text{generalized } SU(2) \text{ cons on } \mathcal{M}\}) \tag{H.1}$$

as defned in the loop quantum gravity. But we can save time by modding out by $SU(2)$ gauge transforms that reduce to identity on S, S_∞.

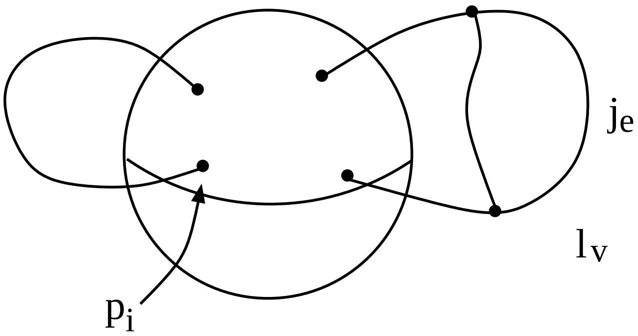


Figure H.5: Bulk states.

The smaler "$\mathcal{H}_{bulk}$ has a basis given by spin networks in $\mathcal{M}$ with" loose ends" at points $p_i \in S$:

• edges labeled by spins j_e

• vertices labelled by interwiners l_v

• punctures p_i: labelled by vectors $|m_i>$ in spin-j_i representation, where j_i is spin of incident edge:

$$m_i = -j_i, -j_i + 1, \ldots, j_i. \tag{H.1}$$

If $P = \{p_1, \ldots, p_n\}$ of bulks states with loose ends at points in P.

H.3.5 Geometric Interpretation of Bulk States

Consider $\psi \in \mathcal{H}^P_{\text{bulk}}$:

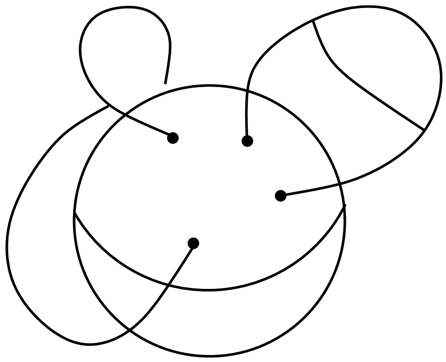


Figure H.6: geobulk1.

punctures p_i labelled by representations j_i and vectors $|m_i>$ in these representations.

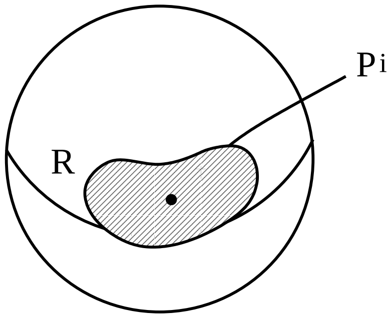


Figure H.7: geobulk2.

Then Area $(R)\gamma \int_R \sqrt{\vec{E} \cdot \vec{E}}$ has eigenvalue

$$8\pi\gamma\sqrt{j_i(j_i+1)} \tag{H.1}$$

and $\int_R \sqrt{\vec{E} \cdot \vec{r}}$ has eigenvalue

$$8\pi m_i \tag{H.1}$$

in the state ψ.

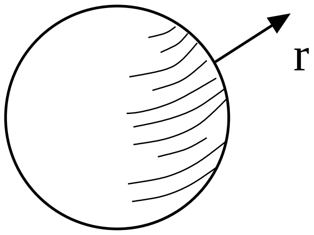


Figure H.8: geobulk3 $U(1)$ fixes internal vector r^i.

H.3.6 Surface States

Since we will impose constraint $F = -\frac{2\pi\gamma}{\mathcal{A}} E \cdot r$ on S, and $E \cdot r$ vanishes except at points $p_i \in P$ in states in $\mathcal{H}^P_{surface}$, we take as our phase space:

$X^P_{surface} = \{$ generalized $U(1)$ conns on S, flat except at $p_i \in P$, mod gauge except at $p_i \in P$ $\}$

A point in $X^P_{surface}$ is described by $2(n-1)$ holonomies:

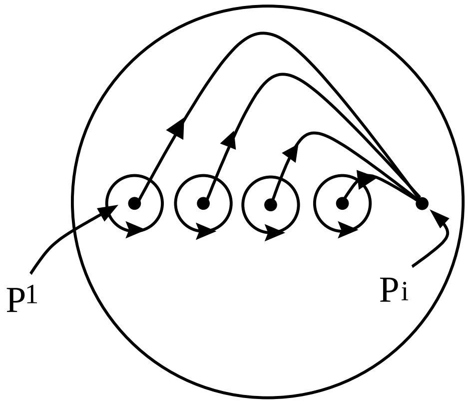


Figure H.9: surfacestate.

so $X^P_{surface} \sim (U(1) \times U(1))^{n-1}$ with sympletic structure equal to $\frac{k}{2\pi}$ times that comming from usual sympletic structure on

$$U(1) \times U(1) = \frac{R}{2\pi Z} \times \frac{R}{2\pi Z} \tag{H.1}$$

??The U(1) Chern-Simons theory action gives this phase space when we take spacetime to be R x M and M is a 2-sphere with two holes removed. If you put suitable boundary conditions at the holes, a flat U(1) connection on M is determined (up to gauge transformations) by its holonomies around one hole and along a path from one hole to the other. So we get two elements of U(1), i.e., a point on the torus. ??

Applying geometric quantization to $X^P_{surface}$ get $\mathcal{H}_{surface}$. Basis of states given by labelling punctures with numbers $m_i = -\frac{k}{2}, -\frac{k}{2}+1, \ldots, \frac{k}{2}$. Really m_i defined only mod k.

$$W = W_1 + \sum_{i=1}^{n} c_i \frac{(x-i)dy - ydx}{(x-i)^2 + y^2} \tag{H.1}$$

where W_1 is a bounded smooth 1-form on $U - \mathcal{P}$. Since W is flat except at the punctures, W_1 must be closed. Note that the constants are not idependent: they must sum to zero modulas 2π, because the holonomy W around a loop enclosing all the punctures must be trivial,

Details: Surface states

(a) X_i satisfies

$$\int_{\gamma_j} X_i = \delta_{ij}, \quad \oint_{\eta_j} X_i = 0. \tag{H.1}$$

(b) Y_i satisfies

$$\oint_{\eta_j} Y_i = \delta_{ij}, \quad \int_{\gamma_j} Y_i = 0. \tag{H.1}$$

(c) X_i and Y_i satisfy

$$\int_{S^2} X_i \wedge X_j = 0, \quad \int_{S^2} Y_i \wedge Y_j = 0, \quad \int_{S^2} X_i \wedge Y_j = \delta_{ij}. \tag{H.1}$$

$$\int_{S^2} d^2x \epsilon^{ij} X_i X_j = 0, \quad \int_{S^2} d^2x \epsilon^{ij} Y_i Y_j = 0, \quad \int_{S^2} d^2x \epsilon^{ij} X_i Y_j = \delta_{ij}. \tag{H.1}$$

(d)

$$W = \tilde{W} + \sum_{i=1}^{n-1} (x_i X_i + y_i Y_i) \tag{H.1}$$

Note property (d) implies the 1-forms X_i and Y_i are closed $X_i = dX$

We actually define X_i and Y_i on regions slightly larger than $S - \mathcal{P}$. Define X_i on all of S by

$$X_i = df_i = dx^a \frac{\partial f_i}{\partial x^a} \tag{H.1}$$

where f_i is any smooth real-valued function on S with $f(p_j) = i$ for $j \neq i$ and $f_i = 0$ in an open disc containing p_i and the loop η_i. To define Y_i, first set

$$Y_i = \frac{1}{2\pi} \frac{(x-i)dy - ydx}{(x-i)^2 y^2} \tag{H.1}$$

in an open disc containing p_i and the loop η_i, and

$$Y_i = \frac{1}{2\pi} \frac{(x-n)dy - ydx}{(x-n)^2 y^2} \tag{H.1}$$

in an open disc containing p_i. Then extend Y_i smoothly to a closed 1-form on all of $S - \{p_i, p_n\}$.

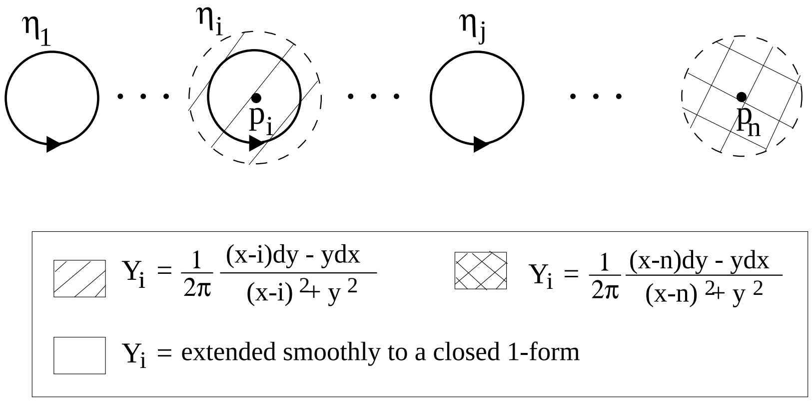


Figure H.10: surfaceY.

$$\int_{\gamma_j} X_i = \int_{p_j}^{p_n} df_i = f_i(p_n) - f_i(p_j) = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases} \tag{H.1}$$

Stoke's theorem and that $dX_i = 0$

$$\oint_{\eta_j} X_i = \int_{S^2} dX_i = 0. \tag{H.1}$$

Consider the phase space $\mathbb{R}^{2(N-1)}$ with canonical brackets $\{y_I, x_J\} = \delta_{IJ}\kappa'$, $\{x_I, x_J\} = \{y_I, y_J\} = 0$. In order to obtain the torus $U(1)^{2(N-1)}$ we devide $\mathbb{R}^{2(N-1)}$ by the action of the discrete translation group, that is, we identify points x_I, y_I up to translations by integer multiples of 2π.

H.3.7 Geometric Quantisation of Surface Degree's of Freedom

Before we proceed to quantise by means of geometric quantisation we must ensure that Weil's integrality criterion is satisfied...

Hence the number K, called the level of the Chern-Simos theorey, must be integral.

We choose a polarisation:

$$\Omega = \frac{\hbar K}{2\pi} dy^I \wedge dx^I \tag{H.1}$$

Let us set $z^I := x^I + iy^I$

We first define the quauntum theorey for $\mathbb{R}^{2(N-1)}$ and then pass to the torus.

$$\begin{aligned} \hat{x}^I &= i\hbar_{\chi_{x^I}} - \Theta(\chi_{x^I}) + x^I \\ \hat{y}^I &= i\hbar_{\chi_{y^I}} - \Theta(\chi_{y^I}) + y^I \end{aligned} \tag{H.1}$$

Moreover

$$\nabla_{\partial/\partial\overline{z}^I} = \partial/\partial\overline{z}^I - \frac{1}{i\hbar}\Theta(\partial/\partial\overline{z}^I) = \partial/\partial\overline{z}^I + \frac{K}{4\pi} z^I \tag{H.1}$$

Therefore polarised states satify

$$\left[\nabla_{\partial/\partial\overline{z}^I}\Psi\right](z, \overline{z}) = 0$$

whose solution is

$$\Psi(z, \overline{z}) = e^{-K\overline{z}^I z^I/(4\pi)}\psi(z) \tag{H.1}$$

H.3.8 Geometric Interpretation of Surface States

Consider $\psi \in \mathcal{H}_{surface}$:

punctures p_i labelled by $m_i = -\frac{k}{2}, \dots \frac{k}{2}$

Suppose only p_i lies in the region RS:

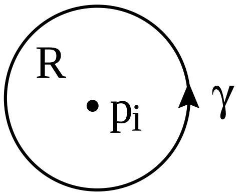


Figure H.11: surfaceR.

then the holonomy around γ has eigenvalue

$$e^{-4\pi i \frac{m_i}{k}} \tag{H.1}$$

in the state ψ. Classically

$$e^{i \oint_\gamma A} = e^{i \int_R F} \tag{H.1}$$

so we may say:

$\int_R F$ has eigenvalue $-4\pi i \frac{m_i}{k}$ mod 4π. Thus the geometry of S is flat except at p_i, where there are canonical singularities:

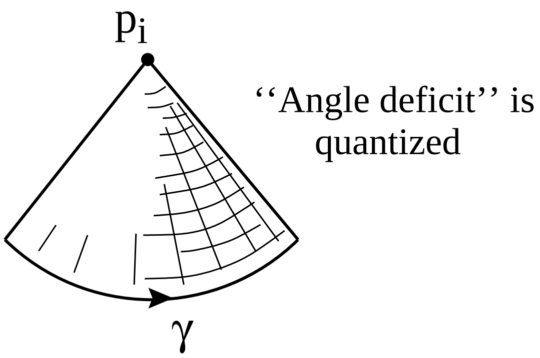


Figure H.12: cone.

H.3.9 Surface and Bulk

Having constructed the volume and surface Hilbert spaces, we now wish to impose the quantum bourndary condition, in order to pick out the kinemetical Hilbert space $\mathcal{H}$ as a subspace of $\mathcal{H}_V \otimes \mathcal{H}_S$.

$$\left(I \otimes e^{\int_R F}\right) \psi_{\text{bulk}} \otimes \psi_{\text{surface}} = \left(e^{-\frac{2\pi i \gamma}{2} \int_R E \cdot r} \otimes I\right) \psi_{\text{bulk}} \otimes \psi_{\text{surface}} \tag{H.1}$$

eigenvalue $e^{-4\pi i \frac{m_i}{k}}$ and $e^{-\frac{2\pi i\gamma}{\mathcal{A}}\cdot 8\pi m_i}$

These eignvalues match if the puncture labels m_i for surface and bulk states agree - since

$$k = \frac{\mathcal{A}}{4\pi\gamma}!! \tag{H.1}$$

$m_i = -j_i, \ldots, j_i$

H.3.10 Entropy Calculation

Recall that in classical general relativity in the Hamiltonian formulation, the bulk Hamiltonian is a first class constraint, so that the entire Hamiltonian consists of the boundary contribution HS on the constraint surface. In the quantum domain, the Hamiltonian operator can be written as

Other stuff

$$\frac{A \times B}{G} \sim \frac{A}{G} \times B \tag{H.1}$$

In $H_{phys} = \lim_p \frac{H^p_{bulk} \otimes H^p_{surface}}{Gauge}$ we form a density matrix from projection onto subspace of states where the horizon has area $\mathcal{A} \pm i_p^2$. Then we trace out ti get a density matrix ρ on $H_{surface}/Gauge$. If for every at least one solution of hte Hamiltonian constraint for any p_i, j_i, then

$$S = \mathrm{tr}(\rho \ln \rho) = \frac{\ln 2}{4\pi\sqrt{3}\gamma} \frac{\mathcal{A}}{l_P^2} + \mathcal{O}(\sqrt{\mathcal{A}}) \tag{H.1}$$

Get agreement with $S = \mathcal{A}/4l_P^2$ if:

$$\boxed{\gamma = \frac{\ln 2}{\pi\sqrt{3}}} \tag{H.2}$$

Heuristic estimate: $j = 1/2$ punctures dominate. These give an area of

$$8\pi\gamma\sqrt{j(j+1)}l_P^2 = 4\pi\sqrt{3}\gamma l_P^2 \tag{H.2}$$

and entropy $\ln 2$, so

$$\boxed{S \sim \frac{ln2}{4\pi\sqrt{3}\gamma}\,\mathcal{A}} \tag{H.3}$$

$$\boxed{S \approx \frac{1}{4}\mathcal{A} \qquad \text{(H.4)}}$$

It was found, with some surprise, that the same calculation with Maxwell/dilaton fields gives $S = \mathcal{A}/4$ with same γ!

It is only those microscopic degrees of freedom that affect the energy exchanged between the blackhole and exteriorthat contribute to the entropy.

The interior degrees of freedom don't come into it.

Examples of normed spaces.

H.4 Maths

$$W(c)\psi = \psi$$

$$V(b)\sum \psi_l e^{il\cdot z} = \sum \psi_l e^{il\cdot z}$$

$$\begin{aligned} V(b)\sum \psi_l e^{il\cdot z} &= \exp\left(\frac{K}{2\pi}[ib\cdot z - b\cdot b/2]\right)\psi(z+ib) \\ &= \exp\left(\frac{K}{2\pi}[ib\cdot z - b\cdot b/2]\right)\sum \psi_l e^{il\cdot z} \\ &= \sum\left[\psi_l e^{-l\cdot b} e^{-\frac{1}{2}\frac{K}{2\pi}}\right] e^{i(l-\frac{Kb}{2\pi})\cdot z} \end{aligned} \tag{H.3}$$

So

$$\psi_{l+\frac{Kb}{2\pi}} = \psi_l e^{-l\cdot b} e^{-\frac{1}{2}\frac{K}{2\pi}b\cdot b}$$

or

$$\psi_l = \psi_{l-\frac{Kb}{2\pi}} e^{-l\cdot b} e^{\frac{1}{2}\frac{K}{2\pi}b\cdot b}$$

with solution

$$\psi_{l+nK} = \psi_l e^{-2\pi l\cdot n} e^{-2\pi\frac{K}{2}n\cdot n}$$

$$\begin{aligned}
\psi(z) &= \sum_l \psi_l e^{il\cdot z} \\
&= \sum_{n\in\mathbb{Z}} \sum_{\prod_I l_I=1}^{K} \psi_{l+nK} e^{i(l+nK)\cdot z} \\
&= \sum_{l\in D_K}^{K} \sum_{n\in\mathbb{Z}} \psi_l e^{-2\pi l\cdot n} e^{-2\pi\frac{K}{2}n\cdot n} \exp(i(l+nK)\cdot z) \\
&= \sum_{l\in D_K} \psi_l \vartheta_l^{K,\mathcal{P}}(z) \qquad \text{(H.1)}
\end{aligned}$$

where

$$\vartheta_l^{K,\mathcal{P}}(z) = \sum_{n\in\mathbb{Z}^{N-1}} e^{-2\pi l\cdot n} e^{-2\pi\frac{K}{2}n\cdot n} \exp\left(i\left(l+nK\right)\cdot z\right) \qquad \text{(H.1)}$$

H.5 Entropy of Rotating and Axisymmetric Distorted Black Holes

With the notion employed, given a type II isolated horizon, its image under any diffeomorphism on Δ is again a type II isolated horizon. The diffeomorphism invariance on Δ remains intact.

In terms of these fields ψ, the surface part of the symplectic strucure is given by:

$$\Omega_S(\delta_1,\delta_2) = \frac{1}{8\pi\gamma G}\oint_S \left[\delta_1\psi\delta_2(\Sigma^i r_i) - \delta_2\psi\delta_1(\Sigma^i r_i)\right] \qquad \text{(H.1)}$$

new connection W as

$$dW = -\frac{2\pi\gamma}{a_0}\underline{\Sigma}^i r_i \epsilon \qquad \text{(H.1)}$$

the sympletic structure (H.5) reduces to the Chern-Simons form:

$$\Omega(\delta_1,\delta_2) = \frac{1}{8\pi G}\frac{a_0}{\gamma\pi}\oint_S \delta_1 W\wedge\delta_2 W \qquad \text{(H.1)}$$

H.5.1 Quasi-normal Modes of Black Hole

H.6 Quantum Black Holes

horizon operators

$$\hat{\theta}_{\pm} = \frac{1}{2i\lambda}\left(\hat{U}_{\lambda} - \hat{U}_{\lambda}^{\dagger}\right) \pm \frac{2}{\epsilon l_P^2}\left(\hat{R}_{\epsilon} - \hat{R}_0\right). \tag{H.1}$$

H.7 Biblioliographical notes

In this chapter I have relied on the following references: Rovelli's paper *Loop Quantum Gravity and Black Hole Physics*.

H.7.1 Review of Chomology Group of Spherical Horizon

In the case of spherical horizons, one has a sphere with N-punctures due to the gravitational spin-network. The first cohomology group of the N-punctured sphere, denoted as $H^1(S - P_N)$, is $(N-1)$-dimensional which is one less than the number of punctures. $(N-1)$ pairs of forms are defined on the punctured sphere to yield the required symplectic structure (see figure ??, which is similar to the figure originally produced in [??]). These forms are constructed via their duality with chains on a punctured sphere as depicted in figure ??.

FIGURE HERE

There exist $N-1$ η paths and $N-1$ conjugate γ paths on this sphere. A basis for all the paths based at p_N

$$\{\gamma_1^{-1}\eta_1\gamma_1,\ \gamma_2^{-1}\eta_2\gamma_2,\ \cdots,\ \gamma_{N-1}^{-1}\eta_{N-1}\gamma_{N-1}\}. \tag{H.1}$$

there exists a fundamental relation

$$\eta_1 \cdot \eta_2 \cdot\ \ldots\ \cdot \eta_N = 1, \tag{H.1}$$

which is a mathematical relation indicating that a loop around all punctures can be shrunk to a point on the sphere. Another way to look at this relation is that a loop around all the $N-1$ punctures is equivalent to a loop around the N-th puncture but in reverse. In other words, η_N is expressible in terms of the other η paths.

Appendix I

Loop Quantum Cosmology

I.1 Introduction

I.2 Quantum Cosmology

I.2.1 Classical theory

$$S[q_{ab}, N, N^a] = \int dt \int d^3x N(\,^3q)^{1/2}(K_{ab}K^{ab} - K^2 + \,^3R - 2\Lambda) \tag{I.0}$$

$$G^{abcd} = \frac{1}{4}(\,^3q)^{1/2}(q^{ac})q^{bd} + q^{ad})q^{bc} - 2q^{ab})q^{cd} \tag{I.0}$$

$$p^{ab} = -2G^{abcd}K_{cd} \tag{I.0}$$

we can write the action as

$$S[q_{ab}, p^{ab}N, N^a] = \int dt \int d^3 \left(p^{ab}\frac{dq_{ab}}{dt} - N\mathcal{C} - N^a\mathcal{C}\right) \tag{I.0}$$

where

$$\begin{aligned}
\mathcal{C} &= \frac{1}{2}G_{abcd}p^{ab}p^{cd} - (q)^{1/2}(\,^3R - 2\Lambda), && \text{(I.1)}\\
\mathcal{C}_a &= -2g_{ac}\nabla_d p^{cd}, && \text{(I.2)}\\
G_{abcd} &= (q)^{-1/2}(g_{ac}g_{bd} + g_{ad}g_{ad} - 2g_{ad}g_{cd}). && \text{(I.3)}
\end{aligned}$$

I.2.2 Minsuperspace

The collection of all admissible three-metrics ia called superspace (nothing to do with supersymmetry!). This space can be given properties such as a “metric” to calculate. Is the DeWitt supermetric given by

$$ds^2 = N^2\, dt^2 - q_{ab}(\, dx^a + N^a\, dt)(\, dx^b + N^b\, dt) \tag{I.3}$$

$$ds^2 = N^2\, dt^2 - q_{ab}\, dx^a\, dx^b \tag{I.3}$$

Shift N^a is zero and is lapse function N is homogenious

$$S[q^i, N] = \int_{t_1}^{t_2} N \left(\frac{1}{2N^2} G_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} - V(q) \right) \, dt \tag{I.3}$$

The involving the Hamiltonian is

$$S[q^i, N] = \int_{t_1}^{t_2} \left(p_i \frac{dq^i}{dt} - N\mathcal{H} \right) \, dt \tag{I.3}$$

where

$$\mathcal{H} = G^{ij} p_i p_j + V(q). \tag{I.3}$$

momenta are proportional to the coordinates:

$$p_i = \frac{1}{N} G_{ij} \frac{dq^i}{dt}. \tag{I.3}$$

$$\frac{dq^i}{dt} = N[q^i, \mathcal{H}], \qquad \frac{dp_i}{dt} = N[p_i, \mathcal{H}] \tag{I.3}$$

$$\mathcal{H} \approx 0 \tag{I.3}$$

(weak equality means it is valid only if you restrict yourself to the constraint surface).

The Wheeler-DeWitt equation of cosed isostropic universe with scalar field ϕ and zero cosmological constant. With generic dependence of the potential on ϕ, namely $V(\phi)$.

$$\left(\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\phi^2} V(\phi) e^{6\Omega} - e^{4\Omega}\right) \Psi(\Omega, \phi) = 0. \tag{I.3}$$

WKB solutions

$$\mathcal{H} = -\frac{3aK}{8\pi G} - \frac{8\pi G p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) = 0. \tag{I.3}$$

Quantization $[\hat{a}, \hat{p}_a] = i$, $[\hat{\phi}, \hat{p}_\phi] = i$ we have the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\psi(a, \phi) = 0. \tag{I.3}$$

I.2.3 Mathematical Excurtion: Symmetry

One way to characterize the invariance of the metric under spacial transformations is to consider

$$e^i_m(x)\, dx^m = e^i_m(x')\, dx'^m \tag{I.3}$$

$$dl^2 = \eta_{ij}(e^i_m(x)\, dx^m)(e^j_n(x)\, dx^n) \tag{I.3}$$

that is, the three metric tensor is given by

$$q_{mn} = \eta_{ij} e^i_m e^j_n \tag{I.3}$$

$$e^m_i e^j_m = \delta^j_i, \quad e^m_i e^j_n = \delta^m_n. \tag{I.3}$$

$$e^m_i \frac{\partial e^n_j}{\partial x^m} - e^m_j \frac{\partial e^n_i}{\partial x^m} = C^k_{ij} e^n_k \tag{I.3}$$

These are the structure constants of the groups of transformations. Ifwe denote by X_i the following differential operator:

$$X_i = e^m_i \frac{\partial}{\partial x^m}, \tag{I.3}$$

then (N.-19) can be written as

$$[X_i, X_j] = C_{ij}^k X_k. \tag{I.3}$$

Some of the geometrical properties of a manifold $\mathcal{M}$ can most easily be examined by constructing a fibre bundle, which is locally a direct product of $\mathcal{M}$ and a suitable space.

The three dimensional rotation gorup $O(3)$ is the isometry group for of the ordinary round sphere S^2.

A group that is also a manifold and for which the group operations are continuous a Lie group.

Now any Lie group is a manifold and can be made into a group of transformations acting on itself as follows: the element g of G defines the transformation

$$L_g(h) = gh. \tag{I.3}$$

left-invariant vector field

$$A_a^i = c\Lambda_I^i \omega_a^I, \quad E_i^a = p\Lambda_i^I X_I^a \tag{I.3}$$

$$A = A_x(x)\Lambda_3 dr + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2)d\theta + (A_1(x)\Lambda_2 - A_2(x)\Lambda_1)\sin\theta d\varphi + \Lambda_3 \cos\theta d\varphi \tag{I.3}$$

$$E = E^x(x)\Lambda_3 \sin\theta \frac{\partial}{\partial x} + (E^1(x)\Lambda 1 + E^2(x)\Lambda_2)\sin\theta \frac{\partial}{\partial \theta} + (E^1(x)\Lambda_2 E^2(x)\Lambda_1)\frac{\partial}{\partial \varphi} \tag{I.3}$$

I.2.4 Symmetries and Backgrounds

It is important to realize that the action of the symmetry group on a space manifold provides a partial background such that the situation is always slightly different from the full theory.

It is impossible to introduce symmetries in a completely background independent manner.

I.2.5 Loop Quantum Cosmology

We don't use metrics

Enough loops so that for any two different connections, one can find some holonomy on these two different connections. We need only one loop because of the symmetry.

There is now another gauge degree of freedom, upon which physical observables should not depend, and so under the action of the Euclidean group on the gauge field and electric field should be

$$A^i_\mu \mapsto g^{-1}A^i_\mu g + g^{-1}\frac{\partial g}{\partial x^\mu}, \qquad E^\mu_i \mapsto g^{-1}E^\mu_i g \tag{I.3}$$

the Poisson brackets of any two functions f and g on this phase space is given by:

$$\{f, g\} = \frac{\kappa\gamma}{3}\left(\frac{\partial f}{\partial c}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial c}\frac{\partial f}{\partial p}\right) \tag{I.3}$$

because of homogeneity and isotropy, we do not need all edges e and surfaces S. Symmetric connections A in $\mathcal{A}$ can be recovered knowing holonomies $h(e)$ along straight lines in $\mathcal{M}$. Similary,

symmetric states exist as disributions supported on invariant connections only

isotropic connection / triad: $A^i_a = c\Lambda^i_I\omega^I_a$, $E^a_i = p\Lambda^I_i X^a_I$ with ω^I_a, X^a_I invariant 1-forms/vector fields.

Λ^i_I internal $su(2)$-triad (purely gauge)

$$h_e(A) = \mathcal{P}\exp\left(\int_e A^i_a(e(t))\dot{e}^a\tau_i\, dt\right) \in SU(2) \tag{I.3}$$

given a surface $S : [0, 1] \to \Sigma$ we can form a flux as a function of the triads

$$E(S) = \int_S E^a_i(y)n_a(y)\tau^i\, d^2y \tag{I.3}$$

where n_a is co-normla to the surface S. The co-normal is defined as

$$n_a = \frac{1}{2}\epsilon_{abc}\epsilon^{de}\frac{\partial x^b}{\partial y^d}\frac{\partial x^c}{\partial y^e} \tag{I.3}$$

without using a background metric, where x^a are coordinates of Σ and y^a coordinates of the surface S.

$$A^i_a(x)\,dx^a = c\omega^i, \quad E^a_i \frac{\partial}{\partial x^a} = pX_i \tag{I.3}$$

where ω^i are invariant 1-forms and X_i invariant vector fields. For spacially flat configuration, $\omega^i = dx^i$ are just coordinate differentials, while X_i are the derivatives.

The symmetry condition can be implemented by using onlt invariant connections (I.2.5) in holonomies as creation operators, i.e.

$$\begin{aligned}
h_i(c) &= \exp(c\tau_i) = \exp(-i\frac{c}{2}\sigma_i) \\
&= 1 + \left(\frac{-ic}{2}\right)\sigma_i + \frac{1}{2!}\left(\frac{-ic}{2}\right)^2\sigma_i^2 + \frac{1}{3!}\left(\frac{-ic}{2}\right)^3\sigma_i^3 + \ldots \\
&= I\left(1 - \frac{1}{2!}\left(\frac{c}{2}\right)^2 + \ldots\right) - i\sigma_i\left(\frac{c}{2} - \frac{1}{3!}\left(\frac{c}{2}\right)^3 + \ldots\right) \quad (\text{using } \sigma^2 = I) \\
&= \cos(c/2) + 2\tau_i\cos(c/2) && \text{(I.1)}
\end{aligned}$$

physical components $c = \frac{1}{6}\dot{a}$ extrinsic curvature (flat model) $p = \epsilon a^2$ a: scale factor , ϵ : orientation

$$A_a = cV_0^{-\frac{1}{3}}\,{}^0\omega^i_a\tau_i, \quad E^a = pV_0^{-\frac{2}{3}}\sqrt{q}e^a_i\tau^i. \tag{I.1}$$

Symmetric States

$$A(e) = \cos\frac{l_C}{2} + 2[\sin\frac{l_C}{2}](\dot{e}^{a0}\omega^i_a)\tau^i \tag{I.1}$$

$$F(A) = \sum_j \xi_j e^{il_jc} \tag{I.1}$$

These are precisely the **almost periodic functions**

$$\{F(A), p\} = \frac{8\pi\gamma G}{6}\sum_I (il_j\xi_j)e^{il_jc} \tag{I.1}$$

$$\mathcal{N}_l(\overline{A}) = e^{il_jc} \tag{I.1}$$

the space of square integrable fucntions on a suitable completion $\overline{\mathcal{A}}_S$ of the classical configuration space.

$\mathcal{A}_S = R$ and $\overline{\mathcal{A}}_S$ is the Gel'fand spectrum of the C* algebra of almost periodic functions on $\mathcal{A}_S$.

Say X is a measurable space and Y is a topological space. A function $f : X \to Y$ is a measuable function if the pre-image $f^{-1}(V)$ of every open set of Y is a measure subset of X.

orthonormal states $|m>$ in connection representation:

$$< c|m >= \frac{\exp\left(\frac{1}{2}imc\right)}{\sqrt{2}\sin\frac{c}{2}} \qquad n \in \mathbf{Z} \tag{I.1}$$

isotropic spin network states $|m| = 2j + 1$

geometric operators

$$\hat{p}|m >= \frac{1}{6}\gamma l_P^2 m|m > \tag{I.1}$$

$$\hat{V}|m >= \left(\frac{1}{6}\gamma l_P^2\right)^{3/2} \sqrt{(|m| - 1)|m|(|m| + 1)}|m > \tag{I.1}$$

sgn m: orientation

Poisson brackets.

$$\begin{aligned} \{F(A), p\} &= \{\sum_j \xi_j \exp(il_j c), p\} \\ &= \sum_j \xi_j \{\exp(il_j c), p\} \\ &= \frac{8\pi\gamma G}{6} \sum_I (il_j \xi_j) \exp(il_j c) \end{aligned} \tag{I.0}$$

Details Hamiltonian.

Quantization of $1/a$ (Test curvature singularity)

$\hat{p}$, $\hat{v}$ have eigenvalues zero implies no inverse

rewrite

$$\frac{1}{a}\delta_{IJ} = \frac{q_{IJ}}{\sqrt{\det q}} = \frac{e_I^i e_I^i}{\det e} =: m_{IJ} \tag{I.0}$$

with cotriad

$$e_I^i = \sqrt{\det q}(E^{-1})_I^i = \frac{2}{\gamma\kappa}\{A_I^i, V\} \tag{I.0}$$

$$m_{IJ} = \frac{16}{\gamma^2\kappa^2}\{A_I^i, \sqrt{V}\}\{A_J^i, \sqrt{V}\} \tag{I.0}$$

quantized:

$$\hat{m}_{IJ} = \frac{64}{\gamma^2 l_P^4}\left[(\sqrt{\hat{V}} - \cos\frac{c}{2}\sqrt{\hat{V}}\cos\frac{c}{2} - \sin\frac{c}{2})^2 - \delta_{IJ}(\sin\frac{c}{2}\sqrt{\hat{V}}\cos\frac{c}{2} - \cos\frac{c}{2}\sqrt{\hat{V}}\sin\frac{c}{2})^2\right] \tag{I.0}$$

eigenvalues bounded, finite even if $V = 0$ rapidly approach the classical behaviour

upper bound:

$$\hat{m}_{IJ,z} = \frac{32(2-\sqrt{2})}{3\sqrt{\gamma}l_P} \tag{I.0}$$

$$\hat{m}_{IJ}|0>= 0 \tag{I.0}$$

due to

$$m_{IJ} = \frac{sgn(a)^2}{|a|}\delta_{IJ} \tag{I.0}$$

evolution equations don't breakdown even though the volume of the universe goes to zero!

$$\mathcal{H} = -12\gamma^{-2}\kappa^{-1}(c(c-k) + (1+\gamma^2)k^2/4)\sqrt{|p|} \tag{I.0}$$

One has $|p| = a^2$ while $c = (k - \gamma\dot{a}$. If we insert this constraint equation $\mathcal{H} + \mathcal{H}_{\text{matter}}$

• All except a finite number of degrees of freedom are "frozen". The shift function is zero and the lapse is homogeneous. N is still a function of τ, so that separation beween two succesive three-surfaces is still undetermined. Reparametrization invariance is what remains of general covariance of the full theory.

• Are there corresponding solutions in full theory?

• Only considering states that are symmetric at the microscopic level.

I.2.6 Continuum Limit

Pre-classicality

$$\frac{3}{\gamma^3 l_P^2}\left[\left(V_{\frac{|m+1|}{2}} V_{\frac{|m+1|}{2}-1}\right) s_{m+4}(\phi) - \left(V_{\frac{|m|}{2}} V_{\frac{|m|}{2}-1}\right) s_m(\phi) + \left(V_{\frac{|m-1|}{2}} V_{\frac{|m-1|}{2}-1}\right) s_{m-4}(\phi)\right] = -\hat{\mathcal{H}}_\phi(m) s_m(\phi) \tag{I.0}$$

at large volume ($m >> 1$) assume s_m to be only mildy varying at small scales (from m to $m+1$) conjtinuum approximation

$$\psi(p,\phi) := s_{s(p)}(\phi) \tag{I.0}$$

with

$$n(p) = \frac{6p}{\gamma l_P^2} \tag{I.0}$$

and interpolation

I.2.7 Inflation from Loop Quantum Cosmology

effective Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{16\pi}{3} G a^{-3}\left(\frac{1}{2} a^{-3} p (3a^2/j l_P^2)^6 p_\phi^2 + a^3 V(\phi)\right). \tag{I.0}$$

Since the right hand side now depends on a for small a the classical behaviour, the dynamics is clearly modified.

I.3 Quantum Configuration Space of LQC

I.3.1 Review of the Schrodinger Representation

In the Schrodinger representation, the Weyl operators

$$W(\alpha, \beta) := e^{i(\alpha q + \beta p)} \quad \text{(I.0)}$$

with standard momentum and position operators p and q and $\alpha, \beta \in \mathbb{R}$, satisfy the Weyl relation

$$W(\alpha_1, \beta_1) W(\alpha_2, \beta_2) := e^{-\frac{i}{2}(\alpha_1 \beta_2 - \alpha_2 + \beta_1)} W(\alpha_1 + \alpha_2, \beta_1 + \beta_2). \quad \text{(I.0)}$$

Together with the unitarity condition

$$W(\alpha, \beta)^* = W(-\alpha, -\beta) \quad \text{(I.0)}$$

these relations alone define a unique simple C^*-algebra, the Weyl algebra. The Schrodinger representation is (up to unitary equivalence) the only irreducible representation of the Weyl algebra in which the Weyl operators are continuous fucntions of α and β with respect to the weak operator topology:

$$\lim_{\alpha' \to \alpha} < W(\alpha', \beta)\phi | \psi >=< W(\alpha, \beta)\phi | \psi > \quad \text{(I.0)}$$

for all $\alpha \in \mathbb{R}$ and for all $\phi, \psi \in \mathcal{H}$ and similarly for β.

$$[q, p] = i. \quad \text{(I.0)}$$

$$\int_{-\infty}^{\infty} < p | \psi(p) dp \;\; \text{with} \;\; \int_{-\infty}^{\infty} |\psi(p)|^2 dp < \infty. \quad \text{(I.0)}$$

The momentum operator acting to the left on a coordinate eigenstate has the realization as the derivative with reaspect to the eigenvalue q'

Let us define

$$U = 1 + i\delta q' p \quad \text{(I.0)}$$

To infinitesimal order, this is a unitary operator,

$$U^{\dagger}U = 1,$$

and hence when acts on a state vector it preserves the norm of the vector.

the canonical commutation relation

$$UqU^{-1} = q + i[p,q]\delta q', \tag{I.0}$$

and so U is the operator which performs infinitesimal tasnlations on the coordinate operator $\hat{q}$ (but leaves p unchanged).

$$\begin{aligned} < q'|Uq &= < q'|UqU^{-1}V \\ &= < q'|(q' + \delta q')U \\ &= (q' + \delta q') < q'|U, \end{aligned} \tag{I.-1}$$

which implies

$$< q'|U =< q' + \delta q'|. \tag{I.-1}$$

The action of U on $< q'|$ is to infintesimal trasnslate the eigenvalue q' to $q' + \delta q'$, with $< q'|$ and $< q' + \delta q'|$ having the same norm.

$$< p'|p >= \delta(p' - p) \tag{I.-1}$$

Transformation function $< q'|p' >$

$$< q'|p' > p' =< q'|p|p' >= \frac{1}{i}\frac{\partial}{\partial q'} < q'|p' >, \tag{I.-1}$$

integrating this differential equation giving

$$< q'|p' >= \frac{1}{\sqrt{2}\pi} e^{iq'p'}, \tag{I.-1}$$

$$\begin{aligned}\int dp' < q'|p' >< p'|q'' > &= \int \frac{dp'}{2\pi} e^{ip'(q'-q'')} \\ &= \delta(q'-q'') \\ &= < q'|q' > \end{aligned} \tag{I.-2}$$

the resolution of identity

$$\int dp'|p' >< p'| = 1. \tag{I.-2}$$

Similarly

$$\int dq' < p'|q' >< q'|p'' > = < p'|p'' >, \tag{I.-2}$$

and

$$\int dq'|q' >< q'| = 1. \tag{I.-2}$$

I.3.2 Polymer Representation

Let $\mathcal{H}_{\mathcal{P}}$ be a non-separable Hilbert space spanned by mutually orthoganol vectors $|p >$, $p \in \mathbb{R}$, $< p'|p >= \delta_{p'p}$, where $\delta_{p'p}$ is the Koneckera delta. A general element of $\mathcal{H}_{\mathcal{P}}$ is of the form

$$\sum_{p\in\mathbb{R}} \psi(p)|p > \text{ with } \sum_{p\in\mathbb{R}} |\psi(p)|^2 < \infty. \tag{I.-2}$$

Thus the polymer Hilbert spae $\mathcal{H}_{\mathcal{P}}$ can also be defined as the space of complex functions on $\mathbb{R}$ that are square integrable with respect to the discrete measure. Necessarily, any wave fucntion $\psi(p)$ can be non-zero only on a countable subset of $\mathbb{R}$, as the uncountable sum of finite terms is divergent (for example the sum of numbers between 0 and ϵ is greater than $\frac{\epsilon}{2} + \frac{\epsilon}{2} + \cdots = \frac{\epsilon}{2}\infty = \infty$).

The momentum operator $\hat{p}$ is defined in this representation by:

$$\hat{p}(k)|p >= p|p > \quad \text{or} \quad \hat{p}\psi(p) = p\psi(p). \tag{I.-2}$$

Since the discrete measure is translation invariant, there are also well defined unitary operators $\mathcal{U}(k)$ implimenting translaions in momentum space:

$$\mathcal{U}(k)|p >= |p + k > \quad \text{or} \quad \mathcal{U}(k)\psi(p) = \psi(p - k), \ k \in \mathbb{R}. \tag{I.-2}$$

The exponentiation of the operator $\hat{p}$ together with the operators $\mathcal{U}(k)$ provide a representation of the Weyl relations:

$$\begin{aligned} e^{iap}e^{ibp}\psi(p) &= e^{i(a+b)p}\psi(p) \\ \mathcal{U}(a)\mathcal{U}(b)\psi(p) &= \psi(p - a - b) \\ &= \mathcal{U}(a + b)\psi(p) \end{aligned} \tag{I.-3}$$

$$\begin{aligned} \mathcal{U}(a)e^{ibp}\psi(p) &= e^{ib(p-a)}\psi(p - a) \\ &= e^{-iab}e^{ibp}\mathcal{U}(a)\psi(p) \end{aligned} \tag{I.-3}$$

Definition of weakly continuous

We say that a sequence k_n converges to k weakly if for all $p \in \mathcal{H}$

$$\lim_{n\to\infty} < k_n|p > = < k|p >$$

We say that $\mathcal{U}(p)$ is weakly continuous if

$$\lim_{n\to\infty} < \mathcal{U}(k_n)p|p > = < \mathcal{U}(k)p|p >$$

for all $p \in \mathcal{H}$ as k_n converges weakly to k.

$k \mapsto \mathcal{U}(k)$ is weakly continuous if

$$\lim_{\tau\to t} < \mathcal{U}(\tau)p'|p > = < \mathcal{U}(t)p'|p >$$

for all $t \in \mathbb{R}$ and $p', p \in \mathcal{H}$

However, the representation of $\mathbb{R}$ given by $k \mapsto \mathcal{U}(k)$ is not continuous as the general element of the Hilbert space $\mathcal{H}_{\mathcal{P}}$ is non-zero only on a countable subset of $\mathbb{R}$. In fact, for arbitrary small k, a vector $|p >$ is mapped by $\mathcal{U}(k)$ to an orthogonal one $|p + k >$ so that

$$< p|\mathcal{U}(k_n)p >= 0 \quad \text{for all } n. \tag{I.-3}$$

However $\mathcal{U}(k)$ is not weakly continuous, since $|p>$ and $|p+k>$ are orthogonal to each other no matter how small the parameter t is. So one always has

$$| < p|\mathcal{U}(k)|p > - < p|p > | = < p|p > \neq 0, \tag{I.-3}$$

even in the limit that k goes to zero. Therefore, the infintesimal generator

Thus, the generator that would correspond to the configuration operator $\hat{x}$ is not defined on $\mathcal{H}_{\mathcal{P}}$. The best we can do is:

$$-i\frac{(\mathcal{U}(\delta k) - 1)}{\delta k}|p > \quad \text{or} \quad -i\frac{(1 - \mathcal{U}(\delta k))}{\delta k}\psi(p) \tag{I.-2}$$

where δk coresponds to the nearest point above p for which $\psi(p)$ is non-zero. The operators $\mathcal{U}(k)$ can then nevertheless be seen as giving a quantization of the classical configuration functions e^{ikx}. Reality conditions for these "approximated position operators" are satified, since $\mathcal{U}^\dagger(k) = \mathcal{U}(-k)$ Thus, the polymer representation provides a quantization of the space of the Poisson algebra of phase space functions made out of finite linear combinations of the functions

$$p \quad \text{and} \quad e^{ikx}, \;\; k \in \mathbb{R}. \tag{I.-2}$$

In particular, the configuration part of the Poison algebra of phase space algebra is the linear space of continuous and bounded complex functions in $\mathbb{R}$ of the form

$$f(x) = \sum_j c_j e^{ik_j x}, \tag{I.-2}$$

where the sums are finite, k_j are arbitrary real numbers and c_j are complex coefficents. The set of functions (I.3.2) clearly separates points in $\mathbb{R}$, i.e. given $x, x' \in \mathbb{R}$, $x \neq x'$, one can find a function f such that $f(x) \neq f(x')$. Infact, two functions are sufficient to separate points, e.g. $e^{ik_1 x}$ and $e^{ik_2 x}$, with k_1/k_2, an irrational number. To see this consider the case when k_1/k_2 is a rational number fig.(I.3.2). Neither function separates the points x_0 and x_1 as both functions are periodic over $x_1 - x_0$. However, if k_1/k_2 is an irrational number there are no finite intervals over which both functions are periodic.

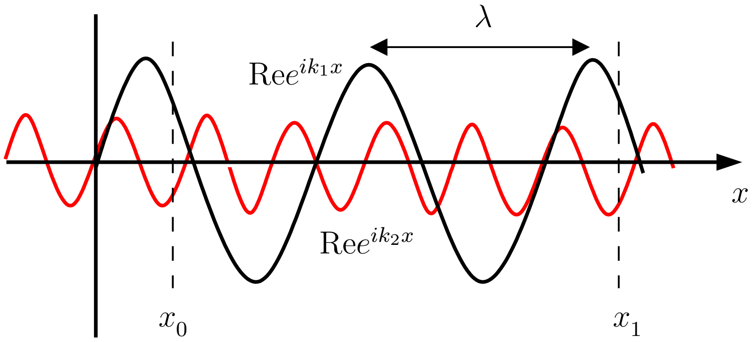


Figure I.1: separatenot. As $k_1/k_2 = 2.5$ both e^{ik_1x} and e^{ik_2x} are periodic over intervals of lengh 2λ, these two functions do not separate points an integer number of 2λ apart. However, if k_1/k_2 were irrational there would be no finite interval over which both functions were periodic.

I.3.3 Quantum Configuration Space as a Compact Group

The space $\overline{\mathbb{R}}$ is introduced as a set of homomorphisms, corresponding to a similar charachterization of $\overline{\mathcal{A}}$, section ??. The role of the group of hoops (or the groupoid of paths) is here played by the discrete group $\mathbb{R}$. The group $SU(2)$ is replaced by T, the unit circle in the complex plane $\mathbb{C}$.

The quantum configuration space of LQG includes all those connections which are discontinuous but all the same asign well defined holonomies (section ??), specifically:

$$\overline{A}(\gamma^{-1}) = (\overline{A}(\gamma)){-}1 \quad \text{and} \quad \overline{A}(\gamma_2 \cdot \gamma_1) = \overline{A}(\gamma_2)\overline{A}(\gamma_1) \tag{I.-2}$$

We can understand these connections as the homomorphisms from $\mathcal{G}$ to the $SU(2)$ group.

$$\overline{\mathcal{A}} \equiv \mathrm{Hom}[\mathcal{G}??, SU(2)]. \tag{I.-2}$$

This compactification can be imagined as being obtained from enlarging the classical configuration space $\mathbb{R}$ by adding points, and thus more continuity conditions, until only functions of the given algebra survive as continuous ones.

Let us consider the real line $\mathbb{R}$ equipped with the commutative group structure given by addition of real numbers. The Bohr compactification $\overline{\mathbb{R}}$ can be described as the set $[\mathbb{R}, T]$ of all, not necessary continuous, group homomorphisms from the group $\mathbb{R}$ to the multiplication group T of the unit circle of $\mathbb{C}$. we identify with the $\mathrm{Hom}[\mathbb{R}, T]$

$$\overline{\mathbb{R}} \equiv \mathrm{Hom}[\mathbb{R}, T]. \tag{I.-2}$$

The generic element of $\bar{\mathbb{R}}$ will be denoted by $\bar{x}$. So, every $\overline{x} \in \overline{\mathbb{R}}$ is a map, $\overline{x} : \mathbb{R} \to T$ such that

$$\overline{x}(0) = 1 \quad \text{and} \quad \overline{x}(k_1 + k_2) = \overline{x}(k_1)\overline{x}(k_2), \text{ for all } k_1, k_2 \in \mathbb{R}. \tag{I.-2}$$

Since T is a commutative group, it is clear that

$$\overline{x}\,\overline{x}'(k) := \overline{x}(k)\overline{x}'(k) \tag{I.-2}$$

defines a group structure on $\overline{\mathbb{R}}$

From the fact that $\bar{\mathbb{R}}$ contains only homomorphisms, it is a closed subset of $\times_{k\in\mathbb{R}}T$, and it is therefore compact. Verify this.

PROOF HERE

□

I.3.4 Projective Aspects

For arbitrary $n\ in \mathbb{N}$, a finite set of real numbers $\gamma = \{k_1, \ldots, k_n\}$ will said to be independent if $k_1, \ldots, k_n$ are algebraically independent if

$$\sum_{i=1}^{n} m_i k_i = 0, \quad m_i \in \mathbb{Z} \tag{I.-2}$$

implies $m_i = 0$, for all i. The set of all such independent sets γ will be denoted by Γ.

Let G_γ denote the subgroup of $\mathbb{R}$ freely generated by the set $\gamma = \{k_1, \ldots, k_n\}$:

$$G_\gamma := \{\sum_{i=1}^{n} m_i k_i,\ m_i \in \mathbb{Z}\}. \tag{I.-2}$$

By (I.3.4) we have an unique unit element corresponding to $m_i = 0$, for all i and a unique inverse of $\sum_i n_i k_i$ given by $\sum_i (-n_i) k_i$.

From this group structure we have a partial order relation making Γ a directed set: a set γ' is said to be greater than γ, and we write $\gamma' \geq \gamma$, if G_γ is a subgroup of $G_{\gamma'}$. It is clear that given γ and γ' one can always find γ'' such that $\gamma'' \geq \gamma$ and $\gamma'' \geq \gamma'$, and so Γ becomes a directed set. Also, Γ has no maximal element.

$$\mathbb{R}_\gamma := \mathrm{Hom}[G_\gamma, T]. \qquad \text{(I.-2)}$$

For any pair γ, γ' such that $\gamma' \geq \gamma$ there are surjective projections

$$p_{\gamma\gamma'}; \mathbb{R}_{\gamma'} \to \mathbb{R}_\gamma \qquad \text{(I.-2)}$$

I.3.5 C^*-algebra Aspects

We will now see explicitely that $\overline{\mathbb{R}}$ is in fact the spectrum of the C^*-algebra of almost periodic functions in $\mathbb{R}$. This characterization of the quantum configuration space corresponds to the original introduction of the space of generalized connections $\overline{\mathcal{A}}$ as the spectrum of the holonomy algebra section(??).

Let us consider the configuration $*-$algebra $\mathcal{C}$ of functions given by finite sums of the form

$$f(x) = \sum_j c_j e^{ik_j x}, \qquad \text{(I.-2)}$$

with respect to the supremum norm, $\|f(x)\| = \sup_{x\in\mathbb{R}} |f(x)|$ this becomes a C^*-algebra (as $\|f^*(x)\| = \|f(x)\|$ and $\|f^*(x)f(x)\| = \|f(x)\|^2$). We form the C^*-completion $\overline{\mathcal{C}}$ with respect to this norm.

The spectrum $\Delta(\overline{\mathcal{C}})$ of the algebra $\overline{\mathcal{C}}$ is the set of all non-zero multipliactive linear functionals on $\overline{\mathcal{C}}$, i.e. non-zero linear functionals $\varphi : \overline{\mathcal{C}} \to \mathbb{C}$ such that

$$\varphi(fg) = \varphi(f)\varphi(g), \text{ for all } f, g \in \overline{\mathcal{C}}. \qquad \text{(I.-2)}$$

such functions are necessarily continuous as . One can check that $\varphi(e^{ikx})$ takes values in T, for all $\varphi \in \Delta(\overline{\mathcal{C}})$, for all k

$$|\varphi(e^{ikx})| =$$

$\{k_1, \ldots, k_n\}$, let $\mathcal{C}_\gamma \subset \mathcal{C}$ denote the $*-$subalgebra generated by the set of fucntions $\{e^{ik_1x}, \ldots, e^{ik_nx}\}$, whose elements are finite sums of the form

$$f(x) = \sum_k c_k e^{ikx} \quad \text{with} \quad k \in G_\gamma. \qquad \text{(I.-2)}$$

I.3.6 The Schrodinger Representation

We will now see how the usual Schrodinger representation is obtained in the present context. It is given by a different measure on $\overline{\mathbb{R}}$. The introduction of this measure corresponds to the so called $r-$Fock measwures in the loop quantization of $U(1)$ connections [??].

consider the Gausian measure $d\nu_G = e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ in $\mathbb{R}$, and the corresponding space of square integrable fucntions $L^2(\mathbb{R}, \nu_G)$. The Hilbert space $L^2(\mathbb{R}, \nu_G)$ carries (a representation unitarily equivalent to) the usual Schrodniger representation of the Weyl relations.

Hilbert space from measure

The standard choice is to select the Hilbert space to be,

$$\mathcal{H} = L^2(\mathbb{R}, dq)$$

the space of square-integrable functions with respect to the Lebesgue measure dq (invariant under constant translations) on $\mathbb{R}$.

There is a representation of the Weyl algebra that can be called the 'Fock type'.

the measure in the Schrodinger representation becomes non trivial and thus the momentum operator acquires an extra term in order to render the operator self-adjoint.

$$(\hat{q} \cdot \psi)(q) = q\psi(q) \tag{I.-2}$$

and

$$(\hat{p} \cdot \psi)(q) = -i\hbar\frac{d\psi}{dq} + \text{ multiplicative term} \tag{I.-2}$$

where the second term in (I.3.6), depending on the configuration, is precisely there to render the operator self-adjoint when the measure is different from the "dx" measure, and depends on the details of the measure.

First we need to find the measure $d\mu$ on the quantum configuration space in order to get the Hilbert space $\mathcal{H}_s$ and second we need to find the multiplicative term of the basic operator ((I.3.6).

Quantum algebra and states

Let $\mathcal{A}$ be a C^*-algebra with unit and let $\omega : \mathcal{A} \to \mathbb{C}$ be a stste. Then there exists a Hilbert space $\mathcal{H}$, a representation $\pi : \mathcal{A} \to L(\mathcal{H})$ and a vector $|\Psi_0 > \in \mathcal{H}$ such that

$$\omega(A) = < \Psi_0, \pi(A)|\Psi_0 >_{\mathcal{H}} . \tag{I.-2}$$

Furthermore, the vector $|\Psi_0>$ is cyclic. The triplet $(\mathcal{H}, \pi(A)|\Psi_0>)$ with these properties is unique (up to unitary equivalence).

One key aspect of the theorem is that one may have different, but unitarily equivalent, representations of the Weyl algebra, which yeild equivalent quantum theories. This is the precise sense in which the Fock and Schrodinger representations are related to each other.

Thus, by vertue of the GNS construction, the value of the state ω_{Fock} acting on the Weyl generators $\hat{W}()$ is interpreted as the expectation value of the corresponding operators $R_{Fock}(\hat{W}())$ on the vacuum state $\Omega_{\mathcal{F}}$

we can now compute the expectation values of the Weyl operators of the Fock vacuum and thus obtain a positive linear functional ω_{Fock} on the algebra $\mathcal{A}$. Now, the Schrodinger representation that will be equivalent to the Fock construction will be the one that the GNS construction provides for the *same* algebraic state ω_{Fock}. What we must do to complete the Schrodinger construction such that the expectation value of the corresponding Weyl operators coincide with those of the Fock representation.

$$\omega_{Fock}(\hat{W}(\lambda)) = e^{-\frac{1}{4}\mu(\lambda,\lambda)} \tag{I.-2}$$

Construction of the Fock representation

The relations

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0 \tag{I.-1}$$
$$\{q_\mu, q_\nu\} = \delta_{\mu\nu} \tag{I.0}$$

$$\Omega(y_1, y_2) = \sum_\mu (p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}) \tag{I.0}$$

$$W(y) = \exp[i\Omega(y, \cdot)] \tag{I.0}$$

every point of Γ uniquely determines a solution. We define $\mathcal{B}$ to be the space of solutions which arise from the initial data in Γ.

The fundamental Poisson brackets on Γ can be expressed as

$$\{\Omega([q_1, p_1], \cdot), \Omega([q_2, p_2], \cdot)\} = -\Omega([q_1, p_1], [q_2, p_2]). \tag{I.0}$$

the CCR read

$$\{\int f_1\phi, \int f_2\pi\} = \int f_1 f_2 \tag{I.0}$$

We specify a real inner product $\mu : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ satisfying, for all $\psi_1 \in \mathcal{B}$,

$$\mu(\psi_1, \psi_1) = \frac{1}{4} l.u.b._{\psi_2 \neq 0} \frac{[\Omega(\psi_1, \psi_1)]^2}{\mu(\psi_2, \psi_2)} \tag{I.0}$$

We define an operator $J : \mathcal{B}_\mu \to \mathcal{B}_\mu$ defined by

$$\mu(\psi_1, \psi_2) = 2\mu(\psi_1, J\psi_2) = (\psi_1, J\psi_2) \tag{I.0}$$

From the antisymmetry of Ω, it follows that

$J^\dagger = -J$. $J^2 = -I$. the specification of an inner product, μ, saticfying (??) gives rise to a complex structure, J, on $\mathcal{B}$.

we now complexify $\mathcal{B}_\mu$ and extend the actions of Ω, μ, and J from $\mathcal{B}_\mu$ to $\mathcal{B}_\mu^{\mathbb{C}}$. We define an inner product on $\mathcal{B}_\mu^{\mathbb{C}}$ by

$$(\psi_1, \psi_2) = 2\mu(\overline{\psi}_1, J\psi_2) \tag{I.0}$$

for $\psi_1, \psi_2 \in \mathcal{B}_\mu^{\mathbb{C}}$, thus making $\mathcal{B}_\mu^{\mathbb{C}}$ into a (complex) Hlbert space.

define the map $K : \mathcal{B}_\mu^{\mathbb{C}} \to \mathcal{H}$ to be the orthogonal projection onto the subspace, $\mathcal{H}$, of $\mathcal{B}_\mu^{\mathbb{C}}$.

$$(K\psi_1, K\psi_2) = -i\Omega(\overline{K\psi}_1, K\psi_2) = \mu(\psi_1, \psi_2) - \frac{i}{2}\Omega(\psi_1, \psi_2). \tag{I.0}$$

we have

$$\mathrm{Im}(K\psi_1, K\psi_2)_{\mathcal{H}} = -\frac{1}{2}\Omega(\psi_1, \psi_2). \tag{I.0}$$

Functional representation

the Schrodinger representation - find the measure $d\mu$ and the multiplicative term in (I.3.6), that corresponds to the given Fock representation.

$$R_{Sch}(\hat{W}(\lambda)) = e^{i\hat{p}[f]}. \tag{I.0}$$

Now, the equation (??) tells us that the state ω_{Sch} should be such that,

$$R_{Sch}(\hat{W}(\lambda)) = \exp\left[-\frac{1}{4}\mu(\lambda,\lambda)\right] = \exp\left[-\frac{1}{4}fBf\right] \tag{I.0}$$

where we have used (??) in the last step. On the other hand, the left hand side of (I.3.6) is the vacuum expectation value of the $\hat{W}(\lambda)$ operator. That is,

$$\omega_{Sch}(\hat{W}(\lambda)) = \int_{\overline{\mathcal{C}}} d\mu \overline{\Psi_0}(R_{Sch}(\hat{W}(\lambda)) \cdot \Psi_0) = \int_{\overline{\mathcal{C}}} d\mu e^{if\psi} \tag{I.0}$$

Let us compare (??) and (??),

$$\int_{\overline{\mathcal{C}}} d\mu e^{i\psi} = \exp\left[-\frac{1}{4}fBf\right] \tag{I.0}$$

The meaning of this is - the fourier tgransform of the measure $\tilde{\mu}$ is defined as

$$\chi_{\tilde{\mu}}(f) := \int d\tilde{\mu} e^{if\varphi}, \tag{I.0}$$

where f is an arbitrary continuous function(al) on $\mathcal{V}$,...Then the measure of the Gausian

I.4 Path Integral

I.5 Limiting values of U when $N \to \infty$

It is convenient to rewrite $A_N(\nu_M, \ldots, \nu_0; \alpha)$ defined in () in the following way

$$\begin{aligned} A_N(\nu_M, \ldots, \nu_0; \alpha) &= U_{\nu_M \nu_{M-1}} \ldots U_{\nu_1 \nu_0} \left[U_{\nu_M \nu_M}\right]^N \left[U_{\nu_M \nu_M} \ldots U_{\nu_0 \nu_0}\right]^{-1} \times \\ & \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \cdots \sum_{N_1=1}^{N_2-1} \left[\frac{U_{\nu_{M-1}\nu_{M-1}}}{U_{\nu_M \nu_M}}\right]^{N_M} \cdots \left[\frac{U_{\nu_0 \nu_0}}{U_{\nu_1 \nu_1}}\right]^{N_1}. \end{aligned} \tag{I.0}$$

Here we give the details of calculating the limit $N \to \infty$ of () and show that is given by $A(\nu_M, \ldots, \nu_0; \alpha)$, of () which we rewrite as

$$\begin{aligned} A_N(\nu_M,\dots,\nu_0;\alpha) &= (-i\alpha)^M\,\Theta_{\nu_M\nu_{M-1}}\dots\Theta_{\nu_1\nu_0}e^{-i\alpha\Theta_{\nu_M\nu_M}} \\ &\qquad \int_0^1 d\tau_M\int_0^{\tau_M}d\tau_{M-1}\dots\int_0^{\tau_2}d\tau_1 e^{i\tau_M b_M}\cdots e^{i\tau_1 b_1} \end{aligned} \tag{I.0}$$

where

$$b_m := -i\alpha\left(\Theta_{\nu_{m-1}\nu_{m-1}} - \Theta_{\nu_m\nu_m}\right). \tag{I.0}$$

We start by calculating the $N \gg 1$ behaviour of the terms appearing in (). These are:

$$U_{\nu_{m+1}\nu_m} = -\frac{i\alpha}{N}\Theta_{\nu_{m+1}\nu_m} + \mathcal{O}\left(N^{-2}\right), \tag{I.0}$$

$$\begin{aligned} \left[U_{\nu_M\nu_M}\right]^N &= e^{NlogU_{\nu_M\nu_M}} \\ &= e^{N\left(-\frac{i\alpha}{N}\Theta_{\nu_M\nu_M}+\mathcal{O}\left(N^{-2}\right)\right)} \\ &= e^{-i\alpha\Theta_{\nu_M\nu_M}} + \mathcal{O}\left(N^{-1}\right), \end{aligned} \tag{I.-1}$$

$$\begin{aligned} \left[U_{\nu_M\nu_M}\dots U_{\nu_0\nu_0}\right]^{-1} &= e^{(logU_{\nu_M\nu_M}+\dots+logU_{\nu_0\nu_0})} \\ &= e^{-\frac{i\alpha}{N}(\Theta_{\nu_M\nu_M}+\dots+\Theta_{\nu_0\nu_0})+\mathcal{O}\left(N^{-2}\right)} \\ &= 1+\mathcal{O}\left(N^{-1}\right), \end{aligned} \tag{I.-2}$$

$$\begin{aligned} \left[\frac{U_{\nu_{M-1}\nu_{M-1}}}{U_{\nu_M\nu_M}}\right]^{N_M} &= e^{N_m\left(logU_{\nu_{m-1}\nu_{m-1}}-logU_{\nu_m\nu_m}\right)} \\ &= e^{N_M\left(b_m/N+\mathcal{O}\left(N^{-2}\right)\right)} \\ &= e^{\frac{N_m}{N}b_m} + \mathcal{O}\left(N_m N^{-2}\right), \end{aligned} \tag{I.-3}$$

with b_m given by (I.5). In () and () we have used the fact that the multivalued nature of the log function does not affect the final result: $e^{N(logx+2\pi k)} = e^{Nlogx}$ where $k \in \mathbb{Z}$ reflects the multiple values that log can take.

Substituting these expression into (I.0) to obtain

$$
\begin{aligned}
& A_N(\nu_M, \ldots, \nu_0; \alpha) \\
= & \left[(-i\alpha)^M \Theta_{\nu_M \nu_{M-1}} \ldots \Theta_{\nu_1 \nu_0} N^{-M} + \mathcal{O}(N^{-M-1})\right] \times \\
& \left[e^{-i\alpha\Theta_{\nu_M \nu_M}} + \mathcal{O}(N^{-1})\right] \left[1 + \mathcal{O}(N^{-1})\right] \times \\
& \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \cdots \sum_{N_1=1}^{N_2-1} \left[e^{\frac{N_M}{N} b_M} + \mathcal{O}(N_M N^{-2})\right] \ldots \left[e^{\frac{N_1}{N} b_1} + \mathcal{O}(N_1 N^{-2})\right] \\
= & \left[(-i\alpha)^M \Theta_{\nu_M \nu_{M-1}} \ldots \Theta_{\nu_1 \nu_0} e^{-i\alpha\Theta_{\nu_M \nu_M}} N^{-M} + \mathcal{O}(N^{-M-1})\right] \times \\
& \prod_{m=1}^{M} \left[\sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N} b_m} + \ \mathcal{O}(N_m N^{-2})\right]. \qquad \text{(I.-7)}
\end{aligned}
$$

Each sum has two terms. The first one gives a contribution of $\sum_{N_m} e^{\frac{N_m}{N} b_M} \sim \mathcal{O}(N)$ while the second one is $\sum_{N_m} \mathcal{O}(N_m N^{-2}) \sim \mathcal{O}(1)$. The M sums then $[\mathcal{O}(N) + \mathcal{O}(1)]^M \sim \mathcal{O}(N^M) + \mathcal{O}(N^{M-1})$. With this, the non-vanishing contribution

$$
\begin{aligned}
A_N(\nu_M, \ldots, \nu_0; \alpha) \ = \ & (-i\alpha)^M \Theta_{\nu_M \nu_{M-1}} \ldots \Theta_{\nu_1 \nu_0} e^{-i\alpha\Theta_{\nu_M \nu_M}} \times \\
& N^{-M} \prod_{m=1}^{M} \left[\sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N} b_m}\right] + \mathcal{O}(N^{-1}) \qquad \text{(I.-7)}
\end{aligned}
$$

This has the pre-factors appearing in (I.0). It remains to show that N^{-M} times the sum in (I.-7) limits to the integrals in (I.0) as $N \to \infty$. We have

$$
\begin{aligned}
& \lim_{N\to\infty} N^{-M} \prod_{m=1}^{M} \left[\sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N} b_M}\right] \\
= & \lim_{N\to\infty} N^{-M} \left(\sum_{N_M=0}^{N} \sum_{N_{M-1}=0}^{N_M} \cdots \sum_{N_1=0}^{N_2} e^{\frac{N_M}{N} b_M} \ldots e^{\frac{N_1}{N} b_1} \right. \\
& \left. - \sum_{N_M=0}^{M-1} \sum_{N_{M-1}=0}^{M-2} \cdots \sum_{N_1=0}^{0} e^{\frac{N_M}{N} b_M} \ldots e^{\frac{N_1}{N} b_1} - e^{b_M} \sum_{N_{M-1}=0}^{N_M} \cdots \sum_{N_1=0}^{N_2} e^{\frac{N_{M-1}}{N} b_{M-1}} \ldots e^{\frac{N_1}{N} b_1} - \ldots \right) \\
= & \lim_{N\to\infty} \left(N^{-M} \sum_{N_M=0}^{N} \sum_{N_{M-1}=0}^{N_M} \cdots \sum_{N_1=0}^{N_2} e^{\frac{N_M}{N} b_M} \ldots e^{\frac{N_1}{N} b_1} + \mathcal{O}(N^{-1}) \right) \\
= & \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 \; e^{\tau_M b_M} \ldots e^{\tau_1 b_1}. \qquad \text{(I.-10)}
\end{aligned}
$$

This concludes the proof of (5.2).

I.6 General Integrals in Eq ()

The amplitude for a single discrete path was given as

$$
\begin{aligned}
A(\nu_M, \ldots, \nu_0, \alpha) \; &= \; \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \cdots \int_0^{\tau_2} d\tau_1 e^{-i(\Delta\tau - \tau_M)\alpha\Theta_{\nu_M\nu_M}} (-i\alpha\Theta_{\nu_M\nu_{M_1}}) \times \\
& e^{-i(\tau_M - \tau_{M-1})\alpha\Theta_{\nu_{M-1}\nu_{M-1}}} \ldots e^{-i(\tau_2-\tau_1)\alpha\Theta_{\nu_1\nu_1}} (-i\alpha\Theta_{\nu_1\nu_0}) e^{i\tau_1\alpha\Theta_{\nu_0\nu_0}} \qquad \text{(I.-10)}
\end{aligned}
$$

This expression can be written in terms of the following integral

$$
\begin{aligned}
I(x_M, \ldots, x_0, \Delta\tau) \; &= \; \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \cdots \int_0^{\tau_2} d\tau_1 \; (i)^M e^{i(\Delta\tau - \tau_M)x_M} e^{i(\tau_M - \tau_{M-1})x_{M-1}} \\
& \ldots e^{i(\tau_2 - \tau_1)x_1} e^{i\tau_1 x_0} \qquad \text{(I.-10)}
\end{aligned}
$$

We will first evaluate this integral for the case where all x_i are distinct. By induction on M-th number of vertices or the number of times that x changes value, we will show that when the x_i are all distinct the integral is

$$
I(x_M, \ldots, x_0, \Delta\tau) = \sum_{i=0}^{M} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M}(x_i - x_j)} \qquad \text{(I.-10)}
$$

This is true by inspection for $M = 0$. If we assume the result (I.6) holds for M we can evaluate the integral with $M + 1$ vertices.

$$
\begin{aligned}
I(x_{M+1}, x_M, \ldots, x_0, \Delta\tau) &= \int_0^{\Delta\tau} d\tau_{M+1}\, i e^{i(\Delta\tau - \tau_{M+1})x_{M+1}} I(x_M, \ldots, x_0, \tau_{M+1}) \\
&= \int_0^{\Delta\tau} d\tau_{M+1}\, i e^{i(\Delta\tau - \tau_{M+1})x_{M+1}} \sum_{i=0}^{M} \frac{e^{ix_i\tau_{M+1}}}{\prod_{j\neq i}^{M}(x_i - x_j)} \\
&= i e^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M} \frac{1}{\prod_{j\neq i}^{M}(x_i - x_j)} \int_0^{\Delta\tau} d\tau_{M+1}\, e^{i(x_i - x_{M+1})\tau_{M+1}} \\
&= i e^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M} \frac{1}{\prod_{j\neq i}^{M}(x_i - x_j)} \left[\frac{e^{i(x_i - x_{M+1})\Delta\tau} - 1}{i(x_i - x_{M+1})}\right] \\
&= \sum_{i=0}^{M} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M+1}(x_i - x_j)} - e^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M} \frac{1}{\prod_{j\neq i}^{M+1}(x_i - x_j)} \\
&= \sum_{i=0}^{M+1} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M+1}(x_i - x_j)} - e^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M+1} \frac{1}{\prod_{j\neq i}^{M+1}(x_i - x_j)}
\end{aligned} \tag{I.-15}
$$

We use the identity

$$
\sum_{i=0}^{M+1} \frac{1}{\prod_{j\neq i}^{M+1}(x_i - x_j)} = 0 \tag{I.-15}
$$

which can be derived as follows. Define

$$
p(x) := -1 + \sum_{i=0}^{M+1} \prod_{j\neq i} \frac{x - x_j}{x_i - x_j}
$$

then

$$
\begin{aligned}
p(x_l) &= -1 + \sum_{i=0}^{M+1} \prod_{j\neq i} \frac{x_l - x_j}{x_i - x_j} \\
&= -1 + \sum_{i=0}^{M+1} \delta_{li} \\
&= 0
\end{aligned} \tag{I.-16}
$$

for $l = 0, 1, 2, \ldots, M+1$. We have the situation that there are apparently $M+2$ distinct roots, however $p(x)$ is blatantly a polynomial of order $M+1$ at most. The only resolution to this is that $p(x)$ is identically zero. Therefore we have

$$\sum_{i=0}^{M+1} \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)} - 1 \equiv 0.$$

Noting that the coefficient of x^{M+1} on the LHS must be zero gives the identity (I.6). So finally the integral (I.-15) can be written

$$I(x_{M+1}, x_M, \ldots, x_0, \Delta\tau) = \sum_{i=0}^{M+1} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M+1}(x_i - x_j)} \tag{I.-16}$$

Therefore if (I.6) holds for M it also holds for $M+1$, thus by induction it holds for all $M \geq 0$.

If the x_i are not distinct, if there exist i, j such that $x_i = x_j$, then the proof follows in a similar way. Key to this is that the integral $I(x_M, \ldots, x_0)$ is independent of the order of the x_i's. This can be seen from the fact that the integral can be rewriten as

$$\begin{aligned} I(x_M, \ldots, x_0, \Delta\tau) \;\; &= \;\; \int_0^{\Delta\tau} d\Delta\tau_M d\Delta\tau_{M-1} \ldots d\Delta\tau_0 \; \delta(\Delta\tau_M + \cdots + \Delta\tau_0 - \Delta\tau) \\ &\qquad (i)^M e^{i\Delta\tau_M x_M} e^{i\Delta\tau_{M-1}x_{M-1}} \cdots e^{i\Delta\tau_1 x_1} e^{i\Delta\tau_0 x_0}. \end{aligned} \tag{I.-16}$$

We take some time to understand how this derived. We first consider the simplest case. Note

$$\begin{aligned} &I(x_2, x_1, x_0, \Delta\tau) \\ = \;\; &\int_0^{\Delta\tau} d\tau_2 \int_0^{\tau_2} d\tau_1 (i)^2 e^{i(\Delta\tau-\tau_2)x_2} e^{i(\tau_2-\tau_1)x_1} e^{i\tau_1 x_0} \\ = \;\; &\int_0^{\Delta\tau} d\tau_2 \int_0^{\tau_2} d\tau_1 (i)^2 \left(\int_0^{\Delta\tau} d\Delta\tau_2 \delta(\Delta\tau_2 + \tau_2 - \Delta\tau) e^{i\Delta\tau_2 x_2} \right) e^{i(\tau_2-\tau_1)x_1} e^{i\tau_1 x_0} \\ = \;\; &\int_0^{\Delta\tau} d\Delta\tau_2 \int_0^{\Delta\tau} d\tau_2 \int_0^{\tau_2} d\tau_1 \delta(\Delta\tau_2 + (\tau_2 - \tau_1) + \tau_1 - \Delta\tau)(i)^2 e^{i\Delta\tau_2 x_2} e^{i(\tau_2-\tau_1)x_1} e^{i\tau_1 x_0} \end{aligned} \tag{I.-19}$$

We now want to make the change of variables

$$\begin{aligned} \Delta\tau_0 \;\; &= \;\; \tau_1 \\ \Delta\tau_1 \;\; &= \;\; \tau_2 - \tau_1. \end{aligned} \tag{I.-19}$$

Inverting the transformation gives

$$\begin{aligned}\tau_1 &= \Delta\tau_0 \\ \tau_2 &= \Delta\tau_0 + \Delta\tau_1.\end{aligned} \tag{I.-19}$$

The Jacobian is then

$$\begin{vmatrix} \frac{\partial\tau_1}{\partial\Delta\tau_0} & \frac{\partial\tau_1}{\partial\Delta\tau_1} \\ \frac{\partial\tau_2}{\partial\Delta\tau_0} & \frac{\partial\tau_2}{\partial\Delta\tau_1} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1. \tag{I.-19}$$

The region of integration is inside the lines

$$\tau_1 = 0,\ \tau_2 - \tau_1 = 0,\ \tau_2 = \Delta\tau.$$

Substituting in (I.-19) the lines in the $\Delta\tau_0\Delta\tau_1$-plane are

$$\Delta\tau_0 = 0, \Delta\tau_1 = 0, \Delta\tau_0 + \Delta\tau_1 = \Delta\tau.$$

The integral is then

$$\begin{aligned}&I(x_2, x_1, x_0, \Delta\tau) \\ = &\int_0^{\Delta\tau} d\Delta\tau_2 \int\int_R d\Delta\tau_1 d\Delta\tau_0 \delta(\Delta\tau_2 + \Delta\tau_1 + \Delta\tau_0 - \Delta\tau)(i)^2 e^{i\Delta\tau_2 x_2} e^{i\Delta\tau_1 x_1} e^{i\Delta\tau_0 x_0}\end{aligned} \tag{I.-20}$$

where R stands for the region of integration defined by $\Delta\tau_1, \Delta\tau_0 \geq 0$ and $\Delta\tau_1 + \Delta\tau_0 \leq \Delta\tau$. But as

$$\delta(\Delta\tau_2 + [\Delta\tau_1 + \Delta\tau_0] - \Delta\tau) = 0$$

for $\Delta\tau_1 + \Delta\tau_0 > \Delta\tau$ we can extend the region of integration to $0 \leq \Delta\tau_1, \Delta\tau_0 \leq \Delta\tau$. Hence we may finally write

$$\begin{aligned}&I(x_2, x_1, x_0, \Delta\tau) \\ = &\int_0^{\Delta\tau} d\Delta\tau_2 \int_0^{\Delta\tau} d\Delta\tau_1 \int_0^{\Delta\tau} d\Delta\tau_0 \delta(\Delta\tau_2 + \Delta\tau_1 + \Delta\tau_0 - \Delta\tau)(i)^2 e^{i\Delta\tau_2 x_2} e^{i\Delta\tau_1 x_1} e^{i\Delta\tau_0 x_0}\end{aligned} \tag{I.-21}$$

which is the desired result. We now look at the general case. We write

$$
\begin{aligned}
& I(x_M, \dots, x_0, \Delta\tau) \\
= & \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 \, (i)^M \left(\int_0^{\Delta\tau} d\Delta\tau_M \delta(\Delta\tau_M + \tau_M - \Delta\tau) e^{i\Delta\tau_M x_M} \right) \\
& \qquad e^{i(\tau_M - \tau_{M-1})x_{M-1}} \dots e^{i(\tau_2 - \tau_1)x_1} e^{i\tau_1 x_0} \\
= & \int_0^{\Delta\tau} d\Delta\tau_M \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 \, (i)^M \delta(\Delta\tau_M + (\tau_M - \tau_{M-1}) + \\
+ & (\tau_{M-1} - \tau_{M-2}) + \dots + (\tau_2 - \tau_1) + \tau_1 - \Delta\tau) e^{i\Delta\tau_M x_M} e^{i(\tau_M - \tau_{M-1})x_{M-1}} \dots e^{i(\tau_2 - \tau_1)x_1} e^{i\tau_1 x_0}
\end{aligned}
\tag{I.-25}
$$

We now want to make the change of variables

$$
\begin{aligned}
\Delta\tau_0 &= \tau_1 \\
\Delta\tau_1 &= \tau_2 - \tau_1 \\
&\vdots \\
\Delta\tau_{M-2} &= \tau_{M-1} - \tau_{M-2} \\
\Delta\tau_{M-1} &= \tau_M - \tau_{M-1}.
\end{aligned}
\tag{I.-28}
$$

Inverting the transformation gives

$$
\begin{aligned}
\tau_1 &= \Delta\tau_0 \\
\tau_2 &= \Delta\tau_0 + \Delta\tau_1 \\
&\vdots \\
\tau_{M-1} &= \Delta\tau_0 + \Delta\tau_1 + \dots + \Delta\tau_{M-2} \\
\tau_M &= \Delta\tau_0 + \Delta\tau_1 + \dots + \Delta\tau_{M-2} + \Delta\tau_{M-1}.
\end{aligned}
\tag{I.-31}
$$

The Jacobian is then

$$
\begin{vmatrix}
\frac{\partial\tau_1}{\partial\Delta\tau_0} & \frac{\partial\tau_1}{\partial\Delta\tau_1} & \dots & \frac{\partial\tau_1}{\partial\Delta\tau_{M-1}} \\
\frac{\partial\tau_2}{\partial\Delta\tau_0} & \frac{\partial\tau_2}{\partial\Delta\tau_1} & \dots & \frac{\partial\tau_2}{\partial\Delta\tau_{M-1}} \\
\vdots & \vdots & & \vdots \\
\frac{\partial\tau_{M-1}}{\partial\Delta\tau_0} & \frac{\partial\tau_{M-1}}{\partial\Delta\tau_1} & \dots & \frac{\partial\tau_{M-1}}{\partial\Delta\tau_{M-1}} \\
\frac{\partial\tau_M}{\partial\Delta\tau_0} & \frac{\partial\tau_M}{\partial\Delta\tau_1} & \dots & \frac{\partial\tau_M}{\partial\Delta\tau_{M-1}}
\end{vmatrix}
=
\begin{vmatrix}
1 & 0 & \dots & 0 \\
1 & 1 & \dots & 0 \\
\vdots & \vdots & & \vdots \\
1 & 1 & \dots & 0 \\
1 & 1 & \dots & 1
\end{vmatrix}
= 1.
$$

The integral is then

$$
\begin{aligned}
& I(x_M, \ldots, x_0, \Delta\tau) \\
= & \int_0^{\Delta\tau} d\Delta\tau_M \int\int \cdots \int_R d\Delta\tau_{M-1} d\Delta\tau_{M-2} \ldots d\Delta\tau_0 \; (i)^M \delta(\Delta\tau_M + \Delta\tau_{M-1} + \\
+ & \Delta\tau_{M-2} + \cdots + \Delta\tau_1 + \Delta\tau_0 - \Delta\tau) e^{i\Delta\tau_M x_M} e^{i\Delta\tau_{M-1} x_{M-1}} \ldots e^{i\Delta\tau_1 x_1} e^{i\Delta\tau_0 x_0}
\end{aligned}
\tag{I.-34}
$$

where R stands for the region of integration defined by $\Delta\tau_{M-1}, \ldots, \Delta\tau_0 \geq 0$ and $\Delta\tau_{M-1} + \Delta\tau_{M-2} + \cdots + \Delta\tau_0 \leq \Delta\tau$, as the reader can verify. But as

$$
\delta(\Delta\tau_M + [\Delta\tau_{M-1} + \Delta\tau_{M-2} + \cdots + \Delta\tau_0] - \Delta\tau) = 0
$$

for $\Delta\tau_{M-1} + \Delta\tau_{M-2} + \cdots + \Delta\tau_0 > \Delta\tau$ we can extend the region of integration to $0 \leq \Delta\tau_{M-1}, \ldots, \Delta\tau_0 \leq \Delta\tau$. Hence we may finally write

$$
\begin{aligned}
& I(x_M, \ldots, x_0, \Delta\tau) \\
= & \int_0^{\Delta\tau} d\Delta\tau_M \int_0^{\Delta\tau} d\Delta\tau_{M-1} \int_0^{\Delta\tau} d\Delta\tau_{M-2} \cdots \int_0^{\Delta\tau} d\Delta\tau_0 \; (i)^M \delta(\Delta\tau_M + \Delta\tau_{M-1} + \\
+ & \Delta\tau_{M-2} + \cdots + \Delta\tau_1 + \Delta\tau_0 - \Delta\tau) e^{i\Delta\tau_M x_M} e^{i\Delta\tau_{M-1} x_{M-1}} \ldots e^{i\Delta\tau_1 x_1} e^{i\Delta\tau_0 x_0}
\end{aligned}
\tag{I.-36}
$$

which is the desired result.

It is clear from (I.-16) that this is symmetric under the interchange of x_i and x_j for all i, j, so the integral is independent of the order of the sequence x_i. Since the integral is idependent of the order of the values x_i it should be characterised by the distinct values, labelled by y_i, and their multiplicity n_i. Where $n_1 + \cdots + n_p = M + 1$. Given a set of values x_i we will evaluate the integral for the case where they are organised such that any x_i sharing the same value are grouped together.

Let us take a simple example to illustrate how the integrand simplifies upon grouping. Say $x_0 = x_1 = x_2 = y_1$, $x_3 = x_4 = y_2$, $x_5 = x_6 = y_3$ and $x_7 = x_8 = y_4$, then

$$
\begin{aligned}
& (i)^8 e^{i(\Delta\tau - \tau_8)x_8} e^{i(\tau_8 - \tau_7)x_7} e^{i(\tau_7 - \tau_6)x_6} e^{i(\tau_6 - \tau_5)x_5} e^{i(\tau_5 - \tau_4)x_4} e^{i(\tau_4 - \tau_3)x_3} e^{i(\tau_3 - \tau_2)x_2} e^{i(\tau_2 - \tau_1)x_1} e^{i\tau_1 x_0} \\
= & (i)^8 e^{i(\Delta\tau - \tau_8)y_4} e^{i(\tau_8 - \tau_7)y_4} e^{i(\tau_7 - \tau_6)y_3} e^{i(\tau_6 - \tau_5)y_3} e^{i(\tau_5 - \tau_4)y_2} e^{i(\tau_4 - \tau_3)y_2} e^{i(\tau_3 - \tau_2)y_1} e^{i(\tau_2 - \tau_1)y_1} e^{i\tau_1 y_1} \\
= & (i)^8 e^{i(\Delta\tau - \tau_7)y_4} e^{i(\tau_7 - \tau_5)y_3} e^{i(\tau_5 - \tau_3)y_2} e^{i\tau_3 y_1}
\end{aligned}
$$

or

$$(i)^8 e^{i(\Delta\tau - \tau_{n_1+n_2+n_3})y_4} e^{i(\tau_{n_1+n_2+n_3} - \tau_{n_1+n_2})y_3} e^{i(\tau_{n_1+n_2} - \tau_{n_1})y_2} e^{i\tau_{n_1} y_1}$$

where $n_1 = 3$, $n_2 = 2$, $n_3 = 2$ and $n_4 = 2$.

The integral (I.-10) simplifies to

$$\begin{aligned} I(y_p, n_p, \ldots, y_1, n_1, \Delta\tau) &= \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 \ (i)^M e^{i(\Delta\tau - \tau_{n_1+\cdots+n_{p-1}})y_p} \\ &\qquad e^{i(\tau_{n_1+\cdots+n_{p-1}} - \tau_{n_1+\cdots+n_{p-2}})y_{p-1}} \ldots e^{i(\tau_{n_1+n_2} - \tau_{n_1})y_2} e^{i\tau_{n_1} y_1} \end{aligned}$$

upon grouping x_i that share the same value.

Let us first consider the case $p = 1$. We need to perform integral

$$I(y_1, n_1) = \int_0^{\Delta\tau} d\tau_{n_1-1} \ldots \int_0^{\tau_2} d\tau_1 (i)^{n_1-1} e^{iy_1\Delta\tau} \tag{I.-41}$$

Using induction on n_1 we will prove that

$$I(y_1, n_1) = \frac{(i\Delta\tau)^{n_1-1}}{(n_1-1)!} e^{iy_1\Delta\tau}. \tag{I.-41}$$

We have

$$\begin{aligned} I(y_1, n_1+1) &= \int_0^{\Delta\tau} d\tau_{n_1} \ldots \int_0^{\tau_2} d\tau_1 \ (i)^{n_1} e^{iy_1\Delta\tau} \\ &= (i)^{n_1} e^{iy_1\Delta\tau} \int_0^{\Delta\tau} d\tau_{n_1} \frac{(\tau_{n_1-1})^{n_1-1}}{(n_1-1)!} \\ &= \frac{(i\tau_{n_1})^{n_1}}{(n_1)!} e^{iy_1\Delta\tau} \end{aligned} \tag{I.-42}$$

The integral $I(y_1, n_1)$ can be written as

$$I(y_1, n_1) = \left(\frac{\partial}{\partial y_1}\right)^{n_1-1} \frac{1}{(n_1-1)!} e^{iy_1\Delta\tau} \tag{I.-42}$$

By induction on p, the number of distinct values, we show that the integral (I.-40) is given by

$$
\begin{aligned}
I(y_p, n_p, \ldots, y_1, n_1, \Delta\tau) &= \frac{1}{(n_p-1)!}\left(\frac{\partial}{\partial y_p}\right)^{n_p-1} \cdots \frac{1}{(n_1-1)!}\left(\frac{\partial}{\partial y_1}\right)^{n_1-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \\
&= \prod_{k=1}^{p} \frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \qquad \text{(I.-42)}
\end{aligned}
$$

We have already proven the case for $p=1$. If we assume that (I.-42) holds for p distinct values then we can evaluate it for $p+1$ distinct values as follows

$$
\begin{aligned}
I(y_{p+1}, n_{p+1}, y_p, n_p \ldots, y_1, n_1, \Delta\tau) = \int_0^{\Delta\tau} d\tau_M \cdots \int_0^{\tau_{M-n_{p+1}+2}} d\tau_{M-n_{p+1}+1} \\
(i)^{n_{p+1}-1} e^{i(\Delta\tau-\tau_{M-n_{p+1}+1})y_{p+1}} I(y_p, n_p, \ldots, y_1, n_1, \tau_{M-n_{p+1}+1}) \qquad \text{(I.-42)}
\end{aligned}
$$

Plugging in the assumed result for p distint values

$$
\begin{aligned}
I(y_{p+1}, n_{p+1}, y_p, n_p, \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p} \frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \\
(i)^{n_{p+1}-1} e^{i\Delta\tau y_{p+1}} \int_0^{\Delta\tau} d\tau_M \cdots \int_0^{\tau_{M-n_{p+1}+2}} d\tau_{M-n_{p+1}+1} e^{i(y_i-y_{p+1})\tau_{M-n_{p+1}+1}} \\
\text{(I.-43)}
\end{aligned}
$$

We are interested in evaluating the integral

$$
I(n,t) = (i)^n e^{ity} \int_0^t dt_n \cdots \int_0^{t_2} dt_1 e^{i(y_i-y)t_1}
$$

we prove by induction in n that the answer is

$$
\frac{e^{iy_i t}}{(y_i-y)^n} - \sum_{m=1}^{n} \frac{e^{iyt}}{(y_i-y)^m} \frac{(it)^{n-m}}{(n-m)!}. \qquad \text{(I.-43)}
$$

Assuming the result for n we find for $I(n+1,t)$:

$$
\begin{aligned}
I(n+1,t) &= (i)^{n+1}e^{ity}\int_0^t dt_{n+1}\int_0^{t_{n+1}} dt_n \ldots \int_0^{t_2} dt_1 e^{i(y_i-y)t_1} \\
&= (i)^{n+1}e^{ity}\int_0^t dt_{n+1}(i)^{-n}e^{-it_{n+1}y}\left[(i)^n e^{it_{n+1}y}\int_0^{t_{n+1}} dt_n \ldots \int_0^{t_2} dt_1 e^{i(y_i-y)t_1}\right] \\
&= ie^{ity}\int_0^t dt_{n+1}e^{-it_{n+1}y}I(n,t_{n+1}) \\
&= ie^{ity}\int_0^t dt_{n+1}e^{-it_{n+1}y}\left[\frac{e^{iy_it_{n+1}}}{(y_i-y)^n}-\sum_{m=1}^{n}\frac{e^{it_{n+1}y}}{(y_i-y)^m}\frac{(it_{n+1})^{n-m}}{(n-m)!}\right] \\
&= ie^{ity}\left[\frac{e^{i(y_i-y)t}}{i(y_i-y)^{n+1}}-\frac{1}{i(y_i-y)^{n+1}}-\sum_{m=1}^{n}\frac{1}{(y_i-y)^m}\frac{(i)^{n-m}(t)^{n-m+1}}{(n-m+1)!}\right] \\
&= \frac{e^{iy_it}}{(y_i-y)^{n+1}}-\sum_{m=1}^{n+1}\frac{e^{ity}}{(y_i-y)^m}\frac{(it)^{n-m+1}}{(n-m+1)!}
\end{aligned}
$$

Now as

$$
\begin{aligned}
& \frac{1}{n-1}\frac{\partial}{\partial y}I(n-1,t;y) \\
= & \frac{1}{n-1}\frac{\partial}{\partial y}\left[\frac{e^{iy_it}}{(y_i-y)^{n-1}}-\sum_{m=1}^{n-1}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m-1}}{(n-m-1)!}\right] \\
= & \frac{e^{iy_it}}{(y_i-y)^n}-\frac{1}{n-1}\sum_{m=1}^{n-1}\left[\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m-1)!}+\frac{e^{iyt}}{(y_i-y)^{m+1}}\frac{(it)^{n-m-1}m}{(n-m-1)!}\right] \\
= & \frac{e^{iy_it}}{(y_i-y)^n}-\frac{1}{n-1}\sum_{m=1}^{n-1}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m-1)!}+\sum_{m=1}^{n}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}(m-1)}{(n-m)!} \\
= & \frac{e^{iy_it}}{(y_i-y)^n}-\frac{1}{n-1}\sum_{m=1}^{n-1}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m)!}[(n-m+(m-1)]-\frac{1}{n-1}\frac{e^{iyt}}{(y_i-y)^n}\frac{n-1}{0!} \\
= & \frac{e^{iy_it}}{(y_i-y)^n}-\sum_{m=1}^{n}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m)!} \\
= & I(n,t;y) \qquad \text{(I.-54)}
\end{aligned}
$$

we have

$$
\begin{aligned}
I(n,t;y) &= \frac{1}{(n-1)!}\left(\frac{\partial}{\partial y}\right)^{n-1}I(1,t;y) \\
&= \frac{1}{(n-1)!}\left(\frac{\partial}{\partial y}\right)^{n-1}\left(\frac{e^{iy_it}}{y_i-y}-\frac{e^{iyt}}{y_i-y}\right) \qquad \text{(I.-54)}
\end{aligned}
$$

Using this in (I.-43)

$$
\begin{aligned}
I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta\tau) = & \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \\
& \left[\frac{1}{(n_{p+1}-1)!} \left(\frac{\partial}{\partial y_{p+1}}\right)^{n_{p+1}-1} \left(\frac{e^{iy_i\Delta\tau}}{y_i-y_{p+1}} - \frac{e^{iy_{p+1}\Delta\tau}}{y_i-y_{p+1}}\right)\right] \\
= & \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \left[\sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \left(\frac{e^{iy_i\Delta\tau}}{y_i-y_{p+1}} - \frac{e^{iy_{p+1}\Delta\tau}}{y_i-y_{p+1}}\right)\right] \\
= & \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \left[\left(\sum_{i=1}^{p} \frac{e^{iy_i\Delta\tau}}{\prod_{j\neq i}^{p+1}(y_i-y_j)} - e^{iy_{p+1}\Delta\tau} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p+1}(y_i-y_j)}\right)\right] \\
= & \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \left[\left(\sum_{i=1}^{p+1} \frac{e^{iy_i\Delta\tau}}{\prod_{j\neq i}^{p+1}(y_i-y_j)} - e^{iy_{p+1}\Delta\tau} \sum_{i=1}^{p+1} \frac{1}{\prod_{j\neq i}^{p+1}(y_i-y_j)}\right)\right]
\end{aligned}
\tag{I.-58}
$$

Now we use the identity

$$
\sum_{i=1}^{p+1} \frac{1}{\prod_{j\neq i, j=1}^{p+1}(y_i-y_j)} = 0
$$

in (I.-58) to give

$$
I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p+1} \frac{e^{iy_i\Delta\tau}}{\prod_{j\neq i}^{p+1}(y_i-y_j)} \tag{I.-58}
$$

Thus if (I.-42) holds for p then it also holds for $p+1$, so it is true for $p \geq 1$. Using this result we find that the contribution due to each discrete path is

$$
\begin{aligned}
A(\nu_M, \ldots, \nu_0, \alpha) = & (\Theta_{\nu_M\nu_{M-1}})(\Theta_{\nu_{M-1}\nu_{M-2}}) \ldots (\Theta_{\nu_2\nu_1})(\Theta_{\nu_1\nu_0}) \\
& \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial \Theta_{w_kw_k}}\right)^{n_k-1} \sum_{i=1}^{p} \frac{e^{-i\alpha\Theta_{w_iw_i}\Delta\tau}}{\prod_{j\neq i}^{p}(\Theta_{w_iw_i}-\Theta_{w_jw_j})}
\end{aligned}
\tag{I.-58}
$$

I.6.1 Eigenstates of Θ

Recall

$$
\begin{aligned}
(\Theta\Psi)(\nu) \;&:=\; -\frac{3\pi G}{4\ell_0^2}\Big[\sqrt{|\nu(\nu+4\ell_0)|}(\nu+2\ell_0)\Psi(\nu+4\ell_0)-2\nu^2\Psi(\nu)+ \\
&\sqrt{|\nu(\nu-4\ell_0)|}(\nu-2\ell_0)\Psi(\nu-4\ell_0)\Big]
\end{aligned}
$$

$$
\chi(b) := \sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0} e^{\frac{i}{2}\nu b}\frac{\Psi(\nu)}{\sqrt{|\nu|}} \qquad \text{(I.-60)}
$$

$$
\begin{aligned}
(\Theta\chi)(b) &= \sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0} e^{\frac{i}{2}\nu b}\frac{(\Theta\Psi)(\nu)}{\sqrt{|\nu|}} \\
&= -\frac{3\pi G}{4\ell_0^2}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0}\frac{e^{\frac{i}{2}\nu b}}{\sqrt{|\nu|}}\Big[\sqrt{|\nu(\nu+4\ell_0)|}(\nu+2\ell_0)\Psi(\nu+4\ell_0)-2\nu^2\Psi(\nu)+ \\
&\sqrt{|\nu(\nu-4\ell_0)|}(\nu-2\ell_0)\Psi(\nu-4\ell_0)\Big] \\
&= -\frac{3\pi G}{4\ell_0^2}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0}\left[e^{\frac{i}{2}(\nu-4\ell_0)b}\sqrt{|\nu|}(\nu-2\ell_0)-2\frac{e^{\frac{i}{2}\nu b}}{\sqrt{|\nu|}}\nu^2+e^{\frac{i}{2}(\nu+4\ell_0)b}\sqrt{|\nu|}(\nu+2\ell_0)\right]\Psi(\nu) \\
&= -\frac{3\pi G}{4}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0} e^{\frac{i}{2}\nu b}\left[\sqrt{|\nu|}\nu(e^{2i\ell_0 b}+e^{-2i\ell_0 b})-\frac{2\nu^2}{\sqrt{|\nu|}}+\sqrt{|\nu|}2\ell_0(e^{2i\ell_0 b}-e^{-2i\ell_0 b})\right]\Psi(\nu) \\
&= -\frac{3\pi G}{4\ell_0^2}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0}\frac{e^{\frac{i}{2}\nu b}}{\sqrt{|\nu|}}\left[2\nu^2\left(\frac{\cos\ell_0 b}{\ell_0}\right)^2+4i\nu\frac{\sin\ell_0 b}{\ell_0}\cos\ell_0 b\right]\Psi(\nu) \\
&= \qquad \text{(I.-65)}
\end{aligned}
$$

This can then be wriitten as a simple differntial equation

$$
(\Theta\chi_k)(b) = -12\pi G\left(\frac{\sin\ell_0 b}{\ell_0}\partial_b\right)^2\chi_k(b) = \omega_k^2\chi_k(b) \qquad \text{(I.-65)}
$$

which have the solutions

$$
\chi_k(b) = A(k)\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}. \qquad \text{(I.-65)}
$$

Let us check this:

$$
\begin{aligned}
-12\pi G\left(\frac{\sin\ell_0 b}{\ell_0}\partial_b\right)^2\chi_k(b) &= -A(k)\frac{12\pi G}{\ell_0^2}\sin\ell_0 b\,\partial_b\left[\frac{ik\sin\ell_0 b\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}}{\tan\frac{\ell_0 b}{2}}\frac{\ell_0}{2}\frac{1}{\cos^2\frac{\ell_0 b}{2}}\right]\\
&= -A(k)\frac{12\pi G}{\ell_0^2}\sin\ell_0 b\,\partial_b[ik\ell_0\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}]\\
&= A(k)12\pi Gk^2\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}\\
&= 12\pi Gk^2\chi_k(b).
\end{aligned}
$$

We see that the eigenvalues are

$$\omega_k^2 = 12\pi Gk^2 \tag{I.-69}$$

To express these eigenvectors in the ν representation we need the inverse transformation of (I.6.1). To this end let us write (I.6.1) as

$$
\begin{aligned}
\chi(b) &= \sum_{-\infty}^{\infty} e^{in(2\ell_0 b)}\left(\sqrt{\frac{\ell_0}{\pi}}\frac{\Psi(4\ell_0 n)}{\sqrt{4\ell_0|n|}}\right)\\
&=: \sum_{-\infty}^{\infty} e^{inb'}\psi(n)
\end{aligned}
$$

Using the Fourier inverse formula

$$
\begin{aligned}
\psi(n) &= \frac{1}{2\pi}\int_0^{2\pi} db'\; e^{-inb'}\chi(b'/2\ell_0)\\
&= \frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\; e^{-in2\ell_0 b}\chi(b)
\end{aligned}
$$

we obtain

$$\sqrt{\frac{\ell_0}{\pi}}\frac{\Psi(\nu)}{\sqrt{|\nu|}} = \frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db e^{-i\frac{\nu}{2}b}\chi(b)$$

or

$$\Psi(\nu) = \sqrt{\frac{\ell_0|\nu|}{\pi}} \int_0^{\pi/\ell_0} dbe^{-i\frac{\nu}{2}b}\chi(b) \tag{I.-73}$$

Eigenvectors with non-zero eigenvalues can also be expressed in the ν representation by applying the inverse transformation

$$e_k(\nu) = A(k)\sqrt{\frac{\ell_0|\nu|}{\pi}} \int_0^{\pi/\ell_0} db\, e^{-\frac{i}{2}\nu b} e^{ik \ln(\tan\frac{\ell_0 b}{2})} \tag{I.-73}$$

Put $\theta = b\ell_0$ then

$$\frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\, e^{-2ib\ell_0 n} e^{ik\ \ln(\tan\frac{\ell_0 b}{2})} = \frac{1}{\pi}\int_0^{\pi} d\theta\, e^{-2i\theta n} e^{ik\ \ln(\tan\frac{\theta}{2})}$$

now

$$\begin{aligned}
\tan\frac{x}{2} = \frac{e^{i\theta/2} - e^{-i\theta/2}}{i(e^{i\theta/2} + e^{-i\theta/2})} &= \frac{i(1-e^{i\theta})}{1+e^{i\theta}} \\
&= \exp[\ln i + \ln\frac{1-e^{i\theta}}{1+e^{i\theta}}] \\
&= \exp[e^{i\pi/2} + \ln\frac{1-e^{i\theta}}{1+e^{i\theta}}]
\end{aligned}$$

$$\frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\, e^{-2ib\ell_0 n} e^{ik\ \ln(\tan\frac{\ell_0 b}{2})} = \frac{e^{-\pi k/2}}{\pi}\int_0^{\pi} (e^{i\theta})^{-2n}\left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{ik} d\theta =: J(k,n) \tag{I.-76}$$

$$d\theta = \frac{dz}{iz}$$

$$J(k,n) = \frac{e^{-\pi k/2}}{\pi i}\int_{\mathcal{C}} (z)^{-2n-1}\left(\frac{1-z}{1+z}\right)^{ik} dz \tag{I.-76}$$

where $\mathcal{C}$ is the unit semicircle in counterwise direction (note that $\mathcal{C}$ not a closed contour) in the upper half plane, Im$z > 0$, of the complex plane.

The second independent eigenfunction $e_{-k}(\nu)$ with the same eigenvalue ω_k^2 can be represented in a similar fashion by replacing $e^{i\theta} \to -e^{i\theta}$:

$$J(-k,n)$$
$$= \frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\, e^{-2ib\ell_0 n} e^{-ik\ \ln(\tan\frac{\ell_0 b}{2})} = -\frac{e^{\pi k/2}}{\pi}\int_{-\pi}^{0}(-e^{i\theta})^{-2n}\left(\frac{1+(-e^{i\theta})}{1-(-e^{i\theta})}\right)^{-ik} d\theta$$

Set $z = -e^{i\theta}$

$$d\theta = -\frac{dz}{iz}$$

The result is a contour integral along the unit semicircle in the counterclockwise direction in the lower half, Im$z < 0$, of the complexplane.

$$J(-k,n) = \frac{e^{\pi k/2}}{\pi i}\int_{-\mathcal{C}}(z)^{-2n-1}\left(\frac{1-z}{1+z}\right)^{ik} dz \tag{I.-78}$$

Combining the integrals

$$\frac{1}{2}\left(e^{\pi k/2}J(k,n) + e^{-\pi k/2}J(-k,n)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{ix})^{-2n}\left(\frac{1-e^{ix}}{1+e^{ix}}\right)^{ik} dx =: I(k,n) \tag{I.-78}$$

Being a linear combination of $e_k(\nu)$ and $e_{-k}(\nu)$, this $I(k,n)$ gives also an eigenfunction of Θ with eigenvalue ω_k^2. Using complex analysis we can evaluate $I(k,n)$.

$$\frac{1}{2\pi i}\oint z^{-2n-1}\left(\frac{1-z}{1+z}\right)^{ik} dz \tag{I.-78}$$

Recall from basic complex anaylsis that

$$\oint_C f(z)dz = 2\pi i \sum Res[f(z_0)]$$

where, if a function has an m-th pole at z_0,

$$Res[f(z_0)] = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^{m-1}f(z)]\right)$$

Therefore

$$I(k,n) = \begin{cases} \frac{1}{(2n)!}\frac{d^{2n}}{z^{2n}} \left(\frac{1-s}{1+s}\right)^{ik}\Big|_{s=0} & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{I.-78}$$

We repeat the same argument but taking $z = e^{-b\ell_0}$ and $z = -e^{-b\ell_0}$

Setting $z = e^{-ix}$ we get

$$J(-k,n) = \frac{e^{\pi k/2}}{\pi} \int_{-\pi}^{0} (e^{-ix})^{2n} \left(\frac{1-e^{ix}}{1+e^{ix}}\right)^{ik} dx \tag{I.-78}$$

$$\frac{1}{2}(e^{-\pi k/2} J(k,n) + e^{\pi k/2} J(-k,n)) = \frac{1}{2\pi i} \oint z^{2n-1} \left(\frac{1-z}{1+z}\right)^{ik} dz = I(k,-n) \tag{I.-78}$$

the basis is

$$\begin{aligned} e_k^{\pm}(\nu) &:= \frac{1}{2}(e^{\pm\pi k/2} e_k(\nu) + e^{\mp\pi k/2} e_k(\nu)) \\ &= A(k)\sqrt{\frac{\pi|\nu|}{\ell_0}}\; I(k, \pm\frac{\nu}{4\ell_0}) \end{aligned} \tag{I.-78}$$

Normalisation of the vectors

There the functions describing the states are

$$\chi_k^{\pm}(b) = \frac{A(k)}{2}\left(e^{\pm\pi k/2} e^{ik\ln(\tan\frac{\ell_0 b}{2})} + e^{\mp\pi k/2} e^{-ik\ln(\tan\frac{\ell_0 b}{2})}\right) \tag{I.-78}$$

$$
\begin{aligned}
< k' + |k+ > &= \int_0^{\pi/\ell_0} db\, 2i\, \overline{\chi^+_{k'}}(b) \partial_b \chi^+_k(b) \\
&= -\frac{1}{2} A(k)\overline{A(k')}\, k\ell_0 \int_0^{\pi/\ell_0} db\, \left(e^{+\pi k'/2} e^{-ik' \ln(\tan\frac{\ell_0 b}{2})} + e^{-\pi k'/2} e^{ik' \ln(\tan\frac{\ell_0 b}{2})} \right) \\
&\quad \frac{1}{\sin \ell_0 b} \left(-e^{+\pi k/2} e^{ik \ln(\tan\frac{\ell_0 b}{2})} + e^{-\pi k/2} e^{-ik \ln(\tan\frac{\ell_0 b}{2})} \right) \\
&= -\frac{1}{2} A(k)\overline{A(k')}\, k\ell_0 \int_0^{\pi/\ell_0} \frac{\ell_0 db}{\sin \ell_0 b} \\
&\quad \left(-e^{+\pi(k+k')/2} e^{i(k-k') \ln(\tan\frac{\ell_0 b}{2})} + e^{-\pi(k+k')/2} e^{-i(k-k') \ln(\tan\frac{\ell_0 b}{2})} \right. \\
&\quad \left. + e^{+\pi(-k+k')/2} e^{-i(k+k') \ln(\tan\frac{\ell_0 b}{2})} - e^{+\pi(k-k')/2} e^{i(k+k') \ln(\tan\frac{\ell_0 b}{2})} \right)
\end{aligned}
$$

Make the change of variables $y = \ln(\tan \frac{\ell_0 b}{2})$

$$
dy = \frac{\ell_0 db}{4 \sin \ell_0 b}
$$

then

$$
\begin{aligned}
< k' + |k+ > &= -\frac{1}{2} A(k)\overline{A(k')}\, k \int_{-\infty}^{\infty} dy \\
&\quad \left(-e^{+\pi(k+k')/2} e^{i(k-k')y} + e^{-\pi(k+k')/2} e^{-i(k-k')y} \right. \\
&\quad \left. + e^{+\pi(-k+k')/2} e^{-i(k+k')y} - e^{+\pi(k-k')/2} e^{i(k+k')y} \right) \\
&= A(k)\overline{A(k')}\, k 2\pi\, \frac{e^{k\pi} - e^{-k\pi}}{2} [\delta(k',k) + \delta(k',-k)] \\
&= \\
&= |A(k)|^2\, 2\pi k \sinh(\pi k)\, \delta(k',k) \qquad \text{(I.-89)}
\end{aligned}
$$

Using this we can find the normalisation of the eigenvectors $e_k(\nu)$:

From

$$
|k\pm >= \frac{1}{2}(e^{\pm\pi k/2}|k > + e^{\mp\pi k/2}| - k >)
$$

we get

$$|k>= \frac{2(e^{\pi k/2}|k+> - e^{-\pi k/2}|k->)}{e^{\pi k} - e^{-\pi k}}$$

and so

$$\begin{aligned}
<k'|k> &= \frac{2(e^{\pi k/2} <k+| - e^{-\pi k/2} <k-|)}{e^{\pi k} - e^{-\pi k}} \frac{2(e^{\pi k/2}|k+> - e^{-\pi k/2}|k->)}{e^{\pi k} - e^{-\pi k}} \\
&= 4 \frac{e^{\pi k} <k+|k+> + e^{-\pi k} <k-|k->}{(e^{\pi k} - e^{-\pi k})^2} \\
&= |A(k)|^2 \, 4\pi k \sinh(\pi k) \frac{\cosh(\pi k)}{\sinh^2(\pi k)} \, \delta(k', k) \\
&= |A(k)|^2 \, 4\pi k \coth(\pi k) \, \delta(k', k) \qquad \text{(I.-91)}
\end{aligned}$$

I.6.2 Matrix Elements for $f(\Theta)$

I.7 Reminder of Perturbation Theory

I.7.1 The Schrödinger Picture

The time-dependent Schrödinger equation is

$$\frac{i}{\lambda} \frac{\partial}{\partial t} |\Psi(t)\rangle_S = H_S |\Psi(t)\rangle_S.$$

We discuss the time-evolution operator in the Schr̈dinger picture. This is defined by the equation

$$|\Psi(t)\rangle_S := U(t, t_0)_S |\Psi(t_0)\rangle_S \qquad \text{(I.-91)}$$

for an arbitrary initial wave function $|\Psi(t_0)\rangle_S$ in the Hilbert space of the problem in question. Inserting (I.7.1) into the Schrödinger equation yields

$$\frac{i}{\lambda} \frac{\partial}{\partial t} U(t, t_0)_S |\Psi(t_0)\rangle_S = H_S U(t, t_0)_S |\Psi(t_0)\rangle_S.$$

Since it holds for any intial function, the operator differential equation

$$\frac{i}{\lambda}\frac{\partial}{\partial t}U(t,t_0)_S = H_S U(t,t_0)_S \tag{I.-91}$$

together with the intial condition $U(t_0,t_0) = 1$ follows.

We prove that the operator $U(t,t_0)$ is unitary. If we use the defining equation (I.7.1) twice, we arrive at

$$\begin{aligned} |\Psi(t)\rangle_S &= U(t,t')_S|\Psi(t')\rangle_S \\ &= U(t,t')_S U(t,t_0)_S|\Psi(t)\rangle_S. \end{aligned}$$

It follows that

$$U(t,t')_S = U(t,t')_S U(t,t_0)_S.$$

for $t = t_0$ we have

$$1 = U(t_0,t')_S U(t',t_0)_S.$$

Hence

$$U(t,t_0)_S^{-1} = U(t_0,t)_S.$$

Furthermore, from the operator differential equation (I.7.1) and its adjoint equation

$$\begin{aligned} \frac{\partial U_S}{\partial t} &= -i\lambda H_S U_S \\ \frac{\partial U_S^\dagger}{\partial t} &= i\lambda U_S^\dagger H_S \end{aligned}$$

it follows that

$$\frac{\partial}{\partial t}\left[U_S^\dagger U_S\right] = \frac{\partial U_S^\dagger}{\partial t}U_S + U_S^\dagger\frac{\partial U_S}{\partial t} = 0$$

which implies that

$$U^\dagger(t,t_0)_S U(t,t_0)_S = constant.$$

Since $U(t_0, t_0)_S = 1$, we have $U^\dagger(t_0, t_0) = 1$, so $U^\dagger(t, t_0)_S U(t, t_0) = 1$. We already have shown that U_S is invertible, hence, the unitarity folls,

$$U^\dagger(t, t_0)_S = U^{-1}(t, t_0)_S.$$

If the Hamiltonian is explicitly time-idependent in the Schrödinger picture, the operator differential equation (I.7.1) has the solution

$$U(t, t_0)_S = \exp\left[-i\lambda H_S(t - t_0)\right]. \qquad \text{(I.-95)}$$

I.7.2 The Heisenberg Picture

In this picture the wavefunctions are constant in time:

$$|\Psi(t)\rangle_H = U(t, t_0)_S |\Psi(t)\rangle_S = |\Psi(t_0)\rangle_S = \mathit{constant}. \qquad \text{(I.-95)}$$

The operators in the Heisenberg picture are given by

$$O(t)_H = U^\dagger(t, t_0)_S O(t)_S U(t, t_0)_S. \qquad \text{(I.-95)}$$

If H_S is explicitly time-independent, we have from equation (I.7.1) (if we also fix $t_0 = 0$)

$$O(t)_H = e^{i\lambda H_S t} O(t)_S U(t, t_0)_S e^{-i\lambda H_S t}. \qquad \text{(I.-95)}$$

Since the wavefunctions are constant in the Heisenberg picture, the dynamics is contained in the operators. We now derive an equation of motion for operators in the Heisenberg picture,

$$\begin{aligned}\frac{i}{\lambda}\frac{d}{dt}O(t)_H &= \frac{i}{\lambda}\frac{d}{dt}\left(U_S^\dagger O_S U_S\right) \\ &= \frac{i}{\lambda}\left(\frac{\partial U_S^\dagger}{\partial t}\right) O_S U_S + \frac{i}{\lambda} U_S^\dagger \left(\frac{\partial O_S}{\partial t}\right) U_S + \frac{i}{\lambda} U_S^\dagger O_S \left(\frac{\partial U_S}{\partial t}\right).\end{aligned}$$

If we then use

$$\frac{i}{\lambda}\frac{\partial U_S}{\partial t} = H_S U_S$$

and

$$-\frac{i}{\lambda}\frac{\partial U_S^\dagger}{\partial t} = U_S^\dagger H_S$$

we can write

$$\begin{aligned}
\frac{i}{\lambda}\frac{d}{dt}O(t)_H &= -U_S^\dagger H_S U_S U_s^\dagger O_S U_S + \frac{i}{\lambda}U_S^\dagger \left(\frac{\partial O_S}{\partial t}\right) U_S + U_S^\dagger O_S U_S U_S^\dagger H_S U_S \\
&= -H_H O_H + O_H H_H + \frac{i}{\lambda}U_S^\dagger \left(\frac{\partial O_S}{\partial t}\right) U_S.
\end{aligned}$$

If we set

$$\left[\frac{\partial O}{\partial t}\right]_H := U_S^\dagger \left(\frac{\partial O_S}{\partial t}\right) U_S$$

we finally can write

$$\frac{i}{\lambda}\frac{d}{dt}O(t)_H = [O_H, H_H] + \frac{i}{\lambda}\left[\frac{\partial O}{\partial t}\right]_H. \qquad \text{(I.-99)}$$

I.7.3 The Interaction Picture

Here, we start from partioning the Hamiltonian according to

$$H_S = H_0 + V_S, \qquad H_0 \neq H_0(t)$$

In the interaction picture

$$|\Psi(t)\rangle_I = e^{i\lambda H_0 t}|\Psi(t)\rangle_I \qquad \text{(I.-99)}$$

and

$$O(t)_I = e^{i\lambda H_0 t} O(t)_S \, e^{-i\lambda H_0 t}. \qquad \text{(I.-99)}$$

In particular, the operator H_0 is the same in both the Schrödinger picture and the interation picture:

$$[H_0]_I = e^{i\lambda H_0 t}[H_0]_S\, e^{-i\lambda H_0 t} = [H_0]_S \equiv H_0.$$

Both the operators and the wavefunctions are time-dependent in the interaction picture. It follows that there are equations of motion for both. For operators, this equation (the derivation is completely analogous to the derivation of the Heisenberg equation of motion):

$$\frac{i}{\lambda}\frac{d}{dt}O(t)_I = [O(t)_I, H_0] + i\lambda\left[\frac{\partial O}{\partial t}\right]_I \tag{I.-99}$$

where

$$\left[\frac{\partial O}{\partial t}\right]_I := e^{i\lambda H_0 t}\left[\frac{\partial O_S}{\partial t}\right] e^{-i\lambda H_0 t}$$

Next, we will derive the equation of motion for the wavefunction in the interaction picture. We start with the Schrödinger equation

$$\frac{i}{\lambda}\frac{\partial}{\partial t}|\Psi(t)\rangle_S = H_S|\Psi(t)\rangle_S$$

which implies

$$\begin{aligned}
\frac{i}{\lambda}\frac{\partial}{\partial t}\left(e^{-i\lambda H_0 t}|\Psi(t)\rangle_I\right) &= e^{-i\lambda H_0 t}\left[H_0|\Psi(t)\rangle_I + \frac{i}{\lambda}\frac{\partial}{\partial t}|\Psi(t)\rangle_I\right] \\
&= H_S\, e^{-i\lambda H_0 t}|\Psi(t)\rangle_I
\end{aligned}$$

which in turn implies

$$\begin{aligned}
\frac{i}{\lambda}\frac{\partial}{\partial t}|\Psi(t)\rangle_I &= e^{i\lambda H_0 t}H_S\, e^{-i\lambda H_0 t}|\Psi(t)\rangle_I - H_0|\Psi(t)\rangle_I \\
&= [H(t)_I - H_0]\,|\Psi(t)\rangle_I
\end{aligned}$$

and finally we have

$$\frac{i}{\lambda}\frac{\partial}{\partial t}|\Psi(t)\rangle_I = V(t)_I\,|\Psi(t)\rangle_I \tag{I.-103}$$

where

$$V(t)_I := e^{i\lambda H_0 t} V(t)_S \, e^{-i\lambda H_0 t} \tag{I.-103}$$

Equation (I.7.3) is the Tomonaga-Schwinger equation. analogous to the Schrödinger picture, we now define a time-evolution operator in the interaction picture through

$$|\Psi(t)\rangle_I = U(t,t')|\Psi(t')\rangle_I. \tag{I.-103}$$

Just as for U_S, we can prove the following properties:

$$\begin{aligned} U(t,t') &= U(t,t'')U(t'',t') \\ U^\dagger(t,t') &= U^{-1}(t,t') = U(t',t). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} |\Psi(t)\rangle_I &= e^{i\lambda H_0 t}|\Psi(t)\rangle_S \\ &= e^{i\lambda H_0 t} U(t,t')_S |\Psi(t')\rangle_S \\ &= e^{i\lambda H_0 t} U(t,t')_S e^{-i\lambda H_0 t'} |\Psi(t')\rangle_I. \end{aligned}$$

From which we obtain the following relation between the time-evolution operators in the Schrödinger picture and in the interaction picture:

$$U(t,t') = e^{i\lambda H_0 t} U(t,t')_S e^{-i\lambda H_0 t'}. \tag{I.-108}$$

If we insert (I.7.3) into the Tomonaga-Schwinger equation, we obtain an equation of motion for U:

$$\frac{i}{\lambda}\frac{\partial}{\partial t} U(t,t') = V(t)_I U(t,t') \tag{I.-108}$$

with $U(t',t') = 1$. This differential equation with its intial condition is equaivalent to the following integral equation:

$$U(t,t') = 1 - i\lambda \int_{t'}^{t} dt_1 V(t_1)_I U(t_1,t'). \tag{I.-108}$$

This integral equation is suitable for successive approximations:

$$\begin{aligned}
U_0(t,t' &= 1, \\
U_1(t,t') &= 1 - i\lambda \int_{t'}^{t} dt_1 V(t_1)_I \\
U_2(t,t') &= 1 - i\lambda \int_{t'}^{t} dt_1 V(t_1)_I U_1(t_1,t') \\
&= 1 - i\lambda \int_{t'}^{t} dt_1 V(t_1)_I + (-i\lambda)^2 \int_{t'}^{t} dt_2 \int_{t'}^{t_2} dt_1 V(t_2)_I V(t_1)_I.
\end{aligned}$$

Correspondingly, we have for the N-th approximation

$$U_N(t,t') = 1 + \sum_{n=1}^{N} \lambda^n \int_{t'}^{t} t_n \int_{t'}^{t_n} dt_{n-1} \dots \int_{t'}^{t_2} dt_1 [-iV(t_n)_I] \dots [-iV(t_1)_I].$$

If the successive approximations converge, we finally arrive at

$$U(t,t') = 1 + \sum_{n=1}^{\infty} \lambda^n \int_{t'}^{t} t_n \int_{t'}^{t_n} dt_{n-1} \dots \int_{t'}^{t_2} dt_1 [-iV(t_n)_I] \dots [-iV(t_1)_I]. \qquad \text{(I.-112)}$$

We then have for $U(t,t')_S$:

$$U_S(t,t') = e^{-i\lambda H_0 t} U(t,t') e^{i\lambda H_0 t'}. \qquad \text{(I.-112)}$$

We are interested in $t' = 0$, in which case:

$$U(t,t') = 1 + \sum_{n=1}^{\infty} \lambda^n \int_{t'}^{t} t_n \int_{0}^{t_n} dt_{n-1} \dots \int_{0}^{t_2} dt_1 [-iV(t_n)_I] \dots [-iV(t_1)_I] \qquad \text{(I.-112)}$$

and

$$U_S(t,0) = e^{-i\lambda H_0 t} U(t,0).. \qquad \text{(I.-112)}$$

I.8 Summary

• Microscopic source for blackhole entropy. Once this parameter is fixed the correct formula for any no extremum blackhole (except rotating ones maybe)

• Removal of cosmological singurlarity. Evolutional equations don't break down at the place where the classical singularity is.

• Initial conditions are derived rather than guessed at. Not suprising as the constraint equations are admissible conditions on the initial data.

• First direct derivation of inflation from a candidate for quantum gravity. Due to a quatnum geometry effect in the early kinematic dominated universe.

• Some features in the minisuperspace are shared with the full theory. Allows proper investigation of the dynamics of minisuperspace that could shed some light on the dynamics of the full theory.

Appendix J

Functional Analysis

J.1 Introduction

introducing heavy mathematical tools, often unfamiliar to the average physicist

a lot of underlying structure is assumed, and taken granted for, even if the physicist is unaware of this. The physicist will not worry so much about the possibilities in the underlying structure if he obtains experimental results confirming all the assumptions made. Field theory used to make predictions with QED which agree with experiment with astounding accuracy - and this is even though the quantum field theory may not exist in a mathematically rigorous way.

in the absence of any experimental observation (at least for the moment) by having a consistent theory

to achieve certainty is to work at a high level of mathematical precision.

we search for a mathematical precision is that in quantum gravity

The study of the spaces of functions. applied mathematics, and in physics, engineering and statistics.

Our reason for studying functions is as a model of the physical world. In selecting a certain class of functions... need to find a balance

general theory of topological spaces. The need to study of ... comes from that the class of .. is too restrictive for our purposes

investigate infinite dimensional representations arise when we consider eigenvalues of operators with continuous spectra. These are already present in undergraduate quantum mechanics - position and momentum operators.

However, now the vector space is infinitely large, the analysis is much more involved and the intuitive expectations coming from the study of finite dimensional vector spaces have to be suitably modified.

A blow-by-blow account of much of the material presented in the *Part II Mathematical tools* [32] T. Thiemann, Introduction to Modern Canonical Quantum General Relativity, [gr-qc/0110034].

J.2 Finite Dimensional Vector Spaces

• Linear Operators transform one vector into another: $\hat{T}f = g$

$$\hat{T}(c_1 f_1 + c_2 f_2) = c_1(\hat{T}f_1) + c_2(\hat{T}f_2)$$

$\hat{T}$ is a linear transformation in the Hilbert space Examples:

1. multiplication
$$\hat{T}_1 = x : \ \ \hat{T}_1 f(x) = x f(x)$$
2. differentiation
$$\hat{T}_2 = \frac{d}{dx} : \ \ \hat{T}_2 f(x) = \frac{d}{dx} f(x)$$
3. combined multiplication and differentiation
$$\hat{T}_1 \hat{T}_2 = x\frac{d}{dx} : \ \ \hat{T}_1 \hat{T}_2 \neq \hat{T}_2 \hat{T}_1$$

J.2.1 Linear Operator and Functionals on Finite Dimensional Spaces

As we saw in section ?? linear operators can be represented in terms of matrices.

Linear functionals where $\dim V = n$ and $\{e_1, \ldots, e_n\}$ is a basis for V, as before. These functionals constitute teh algebraic dual space V^* of V. For every such functional f and every $x = \sum \zeta_j e_j \in V$ we have

$$f(x) = f\left(\sum_{j=1}^{n} \xi_j e_j\right) = \sum_{j=1}^{n} \xi_j f(e_j) = \sum_{j=1}^{n} \xi_j \alpha_j \tag{J.0}$$

where

$$\alpha_j = f(e_j) \qquad j = 1, \ldots, n \tag{J.0}$$

and f is uniquely determined by its values α_j at the n basis vectors of V.

Conversely, every $n-$tuple of scalars $\alpha_1, \dots, \alpha_n$ determines a linear functional on V.

Theorem J.2.1 *Let V be an $n-$dimensional vector space and $\{e_1, \dots, e_n\}$ a basis for V. Then $\{f_1, \dots, f_n\}$ given by*

$$f_k(e_j) = \delta_{jk} \tag{J.0}$$

a basis for the algebraic dual V^* of V, and $dimV^* = dimV = n$.

Proof: $\{f_1, \dots, f_n\}$ is a linear independent set since

$$\sum_{k=1}^{n} \beta_k f_k(x) = 0 \tag{J.0}$$

as we can see by setting $x = e_j$,

$$\sum_{k=1}^{n} \beta_k f_k(e_j) = \sum_{k=1}^{n} \beta_k \delta_{jk} = \beta_j = 0.$$

We show that every $f \in V^*$ can be represented as a linear combination of elements of $\{f_1, \dots, f_n\}$ in a unique way. We write $f(e_j) = \alpha_j$. We know that

$$f(x) = \sum_{j=1}^{n} \xi_j \alpha_j$$

for every $x \in V$. On the other hand

$$f_j(x) = f_j(\xi_1 e_1 + \cdots + \xi_n e_n) = \xi_j.$$

Together,

$$f(x) = \sum_{j=1}^{n} \alpha_j f_j(x).$$

Hence the unique representaion of the arbitrary linear functional f on V in terms of the functionals $f_1, \dots, f_n$ is

$$f = \alpha_1 f_1 + \cdots + \alpha_n f_n.$$

□

J.3 Finite Hilbert Space.

For a finite-dimensional space every Hermitian or unitary operator can be described completely by its eigenvalues and eigenvectors.

For a finite-dimensional space the eigenvectors span the whole space.

This comprises the so-called spectral theorem.

J.3.1 The Cayley-Hamilton Theorem

Let $\mathbf{A}$ be a square $N \times N$ matrix representing in some basis the operator A , and let λ be a parameter. The equation

$$\varphi(\lambda) := \det(\lambda \mathbb{1} - \mathbf{A}) = 0 \tag{J.0}$$

is called the characteristic equation of the operator A (or of the matrix $\mathbf{A}$). It is evident that $\varphi(\lambda)$ is a polynomial of the Nth degree in λ with numerical coefficients the leading (that of λ^N being equal to 1

$$\varphi(\lambda) = \varphi_0 + \varphi_1 \lambda + \varphi_2 \lambda^2 + \cdots + \varphi_{N-1} \lambda^{N-1} + \varphi_N \lambda^N \tag{J.0}$$

The form of the characteristic (and thus the numerical values of the coefficients, φ_i) does not depend on the choice of basis, since the determinant of a matrix, in our case, the matrix $\det(\lambda \mathbb{1} - \mathbf{A})$, is a scalar.

Replacing λ by the operator A we get the operator

$$\varphi(A) = \varphi_0 + \varphi_1 A^2 + \varphi_2 A^2 + \cdots + \varphi_{N-1} A^{N-1} + \varphi_N A^N$$

The Cayley-Hamilton Theorem stats that

$$\varphi(A) = \varphi_0 + \varphi_1 A^2 + \varphi_2 A^2 + \cdots + \varphi_{N-1} A^{N-1} + \varphi_N A^N = 0. \tag{J.0}$$

That is, $\varphi(A)$ is equal to the zero matrix.

Proof:

Recall that we have for an arbitrary matrix A that

$$\mathrm{adj}(M) \cdot M = \det(M)\mathbb{1}_N = M \cdot \mathrm{adj}(M). \tag{J.0}$$

Consider the adguagte

$$B = \mathrm{adj}(\lambda \mathbb{1}_N - A).$$

According to (J.3.1)

$$(\lambda \mathbb{1}_N - A)B = \det(\lambda \mathbb{1}_N - A)\mathbb{1}_N = \varphi(\lambda)\mathbb{1}_N. \tag{J.0}$$

Since B is a matrix with polynomials as entries, we can write

$$B = \sum_{i=0}^{d-1} \lambda^i B_i \tag{J.0}$$

where B_i are N constant matrices.

Now we expand

$$\begin{aligned}
\varphi(\lambda)\mathbb{1}_N &= (\lambda \mathbb{1}_N - A)B \\
&= (\lambda \mathbb{1}_N - A) \sum_{i=0}^{N-1} \lambda^i B_i \\
&= \sum_{i=0}^{N-1} \lambda^{i+1} B - \sum_{i=0}^{N-1} \lambda^i A B_i \\
&= \lambda^N B_{N-1} + \sum_{i=1}^{N-1} \lambda^i (B_{i-1} - AB_i) - AB_0.
\end{aligned} \tag{J.-2}$$

Writing

$$\varphi(\lambda)\mathbb{1}_N = \lambda^N \mathbb{1}_N + \lambda^{N-1} c_{N-1} \mathbb{1}_N + \cdots + \lambda c_1 \mathbb{1}_N + c_0 \mathbb{1}_N.$$

and comparing this to (J.-2) we find

$$B_{N-1} = \mathbb{1}_N, \quad B_{i-1} - AB_i = c_i \mathbb{1}_N \quad \text{for } 1 \leq i \leq N-1, \quad -AB_0 = c_0 \mathbb{1}_N. \tag{J.-2}$$

Substituting these expressions into

$$\varphi(A) = A^N \mathbb{1}_N + A^{N-1} c_{N-1} \mathbb{1}_N + \cdots + A c_1 \mathbb{1}_N + c_0 \mathbb{1}_d$$

we obtain

$$\begin{aligned}
\varphi(A) &= A^N \mathbb{1}_N + A^{N-1} c_{N-1} \mathbb{1}_N + \cdots + A c_1 \mathbb{1}_N + c_0 \mathbb{1}_N \\
&= A^N B_{N-1} + \sum_{i=1}^{N-1} (A^i B_{i-1} - A^{i+1} B_i) - AB_0 \\
&= A^N B_{N-1} + (AB_0 - A^2 B_1 \\
&\qquad + A^2 B_1 - A^3 B_2 \\
&\qquad + A^3 B_2 - A^4 B_3 \\
&\qquad \vdots \\
&\qquad + A^{N-2} B_{N-3} - A^{N-1} B_{N-2} \\
&\qquad + A^{N-1} B_{N-2} - A^d B_{N-1}) - AB_0 \\
&= 0.
\end{aligned}$$

Therefore,

$$\varphi(A) = A^N + c_{N-1} A^{N-1} + \cdots + c_1 A + c_0 \mathbb{1}_N = 0.$$

J.3.2 Projection Operators

In chapter 3 we saw that a projection operator is equivalent to operators that are self-adjoint and satify $P^2 = P$.

We say that a subspace $\mathcal{M}$ **reduces** a linear operator T if $T\psi$ is in $\mathcal{M}$ for every $\psi \in \mathcal{M}$ and $T\phi$ is in $\mathcal{M}^\perp$ for every ϕ in $\mathcal{M}^\perp$. Let P be the projection operator onto $\mathcal{M}$. We say that a subspace $\mathcal{M}$ is **invariant** under an operator if $T\psi$ is in $\mathcal{M}$ for every vector in $\mathcal{M}$.

Theorem J.3.1 *Let P be the projection operator onto the subspace $\mathcal{M}$. The following statements are equivalent:*

i) $\mathcal{M}$ reduces T;

ii) $PT = TP$

Proof: First we show i) implies ii). For any vector

$$\psi = \psi_{\mathcal{M}} + \psi_{\mathcal{M}^\perp}$$

we have

$$T\psi = T\psi_{\mathcal{M}} + T\psi_{\mathcal{M}^\perp}.$$

If $\mathcal{M}$ reduces T, then $T\psi_{\mathcal{M}}$ is in $\mathcal{M}$ and $T\psi_{\mathcal{M}^\perp}$ is in $\mathcal{M}^\perp$. Therefore

$$\begin{aligned} PT\psi &= PT(\psi_{\mathcal{M}} + \psi_{\mathcal{M}^\perp}) = PT\psi_{\mathcal{M}} \\ &= T\psi_{\mathcal{M}} = TP\psi_{\mathcal{M}} \\ &= TP\psi. \end{aligned} \tag{J.-12}$$

Thus i) implies ii). We now show the converse. If ψ is in $\mathcal{M}$ and $PT = TP$, then $P\psi = \psi$ and

$$T\psi = TP\psi = PT\psi$$

so $T\psi$ is in $\mathcal{M}$. It is easy to see that $PT = TP$ and $(1 - P)T = T(1 - P)$ are equivalent. If ϕ is in $\mathcal{M}^\perp$ and $(1 - P)T = T(1 - P)$, then

$$T\phi = T(1 - P)\phi = (1 - P)T\phi$$

so $T\phi$ is in $\mathcal{M}^\perp$.

□

Theorem J.3.2 *If a subspace is invariant under T and $T^\dagger$ then it reduces T.*

Proof: Let $\mathcal{M}$ be a subspace that is invariant under T and $T^\dagger$ and let P be the projection operator onto $\mathcal{M}$. Then $TP\psi$ is in $\mathcal{M}$, so

$$TP\psi = PTP\psi$$

for every vector ψ, or

$$TP = PTP.$$

Because $\mathcal{M}$ is also invariant under $T^\dagger$, we have

$$T^\dagger P = PT^\dagger P.$$

Taking the adjoint, we have

$$PT = PTP.$$

Threfore

$$PT = TP.$$

This implies that $\mathcal{M}$ reduces T by theorem J.3.1.

□

Theorem J.3.3 *If P and Q are the projections on closed linear subspaces M and N of H, then $M \perp N$ if and only if $PQ = 0$ or equivalently $QP = 0$.*

Proof: First note that $PQ = 0$ is equivalent to $QP = 0$ through taking adjoints. If $M \perp N$, so that $N \subseteq M^\perp$, as $Q\psi$ is in N for every x we have $PQ\psi = 0$, or $PQ = 0$. Conversely, if $PQ = 0$ then for every ψ in N we have $P\psi = PQ\psi = 0$, so $N \subseteq M^\perp$ and $M \perp N$.

□

We say that two projections P and Q are orthogonal if $PQ = 0$.

Theorem J.3.4 *If $P_1, P_2, \ldots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of H, then $P = P_1 + P_2 + \cdots + P_n$ is a projection if and only if the P_i's are pairwise orthognal ($P_i P_j = 0$ whenever $i \neq j$). In this case P is the projection onto*

$$M = M_1 + M_2 + \cdots + M_n.$$

Proof: Since P is self-adjoint, to prove it is a projection, we need only show it is idempotent, that is, $P^2 = P$. As the P_i's are pairwise orthogonal,

$$\begin{aligned} P^2 &= (P_1 + P_2 + \cdots + P_n)(P_1 + P_2 + \cdots + P_n) \\ &= P_1^2 + P_2^2 + \cdots + P_n^2 \\ &= P_1 + P_2 + \cdots + P_n = P. \end{aligned} \tag{J.-13}$$

Therefore P is a projection. Conversely, assume that P is idempotent. Let ψ be a vector in the range of P_i then

$$\begin{aligned} \|\psi\|^2 &= \|P_i\psi\|^2 \leq \sum_{j=1}^{n} \|P_j\psi\|^2 \\ &= \sum_{j=1}^{n} (P_j\psi, \psi) = (P\psi, \psi) \\ &= \|P\psi\|^2 \leq \|\psi\|^2 \end{aligned}$$

Given we started with $\|\psi\|^2$ and ended with $\|\psi\|^2$, equality must hold throught, so

$$\sum_{j=1}^{n} \|P_j\psi\| = \|P_i\psi\|$$

and

$$\|P_j\psi\| = 0 \;\text{ for }\; j \neq i.$$

Therefore the range of P_i is contained in the null space of P_j, that is, $M_i \subseteq M_j^{\perp}$, for every $j \neq i$. So $M_i \perp M_j$ whenever $j \neq i$, and by the previous theorem we conclude that the P_i's are pairwise orthogonal. We now prove the final statement. Denote the range of P by Ran(P). First note that since $\|P\psi\| = \|\psi\|$ for every ψ in M_i, each M_i is contained in the range of P, and therefore

$$M \subseteq Ran(P). \tag{J.-16}$$

Next, if ψ is in the range of P, then

$$\begin{aligned}\psi &= P\psi \\ &= P_1\psi + P_2\psi + \cdots P_n\psi\end{aligned}$$

is obviously in M, and so ψ is in M, hence

$$Ran(P) \subseteq M. \tag{J.-18}$$

Comparing (J.3.2) and (J.3.2) implies $M = Ran(P)$.

□

J.3.3 Spectral Theorem for Finite Spaces

An operator on H is said to be **normal** if it commutes with its adjoint, that is, $NN^\dagger = N^\dagger N$. Self-adjoint and unitary operators are examples of normal operators. The spectral theorem states that for each normal operator N on H has a spectral resolution, that is, there exist distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and non-zero pairwise orthogonal projections $P_1, P_2, \ldots, P_m$ such that $\sum_{i=1}^m P_i = I$, and

$$N = \sum_{i=1}^{m} \lambda_i P_i.$$

Before coming on to the spectral theorem, we prove results for normal operators.

Lemma J.3.5 *If T ia normal, then ψ is an eigenvector of T with eigenvalue λ if and only if ψ is an eigenvector of $T^\dagger$ with eigenvalue λ^*.*

Proof:

Firsrt we show that an operator T is normal if and only if $\|T^\dagger\psi\| = \|T\psi\|$ for every ψ. Obviously $\|T^\dagger\psi\| = \|T\psi\|$ is equivalent to $\|T^\dagger\psi\|^2 = \|T\psi\|^2$, that is, $(T^\dagger, \psi T^\dagger\psi) = (T\psi, T\psi)$. This in turn is equivalent to $(TT^\dagger\psi, \psi) = (T^\dagger T\psi, \psi)$. Finally this is eqivalent to $([TT^\dagger - T^\dagger T]\psi, \psi) = 0$.

Since T is normal, it is obvious that also is $T - \lambda I$ for any scalar λ,

$$(T - \lambda I)(T^\dagger - \lambda^* I) = TT^\dagger - \lambda^* T - \lambda T^\dagger + |\lambda|^2 I = (T^\dagger - \lambda^* I)(T - \lambda I).$$

Hence we have

$$\|T - \lambda I\| = \|T^\dagger - \lambda^* I\|$$

for all ψ, and the lemma follows at once.

□

Lemma J.3.6 *If T ia normal, then each M_i reduces T.*

Proof:

It is obvious that each M_i is mapped onto itself under T. By theorem J.3.2 it suffices to show that each M_i is invariant under $T^\dagger$. But this follows from lemma J.3.5, for if ψ_i is a vector in M_i, so $T\psi_i = \lambda_i \psi_i$, then $T^\dagger \psi_i = \lambda_i^* \psi_i$ is also in M_i.

□

We are now in a position to state the spectral theorem (the proof can be found in vol I, appendix F).

Theorem J.3.7 *Let T be an arbitrary operator on H. By ... we know that the distinct eigenvalues of T form a non-empty finite set of complex numbers. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be these eigenvalues, let $M_1, M_2, \ldots, M_n$ be their corresponding eigenspaces, and let $P_1, P_2, \ldots, P_n$ be the projections onto these eigenspaces. The following statements are equivalent:*

i) The M_i's are pairwise orthogonal and span H.

ii) The P_i's orthogonal, $I = \sum_i^m P_i$ and $T = \sum_i^m \lambda_i P_i$.

iii) T is normal.

The expression

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m. \tag{J.-18}$$

for T, when it exists, is called the **spectral resolution** of T. The spectral theorem in particular proves that if T is normal, then it has the spectral resolution (J.3.3).

J.3.4 Functional Calculus for Finite Dimension Vector Spaces

$$Tv_n = \lambda_n v_n \tag{J.-18}$$

The collection of eigenvalues of T is denoted $\sigma(T)$,

$$\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \tag{J.-18}$$

an alternative (equivalent for finite dimensional vector spaces) definition is collection of λ's such that $T - \lambda\mathbf{1}$ is not invertible:

$$\sigma(T) = \{\lambda : T - \lambda\mathbf{1} \text{ is not inveretable}\} \tag{J.-18}$$

$$I = P_1 + P_2 + \cdots + P_m. \tag{J.-18}$$

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m \tag{J.-18}$$

This is known as the spectral decomposition of the operator T. The space of functions on the spectrum $\sigma(T)$. Define $\Phi : C(\sigma(T)) \to \mathcal{B}(\mathcal{H})$

$$\Phi(f) = f(\lambda_1)P_1 + \cdots + f(\lambda_m)P_m. \tag{J.-18}$$

Obviously we have

$$f(\sigma(T)) = \sigma(f(T)) \tag{J.-18}$$

This relation does not hold in general for infinite vector spaces but only under certain conditions - the spectral mapping theorem??. The space of these functions can be equipped with a norm

$$\|f\|_\infty = \sup_{\lambda_i \in \sigma(T)} |f(\lambda_i)|. \tag{J.-18}$$

$$\begin{array}{ll} \text{(a)} & \|\Phi(f)\| = \|f\|_\infty \\ \text{(b)} & \Phi(f+g) = \Phi(f) + \Phi(g) \\ \text{(c)} & \Phi(fg) = \Phi(f)\Phi(g) \\ \text{(d)} & \Phi(\overline{f}) = \Phi(f)^* \end{array} \tag{J.-20}$$

(a)

$$\begin{aligned}\|\Phi(f)\| &= \sup_{\|v\|\le 1}\|f(T)v\| \\ &= \sup_{\lambda_i\in\sigma(T)}|f(\lambda_i)| \\ &= \|f\|_\infty \end{aligned} \tag{J.-21}$$

$$\|f(T)\| = \sup\{|\lambda|\ ;\ \lambda\in\sigma(f(T))\} = \sup\{|\lambda|\ ;\ \lambda\in f(\sigma(T))\} = \|f\|. \tag{J.-21}$$

(b)

$$\begin{aligned}\Phi(f+g) &= \big(f(\lambda_1)+g(\lambda_1))P_1+\cdots+(f(\lambda_m)+g(\lambda_m))P_m \\ &= f(\lambda_1)P_1+\cdots+f(\lambda_m)P_m+g(\lambda_1)P_1+\cdots+g(\lambda_m)P_m \\ &= \Phi(f)+\Phi(g)\end{aligned} \tag{J.-22}$$

(c)

$$\begin{aligned}\Phi(fg) &= f(\lambda_1)g(\lambda_1)P_1+\cdots+f(\lambda_m)g(\lambda_m)P_m \\ &= \big(f(\lambda_1)P_1+\cdots+f(\lambda_m)P_m\big)\big(g(\lambda_1)P_1+\cdots+g(\lambda_m)P_m\big) \quad \text{as } P_iP_j=\delta_{ij}P_i \\ &= \Phi(f)\Phi(g)\end{aligned} \tag{J.-23}$$

(d)

$$\begin{aligned}\Phi(\overline{f}) &= \overline{f}(\lambda_1)P_1+\cdots+\overline{f}(\lambda_m)P_m \\ &= (f(\lambda_1)P_1+\cdots+f(\lambda_m)P_m)^* \qquad (P_i)^*=P_i\end{aligned} \tag{J.-23}$$

we can go from $f\in C(\sigma(T))$ to $f(T)$. Functional calculus can be used to construct great many interesting operators operators for example, square roots, logs, and exponentials of operators, difference equations.

$$\sqrt{T}=\sqrt{\lambda_1}P_1+\cdots+\sqrt{\lambda_m}P_m \tag{J.-23}$$

More standard notation

for each $A \subseteq \sigma(T)$, let

$$P(A) = \sum_{\lambda_i \in A} P_i. \tag{J.-23}$$

for $v, w \in \mathcal{H}$

$$\mu_{v,w} = (P(A)v, w) \tag{J.-23}$$

a measure on $\sigma(T)$

$$\int_{\sigma(T)} f(\lambda) dP(\lambda) \tag{J.-23}$$

J.3.5 Decomposition of an N-Dimensional Space.

It follows from that the decomposition of an arbitrary vector $| >$ into vectors $\varphi(A)| >$ is unique. Therefore, an arbitrary vector either belongs entirely to one of the subspaces $\mathbf{S}^{(i)}(i = 1, 2, \ldots, L)$ or can be decomposed uniquely into vectors belonging to different subspaces $\mathbf{S}^{(i)}$. One says that $\mathbf{S}^{(i)}$ is a direct sum of the subspaces $\mathbf{S}^{(i)}$ and one writes

$$\mathbf{S}_N = \mathbf{S}^{(1)} \oplus \mathbf{S}^{(2)} \oplus \cdots \oplus \mathbf{S}^{(L)} \tag{J.-23}$$

$$A = \sum_{i=1}^{m} \lambda_i P_i. \tag{J.-23}$$

Continuity

Naive arguments can quickly lead to errors. It is complete - We need "enough" functions to do what we want.

J.3.6 Complications in Infinite Dimensional Spaces

The Spectral Theorem

$\psi_n^* \psi_n \not< \infty$ in general.

Even if $\sum_n \psi_n^* \psi_n < \infty$, there could be operators $\hat{A}$ such that

$$\phi_n := \hat{A}|\psi_n > \qquad \sum_n \phi_n^* \phi_n \to \infty \tag{J.-23}$$

The maximal domain $\mathcal{D}_A$ is:

$$\mathcal{D}_A := \{|\psi > \in \mathcal{H} \; : \; \hat{A}|\psi > \in \mathcal{H}\} \tag{J.-23}$$

scalars

$$\lim_{N\to\infty} \lim_{M\to\infty} \varphi_n^* A_{nm} \psi_m \neq \lim_{N\to\infty} \lim_{M\to\infty} \varphi_n^* A_{nm} \psi_m \tag{J.-23}$$

making possible a mathematically rigorous treatment of the subject of Dirac's

finite dimensional spaces are automatically complete.

The norm enables us to define a distance, $d(\phi, \psi) := \|\phi - \psi\|$, between vectors. the state space $\mathcal{H}$ should be complete in the following sense: whenever $\{\psi_n \in \mathcal{H}\}$ is a sequence of vectors

$$\|\psi_m - \psi_n\| < \epsilon \tag{J.-23}$$

for all $m, n > N_\epsilon$, then there exists a limit vector ψ such that

$$\|\psi_n - \psi\| \to 0 \tag{J.-23}$$

Completedness is an important property to have around if you are counting on an equation having a solution.

$\psi_n := \sum_{r=0}^{n} (a_r \cos nx + b_r \sin nx)$

A metric space is **complete** if every Cauchy sequence in M converges to a point in M.

J.3.7 Cauchy Sequences

Proposition J.3.8 *A convergent sequence has a unique limit.*

Proof: Suppose $\{x_n\}$ converges to x and also to x', where $x \neq x'$. Put

$$\epsilon = \frac{1}{2}|x - x'|.$$

Since $\{x_n\}$ converges to x, there exists N_1 such that $|x_n - x| < \epsilon$ for all $n \geq N_1$. Similarly there exists N_2 such that $|x_n - x'| < \epsilon$ for all $n \geq N_2$. Put $N = \max\{N_1, N_2\}$. Then

$$|x - x'| = |x - x_N + x_N - x'| \leq |x - x_N| + |x_N - x'| < 2\epsilon = |x - x'|.$$

This contradiction shows that $x = x'$.

$\square$

Theorem J.3.9 *Every bounded monotonic sequence converges.*

Proof: Suppose that $\{s_n\}$ is a monontonic increasing and bounded from above. By the completeness axiom, for the set S of its memebers $s = \sup S$ exists. We prove that $\{s_n\}$ converges to s. Let $\epsilon > 0$. Then $s - \epsilon$ is not an upper bound for S so

$$s_N > s - \epsilon$$

for some N. Since $\{s_n\}$ is a monontonic increasing,

$$s_n \geq s_N \text{ for all } n \geq N.$$

But

$$s_n \leq s \text{ for all } n,$$

since s is an upper bound for S. Hence for all $n \geq N$,

$$s \geq s_n \geq s_N > s - \epsilon,$$

and in particular

$$|s - s_n| < \epsilon.$$

The proof for $\{s_n\}$ is a monontonic decresing can be obtained from that proved to get the convergence of $-s_1, -s_2, -s_3, \ldots$, which implies the convergence of $s_1, s_2, s_3, \ldots$.

$\square$

Definition A sequence $\{x_n\}$ is a Cauchy sequence if given $\epsilon > 0$, there exists N such that $m, n \geq N$ imply $|x_n - x_m| < 0$.

Lemma J.3.10 *Any Cauchy sequence is bounded.*

Proof: Take $\epsilon = 1$ in the Cauchy sequence. Thus there exists N such that $m, n \geq N$ imply

$$|x_n - x_m| < 1,$$

so for any $n \geq N$,

$$|x_n - x_N| < 1,$$

and hence

$$|x_n| < 1 + |x_N|.$$

Now

$$|x_n| \leq \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x_N|\}$$

for all n.

□

Theorem J.3.11 *A sequence $\{s_n\}$ of real numbers conveges if and only if it is a Cauchy sequence.*

Proof: Suppose that $\{s_n\}$ converges to s. Then given $\epsilon > 0$, there exists N such that $|s_n - s| < 0$ for $n \geq N$, so for $m, n \geq N$ we have

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| < 2\epsilon.$$

Hence $\{s_n\}$ is a Cauchy sequence.

Suppose conversely that $\{s_n\}$ is a Cauchy sequence. We construct a monotonic sequence from the sequence $\{s_n\}$. For each m, let S_m denote a set of members of the sequence

$$S_m := \{s_n : n \geq m\}.$$

Obviously, each S_m is bounded as the whole set $S = S_1$ is. Hence supS_m exists. Let $u_m := \sup S_m$. Since $S_{m+1} \subset S_m$, it follows that $\sup S_{m+1} \leq \sup S_m$. Thus $\{u_m\}$ is monontonic decreasing.

By theorem J.3.9, converges, to s say.

Finally, we show that $\{s_m\}$ also converges to s. Given ϵ, there exists N_1 such that

$$|s_m - s_n| < \epsilon$$

for $m, n \geq N_1$ and N_2 such that

$$|s - u_m| < \epsilon$$

for $m \geq N_2$. Let $N = \max\{N_1, N_2\}$. By definition of u_N, $u_N - \epsilon$ is not an upper bound of S_N, so there exists $M \geq N$

Now for any $n \geq N$,

$$\begin{aligned} |s_n - s| &= |s_n - s_M + s_M - s_N + s_N - s| \\ &\leq |s_n - s_M| + |s_M - s_N| + |s_N - s| \\ &< 3\epsilon. \end{aligned}$$

Hence $\{s_n\}$ converges to s.

□

Definition A sequence of vectors $\{f_n\}$ in a normed vector space V is called a Cauchy sequence if given $\epsilon > 0$, there exists N such that $m, n \geq N$ imply $\|f_n - f_m\| < 0$.

J.3.8 Functions on Metric Spaces

First, $d(x, y)$ should be non-negative for any x, y in A, and it should only be zero if the x and y coincide. The distance should be the same whether we "measure" it from x to y or from y to x. The sum of the lengths of two sides of any triangle is not less than the length of the third. is the formal version of the triangle inequality.

(M1) $d(x, y) \geq 0; \quad d(x, y) = 0 \leftrightarrow x = y$

(M2) $d(x, y) = d(y, x)$ for all x, y in A,

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in A.

Sequences of Functions

Definition The sequence (T_n) is said to *converge pointwise* to T provided

Obviously, uniform convergence is also pointwise convergence (to the same limit). The converse is generally false as can be seen from the example, so uniform convergence is **stronger** than pointwise convergence.

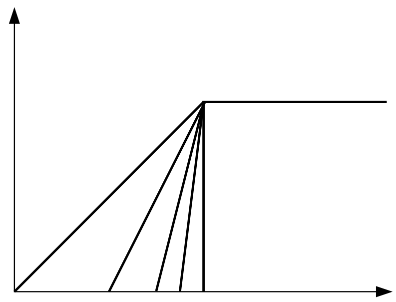


Figure J.1: The limit function of a sequence of continuous functions need not be continuous.

The primary use of uniform convergence is to ensure the continuity of the limit function.

Theorem Suppose that each T_n is continuous and (T_n) converges uniformally to the limit function T. Then, T will be continuous.

Proof: Take any point x in ??For all n and y

$$d(Tx, Ty) \leq d(Tx, Tx_n x) + d(T_n x, T_n y) + d(T_n y, Ty). \tag{J.-26}$$

J.3.9 Topologies

We begin our exposition of topology by attaching names to properties that will be used very often and then introduce the shorthand that has been developed in the subject for talking about such things.

J.4 Function Spaces.

Length of a vector. The vector space consisting of position vectors $\mathbf{r}$ in 3-d Euclidian space with a distance

$$||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2} \tag{J.-26}$$

Although each point of $\mathcal{C}$ is a function, in many ways $\mathcal{C}$ is like a Euclidean space. We may for instance, define a "distance" on on the space $\mathcal{C}$ as follows

$$\|f\| = \max_{x \in A} |f(x)| \tag{J.-26}$$

The symbol $\psi \in V$ means that ψ is a member of the set of vectors in V.

$$\begin{aligned}
&\|\alpha f\| = \max_{x \in A} |\alpha f(x)| = |\alpha| \max_{x \in A} |f(x)| = |\alpha| \|\alpha f\|, \\
&\|f\| = \max_{x \in A} |f(x)| = 0 \ \Rightarrow \ f(x) = 0, \\
&\|f + g\| = \max_{x \in A} |f(x) + g(x)| \leq \max_{x \in A} |f(x)| + \leq \max_{x \in A} g(x)| = \|f\| + \|g\|.
\end{aligned} \tag{J.-28}$$

J.4.1 Normed Spaces and Banach Spaces

Definition (Normed Space). A normed space X is a vector space with a norm defined on it. Here a norm is a (real or complex) vector is a real-valued function on X whose value at an $x \in X$ is denoted by

$$\|x\| \tag{J.-28}$$

and which has the following properities

(i) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$

(ii) For $c \in k$ and $\psi \in E$, $||c\psi|| = |c| \; ||\psi||$.

(iii) $||x + y|| \leq ||x|| + ||y||$. (triangle inequality)

If is endowed with a given norm we call a normed linear (or vector space) space.

A norm defines a metric induced by the norm which is given by

$$d(x, y) := \|x - y\| \tag{J.-28}$$

and is called the *metric induced by the norm.* As a nice example of the space of continuous functions $\mathcal{C}$.

A special case of a normed space.

There is a general theorem in topology that tells us that any metric space can be completed by a procedure analogous to that for constructing the real numbers from the rationals.

Definition (Banach Space) A is a complete normed space (complete in the metric (J.4.1))

Examples of normed spaces.

The vector space consisting of position vectors $\mathbf{r}$ in 3-d Euclidian space with the norm

$$||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2} \tag{J.-28}$$

forms a normed space. That $||\mathbf{r}||$ satisfies the conditions to be a norm is obvious. The triangle inequality is obvious from fig (J.4.3).

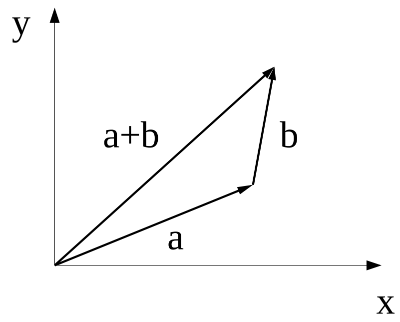


Figure J.2: Euclidean triangle inequaltiy.

This can be thought of as the prototype normed space.

Example 2 Space $\mathbf{l}^p$. Is a Banach space with norm given by

$$\|x\| := \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p} \tag{J.-28}$$

this norm induces the metric

$$d(x,y) \equiv \|x - y\| := \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p} \tag{J.-28}$$

The set of sqaure integrable functions, that is functions for which $||\psi(x)|| < \infty$ where the norm is defined by $:= \int_{-\infty}^{\infty} dx|\psi(x)|^2$ forms a normed space.

The integrand is positive

$$||c\psi|| = \int_{-\infty}^{\infty} dx|c\psi(x)|^2 = |c|^2 \int_{-\infty}^{\infty} dx|\psi(x)|^2 = |c|^2 \; ||\psi|| \tag{J.-28}$$

Consider the quantity

$$\begin{aligned} \int_{-\infty}^{\infty} dx|\psi(x) + \lambda\phi(x)|^2 &= \int_{-\infty}^{\infty} dx(\psi(x) + \lambda\phi(x))^*(\psi(x) + \lambda\phi(x)) \\ &= \int_{-\infty}^{\infty} dx \left[|\psi(x)|^2 + \lambda\psi(x)^*\phi(x) + \lambda^*\psi(x)\phi(x)^* + \lambda\lambda^*|\phi(x)|^2\right] \end{aligned} \tag{J.-29}$$

$$\int_{-\infty}^{\infty} dx|\psi(x) + \phi(x)|^2 \leq \int_{-\infty}^{\infty} dx|\psi(x)|^2 + \int_{-\infty}^{\infty} dx|\phi(x)|^2 \tag{J.-29}$$

There is a general theorem in topology that tells us that any metric space can be completed by a procedure analogous to that for constructing the real numbers from the rationals.

Strong convergence (or convergence in the mean)

$$||\psi - \psi_n|| \to 0 \tag{J.-29}$$

So that, for any $\epsilon > 0$ there exists an integer n large enough that we can have

$$||\psi - \psi_n|| < \epsilon/2 \tag{J.-29}$$

By the triangle inequality

$$||\psi_m - \psi_n|| \leq ||\psi - \psi_m|| + ||\psi - \psi_n|| < \epsilon \tag{J.-29}$$

$$||\psi_m|| = ||(\psi_m - \psi_n) + \psi_n|| \leq ||\psi_m - \psi_n|| + ||\psi_n|| \tag{J.-29}$$

$$\begin{aligned} ||\psi_m - \psi_n|| &\geq ||\psi_m|| - ||\psi_n|| \\ ||\psi_m - \psi_n|| &\geq ||\psi_n|| - ||\psi_m|| \end{aligned} \tag{J.-29}$$

$$\boxed{||\psi_m - \psi_n|| \geq \big|\, ||\psi_m|| - ||\psi_n||\, \big|} \tag{J.-29}$$

$$|\, ||\psi_m|| - ||\psi_n||\, | < \epsilon \tag{J.-29}$$

$$|\, ||\psi|| - ||\psi_n||\, | < \epsilon \tag{J.-29}$$

"if and only if" is a phrase used a lot in mathematics means you cannot have one without the other, in other words, they are equivalent.

Definition A metric space X is said to be **complete** if every Cauchy sequence in X converges to an element of X. Otherwise X is said to be **incomplete**.

Lemma J.4.1 *If a subsequence of a Cauchy sequence converges, then the whole sequence converges.*

Proof: Let $\{x_n\}$ be a Cauchy sequence in a normed vector space X, and let $\epsilon > 0$ be given. Then there is an N such that

$$\|x_n - x_m\| < \epsilon, \quad m, n > N.$$

Now if $\{x_n\}$ has a subsequence converging to $x \in X$, then there is an $m > N$ such that

$$\|x_m - x\| < \epsilon.$$

Thus,

$$\|x_n - x\| \leq \|x_n - x_m\| + \|x_m - x\| < 2\epsilon.$$

for all N.

□

Definition A normed linear space X is said to be complete if it is complete as a metric space in the induced metric.

Definition A **Banach space** is a complete normed space.

Definition A **bounded linear transformation** from a normed linear space $(V_1, \|.\|_1)$ to a normed linear space $(V_2, \|.\|_2)$ is a function, T, from V_1 to v_2 satisfying

i) $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$ for all $v, w \in V$ and for all $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$.

ii) For some $C \geq 0$, $\|Tv\|_2 \leq C\|v\|_1$.

The smallest such C is called the **norm of** T, $\|T\|$. Thus

$$\|T\| := inf_{\|v\|_1=1} \|Tv\|_2.$$

Definition The **closure** $\overline{C}$ of a subset C in a topological space T is the union of C and all the limit points of C in T.

Lemma J.4.2 *A point $x \in T$ is in $\overline{C}$ if and only if every open set containing x contains a point of C.*

Proof:

□

Lemma J.4.3 *Let C be a subset of a metric space X and let $x \in X$. Then $x \in \overline{C}$ if and only if there exists a sequence $\{x_n\}$ in C converging to x.*

Proof: First suppose that $\{x_n\}$ is a sequence in C converging to x. Given $\epsilon > 0$, there exists N such that $x_n \in B_\epsilon(x)$ for al $n \geq N$. Since $x_n \in C$ for all n, this means that for every $\epsilon > 0$, $B_\epsilon(x) \cup C \neq \emptyset$. Hence $x \in \overline{C}$.

Now suppose that $x \in \overline{C}$. By definition of closure, there exists a point x_n in $C \cap B_{1/n}(x)$ for every $n \in \mathbb{N}$, and clearly $\{x_n\}$ converging to x.

□

J.4.2 Completion of a metric space

For example, the space of all real numbers is the completion of the space of all rational numbers.

Theorem J.4.4 *Every metric space of X has a completion. This completion is unique up to an isometric mapping carrying every point $x \in X$ into itself.*

Proof: Proof of the existence of the completion of X:

We call two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X equivalent and write $\{x_n\} \sim \{y_n\}$ if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Let the new space X^* be the family of equivalence classes of Cauchy sequences under this equivalence relation. Then define the distance between two arbitrary points $x^*, y^* \in X^*$ by the formular

$$d_1(x^*, y^*) = \lim_{n\to\infty} d(x_n, y_n), \tag{J.-29}$$

where $\{x_n\}$ and $\{y_n\}$ are any representatives of x^* and y^* respectively. We must establish existence, independent of the choice of sequences $\{x_n\} \in x^*$, $\{y_n\} \in y^*$ and that it satisfies the three properties of a distance. Given any $\epsilon > 0$, it follows from the triangle inequality in X that

$$\begin{aligned} |d(x_n, y_n) - d(x_{n'}, y_{n'})| &\leq |d(x_n, y_n) - d(x_{n'}, y_n)| + |d(x_{n'}, y_n) - d(x_{n'}, y_{n'})| \\ &\leq d(x_n, x_{n'}) + d(y_n, y_{n'}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned} \tag{J.-30}$$

for sufficiently large n and n'. Therefore the sequence of real numbers $\{d(x_n, y_n)\}$ is Cauchy and hence has a limit. We show that it is independent of the representatives. Suppose

$$\{x_n\}, \{\tilde{x}_n\} \in x^*, \qquad \{y_n\}, \{\tilde{y}_n\} \in y^*.$$

Then

$$|d(x_n, y_n) - d(\tilde{x}_n, \tilde{y}_n)| \leq d(x_n, \tilde{x}_n) + d(y_n, \tilde{y}_n).$$

But

$$\lim_{n\to\infty} d(x_n, \tilde{x}_n) = \lim_{n\to\infty} d(y_n, \tilde{y}_n) = 0,$$

since $\{x_n\} \sim \{\tilde{x}_n\}$, $\{y_n\} \sim \{\tilde{y}_n\}$, and hence

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(\tilde{x}_n, \tilde{y}_n).$$

Now we move onto proving the properties of a metric. It is obvious that $d_1(x^*, y^*) = d_1(y^*, x^*)$. That $d_1(x^*, y^*) = 0$ if and only if $x^* = y^*$ is an immediate consequence of the definition of equivalent Cauchy sequences. To verify the triangle inequality in X^* we start from the trianngle inequality

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$$

in the original space X and then take the limit as $n \to \infty$, obtaining

$$\lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n)$$

i.e.,

$$d_1(x^*, z^*) \leq d_1(x^*, y^*) + d_1(y^*, z^*).$$

Now we prove that X^* is a completion of X. First we show that the mapping of X to X^* is isometric. Suppose that every point $x \in X$, we associate the class $x^* \in X^*$ of al Cauchy sequences converging to x. Let

$$x = \lim_{n\to\infty} x_n, \qquad y = \lim_{n\to\infty} y_n.$$

Then

$$d(x, y) = \lim_{n\to\infty} d(x_n, y_n),$$

while on the other hand

$$d_1(x^*, y^*) = \lim_{n\to\infty} d(x_n, y_n),$$

by definition. Therefore

$$d(x, y) = d_1(x^*, y^*),$$

and hence the mapping from X into X^* carrying x into x^* is isometric.

X can be regarded as a subset of X^*. It must be proved that

i) X is everywhere dense in X^*, i.e., $\overline{X} = X^*$; and

ii) X^* is complete.

To this end, given any point $x^* \in X^*$ and any $\epsilon > 0$, choose a Cauchy sequence $\{x_n\} \in x^*$. Let N be such that $d(x_n, x_{n'}) < \epsilon$ for all $n, n' > N$. Then

$$d(x_n, x^*) = \lim_{n' \to \infty} d(x_n, x_{n'}) \leq \epsilon$$

if $n > N$, i.e., every neighbourhood of the point x^* intersects X. If follows that $\overline{X} = X^*$.

Finally we show that X^* is complete, that is, we show that every Cauchy sequence $\{x_n^*\}$, consisting of points in X^*, converges to a point $x^* \in X^*$. Note by the very definition of X^*, any Cauchy sequence $\{x_n\}$ consisting of points in X converges to some point $x^* \in X^*$. As X is dense in X^*, for any $\{x_n^*\}$ we can find an equivalent sequence $\{x_n\}$ in X. Choose x_n to be any point of X such that $d(x_n, x_n^*) < 1/n$, the resulting sequence $\{x_n\}$ is equivalent to $\{x_n^*\}$. Now as

$$\begin{aligned} d(x_n, x_{n'}) &\leq d(x_n, x_n^*) + d(x_n^*, x_{n'}) \\ &\leq d(x_n, x_n^*) + d(x_n^*, x_{n'}^*) + d(x_{n'}^*, x_{n'}) \\ &< \frac{1}{n} + \frac{1}{n'} + d(x_n^*, x_{n'}^*) \end{aligned}$$

$\{x_n\}$ is Cauchy and so converges to a point $x^* \in X^*$. But then the sequence $\{x_n^*\}$ also converges to x^*.

Proof of uniqueness:

Say X^* and $X^{*\prime}$ are two completions of X. We wish to show that there is a one-to-one mapping $i : X^* \to X^{*\prime}$ such that $i(x) = x$ for all $x \in X$ and

$$d_1(x^*, y^*) = d_2(x^{*\prime}, y^{*\prime}) \tag{J.-33}$$

where d_1 is the distance in X^* and d_2 is the distance in $X^{*\prime}$. The required mapping i is constructed as follows: Let x^* be an arbitrary point of X^*. Then, by definition of completion, there is a Cauchy sequence $\{x_n\}$ of points converging to x^*. The points of the sequence $\{x_n\}$ also belong to $X^{*\prime}$, where they form a Cauchy sequence. Therefore $\{x_n\}$ converges to a point $x^{*\prime} \in X^{*\prime}$, since $X^{*\prime}$ is complete. We see that $x^{*\prime}$ is independent of the choice of sequence.

If we set $i(x^*) = x^{*\prime}$, then i is the required mapping. In fact, $i(x) = x$ for all $x \in X$, since $x_n \to x \in X$, then obviously $x = x^* \in X^*$, $x^{*\prime} = x$. Moreover, suppose $x_n \to x^*$, $y_n \to y^*$ in X^*, while $x_n \to x^{*\prime}$, $y_n \to y^{*\prime}$ in $X^{*\prime}$. Then, if d is the distance in X,

$$d_1(x^*, y^*) = \lim_{n\to\infty} d_1(x_n, y_n) = \lim_{n\to\infty} d(x_n, y_n) \tag{J.-33}$$

Now we prove that if $x_n \to x$, $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$. So at the same time

$$d_2(x^{*\prime}, y^{*\prime}) = \lim_{n\to\infty} d_2(x_n, y_n) = \lim_{n\to\infty} d(x_n, y_n) \tag{J.-33}$$

But (J.4.2) and (J.4.2) imply (J.4.2).

$\square$

Definition Let $\mathcal{B}(X, Y)$ be the set of bounded linear operators from X to Y.

Theorem J.4.5 *If Y is a Banach space, so is $\mathcal{B}(X, Y)$.*

Proof: Suppose $\{A_n\}$ is a Cauchy sequence of operators in $\mathcal{B}(X, Y)$. Then for each $\epsilon > 0$ there is an N such that for all $m, n > N$

$$\|A_n - A_m\| < \epsilon.$$

Thus for each $x \neq 0$,

$$\|A_n x - A_m x\|_2 < \epsilon \|x\|_1, \quad m, n > N. \tag{J.-33}$$

This shows that $\{A_n x\}$ is a Cauchy sequence in Y. Since Y is complete, there is a $y_x \in Y$ such that $A_n x \to y_x$ in Y. Define the operator A from X to Y by $Ax = y_x$.

Then A is linear. Note that we have

$$\begin{aligned}
\|A(\alpha x + \beta x') - \alpha A(x) - \beta A(x)'\|_2 &\leq \|A(\alpha x + \beta x') - A_n(\alpha x_n + \beta x'_n)\|_2 \\
&\quad + \|\alpha A_n(x_n) + \beta A_n(x'_n) - \alpha A(x) - \beta \tilde{T}(x')\|_2 \\
&\leq \|A(\alpha x + \beta x') - A_n(\alpha x + \beta x')\|_2 \\
&\quad + |\alpha| \, \|A_n(x) - A(x)\|_2 \\
&\quad + |\beta| \, \|A_n(x') - A(x')\|_2.
\end{aligned}$$

Since the right-hand-side tends to zero as $n \to \infty$, we conclude that $\tilde{T}$ is linear.

Let $m \to \infty$ in (J.4.2). Then

$$\|A_n x - Ax\| < \epsilon \|x\|, \quad n > N.$$

Hence,

$$\|Ax\| \leq \epsilon \|x\| + \|A_n x\| \leq (\epsilon + \|A_n\|)\|x\|, \quad n > N.$$

This shows that A is bounded. Moreover,

$$\|A_n - A\| \leq \epsilon, \quad n > N.$$

Hence,

$$\|A_n - A\| \to 0 \text{ as } n \to \infty.$$

□

Theorem J.4.6 *(The* **bounded linear transformation theorem***). Suppose T is a bounded linear transformation from a normed linear space $(V_1, \|.\|_1)$ to a complete normed linear space $(V_2, \|.\|_2)$. Then T can be uniquely extended to a bounded linear transformation (with the same bound), $\tilde{T}$, from the completion of V_1 to $(V_2, \|.\|_2)$.*

Proof: Proof of existence: Let $\overline{V}_1$ be the completion of V_1. For each $x \in \overline{V}_1$, there is a sequence of elements $\{x_n\}$ in V_1 with $x_n \to x$ as $n \to \infty$. Since x_n converges, it is Cauchy so given $\epsilon > 0$, we can find N so that for $n, m > N$ such that

$$\|x_n - x_m\|_1 \leq \epsilon / \|T\|$$

.

Then

$$\|Tx_n - Tx_m\|_2 = \|T(x_n - x_m)\|_2 \ \leq \|T\| \ \|x_n - x_m\|_1 \ \leq \epsilon$$

which proves that Tx_n is a Cauchy sequence in V_2. Since V_2 is complete, $Tx_n \to y$ for some $y \in V_2$. Set $\tilde{T}x = y$.

We must show that this defintion is independent of the sequence $x_n \to x$ chosen. We know that $x \in \overline{V}_1$ is independent of the Cauchy sequence $\{x_n\}$. $\{Tx_n\}$ is a Cauchy sequence in V_2. We must show that $\{Tx_n\}$ and $\{Tx'_n\}$ have the same convergence point in V_2. As $\{x_n\}$ and $\{x'_n\}$ are Cauchy, there is N such that for all $n > N$, $\|x_n - x\|_1 \leq \epsilon/2\|T\|$ and $\|x'_n - x\|_1 \leq \epsilon/2\|T\|$. Therefore we have

$$\begin{aligned}
\|Tx_n - Tx'_n\|_2 &= \|T(x_n - x'_n)\|_2 \\
&\leq \|T\| \, \|x_n - x'_n\|_1 \\
&\leq \|T\| \, (\|x_n - x\|_1 + \|x - x'_n\|_1) \\
&\leq \epsilon.
\end{aligned}$$

We conclude that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tx'_n$.

The operator $\tilde{T}$ is bounded with same bound as

$$\begin{aligned}
\|\tilde{T}x\|_2 &= \lim_{n\to\infty} \|\tilde{T}x_n\|_2 \\
&\leq \lim_{n\to\infty} C\|x_n\|_1 \\
&= C\|x\|_1. \qquad \text{(J.-43)}
\end{aligned}$$

Proof of linearity: Note that we have

$$\begin{aligned}
\|\tilde{T}(\alpha x + \beta x') - \alpha\tilde{T}(x) - \beta\tilde{T}(x')\|_2 &\leq \|\tilde{T}(\alpha x + \beta x') - T(\alpha x_n + \beta x'_n)\|_2 \\
&\quad + \|\alpha T(x_n) + \beta T(x'_n) - \alpha\tilde{T}(x) - \beta\tilde{T}(x')\|_2 \\
&\leq \|\tilde{T}(\alpha x + \beta x') - T(\alpha x_n + \beta x'_n)\|_2 \\
&\quad + |\alpha| \, \|T(x_n) - \tilde{T}(x)\|_2 \\
&\quad + |\beta| \, \|T(x'_n) - \tilde{T}(x')\|_2.
\end{aligned}$$

Since the right-hand-side tends to zero as $n \to \infty$, we conclude that $\tilde{T}$ is linear.

Proof of uniqueness:

$$\begin{aligned}
\|\tilde{T} - \tilde{T}'\| &= \sup_{x \in \overline{V}_1 - \{0\}} \frac{\|(\tilde{T} - \tilde{T}')x\|_2}{\|x\|_1} \\
&= \sup_{x \in \overline{V}_1 - \{0\}} 0 \\
&= 0
\end{aligned}$$

We conclude that $\tilde{T}$ is unique.

$\square$

J.4.3 Inner Product Spaces and Hilbert spaces

Definition: A linear vector space upon which has an inner product

and which has the folowing properities

(i) $< x, x > \geq 0$ and $||x|| = 0$ if and only if $x = 0$

(ii) For $c \in k$ and $\psi \in E$, $||c\psi|| = |c|\ ||\psi||$.

(iii) $||x + y|| \leq ||x|| + ||y||$. (triangle inequality)

If is endowed with a given norm we call a normed linear (or vector space) space.

Definition: A *Hilbert space* is a complex inner product space which is complete. The state space of QM is aussumed to be a Hilbert space.

Definition: A *separable Hilbert space* is a Hilbert space that contains a countable dense subset.

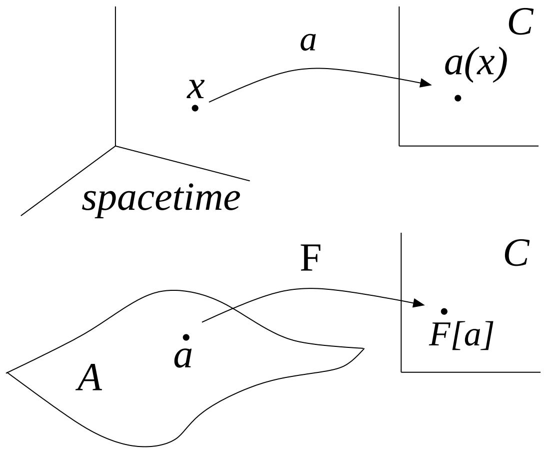


Figure J.3: Normed, Banach, inner product space, Hilbert space.

Orthogonal complements

Lemma J.4.7 *Let M be a linear subspace of an inner product space $\mathcal{H}$ and let $x \in \mathcal{H}$. Then $x \in M^{\perp}$ if and only if*

$$\|x - y\| \leq \|x\| \tag{J.-51}$$

for all $y \in M$.

Suppose that (J.4.7) holds. Pick any $y \in M$ and $\alpha \in \mathbb{C}$. Then $\alpha y \in M$

$\square$

Lemma J.4.8 *Let M be a subspace of a separable Hilbert space. The set of all vectors which are orthogonal to every vector in M is called the* **orthogonal complement** $M^{\perp}$ *of M.*

Proof:

Evidently $M^{\perp}$ is a linear manifold; if ψ and ϕ are vectors in $M^{\perp}$ and α is a scalar, then for any vector χ in M

$$(\chi, \psi + \phi) = (\chi, \psi) + (\chi, \phi) = 0$$

and

$$(\chi, \alpha\phi) = \alpha(\chi, \psi) = 0$$

so $\psi + \phi$ and $\alpha\psi$ are in $M^{\perp}$. In fact $M^{\perp}$ is a closed subspace, for if a sequence of vectors ψ_n in $M^{\perp}$ converges to a limit vector ψ, then for any χ in M as the inner product is continuous

$$(\chi, \psi) = \lim_{n \to \infty} (\chi, \psi_n) = 0,$$

so ψ is in $M^{\perp}$.

$\square$

This implies $\overline{M} = M^{\perp\perp}$

Proof: If $x \in M^{\perp}$ then, for any $y \in M$, x and y are orthogonal so that

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 \geq \|x\|^2.$$

$\square$

Corollary J.4.9 *Let M be a linear subspace of a Hilbert space $\mathcal{H}$, then M is dense in $\mathcal{H}$ (that is $\overline{M} = \mathcal{H}$) if and only if $M^{\perp} = \{0\}$.*

J.4.4 Linear Functionals

A linear functional F assigns a scalar $F(\phi)$ to each vector ϕ such that for any to vectors ϕ and ψ and a scalar A

$$F(\phi + \psi) = F(\phi) + F(\psi) \tag{J.-51}$$

$$F(a\phi) = aF(\phi). \tag{J.-51}$$

An example of a functional would be

$$F(\phi) = (\varphi, \phi), \tag{J.-51}$$

(but not all functionals will be of this form!)

Linear functionals form a linear space with the sum $F_1 + F_2$

To get a similar result for infine-dimensional spaces we need the concept of continuity. Recall that a function f of a complex variable is continuous if $f(z_n) \to f(z)$ whenever $z_n \to z$. The same property defines continuity for linear functionals; a linear functional F is *continuous* if $F(\chi_n)$ converges to $F(\chi)$ for any sequence of vectors χ_n which converges to a limit vector χ.

J.4.5 Complete Orthonormal Systems of Vectors in $\mathcal{H}$

Definition (Gram-Schmidt orthogonalization process). Let $X = \{x_1, \ldots, x_n\}$ be any basis in V. We shall construct a complete orthornormal set $Y = \{y_1, \ldots, y_n\}$ such that each y_j is a linear combination of the x_j's.

As X is linearly independent, $x_1 \neq 0$ and we may take $y_1 = x_1/\|x_1\|$. Now put

$$y_2 = \frac{(x_2 - \alpha_1 y_1)}{\|x_2 - \alpha_1 y_1\|}.$$

Thus setting

$$(y_1, y_2) = \frac{(y_1, x_2) - \alpha_1}{\|x_2 - \alpha_1 y_1\|} = 0$$

implies $\alpha_1 = (y_1, x_2)$. Therefore

$$y_2 = \frac{x_2 - (y_1, x_2)y_1}{\|x_2 - \alpha_1 y_1\|}.$$

Note that

$$x_2 - (y_1, x_2)y_1 = x_2 - \frac{(y_1, x_2)}{\|x_1\|} x_1 \neq 0,$$

since X is linearily independent. Thus we have found two vectors, y_1 and y_2, such that $(y_i, y_j) = \delta_{ij}$, $i, j = 1, 2$. We continue inductively. When this process is repeated n times, we shall have generated an orthonormal set of n basis vectors.

An orthonormal sequence $\{e_n\}$ in a Hilbert space H is complete if the only member of H which is orthogonal to every e_n is the zero vector.

Lemma J.4.10 *Every separable Hilbert space has a complete orthonomal sequence.*

Proof: Let $\mathcal{H}$ be a separable Hilbert space, and let $\{g_n\}$ be a dense sequence in H. Remove from this sequence any element which is a linear combination of the proceding g_j's via a Gram-Schmidt process. Let S_n be a subspace of spanned by $g_1, \ldots, g_n$, and let φ_n be an element of S_n having norm one and orthogonal to S_{n-1}. Such an element exists by a corollary of the Riesz lemma, corollary J.7.4. Clearly, the sequence $\{\varphi_n\}$ is orthonormal. It is complete since the original sequence $\{g_n\}$ is contained in

$$W = \bigcup_{n=1}^{\infty} S_n,$$

and each element in W is a linear combination of a finite number of the φ_n's.

$\square$

Theorem J.4.11 *For each continuous linear functional F on a separable Hilbert space there is a unique vector ψ_F in the space of such states that $F(\phi) = (\psi_f, \phi)$ for every vector ϕ.*

Proof: Let the set of vectors $\phi_1, \phi_2, \ldots, \phi_k, \ldots$ be an orthonormal basis. Consider the infinite linear combination

$$\psi_F = \sum_{n=1}^{\infty} F(\phi_n)^* \phi_n. \tag{J.-51}$$

If this converges to a vector ψ_F, then for any vector $\phi = \sum_{k=1}^{\infty} (\phi, \phi_k)\phi_k$ we have

$$\begin{aligned} F(\phi) &= \sum_{k=1}^{\infty} (\phi_k, \phi) F(\phi_k) \\ &= (\sum_{k=1}^{\infty} F(\phi_k)^* \phi_k, \phi) \\ &= (\psi_F, \phi) \end{aligned} \tag{J.-52}$$

We must prove that ψ_F defines a vector, that is, we must prove $\|\psi_F\|$ is finite. We will then know what the vector is and it will be unique. As

$$\begin{aligned} \|\psi_F\|^2 &= \|\sum_{k=1}^{\infty} F(\phi_k)^* \phi_k\|^2 \\ &= (\sum_{j=1}^{\infty} F(\phi_j)^* \phi_j, \sum_{k=1}^{\infty} F(\phi_k)^* \phi_k) \\ &= \sum_{k=1}^{\infty} |F(\phi_k)|^2 \end{aligned} \tag{J.-53}$$

we must show that $\sum_{k=1}^{\infty} |F(\phi_k)|^2$ converges. Let

$$\psi_n := \sum_{k=1}^{n} F(\phi_k)^* \phi_k.$$

Then

$$F(\psi_n) = \sum_{j=1}^{\infty} \big(\phi_j, \sum_{k=1}^{n} F(\phi_k)^* \phi_k\big) F(\phi_j) = \sum_{k=1}^{n} |F(\phi_k)|^2 = \|\psi_n\|^2.$$

We show that this remains finite as $n \to \infty$. We do this by proving if it doesn't then the continuity of F is violated. To this end set $\chi_n := (1/\|\psi_n\|^2)\psi_n$. Then $\|\chi_n\| = 1/\|\psi_n\|$ and

$$F(\chi_n) = \frac{1}{\|\psi_n\|^2} F(\psi_n) = 1.$$

But $F(0) = 0$ because F is linear. Therefore continuity would be violated if $\chi_n \to 0$ as $n \to \infty$. Since F is continuous we cannot have that $\|\chi_n\|$ converge to zero and hence $\|\psi_n\|$ does not diverge as $\to \infty$. Therefore ψ_F exists.

$\square$

Theorem J.4.12 *Any two complete, orthonormal systems in a Hilbert space have the same cardinality.*

Proof: We need only consider the case where the space $\mathcal{H}$ is a non-separable.

Let $\mathfrak{m}$ and $\mathfrak{n}$ be any two cardinal numbers (finite or infinite). The statement that $\mathfrak{m} \leq \mathfrak{n}$ is defined to mean the following: if X and Y are sets with $\mathfrak{m}$ and $\mathfrak{n}$ elements, then there exits a one-to-one mapping of X into Y.

Suppose that two orthonormal systems M and N of cardinality $\mathfrak{m}$ and $\mathfrak{n}$ respectively are complete.

For every element $f \in N$ an element $e \in M$ can be found such that $(e, f) \neq 0$. Having found such an element e for the element f, we pick out from N all the elements which are not orthogonal to e. There can be no more than a countable set of these elements by the inequality

$$|(e, f')|^2 + |(e, f'')|^2 + |(e, f''')|^2 + \cdots \leq \|e\|^2,$$

which holds for any set of elements in N, for if there were more than a countable number of terms the summation would not converge. We enumerate the elements which are not orthogonal to e by

$$f_1, f_2, \ldots, f_n \qquad (1 \leq n \leq \infty).$$

We denote the set of these elements by S. Since the set N is not countable, an eleement f' can be found in it which is different from all the elements of the set S and which is therefore orthogonal to e. We choose an element e' of M for which $(e', f') \neq 0$, and then pick out from N all the elements which are not orthogonal to e' and which do not appear in S. Let us denote them as

$$f'_1, f'_2, \ldots, f'_n \qquad (1 \leq n' \leq \infty)$$

and the set of these elements by S'.

This process may be continued indefinitely - transfinitely - and brings us a collection of sets $S_\alpha \subset N$ which have the following properties:

a) each element of N appears in just one of the sets S_α;

b) each set S_α corresponds to some element $e_\alpha \in M$, and different sets S_α, S_β correspond to different elements e_α, e_β;

c) each set S_α contains at least one element and does not contain more than a countable set of elements.

Let $\mathfrak{p}$ be the cardinality of the collection of sets S_α (by property a), $\mathfrak{n}$ is the same as the cardinality of the index set α). There is a theroem of set theory which tells us that the cardinality of the collection of a disjoint class of sets, each of which is countably infinite, is equal to the cardinaity of the index set of the class. If each set S_α is infinite then,

$$\mathfrak{n} = \mathfrak{p}.$$

Since the equation $\mathfrak{n} = \mathfrak{p}$ clearly holds in the other extreme case where each set S_α contains just one element, it follows that it holds in general.

On the other hand, it is clear that

$$\mathfrak{p} \leq \mathfrak{m}.$$

Therefore

$$\mathfrak{n} \leq \mathfrak{m}.$$

Interchanging the roles of M and N, we find in the same way that

$$\mathfrak{m} \leq \mathfrak{n}.$$

Consequently

$$\mathfrak{m} = \mathfrak{n}.$$

$\square$

J.4.6 Projection Operators

From the defintion of a projection operator it is easy to see that

i) $P^2 = P$,

ii) $P^\dagger = P$.

Proof:

$\square$

Theorem J.4.13 *If P is an operator defined everywhere in $\mathcal{H}$, and such that, for all $h_1, h_2 \in \mathcal{H}$,*

i) $(P^2 h_1, h_2) = (P h_1, h_2)$,

ii) $(P h_1, h_2) = (h_1, P h_2)$,.

then there is a subspace $G \subseteq \mathcal{H}$ such that P is the operator of projection on to G.

Proof: The operator P is bounded. This follows from the Hellinger-Toeplitz theorem, but is proved more easily by a direct argument,

$$\|Ph\|^2 = (Ph, Ph) = (P^2 h, h) = (Ph, h),$$

and therefore

$$\|Ph\|^2 \leq \|Ph\| \, \|h\|$$

and so $\|Ph\| \leq \|h\|$, i.e., the operator P is bounded and its norm is not greater than 1.

Let G denote the set of all vectors $g \in \mathcal{H}$ for which

$$Pg = g$$

It is clear that G is a linear manifold, and we now show that it is closed, so that it is also a susbspace. Let $g_n \in G$ $(n = 1, 2, 3, \dots)$, so that

$$g_n = P g_n,$$

and therefore, for any $g \in \mathcal{H}$,

$$Pg - g_n = Pg - Pg_n = P(g - g_n),$$

from which

$$\|Pg - g_n\| \leq \|g - g_n\|.$$

We thus see that $g_n \to g$ implies that $Pg = g$, and this shows that G is closed.

For any $h \in \mathcal{H}$ the vector $Ph = g$ belongs to G because $P(Ph) = Ph$. The element $P_G h$ also belongs to G. So it suffices to prove that

$$(Ph - P_G h, g') = 0$$

or

$$(Ph, g') = (P_G h, g')$$

for all $g' \in G$. But this follows from the fact that

$$\begin{aligned} (Ph, g') &= (h, Pg') = (h, g') \\ (P_G h, g') &= (h, P_G g') = (h, g'), \end{aligned}$$

therefore $P = P_G$.

□

Theorem J.4.14 *If $\{P_k\}_{k=1}^{\infty}$ is an infinite monotonic sequence of projection operators, then as $k \to \infty$, P_k converges strongly to some projection operator P*

Proof:

$$\lim_{k \to \infty} P_k = P$$

the limit exists in the sense of strong convergence. Moreover, for any k and for any $f, g \in \mathcal{H}$,

$$(P_k f, P_k g) = (P_k f, g) = (f, P_k g).$$

Therefore in the limit $k \to \infty$

$$(Pf, Pg) = (Pf, g) = (f, Pg).$$

This implies $P = P^\dagger = P^2$, and so P is a projection operator.

□

Theorem J.4.15 *If a sequence $\{P_k\}_{k=1}^\infty$ of projection operators converges weakly to some projection operator P, then it also converges strongly to P*

Proof: By assumption, for any $h \in \mathcal{H}$,

$$(P_k h, h) \to (Ph, h).$$

Therefore

$$\|P_k h\| \to \|Ph\|,$$

and since the sequence of vectors $\{P_k h\}_{k=1}^\infty$ converges weakly to Ph, it must also converge strongly to Ph, by theorem ??.

□

Definition The **aperture** of two linear subspaces M_1, M_2 in $\mathcal{H}$ is the norm of the difference of the operators P_1, P_2 which project $\mathcal{H}$ on to the closure of these subspaces, denoted by $\theta(M_1, M_2)$. Thus

$$\theta(M_1, M_2) := \|P_1 - P_2\| \equiv \|P_2 - P_1\|.$$

□

Since for any $h \in \mathcal{H}$ the vectors $(\mathbb{I} - P_1)h$ and $P_1 h$ are orthogonal it is easy to see that the vectors $P_2(\mathbb{I} - P_1)h$ and $(\mathbb{I} - P_2)P_1 h$ are also orthogonal. So by the identity

$$P_2 - P_1 = P_2(\mathbb{I} - P_1) - (\mathbb{I} - P_2)P_1$$

it follows that

$$
\begin{aligned}
\|(P_2 - P_1)h\|^2 &= \|P_2(\mathbb{I} - P_1)h\|^2 + \|(\mathbb{I} - P_2)P_1 h\|^2 \\
&\le \|(\mathbb{I} - P_1)h\|^2 + \|P_1 h\|^2 \\
&= \|h\|^2.
\end{aligned}
$$

This inequality shows that the aperature of two subspaces does not exceed 1:

$$\theta(M_1, M_2) \le 1.$$

Moreover, we see that $\|(P_2 - P_1)h\|$ attains its maximum value of 1 if and only if one subspace contains a non-zero vector orthogonal to the other subspace. This fact is employed in the following theorem.

Theorem J.4.16 *If the aperture of two subspaces M_1, M_2 is less than 1, then the two subspaces have the same dimension:*

$$dim\, M_1 = dim\, M_2.$$

Proof: It suffices to show the inequality

$$\dim M_2 > \dim M_1.$$

implies that there is a non-zero vector in $\overline{M}_2$ which is orthogonal to $\overline{M}_1$. Consider the linear subspace

$$G = P_2 \overline{M}_1$$

Obviously

$$\dim G \le \dim \overline{M}_1 < \dim \overline{M}_2$$

Therefore there is a non-zero vector in $\overline{M}_2$ orthogonal to G. This vector will be orthogonal to the whole subspace $\overline{M}_1$, since the subspace $\overline{M}_2$ is orthogonal to $\overline{M}_1 \ominus G$.

□

An alternative defintion of the aperture of two linear subspaces M_1, M_2 is:

$$\theta(M_1, M_2) = \max\left\{ \sup_{f\in\overline{M}_2\ ,\ \|f\|=1} \|(\mathbb{I} - P_1)f\|, \sup_{g\in\overline{M}_1\ ,\ \|g\|=1} \|(\mathbb{I} - P_2)g\| \right\}. \tag{J.-58}$$

Proof:

Proof of (J.4.6). By the original defintion of the aperture,

$$\begin{aligned}
\theta(M_1, M_2) &= \sup_{h\in\mathcal{H}} \frac{\|(P_2 - P_1)h\|}{\|h\|} \\
&= \sup_{h\in\mathcal{H}} \frac{\sqrt{\|P_2(\mathbb{I} - P_1)h\|^2 + \|(\mathbb{I} - P_2)P_1h\|^2}}{\|h\|}.
\end{aligned} \tag{J.-58}$$

Say we allow h to only transverse $\overline{M}_1$ instead of the whole of $\mathcal{H}$, then the right-hand side of (J.-58) is either unchanged or is decreased, and so

$$\begin{aligned}
\theta(M_1, M_2) &\geq \sup_{h\in\overline{M}_1} \frac{\sqrt{\|P_2(\mathbb{I} - P_1)h\|^2 + \|(\mathbb{I} - P_2)P_1h\|^2}}{\|h\|}. \\
&= \sup_{h\in\overline{M}_1} \frac{\|(\mathbb{I} - P_2)h\|}{\|h\|} =: \rho_2.
\end{aligned}$$

Similarly it may be shown that

$$\theta(M_1, M_2) \geq \sup_{h\in\overline{M}_2} \frac{\|(\mathbb{I} - P_1)h\|}{\|h\|} =: \rho_1.$$

Thus

$$\theta(M_1, M_2) \geq \max\{\rho_1, \rho_2\}.$$

It remains to prove

$$\theta(M_1, M_2) \leq \max\{\rho_1, \rho_2\}.$$

Notice that, by definition of ρ_2,

$$\|(\mathbb{I} - P_2)P_1h\|^2 \leq \rho_2^2\|P_1h\|^2. \tag{J.-60}$$

On the other hand,

$$\begin{array}{rcl}\|P_2(\mathbb{I}-P_1)h\|^2 & = & (P_2(\mathbb{I}-P_1)h, P_2(\mathbb{I}-P_1)h) \\ & = & (P_2(\mathbb{I}-P_1)h, (\mathbb{I}-P_1)h) \\ & = & ((\mathbb{I}-P_1)P_2(\mathbb{I}-P_1)h, (\mathbb{I}-P_1)h) \\ & \leq & \|(\mathbb{I}-P_1)P_2(\mathbb{I}-P_1)h\| \cdot \|(\mathbb{I}-P_1)h\|\end{array}$$

and therefore by definition of ρ_1,

$$\|P_2(\mathbb{I}-P_1)h\|^2 \leq \rho_1 \|P_2(\mathbb{I}-P_1)h\| \cdot \|(\mathbb{I}-P_1)h\|$$

i.e.,

$$\|P_2(\mathbb{I}-P_1)h\| \leq \rho_1 \|(\mathbb{I}-P_1)h\|. \tag{J.-64}$$

From (J.4.6) and (J.4.6) we conclude that

$$\begin{array}{rcl}\|(\mathbb{I}-P_2)P_1h\|^2 + \|P_2(\mathbb{I}-P_1)h\|^2 & \leq & \rho_2^2\|P_1h\|^2 + \rho_1^2\|(\mathbb{I}-P_1)h\|^2 \\ & \leq & \max\{\rho_1,\rho_2\}\ [\|P_1h\|^2 + \|(\mathbb{I}-P_1)h\|^2] \\ & = & \|h\|^2 \cdot \max\{\rho_1,\rho_2\},\end{array}$$

and (J.-58) gives the required result

$$\theta(M_1, M_2) \leq \max\{\rho_1, \rho_2\}.$$

□

J.4.7 Spectral Decomposition

We now reexpress these results in a way more to make convenient the transition to infinite-dimension case. For a self-adjoint and unitary operators the eigenvalues are ordered in a natural way. For a self-adjoint operator this is so because the eigenvalues are real numbers and for a unitary operators the eigenvalues are represented by points on the unit circle of the complex plane.

Let us first consider self-adjoint operators. We assume that $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, and we use the P_i's to define new projections:

$$
\begin{aligned}
E_{\lambda_0} &= 0 \\
E_{\lambda_1} &= P_1 \\
E_{\lambda_2} &= P_1 + P_2; \\
&\vdots \\
E_{\lambda_m} &= P_1 + P_2 + \cdots + P_m.
\end{aligned} \tag{J.-70}
$$

$$
\begin{aligned}
A &= \lambda_1 P_1 + \lambda_2 P_2 + \cdots \lambda_m P_m \\
&= \lambda_1(E_{\lambda_1} - E_{\lambda_0}) + \lambda_2(E_{\lambda_2} - E_{\lambda_1}) + \cdots \lambda_m(E_{\lambda_m} - E_{\lambda_{m-1}}) \\
&= \sum_{i=1}^{m} \lambda_i (E_{\lambda_i} - E_{\lambda_{i-1}}).
\end{aligned} \tag{J.-71}
$$

$$
A = \sum_{i=1}^{m} \lambda_i \Delta E_{\lambda_i} \tag{J.-71}
$$

Proof:

□

Theorem J.4.17 *For each self-adjoint operator there is a unique spectral family of projection operators P_x such that*

Proof:

□

Theorem

$$
(\phi, A\varphi) = \int_{\infty}^{\infty} x d(\phi, P_x \varphi) \tag{J.-71}
$$

for all vectors φ and ϕ.

The spectral decomposition of A

$$
A = \int_{-\infty}^{\infty} dE_x \tag{J.-71}
$$

$$
\begin{aligned}
(\phi, \psi) &= (\phi, P_1\psi) + (\phi, P_2\psi) + \cdots (\phi, P_m\psi) \\
&= (\phi, (E_{\lambda_1} - E_{\lambda_0})\psi) + (\phi, (E_{\lambda_2} - E_{\lambda_1})\psi) + \cdots (\phi, (E_{\lambda_m} - E_{\lambda_{m-1}})\psi) \\
&= \sum_{i=1}^{m} (\phi, \Delta E_{\lambda_i}\psi). \qquad \text{(J.-72)}
\end{aligned}
$$

$$
\begin{aligned}
(\phi, A\psi) &= (\phi, \lambda_1 P_1\psi) + (\phi, \lambda_2 P_2\psi) + \cdots (\phi, \lambda_m P_m\psi) \\
&= (\phi, (E_{\lambda_1} - E_{\lambda_0})\psi) + (\phi, (E_{\lambda_2} - E_{\lambda_1})\psi) + \cdots (\phi, (E_{\lambda_m} - E_{\lambda_{m-1}})\psi) \\
&= \sum_{i=1}^{m} \lambda_i (\phi, \Delta E_{\lambda_i}\psi). \qquad \text{(J.-73)}
\end{aligned}
$$

for any vectors φ and ϕ we have

$$
(\phi, \psi) = \int_{-\infty}^{\infty} d(\phi, E_x\psi) \qquad \text{(J.-73)}
$$

and

$$
(\phi, A\psi) = \int_{-\infty}^{\infty} x d(\phi, E_x\psi). \qquad \text{(J.-73)}
$$

Unitary operators can be treated in a similarly. For a unitary operator U let its eigenvalues be $u_i = e^{i\theta_i}$ labelled in the order

$$
0 < \theta_1 < \theta_2 < \cdots < \theta_{m-1} < \theta_m \leq 2\pi \qquad \text{(J.-73)}
$$

For each real number x let

$$
E_x = \sum_{\theta_i \leq x} P_i \qquad \text{(J.-73)}
$$

This is the progection operator onto the space spanned by all eigenvectors for eigenvalues $e^{i\theta_i}$ with $\theta_i \leq x$. If $x \leq 0$, then $E_x = 0$. If $x \geq 2\pi$, then $E_x = 1$. Evidently E_x increses by incriments P_i the same as for the Hermitian operator with eigenvalues θ_i. For

$$
U = \sum_{i=1}^{m} u_i P_i = \sum_{i=1}^{m} e^{i\theta_i} P_i \qquad \text{(J.-73)}
$$

we can write

$$U = \int_0^{2\pi} e^{ix} dE_x. \tag{J.-73}$$

For any vectors ψ and ϕ we have

$$(\phi, U\psi) = \int_0^{2\pi} e^{ix} d(\phi, E_x\psi). \tag{J.-73}$$

Theorem J.4.18 *Let A be a self-adjoint operator with spectral decomposition*

$$A = \int_{-\infty}^{\infty} x dE_x.$$

Then E_x jumps in value at $x = a$ if and only if a is an eigenvalue of A.

Proof: First we prove that if there is a discontinuity from the right at $x = a$ that a is an eigenvalue. Take $\epsilon < \epsilon' < 0$ and any vector ψ,

$$\begin{aligned}
\|(E_a - E_{a-\epsilon})\psi - (E_a - E_{a-\epsilon'})\psi\|^2 &= \|(E_a - E_{a-\epsilon})\psi\|^2 + \|(E_a - E_{a-\epsilon'})\psi\|^2 \\
&\quad -\big(\psi, (E_a - E_{a-\epsilon})(E_a - E_{a-\epsilon'})\psi\big) \\
&\quad -\big(\psi, (E_a - E_{a-\epsilon'})(E_a - E_{a-\epsilon})\psi\psi\big) \\
&= \|(E_a - E_{a-\epsilon})\psi\|^2 + \|(E_a - E_{a-\epsilon'})\psi\|^2 \\
&\quad -2\big(\psi, (E_a - E_{a-\epsilon'})\psi\big) \\
&= \|(E_a - E_{a-\epsilon})\psi\|^2 - \|(E_a - E_{a-\epsilon'})\psi\|^2
\end{aligned}$$

because $E_a - E_{a-\epsilon}$ and $E_a - E_{a-\epsilon'}$ are projection operators and $E_a - E_{a-\epsilon} \geq E_a - E_{a-\epsilon'}$. This shows that $\|(E_a - E_{a-\epsilon})\psi\|^2$ is monotonically decreasing as $\epsilon \to 0$ and by ... shows that the vectors $(E_a - E_{a-\epsilon})\psi$ have the Cauchy property. Because a Hilbert space is complete by definition, $(E_a - E_{a-\epsilon})\psi$ must converge to a limit vector as $\epsilon \to 0$. We denote the limit vector by ψ_a. We are supposing that there is a discontinuity and hence take ψ_a to be non-zero. If $x \geq a$, then

$$E_x(E_a - E_{a-\epsilon}) = E_xE_a - E_xE_{a-\epsilon} = E_a - E_{a-\epsilon}.$$

If $x < a$, then for ϵ small enough that $x < a - \epsilon$

$$E_x(E_a - E_{a-\epsilon}) = E_xE_a - E_xE_{a-\epsilon} = E_x - E_x = 0.$$

Therefore $E_x\psi_a = 0$ for $x < a$ and $E_x\psi_a = \psi_a$ for $x \geq a$, so

$$\begin{aligned}
(\phi, A\psi_a) &= \int_{-\infty}^{\infty} x d(\phi, E_x\psi_a) \\
&= \int_{-\infty}^{\infty} x \frac{d}{dx}(\phi, E_x\psi_a) dx \\
&= \int_{-\infty}^{\infty} x(\phi, E_x\psi_a)\delta(x-a) dx \\
&= (\phi, a\psi_a)
\end{aligned}$$

for any vector ϕ. Putting $\phi = A\psi_a - a\psi_a$ we have

$$(A\psi_a - a\psi_a, A\psi_a - a\psi_a) = \|A\psi_a - a\psi_a\|^2 = 0$$

so

$$A\psi_a = a\psi_a.$$

Thus a is an eigenvalue and ψ_a is an eigenvector.

Now we prove that if a is an eigenvalue that E_x jumps in value at $x = a$. For any vectors ψ and ϕ we have

$$
\begin{aligned}
(\phi, A^2\psi) &= (\phi, A(A\psi)) \\
&= \int_{-\infty}^{\infty} xd(\phi, E_x A\psi) \\
&= \int_{-\infty}^{\infty} xd(E_x\phi, A\psi) \\
&= \int_{-\infty}^{\infty} xd_x \int_{-\infty}^{\infty} yd_y(E_x\phi, E_y\psi) \\
&= \int_{-\infty}^{\infty} xd_x \int_{-\infty}^{\infty} yd_y(\phi, E_x E_y\psi) \\
&= \int_{-\infty}^{\infty} xd_x \int_{-\infty}^{x]} yd_y(\phi, E_x E_y\psi) + \int_{-\infty}^{\infty} xd_x \int_{(x}^{\infty} yd_y(\phi, E_x E_y\psi) \\
&= \int_{-\infty}^{\infty} xd_x \int_{-\infty}^{x]} yd_y(\phi, E_y\psi) + \int_{-\infty}^{\infty} xd_x \int_{(x}^{\infty} yd_y(\phi, E_x\psi) \\
&= \int_{-\infty}^{\infty} x^2 d(\phi, E_x\psi)
\end{aligned}
$$

because $E_x E_y = E_y$ for $y \leq x$, and for $y > x$

$$(\phi, E_x E_y) = (\phi, E_x\psi)$$

which is independent of y.

By assumption a is an eigenvalue of A and let I_a be the projection onto the subspace of eigenvectors. Then

$$\int_{-\infty}^{\infty} (x^2 - 2ax - a^2) d(\phi, E_x I_a \psi) = (\phi, [A^2 - 2aA + a^2] I_a \psi) = 0$$

for any vectors ψ and ϕ. For $\phi = I_a\psi$ we have

$$(\phi, E_x I_a \psi) = (I_a\psi, E_x^2 I_a \psi) = \|E_x I_a \psi\|^2$$

and

$$\int_{-\infty}^{\infty} (x-a)^2 d\|E_x I_a \psi\|^2 = 0.$$

Because $\|E_x I_a \psi\|^2$ is a monotonically increasing function of x, we see that it can only change value at $x = a$. But $\|E_x I_a \psi\|^2 \to 0$ as $x \to -\infty$ and $\|E_x I_a \psi\|^2 \to \|I_a \psi\|^2$ as

$x \to +\infty$. Therefore $\|E_x I_a \psi\|^2 = 0$ if $x < a$, and $\|E_x I_a \psi\|^2 = \|I_a \psi\|^2$ if $x \geq a$ for any vector ψ. This implies that $E_x I_a = 0$ for $x < a$ and $E_x I_a = I_a$ for $x \geq a$. For any vector ψ we have

$$(E_a - E_{a-\epsilon}) I_a \psi = (E_a I_a - E_{a-\epsilon} I_a) = I_a \psi.$$

Thus E_x jumps in value at $x = a$.

$\square$

Dirac Notation

A vector $|\Psi> \in \mathcal{H}$ is called a **ket** and to this vector we can associate a linear functional $\omega_{|\Psi>} \equiv <\Psi|$ called a **bra** and defined by means of a scalar product. This acts on elements of the Hilbert space to give a complex number.

$$\omega_{|\Psi>} : \mathcal{H} \to C \tag{J.-91}$$

and for typical ket Φ we write

$$\omega_{|\Psi>}(|\Phi>) < |\Psi>, |\Phi>>_{\mathcal{H}} \equiv <\Psi|\Phi> . \tag{J.-91}$$

Conversely, to each bra $<\Psi| \in \mathcal{H}^*$ we can associate a ket $|\Psi> \in \mathcal{H}$. Moreover, by virtue of the Theorem ?? just proven, every element $\omega \in \mathcal{H}^*$ uniquely determines a vector $\Psi_\omega \in \mathcal{H}$ such that

$$\omega(|\Phi>) = <\Psi_\omega|\Phi> \quad \text{for all } \ |\Phi> \in \mathcal{H}. \tag{J.-91}$$

The vector

we thus have a one-to-one correspondence between $|\Psi> \in \mathcal{H}$ and $<\Psi| \in \mathcal{H}^*$. We can identify (isometry)

• **Every ket has a corresponding bra**

If one defines the norm of $\omega \in \mathcal{H}^*$ by $||\omega|| := \sup |\omega(f)|$, where the supremum is taken over all unit vectors $f \in \mathcal{H}$, then .... one-to-one and *onto* i.e. a **bijection** $\mathcal{H} \to \mathcal{H}^*$ is antilinear and is norm-preserving, that is it represents an **isometry**.

The idea of an operator $\hat{A}$ is something tht acts on a ket $|\psi>$ and transforms it into another ket $|\psi'>$ of the same space and when it acts on a bra it transforms it into another bra:

$$\hat{A}|\psi >= |\psi' >, \quad < \psi|\hat{A} =< \psi'|. \tag{J.-91}$$

The introduction of a dual vector space is only necessary for generalized vectors.

J.5 Distributions (or Generalized Functions)

J.5.1 Introduction

The Dirac delta-function can not be defined as a function but as a functional on a suitable space of functions.

We note that distributions play an essential role in in quantum field theory, where quantum fields are defined as operator-valued distributions. According to the LQG approach, the surface states of a black hole, which are the statistical mechanical origin of black hole entropy, must be the pull-back of distributional fields of the bulk spacetime.

J.5.2 Mathematical Formulism of Quantum Mechanics

Schwartz space

$$h_\alpha = \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}} Z^{*\alpha} h_0$$

$$\begin{aligned} Z_j^* f(x) &= x_j f(x) - \frac{1}{2\pi} \frac{\partial f}{\partial x_j} \\ &= -\frac{1}{2\pi} e^{\pi x^2} \frac{\partial}{\partial x_j} (e^{-\pi x^2} f(x)) \end{aligned}$$

so that

$$\begin{aligned} Z_j^{*2} f(x) &= \left(x_j - \frac{1}{2\pi} \frac{\partial}{\partial x_j} \right) \frac{-1}{2\pi} e^{\pi x^2} \frac{\partial}{\partial x_j} (e^{-\pi x^2} f(x)) \\ &= \left(-\frac{1}{2\pi} \right)^2 e^{\pi x^2} \left(\frac{\partial}{\partial x_j} \right)^2 (e^{-\pi x^2} f(x)) \end{aligned}$$

and

$$Z_j^{*\alpha} f(x) = \left(-\frac{1}{2\pi}\right)^{|\alpha|} e^{\pi x^2} \left(\frac{\partial}{\partial x}\right)^{\alpha} (e^{-\pi x^2} f(x))$$

explicityly

$$h_\alpha(x) = \frac{2^{n/4}}{\sqrt{\alpha!}} \left(\frac{-1}{2\sqrt{\pi}}\right)^{|\alpha|} e^{\pi x^2} \left(\frac{\partial}{\partial x}\right)^{\alpha} e^{-\pi x^2} \tag{J.-95}$$

The eigenfunctions of $\mathcal{F}$ are obtained by taking tensor products of the one dimensional Hermite functions.

For each $\alpha \in \mathbb{N}^n$ we define

$$\Phi_\alpha(x) = \varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2)\ldots\varphi_{\alpha_n}(x_n).$$

Since

$$\begin{aligned} [Z_j, Z_k^*]f(x) &= \Big[x_j + \frac{1}{2\pi}\frac{\partial}{\partial x_j}, x_k - \frac{1}{2\pi}\frac{\partial}{\partial x_k}\Big] f(x) \\ &= \pi^{-1}\delta_{jk} \end{aligned}$$

we obviouslsy have

$$[Z_j, Z^{*\alpha}] = \pi^{-1}\alpha_i Z^{*(\alpha-1_j)}$$

$$Z_j h_0 = 2^{n/4} Z_j e^{\pi x^2} = 2^{n/4}(x_j + \frac{1}{2\pi}\frac{\partial}{\partial x_j})e^{-\pi x^2} = 0$$

Lemma J.5.1 *$\{h_\alpha\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.*

Proof: We have

$$(h_\alpha, h_\beta) = \sqrt{\frac{\pi^{|\alpha|+|\beta|}}{\alpha!\beta!}}(h_0, Z^\alpha Z^{*\beta} h_0).$$

Consider $\alpha_j > \beta_j$

$$\begin{aligned}
Z_j^{\alpha_j} Z_j^{*\beta_j} h_0 &= Z_j^{\alpha_j - 1} Z_j^{*\beta_j} Z_j h_0 + \pi^{-1}\alpha_j Z_j^{\alpha_j-1} Z_j^{*\beta_j - 1} h_0 \\
&= \pi^{-2}\alpha_j(\alpha_j - 1) Z_j^{\alpha_j-2} Z_j^{*\beta_j-2} h_0 \\
&= \pi^{-\beta_j}\alpha_j(\alpha_j-1)\dots(\alpha_j-\beta_j+1) Z_j^{\alpha_j-\beta_j} h_0 = 0
\end{aligned}$$

If $\alpha_j = \beta_j$

$$Z_j^{\alpha_j} Z_j^{*\alpha_j} h_0 = \pi^{-\alpha_j}\alpha_j! h_0$$

So we have

$$(h_\alpha, h_\beta) = \delta_{\alpha\beta}\|h_0\|_2^2 = \delta_{\alpha\beta}.$$

Now we prove completeness. If $g \in L^2(\mathbb{R}^n)$ and

$$(g, h_\alpha) = 0 \quad \text{for all} \quad \alpha,$$

Now the function $H_n(x) = e^{\pi x^2} h_\alpha(x)$ is a polynomial of degree $|\alpha|$ called the αth Hermite polynomial. Obviously, every polynomial of degree less than equal to k on $\mathbb{R}^n$ is some linear combination of Hermite polynomials of degree less than equal to k, hence the condition $(g, h_\alpha) = 0$ for all α is equivalent to $(g, P(x)e^{-\pi x^2}) = 0$ for all polynomials P. But it then follows that

$$\int g(x)e^{-\pi x^2} e^{2\pi i x\zeta} dx = \sum_{j=1}^{\infty} \int g(x) e^{-\pi x^2} \frac{(2\pi i x\zeta)^j}{j!} dx = 0.$$

By the Fourier uniqueness, $g(x)e^{-\pi x^2} = 0$ a.e., and hence $g = 0$.

□

$$\begin{aligned}
\mathcal{F} Z_j^* f &= \int dx e^{2\pi i k x} (X_j - iD_j) f(x) \\
&= \int dx e^{2\pi i k x} (x_j - \frac{1}{2\pi}\frac{d}{dx_j}) f(x) \\
&= (i\frac{1}{2\pi}\frac{d}{dk_j} - ik_j) \int dx e^{2\pi i k x} f(x) \\
&= -i Z_j^* \mathcal{F} f
\end{aligned}$$

Recalling

$$\mathcal{F}h_0 = h_0$$

$$\begin{aligned}\mathcal{F}h_\alpha &= \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}}\mathcal{F}Z^{*\alpha}h_0 \\ &= (-i)^{|\alpha|}\sqrt{\frac{\pi^{|\alpha|}}{\alpha!}}Z^{*\alpha}h_0 \\ &= (-i)^{|\alpha|}h_\alpha\end{aligned}$$

Then it follows that

$$\mathcal{F}\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha$$

where $|\alpha| = \sum_{j=1}^n \alpha_j$. Moreover, $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Thus every $f \in L^2(\mathbb{R}^n)$ has the expansion

$$f = \sum_{\alpha\in\mathbb{N}^n}(f, \Phi_\alpha)\Phi_\alpha$$

the series being convergent in the L^2 norm.

Definition A subspace W of $L^2(\mathbb{R}^n)$ is said to be invariant under $\mathcal{F}$ if $\mathcal{F}f \in W$ whenever $f \in L^2(\mathbb{R}^n)$.

$\square$

As $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ it follows that $f \in L^2(\mathbb{R}^n)$ if and only if

$$\sum_{\alpha\in\mathbb{N}^n} |(f, \Phi_\alpha)|^2 < \infty.$$

Since

$$(\mathcal{F}f, \Phi_\alpha) = (-i)^{|\alpha|}(f, \Phi_\alpha)$$

any subspace of $L^2(\mathbb{R}^n)$ defined in terms of its behaviour of $|(f, \Phi_\alpha)|$ will be invariant under the Fourier transform. We can define a whole family of invariant subspaces. Indeed, for each $s > 0$ define $W_H^s(\mathbb{R}^n)$ to be a subspace of $L^2(\mathbb{R}^n)$ consisting of functions f for which

$$\|f\|_{2,s}^2 = \sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^2 |(f, \Phi_\alpha)|^2 < \infty.$$

Definition Let $\mathcal{S}(\mathbb{R}^n) = \cap_{s>0} W_H^s(\mathbb{R}^n)$. This is called the Schwartz space and its members are called Schwartz functions.

□

Theorem J.5.2 *The Schwartz space*

(i) is a dense subspace of $L^2(\mathbb{R}^n)$*;*

(ii) $\mathcal{S}(\mathbb{R}^n)$ *is invariant under* $\mathcal{F}$*;*

(iii) $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is one to one and onto.*

Proof: The density follows from the fact that finite linear combinations of Hermite functions form a subspace of $\mathcal{S}(\mathbb{R}^n)$ which is dense in $L^2(\mathbb{R}^n)$.

The inverse follows from that of each $W_H^s(\mathbb{R}^n)$.

As $\mathcal{F}(\mathcal{F}^* f) = f$ surjectivity follows.

□

The original motivation for RHS was to provide a rigorous mathematical context for Dirac's bra-and-ket formulation of quantum mechanics.

$$\sqrt{\frac{n}{\pi}} e^{-nx^2} \quad \text{for } n = 1, 2, \ldots \tag{J.-107}$$

$$\lim_{n\to\infty} D_n(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{J.-107}$$

$$\int_{-\infty}^{\infty} D_n(x) dx = 1 \tag{J.-107}$$

for all n

Dirac formulism on concrete mathematical

A distribution $\chi(x)$ is the limit of a sequence of good functions $h(x)$ such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h_n(x)g(x)dx := \int_{-\infty}^{\infty}\chi(x)g(x)dx \tag{J.-107}$$

exists i.e. $< \infty$. Lebesgue integral?????

The product of two generalized functions is not defined. the convergence of

$$\int_{-\infty}^{\infty} a_n(x)g(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} a_n(x)g(x)dx \tag{J.-107}$$

as $n \to \infty$, does not imply the convergence of

$$\int_{-\infty}^{\infty} [a_n(x)b_n(x)]g(x)dx \tag{J.-107}$$

products of distributional valued operators in Field theory need to be regularized to make mathematical sence.

Theorem Let the sequnce $f_n(x)$ $(n = 1, 2, \dots$ define a generalized function $\varphi(x)$. Then the sequence of Fourier transforms of the sequence $f_n(x)$

$$F_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_n(x)dx \tag{J.-107}$$

defines a generalized function $\Phi(t)$, which is called the Fourier transform of $\varphi(x)$

$$\int_{-\infty}^{\infty} F_n(t)g(t)dt \tag{J.-107}$$

Let

$$g(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x)e^{itx}dt \tag{J.-107}$$

$$\int_{-\infty}^{\infty} F_n(t)g(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_n(x)g(t)e^{itx}dtdx \tag{J.-107}$$

and also

$$\int_{-\infty}^{\infty} f_n(x)G(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_n(x)g(t)e^{itx}dtdx \tag{J.-107}$$

Since the RHS of exist, we have

$$\int_{-\infty}^{\infty} F_n(t)g(t)dt = \int_{-\infty}^{\infty} f_n(x)G(x)dx \tag{J.-107}$$

Formally

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(x)e^{-itx}dt \tag{J.-107}$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Phi(t)e^{itx}dt \tag{J.-107}$$

$$G(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-itx}dx \tag{J.-107}$$

a derivative of the mth order of $G(t)$ is

$$\frac{d^mG(t)}{dt^m} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-ix)^m g(x)e^{-itx} \tag{J.-107}$$

$$\begin{aligned}\left|\frac{d^mG(t)}{dt^m}\right| &= \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-ix)^m g(x)e^{-itx}dx\right| \\ &\le \frac{1}{\sqrt{2\pi}}\frac{1}{|t|^n}\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}[x^m g(x)]\right|dx \end{aligned} \tag{J.-107}$$

Dirac δ function

$$D_n(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2} \quad (n = 1, 2, \dots) \tag{J.-107}$$

for a good function $g(x)$,

$$\int_{-\infty}^{\infty} \delta(x)g(x)dx = g(0) \tag{J.-107}$$

$$\delta(y(x)) = \sum_{i=1} \frac{\delta(x - x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}} \tag{J.-107}$$

We define distributions to be abstract linear functionals defined on Schwartz space, i.e. linear maps from $\mathcal{S}(\mathbb{R}^n)$ to the complex numbers. The delta function is then the linear functional

$$\phi \mapsto \phi(0).$$

The Fourier transform has a natural extension to $\mathcal{S}'(\mathbb{R}^n)$. If f is a Schwartz function, then

$$(\mathcal{F}u, f) = \int\int f(\zeta)e^{ix\cdot\zeta}\phi(x)dxd\zeta = (u, \mathcal{F}f)$$

Thus we define, for any distribution, the Fourier transform $\mathcal{F}u$ by

$$(\mathcal{F}u, f) = (u, \mathcal{F}f)$$

where the brackets here stands for the action of a tempered distribution on a Schwartz function.

$\delta-$sequences

convolutions of $\delta-$sequences.

$$\sigma_{m,n}(x) := \int_{-\infty}^{\infty} \alpha_n^*(t)\beta_m(x - t)dt \tag{J.-107}$$

$$\lim_{m,n\to\infty} \int_{-\infty}^{\infty} \sigma_{m,n}(x)g(x)dx = g(0)? \tag{J.-107}$$

$$\begin{aligned}
\lim_{m,n\to\infty} \int_{-\infty}^{\infty} \sigma_{m,n}(x)g(x)dx &= \lim_{m,n\to\infty} \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} \alpha_n^*(t)\beta_m(x - t)dt)\ g(x)dx \\
&= \lim_{m\to\infty} \int_{-\infty}^{\infty} \left(\lim_{n\to\infty} \int_{-\infty}^{\infty} \alpha_n^*(t)\beta_m(x - t)dt\right)\ g(x)dx \\
&= \lim_{m\to\infty} \int_{-\infty}^{\infty} \beta_m(x)g(x)dx \\
&= g(0).
\end{aligned} \tag{J.-109}$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty}\lim_{n\to\infty} f_n(x)dx \qquad \text{(J.-109)}$$

where

$$f_n(x) := g(x)\int_{-\infty}^{\infty} \alpha_n^*(t)\beta_m(x-t)dt \qquad \text{(J.-109)}$$

Is $f_n(x)$ uniformally continuous.

J.5.3 Gel'fand triple

Most of the operators one wants to use are in practice are defined on a common dense domain, S, consisting of the space of infinetly differentiable wavefunctions, ψ, for which $\hat{x}^j\hat{p}^k\psi$ is normalizable for all positive integers j and k. This actually too small to be of much use, but it has a very large dual space, S', called the space of *tempered distributions*, to which most interesting operators can be transposed.

J.6 Linear Operators

We have seen that in a particular basis $\mathbf{e}_i$ linear operators can be described in terms of their components with respect ot the basis. These components can be displayed as an array of numbers - a matrix.

A *linear operator* A on a vector space assigns to each vector ψ a vector $A\psi$ such that

$$A(\psi+\phi) = A\psi + A\phi \qquad \text{(J.-109)}$$

and

$$A(a\psi) = aA\psi \qquad \text{(J.-109)}$$

for any vectors ψ and ϕ and a scalar a.

Linear operators form a with sum $A+B$ of two operators A and B and a scalar multiple aA of an operator A and a scalar A defined by

$$(A+B)\psi = A\psi + B\psi \qquad \text{(J.-109)}$$

and

$$(aA)\psi = aA\psi \tag{J.-109}$$

for every vector ψ.

J.6.1 Mappings

Let X and Y be sets and $A \subset X$ any subset. A mapping T from A into Y is defined by an association with each $x \in A$ a single $y \in Y$, written and called the image of x with respect to T.

Definition The set A is called the domain of defintion of T or just the **domain** of T and is denoted $D(T)$, and we write

□

$$\begin{aligned} T : D(T) &\rightarrow Y \\ x &\mapsto Tx \end{aligned}$$

Definition The **range** $\mathrm{Ran}(T)$ of T is the set of all images, that is,

$$\mathrm{Ran}(T) = \{y \in Y : y = Tx \text{ for some } x \in D(T)\}.$$

□

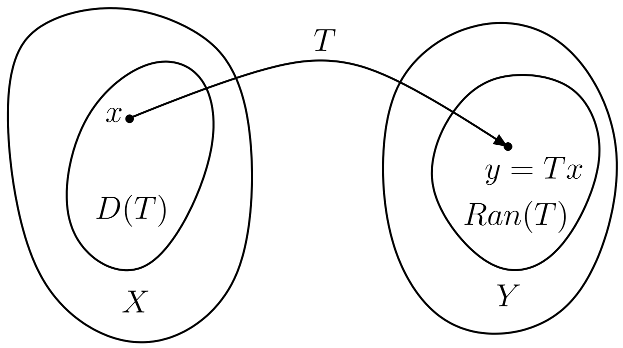


Figure J.4: Visualization of a mapping.

Definition The **inverse image** of a $y_0 \in Y$ is the set of all $x \in D(T)$ such that $Tx = y_0$.

$\square$

Definition The mapping T^{-1} is **injective**, an **injection**, or **one-to-one** if for every $x_1, x_2 \in D(T)$,

$$x_1 \neq x_2 \qquad \text{implies} \qquad Tx_1 \neq Tx_2;$$

that is, different points in $D(T)$ have different images, so that the inverse image of any point in $\mathrm{Ran}(T)$ is a single point.

$\square$

If a mapping is injective different points in the domain have different images, that is, if for any $x_1, x_2 \in D(T)$,

$$x_1 \neq x_2 \qquad \Rightarrow \qquad Tx_1 \neq Tx_2; \tag{J.-111}$$

equivalently,

$$Tx_1 = Tx_2 \qquad \Rightarrow \qquad x_1 = x_2. \tag{J.-111}$$

In this case there exists a maping $T^{-1} : \mathrm{Ran}(T) \to D(T)\ y_0 \mapsto x_0$, which maps every $y_0 \in \mathrm{Ran}(T)$ onto that $Tx_0 = y_0$.

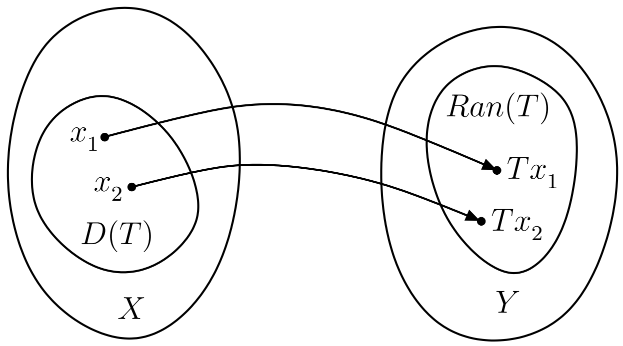


Figure J.5: Injective mapping.

Definition The mapping T^{-1} is **surjective**, a **surjection**, or a mapping of $D(T)$ **onto** Y if $\mathrm{Ran}(T) = Y$. Clearly,

$$\begin{array}{rcl} D(T) & \to & \mathrm{Ran}(T) \\ x & \mapsto & Tx \end{array}$$

$\square$

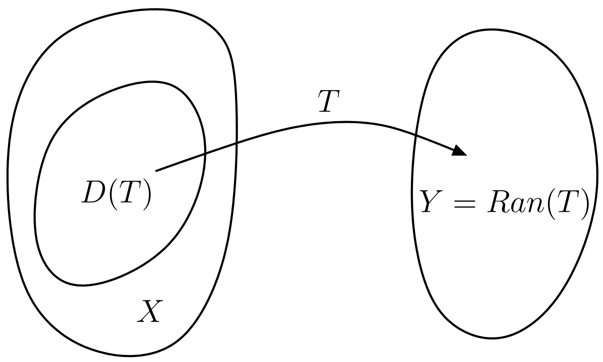


Figure J.6: Surjective mapping.

Definition The mapping T^{-1} is called the **inverse** of T.

$\square$

Theorem J.6.1 *Let X and Y be vector spaces. Let $T : D(T) \to Y$ be a linear operator with domain $D(T) \subset X$ and range $Ran(T) \subset Y$. Then*

The inverse $T^{-1} : Ran(T) \to D(T)$ exists if and only if

$$Tx = 0 \qquad \Rightarrow \qquad x = 0.$$

Proof: Suppose that $Tx = 0$ implies $x = 0$. Let $Tx_1 = Tx_2$. Since T is linear,

$$T(x_1 - x_2) = Tx_1 - Tx_2 = 0,$$

so that $x_1 - x_2 = 0$ by assumption. Hence $Tx_1 = Tx_2$ implies $x_1 = x_2$, and T^{-1} exists by (J.6.1).

Conversely, if T^{-1} exists, then by (J.6.1) holds. Putting $x_2 = 0$ in (J.6.1) we obtain

$$Tx_1 = T0 = 0 \qquad \Rightarrow \qquad x_1 = 0.$$

□

J.6.2 Bounded Operators

Definitions. A linear operator is *continuous* if $A\psi_n \to A\psi$ for any sequence of vectors ψ_n which converge to a limit vector ψ. A linear operator is *bounded* if there is a positive number b such that $||A|| \leq b||\psi||$ for every vector ψ ; the smallest number with this property is called the *norm* of A and is denoted by $||A||$.

Theorem A linear operator is continuous if and only if it is bounded.

Proof. Let A be a bounded linear operator. If the sequqnce of vectors ψ_n converges to a limit vector ψ, then

$$||A\psi - A\psi_n|| = ||A(\psi - \psi_n)|| \leq ||A||\ ||\psi - \psi_n|| \to 0 \qquad \text{(J.-113)}$$

so $A\psi_n \to A\psi$ as $n \to \infty$. Thus A is continuous.

To complete the proof we suppose a linear operator A is not bounded and show that it is not continuous. As A is not bounded there does not exist a number b such that $||A|| \leq b||\phi||$ for every vector ϕ. Therefore for each positive integer n there must be a vector ψ_n such that $||A\psi_n|| > n||\psi_n||$. Let $\chi_n = (1/n||\psi_n||)\psi_n$. Then $||\chi_n|| = 1/n$ so $\chi_n \to 0$ as $n \to \infty$. But $||A\chi_n|| > 1$ so $A\chi_n \not\to o$ as $n \to \infty$. Thus A is not continuous.

Adjoints

The *adjoint* of a bounded linear operator A is defined by letting

$$(\phi, A^\dagger\psi) = (A\phi, \psi) \qquad \text{(J.-113)}$$

for all vectors ψ and ϕ. For each vector ψ a linear functional F is defined by $F(\phi) = (\psi, A\phi)$. This linear functional is continuous because A is bounded; if a sequence of vectors χ_n converges to a limit vector χ, then

$$|F(\chi) - F(\chi_n)| = |(\psi, A[\chi - \chi_n])| \leq ||\psi||\ ||A||\ ||\chi - \chi_n|| \to 0 \qquad \text{(J.-113)}$$

as $n \to \infty$. This implies (by theorem) that there exists a unique vector, which we call $A^\dagger\psi$, such that $F(\phi) = (A^\dagger\psi, \phi)$ or $(A^\dagger\psi, \phi) = (\psi, A\phi)$.

That $A^\dagger$ is a linear operator is evident form the definition. It is also bounded.

$$\hat{A}(a_1|\psi_1 > +a_2|\psi_2 >) = a_1\hat{A}|\psi_1 > +a_2\hat{A}|\psi_2 >, \qquad \text{(J.-113)}$$

J.7 Three Main Theorems in Functional Analysis

Hahn-Banach theorem guarantees there exists a linear functional over the whole vector space.

J.7.1 Uniform Boundedness Theorem

Theorem J.7.1 (Uniform boundedness Theorem) *Let X be a Banach space and Y be a normed space. Let the family $\{A_i\}_{i\in I} \subset L(X,Y)$ be pointwise bounded, i.e. for every $x \in X$ there exists a constant $C(x) > 0$ such that $\|A_i x\|_Y \leq C(x)$ for all $i \in I$. The family of norms $\{\|\{A_i\|\}_{i\in I}$ is bounded.*

J.7.2 The Open Mapping Theorem

Theorem J.7.2 *Let F and G be two Banach spaces. A continuous linear map $L :\to F$ which takes an open subset to an open subset. If it is continuous and one to one, it is a homeomorphism*

J.7.3 The Hahn-Banach Theorem

dual spaces. guarantees rich supply of functionals for Banach (and normed) spaces

The closest point property

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \qquad \text{(J.-113)}$$

Definition: A subset $A$of a real or complex vector space is *convex* for if, for all$a, b \in A$ and all λ such that $0 < \lambda < 1$, the point $\lambda a + (1-\lambda)b$ belongs to A.

There is a unique point in $y \in A$ such that

$$\|x-y\| = \inf_{a\in A} \|x-a\| \qquad \text{(J.-113)}$$

Theorem: (The closest point property)

Let

$$M = \inf_{a \in A} \|x - a\| \tag{J.-113}$$

$$\|x - y_n\|^2 < M^2 + \frac{1}{n}. \tag{J.-113}$$

We are interested in establishing an inequality for the quantity $\|y_n - y_m\|$ when we know that $\|x - y_n\|$

$$\begin{aligned}
\|y_n - y_m\|^2 + \|2x - y_n - y_m\|^2 &\equiv \|x - y_n - (x - y_m)\| + \|x - y_n + (x - y_m)\| \\
&= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 \\
&< 4M^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right)
\end{aligned} \tag{J.-114}$$

$$\|y_n - y_m\|^2 < 4M^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right) - 4\left\|x - \frac{y_n + y_m}{2}\right\|^2. \tag{J.-113}$$

Since A is convex and $y_n, y_m \in A$, $(y_n + y_m)/2 \in A$, therefore

$$4\left\|x - \frac{y_n + y_m}{2}\right\|^2 \geq M^2 \tag{J.-113}$$

$$\|y_n - y_m\|^2 < 2\left(\frac{1}{n} + \frac{1}{m}\right) \tag{J.-113}$$

it is a Cauchy sequence.

Suppose that $w \in A$ and $\|x - w\| = M$. Then $(y+)/2 \in A$, so that

$$\left\|x - \frac{y + w}{2}\right\| \geq M. \tag{J.-113}$$

On applying the parallelogram law to $x - w$, $x - y$ we obtain

$$\begin{aligned}
\|y - w\|^2 &= 2\|x - w\| + 2\|x - y\| - \left\|x - \frac{y + w}{2}\right\|^2 \\
&\leq 2M^2 + 2M^2 - 4M^2 = 0.
\end{aligned} \tag{J.-113}$$

Thus $y = w$, proving uniqueness.

Definition The space $\mathcal{L}(\mathcal{H}, \mathbf{C})$ is called the **algebraic dual space** of $\mathcal{H}$ and is denoted $\mathcal{H}^*$. The elements of $\mathcal{H}^*$ are **linear functionals**.

the Riesz lemma

Lemma J.7.3 *For each $T \in \mathcal{H}^*$, there is a unique $y_T \in \mathcal{H}$ such that $T(x) = (y_t, x)$ for all $x \in \mathcal{H}$. In addition $\|y_T\| = \|T\|_{\mathcal{H}*}$.*

Proof:

Define $y_T := \overline{T(x_0)}\|T\|$

$$\|T\| = \sup_{\|x\|=1} \|T(\mathbf{x})\| \tag{J.-113}$$

Note that any element $y \in \mathcal{H}$ can be written

$$y = \underbrace{\left(y - \frac{T(y)}{T(x_0)}x_0\right)}_{\in \mathcal{N}} + \underbrace{\frac{T(y)}{T(x_0)}x_0}_{\sim x_0} \tag{J.-113}$$

Since the functions $(y_T, .)$ and $T(.)$ are linear and agree on $\mathcal{N}$ and x_0, they must agree for any $y \in \mathcal{H}$. Thus $T(x) = (y_T, x)$ for all $x \in \mathcal{H}$.

To prove $\|T\|_{\mathcal{H}^*} = \|y_T\|_{\mathcal{H}}$

$$\begin{aligned} \|\mathbf{T}\| = \sup_{\|x\|=1} |T(x)| &\equiv \sup_{\|x\|=1} |(y_T, x)| \\ &\leq \sup_{\|x\|=1} \|y_T\|\|x\| = \|y_T\| \end{aligned} \tag{J.-113}$$

□

Example

$$\|x\| := \sqrt{(x^2 + y^2 + z^2)} \tag{J.-113}$$

$$T(\mathbf{x}) \Longleftrightarrow \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{J.-113}$$

$$\|T\| = \sup_{\sqrt{(x^2+y^2+z^2)}=1} \sqrt{[(ax+by)^2 + (cx+dy)^2]} \tag{J.-113}$$

Corollary J.7.4 *If N is a closed subspace of a Hilbert space H but is not the whole of H, then there is an element $y \neq 0$ in H which is orthogonal to N.*

Proof:

$\square$

Hahn-Banach Theorem

Theorem J.7.5 (The real Hahn-Banach Theorem)

$$F(x_1) \geq \frac{1}{\alpha}[p(\alpha x_1 + x) - f(x)] = p(x_1 + \frac{x}{\alpha}) - f(\frac{x}{\alpha}) = p(x_1 + z) - f(z) \tag{J.-113}$$

we need Zorn's lamma. Let E be the collection of all extensions $\tilde{\ell}(x) < \varphi(x)$. Then E can be partially ordered by inclusion with respect to the domain and every linear chain has an upper bound (defined on the union of all domains). Hence there is a maximal element $\tilde{\ell}$ by Zorn's lemma.

Theorem J.7.6 (The complex Hahn-Banach Theorem)

functional f on M of V such that

$$Re\ f(u) \leq p(u), \quad u \in M. \tag{J.-113}$$

There is a linear functional F on the whole of V such that

$$\begin{aligned} F(u) &= f(u), \quad u \in M. && \text{(J.-112)} \\ |F(u)| &\leq p(u), \quad u \in V. && \text{(J.-111)} \end{aligned}$$

Proof.

define the real valued functional

$$f_1(u) = \text{Re } f(u) \tag{J.-111}$$

Then by (J.7.6),

$$f_1(u) \le p(u), \quad u \in M. \tag{J.-111}$$

By the real Hahn-Banach theorem there is a real functional $F_1(u)$ on V such

$$\begin{aligned} F_1(u) &= f_1(u), \quad u \in M. && \text{(J.-110)} \\ F_1(u) &\le p(u), \quad u \in V. && \text{(J.-109)} \end{aligned}$$

$$f_1(iu) = \text{Re } f(iu) = \text{Re } if(u) = -\text{Im } f(u) \tag{J.-109}$$

$$f(u) = f_1(u) - if_1(iu). \tag{J.-109}$$

Let

$$F(u) = F_1(u) - iF_1(iu), \quad u \in V. \tag{J.-109}$$

For it to be linear with respect to complex scalars, i.e. $F((\alpha + i\beta)u) = (\alpha + i\beta)F(u)$, we need only check that $F(iu) = iF(u)$,

$$F(iu) = F_1(iu) - iF_1(-u) = i[F_1(u) - iF_1(iu)] = iF(u) \tag{J.-109}$$

Note $F(u) = f(u)$ for $u \in M$. We must now show (J.-111)

$$|F(u)| = e^{-i\theta}F(u) = F(e^{-i\theta}u) = F_1(e^{-i\theta}u) \le p(e^{-i\theta}u) = p(u). \tag{J.-109}$$

as the imaginary part of $F(e^{-i\theta}u)$ is zero and $|e^{-i\theta}| = 1$.

Theorem J.7.7 (Hahn-Banach) *Let Y be a subspace of a normed linear space X and f a bounded linear functional on Y. There exists a bounded linear extension F of f on X, such that $\|F\| = \|f\|$.*

J.7.4 Adjoint of Bounded Operators

Theorem J.7.8 *Let A be a bounded operator in $\mathcal{B}(X,Y)$ where X,Y are Hilbert spaces. There exists a unique operator $A^\dagger \in \mathcal{B}(Y,X)$ such that*

$$(x, A^\dagger y)_X = (Ax, y)_Y \tag{J.-109}$$

for all $x \in X$ $y \in Y$.

Proof: Consider any $y \in Y$. The linear functional on X

$$L(x) = (Ax, y)_Y$$

is continuous by the Schwartz inequality: If a sequence of vectors x_n converges to a limit vector x, then

$$\begin{aligned} |L(x) - L(x_n)| &= |L(x - x_n)| \\ &= |(A(x - x_n), y)_Y| \\ &\leq \|A\| \, \|x - x_n\| \, \|y\| \end{aligned}$$

so $L(x_n) \to L(x)$ as $n \to \infty$. Thus L is continuous. So by the Riesz lemma, there is a unique element $z \in X$ such that

$$(Ax, y)_Y = (x, z)_X$$

for all $x \in X$. We define $A^\dagger y$ to be z.

Let us show that $A^\dagger$ is linear. For any $y, z \in Y$ and $\alpha, \beta \in \mathbb{C}$ we have, for any $x \in X$,

$$\begin{aligned} (x, A^\dagger(\alpha y + \beta z))_X &= (Ax, \alpha y + \beta z)_Y \\ &= \overline{\alpha}(Ax, y)_Y + \overline{\beta}(Ax, z)_Y \\ &= \overline{\alpha}(x, A^\dagger y)_X + \overline{\beta}(x, A^\dagger z)_X \\ &= (x, \alpha A^\dagger y + \beta A^\dagger z)_X \end{aligned}$$

implying

$$A^\dagger(\alpha y + \beta z) = \alpha A^\dagger y + \beta A^\dagger z$$

To see that $A^\dagger$ is bounded note that for any $y \in Y$,

$$\begin{aligned} \|A^\dagger y\|^2 &= (A^\dagger y, A^\dagger y) \\ &= (AA^\dagger y, y) \\ &\leq \|AA^\dagger y\| \, \|y\|, \end{aligned}$$

by the Cauchy-Schwartz inequality. Since $\|AA^\dagger y\| \leq \|A\| \, \|A^\dagger y\|$, this implies

$$\|A^\dagger y\|^2 \leq \|A\| \, \|A^\dagger y\| \, \|y\|.$$

If $\|A^\dagger y\| > 0$ we may divide through to obtain

$$\|A^\dagger y\| \leq \|A\| \, \|y\|.$$

and since the inequality holds trivially if $\|A^\dagger y\| = 0$, it is always true. It follows that $A^\dagger$ is bounded, and

$$\|A^\dagger\| \leq \|A\|.$$

Finally we show that $A^\dagger$ is determined uniquely by (J.7.8). Say we have $B, B' \in \mathcal{B}(X, Y)$ such that

$$(x, By)_X = (x, B'y)_X$$

for all $x \in X$, $y \in Y$, then

$$\begin{aligned} 0 &= (x, By)_X + (-1)(x, B'y)_X \\ &= (x, By)_X + (x, -B'y)_X \\ &= (x, By - B'y)_X \end{aligned}$$

holds for all $x \in X$ then in particular it holds when $x = By - B'y$. Thus $(By - B'y, By - B'y) = 0$ and we conclude $By = B'y$ for all $y \in Y$, so $B = B'$.

□

Theorem J.7.9 *For any $A \in \mathcal{B}(X, Y)$, $A^{\dagger\dagger} = A$ and $\|A^\dagger\| = \|A\|$ for any pair X, Y of Hilbert spaces.*

Proof: Set $B = A^\dagger$ then $A^{\dagger\dagger} \equiv B^\dagger$. We have $B \in \mathcal{B}(Y, X)$ and $B^\dagger \in \mathcal{B}(X, Y)$. On applying the general defintion of the adjoint, we have for $x \in Y$, $y \in X$,

$$\begin{aligned}
(x, B^\dagger y)_Y &= (Bx, y)_X \\
&= (A^\dagger x, y)_X \\
&= (y, A^\dagger x)^*_X \\
&= (Ay, x)^*_Y \\
&= (x, Ay)_Y
\end{aligned}$$

and hence

$$A^{\dagger\dagger} y = B^\dagger y = Ay,$$

so that $A^{\dagger\dagger} = A$.

We know we have

$$\|A^\dagger\| \leq \|A\|.$$

Since this applies to any bounded linear operator we may replace A by $A^\dagger$ to obtain

$$\|A^{\dagger\dagger}\| \leq \|A^\dagger\|.$$

and, as we have established, $A^{\dagger\dagger} = A$,

$$\|A\| \leq \|A^\dagger\|.$$

Hence

$$\|A\| = \|A^\dagger\|.$$

$\square$

J.7.5 Unbounded Operators

In quantum mechanics we often work with operators which are not bounded. For example of such an operator consider the space of $L^2(-\infty, \infty)$ of sqaure-integrable functions $\psi(x)$ for $-\infty < x < \infty$, and let $\mathcal{Q}$ be the linear operator defined by $(\mathcal{Q}\psi)(x) = x\psi(x)$. This operator has the Hermitian property that

$$(\phi, \mathcal{Q}\psi) = \int_{-\infty}^{\infty} \phi(x)^* x\psi(X)dx = \int_{-\infty}^{\infty} [x\phi(x)]^* \psi(X)dx = (\mathcal{Q}\phi, \psi) \tag{J.-127}$$

when the integrals converge. It as not bounded, because

$$||\mathcal{Q}\psi||^2 = \int_{-\infty}^{\infty} |x\psi(x)|^2 dx \tag{J.-127}$$

can be any number of times larger than

$$||\psi||^2 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx. \tag{J.-127}$$

In fact $\mathcal{Q}$ is not even defined for all vectors; there are vectors ψ for which $x\psi(x)$ does not define a vector because $\int_{-\infty}^{\infty} |x\psi(x)|^2 dx$ is not finite.

It is impossible to define an unbounded Hermitian operator for all vectors. That this is true is a general fact shown by the following.

Theorem J.7.10 **(Hellinger-Toeplitz theorem).** *If a linear operator T is defined on all of a complex Hilbert space $\mathcal{H}$ satisfying*

$$(Tx, y) = (x, Ty)$$

for all $x, y \in \mathcal{H}$, then T is bounded.

Proof: Suppose T is not bounded so that we can find a sequence $\{y_n\}$ such that

$$\|y_n\| = 1 \qquad \text{and} \qquad \|Ty_n\| \to \infty.$$

Consider the functional f_n defined by

$$f_n := (Tx, y_n) = (x, Ty_n)$$

where $n = 1, 2, \ldots$. Each f_n is defined on all of $\mathcal{H}$ and is linear Note each f_n is a linear operator $f_n : \mathcal{H} \to \mathbb{C}$. For each n the functional f_n is bounded since

$$|f_n(x)| = |(x, Ty_n)| \leq \|Ty_n\| \, \|x\|.$$

Therefore $\{f_n\}$ is a sequence of bounded linear operators $f_n : \mathcal{H} \to \mathbb{C}$. Moreover, since

$$|f_n(x)| = |(Tx, y_n)| \leq \|Tx\|.$$

for every fixed $x \in \mathcal{H}$, the sequence $\{f_n(x)\}$ is bounded, i.e.,

$$\|f_n x\| := \sup_{\|x\|=1} |f_n(x)| \leq k_x$$

By the uniform boundedness theorem J.7.1 we can conclude that $\{\|f_n\|\}$ is bounded, say,

$$\|f_n\| \leq k \qquad \text{for all } \ n.$$

This implies that for every $x \in \mathcal{H}$ we have

$$|f_n(x)| \leq \|f_n\| \, \|x\| \leq k\|x\|$$

and, taking $x = Ty_n$, we arrive at

$$\|Ty_n\|^2 = (Ty_n, Ty_n) = |f_n(Ty_n)| \leq k\|Ty_n\|.$$

Hence $\|Ty_n\| \leq k$, which contradicts the original assumption $\|Ty_n\| \to \infty$.

□

Definition of the adjoint of an unbounded operator

Theorem J.7.11 *Every self-adjoint operator is a maximal symmetric operator.*

The domain of an unbounded operator

On the other hand an unbounded operator can not be deifned as self-adjoint for all vectors.

The set of all vectors ψ for which $A\psi$ is defined is called the *domain* of the operator A. We can assume that this is a linear manifold; if $A\psi$ $A\phi$ are defined, then

$$A(\psi + \phi) = A\psi + A\phi \tag{J.-127}$$

and

$$A(b\psi) = bA\psi \tag{J.-127}$$

define $A(\psi + \phi)$ and $A(b\psi)$.

Until now we have assumed that the domain of every operator being considered is the whole space. In the absence of this assumption some of our definitions have to be refined.

Thus $A = B$ implies that A and B have the same domain. The domain of $A + B$ is the set of all vectors which are both in the domain of A and the domain of B.

The sum of two operators A and B is given by $(A + B)u = Au + Bu$,

$$D(A + B) = D(A) \cap D(B).$$

The domain of AB is the set of all vectors ψ in the domain of B such that $B\psi$ is in the domain of A.

The composition of an operator A with an operator B is given by $(AB)u = A(Bu)$, where $D(AB) = \{u \in D(B)|Bu \in D(A)\}$

Two operators A and B are said to be equal if $D(A) = D(B)$ and for $u \in D(A)$, $Au = Bu$. A is said to be a restriction of B if $D(A) \subseteq D(B)$ and for $u \in D(A)$, $Au = Bu$.

If $D(B) \subseteq D(A)$ and for $u \in D(B)$ $Au = Bu$, then we say that A is an extension of B.

Definition A set of vectors, for example the domain of an operator, is called *dense* if for every vector ψ there is a sequence of vectors ψ_n in the set such that $\psi_n \to \psi$.

Definition If an operator has a dense domain, we can define $A^\dagger$ as follows. The domain of $A^\dagger$ is the set of all vectors ψ for which there is a vector $A^\dagger\psi$ such that

$$(\phi, A^\dagger\psi) = (A\phi, \psi) \tag{J.-127}$$

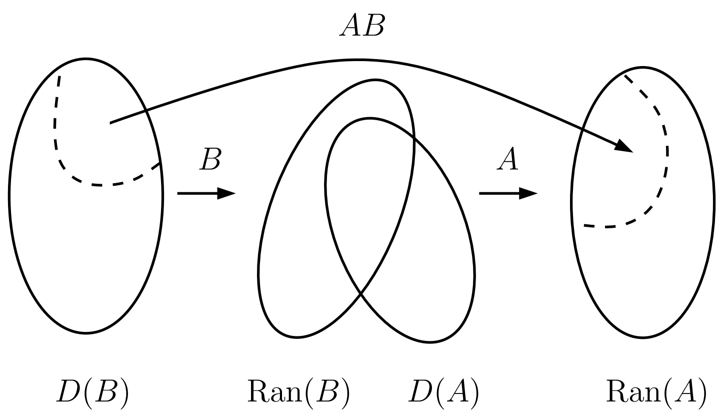


Figure J.7: $\mathrm{Ran}(B) \cap D(A)$

for every vector ϕ in the domain of A.

This defines $A^{\dagger}\psi$. For suppose χ and χ' are vectors that

$$(\phi, \chi) = (\phi, \chi') \tag{J.-127}$$

for ever vector ϕ in the domain of A. Then, because the domain of A is dense, there is a sequence of vectors ϕ_n such that $\phi_n \to \chi - \chi'$ and

$$(\phi_n, \chi - \chi') = 0 \tag{J.-127}$$

Therefore $\chi - \chi' = 0$, because

$$(\phi_n, \chi - \chi') \to (\chi - \chi', \chi - \chi'). \tag{J.-127}$$

Example: A dense operator.

We consider the operator $\mathcal{Q}$ of Example. Let the domain of $\mathcal{Q}$ be the set of all vectors $\psi(x)$ such that $x\psi(x)$ is square-integrable. We prove this to be dense.

For any vector $\psi(x)$ let the sequance of vectors $\psi_n(x)$ be defined by

$$\begin{aligned} \psi_n(x) = \psi(x) \ \text{ for } \ -n \le x \le n \\ \psi_n(x) = 0 \ \text{ for } \ |x| > n. \end{aligned} \tag{J.-127}$$

Then $x\psi_n(x)$ is square integrable, so $\psi_n(x)$ is in the domain of $\mathcal{Q}$ for each $n = 0, 1, 2, \ldots$, and

$$||\psi - \psi_n||^2 = \int_{-\infty}^{-n} |\psi(x)|^2 dx + \int_{n}^{-\infty} |\psi(x)|^2 dx \to 0 \qquad \text{(J.-127)}$$

as $n \to \infty$, so $\psi - \psi_n$. Equation() in example ?? shows that $\mathcal{Q}$ symmetric. Therefore $\mathcal{Q}^\dagger$ is an extension of $\mathcal{Q}^\dagger$.

The example above demonstrates that a symmetric operator A can fail to be self-adjoint only if $A^\dagger$ is an extension of A to a larger domain.

Theorem J.7.12 *Let A be a Hermitian operator on a Hibert space $\mathcal{H}$. All eigenvalues of A are real, and eigenvectors of A corresponding to distinct eigenvalues are orthogonal.*

Proof: Suppose that λ is an einvalue of A and φ a corresponding eigenvector: then $A\varphi = \lambda\varphi$ and $\varphi \neq 0$. By self-adjointness of A,

$$\begin{aligned} 0 &= (A\varphi, \varphi) - (\varphi, A\varphi) \\ &= (\lambda\varphi, \varphi) - (\varphi, \lambda\varphi) \\ &= (\lambda - \overline{\lambda})\|\varphi\|^2 \end{aligned}$$

Since $\varphi \neq 0$, $\lambda = \overline{\lambda}$, and so λ is real. Now let λ, μ be distinct eigenvalues of A and let φ, ψ be corresponding eigenvectors. Then

$$\begin{aligned} 0 &= (A\varphi, \psi) - (\varphi, A\psi) \\ &= (\lambda\varphi, \psi) - (\varphi, \mu\psi) \\ &= (\lambda - \mu)(\varphi, \psi) \end{aligned}$$

Since μ is real and $\lambda \neq \mu$,

$$\lambda - \overline{\mu} = \lambda - \mu \neq 0.$$

Hence $(\varphi, \psi) \neq 0$, as required.

□

However note that A need not have any eigenvalues!

J.7.6 Operators of Finite Rank

Definition The **rank** of an operator is defined as the dimension of its range.

As an example, if $A \in \mathcal{L}(X, Y)$, ($\mathcal{L}(X, Y)$ being the space of all linear operators from X to Y), it is of finite rank if it maps X into some finite dimensional subspace Z of Y. Z is a normed space with respect to the restriction of the norm of Y.

□

Finite rank operators have the characteristic property that they can be expressed in the form

$$Tf = \sum_{k=1}^{n} (f, g_k) h_k, \tag{J.-133}$$

where n is the dimension of Ran$T$, $\{h_k\}_{k=1}^{n}$ is some basis in Ran$T$, and $\{g_k\}_{k=1}^{n}$ is some finite system of vectors which does not depend on f.

We prove the part that every finite rank operator T can be expressed in the form (J.7.6). Choose any orthonormal basis $\{h_k\}_{k=1}^{n}$ in Ran$T$. For any $f \in \mathcal{H}$ we shall have

$$Tf = \sum_{k=1}^{n} \alpha_k h_k,$$

where the numbers α_k can be found from

$$\alpha_k = (Tf, h_k).$$

Thus the α_k's are linear functionals of f and by Riesz's lemma, elements g_k ($k = 1, 2, \ldots, n$) can be found such that the α_k's can be expressed as

$$\alpha_k = (f, g_k).$$

□

J.7.7 Square Roots of a Positive operator

Define

$$T_1 \leq T_2 \quad \text{if and only if} \quad (T_1 x, x) \leq (T_2 x, x)$$

for all $x \in \mathcal{H}$.

Definition A bounded self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$ is said to be **positive**, written $T \geq 0$ if and only if

$$(Tx, x) \geq 0.$$

$\square$

Lemma J.7.13 *If two bounded self-adjoint linear operators A and B on a Hilbert space $\mathcal{H}$ are positive and commute ($AB = BA$), then their product AB is positive.*

Proof: We prove the lemma by introducing a sequence of opeators defined by

$$A_1 = \frac{A}{\|A\|}, \qquad A_{n+1} = A_n - A_n^2 \qquad (n = 1, 2, \dots). \tag{J.-133}$$

First we porve by induction that

$$0 \leq A_n \leq I. \tag{J.-133}$$

We see that the equality holds for $n = 1$,

$$(A_1 x, x) = \frac{1}{\|A\|}(Ax, x) \leq \frac{1}{\|A\|}\|Ax\| \, \|x\| \leq \|x\|^2 = (Ix, x).$$

Suppose (J.7.7) holds for fixed n,

$$0 \leq A_n \leq I, \quad \text{or} \quad 0 \leq I - A_n \leq I.$$

Since A_n is self-adjoint, for every $x \in \mathcal{H}$ and $y \in A_n x$ we obtain

$$
\begin{aligned}
(A_n^2(I - A_n)x, x) &= ((I - A_n)A_n x, A_n x) \\
&= ((I - A_n)y, y) \geq 0.
\end{aligned}
$$

This proves

$$A_n^2(I - A_n) \geq 0.$$

Similarly,

$$A_n(I - A_n)^2 \geq 0.$$

Addition results in

$$0 \leq A_n^2(I - A_n) + A_n(I - A_n)^2 = A_n - A_n^2 = A_{n+1}$$

Hence $0 \leq A_{n+1}$. That $A_{n+1} \leq I$ follows from $A_n^2 \geq 0$ and $I - A_n \geq 0$,

$$0 \leq I - A_n + A_n^2 = I - A_{n+1}.$$

Thus we have proved $0 \leq A_{n+1} \leq I$.

Now we move on to show that $(ABx, x) \geq 0$ for all $x \in \mathcal{H}$.

$$
\begin{aligned}
A_1 &= A_1^2 + A_2 \\
&= A_1^2 + A_2^2 + A_3 \\
&\cdots \\
&= A_1^2 + A_2^2 + \cdots + A_n^2 + A_{n+1}
\end{aligned}
$$

Since $A_{n+1} \geq 0$, this implies

$$A_1^2 + \cdots + A_n^2 = A_1 - A_{n+1} \leq A_1. \tag{J.-139}$$

By the definition of $\leq$ and the self-adjointsness of the A_j's implies

$$\sum_{j=1}^{n} \|A_j x\|^2 = \sum_{j=1}^{n} (A_j x, A_j x) = \sum_{j=1}^{n} (A_j^2 x, x) \leq (A_1 x, x).$$

Since n is arbitrary, the infinite series $\|A_1x\|^2+\|A_2x\|^2+\cdots$ converges. Hence $\|A_nx\| \to 0$ and so $A_nx \to 0$.

$$\left(\sum_{j=1}^{n} A_j^2\right) x = (A_1 - A_{n+1})x \to A_1x \quad (n \to \infty). \tag{J.-139}$$

Set $z_n = \sum_{j=1}^{n} A_j^2 x$ and $z = \lim_{n\to\infty} z_n$, by the continuity of the inner product we have

$$\lim_{n\to\infty} \sum_{j=1}^{n} (BA_j^2 x, x) = \lim_{n\to\infty} (Bz_n, x) = (Bz, x) = (BA_1x, x).$$

All the A_j's commute with B since they are sums and products of $A_1 = \|A\|^{-1}A$, and A and B commute .For every $x \in \mathcal{H}$ and $y_j = A_jx$,

$$\begin{aligned}
(ABx, x) &= \|A\|(BA_1x, x) \\
&= \|A\| \lim_{n\to\infty} \sum_{j=1}^{n} (BA_j^2 x, x) \\
&= \|A\| \lim_{n\to\infty} \sum_{j=1}^{n} (A_jBA_jx, x) \\
&= \|A\| \lim_{n\to\infty} \sum_{j=1}^{n} (By_j, y_j) \geq 0,
\end{aligned}$$

that is, $(ABx, x) \geq 0$.

$\square$

Lemma J.7.14 *Let $\{A_n\}$ be a sequence of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ such that*

$$A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq K \tag{J.-143}$$

where K is a bounded self-adjoint operator on $\mathcal{H}$. Suppose that any A_j commutes with K and every A_k. Then $\{A_n\}$ strongly convergent ($A_nx \to Ax$ for all $x \in \mathcal{H}$) with the limit operator linear, bounded and self-adjoint and satisfying $A \leq K$

Proof: Consider the operators

$$B_n = K - A_n.$$

Clearly B_n is self-adjoint. We have

$$B_m^2 - B_n B_m = (B_m - B_n)B_m = (A_n - A_m)(K - A_m).$$

Let $m < n$. Then $A_n - A_m$ and $K - A_m$ are positive by (J.7.14). Sine these operators commute, their product is also positive by lemma J.7.13. Hence $B_m^2 \geq B_n B_m$ for $m < n$. In a similary manner we can show that $B_n B_m \geq B_n$ for $m < n$.

Together

$$B_m^2 \geq B_n B_m \geq B_n^2 \qquad (m < n).$$

By the self-adjointnes of the B_n's we thus have

$$(B_m^2 x, x) \geq (B_n B_m x, x) \geq (B_n^2 x, x) = (B_n x, B_n x) = \|B_n\|^2 \geq 0. \tag{J.-143}$$

This shows that $(B_m^2 x, x)$ with fixed x is a monotone decreasing sequence of nonnegative numbers. Hence it converges.

We now show that $\{A_n\}$ converges. By assumption, every A_n commutes with every A_m and with K. Hence the B_j's all commute. These operators are self-adjoint. Since $-2(B_m B_n x, x) \leq -2(B_n^2 x, x)$, where $m < n$, we obtain

$$\begin{aligned}
\|B_m x - B_n x\|^2 &= ((B_m x - B_n)x, (B_m x - B_n)x) \\
&= ((B_m x - B_n)^2 x, x) \\
&= (B_m^2 x, x) - 2(B_n B_m x, x) + (B_n^2 x, x) \\
&\leq (B_m^2 x, x) - (B_n^2 x, x)
\end{aligned}$$

From thus we see that the sequence $\{B_n\}$ is Cauchy. It converges since $\mathcal{H}$ is complete. Now $A_n = K - B_n$. Hence $\{A_n\}$ also converges.

Since $\{A_n x\}$ converges, it is bounded for every $x \in \mathcal{H}$. The uniform boundednes theorem implies that A is bounded.

$\square$

Definition Let $T : \mathcal{H} \to \mathcal{H}$ be a positive bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then a bounded self-adjoint operator A satisfying

$$A^2 = T$$

is called the **square root** of T. If in addition $A \geq 0$, then A is called a **positive square root** of T and is denoted by

$$A = T^{1/2}.$$

$\square$

In the following theorem it is shown that $T^{1/2}$ exists and is unique.

Theorem J.7.15 *Every positive bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ has a unique posititve square root.*

Proof: First we will prove that if the theorem holds for $T \leq I$, it holds in general. If $T = 0$, we can take $A = T^{1/2} = 0$. Let $T \neq 0$. By the Schwarz inequality

$$(Tx, x) \leq \|Tx\| \, \|x\| \leq \|T\| \, \|x\|^2.$$

Dividing by $\|T\| \neq 0$ and setting

$$S = \frac{1}{\|T\|} T,$$

we obtain

$$(Sx, x) \leq \|x\|^2 = (Ix, x),$$

that is, $S \leq I$. Asumming that S has a unique positive square root $B = S^{1/2}$, we have $B^2 = S$ and we see that the unique square root of $T = \|T\| S$ is $\|T\|^{1/2} B$ because

$$(\|T\|^{1/2} B)^2 = \|T\| B^2 = \|T\| S = T.$$

We now establish the theorem under the condition $T \leq I$. To prove the existence of the operator $A = T^{1/2}$ we introduce a sequence of operators defined by

$$A_{n+1} = A_n + \frac{1}{2}(T - A_n^2), \qquad n = 0, 1, \dots \tag{J.-147}$$

where $A_0 = 0$. We will prove $A_n x \to T^{1/2} x$.

Since $A_0 = 0$ $A_1 = \frac{1}{2}T$, $A_2 = T - \frac{1}{8}T^2$, etc. Each A_n is a polynomial in T. Hence the A_n's are self-adjoint and all commute, and they also commute with every operator that commutes with T. We will prove

$$\begin{aligned} A_n &\leq I, & n = 0, 1, \dots \quad & \text{(J.-146)} \\ A_n &\leq A_{n+1}, & n = 0, 1, \dots \quad & \text{(J.-145)} \\ A_n x \to Ax, & \quad A = T^{1/2} & & \text{(J.-144)} \end{aligned}$$

We have that $A_0 \leq I$. Let $n > 0$. Since $I - A_{n-1}$ is self-addjoint,

$$(I - A_{n-1})^2 \geq 0.$$

But $I - T \geq 0$ so we have

$$\begin{aligned} 0 &\leq \frac{1}{2}(I - A_{n-1})^2 + \frac{1}{2}(I - T) \\ &= I - A_{n-1} - \frac{1}{2}(T - A_n^2) \\ &= I - A_n. \end{aligned}$$

We now prove by induction that $A_n \leq A_{n+1}$ for all $n \geq 0$. From (J.7.7) we have $0 = A_0 \leq A_1 = \frac{1}{2}T$. We assum that $A_{n-1} \leq A_n$ for fixed n and show it implies $A_n \leq A_{n+1}$. By (J.7.7)

$$\begin{aligned} A_{n+1} - A_n &= A_n + \frac{1}{2}(T - A_n^2) - A_{n-1} - \frac{1}{2}(T - A_{n-1}^2) \\ &= (A_n - A_{n-1})[I - \frac{1}{2}(A_n + A_{n-1})]. \end{aligned}$$

It follows from (J.-146) that $A_n \leq A_{n+1}$.

As $\{A_n\}$ satisfies $A_n \leq A_{n+1}$ and $A_n \leq I$, lemma J.7.14 implies the existence of a bounded self-adjoint linear operator A such that $A_n x \to Ax$ for all $x \in \mathcal{H}$. Since $\{A_n x\}$ converges, (J.7.7) gives

$$A_{n+1}x - A_n x = \frac{1}{2}(Tx - A_n^2 x) \to 0$$

as $n \to \infty$. Hence

$$Tx - A^2 x = 0 \qquad \text{for all} \qquad x \in \mathcal{H}$$

$\square$

J.7.8 Polar decomposition

Recall, a linear operator $A : X \to Y$ is called an isometry if and only if for all $x, y \in X$ the inner product is invariant: $< x, y >=< Ax, Ay >$. If A is a surjective isometry, then for every $y \in Y$ the equation $Ax = y$ has a solution. Then

$$AA^*y = A(A^*A)x = Ax = y,$$

hence $AA^* = \mathbb{I}$.

We are interested in the case where $X = Y = \mathcal{H}$.

Definition An operator $W \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if there are closed subspaces $\mathcal{K}, \mathcal{L} \subset \mathcal{H}$ such that $W : \mathcal{K} \to \mathcal{L}$ is an isometry onto $\mathcal{L}$ and $W : \mathcal{K}^\perp \to \{0\}$. $\mathcal{K}$ is called the initial subspace and $\mathcal{L}$ the final subspace of W.

Theorem J.7.16 (Polar decomposition) *Let $A \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry*

$$W : \overline{Ran|A|} \to \overline{RanA}$$

(where Ran stands for the range), such that A can be written uniquely as $A = W|A|$ where $|A|$ is the positive square root of A^*A.

Proof:

$$\|Ax\|^2 = < Ax, Ax > = < x, A^*Ax > = < x, |A|^2 x >= \| \, |A|x\|^2.$$

$$|A|(x - y) = 0 \quad \text{if and only if} \quad A(x - y) = 0.$$

Thus there is a unique unitary operator W_0 from the range of $|A|$ to the range of A and

$$Ax = W_0|A|x$$

an isometric operator $W_0 : \mathrm{Ran}|A| \to imA$ which can be extended uniquely to an isometric operator $W : \overline{\mathrm{Ran}|A|} \to \overline{\mathrm{Ran}A}$ by

$$Wx = \lim W_0 x_k$$

for $x = \lim x_k$. Hence

$$A = W|A|,$$

We extend W to a partial isometry on $\mathcal{H}$ by defining $Wy = 0$ if y is orthogonal to the range of $|A|$.

□

The uniqueness of the polar decomposition follows from $A = VB$, hence, $A^* = B^*V^*$ and $|A|^2 = A^*A = VB^*V^* = |B|^2 = B^2$, hance $B = |A|$, since the root is unique.

Suppose that U is unitary and commutes with A. Then U also commutes with $|A|$ and we have

$$A = UAU^* = UWU^*U|A|U^* = UWU^*|A|.$$

This is another polar decomposition of A and so, by uniqueness, we see that $W = UWU^*$, that is, U also commutes with W.

J.7.9 Compact Operators

Definition A linear operator K from X to Y is said to be **compact** if it transforms every bounded sequence of vectors $\{x_n\} \in X$ (i.e., vectors whose norms are all less than some fixed constant) into a sequence $Kx_n \in Y$ containing a convergent subsequence. The set of compact operators from X to Y is denoted $K(X,Y)$.

Notice that compact operators are always bounded. Otherwise, there would be a sequence $\{x_n\}$ such that $\|x_n\| \leq C$, while $\|Kx_n\| \to \infty$. Then $\{Kx_n\}$ could not have a convergent subsequence. The sum of two compact operators is always compact, as is the product of a scalar and a compact operator. Hence, $K(X,Y) \subset \mathcal{B}(X,Y)$.

All finite-rank operators are compact, for they take a bounded sequence into a bounded sequence in a finite-dimensional Banach space.

The product of a compact operator with a bounded operator is a compact operator. Suppose that K is compact and B is a bounded operator. Consider KB acting on $\{x_n\}$, where $\|x_n\| < C$ for all n. Then K acts on the sequence $\{Bx_n\}$, which is bounded because B is a bounded operator. Since K is compact, there must be a subsequence of $\{KBx_n\}$ which converges to an element of the space, so KB is compact. Now consider BK acting on $\{x_n\}$. Since K is compact, $\{Kx_n\}$ posses a subsequence which converges. Since B is bounded, it is continuous, so $\{BKx_n\}$ has a related convergent subsequence.

Lemma J.7.17 *If $\{g_k\}_{k=1}^{\infty}$ is an infinite orthonormal sequences of vectors in X, and if*

$$Kg_k = \beta_{k0}g_0 + \beta_{k1}g_1 + \cdots + \beta_{kk}g_k \tag{J.-149}$$

where K is a compact operator in X, then

$$\lim_{k\to\infty} \beta_{kk} = 0.$$

Proof: Let $n > m$; then

$$\|Kg_n - Kg_m\|^2 \geq |\beta_{kk}|^2.$$

If there were only a finite number of β_{kk}'s such that $|\beta_{kk}| \geq \epsilon$ for all $\epsilon > 0$ then β_{kk} would tend to 0 as $k \to \infty$. Say β_{kk} does not tend to 0 as $k \to \infty$, then there must be an infinite sequence of subscripts

$$n_1 < n_2 < n_3 < \ldots$$

such that

$$|\beta_{n_j n_j}| \geq \delta > 0, \qquad j = 1, 2, 3, \ldots$$

and therefore

$$\|Kg_{n_k} - Kg_{n_j}\|^2 \geq \delta^2 > 0.$$

This implies that the infinite sequence of vectors $\{Kg_{n_j}\}_{j=1}^{\infty}$ does not contain any convergent subsequences, and as the sequence $\{g_{n_j}\}_{j=1}^{\infty}$ is bounded, (of course, $\|g_{n_j}\| = 1$ for all j), this contradicts the defintion of a compact operator.

□

Lemma J.7.18 *If $\lambda \neq 0$, if K is a compact operator and if $\{f_n\}_{n=0}^{\infty}$ is an infinite sequence of vectors which satisfy the relations*

$$Kf_0 = \lambda f_0 = 0, \quad Kf_n = \lambda f_n = f_{n-1} \qquad (n = 1, 2, 3, \dots),$$

then $f_0 = 0$ (and hence also $f_n = 0$ for $n = 1, 2, 3, \dots$).

Proof: First we show that if $f_0 \neq 0$ implies there are no linearly dependent vectors among the f_k's. For suppose that among the vectors

$$f_0, f_1, f_2, \dots, f_k, \dots$$

there are some elements which are linear combinations of the earlier ones. Let f_n be the first such element, i.e. the first of the f_k's such that

$$f_n = \alpha_0 f_0 + \alpha_1 f_1 + \dots + \alpha_{n-1} f_{n-1}, \tag{J.-149}$$

then, applying the operator K to both sides, we obtain

$$\lambda f_n + f_{n-1} = \alpha_0 \lambda f_0 + \alpha_1(\lambda f_1 + f_0) + \dots + \alpha_{n-1}(\lambda f_{n-1} + f_{n-2}).$$

But by (J.7.9), this implies

$$f_{n-1} = \alpha_1 f_0 + \alpha_2 f_1 + \dots + \alpha_{n-1} f_{n-2}$$

which contradicts that f_n is te first elements that is a linear combination of its predecessors.

Now, we assume the lemma is false by taking the vectors $f_0, f_1, f_2, \dots$ to be linearly independent, and so we can orthonormalize this sequence. Let the orthonormalized vectors be

$$
\begin{array}{rcl}
g_0 & = & \alpha_{00} f_0, \\
g_1 & = & \alpha_{10} f_0 + \alpha_{11} f_1, \\
\vdots & \vdots & \vdots \\
g_k & = & \alpha_{k0} f_0 + \alpha_{k1} f_1 + \cdots + \alpha_{kk} f_k \\
\vdots & \vdots & \vdots \qquad \vdots
\end{array}
\tag{J.-152}
$$

We observe that

$$
\begin{aligned}
K g_k &= \alpha_{k0} \lambda f_0 + \alpha_{k1}(\lambda f_1 + f_0) + \cdots + \alpha_{kk}(\lambda f_k + f_{k-1}) \\
&= \alpha_{k1} \lambda f_0 + \alpha_{k2} \lambda f_1 + \cdots + \alpha_{kk} f_{k-1} + \lambda(\alpha_{k0} f_0 + \alpha_{k1} f_1 + \cdots + \alpha_{kk} f_k) \\
&= \alpha_{k1} \lambda f_0 + \alpha_{k2} \lambda f_1 + \cdots + \alpha_{kk} f_{k-1} + \lambda g_k \\
&= \beta_{k0} g_0 + \beta_{k1} g_1 + \cdots + \beta_{k,k-1} g_{k-1} + \lambda g_k.
\end{aligned}
$$

contradicting lemma J.7.17, showing that the hypothesis that $f_0 \neq 0$ is impossible.

□

Theorem J.7.19 *Every compact operator K in X can have, for any $\rho > 0$, only a finite nunmber of linearly independent eigenvectors corresponding to eigenvalues greater than $|\rho|$.*

Proof: Assuming the ocntrary, we suppose that there is an infinite set of linearly independent vectors f_n $(n = 1, 2, 3, \dots)$ for which

$$
K f_n = \lambda_n f_n, \qquad |\lambda_n| > \rho > 0 \qquad n = 1, 2, 3, \dots
$$

Orthonormalizing the sequence $\{f_n\}_{n=1}^{\infty}$, we obtain an orthonormal sequence of vectors

$$
\begin{array}{rcl}
g_1 & = & \alpha_{11} f_1, \\
g_2 & = & \alpha_{21} f_0 + \alpha_{22} f_2, \\
\vdots & \vdots & \vdots \\
g_k & = & \alpha_{k1} f_1 + \alpha_{k1} f_2 + \cdots + \alpha_{kk} f_k \\
\vdots & \vdots & \vdots \qquad \vdots
\end{array}
$$

Then

$$
\begin{aligned}
Kg_k - \lambda_k g_k &= \alpha_{k1}(\lambda_1 - \lambda_k) + \cdots + \alpha_{k1}(\lambda_{k-1} - \lambda_k)\alpha_{k,k-1} f_{k-1} \\
&= \beta_{k1} g_1 + \beta_{k2} g_2 + \cdots + \beta_{k,k-1} g_{k-1}, \qquad \text{(J.-161)}
\end{aligned}
$$

and thus

$$
Kg_k = \beta_{k1} g_1 + \beta_{k2} g_2 + \cdots + \beta_{k,k-1} g_{k-1} + \lambda_k g_k.
$$

Thus the assumption that $|\lambda_k| > \rho > 0$ $(k = 1, 2, 3, \dots)$ contradicts lemma J.7.17.

□

Corollary J.7.20 *The eigenvalues of a compact operator K in X can have a limit point only at the point 0.*

Say we have an infinite sequence of eiegenvalues $\{\lambda_n\}$ of a compact operator K tending to λ where $|\lambda| > 0$. Then for any ϵ satifying $|\lambda| > \epsilon > 0$ there is some n_0 such that $|\lambda - \lambda_n| < \epsilon$ for all $n \geq n_0$. However this contradicts theorem J.7.19, for pick any ρ such that $0 < \rho < |\lambda| - \epsilon$, by the theorem there can only be a finite number of eigenvalues with modulus greater than $|\rho|$. Thus, if the eigenvalues of K are infinite in number, they must comprise a sequence tending to zero.

□

Corollary J.7.21 *The number of independent eigenvectors corresponding to any non-zero eigenvalue of a compact operator in X is finite. In other words, the multiplicity of each non-zero eigenvalue of a compact operator in X is finite.*

Given any non-zero eigenvalue λ, by theorem J.7.19, for any ρ such that $|\lambda| < \rho < 0$ there are only a finite number of linearly independent eigenvectors with eigenvalues whose modulus is greater than ρ. Hence the mutilplicity of λ must be finite.

□

Corollary J.7.22 *Every compact operator in X has not more than a countable set of linearly independent eigenvectors corresponding to non-zero eigenvalues.*

Choose $\rho_n = 1/n$ for $n \in \mathbb{N}$. Let N_n be the *finite* (possibly zero) number of linearly independent eigenvectors with eigenvalue of modulus greater than ρ_n take away the number with modulus greater than ρ_{n-1}. Then the number of linearly independent eigenvectors corresponding to non-zero eigenvalues is $\sum_{n\in\mathbb{N}} N_n$ which is at most a countable set.

□

Theorem J.7.23 *If $\|K_j - K\| \to 0$ as $j \to \infty$, and if K_j is a compact transformation for all n, then K is a compact operator.*

Proof:

$$\begin{aligned}
\|K\phi_n - K\phi_m\| &\leq \|K\phi_n - K_j\phi_n\| + \|K_j\phi_n - K_j\phi_n\| + \|K_j\phi_m - K\phi_m\| \\
&\leq \|K - K_j\| \cdot \|\phi_n\| + \|K_j\phi_n - K_j\phi_n\| + \|K_j - K\| \cdot \|\phi_m\|.
\end{aligned}$$

Since the ϕ_n are bounded,

$$\|K\phi_n - K\phi_m\| \leq C\|K - K_j\| + \|K_j\phi_n - K_j\phi_n\|.$$

Because $\|K_j - K\| \to 0$, we can pick some j sufficiently larlge so that

$$\|K_j - K\| < \epsilon/(2C).$$

Then when $m, n > N$ the second term on the right above can be made smaller than $\epsilon/2$. Thus for $m, n > N$, we have

$$\|K\phi_n - K\phi_m\| < \epsilon.$$

□

Any transformation with square-integrable kernel can be approximated arbitrarily closely in norm by a kernel of finite rank.

Things are especially nice if $X = Y$ is a separable Hilbert space.

Theorem J.7.24 *For each compact operator K of a Hilbert space, on can find a sequence of operators of finite rank converging to K in norm.*

Proof:

Define the operator

$$P_n u = \sum_{i=1}^{n} (u, \varphi_i)\varphi_i \tag{J.-163}$$

If the assertion were false, there would be an operator $K \in K(H)$ and number $\delta > 0$ such that

$$\|K - F\| \geq \delta. \tag{J.-163}$$

for all operators F of finite rank. Now

$$F_n u := P_n K u = \sum_{i=1}^{n} (Ku, \varphi_i)\varphi_i$$

is an operator of finite rank. By (J.7.9) for each n there is a $u_n \in H$ satifying

$$\|u_n\| = 1, \quad \|(K - F_n)u_n\| \geq \delta/2. \tag{J.-163}$$

Since the sequence $\{u_n\}$ is bounded, Ku_n has a subsequence converging to an element $w \in H$. But

$$\begin{aligned} \|(K - F_n)u_n\| &\leq \|(1 - P_n)(Ku_n - w)\| + \|(1 - P_n)w\| \\ &\leq \|(Ku_n - w)\| + \|(1 - P_n)w\|. \end{aligned} \tag{J.-163}$$

□

We see that compact operators are in a sence the closest analogue of finite dimensional matrices and in fact the finite rank operators form a dense subset with respect to the operator norm $\|T\|$.

Compact Self-Adjoint Operators

Lemma J.7.25 *If A is a self-adjoint operator on a Hilbert spacce $\mathcal{H}$ then*

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|.$$

Proof: for any $x \in \mathcal{H}$ such that $\|x\| = 1$,

$$\begin{aligned}
|(Ax, x)| &\leq \|Ax\|\,\|x\| \\
&\leq \|A\|\,\|x\|^2 \\
&\leq \|A\|.
\end{aligned}$$

Hence

$$\|A\| \geq \sup_{\|x\|=1} |(Ax, x)|.$$

Let $m \equiv \sup_{\|x\|=1} |(Ax, x)|$, so that

$$|(Au, u)| \leq m\|u\|^2$$

for any $u \in \mathcal{H}$. Then for any x, y,

$$(A(x \pm y), (x \pm y)) = (Ax, x) \pm 2Re(Ax, y) + (Ay, y).$$

Thus

$$\begin{aligned}
4Re(Ax, y) &\leq (A(x + y), (x + y)) - (A(x - y), (x - y)) \\
&\leq m(\|x + y\|^2 + \|x - y\|^2) \\
&= 2m(\|x\|^2 + \|y\|^2) \qquad \text{(J.-167)}
\end{aligned}$$

by the parallelogram law. Replace x by λx where $|\lambda| = 1$ and λ is chosen so that $\lambda(Ax, y) \geq 0$. We obtain

$$|(Ax, y)| \leq \frac{1}{2} m(\|x\|^2 + \|y\|^2).$$

Suppose that $Ax \neq 0$. Since the above holds for all $x, y \in \mathcal{H}$ we may choose y to be given by

$$y = \frac{\|x\|}{\|Ax\|} Ax.$$

Since $\|y\| = \|x\|$ the inequality yields

$$\|x\| \, \|Ax\| \leq m\|x\|^2.$$

This holds trivially when $Ax = 0$, hence is true for all x. From the definition of the operator norm

$$\begin{aligned} \|A\| &\leq m \\ &= \sup_{\|x\|=1} |(Ax, x)|. \end{aligned}$$

□

Theorem J.7.26 *Let K be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. Either $\|K\|$ or $-\|K\|$ is an eigenvalue of K.*

Proof: We have by the above

$$\|K\| = \sup_{\|x\|=1} |(Kx, x)|.$$

Since the inner product (Kx, x) is real, by definition of the least upper bound a sequence of (normalized) vectors $\{g_n\}_{n=1}^{\infty}$ can be found such that the limit

$$\lim_{n\to\infty} (Kg_n, g_n)$$

exists and is equal to $\|K\|$ or -$\|K\|$. We shall call this limit λ.

Since K is a compact operator there is a subsequence $\{g_{n_i}\}_{i=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that $\{Kg_{n_i}\}_{i=1}^{\infty}$ is a convergent sequence. Denote its limit by y,

$$Kg_{n_i} \to y.$$

As $\{(Kg_n, g_n)\}$ is Cauchy

$$|(Kg_n, g_n) - (Kg_m, g_m)| < \epsilon, \quad m, n > N.$$

The sequence $\{(Kg_n, g_n)\}$ has a subsequence converging to $\lim_{i\to\infty}(y, g_{n_i}) =: \mu$, then there is an $n_i > N$ such that

$$|(Kg_{n_i}, g_{n_i}) - (Kg_n, g_n)| < \epsilon.$$

Thus,

$$|(Kg_n, g_n) - \mu| \leq |(Kg_n, g_n) - (Kg_{n_i}, g_{n_i})| + |(Kg_{n_i}, g_{n_i}) - \mu| < 2\epsilon.$$

for all $n > N$. It follows $\mu = \lambda$.

Now since

$$\|Kg_{n_i} - \lambda g_{n_i}\|^2 = \|Kg_{n_i}\|^2 - 2\lambda(Kg_{n_i}, g_{n_i}) + \lambda^2, \tag{J.-169}$$

we have

$$\lim_{i\to\infty} \|Kg_{n_i} - \lambda g_{n_i}\|^2 = \|y\|^2 - 2\lambda^2 + \lambda^2 = \|y\|^2 - \lambda^2 \tag{J.-169}$$

But

$$\|Kg_{n_i}\| \leq \|K\| \, \|g_{n_i}\| = \|K\| = |\lambda|,$$

and consequently

$$\|y\| \leq |\lambda|.$$

But since the left-hand side of (J.7.9) is non-negative $\|y\| \geq |\lambda|$, it follows that $\|y\| = |\lambda|$, and therefore

$$\lim_{i\to\infty} \|Kg_{n_i} - \lambda g_{n_i}\| = 0. \tag{J.-169}$$

But

$$\|y - \lambda g_{n_i}\| \leq \|Kg_{n_i} - y\| + \|Kg_{n_i} - \lambda g_{n_i}\|$$

Thus by (J.7.9) and (J.7.9) it follows that $\lim_{i\to\infty} g_{n_i}$ exists and is equal to y/λ. Then (J.7.9) can be rewritten as

$$Ky - \lambda y = 0.$$

$\square$

Lemma J.7.27 *Let M be a closed linear subspace of a Hilbert space $\mathcal{H}$ and let M be invariant under a bounded linear operator T on $\mathcal{H}$ (that is, $TM \subseteq M$). Then $M^{\perp}$ is invariant under $T^{\dagger}$.*

Proof: Consider $x \in M^{\perp}$ and $m \in M$. Then $Tm \in M$ and hence $(Tm, x) = 0$. Thus $(m, T^{\dagger}x) = 0$ for all $m \in M$, so that $T^{\dagger}x \in M^{\perp}$.

□

Theorem J.7.28 *Let K be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. There exists a finite or infinite orthonormal sequence $\{e_n\}$ of eigenvectors of K, corresponding to non-zero eigenvalues*

$$\lambda_1, \lambda_2, \lambda_3, \ldots \qquad (|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots$$

We obtain the eigenvectors inductively. By theorem J.7.26 K has an eigenvalue $\lambda_1 = \pm\|K\|$, and we may pick a corresponding eigenvector e_1 of unit norm.

$$Ke_1 = \lambda_1 e_1$$

Since the linear span of e_1 is invariant under K, by lemma J.7.27, its orthogonal complement $\mathcal{H}_2$ is invariant under $K^{\dagger}(= K)$ (let us redenote K as K_1). Thus $K_1\mathcal{H}_2 \subseteq \mathcal{H}_2$ and so we may define K_2 to be the restriction of K to $\mathcal{H}_2$. Clearly K_2 is compact, and if $x, y \in \mathcal{H}_2$ then

$$\begin{aligned} (K_2^{\dagger}x, y) &= (x, K_2 y) = (x, Ky) \\ &= (K^{\dagger}x, y) = (x, Ky). \end{aligned}$$

Since $Kx \in \mathcal{H}_2$, it follows that $Kx = K_2^{\dagger}x$ for all $x \in \mathcal{H}_2$. Thus $K_2^{\dagger}$ is the restriction of K to $\mathcal{H}_2$, that is, $K_2^{\dagger} = K_2$. Thus K_2 is a self-adjoint operator on $\mathcal{H}_2$. So if K_2 is not identically zero, we may apply theroem J.7.26 again to obtain an eigenvalue $\lambda_2 = \pm\|K_2\|$ of K_2, and hence of K, corresponding to a unit eigenvector e_2 of K_2,

$$K_2 e_2 = \lambda_2 e_2$$

Since $e_2 \in \mathcal{H}_2$, we have $(e_2, e_1) = 0$ and

$$|\lambda_2| = \sup_{\|f\|=1, f\in\mathcal{H}_2} |(K_1 f, f)| \le \sup_{\|g\|=1} |(K_1 g, g)| = |\lambda_1|$$

We continue this inductive construction as long as $K_n \neq 0$. If we encounter m such that $K_m = 0$ the construction stops. Then for an arbitrary $f \in \mathcal{H}$,

$$f - \sum_{j=1}^{m-1} (f, e_j) e_j \in \mathcal{H}_m.$$

Denoting the left-hand side by g, we have

$$\begin{aligned} 0 &= K_m g = K g \\ &= K g - \sum_{j=1}^{m-1} (f, e_j) K e_j \end{aligned} \tag{J.-171}$$

That is, for any $f \in \mathcal{H}$

$$\begin{aligned} K g &= \sum_{j=1}^{m-1} (f, e_j) K e_j \\ &= \sum_{j=1}^{m-1} \lambda_j (f, e_j) e_j, \end{aligned} \tag{J.-171}$$

and the assertion is proved with finite sum.

If the process does not terminate, we obtain an infinite, orthonormal sequence $\{e_k\}_{k=1}^{\infty}$. Consier an arbitray $f \in \mathcal{H}$ and let

$$g_n = f - \sum_{j=1}^{n-1} (f, e_j) e_j.$$

Then $g_n \in \mathcal{H}_n$. Applying Pythagoras' theorem to the relation

$$f = g_n + \sum_{j=1}^{n-1} (f, e_j) e_j$$

we see that

$$\|f\|^2 = \|g_n\|^2 + \sum_{j=1}^{n-1} |(f, e_j)|^2$$

so that

$$\|g_n\| \le \|f\|.$$

Now

$$\begin{aligned} \|Kg_n\| &= \|K_n g_n\| \le \|K_n\| \, \|g_n\| \\ &\le |\lambda_n| \, \|f\|. \end{aligned}$$

That is,

$$\left\| Kf - \sum_{j=1}^{n-1} \lambda_j (f, e_j) e_j \right\| \le |\lambda_n| \, \|f\|$$

By corollary J.7.20, $\lambda_n \to 0$ as $n \to \infty$. Hence

$$Kf = \sum_{j=1}^{\infty} \lambda_j (f, e_j) e_j$$

for all $f \in \mathcal{H}$.

□

Corollary J.7.29 *(Hilbert-Schmidt theorem) Let K be a compact operator on a separable infinite-dimensional Hilbert space $\mathcal{H}$. There exists a complete orthonormal sequence $\{e_n\}_{n \in \mathbb{N}}$ of eigenvectors of K with real eigenvalues λ_n which converge to zero. For any $f \in \mathcal{H}$,*

$$Kf = \sum_{n=1}^{\infty} \lambda_n (f, e_n) e_n.$$

Proof: By theorem J.7.28 there is a finite or infinte orthonormal sequence $\{x_n\}$ such that, for all $f \in \mathcal{H}$,

$$Kf = \sum_n \lambda_n (f, x_n) x_n. \tag{J.-173}$$

We may suppose that each $\lambda_n \neq 0$. Let $\{\varphi_n\}$ be a complete orthonormal system in the Hilbert space Ker K, the kernel of K. Then each φ_m is an eigenvector of K corresponding to the eigenvector 0. As x_n is an eigenvector corresponding to a different eigenvalue, x_n is orthogonal to φ_n by theorem ??. Thus $\{x_n\} \cup \{\varphi_n\}$ is a countable orthonormal set in $\mathcal{H}$. For any $f \in \mathcal{H}$, (J.7.9) implies that

$$f - \sum_n (f, x_n) x_n \in \mathrm{Ker} K,$$

and since $\{\varphi_n\}$ is an orthonrmal basis of Ker K,

$$f - \sum_n (f, x_n) x_n = \sum_m (f, \varphi_m) \varphi_m.$$

Thus $\{x_n\} \cup \{\varphi_n\}$ is a complete orthonormal set in $\mathcal{H}$. Since dim $\mathcal{H} = \infty$ so this set is infinite, so it may be indexed by $\mathbb{N}$, as $\{e_n\}_{n \in \mathbb{N}}$.

□

Theorem J.7.30 *Let K be a compact operator of a Hilbert space $\mathcal{H}$. Then there exist orthonormal but not necessarily complete systems $\{x_n\}_{n=1}^N$ and $\{y_n\}_{n=1}^N$ as well as a sequence of positive real numbers $\{\lambda_n\}_{n=1}^N$ converging to zero, called the singular values of K such that*

$$K = \sum_{n=1}^N \lambda_n (x_n, .) y_n. \tag{J.-173}$$

Proof: Since K is compact so is $K^\dagger K$. To see this, let $\{f_n\}_{n=1}^\infty$ be any sequence of bounded vectors ($\|f_n\| < C$ for all n). Since

$$\begin{aligned}
\|K^\dagger f_n - K^\dagger f_m\|^2 &= (K^\dagger (f_n - f_m), K^\dagger (f_n - f_m)) \\
&= (KK^\dagger (f_n - f_m), (f_n - f_m)) \\
&\leq \|K^\dagger\| \, \|Kf_n - Kf_m\| \, \|f_n - f_m\|,
\end{aligned}$$

$K^\dagger$ is bounded and $\|f_n - f_m\| < 2C$, it follows

$$\lim_{n,m\to\infty} \|K^\dagger f_n - K^\dagger f_m\| = 0,$$

i.e., the sequence $K^\dagger\{f_n\}_{n=1}^\infty$ converges. Since the product of two compact operators is compact $K^\dagger K$ is compact.

As $K^\dagger K$ is compact and self-adjoint, by the Hilbert-Schmidt theorem, there is an orthonormal set $\{x_n\}_{n=1}^N$ such that

$$K^\dagger K x_n = \mu_n x_n \qquad \text{with } \mu_n \neq 0$$

and so that $K^\dagger K$ is the zero operator on the subspace orthogonal to $\{x_n\}_{n=1}^N$.

Since $K^\dagger K$ is positive, each $\mu_n > 0$. Let λ_n be the positive square root of μ_n and set $y_n = Kx_n/\lambda_n$. As

$$\begin{aligned}
(y_n, y_m) &= \frac{1}{\lambda_n\lambda_m}(K^\dagger K x_n, x_m) \\
&= \frac{\mu_n}{\lambda_n\lambda_m}(x_n, x_m) = \frac{\mu_n}{\lambda_n\lambda_m}\delta_{mn} = \delta_{mn}
\end{aligned}$$

the vectors y_n are orthonormal. Now we prove the claimed expansion for Kf. Consider

$$\begin{aligned}
K^\dagger K f &= \sum_n \mu_n (f, x_n) x_n \\
&= \sum_n (f, x_n)(K^\dagger K x_n).
\end{aligned}$$

Set

$$g = f - \sum_n (f, x_n) x_n,$$

then

$$\begin{aligned}
\|Kg\|^2 &= (Kg, Kg) \\
&= (K^\dagger K g, g) \\
&= 0. \qquad \text{(J.-181)}
\end{aligned}$$

Therefore

$$Kf = \sum_n (f, x_n) K x_n = \sum_n \lambda_n (f, x_n) y_n.$$

□

The proof shows that the singular values of K are precisely the eigenvalues of $|K| = \sqrt{K^\dagger K}$.

Theorem J.7.31 *Let X be a separable Hilbert space and T a bounded positive operator. Let $\{b_n\}$ be any orthonormal basis. Then*

$$Tr(T) := \sum_n^\infty < b_n, T b_n >$$

is independent of the orthonormal basis and is called the trace of T. It satisfies the following properties:

(1) $Tr(T_1 + T_2) = Tr(T_1) + Tr(T_2)$,

(2) $Tr(\lambda T) = \lambda Tr\ (T)$,

(3) $Tr(UTU^{-1}) = Tr(T)$ for all unitary operators U and

(4) $T \geq 0$ implies $Tr(T) \geq 0$ for all positive T_I and $\lambda \in \mathbb{C}$.

Proof: Given an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$, define $\mathrm{Tr}_\varphi(A) = \sum_{n=1}^\infty (\varphi_n, A\varphi_n)$. If $\{\psi_n\}_{n=1}^\infty$ is another orthonormal basis then

$$\begin{aligned}
(\psi_l, A^{1/2}\varphi_n) &= (\psi_l, \sum_{m=1}^\infty c_m \psi_m) \\
&= \sum_{m=1}^\infty c_m (\psi_l, \psi_m) = c_l.
\end{aligned}$$

Using this in the trace formula,

$$
\begin{aligned}
\mathrm{Tr}_\varphi(A) &= \sum_{n=1}^{\infty}(\varphi_n, A\varphi_n) = \sum_{n=1}^{\infty}\|A^{1/2}\varphi_n\|^2 \\
&= \sum_{n=1}^{\infty}(A^{1/2}\varphi_n, A^{1/2}\varphi_n) \\
&= \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} c_m\psi_m, A^{1/2}\varphi_n) \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|(\psi_m, A^{1/2}\varphi_n)|^2\right) \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}|(A^{1/2}\psi_m, \varphi_n)|^2\right) \\
&= \sum_{m=1}^{\infty}\|A^{1/2}\psi_m\|^2 \\
&= \sum_{m=1}^{\infty}(\psi_m, A\psi_m) \\
&= \mathrm{Tr}_\psi(A)
\end{aligned}
$$

Properties 1), 2),tracetheorem and 4) are obvious. To prove 3) we note that $\{\varphi_n\}$ is an orthonormal basis, then so is $\{U\varphi_n\}$. Thus,

$$\mathrm{Tr}(UAU^{-1}) = \mathrm{Tr}_{(U\varphi)}(UAU^{-1}) = \mathrm{Tr}_\varphi(A) = Tr(A)$$

□

Two important subsets of compact operators based on the trace are the following.

Definition A bounded operator T on a separable Hilbert space X is called Hilbert-Schmidt if and only if $\mathrm{Tr}(T^\dagger T) < \infty$. We denote the family of Hilbert-Schmidt operators by $B_2(X)$.

□

Definition A bounded operator T on a separable Hilbert space X is called **trace class** if and only if $\mathrm{Tr}(|T|) < \infty$. We denote the family of trace class operators by $B_1(X)$.

For any $T \in B_1(X)$ we extend the trace by $\mathrm{Tr}(T) := \sum_n < b_n, Tb_n >$ which is independent of the orthogonal basis (b_n). The trace satisfies $\mathrm{Tr}(T^\dagger) = \overline{\mathrm{Tr}(T)}$ and $\mathrm{Tr}(AT) = \mathrm{Tr}(TA)$ for $T \in B_1(X)$ and $A \in B_2$.

$\square$

A bounded operator A on a separable Hilbert space $\mathcal{H}$ is said to be trce class if $|A| = \sqrt{A^\dagger A}$ has finite trace, that is $Tr(|A|) := \sum_I < b_I, |A| b_I >< \infty$ where e_I is any orthonormal basis for $\mathcal{H}$. Then $\mathrm{Tr}(A)$ is independent of the basis chosen.

An operator A is called Hilbert-Schmidt if $A^\dagger$ is trace class. Trace class operators are dense in a Hilbert space with inner product $< A, B >= \mathrm{Tr} A^\dagger B$ and its completion are the Hilbert-Schmidt operators.

Mixed states or density matrices are positive trace class operators of unit trace.

Hilbert-Schmidt operators naturally appear in Bogol'ubov transformations

J.7.10 Hilbert-Schmidt operators

We define the Hilbert-Schmidt norm $\|A\|_2$ as

$$\|A\|_2 := \left[\sum_{n=1}^{\infty} \|A f_n\|^2 \right]^{1/2}$$

which, by theorem J.7.31, when finite, is independent of the orthonormal basis in $\mathcal{H}$.

Theorem J.7.32 *Hilbert-Schmidt operators are compact.*

Proof: Let A be a bounded linear operetor $A : X \to Y$ and $\{e_n\}$ be a complete orthnormal sequence in X. We shall show that A is compact by expressing it as a norm limit of a sequence of finite rank operators. Define A_k for $k \in \mathbb{N}$ by

$$A_k \left(\sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^{k} x_n A e_n$$

where

$$x = \sum_{n=1}^{\infty} x_n e_n \tag{J.-191}$$

is an arbitrary element of X. Thus A_k agrees with A in the span of $e_1, \ldots, e_k$ and is zero on the span of the remaining e_n's. the rank of A_k is at most k, and so A_k is compact. For x as in (J.7.10),

$$\begin{aligned}
(A - A_k)x &= \sum_{n=1}^{\infty} x_n e_n - \sum_{n=1}^{k} x_n A e_n \\
&= \sum_{n=k+1}^{\infty} x_n e_n
\end{aligned}$$

Hence

$$\begin{aligned}
\|(A - A_k)x\| &= \sum_{n=k+1}^{\infty} |x_n| \, \|Ae_n\| \\
&\leq \left\{ \sum_{n=k+1}^{\infty} |x_n|^2 \right\}^{1/2} \left\{ \sum_{n=k+1}^{\infty} \|Ae_n\|^2 \right\}^{1/2} \\
&\leq \|x\| \left\{ \sum_{n=k+1}^{\infty} \|Ae_n\|^2 \right\}^{1/2}
\end{aligned}$$

It follows from the definition of the operator norm that

$$\|(A - A_k)x\| \leq \left\{ \sum_{n=k+1}^{\infty} \|Ae_n\|^2 \right\}^{1/2} .$$

The sum is the tail of a convergent series, and hence tends to zero as $k \to \infty$.

□

Lemma J.7.33 *Let A be an operator such that the series $\sum_{n=1}^{\infty} \|Af_n\|^2$ converges for some orthonormal basis $f_1, f_2, \ldots$ in $\mathcal{H}$. Then*

$$\|A\| \leq \|A\|_2$$

Proof: Since we may take any unit vector as f_1,

$$\|Af_1\| \leq \left[\sum_{n=1}^{\infty} \|Af_n\|^2 \right]^{1/2}$$

from which it follows

$$\|A\| = \sup_{\|f\|=1} \|Af\| \leq \|A\|_2.$$

□

Theorem J.7.34 *In order that a the operator A be of Hilbert-Schmidt type, it is necessary and sufficient that $\sum_{n=1}^{\infty} \lambda_n^2$ converge.*

Proof: First we prove necessity. Let $\{e_n\}$ be an orthonormal basis in $\mathcal{H}$ consisting of eigenvectors of $A^{\dagger}A$. Then $\lambda_n = (A^{\dagger}Ae_n, e_n) = \|Ae_n\|$. As the trace is independent of the basis in $\mathcal{H}$, $\sum_{n=1}^{\infty} \|Af_n\|^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{n=1}^{\infty} \lambda_n^2$. As the trace is finite by the definition of a Hilbert-Schmidt operator, the series $\sum_{n=1}^{\infty} \lambda_n^2$ is finite.

Now we prove that convergence of the series $\sum_{n=1}^{\infty} \lambda_n^2$ is sufficient for A to be a Hilbert-Schmidt operator.

First we prove it is compact.

Let us denote by A_k the finite rank operator

$$A_k f = \sum_{n=1}^{k} (Af, e_n) e_n.$$

Then

$$\|A - A_k\|^2 \leq \|A - A_k\|_2^2 = \sum_{n=1}^{\infty} \|(A - A_k)e_n\|^2 = \sum_{n=k+1}^{\infty} \lambda_n^2.$$

From the convergence of the series $\sum_{n=1}^{\infty} \lambda_n^2$ it follows that $\lim_{k\to\infty} \|A - A_k\| = 0$. Therefore A is the limit in the operator norm of a sequence of finite rank operators. Since finite rank operators are compact, the operator A is also compact.

Since $\lambda_n = \|Ae_n\|$ where $e_1, e_2, \ldots$ is an orthonormal basis consisting of eigenvectors of $A^{\dagger}A$, and as the series $\sum_{n=1}^{\infty} \lambda_n^2$ converges, the series $\sum_{n=1}^{\infty} \|Af_n\|^2$, where $f_1, f_2, \ldots$ is any orthonormal basis of $\mathcal{H}$, also converges and so A is a Hibert-Schmidt operator.

□

Definition We call the number $\|A\|_2$ the **Hilbert-Schmidt** norm of A.

$\square$

Obviously the Hilbert-Schmidt norm is finite for Hilbert-Schmidt operators and only for such operators. It satifies the basic properties of norms:

$$\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$$

for

$$\begin{aligned} \|A+B\|_2 &= \left[\sum_{n=1}^{\infty} \|(A+B)f_n\|^2\right]^{1/2} \\ &\leq \left[\sum_{n=1}^{\infty} (\|Af_n\| + \|Bf_n\|)^2\right]^{1/2} \\ &\leq \left[\sum_{n=1}^{\infty} \|Af_n\|^2\right]^{1/2} + \left[\sum_{n=1}^{\infty} \|Bf_n\|^2\right]^{1/2} = \|A\|_2 + \|B\|_2 \end{aligned}$$

and

$$\|\lambda A\|_2 = |\lambda| \|A\|_2.$$

The space of Hilbert-Schmidt operators is closed in the Hilbert-Schmidt norm.

$$\|A\|_2 = \left[\sum_{n=1}^{\infty} \|Af_n\|^2\right]^{1/2} = \left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |(Af_n, h_k)|^2\right]^{1/2}.$$

From this it follows that the space of Hilbert-Schmidt operators is isomophic to the space of infinite matrices $\|a_{nk}\|$ for which the series $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |a_{nk}|^2$ converges. As is well know, this form a Hilbert space.

J.7.11 Trace Class Operators

Lemma J.7.35 *A positive operator is self-adjoint.*

Proof: Let T be positive, that is

$$(Tx, x) \geq 0$$

indicating it is real number, therefore

$$(Tx, x) = (Tx, x)^* = (x, Tx) = (T^\dagger x, x).$$

This implies

$$(T(x + \lambda y), x + \lambda y) = (T^\dagger(x + \lambda y), x + \lambda y)$$

for any complex number λ. The expansion of both sides is displayed below

$$\begin{aligned}
(T(x + \lambda y), x + \lambda y) &= (Tx, x) + |\lambda|^2(Ty, y) + \lambda(Tx, y) + \lambda^*(Ty, x) \\
(T^\dagger(x + \lambda y), x + \lambda y) &= (T^\dagger x, x) + |\lambda|^2(T^\dagger y, y) + \lambda(T^\dagger x, y) + \lambda^*(T^\dagger y, x).
\end{aligned}$$

From which we have

$$\lambda(Tx, y) + \lambda^*(Ty, x) = \lambda(T^\dagger x, y) + \lambda^*(T^\dagger y, x).$$

Taking λ to be 1 then i, and then subtracting the results gives

$$(Tx, y) = (T^\dagger x, y).$$

□

Lemma J.7.36 *In order that a compact positive operator T be trace class, it is necessary and sufficient that $\sum_{n=1}^{\infty} \lambda_n$ converge.*

Proof: Let T be a positive-definte trace class opertor. We introduce the operator $T^{1/2}$, setting

$$T^{1/2} e_n = \lambda_n^{1/2} e_n,$$

where $\{e_n\}$ is an orthonormal basis consisting of eigenvectors of T, and λ_n are corresponding eigenvalues. In view of the self-adjointness of $T^{1/2}$

$$\sum_{n=1}^{\infty} \|T^{1/2} f_n\|^2 = \sum_{n=1}^{\infty} (T^{1/2} f_n, T^{1/2} f_n) = \sum_{n=1}^{\infty} (T f_n, f_n).$$

Since

$$\sum_{n=1}^{\infty} \|T^{1/2}e_n\|^2 = \sum_{n=1}^{\infty}(Te_n, e_n) < \infty,$$

$T^{1/2}$ is a Hilbert-Schmidt operator. Therefore, for any orthonormal basis $f_1, f_2, \ldots$ we have

$$\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \|T^{1/2}e_n\|^2 = \sum_{n=1}^{\infty} \|T^{1/2}f_n\|^2.$$

Therefore for any orthonormal basis $f_1, f_2, \ldots$ in $\mathcal{H}$

$$\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty}(Tf_n, f_n) < \infty$$

from which it follows that $\sum_{n=1}^{\infty} \lambda_n$ is finite for the operator T.

Conversely, let T be a compact positive-definite operator for which $\sum_{n=1}^{\infty} \lambda_n$ is finite. We choose a basis $e_1, e_2, \ldots$ in $\mathcal{H}$ consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$. Then

$$\sum_{n=1}^{\infty}(Te_n, e_n) = \sum_{n=1}^{\infty} \lambda_n < \infty$$

from which it follows that T is a trace class operator.

□

Theorem J.7.37 *The product AB of two Hilbert-Schmidt operators is a trace class operator. Conversely, every trace class operator is the product of two Hilbert-Schmidt operators.*

Proof: Suppose that B maps $\mathcal{H}_1$ into $\mathcal{H}_2$ and A maps $\mathcal{H}_2$ into $\mathcal{H}_3$, and let $AB = UT$ be the decomposition of the operator AB into the product of a positive-definite operator T, acting on the space $\mathcal{H}_1$ and an isometric operator U which maps the range of T into the sapce $\mathcal{H}_3$. We denote by $e_1, e_2, \ldots$ an orthonormal basis in $\mathcal{H}_1$ consisting of eigenvectors of T, and by $h_1, h_2, \ldots$ the orthonormal system in $\mathcal{H}_3$ consisting of the vectors $h_n = Ue_n$, $\lambda_n \neq 0$. Then for $\lambda_n \neq 0$ we have

$$
\begin{aligned}
\lambda_n &= (Te_n, e_n) = (UTe_n, Ue_n) \\
&= (Be_n, A^\dagger h_n) \\
&\leq \|Be_n\| \, \|A^\dagger h_n\| \\
&\leq \frac{1}{2}[\|Be_n\|^2 + \|A^\dagger h_n\|^2]
\end{aligned} \tag{J.-203}
$$

where the last inequality follows from $(\|Be_n\| - \|A^\dagger h_n\|)^2 \geq 0$. If A and B are Hilbert-Schmidt operators, then the series $\sum_{n=1}^{\infty} \|Be_n\|^2$ and $\sum_{n=1}^{\infty} \|A^\dagger h_n\|^2$ converge and so the series $\sum_{n=1}^{\infty} \lambda_n$ converges. Thus we have proved that the product AB of two Hilbert-Schmidt operators is a trace class operator.

Now, let A be a trace class operator, and UT be the decomposition into the product of a positive-definite operator T and an isometric operator U. Then as was shown above, the operator $T^{1/2}$ is a Hilbert-Schmidt operator. Since

$$
\sum_{n=1}^{\infty} \|UT^{1/2} e_n\|^2 = \sum_{n=1}^{\infty} \|T^{1/2} e_n\|^2
$$

$UT^{1/2}$ is also a Hilbert-Schmidt operator. As $A = (UT^{1/2})T^{1/2}$, A is the product of two Hilbert-Schmidt operators.

□

Corollary J.7.38 *The space of trace class operators forms a two-sided* * *ideal in the space of bounded operators, i.e.*

1) The adjoint $A^\dagger$ *of a trace class operator* A *is a trace class operator.*

2) The product AB *of any bounded linear operator* A *with a trace class opertor* B *is a trace class operator. The analogous assertion holds for the product* BA*.*

Proof of 1): With $A = UT^{1/2}T^{1/2}$, then $A^\dagger = T^{1/2}(UT^{1/2})^\dagger$. The operator $(UT^{1/2})^\dagger$, as the adjoint of a Hilbert-Schmidt operator is also Hilbert-Schmidt. Consequently A is a trace class operator.

Proof of 2): If B is a trace class operator then $B = UT^{1/2}T^{1/2}$, where $UT^{1/2}$ and $T^{1/2}$ are Hilbert-Schmidt operators. Consequently,

$$
AB = (AUT^{1/2})T^{1/2}
$$

is the product of two Hilbert-Schmidt operators, and therefore is a trace class operator. The analogous assertion holds for the product BA, since $(A^\dagger B^\dagger)^\dagger$

□

Lemma J.7.39 *Every trace class operator*

$$\sum_{n=1}^{\infty}(ABe_n, e_n) = \sum_{n=1}^{\infty}(BAe_n, e_n) \tag{J.-203}$$

Proof:

□

Corollary J.7.40 *In order that a compact A be trace class, it is necessary and sufficient that $\sum_{n=1}^{\infty} \lambda_n$ converge.*

Proof: We prove that for any unitary opertor V and any trace class operator $A = UT$ we have

□

Corollary J.7.41 *Every trace class operator is a Hilbert-Schimdt operator.*

Proof: Since the converge of the series $\sum_{n=1}^{\infty} \lambda_n^2$ follows from the convergence of $\sum_{n=1}^{\infty} \lambda_n$, every trace class operator is a Hilbert-Schimdt operator.

□

J.8 Convergence of Sequences of Operators and Functionals

Definition Weak convergence in a Hilbert space.

We say that a sequence of vectors $\{f_n\}$ conveges weakly to a vector f, if for every $h \in \mathcal{H}$,

$$\lim_{n\to\infty} (f_n, h) = (f, h).$$

□

Strong convergence implies weak convergence.

Proof:

Consider the sequence $\{h_n\}$ that converges strongly to h (i.e., $\lim_{n\to\infty} \|h_n - h\|$). Schwarz's inequailty gives

$$|\langle h_n - h, f\rangle| \leq \langle h_n - h, h_n - h\rangle \langle f, f\rangle = \|h_n - h\| \, \|f\|.$$

Therefore as $n \to \infty$ the definition of weak convergence is satisfied.

□

Weak convergence does not imply strong convergence.

Proof:

Let $\{e_n\}_{n=1}^{\infty}$ be any infinite orthonormal sequence of vectors in $\mathcal{H}$. Since

$$\sum_{k=1}^{\infty} |(h, e_k)|^2 \leq (h, h) \qquad \text{(by Schwartz inequality } |\langle h, e_n\rangle|^2 \leq \langle h, h\rangle \langle e_n, e_n\rangle\text{)},$$

it follows that, for any $h \in \mathcal{H}$,

$$\lim_{k\to\infty} (e_k, h) = 0.$$

Thus the sequence $\{e_n\}_{n=1}^{\infty}$ converges weakly to the vector 0, but this sequence does not converge strongly, because

$$\|e_m - e_n\|^2 = 2 \qquad (m \neq n)$$

and therefore $\|e_m - e_n\|$ does not tend to 0 as $m, n \to \infty$.

□

Convergence of sequence of operators. Let X and Y be normed spaces. A sequence (T_n) of operators $T_n \in \mathcal{B}(X, Y)$ is said to be

(i) if (T_n) converges in the norm on $\mathcal{B}(X, Y)$

$$\|T_n - T\| \to 0 \tag{J.-202}$$
$$\|T_n x - Tx\| \to 0 \quad \text{for all } x \in X \tag{J.-201}$$
$$\|f(T_n x) - f(Tx)\| \to 0 \quad \text{for all } x \in X \text{ and for all } f \in Y'. \tag{J.-200}$$

T is called the uniform, strong and weak operator limit of T_n, respectively.

for the other four topologies sequencies are not sufficient. the resulting topologies are not first countable, and so the closure of a subset N of $\mathcal{B}(\mathcal{H})$ is generally larger than the set of all limit points of sequencies in N. Rather, the closure of N is the set of all limit points of generalised sequencies (nets) in N.

Definition In terms of nets.

A net $\{T_\alpha\}$ converges to T in the strong operator topology if T_α converges to $T_\alpha \xi$ for every $\xi \in \mathcal{H}$.

A net $\{T_\alpha\}$ converges to T in the weak operator topology if $(T_\alpha \xi, \eta) \to (T\xi, \eta)$ converges to $T_\alpha \xi$ for all $\xi, \eta \in \mathcal{H}$.

$\square$

Definition Strong and weak* convergence of a sequence of functionals. Let (f_n) be a sequence of bounded linear functionals on a normed space X. Then:

(a) Strong convergence of (f_n) means that there is an $f \in X'$ such that $\|f_n - f\| \to 0$.

(b) weak* convergence of (f_n) means that there is an $f \in X'$ such that $f_n(x) \to f(x)$ for all $x \in X$.

$\square$

the norm topology is characterized by convergence of sequences. The other topologies requires the use of a generalization of sequences called nets. This we will come to after we have more topology.

for example when wish to investigate semi-classical issues in LQG when we do not yet have a physical inner product.

Definition The ultrastrong topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A; (x)_1^\infty, \epsilon) = \{B \in \mathcal{B}(\mathcal{H}) : |\sum_{i=1}^{\infty} \|(A - B)x_i\|^2| < \epsilon\} \tag{J.-200}$$

for $A \in \mathcal{B}(\mathcal{H})$, $\epsilon > 0$ and any sequences in $\mathcal{H}$ satisfying $\sum_{i=1}^\infty \|x_i\|^2 < \infty$.

$\square$

J.9 More Topology

we consider projective limits of infinite families of finite dimensional topological and measurable spaces.

J.9.1 Basis and Subbasis for a Topology

The notion of a basis for a finite dimensional vector space: It is a minimal collection of vectors that spans the vector space. If you know a basis, you can always recover the vector space. A similar notion of a basis exists for a topology as a family of open subsets that 'span' the topology.

Definition A basis for a given topology $\mathcal{T}$ on X is a subcollection $\mathcal{B}$ of $\mathcal{T}$ such that any open set in X is a union of members of $\mathcal{B}$.

□

Definition A subbasis for a given topology is $\mathcal{T}$ on X is a subcollection $\mathcal{S}$ of $\mathcal{T}$ such that the collection of all finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{T}$. Hence, any open subset of X is a union of finite intersections of members in $\mathcal{T}$.

□

Definition If $Y \subset X$, and $X, \mathcal{T}$ is a topological space, then relative topology on Y consists of the sets

$$\{A \cap Y | A \in \mathcal{T}\}. \tag{J.-200}$$

□

J.9.2 Hausdorff Spaces

Definition A *Hausdorff space* if the following is true: If $p \in X$, $q \in X$, and $p \neq q$, then p has a neighborhood U and q has a neighborhood V such that $U \cap V$.

□

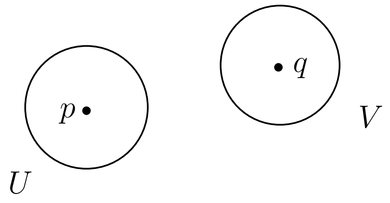


Figure J.8: hausdorff.

Theorem J.9.1 *Metric spaces are Hausdorff.*

Proof: If p and q are distinct points in a metric space, then $r = d(p, q) \neq 0$. Thus the open balls $B_{r/3}(p)$ and $B_{r/3}(q)$ are disjoint neighborhoods of p and q.

□

Theorem J.9.2 *In a Hausdorff space, a convergent sequence has at most one limit point.*

Proof: Suppose we have a convergent sequence $\{x_n\}$ with two different limit points y and z. As $y \neq z$, there are disjoint neighbourhoods U and V of y and z, respectively. Since $x_n \to y$, there is $m \in \mathbb{N}$ such that $x_n \in U$ whenever $n \geq m$. Similarly, there is $m' \in \mathbb{N}$ such that $x_n \in V$ whenever $n \geq m'$. Let $l \in \mathbb{N}$ such that $m \leq l$ and $m' \leq l$. Then $x_l \in U, V$ i.e. $x_l \in U \cap V$, which is impossible since $U \cap V = \emptyset$. It follows that $y = z$.

□

Lemma J.9.3 *In Hausdorff spaces one point sets are closed.*

Proof: One point of a topological space is cloesd if and only if its complement is open which is true if and only if each point x' different from x has an open neighbourhood which does not contain x. Clearly, for any Hausdorff space one point sets are closed.

□

Theorem J.9.4 *The finite product of Hausdorff spaces is Hausdorff.*

Proof: Let $X := \prod_i X_i$ be the product of Hausdorff spaces X_i. If $x \neq x'$ then we must have $x_{i_0} \neq x'_{i_0}$ for at least one index, i_0. Since X_{i_0} is a Hausdorff space, x_{i_0} and x'_{i_0} can be separated by open sets in X_{i_0}. These two disjoint open subsets of X_{i_0} give rise to two disjoint sets in the defining open subbase for X, each of which contains one of the points x and x'.

□

Lemma J.9.5 *In a Hausdorff space, any point and disjoint compact subspace can be separated by open sets, that is, they have disjoint open neighbourhoods.*

Proof: Let X be a Hausdorff space, x a point in X, and C a compact subspace of X which does not contain x. We construct a disjoint pair of open sets U and V such that $x \in U$ and $C \subseteq V$. Let y be a point in C. Since X is Hausdorff, x and y have disjoint open neighbourhoods U_x and V_y. If we allow y to vary over C, we obtain a class of V_y's whose union contains C. Since C is compact, there is some finite subcover, i.e, a subclass $\{V_1, V_2, \dots V_n\}$ such that $C \subseteq \cup_{i=1}^{n} V_i$. If $U_1, U_2, \dots, U_n$ are the open sets of x which correspond to the V_i's, then any V_i is disjoint from the intersection of the U_i's so

$$U := \bigcap_{i=1}^{n} U_i$$

and

$$V := \bigcup_{i=1}^{n} V_i$$

are such that $x \in U$, $C \subseteq V$, and $U \cap V = \emptyset$.

□

Lemma J.9.6 *Every compact subspace of a Hausdorff space is closed.*

Proof: Let C be a compact subspace of a Hausdorff space X. We prove C is closed by showing that X/C is open. Let x be any point in X/C. By lemma J.9.5, x has an open neighbourhood G such that $x \in G \subseteq X/C$. This shows that X/C is a union of open sets and is therefore open itself.

□

Definition X is *locally compact* if every point of X has a neighbourhood whose closure is compact.

□

The compact subsets of a Euclidean space $\mathbb{R}^n$ are precisely those that are closed and bounded. From this it follows easily that $\mathbb{R}^n$ is a localy compact Hausdorff space. Also, every metric space is locally compact.

Definition A collection of functions $\mathcal{C}$ on a (topological) space X is said to separate its points if and only if for any $x_1 \neq x_2$ we can find $f \in \mathcal{C}$ such that $f(x_1) \neq f(x_2)$.

□

The only if part of the definition says given the values assumed by each and every function in the collection $\mathcal{C}$ exists a unique point $p \in X$ for which the functions take their given values.

Lemma J.9.7 *Let X be a topological space and $\mathcal{C} \subset C(X)$ a collection of continuous functions X which separate the points of X. Then the topology of X is Hausdorff.*

Proof of Lemma J.9.7:

Let $x_1, x_2 \in X$ with $x_1 \neq x_2$ be any two distinct points. Since $\mathcal{C}$ separates points we find $f \in \mathcal{C}$ with $f(x_1) \neq f(x_2)$. Let $d := |f(x_2) - f(x_1)| > 0$. Since f is continuous at points x_1 and x_2, for any $\epsilon > 0$ we find a neighbourhood $U_1(\epsilon)$ of x_1 such that

$$|f(x) - f(x_1)| < \epsilon$$

for any $x \in U_1(\epsilon)$ and a neighbourhood $U_2(\epsilon)$ of x_2 such that

$$|f(x) - f(x_2)| < \epsilon$$

for any $x \in U_2(\epsilon)$. Now we consider any one point $x \in X$ and show it cannot be in both $U_1(\epsilon)$ and $U_2(\epsilon)$ at the same time if we take ϵ to be small enough. We have

$$d = |f(x_2) - f(x_1)| \leq |f(x_2) - f(x)| + |f(x_1) - f(x)|$$

for any $x \in X$. Thus

$$d - \epsilon < |f(x_2) - f(x)| \text{ for any } x \in U_1(\epsilon)$$

and

$$d - \epsilon < |f(x_1) - f(x)| \text{ for any } x \in U_2(\epsilon).$$

Take x to be a point in $U_1(\epsilon) \cap U_2(\epsilon)$. If we choose $\epsilon < d/2$ the previous two inequlities give

$$d/2 < |f(x_2) - f(x)| \text{ and } d/2 < |f(x_1) - f(x)|.$$

Now, the point x cannot satisfy both these conditions at the same time, hence

$$U_1(\epsilon) \cap U_2(\epsilon) = \emptyset.$$

□

J.9.3 Urysohns's Lemma

Definition A **normal space** is a space such that one point sets are closed and any two disjoint closed sets A, B are contained in disjoint open sets.

□

Theorem J.9.8 *Every compact Hausdorff space is normal.*

Proof: By lemma J.9.3 in Hausdorff spaces one point sets are closed.

Let X be a compact Hausdorff space, and A and B disjoint closed subsets of X. We must construct a pair of disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Since X is compact, A and B are disjoint compact subspaces of X. Let x be a point of A. By lemma J.9.5, x and B have disjoint neighbourhoods U_x and V_B. If we allow x to vary over A, we obtain a class of U_x's whose union contains A. Since A is compact, there is a finite subcover, i.e., a subclass, $\{U_1, U_2, \ldots, U_n\}$ such that $A \subseteq \cup_{i=1}^n U_i$. Let $V_1, V_2, \ldots, V_n$ be the open neighbourhoods of B which are correspond to the U_i's, it is clear that $U := \cup_{i=1}^n U_i$ and $V := \cap_{i=1}^n V_i$, are disjoint open neighbourhoods of A and B.

□

Lemma J.9.9 *A topological space X is normal if and only if for every closed subset $A \subset X$ and every open subset $B \subset X$ containing A, there exists an open set U such that*

$$A \subset U \subset \overline{U} \subset B.$$

Proof: Suppose that X is normal and A and B are as given in the lemma. Then the sets A and X/B are closed and disjoint. By the normailty of X, there exist open disjoint sets U and V such that $A \subset U$ and $X/B \subset V$. Then

$$\overline{U} \subset X/V \subset B,$$

so that U has the properties of the lemma.

Conversely, let A and B be closed disjoint subsets of X. Then $V = X/B$ is open and $A \subset V$. By hypothesis there exists an open set U such that $A \subset U \subset \overline{U} \subset V$. Then U and $X/\overline{U}$ are disjoint open sets satisfying $A \subset U$ and $B \subset X/\overline{U}$ ($\overline{U} \subset X/B$ implies $B \subset X/\overline{U}$). So X is normal.

□

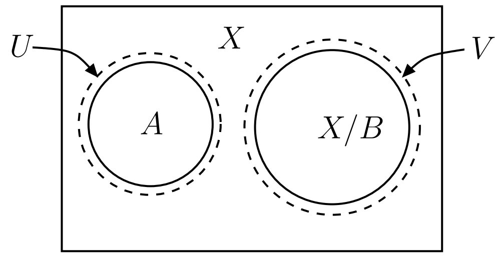


Figure J.9:

Lemma J.9.10 (Urysohn's Lemma). *In a normal topological space* $(X, \mathcal{T})$ *there is for each pair* A, B *of disjoint closed sets a continuous function* $f : X \to [0, 1]$ *that is* 0 *on* A *and 1 on* B*.*

Proof: For the proof recall that a dyadic rational number is a number which can be written in the form $p = \frac{m}{2^n}$ with n, m being integers. Set $V = X/B$, an open set which contains the closed set A. By the previous lemma, there exists an open set $U_{1/2}$ such that

$$A \subset U_{1/2} \subset \overline{U_{1/2}} \subset V. \tag{J.-200}$$

Applying lemma J.9.9 again to the open set $U_{1/2}$ containing A, we obtain an open set $U_{1/4}$ such that

$$A \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_{1/2}$$

and to the open set V containing $\overline{U_{1/2}}$, we obtain the open set $U_{3/4}$ such that

$$\overline{U_{1/2}} \subset U_{3/4} \subset \overline{U_{3/4}} \subset V,$$

that is,

$$A \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_{3/4} \subset \overline{U_{3/4}} \subset V.$$

Continuing in this way, we associate to every such dyadic rational number $p \in (0, 1)$ of the form $p = m/2^n$ (where $n = 1, 2, \ldots$ and $m = 1, 2, \ldots 2^n - 1$) an open subset $U_p \subset U$, such that

$$A \subset U_p \subset \overline{U_p} \subset U_q \subset \overline{U_q} \subset V$$

for $0 < p < q < 1$. Now we construct a function f which us continuous and such that the sets ∂U_p are level sets of f on which f assumes the value p. Define $f(x) = 0$ if x is in every U_p and

$$f(x) = \sup\{p : x \notin U_p\}$$

otherwise. It is clear that the values of f lie in $[0, 1]$, and that $f(A) = 0$ and $f(B) = 1$. It remains to show that f is continuous. All intervals of the form $[0, a)$ and $(a, 1]$, where $0 < a < 1$, constitute an open subbase for $[0, 1]$. It therefore suffices to show that $f^{-1}([0, a))$ and $f^{-1}((a, 1])$ are open. It is easy to see that $f(x) < a$ means that x is in some U_p for $p < a$. From this it follows that

$$f^{-1}([0, a)) = \{x : f(x) < a\} = \bigcup_{p<a} U_p,$$

which is an open set. Similarly, $f(x) > a$ means x is outside of $\overline{U_p}$ for some $p > a$.

$$f^{-1}((a, 1]) = \{x : f(x) > a\} = \bigcup_{p>a} X/\overline{U_p},$$

which is an open set.

□

J.9.4 Products of Topological Spaces

Topological Space in Terms of Closed Sets

From De Morgan's laws and the definition of a topology in terms of opens sets

i) $X \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$

ii) $\mathcal{F}$ is closed under finite union

iii) $\mathcal{F}$ is closed under arbitrary intersection.

DeMorgan'a Laws: Let A and B be subsets of X then

1) $X\backslash(A \cup B) = (X\backslash A) \cap (X\backslash B)$ and

2) $X\backslash(A \cap B) = (X\backslash A) \cup (X\backslash B)$.

open subsets $A_1, A_2, \ldots, A_n$ - finite intersctions of open sets, in terms of closed sets

$$X\backslash(\underbrace{A_1 \cap A_2 \cap \cdots \cap A_n}_{\text{finite intersection of open sets}}) = \underbrace{(X\backslash A_1) \cup (X\backslash A_2) \cup \cdots \cup (X\backslash A_n)}_{\text{finite union of closed sets}} \qquad \text{(J.-200)}$$

$$\bigcap_{\alpha} (X\backslash A_\alpha) = X\backslash(\bigcup_{\alpha} A_\alpha) \qquad \text{(J.-200)}$$

Since $\bigcup_\alpha A_\alpha$ is an open set $\bigcap_\alpha (X\backslash A_\alpha)$ is a closed set.

Compactness in Terms of Closed Sets

Recall that a topological space X is said to be compact if every open covering of X has a finite subcovering. This can be expressed in terms of closed sets instead, based on the notion of the *finite intersection property*:

Definition (**The finite intersection property (FIP)**) A nonempty family of sets $\mathcal{E}$ of a set X is said to have the finite intersection property (FIP for short) if the intersection of each nonempty finite subfamily $\mathcal{F}$ is nonempty, that is,

$$\bigcap_{E\in\mathcal{F}} E \neq \emptyset \quad \text{for every finite family } \mathcal{F} \subset \mathcal{E}. \qquad \text{(J.-200)}$$

Proposition J.9.11 *The space X is compact if every family of closed sets with the finite intersection property has nonempty intersection.*

Proposition J.9.12 *A topological space is compact if and only if every family of closed sets with the FIP has nonempty intersection.*

This means that for a family of sets $\mathcal{E}$

$$\begin{aligned} \bigcap_{E\in\mathcal{F}} E \ \neq \ \emptyset \quad & \text{for every finite family of closed sets } \mathcal{F} \subset \mathcal{E}, \\ & \text{implies } \bigcap_{E\in\mathcal{E}} E \neq \emptyset. \end{aligned} \qquad \text{(J.-200)}$$

X is compact

Proof. We show that the following statements are all equivalent

(i) every open cover of X has a finite subcover

(ii) if $\mathcal{U}$ is a family of open sets such that each finite subfamily fails to cover X, then $\mathcal{U}$ fails to cover X

(iii) if $\mathcal{U}$ is a family of open sets such that $\{X - U : U \in \mathcal{U}\}$ has the FIP, then $\cap\mathcal{U} \neq \emptyset$

(iv) every family of closed sets with the FIP has nonempty intersection.

It is obvious that (i) implies (ii). Note that any collection of opens sets which contains a finite cover of X is again a cover of X. With this trivial observation, it is easy to see (ii) implies the only open covers of X are those which contain finite subcovers. Hence (i) $\Leftrightarrow$ (ii). If a finite subfamily fails to cover X then

$$(X - U_{i_1}) \cap (X - U_{i_2}) \cap \cdots \cap (X - U_{i_n}) \neq \emptyset$$

If a collection of open sets $\mathcal{U}$ fail to cover X then $\cap\mathcal{U} \neq \emptyset$. So (iii) is just a restatement of (ii). Hence (ii) $\Leftrightarrow$ (iii) . Any closed set, by definition, is the complement, $X - U$, of an open set U. Any family of closed sets is given by the collection $\{X - U_i\}$ where the U_i's belongs to certain family of open sets $\mathcal{U}$, and so it follows that (iii) $\Leftrightarrow$ (iv).

$\square$

Proposition J.9.13 *For a family of sets, the finite intersection property implies the complete intersection property for closures.*

This means that for a family of sets $\mathcal{E}$

$$\begin{aligned} \bigcap_{E \in \mathcal{F}} E &\neq \emptyset \quad \text{for every finite family } \mathcal{F} \subset \mathcal{E}, \\ &\text{implies } \bigcap_{E \in \mathcal{E}} \overline{E} \neq \emptyset. \end{aligned} \tag{J.-200}$$

Proof:

Assume (J.-200). If the finite family of sets $\mathcal{F}$ in (J.-200) has nonempty intersection then so does the closure of each of its members. Then by condition (J.-200) we have $\bigcap_{E \in \mathcal{E}} \overline{E} \neq \emptyset$. Hence (J.-200) implies (J.-200). Conversly, (J.-200) is just a special case of (J.-200) where all the sets in the family $\mathcal{F}$ are closed. Hence (J.-200) $\Leftrightarrow$ (J.-200).

Lemma J.9.14 *FIP is closed under union.*

In Tynchonov's theorem we will...

...and we need only show the intersection over closures of elements of D is nonempty

J.9.5 Tychonov's Theorem

Tychonov's Theorem is possibly the deepest theorem in point set topology.

Tychonov Topology

We know the topological space when we know a subbase of open sets by arbitray unions and finite intersections. Since the preimages of open sets under contiuous functions are open by definition, we obtain a topology once we know which functions are continuous.

Lemma J.9.15 *the spaces are topological spaces $(X_\alpha, \mathcal{T}_\alpha)$ are topological spaces and we have maps $f_\alpha : X \to X_\alpha$. Then there is a smallest topology $\mathcal{T}$ on X for which all the maps are contiuous.*

A topology $\mathcal{T}$ on X makes all the functions f_α continuous if and only if it contains

$$\mathcal{D} = \{f_\alpha^{-1}(U) \ : \ U \in \mathcal{T}_\alpha, \ \alpha \in A\}. \tag{J.-200}$$

The weakest topology for which all f_α are continuous is generated by including the finite intersection and arbitray union of the open sets in $\mathcal{D}$, that is, the collection $\mathcal{D}$ forms a subbasis for the topology $\mathcal{T}$.

Definition The product topoloy is the smallest topology on the product for which all projections π_α are continuous.

□

The sets formed by finite intersection of the $\pi_i^{-1}(U_i)$'s are bases for the product topology If we have a family of topological spaces $(X_i, \mathcal{T}_i)_{i\in\mathcal{I}}$, then we define the product topology on $\times_{i\in\mathcal{I}} X_i$ as the topology with a basis

$$\bigcap_{i=1}^{n} \pi_i^{-1}(U_i) \tag{J.-200}$$

where i runs over a finite number of elements of indexing set $\mathcal{I}$.

Note that for $\cap_{i_j}^n \pi^{-1}(U_{i_j})$, without loss of generality we can assume $i_1, i_2, \ldots, i_n$ all different because if there were repetitions like

$$\cdots \cap \pi_{i_k}^{-1}(U) \cap \pi_{i_k}^{-1}(V) \ldots$$

we can replce each by

$$\cdots \cap \pi_{i_k}^{-1}(U \cap V) \cap \ldots$$

and so eliminate repetitions.

Definition A subfamily $\mathcal{C}$ of a family of $\mathcal{E}$ of sets is called a chain if it is totally ordered by inclusion, that is, for any two members C_i, C_j of $\mathcal{C}$ one is contained in the other or they are equal,

$$C_i \subseteq C_j \quad \Longleftrightarrow \quad C_i \succeq C_j \tag{J.-200}$$

□

Definition A family of $\mathcal{E}$ of sets is said to be closed under union if, whenever $\mathcal{C}$ is a chain in $\mathcal{E}$, we have $\cup\mathcal{C} \in \mathcal{E}$.

□

Definition A member M of $\mathcal{E}$ is said to be maximal if for any $X \in \mathcal{E}$ we have $M \subseteq X$ implies $M = X$. That is, a member is maximal if it is not a proper subset of any other memeber in the family.

□

The following Lemma, which can be derived from the Axion of choice,

Lemma J.9.16 (Zorn's Lemma). *Any family of sets closed under unions of chains has a maximal member.*

Zorn's Lemma is used in the proof of Tychonov's Theorem.

Theorem J.9.17 (Tychonov's Theorem). *If all X_α is compact, then $X = \prod_{\alpha \in A} X_\alpha$ is compact in the product topology.*

Proof of Theorem J.9.17:

Let $\{X_i : i \in I\}$ be a family of compact spaces. To show their product X is compact it suffices to show that, if $\mathcal{E}$ is any family of subsets of X with the FIP, then

$$\bigcap \overline{\mathcal{E}} \neq \emptyset, \quad \text{that is,} \quad \bigcap_{E \in \mathcal{E}} \overline{E} \neq \emptyset. \tag{J.-200}$$

Consider the collection of all such families of subsets of X, we denote it $\mathbf{\Omega}$. Take some subcollection $\mathcal{E}_i$ of $\mathbf{\Omega}$, this defines a *subset* of $\mathbf{\Omega}$. We can define a partial order on such subsets. Take any two $\mathcal{E}_1, \mathcal{E}_2 \in \mathbf{\Omega}$, we write $\mathcal{E}_1 \subseteq \mathcal{E}_2$ if for every $E_i \in \mathcal{E}_1$ there exists $E_j \in \mathcal{E}_2$ such that $E_i \subseteq E_j$. A subfamily $\mathcal{C}$ of $\mathbf{\Omega}$ is a chain if it is totally ordered by this inclusion. A member $\mathcal{M}$ of $\mathbf{\Omega}$ is maximal if for any $\mathcal{E} \in \mathbf{\Omega}$ we have $\mathcal{M} \subseteq \mathcal{E}$ implies $\mathcal{M} = \mathcal{E}$.

It is easy to verify that $\mathbf{\Omega}$ is closed under unions of chains, i.e., $\cup\mathcal{C} \in \mathbf{\Omega}$.

By Zorn's Lemma, there exists a maximal family satisfying FIP and containing $\mathcal{E}$. Let $\mathcal{M}$ be the collection of all families of subsets of X which contain $\mathcal{E}$ and have the FIP (- no collection satisfying the FIP properly contains $\mathcal{M}$). We need only show the intersection over closures of elements of $\mathcal{M}$ is nonempty, i.e.

$$\bigcap \overline{\mathcal{M}} \neq \emptyset, \quad \text{that is,} \quad \bigcap_{E \in \mathcal{M}} \overline{E} \neq \emptyset, \tag{J.-200}$$

as this will imply (J.9.5). First of all we want to show that $\mathcal{M}$, being a maximal FIP, satisfies the two following conditions:

1) any finite intersection of elements of $\mathcal{M}$ is contained in $\mathcal{M}$, i.e., $E_1, E_2 \in \mathcal{M}$ implies $E_1 \cap E_2 \in \mathcal{M}$ and

2) any subset A of X intersecting every element of $\mathcal{M}$ is also contained in $\mathcal{M}$, i.e., for $A \subset X$ such that $A \cap E \neq \emptyset$ for all $E \in \mathcal{M}$ implies $A \in \mathcal{M}$.

To verify 1): If $E_1, E_2 \in \mathcal{M}$, then clearly the $\mathcal{M} \cup (E_1 \cap E_2)$ is a member of $\mathbf{\Omega}$; since it includes $\mathcal{M}$, and $\mathcal{M}$ is maximal in $\mathbf{\Omega}$, it must coincide with $\mathcal{M}$, so that $E_1 \cap E_2$ must be a member of $\mathcal{M}$.

To verify 2): Suppose that A is a subset of X which meets every member of $\mathcal{M}$. Then, for each finite subset $\{E_1, \ldots, E_n\}$ of $\mathcal{M}$ we have by 1) $E_1 \cap \cdots \cap E_n \in \mathcal{M}$, so that $A \cap (E_1 \cap \cdots \cap E_n) \neq \emptyset$. Therefore $\mathcal{M} \cup \{E\}$ has the FIP, and so is a member of $\mathbf{\Omega}$ including $\mathcal{M}$. The latter's maximality implies then $\overline{U}\hat{A} \in \mathcal{M}$.

Now for each $\alpha \in I$, FIP for $\mathcal{M}$ implies FIP for $\pi_\alpha(\mathcal{M}) = \{\pi_\alpha(E) \mid E \in \mathcal{M}\}$. Since each X_α is compact we have

$$\bigcap_{E\in\mathcal{M}} \overline{\pi_\alpha(E)} \neq \emptyset.$$

This means for all α there exists $x_\alpha \in X_\alpha$, for all U_α and for all $E \in \mathcal{M}$

$$U_\alpha \cap \pi_\alpha(E) \neq \emptyset. \tag{J.-200}$$

But this is equivalent to $\pi_\alpha^{-1}(U_\alpha) \cap E \neq \emptyset$, therefore from 2) we get $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{M}$. Thus from 1) we get

$$\bigcap_{k=1}^{n} \pi_{\alpha_k}^{-1}(U_{\alpha_k}) \in \mathcal{M} \tag{J.-200}$$

so that for all $E \in \mathcal{M}$

$$\bigcap_{k=1}^{n} \pi_{\alpha_k}^{-1}(U_{\alpha_k}) \bigcap E \neq \emptyset. \tag{J.-200}$$

Comparing with (J.9.5), this means that $x = (x_\alpha)$ belongs to $\cap_{E\in\mathcal{M}}\overline{E}$. Thus

$$x \in \bigcap_{E\in\mathcal{M}} \overline{E} \neq \emptyset. \tag{J.-200}$$

Finally, since $\mathcal{E} \subseteq \mathcal{M}$, it follows that $\bigcap_{E\in\mathcal{M}} \overline{E} \subseteq \bigcap \overline{\mathcal{E}}$, so that $\bigcap \overline{\mathcal{E}} \neq \emptyset$.

□

The proof follows trivially form the properties of nets on product spaces that is technically clearer.

J.10 Nets

A net does for general topological spaces that a sequence does for metric (and metrizable) spaces. The notions of closedness, continuity and compactness can be formulated in terms of nets - nets fully describe the structure of topological spaces.

At first we will see that nets are a straightforward generalsation of sequencies and things proved earlier with sequencies are easily reproved with nets - we will try to avoid much repetion. However, generalisation is not so easy when we need the notion of a *subnet*, for example when we come to compactness.

One easily shows that if a net converges (a fucntion is continuous) in a certain toplogy, then it does so in any weaker (stronger) toplogy. In our applications direct products of topological spaces are of fundamental imprtance.

"eventually greater than" - the concept of a directed set: First we need the notion of partial order.

Definition A binary relation $\preceq$ on a set X is said to be a **pre-order** iff

(i) $p \preceq p$ for all $p \in X$

(ii) $p \preceq q$ and $q \preceq r$ imply $p \leq r$ for all $p, q, r \in X$.

If it is also true that for $p, q \in X$,

$p \preceq q$ and $q \preceq p$ then $p = q$

then X is said to be a **partially ordered set** (or poset).

□

So a poset consists of a set with a binary relation that indicates that, for certain pairs of elements in the set, one of the elements preorders the other. In a poset some pairs of elements bear the preorder relationship but other pairs don't.

Not all pairs of elements of L need to be in relation but if they, L is said to be totally ordered or linearly ordered.

Definition A pre-ordered set X is said to be **directed** if and only if each pair of members of X has an upperbound. That is, if $p, q \in X$, then there exists $s \in X$ such that $p \preceq s, q \preceq s$.

□

Examples of directed sets

Example (Reinmann integrals)

(i) Partial order relation. Two partitions are ordered $P_1 \prec P_2$ if one or more of subintervals in the coarser partition P_1 is a finite union of subintervals of the finer partition P_2. Put another way, a sub-interval of P_2 breaks up some of the subintervals of P_1.

(ii) Common refinement. Given any two partitions P_1, P_2 there is a common refinement; P_3 such that $P_1 \preceq P_3$ and $P_2 \preceq P_3$. This property makes the family of partitions a partially ordered and directed set. (Directed towards refinement of partition.)

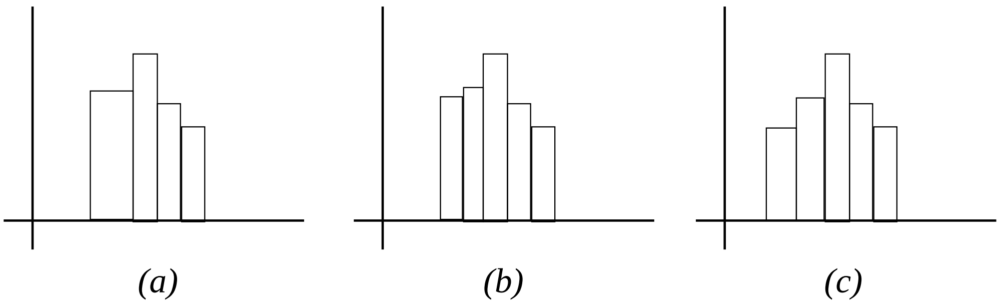


Figure J.10: netsReiman: Partions of Reinmann integrals. The partition in (b), P_2, is a finer partition than that in (a), P_1.

Example Consider the set Γ of all graphs with oriented edges. Whenever each edge of γ can be .... of edges of γ'. The set Γ is a directed set with respect to the relation

$$\gamma' \preceq \gamma \tag{J.-200}$$

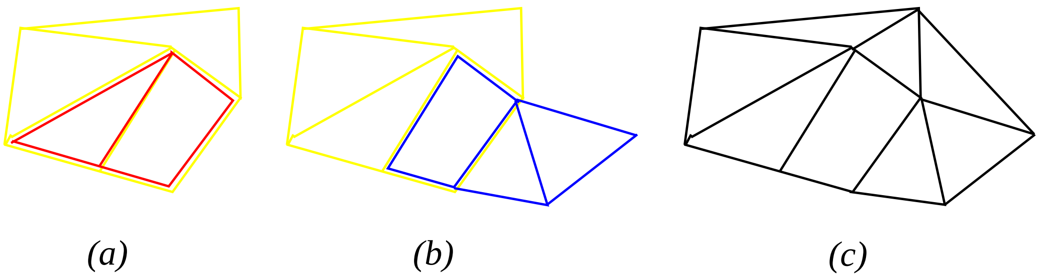


Figure J.11: (a) graph set Γ. (a) The red graph γ_R is contained in the yellow graph γ_Y so that $\gamma_R \prec \gamma_Y$ (b) Neither graph is contained in the other so γ_Y and the blue graph γ_B are incomparible. (c) Both the graphs γ_R and γ_B are contained this larger graph γ_L, i.e., $\gamma_Y \prec \gamma_L$ and $\gamma_B \prec \gamma_L$.

Example (Cellular decomposition)

(i) Partial order relation. Two cellular decompositions are related $C_1 \preceq C_2$ if any cell in the coarser decomposition C_1 is a finite union of cells of the finer decomposition C_2.

(ii) Common refinement. Given any two cellular decompositions C_1, C_2 there is a common refinement; C_3 such that $C_1 \preceq C_3$ and $C_2 \preceq C_3$. This property makes the family of cellular decompositions a partially ordered and directed set. (Directed towards refinement.)

Definition Here we define a **neighbourhood**, N, of a point x in a topological space if and only if there is an ope set in the topology such that $x \in U$ and $U \subseteq N$.

□

A sequence in a set X is a mapping from $\mathbb{N}$ into X, we generalise this to directed sets

Definition A net in a topological space $(X, \mathcal{T})$ is a mapping from a directed set I into X, denoted $(x_\alpha)_{\alpha \in I}$.

□

Examples of a net

A sequence is a net whose domain is the set of positive integers with the usual ordering.

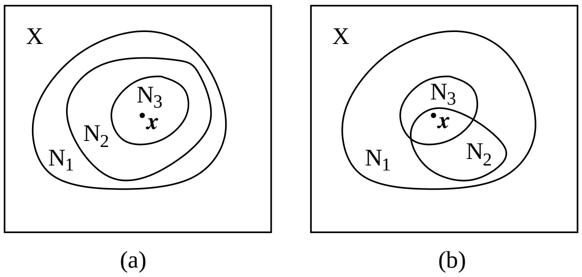


Figure J.12: (a) Inverse set inclusion: $N_3 \subset N_2 \subset N_1$. (b) $N_3 \not\subset N_2$ still so that $N_2 \subset N_1$ and $N_3 \subset N_1$.

Given $x \in (X, \mathcal{T})$ select in any way an element from each neigbourhood N of x; then (x_N) is a net in X.

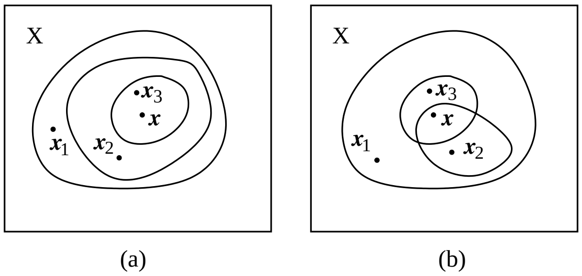


Figure J.13: Example of a net.(a) $x_1 \preceq x_2 \preceq x_3$. (b) We have that x_2 and x_3 are incomparable, $x_3 \| x_2$, but $x_1 \preceq x_2$ and $x_1 \preceq x_3$.

J.10.1 Net Convergence and Closure

We say that $(x_\alpha)_{\alpha \in I}$ converges to the point $x \in X$ if for any neighbourhood U of x $(x_\alpha)_{\alpha \in I}$ is eventually in U. x is called the limit of $(x_\alpha)_{\alpha \in I}$, and we write $x_\alpha \to x$.

Definition A **net** (x_α) in $(X, \mathcal{T})$ *converges to a limit* l if for each neighbourhood N of l, there exists some $\alpha_N \in A$ such that $x_\alpha \in N$ for all $\alpha \succeq \alpha_N$.

□

Proposition J.10.1 *Let $\mathcal{N}_x$ be a neighbourhood base at a point x in a topological space $(X, \mathcal{T})$ and suppose that, for each $U \in \mathcal{N}_x$, x_U is a given point in U. Then the net $(x_U)_{\mathcal{N}_x}$ converges to x, where $\mathcal{N}_x$ is partially ordered by reverse inclusion.*

Proof:

□

Theorem J.10.2 *Given $(X, \mathcal{T})$, $A \subseteq X$, $p \in X : p \in \overline{A}$ if and only if there exists a net in A converging to p.*

Proof: We know that a point $x \in X$ belongs to $\overline{A}$ if and only if every neighbourhood of x meets A. Suppose that $(a_\alpha)_{\alpha \in I}$ is a net in A such that $a_\alpha \to x$. By defintion of convergence, $(a_\alpha)_{\alpha \in I}$ is eventually in every neighbourhood of x, so $x \in \overline{A}$.

Now suppose that $x \in \overline{A}$. Let $\mathcal{U}_x$ be the collection of all neighbourhoods of x ordered by reverse inclusion (i.e. for $U, V \in \mathcal{U}_x$, $U \preceq V$ if and only if $V \subseteq U$). Then $\mathcal{U}_x$ is a directed set. We know that for each $U \in \mathcal{U}_x$ the set $U \cap A$ is non-empty. Let a_U be any element of $U \cap A$. So $a_U \to x$.

□

Theorem J.10.3 *A set E in a topological space $(X, \mathcal{T})$ is closed if and only if no net in E can converge to a point in X/E.*

Proof: Suppose E is closed and let (x_α) be any net in E such that (x_α) is eventually in every neighbourhood of some point p, say. Then for each open set U containing p, there exists α such that $x_\alpha \in U$. Thus $U \cap E \neq \emptyset$ for each open set U containing p. It follows that $p \in \overline{E}$, and since E is closed, $p \in E$.

Now suppose that any net in E can not be eventually in every neighbourhood of a point not belonging to E. Let p be any point in $\overline{E}$. Then for each open set U containing p, $U \cap E \neq \emptyset$. Let $x_U \in U \cap E$ for each open set U containing p, then $(x_U)_{U \in I}$ is a net in E, where I is the family of neighbourhoods of p ordered by reverse inclusion, such that x_U converges to p. But by hypothesis the convergence point must lie in E and we conclude that E is closed.

□

Theorem J.10.4 *In a Hausdorff space, a convergent net has at most one limit point.*

Proof: Similar as the proof of Theorem J.9.2. If $y \neq z$, there are disjoint open neighbourhoods U and V of y and z, respectively. Since $x_\alpha \to y$, there is $\beta \in I$ such that $x_\alpha \in U$ whenever $\alpha \succeq \beta$. Similarly, there is $\beta' \in I$ such that $x_\alpha \in V$ whenever $\alpha \succeq \beta'$. Let $\gamma \in I$ be such that $\beta \preceq \gamma$ and $\beta' \preceq \gamma$. Then $x_\gamma \in U \cap V$, which is impossible since $U \cap V = \emptyset$.

□

Theorem J.10.5 *Let X and Y be topological spaces. A mapping $f : X \to Y$ is continuous if and only if whenever $(x_\alpha)_I$ is a net in X convergent to x then the net $(f(x_\alpha))_I$ converges to $f(x)$.*

Proof: Suppose that $f : X \to Y$ is continuous and suppose that $(x_\alpha)_I$ is a net in X such that $x_\alpha \to x$. We wish to show that $(f(x_\alpha))_I$ converges to $f(x)$. To see this, let V be any open set in Y with $f(x) \in V$. Since f is continuous, $f^{-1}(V)$ is an open set in X containing x. But since $(x_\alpha)_I$ convergess to x, it is eventually in $f^{-1}(V)$, and so $(f(x_\alpha))_I$ is eventually in V. We conclude $f(x_\alpha) \to f(x)$.

Conversely, suppose that $f(x_\alpha) \to f(x)$ whenever $x_\alpha \to x$. Let V be open in Y. We must show that $f^{-1}(V)$ is open in X. If this is not true then there is a point $x \in f^{-1}(V)$ which is not an interior point. So that every open set containing x meets $X/f^{-1}(V)$. This means there is a net $(x_\alpha)_I$ in $X/f^{-1}(V)$ converging to x. By hypothesis, it follows that $f(x_\alpha) \to f(x)$. In particular, $(f(x_\alpha))_I$ is eventually in V, that is, $(x_\alpha)_I$ is eventually in $f^{-1}(V)$. But then $(x_\alpha)_I$ cannot be eventually in $X/f^{-1}(V)$. From this contracdition we conclude that $f^{-1}(V)$ is, indeed, open and therefore f is continuous.

□

J.10.2 Nets and Compactness

A subset of a metric space is compact if and only if any sequence in K has a subsequence which converges to an element of K.

What is the generalization, a subnet, of the notion of the subsequence of a sequence for a net?

Definition A map $F : J \to I$ between directed sets J and I is said to be cofinal if for any $\alpha \in I$ there is some $\beta' \in J$ such that $F(\beta) \succeq \alpha$ whenever $\beta \succeq \beta'$. In other words, F is eventually greater than any given $\alpha \in I$.

□

Definition Suppose that (x^α) is a net indexed by I and that $F : J \to I$ is a cofinal map from the directed set J into I. The net $(y_\beta)_{\beta \in J} = (x_{F(\beta)})_{\beta \in J}$ is said to be a subnet of (x^α)

□

Definition Let $(x_\alpha)_{\alpha \in A}$ be any net and let $\alpha_0 \in A$. The α_0^{th} **tail** of the net is the set $\{x_\alpha : \alpha \succeq \alpha_0\} = x([\alpha_0, \ ))$. We denote it by $x(\alpha_0 \to)$.

□

Definition The net $(x_\alpha)_{\alpha \in I}$ is said to be frequently in the set A if, for any given $\gamma \in I$, $x_\alpha \in A$ for some $\alpha \in I$ with $\alpha \succeq \gamma$.

□

Definition A point x is a cluster point of the net $(x_\alpha)_{\alpha \in I}$ if $(x_\alpha)_{\alpha \in I}$ is frequently in any neighbourhood of x.

□

Lemma J.10.6 *A function $f : X \to Y$ between topological spaces is continuous if for every convergent net (x_α) in X, the net $(f(x_\alpha))$ is convergent in Y.*

Proposition J.10.7 *Let $(x_\alpha)_I$ be a net in the space X and let $\mathcal{A}$ be a family of subsets of X such that*

(i) $(x_\alpha)_I$ is frequently in each member of $\mathcal{A}$;

(ii) for any $A, B \in \mathcal{A}$ there is $C \in \mathcal{A}$ such that $C \subseteq A \cap B$.

Then there is a subnet $(x_{F(\beta)})_J$ of the net $(x_\alpha)_I$ such that $(x_{F(\beta)})_J$ is eventually in each member of $\mathcal{A}$.

Proof: Let us equip $\mathcal{A}$ with ordering by reverse inclusion, i.e for any $A, B \in \mathcal{A}$

$$A \preceq B \quad \text{implies} \quad B \subseteq A.$$

Condition (ii) makes $\mathcal{A}$ equipped with this partial ordering a directed set, as for given any $A, B \in \mathcal{A}$, there is $C \in \mathcal{A}$ with $C \subseteq A \cap B$ meaning $C \succeq A$ and $C \succeq B$.

From this directed set we now construct a second directed set that will be the index set for the subnet. Let $\mathcal{E}$ denote the collection of pairs $(\alpha, A) \in I \times \mathcal{A}$ such that $x_\alpha \in A$, i.e

$$\mathcal{E} = \{(\alpha, A) : \alpha \in I,\ A \in \mathcal{A},\ x_\alpha \in A\}.$$

Define

$$(\alpha', A') \preceq (\alpha'', A'') \quad \text{to mean} \quad \alpha' \preceq \alpha'' \text{ in } I \text{ and } A' \preceq A'' \text{ in } \mathcal{E}.$$

This is a partial ordering for $\mathcal{E}$. We shall now see that condition (i) makes this into a directed set. Given $(\alpha', A'), (\alpha'', A'') \in \mathcal{E}$, there is $\alpha \in I$ with $\alpha \succeq \alpha', \alpha''$, and there is $A \in \mathcal{A}$ with $A \succeq A', A''$ ($\mathcal{A}$ being a directed set). Now, by (i) (x_α) is frequently in A and therefore there is $\gamma \succeq \alpha \in I$ such that $x_\gamma \in A$. Thus $(\gamma, A) \in \mathcal{E}$ and $(\gamma, A) \succeq (\alpha', A')$, $(\gamma, A) \succeq (\alpha'', A'')$ and hence $\mathcal{E}$ is a directed set.

Now we construct a cofinal map from $\mathcal{E}$ to I. Define $F : \mathcal{E} \to I$ by

$$F((\alpha, A)) = \alpha.$$

Let $\alpha \in I$ be given. For any $A_1 \in \mathcal{A}$ there is $\alpha_1 \succeq \alpha$ such that $x_{\alpha_1} \in A_1$ (by (i)). Set $\beta' = (\alpha_1, A_1)$, $\beta' \in \mathcal{E}$ and $F(\beta') = \alpha_1 \succeq \alpha$. So whenever we have $\beta \equiv (\alpha_2, A_2)$ satisfying

$$\beta \succeq \beta'$$

we have by $F(\beta) = \alpha_2 \succeq \alpha_1 \succeq \alpha$,

$$F(\beta) \succeq \alpha.$$

This shows that F is cofinal and therefore $(x_{F((\alpha,A))})_{\mathcal{E}}$ is a subnet of $(x_\alpha)_I$.

It remains to show that this subnet is eventually in every member of $\mathcal{A}$. Let $A \in \mathcal{A}$ be given. Then, by (i), there is $\alpha \in I$ such that $x_\alpha \in A$ and so $(\alpha, A) \in \mathcal{E}$. For any $(\alpha', A') \in \mathcal{E}$ with $(\alpha', A') \succeq (\alpha, A)$, we have

$$x_{F((\alpha',A'))} = x_{\alpha'} \in A' \subseteq A.$$

Thus $(x_{F((\alpha,A))})_{\mathcal{E}}$ is eventually in A.

$\square$

Theorem J.10.8 *A point x in a topological space X is a cluster point of the net $(x_\alpha)_I$ if and only if some subnet converges to x.*

Proof: Suppose that x is a cluster point of the net $(x_\alpha)_I$ and let $\mathcal{N}_x$ denote the collection of all neighbourhoods of x. If $A, B \in \mathcal{N}_x$, so is their intersection $A \cap B$. As the net $(x_\alpha)_I$ is frequently in each member of $\mathcal{N}_x$, by the preceding proposition, there is a subnet $(y_\beta)_J$ eventually in each member of $\mathcal{N}_x$, that is, the subnet $(y_\beta)_J$ converges to x.

Conversely, suppose that $(y_\beta)_{\beta\in J} = (x_{F(x)})_{\beta\in J}$ is a subnet of $(x_\alpha)_I$ converging to x. Let N be any neighbourhood of x. Then there is $\beta_0 \in J$ such that $x_{F(\beta)} \in N$ whenever $\beta \succeq \beta_0$. Since F is cofinal, for any given $\gamma \in I$ there is $\beta' \in J$ such that $F(\beta) \succeq \gamma$ whenever $\beta \succeq \beta'$. Let $\beta \succeq \beta_0$ and $\beta \succeq \beta'$. Then

$$F(\beta) \succeq \gamma \ \text{ and } \ y_\beta = x_{F(\beta)} \in N.$$

Hence $(x_\alpha)_I$ is frequently in N and we conclude that x is a cluster point of the net $(x_\alpha)_I$.

□

Theorem J.10.9 $(X, \mathcal{T})$ *is compact if and only if in X, every net has (at least one) convergent subnet.*

Proof: Suppose every net has a convergent subnet. Let $\{V_\alpha\}_{\alpha\in I}$ be an open cover of X with no finte subcover. Let $\mathcal{F}$ be the collection of all finite subsets of $\{V_\alpha\}_{\alpha\in I}$ ordered by set-theoretical inclusion:

$$\mathcal{U}_1 \preceq \mathcal{U}_2 \ \text{ if and only if } \ \mathcal{U}_1 \subseteq \mathcal{U}_2$$

for $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{F}$. Given any two $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{F}$ and an arbitrary third $\mathcal{U}_3 \in \mathcal{F}$ we have

$$\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \in \mathcal{F} \quad \text{that is} \quad \mathcal{U}_1, \mathcal{U}_2 \preceq \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \in \mathcal{F}$$

therefore it forms a directed set. For each $\mathcal{U} = \{V_{\alpha_1}, \ldots, V_{\alpha_m}\} \in \mathcal{F}$, there exists an

$$x_{\mathcal{U}} \notin \bigcup_{j=1}^{m} V_{\alpha_j}$$

otherwise, $\{V_\alpha\}_{\alpha\in I}$ would have a finite subcover. By hypothesis, the net $(x_{\mathcal{U}})_{x_{\mathcal{U}}\in\mathcal{F}}$ has a convergent subnet, equivalently a cluster point, x, by theorem J.10.8. Now, since $\{V_\alpha\}_{\alpha\in I}$ is a cover of X, there is some α' such that $x \in V_{\alpha'}$, and which $(x_{\mathcal{U}})_{x_{\mathcal{U}}}$ is frequently in.

Thus, by definition of frequently in, for any F' in the directed set $\mathcal{F}$, there is $F \succeq F' \in \mathcal{F}$ such that $x_F \in V_{\alpha'}$. In particular, if we take $F' = \{V_{\alpha'}\}$, we deduce that there is $F = \{V_{\alpha_1}, \ldots, V_{\alpha_k}\}$ such that $F \succeq \{V_{\alpha'}\}$, that is, $\{V_{\alpha'}\} \subseteq F$, and such that $x_F \in V_{\alpha'}$. But $x_F \notin F$, by construction. This contradiction implies that $(X, \mathcal{T})$ is compact.

Now, suppose that $(X, \mathcal{T})$ is compact and let $(x_\alpha)_{\alpha \in I}$ be a net in X. We suppose that $(x_\alpha)_{\alpha \in I}$ has no cluster points. Then, for any $x \in X$, there is an open neighbourhood U_x of x and $\alpha_x \in I$ such that $x_\alpha \notin U_x$ whenever $\alpha \succeq \alpha_x$ As the topological space is compact, the family $\{U_x : x \in X\}$ has an open cover of X for a finite number of points $x_1, \ldots, x_n \in X$, i.e. $X = \cup_{i=1}^n U_{x_i}$. Since I is a directed set there is $\alpha \succeq \alpha_i$ for each $i = 1, \ldots, n$. But then $x_\alpha \notin U_{x_i}$ for $i = 1, \ldots, n$, which is impossible since the U_{x_i}'s are an open cover for X.

□

Corollary J.10.10 *Compactness is preserved by continuous maps.*

J.10.3 Universal Nets

Universal nets will be used in the proof of Tychonov's Theorem.

Definition A net (x^α) on a set X is called *universal* if for every subset Y of X, either (x^α) is eventually either only in Y or only in X/Y.

□

Note that if (x_α) is a universal net, $A \subseteq X$, and (x_α) is frequently in A, then (x_α) is eventually in A.

Proposition J.10.11 *If a universal net has a cluster point, then it converges to the cluster point. In particular, a universal net in a Hausdorff space can have at most one cluster point.*

Proof: Suppose that x is a cluster point of the univrsal net $(x_\alpha)_{\alpha \in I}$, that is, for each neighbourhood N of x, $(x_\alpha)_{\alpha \in I}$ is frequently in N. So $(x_\alpha)_{\alpha \in I}$ is eventually in every neighbourhood of x and so converges to x. The last part of the proposition follows because in a Hausdorff space a net can converge to at most one point.

□

The following lemma is where Zorn's lemma comes into the proof of Tychonoff's theorem via nets.

Lemma J.10.12 *Let $(x_\alpha)_{\alpha\in I}$ be a net in a toplogical space X. Then there is a family $\mathcal{C}$ of subsets of X such that*

(i) $(x_\alpha)_{\alpha\in I}$ is frequently in each member of $\mathcal{C}$;

(ii) if $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$;

(iii) for any $A \subset X$, either $A \in \mathcal{C}$ or $X/A \in \mathcal{C}$

Proof: Let Φ denote the collection of families of subsets of X satisfying the conditions (i) and (ii):

$$\Phi = \{\mathcal{F} : \mathcal{F} \text{ satifies (i) and (ii)}\}.$$

Φ is non-empty since $\{X\} \in \Phi$ ($\emptyset \notin \Phi$). The collection of Φ is partially ordered by set inclusion:

$$\mathcal{F}_1 \preceq \mathcal{F}_2 \text{ if and only if } \mathcal{F}_1 \subseteq \mathcal{F}_2$$

for $\mathcal{F}_1, \mathcal{F}_2 \in \Phi$.

Let $\{\mathcal{F}_\gamma\}$ be a totally ordered family in Φ, (recall a totally ordered family $\mathcal{F}_\gamma$ is a family in which any two subsets $A_1, A_2 \in \mathcal{F}_\gamma$ satisfy either $A_1 \preceq A_2$ or $A_2 \preceq A_1$). Put $\hat{\mathcal{F}} = \bigcup_\gamma \mathcal{F}_\gamma$. We shall show that $\hat{\mathcal{F}}$ itself is in Φ and so Zorn's lemma apllies.

If $A \in \hat{\mathcal{F}}$, then there is some γ such that $A \in \mathcal{F}_\gamma$, and so (x_α) is frequently in A (by (i) for $\mathcal{F}_\gamma$) and so condition (i) holds for $\hat{\mathcal{F}}$. Now, for any $A, B \in \hat{\mathcal{F}}$, there is γ_1 and γ_2 such that $A \in \mathcal{F}_{\gamma_1}$, and $B \in \mathcal{F}_{\gamma_2}$. Suppose, without loss of generality, that $\mathcal{F}_{\gamma_1} \preceq \mathcal{F}_{\gamma_2}$. Then $A, B \in \mathcal{F}_{\gamma_2}$, and therefore $A \cap B \in \mathcal{F}_{\gamma_2} \subseteq \hat{\mathcal{F}}$ (by (ii) for $\mathcal{F}_{\gamma_2}$), and we see that condition (ii) is satisfied for $\hat{\mathcal{F}}$. Thus $\hat{\mathcal{F}} \in \Phi$ as claimed.

By Zorn's lemma J.9.16, we conclude that Φ has a maximal element, $\mathcal{C}$, say. We shall show that $\mathcal{C}$ also satisfies condition (iii).

To see this, first suppose that it is true that (x_α) is frequently in $A \cap B$ for all $B \in \mathcal{C}$. Define $\mathcal{F}'$ by

$$\mathcal{F}' = \{C \subseteq X : A \cap B \subseteq C, \text{ for some } B \in \mathcal{C}\}.$$

We shall show that under the supposition that $\mathcal{F}' \in \Phi$. If $C \in \mathcal{F}'$ implies that $A \cap B \subseteq C$ for some $B \in \mathcal{C}$ and so (x_α) is frequently in C. Also, if $C_1, C_2 \in \mathcal{F}'$, then there are B_1 and B_2 in $\mathcal{C}$ such that $A \cap B_1 \subseteq C_1$ and $A \cap B_2 \subseteq C_2$. Obviously $A \cap (B_1 \cap B_2) \subseteq C_1 \cap C_2$.

Since $B_1 \cap B_2 \in \mathcal{C}$, (as $\mathcal{C}$ satisfies (ii) being in Φ), we deduce that $C_1 \cap C_2 \in \mathcal{F}'$. Thus $\mathcal{F}' \in \Phi$.

However, it is clear that $A \in \mathcal{F}'$ and also that if $B \in \mathcal{C}$ then $B \in \mathcal{F}'$. But $\mathcal{C}$ is maximal in Φ, and so $\mathcal{F}' = \mathcal{C}$ and we conclude that $A \in \mathcal{C}$, and (iii) holds.

Now suppose that it is false that (x_α) is frequently in every $A \cap B$, for $B \in \mathcal{C}$. Then there is some $B_0 \in \mathcal{C}$ such that (x_α) is not frequently in $A \cap B_0$, and so must be frequently in $X/(A \cap B_0)$. Thus for any α_0 there is $\alpha \succeq \alpha_0$ such that $x_\alpha \in X/(A \cap B_0)$ for all $\alpha \succeq \alpha_0$. That is, (x_α) is eventually in $X/A \cap B_0$. Let us set $\tilde{A} \equiv X/A \cap B_0$. As (x_α) is eventually in $\tilde{A}$, (x_α) must be frequently in $\tilde{A} \cap B$ for every $B \in \mathcal{C}$. Define $\mathcal{F}''$ by

$$\mathcal{F}'' = \{C \subseteq X : \tilde{A} \cap B \subseteq C, \text{ for some } B \in \mathcal{C}\}.$$

Thus, as above, we deduce that $\tilde{A} \in \mathcal{C}$. Furthermore, for any $B \in \mathcal{C}$, $B \cap B_0 \in \mathcal{C}$ and so $\tilde{A} \cap B \cap B_0 \in \mathcal{C}$. But

$$\begin{aligned}
\tilde{A} \cap B \cap B_0 &= \big(X/(A \cap B_0)\big) \cap (B \cap B_0) \\
&= \big((X/A) \cup (X/B_0)\big) \cap B \cap B_0 \\
&= \{(X/A) \cap B \cap B_0\} \cup \underbrace{\{(X/B_0) \cap B \cap B_0\}}_{=\emptyset} \\
&= (X/A) \cap B \cap B_0
\end{aligned}$$

and so we see that (x_α) is frequenlty in $(X/A) \cap B \cap B_0$ and hence is frequently in $(X/A) \cap B$ for any $B \in \mathcal{C}$. Define $\mathcal{F}'''$ by

$$\mathcal{F}''' = \{C \subseteq X : (X/A) \cap B \subseteq C, \text{ for some } B \in \mathcal{C}\}.$$

Again, by the above argument, we deduce that $(X/A) \in \mathcal{C}$. However, it is clear that $(X/A) \in \mathcal{F}'''$ and also that if $B \in \mathcal{C}$ then $B \in \mathcal{F}'''$. But $\mathcal{C}$ is maximal in Φ, and so $\mathcal{F}''' = \mathcal{C}$ and we conclude that $(X/A) \in \mathcal{C}$, and (iii) holds. This proves the claim and completes the proof of the lemma.

□

Theorem J.10.13 *Every net has a universal subnet.*

Proof: Let $(x_\alpha)_I$ be any net in X, and let $\mathcal{C}$ be a family of subsets as given by the previous lemma. Then by proposition J.10.7, $(x_\alpha)_I$ has a subnet $(y_\beta)_J$ which is eventually in each member of $\mathcal{C}$. But, by (iii) of the previous lemma, for any $A \subseteq X$, either $(y_\beta)_J$ is in $A \in \mathcal{C}$ or is in $X/A \in \mathcal{C}$, hence the subnet $(y_\beta)_J$ is either eventually in A or eventually in X/A, hence $(y_\beta)_J$ is universal.

□

Lemma J.10.14 *If (x_α) is a universal net in a space X, and $f : X \to Y$ is a function, then $f(x_\alpha)$ is a universal net in Y.*

Take any $A \subseteq Y$ and suppose that the net $f(x_\alpha)$ is frequently in A. Then (x_α) is frequently in $f^{-1}(A)$. It follows that (x_α) is eventually in $f^{-1}(A)$ since (x_α) is universal. Thus $f(x_\alpha)$ is eventually in A and so $f(x_\alpha)$ is universal.

□

Theorem J.10.15 *A topological space is compact if and only if every universal net converges.*

Proof: Suppose $(X, \mathcal{T})$ is a compact toplogical space and that (x_α) is a universal net in X. Since X is compact (x_α) has a convergent subnet, with limit point $x \in X$, say (by theorem J.10.9). The universal net (x_α) is frequently in any neighbourhood of x, for if it wasn't it wouldn't have a subnet converging to x. So x is a cluster point of the univrsal net (x_α) and therefore the net itself (x_α) converges to x (by proposition J.10.11).

Now, suppose that every universal net in X converges. Let (x_α) be any net in X. Then (x_α) has a subnet which is a universal subnet and must therefore converge.

□

Corollary J.10.16 *A subset C of a topological space is compact if and only if every universal net in C converges in C.*

The subset C of the topological space $(X, \mathcal{T})$ is compact if and only if it is compact with respect to the induced topology $\mathcal{T}_C = \{U \cap C : U \in \mathcal{T}\}$ on C. The result now follows by applying the previous theorem to $(C, \mathcal{T}_C)$.

□

J.10.4 Proof of Tychonov's Theorem Using Nets

The following, proven earlier in this section, facts about nets. For a map $f : X \to Y$ between topological spaces the net $f(x_\alpha)$ in Y is universal whenever (x_α) is universal in

X with no restriction on f and a topological space X is compact if and only if every universal net converges.

Proof of Theorem J.9.17

Let $(x_\alpha) = (x_{\alpha,l})_{l\in I}$ be any universal net in $X_\infty = \prod_{l\in I} X_l$. By Lemma J.10.14 the net $(\pi_l(x_\alpha)) = (x_\alpha)$ is universal in X_l. Since X_l is compact, it converges to some x_l. Define $x := (x_l)_{l\in I}$. By definition of the Tychonov topology, $x_\alpha \to x$ if and only if $x_{\alpha,l} \to x_l$ for any $l \in I$ whence (x_α) converges.

J.10.5 Quotient Topologies

"In our discussion of the gauge orbit of connections we will deal with the quotient of connections by the set of gauge tansformations whihc is a topological which is a toological spce again. The resulting quotient space carries a natural topology, the quotient topology."

Definition Let X, Y be a toplogical spaces and $p : X \to Y$ a surjection (onto). The map p is said to be a **quotient map** provided that $V \subset Y$ is open in Y if and only if $p^{-1}(V)$ is open in X.

□

As p is surjective the inverse map p^{-1} is well defined.

Notice the requirement for p to be a quotient map is stronger that that it be continuous which would only require that $p^{-1}(V)$ is open in X whenever V is open in Y (but not vice versa).

Lemma J.10.17 *Let X and Y be two sets, $p : X \to Y$ a surjection, and V_1, V_2 any two subsets of Y then*

$$p^{-1}(V_1) \cap p^{-1}(V_2) = p^{-1}(V_1 \cap V_2) \quad \text{and} \quad p^{-1}(V_1) \cup p^{-1}(V_2) = p^{-1}(V_1 \cup V_2)$$

Proof: Consider any two subsets $V_1, V_2 \subset Y$. Suppose $x \in p^{-1}(V_1) \cap p^{-1}(V_2) \subset X$. Then there exist $y_1 \in V_1$, $y_2 \in V_2$ such that $y_1 = p(x) = y_2$, that is, $y_1 = y_2 \in V_1 \cap V_2$ so that actually $x \in p^{-1}(V_1 \cap V_2)$. We conclude

$$p^{-1}(V_1) \cap p^{-1}(V_2) \subset p^{-1}(V_1 \cap V_2)$$

On the other hand, let $x \in p^{-1}(V_1 \cap V_2) \subset X$ such that $x \in p^{-1}(y)$. Since $y \in V_1 \cap V_2$ we have $p^{-1}(y) \in p^{-1}(V_1)$ and $p^{-1}(y) \in p^{-1}(V_2)$, thus $x \in p^{-1}(V_1) \cap p^{-1}(V_2)$. We conclude

$$p^{-1}(V_1 \cap V_2) \subset p^{-1}(V_1) \cap p^{-1}(V_2),$$

so that

$$p^{-1}(V_1) \cap p^{-1}(V_2) = p^{-1}(V_1 \cap V_2)$$

and by taking complements we have

$$p^{-1}(V_1) \cup p^{-1}(V_2) = p^{-1}(V_1 \cup V_2).$$

□

Theorem J.10.18 *If X is a toplogical space, Y a set and $p : X \to Y$ a surjection then there exists a unique topology on Y with respect to which p is a quotient map.*

Proof: Let

$$\mathcal{T}' = \{V : V \subseteq Y \text{ and } p^{-1}(V) \text{ is open in } X\}. \tag{J.-204}$$

It is immediate that $\emptyset, Y \in \mathcal{T}'$ as $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(Y) = X$.

Let V and W be in $\mathcal{T}'$. Then $p^{-1}(V \cap W) = p^{-1}(V) \cap p^{-1}(W)$ is open in X and hence $V \cap W \in \mathcal{T}'$.

Let $\{V_\alpha : \alpha \in A\}$ be a family of sets in $\mathcal{T}'$. Then $p^{-1}(\bigcup_{\alpha\in A} V_\alpha) = \bigcup_{\alpha\in A} p^{-1}(V_\alpha)$ is open in X and hence $\bigcup_{\alpha\in A} V_\alpha \in \mathcal{T}'$. It follows that $(Y, \mathcal{T}')$ is a toplogical space.

That p is quotient follows directly from the defintion of the topology $\mathcal{T}'$.

□

The topology $\mathcal{T}'$ is called the quotient topology on Y induced by p.

Definition Let X be a topological space and let $[X]$ be a **partition** of X (i.e. a collection of mutually disjoint subsets of X whose union is X). Denote by $[x]$, $x \in X$ the subset of X in that partition of X that contains x. Equip $[X]$ with the quotient toplogy induced by the map $[\,] : X \to [X];\ x \mapsto [x]$. Then $[X]$ is called the **quotient space** of X.

□

Quotient spaces naturally arise if we have a group action $\lambda : G \times X \to X$; $(g, x) \to \lambda_g(x) := \lambda(g, x)$ on the topological space X and define $[x] := \{\lambda_g(x) : g \in G\}$ to be the orbit of x. The orbits clearly define a partition of X.

Lemma J.10.19 *Let X be a compact topological space, Y a set and $p : X \to Y$ a surjection. Then Y is compact in the quotient topology.*

Proof: Let $\mathcal{V}$ be an open cover for Y. By defintion of the quotient topology, $p^{-1}(V)$ is open in X and $\mathcal{U} := \{p^{-1}(V) : V \in \mathcal{V}\}$ covers X because

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{V \in \mathcal{V}} p^{-1}(V) = p^{-1}(\bigcup_{V \in \mathcal{V}} V) = p^{-1}(Y) = X$$

since p is a surjection and $\mathcal{V}$ covers Y. Therefore $\mathcal{U}$ is an open cover of X. Since X is compact there is a finite subcover $\{p^{-1}(V_k)\}_{k=1}^N$ of X so that

$$X = \bigcup_{k=1}^N p^{-1}(V_k) = p^{-1}(\bigcup_{k=1}^N V_k) = p^{-1}(Y)$$

Thus $Y = \bigcup_{k=1}^N V_k$, that is, $\{p^{-1}(V_k)\}_{k=1}^N$ is a finite subcover of $\mathcal{V}$ and so Y is compact.

□

Lemma J.10.20 *Let X be a Hausdorff space and $\lambda : G \times X \to X$ a continupos group action on X (i.e., λ_g defined by $\lambda_g(x) := \lambda(g, x)$ is continuous for any $g \in G$). Then the quotient space $X/G := \{[x] : x \in X\}$ defined by the orbits $[x] = \{\lambda_g(x); g \in G\}$ is Hausdorff in the quotient topology.*

Proof:

□

Theorem J.10.21 *Let X, Y be topological spaces and let G be a group acting (not necessarily continuously) on them via λ, λ' respectively. If $f : X \to Y$ is a homeomorphism with respect to which the actions λ, λ' are equivariant then f extends as a homeomorphism to the quotient spaces $X/G, Y/G$ in their respective quotient topologies.*

Proof:

□

J.10.6 Limit Spaces

Definition A **projective system** $(X_\alpha, \varphi_{\beta\alpha}, D)$ is a directed set D with a collection $\{X_\alpha : \alpha \in D\}$ of Hausdorff spaces, and continuous functions

$$\varphi_{\beta\alpha} : X_\beta \to X_\alpha$$

such that if $\alpha \preceq \beta \preceq \gamma$ in D, then

$$\varphi_{\gamma\alpha} = \varphi_{\beta\alpha} \circ \varphi_{\gamma\beta}, \quad \text{and} \quad \varphi_{\alpha\alpha} = \mathbf{1}_{X_\alpha}$$

for each $\alpha \in D$. Each X_α is called a **factor space**.

□

Definition Let $(X_\alpha, \varphi_{\beta\alpha}, D)$ be a projective system of spaces and let

$$X_\infty = \prod_{\alpha \in D} X_\alpha.$$

For $\alpha \preceq \beta$ in D, let

$$S_\alpha^\beta = \{x \in X_\infty : \varphi_{\beta\alpha} \circ \pi_\beta(x) = \pi_\alpha(x)\}$$

where $\pi_\beta : X_\infty \to X_\beta$ is a projection. Let

$$S_\beta = \bigcap \{S_\alpha^\beta : \alpha \preceq \beta\}$$

for each $\beta \in D$, and let X be the subset of X_∞ defined by

$$X = \bigcap \{S_\beta : \beta \in D\}.$$

The space X with the subspace topology of X_∞ is called the **projective limit** of the system $(X_\alpha, \varphi_{\beta\alpha}, D)$ and is denoted $\overline{X}$.

□

Theorem J.10.22 *If $(X_\alpha, \varphi_{\beta\alpha}, D)$ is a projective system of spaces, then the projective limit of $\overline{X}$ of X_α is a closed subspace of X_∞.*

Proof: Let $\alpha \preceq \beta$ in D. We shall show that S_α^β is closed. Let $x \in X/S_\alpha^\beta$. Then $\varphi_{\beta\alpha}(x_\beta) \neq x_\alpha$. Let U and V be disjoint open subsets of X_α such that $\varphi_{\beta\alpha}(x_\beta) \neq x_\alpha$ and $x_\alpha \in V$. Since $\varphi_{\beta\alpha}$ is a continuous function from X_β to X_α there exists an open subset W in X_β such that $\varphi_{\beta\alpha}(W) \subseteq U$. Let $H \subseteq X$ such that

$$\pi_\gamma(H) = \begin{cases} V & \text{if } \gamma = \alpha : \\ W & \text{if } \gamma = \beta; \text{ and} \\ X_\gamma & \text{otherwise .} \end{cases}$$

□

Theorem J.10.23 *The projective limit of nonempty compact spaces is nonempty and compact.*

Proof: Since closed subspaces of compact spaces are compact in the subspace topology we conclude that $\overline{X}$ is compact in the subspace topology induced by X_∞.

□

Theorem J.10.24 *Both $X_\infty, \overline{X}$ are Haudorff spaces.*

Proof: Let $x \neq x'$ be points in X_∞. There is at least one $\alpha_0 \in D$ such that

$$\pi_{\alpha_0}(x) \neq \pi_{\alpha_0}(x').$$

Since X_{α_0} is Hausdorff there are disjoint open neighbourhoods $U_{\alpha_0}, U'_{\alpha_0} \subset X_{\alpha_0}$ of $\pi_{\alpha_0}(x), \pi_{\alpha_0}(x')$ repectively. Let

$$U := \pi_{\alpha_0}^{-1}(U_{\alpha_0}) \text{ and } U' := \pi_{\alpha_0}^{-1}(U'_{\alpha_0}).$$

Since the topology of X_∞ is generated by the continuous functions $\pi_\alpha : X_\infty \to X_\alpha$ from the topology of X_α, it follows that U and U' are open in X_∞. Obviously U and U' are neighbourhoods of x and x' respectively. Finally, $U \cap U' = \emptyset$ since $\pi_{\alpha_0}(U \cap U') = U_{\alpha_0} \cap U'_{\alpha_0} = \emptyset$ so that U and U' are disjoint open neighbourhoods of $x \neq x'$ and thus X_∞ is Haudorff.

To see that $\overline{X}$ is Hausdorff, let $x \neq x'$ be points in $\overline{X}$, then there are respective disjoint open neighbourhoods $U, U' \in X_\infty$, then $U \cap \overline{X}$ and $U' \cap \overline{X}$ are disjoint open neighbourhoods in $\overline{X}$ by definition of the subspace topology.

$\square$

The compactness of each Hausdorff space will imply that every X_α is compact and Haudorff. Now on a direct product space (independent of the cardinality of the index set) in which each factor space is compact and Haudorff one can naturally define a topology, the Tychonov topology, such that X_∞ is itself compact. If $\overline{X}$ is closed in X_∞ then $\overline{X}$ will be compact and Hausdorff as well in the subspace toplogy. However, for compact Hausdorff spaces powerful measure theoretic theorems hold which enable one to equip to relavent infinite dimensional spaces associated to background independent gauge theories with the structure of the so-called $\sigma-$algebra and to develop measure theory thereon.

J.10.7 Locally Convex Spaces

Locally convex spaces play an important role in the theory of distributions which typically arise as solutions of constraints.

Definition A **seminorm** on a vector space X is a map $\rho : X \to [0, \infty)$ such that

1) $\rho(x + y) \leq \rho(x) + \rho(y)$ sublinerity

and

2) $\rho(\lambda x) = |\lambda| \rho(x)$ homogeneity

for all $x, y \in X$, $\lambda \in (C)$.

$\square$

Unlike a norm, a seminorm is not positive definite, i.e. it does not have the property that $\rho(x) = 0$ implies $x = 0$.

It is possible to characterize seminorms geometrically. Let B_1 be the unit ball,

$$B_1 = \{x \in V : p(x) \leq 1\}$$

Proposition J.10.25 *Suppose that p is homogeneous. Then, it is subliear if and only if its unit ball, B_1, is a convex subset of V.*

Proof:

First, let us suppose that p is sublinear, and prove B_1 must be convex. Let $x, y \in B_1$, and let k be real number between 0 and 1. We show that $kx + (1-k)y$ is also in B_1. By assumption,

$$p(kx + (1-k)y) \leq kp(x) + (1-k)p(y).$$

As this is the weighted average of the two numbers $p(x) \leq 1$, $p(y) \leq 1$, we have

$$kx + (1-k)y \in B_1.$$

Conversely, suppose that the unit ball is convex. Given, $x, y \in V$, we must show that

$$p(x+y) \leq p(x) + p(y).$$

There are three cases we need to consider. First $p(x) = p(y) = 0$ where x and y need not be the zero elements. By homogeneity, for every $K > 0$ we have

$$Kx, Ky \in B_1,$$

and hence as B_1 is assumed to be convex

$$k(Kx) + (1-k)(Ky) \in B_1$$

which for the case where $k = 1/2$ implies

$$\frac{K}{2}x + \frac{K}{2}y \in B_1$$

and so

$$p\left(\frac{K}{2}x + \frac{K}{2}y\right) \leq 1.$$

By homogeneity, again,

$$p(x+y) \leq \frac{2}{K}.$$

Since the above is true for all positive K, we conclude that

$$p(x + y) = 0.$$

Next suppose $p(x) = 0$, but that $p(y) \neq 0$, we show that it turns out that we have

$$p(x + y) = p(y).$$

Due to homogeneity, we may without loss of generality assume that

$$p(y) = 1.$$

For k such that $0 \leq k < 1$ we have

$$kx + ky = (1 - k)\frac{kx}{1 - k} + ky.$$

The right hand side is an element of B_1 as

$$\frac{kx}{1 - k}, y \in B_1$$

and because of convexity. Hence

$$kp(x + y) \leq 1,$$

and since this holds for k arbitrarily close to 1 we conclude

$$p(x + y) \leq p(y).$$

Now consider $x' = -x$ and $y' = x + y$. Note $p(x') = p(x) = 0$. We will prove $p(y') \neq 0$ by showing that $p(y') = 0$ leads to a contradiction. Assuming $p(y') = 0$, then because $p(x') = p(y') = 0$, we must have $p(x' + y') = 0$ or $p(-x + x + y) = p(y) = 0$. But we have assumed that $p(y) = 1$ and hence we have a contradiction.

So for $x' = -x$ and $y' = x + y$ we have that $p(x') = 0$ and $p(y') \neq 0$. This allows us to apply the same argument as above to x' and y' to obtain

$$p(x' + y') \leq p(y')$$

or

$$p(y) \leq p(x+y).$$

Hence

$$p(x+y) = p(y).$$

Finally, suppose that $p(x) \neq 0, p(y) \neq 0$. Then

$$\frac{x}{p(x)}, \frac{y}{p(y)} \in B_1,$$

and hence so is

$$\frac{p(x)}{p(x)+p(y)} \frac{x}{p(x)} + \frac{p(y)}{p(x)+p(y)} \frac{y}{p(y)} = \frac{x+y}{p(x)+p(y)}.$$

because of convexity. By homogeneity, we conclude that

$$p(x+y) \leq p(x) + p(y).$$

□

Definition A family of seminorms $(\rho_I)_{I \in \mathcal{I}}$ are said to **separate points** if $\rho_I = 0$ for all $I \in \mathcal{I}$ implies $x = 0$.

□

Definition A **locally convex space** is a vextor space X with a family of seminorms $(\rho_I)_{I \in \mathcal{I}}$ separating points.

□

Its **natural topology** is the weakest topology in which all the ρ_I and the operation of addition is continuous, i.e. the weakest topology such that

1) if $V \subset [0, \infty)$ is open then $\rho_I^{-1}(V)$ is open in X for all $I \in \mathcal{I}$ or, equivalently in terms of nets, if $x_\alpha \to x$ then $\rho(x_\alpha) \to \rho_I(x)$ for all $I \in \mathcal{I}$;

2) if $x_\alpha \to x$ and $y_\alpha \to y$, then $x_\alpha + y_\alpha \to x + y$

The natural topology is the intersection of all topologies on X with respect to which all the ρ_I's are continuous (equivalently, it is made up of the subsets $\rho_I^{-1}(V)$ where V is any open set in $[0, \infty)$, along with their arbitrary unions and finite intersections). This clearly defines a topology on X, and is weaker than any topology which has this property.

Definition Let $(\rho_I)_{I \in \mathcal{I}}$ be a family of seminorms on a vector space X. An $\mathcal{I}-$**open ball** of center x_0 is the set of points $x \in X$ which satisfy a finite number of inequalities

$$\rho_I(x - x_0) < \epsilon_I, \qquad \epsilon_I > 0, \quad I \in \mathcal{I}.$$

□

Recall that an open base for X is a class of open sets with the property that every open set is a union of sets in this class. The condition can also be expressed in the following equivalent form: if U is an arbitrary non-empty set and x is a point in U, then there is a set B in the open base such that $x \in B \subseteq U$.

Theorem J.10.26 *Let X be a locally convex vector space. Consider subsets U of X satisfying the condition, for each $x_0 \in U$, there are ϵ and $\rho_1, \ldots, \rho_n \in (\rho_I)_{I \in \mathcal{I}}$ such that*

$$\left\{ x \in X : \max_{j=1,\ldots,n} \rho_j(x - x_0) < \epsilon \right\} \subset U.$$

The collection of such subsets of X is a topology, i.e.

(i) $\emptyset$ and X are included;

(ii) if $(U_\alpha)_\alpha$ are in this family of sets, then so is $\cup_\alpha U_\alpha$;

(iii) if $U_1, \ldots, U_n$ are in this family of sets, then so is $U_1 \cap \cdots \cap U_n$.

Proof: We only prove (iii).

Let $x_0 \in U_1 \cap \cdots \cap U_n$. For each $k = 1, \ldots, n$, there are $\epsilon_k > 0$ and $\rho_1^{(k)}, \ldots, \rho_{n_k}^{(k)} \in (\rho_I)_{I \in \mathcal{I}}$ such that

$$V_k := \left\{ x \in X : \max_{j=1,\ldots,n_k} \rho_j^{(k)}(x - x_0) < \epsilon_k \right\} \subset U_k.$$

Let $\epsilon := \min\{\epsilon_1, \ldots, \epsilon_k\}$, and note that

$$\left\{x \in X : \max_{k=1,\ldots,n} \max_{j=1,\ldots,n_k} \rho_j^{(k)}(x - x_0) < \epsilon_k\right\} \subset V_1 \cap \cdots \cap V_n$$
$$\subset U_1 \cap \cdots \cap U_n.$$

Similarly, conditions (i) and (ii) are also satisfied by the subsets U's defined at the beginning of the theorem. Hence, they form a topology and are therefore themselves open sets of the topology.

□

Theorem J.10.27 *Let X be a locally convex vector space. Then a net $(x_\alpha)_\alpha$ in X converges to $x_0 \in X$ in the topology defined in theorem J.10.26 if and only if $\rho_I(x_\alpha - x) \to 0$ for each $I \in (\rho_I)_{I\in\mathcal{I}}$.*

Proof: Suppose that $x_\alpha \to x$ in the topology. Fix $\epsilon > 0$ and $I \in (\rho_I)_{I\in\mathcal{I}}$. Then $U := \{x \in X : \rho(x_\alpha - x_0) < \epsilon\}$ is an open neighbourhood of x_0. Hence, there is an index α_0 such that $x_\alpha \in U$, i.e. $\rho(x_\alpha - x_0) < \epsilon$ for all $\alpha \succ \alpha_0$. Hence, $\rho(x_\alpha - x_0) \to 0$.

Conversely, suppose that $\rho(x_\alpha - x) \to 0$ for all $I \in \mathcal{I}$. Let U be a neighbourhood of x_0, i.e. there is an open set $V \subset U$ with $x_0 \in V$. By the definition, there are $\epsilon > 0$ and $\rho_1, \ldots, \rho_n \in (\rho_I)_{I\in\mathcal{I}}$ such that

$$\left\{x \in X : \max_{j=1,\ldots,n} \rho_j(x - x_0) < \epsilon\right\} \subset V.$$

Since $\rho(x_\alpha - x) \to 0$ for $j = 1, \ldots, n$, there is an index α_0 such that

$$\rho_j(x_\alpha - x_0) < \epsilon \qquad (j = 1, \ldots, n,\ \alpha \succ \alpha_0).$$

Thus $x_\alpha \in V \subset U$ for all $\alpha \succ \alpha_0$.

□

By theorem J.10.27, when $x_\alpha \to x$ and $y_\alpha \to y$, then $x_\alpha + y_\alpha \to x + y$ if and only if when $\rho_I(x_\alpha) \to \rho_I(x)$ and $\rho_I(y_\alpha) \to \rho_I(y)$, then $\rho_I(x_\alpha + y_\alpha) \to \rho_I(x + y)$ for each $I \in (\rho_I)_{I\in\mathcal{I}}$.

Theorem J.10.28 *The topology defined in theorem J.10.26 is the natural topology.*

Proof: This is a direct consequence of the previous theorem. It implies the topology defined in theorem J.10.26 is necessary and sufficient for all the ρ_I's to be continuous.

That addition is continuous in this topology follows from the first property of seminorms ($\rho_I(x+y) \leq \rho_I(x) + \rho_I(y)$). By

$$\rho_I(x_\alpha + y_\alpha - x - y) \leq \rho_I(x_\alpha - x) + \rho_I(y_\alpha - y),$$

we see that $x_\alpha \to x$ and $y_\alpha \to y$ implies $x_\alpha + y_\alpha \to x + y$.

$\square$

Theorem J.10.29 *A family of seminorms which separates points of X defines a Hausdorff topology.*

Proof: Let $x, y \in X$, $x \neq y$. Then there exists a seminorm ρ such that $\rho(x-y) \neq 0$, i.e. such that $\rho(x-y) > \epsilon$. The nieghbourhoods of x and y defined respectively by

$$\rho(z-x) < \epsilon/2 \ \text{ and } \ \rho(z'-y) < \epsilon/2$$

are disjoint for if we had $z = z'$ at all the three inequalities would violate the triangle inequality

$$\rho(x-z+z-y) \leq \rho(z-x) + \rho(z-y).$$

Therefore if the family of seminorms separates points the corresponding topology must be Hausdorff.

Conversely, for the topology of a family of seminorms to be Hausdorff this family must separate points of X. For if there were an $x \in X$, $x \neq 0$, such that $\rho(x) = 0$ for every $\rho \in (\rho_I)_{I \in \mathcal{I}}$, then every ball of centre 0, and hence every neighbourhood of 0, would contain x; thus 0 and x would not be separated.

$\square$

We show that a locally convex space X whose underlying family of seminorms is countable can be equipped with the following metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(x-y)}{1+\rho_n(x-y)} \tag{J.-206}$$

Note that for real number x such that $0 \leq x \leq \infty$ by

$$\frac{x}{1+x} = 1 - \frac{1}{1+x} \tag{J.-206}$$

we have

$$0 \leq \frac{x}{1+x} \leq 1$$

and hence

$$0 \leq \frac{\rho_n(x-y)}{1+\rho_n(x-y)} \leq 1.$$

This is the reason for the factor 2^{-n} in the summation; it guarantees convergence. We must demonstrate that (J.10.7) has the properties of a metric. As $p_n(x) \geq 0$ we obviously have $d(x,y) \geq 0$. If $d(x,y) = 0$ then we must have that $p_n(x-y) = 0$ for all n, but as $\{p_n\}$ seperates points this implies $x = y$. As $p_n(x) = p_n(-x)$ we obviusly have $d(x,y) = d(y,x)$. By (J.10.7) the function $x/(1+x)$ is monotonically increasing and so we can write for real numbers $a, b, c \geq 0$ with $a \leq b + c$ that

$$\begin{aligned} \frac{a}{1+a} &\leq \frac{b+c}{1+b+c} \\ &= \frac{b}{1+b+c} + \frac{c}{1+b+c} \\ &\leq \frac{b}{1+b} + \frac{c}{1+c}. \end{aligned} \qquad \text{(J.-207)}$$

This inequality underlies the definition of the metric (J.10.7), it implies, via $p_n(x-z) = p_n(x-y+y-z) \leq p_n(x-y) + p_n(y-z)$, that

$$\frac{p_n(x-z)}{1+p_n(x-z)} \leq \frac{p_n(x-y)}{1+p_n(x-y)} + \frac{p_n(y-z)}{1+p_n(y-z)}.$$

and hence

$$d(x,z) \leq d(x,y) + d(y,z),$$

completing the properties defining a metric.

Note it is obvious that this metric is translatinally invariant. This metric generates the same toplogy as the family of seminorms.

Proof:

On the one hand, one has for $\epsilon \in [0,1]$ and $N \in \mathbb{N}$:

$$d(x) \leq \epsilon 2^{-N}$$

implies

$$\frac{1}{2^n}\frac{p_n(x)}{1+p_n(x)} \leq \epsilon 2^{-N} \quad \text{for all } n$$

this in turn implies

$$\frac{p_n(x)}{1+p_n(x)} \leq \epsilon \quad \text{for } n \leq N$$

and finally

$$p_n(x) \leq \frac{\epsilon}{1-\epsilon} \quad \text{for } k \leq N$$

We have that $B(0, r = \epsilon 2^{-N})$ corresponds to the points satisfying $p_k(x) \leq \epsilon/(1-\epsilon)$ for $k = 1, \ldots, N$. Therefore we have the following inclusions between base-neighbourdhoods

$$B(0, r) \subset W(p_1, \ldots, p_N; \delta, \ldots, \delta)$$

when δ and N are given, $\epsilon > 0$ is chosen so that $\epsilon/(1-\epsilon) < \delta$.

On the other hand if $2^{-N} < \epsilon/2$, $p_n(x) \leq \epsilon/2$ for $k \leq N$ implies

$$\begin{aligned}
d(x) &= \sum_{n=1}^{N}\frac{1}{2^n}\frac{p_n(x)}{1+p_n(x)} + \sum_{n=N+1}^{\infty}\frac{1}{2^n}\frac{p_n(x)}{1+p_n(x)} \\
&\leq \sum_{n=1}^{N}\frac{1}{2^n}p_n(x) + \sum_{n=N+1}^{\infty}\frac{1}{2^n} \\
&\leq \frac{\epsilon}{2}\sum_{n=1}^{N}\frac{1}{2^n} + \frac{1}{2^N} \\
&\leq \frac{\epsilon}{2}(1-\frac{1}{2^N}) + \frac{\epsilon}{2} \\
&\leq \epsilon
\end{aligned} \tag{J.-210}$$

We have that if $2^{-N} < \epsilon/2$, $p_n(x) \leq \epsilon/2$ for $k \leq N$ corresponds to points satisfying $d(x) \leq \epsilon$. Therefore we have the following inclusions between base-neighbourdhoods

$$W(p_1, \ldots, p_N; \delta, \ldots, \delta) \subset B(0, r)$$

when r is given, N is choosen so that $2^{-N} \leq r/2$, and δ is taken to be equal to $r/2$.

□

Theorem J.10.30 *Let X be a locally convex space. The following are equivalent:*

1) X is metrizable.

2) 0 has a countable neighbourhood base.

3) The topology on X is generated by some countable family of seminorms.

Proof: That 1) implies 2) is a property of any metric space.

□

Definition If the locally convex space X whose underlying family of seminorms is countable is completed in the metric (J.10.7), it is called a **Frechet space**.

□

An important application is the following. Consider the space $\mathbb{R}^n$ with coordinates x_k and let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_k = 0, 1, 2, \ldots$ and $|\alpha| = \sum_{k=1}^n \alpha_k$. Set $\partial_\alpha ;= \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The space of smooth functions on $\mathbb{R}^n$ of rapid decrease $\mathcal{S}(\mathbb{R}^n)$ consists of those smooth functions f for which

$$\rho_{\alpha,\beta}(f) := \sup_x |x^\alpha \partial_\beta f(x)| < \infty$$

for all α, β. This is called the Schwartz space and is denoted $\mathcal{S}(\mathbb{R}^n)$. They fall off together with their derivatives faster than any polynomial at infinity. One can show that this space with the countable family of seminorms $\rho_{\alpha,\beta}$ is a Frechet space.

The family of seminorms is clearly separating, because if $\rho_{\alpha,\beta}(f) = 0$ for all $\alpha, \beta \in \mathbb{N}$ then in particular $\rho_{0,0}(f) = \sup_x |f(x)| = 0$ for all $x \in \mathbb{R}^n$, which implies $f \equiv 0$ on $\mathbb{R}^n$.

Proof:

The sum of two smooth functions is also a smooth function, multiplying a smooth function by a complex number also gives a smooth function.

We establish that $\rho_{\alpha,\beta}(f)$ is a seminorm. Sublinerity: We note that

$$|x^\alpha \partial_\beta \left(f(x) + g(x) \right)| \leq |x^\alpha \partial_\beta f(x)| + |x^\alpha \partial_\beta g(x)|$$

Let $(\overline{x}_1, \ldots, \overline{x}_n)$ be such that $|x^\alpha \partial_\beta \left(f(x) + g(x) \right)|$ is supremum. Note that

$$|\overline{x}^\alpha \partial_\beta f(\overline{x})| \leq \sup_x |x^\alpha \partial_\beta f(x)|, \qquad |\overline{x}^\alpha \partial_\beta g(\overline{x})| \leq \sup_x |x^\alpha \partial_\beta g(x)|.$$

Combining these observations, we obtain

$$\begin{aligned}
\rho_{\alpha,\beta}(f+g) &= \sup_x |x^\alpha \partial_\beta \left(f(x) + g(x) \right)| \\
&= |\overline{x}^\alpha \partial_\beta \left(f(\overline{x}) + g(\overline{x}) \right)| \\
&\leq |\overline{x}^\alpha \partial_\beta f(\overline{x})| + |\overline{x}^\alpha \partial_\beta g(\overline{x})| \\
&\leq \sup_x |x^\alpha \partial_\beta f(x)| + \sup_x |x^\alpha \partial_\beta g(x)| \\
&= \rho_{\alpha,\beta}(f) + \rho_{\alpha,\beta}(g). \qquad \text{(J.-213)}
\end{aligned}$$

Homogeneity: This easily follows from

$$\begin{aligned}
\rho_{\alpha,\beta}(\lambda f) &= \sup_x |x^\alpha \partial_\beta (\lambda f(x))| \\
&= |\lambda| \sup_x |x^\alpha \partial_\beta f(x)|. \qquad \text{(J.-213)}
\end{aligned}$$

Therefore

$$\rho_{\alpha,\beta}(\lambda f) \leq |\lambda| \rho_{\alpha,\beta}(f).$$

The properties of a seminorm mean that

$$\rho_{\alpha,\beta}(\lambda f + g) < \infty.$$

Hence, if f and g belong to the space, we have that $\lambda f + g$ also belongs to the space. Thus the Schwartz space is a vector space.

We must now show that the space is complete. The metric is constructed such that for any sequence of Schwartz functions $(f_n)_{n=0}^\infty$ we have that f_n converges to $f \in \mathcal{S}(\mathbb{R}^n)$ with respect to the metric d if and only if $\lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0$ for any $\alpha, \beta \in \mathbb{N}$. Thus showing convergence in $\mathcal{S}(\mathbb{R}^n)$ it suffices to do computations with $\| \cdot \|_{\alpha,\beta}$.

□

Definition The topological dual of $\mathcal{S}(\mathbb{R}^n)$, denoted $\mathcal{S}'(\mathbb{R}^n)$, is called the space of **tempered distributions**.

$\square$

J.11 Stone-Weierstrass Theorem

Important results are the real and complex Stone-Weierstrass theorems. We start with the Weierstrass approximation theorem as it is used in the proof of the Stone-Weierstrass Theorem.

J.11.1 Weierstrass Approximation Theorem

Theorem J.11.1 (Weierstrass Approximation Theorem) *Let f be a continuous real function defined on a closed interval $[a, b]$ and let $\epsilon > 0$. Then there is a polynomial p such that $\|f(x) - p(x)\|_\infty = sup_{x\in[a,b]}|f(x) - p(x)| < \epsilon$*

Without loss of generality, we let $a = 0$ and $b = 1$. The n-th Berstein polynomials associated with $f \in \mathcal{C}([0, 1])$ are defined as

$$B_n(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{J.-213}$$

First we will find a couple of identities, these will be the main tools for proving the theorem.

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k} = [x + (1-x)]^n = 1 \tag{J.-213}$$

Differentiating (J.11.1) with respect to x gives

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^{k-1}(1-x)^{n-k-1}[k(1-x) - (n-k)x] = 0.$$

and then multiplying by $x(1-x)$ gives us

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k}(k - nx) = 0.$$

We now differentiate this with respect to x, considering $x^k(1-x)^{n-k}$ as one of the two factors in applying the product rule,

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^{k-1}(1-x)^{n-k-1}(k-nx)^2-\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^k(1-x)^{n-k}n=0.$$

We then multiply by $x(1-x)/n^2$ to give us upon simply rearangement

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^k(1-x)^{n-k}\left(x-\frac{k}{n}\right)^2=\frac{x(1-x)}{n}. \qquad \text{(J.-213)}$$

By using (J.11.1), we see that

$$f(x)-B_n(x)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^k(1-x)^{n-k}\left[f(x)-f\left(\frac{k}{n}\right)\right]$$

and hence

$$|f(x)-B_n(x)|\le\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^k(1-x)^{n-k}\left|f(x)-f\left(\frac{k}{n}\right)\right| \qquad \text{(J.-213)}$$

Since f is uniformly continuous, for any $\epsilon>0$ there is $\delta>0$ such that $|f(x)-f(y)|<\epsilon/2$ when $|x-y|<\delta$ for $0\le x,y\le 1$. In particular,

$$|f(x)-f(k/n)|<\epsilon/2 \;\text{ when }\; |x-k/n|<\delta. \qquad \text{(J.-213)}$$

We now spilt the sum on the right of (J.11.1) into two parts, a sum over those terms for which $|x-k/n|<\delta$ and a sum over the remaining terms:

$$\begin{aligned}|f(x)-B_n(x)| &\le \Big(\sum_{|x-k/n|<\delta}+\sum_{|x-k/n|\ge\delta}\Big)\left(\begin{array}{c}n\\k\end{array}\right)x^k(1-x)^{n-k}\left|f(x)-f\left(\frac{k}{n}\right)\right| \\ &\le \Big(\sum_{|x-k/n|<\delta}\frac{\epsilon}{2}+\sum_{|x-k/n|\ge\delta}\left|f(x)-f\left(\frac{k}{n}\right)\right|\Big)\left(\begin{array}{c}n\\k\end{array}\right)x^k(1-x)^{n-k} \\ &\le \frac{\epsilon}{2}+\sum_{|x-k/n|\ge\delta}\left(\begin{array}{c}n\\k\end{array}\right)x^k(1-x)^{n-k}\left|f(x)-f\left(\frac{k}{n}\right)\right| \qquad \text{(J.-214)}\end{aligned}$$

Where we have used (J.11.1) and that $\sum_{|x-k/n|<\delta} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k} \leq 1$ by (J.11.1) (remember $0 \leq x \leq 1$). We complete the proof by showing the second term can be made less than $\epsilon/2$ independent of x. As f is a continuous function of a closed interval on the real line, it is bounded. Since f is bounded there exists a positive number K such that $|f(x)| \leq K$ for all $x \in [0,1]$ so that $|f(x) - f(k/n)| \leq 2K$. It follows that

$$|f(x) - B_n(x)| \leq \frac{\epsilon}{2} + 2K \sum_{|x-k/n|\geq\delta} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k} \tag{J.-214}$$

if n is sufficiently large, the second term can be made less than $\epsilon/2$. Because we are summing over k such that $|x - k/n| \geq \delta$, we have

$$\begin{aligned} 2K\delta^2 \sum_{|x-k/n|\geq\delta} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k} &\leq 2K \sum_{|x-k/n|\geq\delta} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 \\ &\leq 2K \frac{x(1-x)}{n} \end{aligned}$$

where we have used (J.11.1). The maximum value of $x(1-x)$ on $[0,1]$ is $1/4$, so the previous inequality leads to

$$2K \sum_{|x-k/n|\geq\delta} \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k} \leq \frac{K}{2\delta^2 n}. \tag{J.-216}$$

If we take n to be any integer greater than $K/\delta^2\epsilon$, then from (J.11.1) we have

$$|f(x) - B_n(x)| < \epsilon$$

for all x in $[0,1]$, which proves the theorem.

□

Second Proof the Weierstrass Approximation Theorem

The set P of all polynomials on $[a,b]$ is identical with the set of functions which can be built 1 and x by applying multiplication, multiplication by a number, and addition. That is, P is a subalgebra of $\mathcal{C}[a,b]$ generated by $\{1, x\}$.

Instructive in the generalisation to the Stone-Weierstrass theorem.

Given an arbitrary function in $C[a,b]$,

$$g(z) = a\frac{z-y}{x-y} + b\frac{z-x}{y-x}. \quad \text{(J.-216)}$$

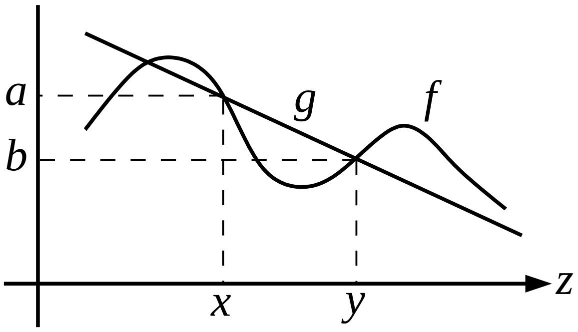


Figure J.14: polyExists.

As it is uniformally continuous, there is $\|f-p\| < \epsilon$

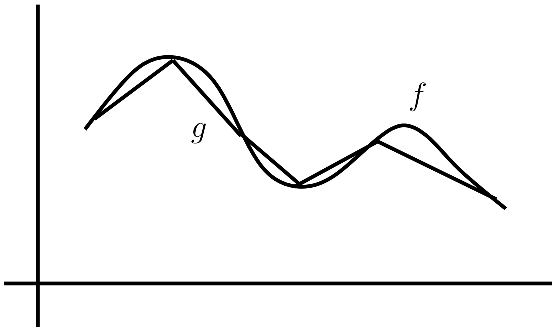


Figure J.15: polygonApp.

□

J.11.2 Stone-Weierstrass Theorem

a generalisation of Theorem J.11.1 to the space of continuous functions on an arbitrary compact Hausdorff space X

Suppose $\mathcal{B}$ is a set of functions on X.

i. $\mathcal{B}$ **separates points** in that given any two different points x and y in X there exists a function g in $\mathcal{B}$ with $g(x) \neq g(y)$.

ii. $\mathcal{B}$ **vanishes at no point** in that for each point $x \in X$ there is a $g \in \mathcal{B}$ such that $g(x) \neq 0$.

Clearly x separates points of $[a, b]$ and the unit function nowhere vanishes on $[a, b]$.

Theorem J.11.2 (Stone-Weierstrass Theorem (real case)) *Suppose $\mathcal{B}$ is an algebra of real functions on a compact Hausdorff space X. If $\mathcal{B}$ separates points on X and if $\mathcal{B}$ vanishes at no point of X, then $\mathcal{B} = C(X, \mathbb{R})$. In other words, if $f \in C(X, \mathbb{R})$ then given $\epsilon > 0$ there is a function $p \in \mathcal{B}$ such that $\|f - p\| < \epsilon$.*

The proof is split up into three lemmas.

$$\begin{aligned}(f \vee g)(x) &= \max\{f(x), g(x)\} \\ (f \wedge g)(x) &= \min\{f(x), g(x)\}\end{aligned} \tag{J.-216}$$

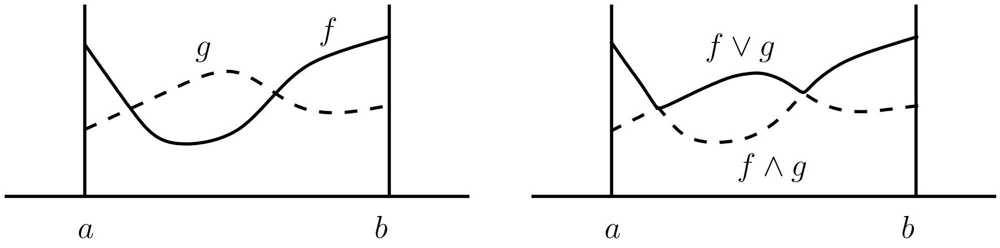


Figure J.16: fgmaxmin. The meaning of $f \vee g$ and $f \wedge g$ for two continuous functions on a closed interval $[a, b]$ of the real line.

$$\begin{aligned}(f \vee g)(x) &= \frac{f + g + |f - g|}{2} \\ (f \wedge g)(x) &= \frac{f + g - |f - g|}{2}\end{aligned} \tag{J.-216}$$

Definition (A lattice of functions) If for any $f, g \in \mathcal{B}$, $\max\{f, g\} \in \mathcal{B}$ and $\min\{f, g\} \in \mathcal{B}$ then $\mathcal{B}$ is called a lattice.

Lemma J.11.3 *Let X be an arbitrary topological space. Then every closed subalgebra of $\mathcal{C}(X, \mathbb{R})$ is also a closed sublattice of $\mathcal{C}(X, \mathbb{R})$.*

By (J.-216) it suffices to prove that if $f \in \mathcal{B}$ then so is $|f|$. The maximum value of $f(x)$ in X is $\|f\|$. By the Weierstrass approximation theorem there exists a polynomial

$p'(t) = a_o + a_1 t + \cdots + p_n t^n$ with the property that $\big||t| - p'(t)\big| < \epsilon/2$ for every t on the closed interval $[-\|f\|, \|f\|]$. In particular $|a_0| = p'(0) < \epsilon/2$. We define the polynomial $p(t) = p'(t) - a_0$, (so that $p(t) = 0$). This satisfies,

$$\begin{aligned} \big|\,|t| - p(t)\big| &< \frac{\epsilon}{2} + |a_0| \\ &< \epsilon. \end{aligned} \tag{J.-216}$$

Now define the function

$$p(f(x)) = a_1 f(x) + \cdots + a_n f^n(x). \tag{J.-216}$$

Since $|f(x)| \leq \|f\|$ or all $x \in X$, we can substitute t for $f(x)$ in (J.-216) and obtain

$$\big|\,|f(x)| - p(f(x))\big| < \epsilon \quad \text{for all } x \in X. \tag{J.-216}$$

The function $p(f) = \sum_{n=1}^{k} a_n f^n$ is in $\mathcal{B}$ since $\mathcal{B}$ is an algebra. Since $\mathcal{B}$ is closed, the fact that $|f|$ can be approximated by the function $p(f)$ shows that $|f| \in \mathcal{B}$.

□

Note, inductively, if $f_1, \ldots, f_n \in \mathcal{B}$ then

$$\begin{aligned} f_1 \vee f_2 \vee \cdots \vee f_n = \max\{f_1, \ldots, f_n\} &\in \mathcal{B} \\ f_1 \wedge f_2 \wedge \cdots \wedge f_n = \max\{f_1, \ldots, f_n\} &\in \mathcal{B}. \end{aligned} \tag{J.-216}$$

Proof of Theorem J.11.2

Let f be an arbitrary function in $C(X, \mathbb{R})$. $\epsilon > 0$. We construct a function g such that $f(z) - \epsilon g(z) < f(z) + \epsilon$ for all points z in X $\|f - g\| < \epsilon$. We split this up into three steps.

Step 1. Suppose x and y are distinct points in X and a and b are constants. Since $\mathcal{B}$ separates points there exists a function g in $\mathcal{B}$ such that $g(x) \neq g(y)$. Let f

$$f(z) = a\frac{g(z) - g(y)}{g(x) - g(y)} + b\frac{g(z) - g(x)}{g(y) - g(x)}. \tag{J.-216}$$

Then $f \in \mathcal{A}$ and $f(x) = a$ and $f(y) = b$. For example in Weierstrass theorem f would be (J.11.1).

In the second part we show that given any function $f \in C(X, \mathbb{R})$, a point $x \in X$, and $\epsilon > 0$, there is a function $g_x \in \mathcal{B}$ such that

$$
\begin{aligned}
g_x(x) &= f(x) \\
g_x(t) &> f(t) - \epsilon, \quad \text{for } t \in X.
\end{aligned} \qquad \text{(J.-216)}
$$

Let $y \in X$ be distinct from x. Given $c_1 = f(x)$ and $c_2 = f(y)$ we know there is a function $h_y \in \mathcal{A}$ with the property that

$$
h_y(x) = f(x) \quad \text{and} \quad h_y(y) = f(y). \qquad \text{(J.-216)}
$$

The function $h'_y(t) := h_y(t) - f(t)$ is continuous and $h_y(y) - f(y) = 0$. There is an open interval G_y about y such that $|h_y(t) - f(t)| < \epsilon$ for all $t \in G_y$. This implies

$$
h_y(t) > f(t) - \epsilon, \quad t \in G_y. \qquad \text{(J.-216)}
$$

Since X is compact there are $y_1, \ldots, y_n$ such that $G_{h_1}, \ldots, G_{h_n}$ covers X. Let

$$
g_x(t) = h_{h_1}(t) \vee \cdots \vee h_{h_n}(t). \qquad \text{(J.-216)}
$$

$$
g_z = f_1 \wedge f_2 \wedge \cdots \wedge f_n \qquad \text{(J.-216)}
$$

is a function in L such that $g_x(x) = f(x)$ and $g_x(z) < f(z) + \epsilon$ for all points z in X.

$$
g = g_1 \vee g_2 \vee \cdots \vee g_m \qquad \text{(J.-216)}
$$

g is a function in $\mathcal{A}$ with the property that

$$
f(z) - \epsilon < g(z) < f(z) + \epsilon \qquad \text{(J.-216)}
$$

□

We next turn to the complex case, which is the most important for our purposes. The conditions under which a subalgebra of $C(X, \mathbb{C})$ equals $C(X, \mathbb{C})$.

A complex algebra involves multiplication and addition of complex functions and multiplication by complex numbers.

the conjugate function $\overline{f}(x) = \overline{f(x)}$. if the algebra contains its conjugate

Theorem J.11.4 (Stone-Weierstrass Theorem (complex case)) *Let X be a compact Hausforff space, and let $\mathcal{A}$ be a closed subalgebra of $C(X, \mathbb{C})$ such that:*

(i) $\mathbf{1} \in \mathcal{A}$

(ii) if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$, that is, closed under complex conjugation;

(iii) if $x, y \in X$ with $x \neq y$, there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Then $\mathcal{A} = C(X, \mathbb{C})$.

Proof of Theorem J.11.4

Take any complex function $f = u+iv \in C(X, \mathbb{C})$, where u and v are real valued continuous functions. We wish to demonstrate that there exists a function $f_1 = u_1 + iv_1 \in \mathcal{A}$ such that given any $\epsilon > 0$

$$\|f - f_1\| \leq \epsilon.$$

Observe

$$\|f - f_1\| = \|u - u_1 + i(v - v_1)\| \leq \|u - u_1\| + \|v - v_1\|.$$

If we show that there exists $u_1, v_1 \in \mathcal{A}$ such that $\|u - u_1\| \leq \epsilon/2$ and $\|v - v_1\| \leq \epsilon/2$, then we will have proved the theorem. The real functions in $\mathcal{A}$ form a closed subalgebra $\mathcal{A}_{\mathbb{R}}$ of $C(X, \mathbb{C})$. Since $\mathcal{A}$ is an algebra which contains the conjugate of each of its functions, $u = (f + \overline{f})/2$ and $v = (f - \overline{f})/2i$ are in $\mathcal{A}_{\mathbb{R}}$. With these remarks, it should be easy to see that to prove the complex Stone-Weierstrass theorem it is sufficient to show that $\mathcal{A}_{\mathbb{R}}$ satisfies the conditions of the real Stone-Weierstrass theorem. First we show that $\mathcal{A}_{\mathbb{R}}$ separates points. Take $x \neq y \in X$. Since $\mathcal{A}$ separates points, there exists a function $f \in \mathcal{A}$ which has different values at the two points,

$$u(x) + iv(x) = f(x) \neq f(y) = u(y) + iv(y)$$

implying $u(x) \neq u(y)$ or $v(x) \neq v(y)$ or both, so $\mathcal{A}_{\mathbb{R}}$ separates points. Next we prove that $\mathcal{A}_{\mathbb{R}}$ contains a non-zero constant function. Since $\mathcal{A}$ is an algebra which contains the conjugate of each of its functions, $g\overline{g} = |g|^2$ is a non-zero constant in $\mathcal{A}_{\mathbb{R}}$.

$\square$

J.12 Introduction to Measure Theory and Lebesgue Integration

The Riemann integral we split the x coordinate into small intervals and approximating $f(x)$ in every interval by its maximum and minimum. The problem with this approach is that the difference between the maximum and minimum will only tend to zero, as the interval gets smaller, if $f(x)$ is sufficiently well behaved.

J.13 Measure

A set A is covered by *intervals* $I_1, \ldots, I_n$ when

$$A \subset \cup_{i=1}^{n} I_i.$$

J.13.1 Null Sets

Definition A null set $A \subseteq \mathbb{R}$ is a set that may be covered

$$A \subset \cup_{i=1}^{n} I_i$$

and

$$\sum_{i=1}^{\infty} |I_i| < \epsilon.$$

□

Theorem J.13.1 *If $(N_i)_{i\geq 1}$ is a sequence of null sets, then their union*

$$N = \bigcap_{i=1}^{\infty} N_i$$

is also null.

Proof:

We take any $\epsilon > 0$. What we wish to prove is that is that we can cover the set N by countably many intervals with total length less than ϵ.

Since N_1 is null, there exist intervals $I_k^1 \geq 1$, such that

$$\sum_{k=1}^{\infty} |I_k^1| < \frac{\epsilon}{2}, \qquad N_1 \subseteq \bigcup_{k=1}^{\infty} I_k^1$$

For N_2 is null, there exist intervals $I_k^2 \geq 1$, such that

$$\sum_{k=1}^{\infty} |I_k^2| < \frac{\epsilon}{4}, \qquad N_2 \subseteq \bigcup_{k=1}^{\infty} I_k^2$$

In general, we cover N_i with intervals I_k^i, $k \geq 1$, whose total length is leass than $\frac{\epsilon}{2^i}$

$$\sum_{k=1}^{\infty} |I_k^i| < \frac{\epsilon}{2^i}, \qquad N_i \subseteq \bigcup_{k=1}^{\infty} I_k^i.$$

The next step is to form the intervals I_k^i into a sequence. We arrange the countable family of intervals $\{I_k^i\}_{k \geq 1, n \geq 1}$ into a sequence J_j, $j \geq 1$. We have

$$N = \bigcup_{i=1}^{\infty} N_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^i = \bigcup_{j=1}^{\infty} J_j.$$

We now compute the total length of J_j. We are summing over numbers with two indices:

$$\sum_{j=1}^{\infty} |J_j| = \sum_{i=1,k=1}^{\infty} |I_k^i|.$$

As the componenets are all positive, if we can show that the summation converges for a particular ording of the terms we will have shown the series to be absolutely convergent. This then will mean that the summation would converge for any rearrangement and that the rearranged series has the same sum. We choose the following arrangement of the series:

$$\sum_{i=1,k=1}^{\infty} |I_k^i| = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |I_k^i| \right) < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon,$$

which completes the proof.

□

J.13.2 Outer Measure

Definition The (Lebesgue) outer measure of a set $A \in \mathbb{R}$ is given by

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |I_i| \right\}$$

where the infimum is taken over all countable covers of A by intervals. (Note that some of the I_i may be empty; this avoids having to be concerned with whether the sequence $\{I_i\}$ has finitely or infinitely many different memebers.)

Let us define Z_A by

$$Z_A = \{\sum_{i=1}^{\infty} |I_i| : \text{where } I_i \text{ are intervals, } A \subseteq \bigcup_{i=1}^{\infty} I_i\}$$

then

$$m^*(A) = \inf Z_A.$$

□

We show that the concept of a null set is consistent with that of outer measure.

Theorem J.13.2 *$A \subseteq \mathbb{R}$ is a null set if and only if $m^*(A) = 0$.*

Proof:

Suppose that A is a null set. We wish to show that $\inf Z_A = 0$. We do this by showing that for any $\epsilon > 0$ we can find an element $z \in Z_A$ such that $z < \epsilon$. Since we can do this for any $\epsilon > 0$, we must have $\inf Z_A = 0$.

By definition of a null set we can find a sequence (I_i) of intervals covering A with $\sum_{i-1}^{\infty} |I_i| < \epsilon$ and so $\sum_{i-1}^{\infty} |I_i|$ is the required element z of Z_A.

Conversely, If $A \subseteq \mathbb{R}$ has $m^*(A) = 0$, then by definition of inf, given any $\epsilon > 0$, there is $z \in Z_A$ with $z < \epsilon$. But a member of Z_A is the total length of some covering of A. That is, there is a covering (I_i) of A with total length less than ϵ, so A is null.

□

Non-Measurable Sets

There exist a non-measurable set $V \in \mathbb{R}$.

Proof: We say that two numbers $x, y \in \mathbb{R}$ are equivalent if $x - y = \mathbb{Q}$, i.e., if they differ by a rational number. This is an equaivalence realtion as $x - x = 0$, $x - y = y - x$, and $x - z = (x - y) + (y - z)$ is a rational number when $x - y$ and $y - z$ are rational numbers. As such the real line $\mathbb{R}$ is partinaed into mutaully disjoint subsets, such that each $x \in \mathbb{R}$ belongs to one subset. Each equivalence class is countable and can be thought of as a "shifted copy" of the set of rational numbers. The collection of classes is uncountable.

Each equivalence class is dense, so they all intersect the unit interval $[0, 1]$. We want to choose one (arbitrary) representation from each equivalence class in $[0, 1]$. The chosen numbers make up a set (a subset of $[0, 1]$). This is called the Vitali set, denoted by V. Note that $V \subset [0, 1]$, and for each equivalence class $C \subset \mathbb{R}$ the intersection $V \cap C$ consists of a single point. Note that for any $x, y \in V$ the difference $x - y$ is irrational (otherwise these points belong to the same class).

Next we argue that the Vitali set cannot be measurable. Let $r_1, r_2, \ldots$ denote all rational numbers in $[-1, 1]$. From the construction of V it follows that the translated sets $V_i = V + r_i$ are pairwise disjoint. For if we had $v_i = v_j$ where $v_i \in V_i$ and $v_j \in V_j$ then we would have $v + r_i = \tilde{v} + r_j$ where $v, \tilde{v} \in V$. But then $v - \tilde{v} = r_j - r_i$ which is a contradiction as $r_j - r_i$ is a ratioanl number.Further note that

$$[0, 1] \subset \coprod_{i=1}^{\infty} V_i \subset [-1, 2].$$

(To see the first inclusion, consider any real number $x \in [0, 1]$. This x belong to one equivalence class C. Let v be the representative in V for the equivalence class containing x; then $x - v = r$ for some rational number $r \in [-1, 1]$.)

Now suppose V is measurable. Its measure $m(V)$ is a nonnegative real number. All the "shifted" copies of V, i.e., the sets $V_i = V + r_i$, are measurable, as well, and their measure is the same as that of V, i.e. $m(V_i) = m(V)$, due to the translation invariance. This implies

$$m\left([0, 1]\right) = 1 \leq \sum_{i=1}^{\infty} m(V_i) \leq 3 = m\left([-1, 2]\right).$$

But the infinite sum in the middle can only be zero (if $m(V) = 0$) or infinity (if $m(V) > 0$), a contradiction.

$\square$

Axiom of Choice.

A subtle and contraversial issue in mathematical logic - Axiom of Choice. How can we choose one representative from each equivalence class? Is there a rule? An algorithm? There are uncountably many classes out there, so no formal procedure can handle all of them.

The Axiom of Choice asserts that such a choice is always possible. This principle doesnot follow from other, standard logical axioms, so has to adopted as a separate one.

J.13.3 Lebesgue-Measure

Definition A set $E \subseteq \mathbb{R}$ is (Lebesgue-) measurable if for every set $A \subseteq \mathbb{R}$ we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{J.-216}$$

where $E^c := \mathbb{R}/E$.

We obviously have $A = (A \cap E) \cup (A \cap E^c)$, hence by countable subadditivitiy we have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

for any A and E. Hence proving (J.13.3) can be simplified to proving the following inequality holds:

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \tag{J.-216}$$

for all $A \subseteq \mathbb{R}$.

□

Theorem J.13.3

(i) $\mathbb{R} \in \mathcal{E}$,

(ii) if $E \in \mathcal{E}$ *then* $E^c \in \mathcal{E}$,

(iii) if $E_n \in \mathcal{E}$ *for all* $n = 1, 2, \ldots$ *then* $\cup_{n=1}^{\infty} E_n \in \mathcal{E}$.

Moreover, if $E_n \in \mathcal{E}$ *for all* $n = 1, 2, \ldots$ *and the* E_n*'s are pairwise disjoint then*

$$m^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m^*(E_n) \tag{J.-216}$$

Proof:

(i) Let $A \subseteq \mathbb{R}$. Note that $A \cap \mathbb{R} = A$, $\mathbb{R}^c = \emptyset$, so that $A \cap \mathbb{R}^c = \emptyset$.

$$\begin{aligned} m^*(A) &= m^*(A) + 0 \\ &= m^*(A) + m^*(\emptyset) \\ &= m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c), \end{aligned}$$

hence $\mathbb{R} \in \mathcal{E}$.

(ii) Suppose $E \in \mathcal{E}$

$$\begin{aligned} m^*(A) &= m^*(A \cap E) + m^*(A \cap E^c) \\ &= m^*(A \cap E^c) + m^*(A \cap E) \\ &= m^*(A \cap E^c) + m^*(A \cap (E^c)^c) \end{aligned}$$

where we have used $(E^c)^c = E$. Hence $E^c \in \mathcal{E}$.

(iii)

Part (a). Proof for pairwise disjoint E_k, $k = 1, 2$.

Suppose

$$E_1 \cap E_2 = \emptyset, \quad E_1, E_2 \in \mathcal{E}.$$

we show that

$$E_1 \cup E_2 \in \mathcal{E}$$

and

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2).$$

We have

$$m^*(A) \leq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

we will prove the inequality the other way round.

As $E_1 \in \mathcal{E}$ we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \tag{J.-222}$$

Apply the same condition for E_2 but with A replaced by $A \cap E_1^c$:

$$\begin{aligned} m^*(A \cap E_1^c) &= m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c) \\ &= m^*(A \cap (E_1^c \cap E_2)) + m^*(A \cap (E_1^c \cap E_2^c)) \end{aligned}$$

Since E_1 and E_2 are disjoint,

$$E_1^c \cap E_2 = E_2.$$

By de Morgan'law

$$E_1^c \cap E_2^c = (E_1 \cup E_2)^c.$$

Therefore we have

$$m^*(A \cap E_1^c) = m^*(A \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c)$$

Inserting this into ()

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c) \tag{J.-224}$$

Now by the subaddativity property of m^*

$$\begin{aligned} m^*(A \cap E_1) + m^*(A \cap E_2) &\geq m^*((A \cap E_1) \cup (A \cap E_2)) \\ &= m^*(A \cap (E_1 \cup E_2)) \end{aligned}$$

So that

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

Hence $E_1 \cup E_2 \in \mathcal{E}$.

Finally put $A = E_1 \cup E_2$ in (J.13.3) to get

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2).$$

Part (b). Proof for pairwise disjoint E_k, $k = 1, 2, \ldots$

We start out by proving that if pairwise disjoint E_k, $k = 1, 2, \ldots$ are in $\mathcal{E}$ then their union is in $\mathcal{E}$ and (J.13.3) holds.

We prove by induction that

$$m^*(A) = \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{n})^c). \tag{J.-226}$$

The case $n = 1$ has already been proved and reads:

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c).$$

Suppose that

$$m^*(A) = \sum_{k=1}^{n-1} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{n-1} E_k)^c). \tag{J.-226}$$

Since $E_n \in \mathcal{E}$, we may apply () with $A \cap (\cup_{k=1}^{n} E_k)^c$ n place of A:

$$m^*(A \cap (\bigcup_{k=1}^{n-1} E_k)^c) = m^*(A \cap (\bigcup_{k=1}^{n-1} E_k)^c \cap E_n) + m^*(A \cap (\bigcup_{k=1}^{n-1} E_k)^c \cap E_n^c). \tag{J.-226}$$

Noting:

$$\begin{aligned}
(\bigcup_{k=1}^{n-1} E_k)^c \cap E_n &= E_n \quad (E_i \text{ are pairwise disjoint}) \\
(\bigcup_{k=1}^{n-1} E_k)^c \cap E_n^c &= (\bigcup_{k=1}^{n} E_k)^c \quad (\text{by de Morgan's law}).
\end{aligned}$$

Inserting these into (J.13.3) we obtain

$$m^*(A \cap (\bigcup_{k=1}^{n-1} E_k)^c) = m^*(A \cap E_n) + m^*(A \cap (\bigcup_{k=1}^{n} E_k)^c),$$

and inserting this into the inductive hypothesis (N.-19)

$$m^*(A) = \sum_{k=1}^{n-1} m^*(A \cap E_k) + m^*(A \cap E_n) + m^*(A \cap (\bigcup_{k=1}^{n} E_k)^c)$$

as required.

We now turn

$$m^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m^*(E_n).$$

Since

$$\left(\bigcup_{k=1}^{n} E_k \right)^c \supseteq \left(\bigcup_{k=1}^{\infty} E_k \right)^c$$

from (J.13.3) we get

$$\begin{aligned} m^*(A) &= \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{n} E_k)^c) \\ &\geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \end{aligned}$$

where we have applied monitonicity. The inequality remains true as we take the limit $n \to \infty$:

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \qquad \text{(J.-230)}$$

By countable subadditivity

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \geq m^*(A \cap (\bigcup_{k=1}^{\infty} E_k))$$

and so

$$m^*(A) \geq m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \tag{J.-230}$$

So we have shown that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{E}$ and hence the two sides of (J.13.3) are equal.

We then have

$$\begin{aligned} m^*(A) &\geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \\ &\geq m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \\ &= m^*(A) \end{aligned}$$

and therefore

$$m^*(A) = \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \tag{J.-233}$$

The equality here is a consequence of the assumption that the E_k are pairwise disjoint. It holds for are set A, so we may insert $A = \bigcup_{j=1}^{\infty} E_j$, we obtain

$$\begin{aligned} m^*(\bigcup_{j=1}^{\infty} E_j) &= \sum_{k=1}^{n} m^*((\bigcup_{j=1}^{\infty} E_j) \cap E_k) + m^*((\bigcup_{j=1}^{\infty} E_j) \cap (\bigcup_{k=1}^{\infty} E_k)^c) \\ &= \sum_{k=1}^{n} m^*(E_k) + m^*(\emptyset) \\ &= \sum_{k=1}^{n} m^*(E_k). \end{aligned}$$

We have thus proven (J.13.3).

Part (c). Proof for pairwise disjoint E_1, E_2, then $E_1 \cup E_2 \in \mathcal{E}$ (not necessarily disjoint)

As $E_1 \in \mathcal{E}$ we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c). \tag{J.-236}$$

As $E_2 \in \mathcal{E}$ we can write

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c).$$

Substituting this into (J.13.3) we obtain

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \tag{J.-236}$$

By de Morgan's law $E_1^c \cap E_1^c = (E_1 \cup E_2)^c$, so as before

$$m^*(A \cap E_1^c \cap E_2^c) = m^*(A \cap (E_1 \cup E_2)^c). \tag{J.-236}$$

By subadditivity we have

$$\begin{aligned} m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) &\geq m^*(A \cap (E_1 \cup (E_1^c \cap E_2))) \\ &= m^*(A \cap (E_1 \cup (E_2 \backslash E_1))) \\ &= m^*(A \cap (E_1 \cup E_2)). \end{aligned} \tag{J.-237}$$

Inserting (J.13.3) and (J.-237) into (J.13.3) we obtain

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

as required.

Part (d). If $E_k \in \mathcal{E}$, $k = 1, \ldots, n$, then $E_1 \cup \cdots \cup E_n \in \mathcal{E}$ (not necessarily disjoint).

We argue by induction.There is nothing to prove for $n = 1$. Suppose the claim is true for $n - 1$. Then

$$E_1 \cup \cdots \cup E_n = (E_1 \cup \cdots \cup E_{n-1}) \cup E_n$$

so the result follows from part (c).

Part (e). If $E_1, E_2 \in \mathcal{E}$, then $E_1 \cap E_2 \in \mathcal{E}$ (not necessarily disjoint).

We have $E_1^c, E_2^c \in \mathcal{E}$ by (ii), that $E_1^c \cup E_2^c \in \mathcal{E}$ by part (c), $(E_1^c \cup E_2^c)^c \in \mathcal{E}$ by (ii) again, but by de Morgan's law the last set is equal to $E_1 \cap E_2$.

Part (f). The general case: if $E_1, E_2, \ldots$ are in $\mathcal{E}$, then $\bigcup_{k=1}^{\infty} E_k$.

Let $E_k \in \mathcal{E}$, $k = 1, 2, \ldots$. We define a sequence of pairwise disjoint sets C_k withe the same union as E_k:

$$\begin{aligned}
C_1 &= E_1 \\
C_2 &= E_2 \backslash E_1 = E_2 \cap E_1^c \\
C_3 &= E_3 \backslash (E_1 \cup E_2) = E_3 \cap (E_1 \cup E_2)^c \\
&\ldots \\
C_k &= E_k \backslash (E_1 \cup \cdots \cup E_{k-1}) = E_k \cap (E_1 \cup \cdots \cup E_{k-1})^c;
\end{aligned}$$

By parts (d) and (e) we know that all the C_k's are in $\mathcal{E}$. By their construction they are pairwise disjoint, so by part (b) their union is in $\mathcal{E}$. We shall show that

$$\bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} E_k.$$

This will complete the proof since the former is in $\mathcal{E}$. The inclusion

$$\bigcup_{k=1}^{\infty} C_k \subseteq \bigcup_{k=1}^{\infty} E_k.$$

is obvious because since for each k, $C_k \subseteq E_k$ by definition. To prove the inverse let $a \in \bigcup_{k=1}^{\infty} E_k$. Let S denote the set $\{n \in \mathbb{N} : a \in E_n\}$ which is non-empty as a belongs to the union. Let $n_0 = \min S \in S$. If $n_0 = 1$, then $a \in E_1 = C_1$. Suposse that $n_0 > 1$. So $a \in E_{n_0}$ and $a \notin E_1, \ldots a \notin E_{n_0-1}$. By the definition of C_{n_0}this means that $a \in C_{n_0}$, and so $a \in \bigcup_{k=1}^{\infty} C_k$.

□

J.14 Intergration

To circumvent these difficulties, the idea is to portion the range instead of the domain. Fig(J.14).

We considering instead of an interval, the set of x for which $f(x)$ lies between two numbers $a < b$. Now we need the size of the set of these x, that is, the size of the preimage $f^{-1}((a, b))$.

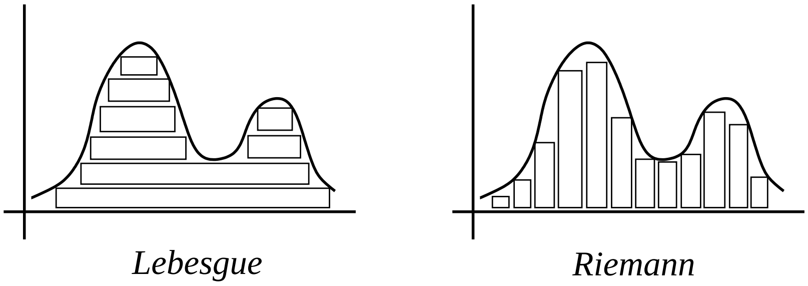


Figure J.17: LebRien.

What are reasonable to take for subsets? There is no reasonable way of adding up an uncountable set of numbers each of which is zero. Subsets that satisfy the union of any two subsets is also in the family, the intersection of any two subsets is also in the family, the completment of any subset is also in the family.

Borel Sets

$$\begin{aligned} [a,b) &= \bigcap_{n=1}^{\infty} (a-\frac{1}{n},b), \\ [a,b] &= \bigcap_{n=1}^{\infty} (a-\frac{1}{n},b+\frac{1}{n}), \\ (a,b] &= \bigcap_{n=1}^{\infty} (a,b+\frac{1}{n}). \end{aligned} \tag{J.-243}$$

The first equality holds as $a \in (a-\frac{1}{n},b)$ for all $n \in \mathbb{N}$.

Characteristic function of a set $A \subset X$ is a mapping defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \tag{J.-243}$$

A finite linear combination of characteristic functions of semi-open intervals

$$f = \sum_{k=1}^{N} \alpha_k \mu(I_k) \tag{J.-243}$$

is called a *step function*.

The Lebesgue integral of a step function is defined

$$\int \sum_{k=1}^{N} \alpha_k \chi_{I_k} dx = \sum_{k=1}^{N} \alpha_k \mu(I_k) \tag{J.-243}$$

Lebesgue integral

$$(1) \quad \sum_{k=1}^{\infty} \int |f_k| dx < \infty, \tag{J.-242}$$

$$(2) \quad f(x) = \sum_{k=1}^{\infty} f_k(x) \text{ for all } x \in \mathbb{R} \text{ such that } \sum_{k=1}^{\infty} |f_k(x)| < \infty. \tag{J.-241}$$

The *Lebesgue integral* of f is then defined by

$$\int f dx := \sum_{k=1}^{\infty} \int f_k dx \tag{J.-241}$$

In R we need a notion of length. We need to define a function $\mu : \mathcal{P}(R) \to [0, \infty]$ to measure length of as many sets as possible, such that

$$\mu((a, b]) = b - a \text{ and } \mu(A \sqcup B) = \mu(A) + \mu(B), \;\; \mu\left(\sqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{J.-241}$$

However this cannot be done for all subsets of $\mathcal{P}$.

$$(a, b] = \bigcup_{n=1}^{\infty} \left[a + \frac{(b-a)}{n}, b\right]. \tag{J.-241}$$

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{(b-a)}{n}\right]. \tag{J.-241}$$

To say a property holds almost everywhere with respect to a measure μ means that there is a set $E \in$ with $\mu(E) = 0$ such that the property holds for all $x \in X \backslash E$.

A measure on a set A is an association of nonnegative numbers to subsets of A

We will denote the union of pairwise disjoint subsets $A_1, \ldots, A_n$

$$\bigsqcup_{i=1}^{n} A_i. \tag{J.-241}$$

Definition Let X be a set. Then a collection of sets $\mathcal{L} \subseteq \mathcal{P}(X)\mathcal{P}$ is a semi-ring if

(i) $\emptyset \in \mathcal{L}$,

(ii) if $A, B \in \mathcal{L}$ then $A \cap B \in \mathcal{L}$,

(iii) if $A, B \in \mathcal{L}$ then there is an $n \in N$ and there are sets $A_1, A_2, \ldots, A_n \in \mathcal{L}$ such that A_i are pairwise disjoint and $A \backslash B = \bigcup_{i=1}^{n} A_i$,

Definition Let X be a set, let $R \subseteq \mathcal{P}(\mathcal{X})$. Then R is a ring of subsets of X if

(i) $\emptyset \in R$;

(ii) if $A, B \in R$ then $A \cap B$, $A \cup B$ and $A \backslash B$ are all in R.

$\square$

Definition Let X be a set and let $\mathcal{F} \subseteq \mathcal{P}(X)$. Then $\mathcal{F}$ is a $\sigma-$**field** of subsets of X if $\mathcal{F}$ satisfies

(i) $\emptyset, X \in \mathcal{F}$,

(ii) for all $A, B \in \mathcal{F}$

(iii) whenever $A_1, A_2, A_3, \cdots \in \mathcal{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Proof:

(i) Each $\sigma-$field contains $\emptyset$ and X. Thus $\emptyset \in$ the intersection and $X \in \cap_{\gamma \in \Gamma} \mathcal{F}_\gamma$.

(ii) for all $A, B \in \mathcal{F}$

(iii) whenever $A_1, A_2, A_3, \cdots \in \cap_{\gamma \in \Gamma} \mathcal{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

$$A_1 \cap A_2 = X \backslash \left(\, (X \backslash A_1) \cup (X \backslash A_2) \, \right), \tag{J.-241}$$

Iterating this we have

$$\bigcap_{k=1}^{n} A_k = X \backslash \left(\bigcup_{k=1}^{n} (X \backslash A_k) \right), \tag{J.-241}$$

Taking the limit $k \to \infty$

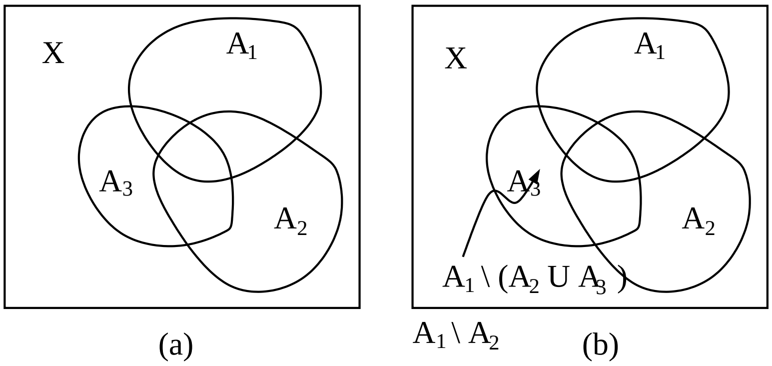


Figure J.18: (a) . (b) $A_1 \cap A_2 = X \backslash (\ (X \backslash A_1) \cup (X \backslash A_2)\ )$

$$\bigcap_{n=1}^{\infty} A_n = X \backslash \left(\bigcup_{n=1}^{\infty} (X \backslash A_n) \right), \tag{J.-241}$$

J.14.1 Measures and Measure Spaces

Definition Let X be a set, let $\mathcal{C} \subseteq \mathcal{P}$ which contains the empty set, $\emptyset \in \mathcal{C}$, and let $\mu : \mathcal{C} \to [0, \infty]$. Then μ is a measure on $\mathcal{C}$ if

(i) $\mu(\emptyset) = 0$,

(ii) if $A_1, A_2, \ldots,$ is a sequence of pairwise disjoint sets in $\mathcal{C}$ such that $\sqcup_{n=1}^{\infty} A_n$ is in $\mathcal{C}$, then

$$\mu \left(\bigsqcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{J.-241}$$

Definition: A *function* $m : \mathcal{U} \to \mathbb{R}_+ \cup \{\infty\}$ *is called a measure if*
1. $m(A) \geq 0$ *for any* $A \in \mathcal{U}$ *and* $\mu(\phi) = 0$,
2. if $(A_i)_{i \geq 1}$ *is a disjoint family of sets in* $\mathcal{U}$ ($A_i \cap A_j = 0$ *for any* $i \neq j$) *such that* $\bigsqcup_{i=1}^{\infty} \in \mathcal{U}$, *then*

$$\mu \left(\bigsqcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{J.-240}$$

This important property is called *countability additivity* or σ*-additivity* of the measure μ.

Definition

A measurable space is a pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F}$ is a $\sigma-$field of subsets of X.

A measure space is a triple $(X, \mathcal{F}, \mu)$ where $\mathcal{F}$ is a $\sigma-$field on X, and

$$\mu : \mathcal{F} \to [0, \infty] \text{ is a measure.} \tag{J.-240}$$

Measures of Rings

(i) Monotonicty of μ: If $A, B \in \mathcal{U}$ and $B \subset A$ then $\mu(A) \leq \mu(A)$.

Proof: $A = (A/B) \sqcup B$ implies that

$$\mu(A) = \mu(A \backslash B) + \mu(B). \tag{J.-240}$$

Since $\mu(A \backslash B) \geq 0$ it follows that $\mu(A) \geq \mu(B)$.

(ii) Subtractivity of μ. If $A, B \in \mathcal{U}$ and $B \subset A$ and $\mu < \infty$ then $\mu(A \backslash B) = \mu(A) - \mu(B)$.

Proof: In (i) we proved that

$$\mu(A) = \mu(A \backslash B) + \mu(B). \tag{J.-240}$$

If $\mu(B) < \infty$ then

$$\mu(A) - \mu(B) = \mu(A \backslash B). \tag{J.-240}$$

with counting measure on N.

Set

$$\begin{aligned} A &= \{2, 4, 6, \ldots\} \\ B &= \{\text{primes}\} \end{aligned} \tag{J.-240}$$

$A \cap B = \{2\},\ \mu(A \cap B) = 1,\ \mu(A) = \mu(B) = \mu(A \backslash B) = \infty$ so that $\mu(A) - \mu(A \backslash B)$ is not defined.

(iii) If $A, B \in \mathcal{U}$ and $\mu(A \cap B) < \infty$ then $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

Proof

$A \cap B \subset A$, $\mu(A \cap B) \subset B$, therefore

$$A \cup B = (A \backslash (A \cap B)) \sqcup B. \tag{J.-240}$$

Since $\mu(A \cap B) < \infty$ one has

$$\mu(A \cup B) = (\mu(A) - \mu(A \cap B)) + \mu(B). \tag{J.-240}$$

Properties of measures

(i) If $\bigcup_{n=1}^{\infty} A_n \in R$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} A_k\right). \tag{J.-240}$$

(ii) If $\mu(A_1) < \infty$ $\bigcap_{n=1}^{\infty} A_n \in R$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu\left(\bigcap_{k=1}^{n} A_k\right). \tag{J.-240}$$

Proof

(i) Set $A := \cup_{n=1}^{\infty} A_n$. As the sets $A_1, A_2, \ldots$ are not pairwise disjoint.

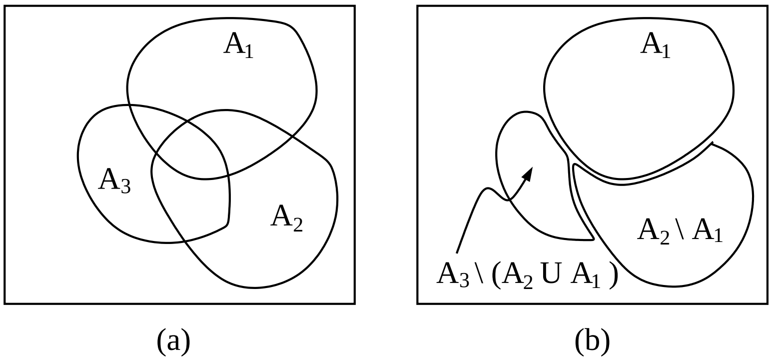


Figure J.19: (a) . (b) $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash (A_2 \cup A_1)$. $\cup_{k=1}^{3} A_k = \sqcup_{k=1}^{3} B_k$

Set $B_1 := A_1$ and $B_n := A_n \backslash \cap_{k=1}^{n-1} A_k$ (for $n > 1$). Then each $B_n \in R$, the sets B_n are pointwise disjoint,

$$\bigcup_{k=1}^{n} A_k = \bigsqcup_{k=1}^{n} B_k \text{ for all } n, \tag{J.-240}$$

and $A = \bigsqcup_{k=1}^{\infty} B_k$. Thus

$$\begin{aligned}
\mu(A) &= \sum_{n=1}^{\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \mu(B_k) \right) \\
&= \lim_{n\to\infty} \left(\mu \left(\bigsqcup_{k=1}^{n} B_k \right) \right) \\
&= \lim_{n\to\infty} \left(\mu \left(\bigcup_{k=1}^{n} A_k \right) \right). \qquad \text{(J.-242)}
\end{aligned}$$

(ii) Suppose that $\mu(A_1) < \infty$.

set

$$C_n = A_1 \backslash A_n \ \text{ for all } n. \qquad \text{(J.-242)}$$

Then $C_n \in R$ and

$$\begin{aligned}
\bigcap_{n=1}^{\infty} A_n &= A_1 \backslash \bigcup_{n=1}^{\infty} (A_1 \backslash A_n) \\
&= A_1 \backslash \bigcup_{n=1}^{\infty} C_n. \qquad \text{(J.-242)}
\end{aligned}$$

If we do not have $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ we can work with $A_1, A_1 \cap A_2, A_1 \cap A_2 \cap A_3, \ldots$

Nice Phrases

Suppose X is a set equipped with a collection $\mathcal{M}$ of subsets $E \subset X$ such that $\mathcal{M}$ is closed under countable unions and complements; that is, whenever $E \subset \mathcal{M}$, so is $X \backslash E$, and whenever we have a sequence E_j of sets in $\mathcal{M}$, then $\cup_{j=1}^{\infty}$ is in $\mathcal{M}$. Then $\mathcal{M}$ is said to be a sigma-algebra on X.

(The word “sigma” refers to sum, meaning union, while the word “algebra” indicates that $\mathcal{M}$ is defined in terms of certain operations, in this case unions and complements of sets).

(AN ENTROPY PRIMER CHRIS HILLMAN)

σ-algebra

Theorem J.14.1 *If $\mathcal{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathcal{M}^*$ in X such that $\mathcal{F} \in \mathcal{M}^*$.*

2^X is an algebra so that there always is an algebra. We need to prove that the intersection of any two algebras also forms and algebra.

Definition A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a *Borel measure* on X.

$\square$

J.14.2 Measurable functions

Let X be a set, $\mathcal{U}$ a σ-algebra on X.

Definition Let X be a measurable space and let Y be a topological space. A function $f : X \to Y$ is said to be measurable provided that the pre-image $f^{-1}(V) \subset X$ of any open set $V \in Y$ is a measurable subset in X.

$\square$

Definition Let f be a function defined on a measurable space $(X, \mathcal{U})$, with values is the extended real number system. The function f is called measurable if the set

$$\{x : f(x) > a\} \tag{J.-242}$$

is a measurable for every real a.

$\square$

Definition A pair $(X, \mathcal{U})$ is called measurable space.

Step functions (simple functions)

Definition A real valued function $s : X \to \mathbb{C}$ is called simple function if it takes only a finite number of distinct values. If $z_k \in \mathbb{C}$, $k = 1, \ldots, N$ are these values and $S_k = s^{-1}(\{z_k\})$ then

$$s = \sum_{k=1}^{N} z_k \chi_{S_k},$$

where

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

is called the characteristic function of the subset $S \subset X$.

□

Theorem J.14.2 *A simple function $f = \sum_{k=1}^{n} z_k \chi_{S_k}$ is measurable if and only if all the sets E_j are measurable.*

□

Lemma J.14.3 *Let $f : X \to [0, \infty]$ be measrable. Then there exists a sequence of measurable simple functions s_n such that*

(a) $0 \leq s_1 \leq s_2 \leq \cdots \leq f$
(b) $\lim_{n\to\infty} s_n(x) = f(x)$ *pointwise in* $x \in X$.

□

J.15 Main Results of Integration Theory

J.15.1 Dominating Convergence

Theorem J.15.1 *(Monotone Convergence Theorem)*

Let (X, f, μ) be a measure space, let

$$f_n : X \to [0, \infty] \tag{J.-242}$$

be a sequence of measurable functions with

$$0 \leq f_1(x) \leq f_2(x) \leq \ldots \text{ for all } x \in X. \tag{J.-242}$$

Suppose

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ for all } x \in X. \tag{J.-242}$$

Then f is measurable and

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu. \tag{J.-242}$$

Lemma J.15.2 (Fatou's Lemma).

Theorem J.15.3 (Dominating Convergence Theorem)

The Lebesgue dominating convergence theorem we can see when the integral of the limit of functions is equal to the limit of integrals,

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n \tag{J.-242}$$

If

$$f(x) \to \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x), \tag{J.-242}$$

then it says we can intergrate term by term, i.e.,

$$\int f(x) d\alpha = \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k(x) d\alpha \tag{J.-242}$$

$$|f(x) - f_n(x)| \leq \epsilon_n \tag{J.-242}$$

proof:

$$\int (f(x) - \epsilon_n) d\alpha \leq \underline{\int} f_n d\alpha \leq \overline{\int} f d\alpha \leq \int (f(x) + \epsilon_n) d\alpha \tag{J.-242}$$

$$0 \leq \overline{\int} f_n d\alpha - \underline{\int} f_n d\alpha \leq 2\epsilon_n[\alpha(b) - \alpha(a)] \tag{J.-242}$$

$$|f(x) - f_n(x)| \leq \epsilon_n[\alpha(b) - \alpha(a)] \tag{J.-242}$$

J.15.2 Fubini

Theorem J.15.4 *(Fubini)*

Let (X_1, τ_1, μ_1) and (X_2, τ_2, μ_2) be two measure spaces in which μ_1 and μ_2 are $\sigma-$finite.

There is a unique measure μ on $X_1 \times X_2$ with the property that

$$\mu(A \times B) = \mu_1(A) \times \mu_2(B) \tag{J.-242}$$

for all measurable rectangles $A \times B$. This measure is called the product measure *and we write $\mu = \mu_1 \times \mu_2$. Fubini's theorem is the most important result for product measures.*

Theorem J.15.5 *If f is an integrable function on $X_1 \times X_2$, then $x \mapsto f(x, y)$ is an integrable function of X for a.e. y, $y \mapsto f(x, y)$ is an integrable function of y for a.e. x, and*

$$\int f \ d(\mu_1 \times \mu_2) = \int \left[\int f \ d\mu_1\right] d\mu_2 = \int \left[\int f \ d\mu_2\right] d\mu_1 \tag{J.-242}$$

$$\int_X f d\mu = \int_{X_1} \left[\int_{X_2} f(\omega_1, \omega_2)\mu_2(d\omega_2)\right] \mu_1(d\omega_1). \tag{J.-242}$$

J.15.3 Radon-Nikodym

a real Borel measure ν and the Lebesgue measure m.

$$A \mapsto \nu(A) = \int_A f \ dm \tag{J.-242}$$

defines a Borel measure ν on $(\mathbb{R}, \mathcal{M})$.

Definition Let X be a set and $\mathcal{F}$ be a $\sigma-$field of its subsets. Suppose that ν and μ are measures on $(X, \mathcal{F})$. We say that ν is absolutely continuous with respect to μ if $\mu(A) = 0$ implies $\nu(A) = 0$ for $A \in \mathcal{F}$

how it connects with the notion of continuity of real functions.

Definition If there is a set $E \in \mathcal{F}$ such that $\lambda(F) = \lambda(E \cap F)$ for every $F \in \mathcal{F}$ then λ is concentrated on E.

If two measures μ, ν are concentrated on disjoint subsets of X, we say that they are mutually singular and write $\mu \perp \nu$.

The two main results will follow as simple consequence of the following result:

Theorem J.15.6 *Let μ, ν be $\sigma-$finite measures. Then there exists a unique (a.e.) non-negative function f and a set N of measure zero, such that*

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu. \tag{J.-242}$$

Proof. We first assume μ, to be finite measures. Let $\alpha = \mu + \nu$ and consider the Hilbert space $L^2(X, d\alpha)$. Then

$$\ell(h) = \int_X h d\nu \tag{J.-242}$$

is a bounded linear functional by Cauchy-Schwarz:

$$\begin{aligned} |\ell(h)|^2 &= \Big| \int_X h d\nu \Big|^2 \leq \Big(\int |1|^2 d\nu \Big) \Big(\int |h|^2 d\nu \Big) \\ &= \nu(X) \Big(\int |h|^2 d\alpha \Big) = \nu(X) \|h\|^2. \end{aligned} \tag{J.-242}$$

Hence by the Riesz lemma (Theorem N.-19) there exists an $g \in L^2(X, d\alpha)$ such that

$$\ell(h) = \int_X hg \, d\alpha \tag{J.-242}$$

By construction

$$\nu(A) = \int \chi_A \, d\nu = \int \chi_A g \, d\alpha = \int_A g \, d\alpha \tag{J.-242}$$

In particular, g must be positive a.e. (to see this take A to be the set where g is negative). Furthermore, let $N = x|g(x) \geq 1$, then

$$\nu(N) = \int_N g \, d\alpha \geq \mu(N) + \nu(N), \tag{J.-242}$$

which shows $\mu(N) = 0$. Now set

$$f = \frac{g}{1-g}\chi_{N'}, \tag{J.-242}$$

$N' = X \backslash N$. Then, since (N.-19) implies $d\nu = g d\alpha$ respectively $d\mu = (1-g)d\alpha$, we have

$$\begin{aligned} \int_A f d\mu &= \int \chi \frac{g}{1-g}\chi_{N'} d\mu \\ &= \int \chi_{A\cap N'} g \, d\alpha \\ &= \nu(A \cap N') \end{aligned} \tag{J.-243}$$

as desired. Clearly f is unique, since if there is a second function $\tilde{f}$, then $\int_A (f - \tilde{f}) d\mu = 0$ for every A shows $f - \tilde{f} = 0$ a.e..

To see the $\sigma-$finite case, observe that $X_n \subseteq X$, $\mu(X_n) < \infty$ and $Y_n \subseteq X, (Y_n) < \infty$ implies $X_n Y_n \subseteq X$ and $\alpha(X_n \, Y_n) < \infty$. Hence when restricted to $X_n \cap Y_n$ we have sets N_n and functions f_n. Now take $N = \cup N_n$ and choose f such that $f|_{X_n} = f_n$ (this is possible since$f_{n+1}|_{X_n} = f_n$ a.e.). Then $(N) = 0$ and

$$\nu(A \cap N') = \lim_{n\to\infty} \nu(A \cap (X_n \backslash N)) \lim_{n\to\infty} \int_{A\cap X_n} f \, d\mu = \int f \, d\mu, \tag{J.-243}$$

which finishes the proof.

□

Theorem J.15.7 (Lebesgue decomposition) *Let λ, μ be $\sigma-$finite measures on $(X, \mathcal{F})$. Then λ can be expressed uniquely as a sum of two measures, $\lambda = \lambda_a + \lambda_s$ where $\lambda_a \gg \mu$ and $\lambda_s \perp \mu$.*

$\square$

Theorem J.15.8 (Radon-Nikodym)

$$\nu(A) = \int_A f \, d\mu \tag{J.-243}$$

for every $A \in \Sigma$.

$\square$

Definition (Radon-Nikodym derivative) The function f of theorem J.15.8 is called the Radon-Nikodym derivative

$$\frac{d\nu}{d\mu} \tag{J.-243}$$

of ν with respect to μ.

Some Definitions and Theorems

Theorem J.15.9 *Let* $(X, \mathcal{U}, \mu)$ *be a measure space. Let* $\mathcal{U}'$ *be a collection of all* $S \subset X$ *such that there exists* $U, V \in \mathcal{U}$ *with* $U \subset S \subset V$ *and* $\mu(V - U) = 0$ *(in particular* $\mathcal{U} \subset \mathcal{U}'$*). Define* $\mu'(S) = \mu(U)$ *in that case. Then* $(X, \mathcal{U}', \mu')$ *is a measure space again, called the completion of* $(X, \mathcal{U}, \mu)$.

Definition A set Y in X a measure space $(X, \mathcal{U}, \mu)$ is called thick or a support for μ provided that for any measurable set $U \in \mathcal{U}$ the condition $U \cap Y = \emptyset$ implies $\mu(U) = 0$, A support for μ will be denoted $\mathrm{supp}(\mu)$.

$\square$

quantization of background independent theories of connecctions with local degrees of freedom for example of above defition!!!

J.15.4 Riesz Representation

Theorem J.15.10 *(Riesz Representation Theorem)*

i)

Let X be a locally compact Hausdorff space and let $\Lambda : C_0(X) \to \mathbb{C}$ be a positive linear functional one the space of continuous, complex-valued functions of compact support in X. Then there exists a $\sigma-$algebra $\mathcal{U}$ on X which contains the Borel $\sigma-$algebra and a unique positive measure μ on $\mathcal{U}$ such that Λ is represented by μ, that is,

$$\Lambda(f) = \int_X d\mu(x) f(x) \text{ for all } f \in C_0(X). \tag{J.-243}$$

Moreover, μ has the following properties:

1) $\mu(K) < \infty$ if $K \subset X$ is compact.

2) For every $S \in \mathcal{U}$ property N.-19 holds.

3) For every open $S \in \mathcal{U}$ with $\mu(S) < \infty$ property N.-19.

4) If $S' \subset S \in \mathcal{U}$ and $\mu(S) = 0$ then $S' \in \mathcal{U}$

ii)

If, in addition to i), X is $\sigma-$compact then μ has the following properties

5) μ is regular

6) For any $S \in \mathcal{U}$ and every $\epsilon > 0$ there exists a closed set C and an open set O such that $C \subset S \subset O$ and $\mu(O - C) < \epsilon$.

7) For any $S \subset \mathcal{U}$ there exist sets C' and O'which ar respectively countable unions and interactions of closed and open sets respectively such that $C' \subset S \subset O'$ and $\mu(O' - C') = 0$.

Proof:

$\square$

J.16 Spectral Decomposition of Infinite-Dimensional Spaces

J.16.1 Spectral Integrals

should we put this in LoopRep???

Recall . In the finite dimensional case the spectrum is $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$.

$$
\begin{aligned}
A &= \lambda_1 P_1 + \lambda_2 P_2 + \cdots \lambda_m P_m \\
&= \lambda_1 (E_{\lambda_1} - E_{\lambda_0}) + \lambda_2 (E_{\lambda_2} - E_{\lambda_1}) + \cdots \lambda_m (E_{\lambda_m} - E_{\lambda_{m-1}}) \\
&= \sum_{i=1}^{m} \lambda_i (E_{\lambda_i} - E_{\lambda_{i-1}}). \qquad \text{(J.-244)}
\end{aligned}
$$

$$
A = \sum_{i=1}^{m} \lambda_i \Delta E_{\lambda_i} \qquad \text{(J.-244)}
$$

for each $B \subseteq \sigma(T)$, let

$$
P(B) = \sum_{\lambda_i \in B} P_i. \qquad \text{(J.-244)}
$$

$$
P : \mathcal{P}(\sigma(T)) \to \mathcal{B}(\mathcal{H}) \qquad \text{(J.-244)}
$$

a "$\mathcal{H}$-projection valued measure" on $\sigma(T)$.

Definition Let $\mathcal{H}$ be a complex Hilbert space, and suppose $(X, \mathcal{F})$ is a measurable space. function P from $\mathcal{F}$ into orthogonal projections in $B(\mathcal{H})$ so that

(1) $P(X) = I$, and
(2) if $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint, then

$$
P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) \qquad \text{(J.-244)}
$$

in the strong operator topology.

$$
\mu_{\xi,\eta}(E) = (P(E)\xi, \eta) \qquad \text{(J.-244)}
$$

Properties:

i) $P(A)P(B) = P(A \cap B)$;

ii) $P(\emptyset) = 0$;

iii) $P(A) \leq P(B)$ for $A \subset B$ where for two projections $P_1 \leq P_2$ means that $P_1\mathcal{H} \subset P_2\mathcal{H}$.

$$
\begin{aligned}
P(A \cup B) + P(A \cap B) &= P(A) + P(B) \\
P(A \cup B)P(A) + P(A \cap B)P(A) &= P(A) + P(B)P(A) \\
P(A) + P(A \cap B) &= P(A) + P(B)P(A) \qquad \text{(J.-245)}
\end{aligned}
$$

Just for ordinary measures, the set of $B \in \mathcal{F}$ for which $P(B) = 0$ is a $\sigma-$algebra - we may enlarge $\mathcal{F}$ if necessary

J.16.2 Spectral Measure

we derived from the projection property $E(B)^2 = E(B)$ that $E(B_1)E(B_2) = E(B_1 \bigcap B_2)$, $E(\emptyset) = 0$ and $E(B_1) \leq E(B_2)$ for $B_1 \subset B_2$ where for two projections $P_1 \leq P_2$ means that $P_1\mathcal{H} \subset P_2\mathcal{H}$.

Given a p.v.m P and a unit vector $\Omega \in \mathcal{H}$ we define the spectral measure

$$
\mu_\Omega(B) := < \Omega, P(B)\Omega >_\mathcal{H} := \int_B d\mu_\Omega(x) \qquad \text{(J.-245)}
$$

Theorem J.16.1 *Let a be a self-adjoint operator on a Hilbert space. Then there exists a* p.v.m. *E on the measure space $(\mathbb{R}, \mathcal{B}_{Borel})$ such that*

$$
a = \int_\mathbb{R} x dE(x) \qquad \text{(J.-245)}
$$

In order to construct E from a we notice that for each measurable, bounded set B the function χ_B has support in a closed, finite interval containing B, therefore it can be approximated pointwise by polynomials due to the Weierstrass theorem

$$
\chi_B(x) \approx \sum x^n
$$

On the other hand, the function x can be approximated arbitrarily well by simple functions of the form $f(x) = \sum_k x_k \chi_{B_k}(x)$ where $\cup_n B_k = \mathbb{R}$ is a collection of disjoint intervals and $x_k \in B_k$.

$$
\int x d\mu_{\Omega_1, a^{n-1}\Omega_2} = x = \lim \sum_k x_k \chi_{B_k}(x)
$$

In order to construct E from a we notice that for each measurable, bounded set B the function χ_B has support in a closed, finite interval containing B, therefore it can be approximated pointwise by polynomial due to Weierstrass theorem. On the other hand, the function x can be approximated arbitrarily well by simple functions of the form $f(x) = \sum_k \chi_{B_k}(x)$ where $\cup_n B_k = \mathbb{R}$ is a collection of disjoint intervals and $x_k \in B_k$. Therefore for Ω_2 in the domain of a^n

$$
\begin{aligned}
<\Omega_1, a^n\Omega_2> &= \int x d\mu_{\Omega_1, a^{n-1}\Omega_2}(x) \\
&= \lim \sum_{k_1} x_{k_1} \; <\Omega_1, E(B_{k_1}) a^{n-1}\Omega_2> \\
&= \lim \sum_{k_1} x_{k_1} \; <E(B_{k_1})\Omega_1, a^{n-1}\Omega_2> \\
&= \lim \sum_{k_1} x_{k_1} \int x \; d\mu_{E(B_{k_1})\Omega_1, a^{n-2}\Omega_2}(x) \\
&= \lim \sum_{k_1,k_1} x_{k_1} x_{k_2} \; <E(B_{k_2})E(B_{k_1})\Omega_1, a^{n-2}\Omega_2> \\
&= \cdots \\
&= \lim \sum_{k_1 \ldots k_n} x_{k_1} \ldots x_{k_n} \; <E(B_{k_1} \cap \cdots \cap B_{k_n})\Omega_1, \Omega_2> \\
&= \lim \sum_{k} x_k^n <\Omega_1, E(B_{k_1})\Omega_2> \\
&= \int x^n d\mu_{\Omega_1,\Omega_2}(x)
\end{aligned}
\tag{J.-252}
$$

we conclude that for every measurable set B

$$
<\Omega_1, \chi_B(a)\Omega_2> = \int \chi_B(x) \; d\mu_{\Omega_1,\Omega_2}(x) = <\Omega_1, E(B)\Omega_2> \tag{J.-252}
$$

for all Ω_1, Ω_2 since $E(B)$ is a bounded operator. Thus

$$
E(B) = \chi_B(a) \tag{J.-252}
$$

are the spectral projections associated with a self-adjoint operator. If we know the representation of a on H then we have to aproximate $\chi_B(a)$ by polynomials and then can construct the $E(B)$. In particular we conclude that for every measurable function and Ω_2 in the domain of $f(a)$

$$
<\Omega_1, f(a)\Omega_2> = \int f(x) \; d\mu_{\Omega_1,\Omega_2}(x) \tag{J.-252}
$$

since $f(E) := f(a)$ if $f(E) = \sum_n z_n E(B_n)$. Formula (J.16.2) is sometimes referred to as the functional calculus. Combining (??) ans (??) we have

$$\chi_{(-\infty,\lambda]}(a) = \theta(\lambda - a) = E((-\infty,\lambda]) = \int_{-\infty}^{\lambda} dE(x) = E(\lambda) - E(\infty) = E(\lambda) \qquad \text{(J.-252)}$$

where the inegration constant $E(-\infty) = 0$ follows from the fact that $E((-\infty,-\infty)) = E(\emptyset) = 0$ by defintion.

We remark that the spcetral theorem holds without making any separability assumptions, that is, it holds also when $\mathcal{H}$ does not have a countable basis.

J.17 Measure Theory on Compact Spaces

There is a one-to-one correspondence between Baire measures and regular Borel measures.

Theorem J.17.1 *(the Riesz-Markov theorem) Let X be a compact Hausdorff space. For any positive linear functional ℓ on $C(X)$ there is a unique Baire measure μ on X with*

$$\ell(f) = \int f d\mu.$$

Proof:

Since μ is inner regular, that is,

$$\mu(Y) = \sup\{\mu(C) : C \subset Y, C \text{ compact}\},$$

we need only find $\mu(C)$ for compact C to recover μ.

It is claimed that

$$\mu(C) = \inf\{\ell_\mu(f) : f \in C(X), f \geq \chi_C\}.$$

Since μ is positive, it is clear that

$$\mu(C) \leq \ell_\mu(f) \qquad \text{with} \qquad f \geq \chi_C;$$

thus we need only show that, given ϵ, we can find

$$f \in C(X) \quad \text{with} \quad \chi_C \leq f \quad \text{and} \quad \ell_\mu(f) \leq \mu(C) + \epsilon.$$

Since μ is outer regular, given ϵ we can find O open with $\mu(O/C) < \epsilon$ and $C \subset O$. By Urysohn's lemma, we can find $f \in C(X)$ with $0 \leq f \leq 1$, $f(x) = 1$ if $x \in C$ and $f(x) = 0$ if $x \in X/O$. Thus

$$\ell(f) \leq \mu(O) < \mu(C) + \epsilon.$$

□

J.17.1 Spectral Theorem for Bounded Self-Adjoint Operators

Definition A vector $\psi \in \mathcal{L}(\mathcal{H})$ is called a **cyclic vector** for A if finite linear combinations of the elemnets $\{A^n\psi\}_{n=1}^{\infty}$ are dense in $\mathcal{H}$.

Theorem J.17.2 *(spectrum theorem - multiplication operator form) Let A be a bounded self-adjoint operator on $\mathcal{H}$, a separable Hilbert space. Then, there exist measures $\{\mu_n\}_{n=1}^{N}$ $(N = 1, 2, \ldots)$ or $\infty)$ on $\sigma(A)$ and a unitary operator*

$$U : \mathcal{H} \to \oplus_{n=1}^{N} L_2(\mathbb{R}, d\mu_n) \tag{J.-252}$$

so that

$$(UAU^{-1}\psi)_n(\lambda) = \lambda\psi_n(\lambda) \tag{J.-252}$$

Proof:

□

J.18 Extensions of Unbounded Operartors

Can they be defined for unbounded operators?

it would be useful if some of the results that have been proved for bounded operators would hold for unbounded ones. many do apparently

J.18.1 Semi-bounded Operators

A symmetric transformation S is said to be *semi-bounded* if there exists a real quantity c such that

$$(S\psi, \psi) \geq c(\psi, \psi) \qquad \text{(J.-252)}$$

for all ψ in $\mathcal{D}[S]$; it is said to be *upper semi-bounded* if the opposite inequality is valid. If in particular,

$$(S\psi, \psi) \geq 0, \qquad \text{(J.-252)}$$

we shall say, following the definition set down for bounded transformations, that S is *positive*.

For a positive self-adjoint transformation A, the spectral decomposition can be deduced from that for a bounded self-adjoint transformation. This is done with the aid of a linear transformation of the semi-axis $\lambda \geq 0$ into a finite segment of the μ-axis, for example, the transformation

$$\mu = \frac{\lambda - 1}{\lambda + 1}, \qquad \text{(J.-252)}$$

which carries the semi-axis $\lambda \geq 0$ into the segment $-1 \leq \lambda < 1$.

$$B = (A - I)(A + I)^{-1} \qquad \text{(J.-252)}$$

Since C is the inverse of a self-adjoint transformation, it is also self-adjoint, and since it is bounded in its domain

$$I - 2C = (A + I)C2C = (A - I)C = B \qquad \text{(J.-252)}$$

is also self-adjoint and bounded, and since $0 \leq C \leq I$, we have $||B| \leq 1$.

Let

$$B \int_{-1-0}^{1} \mu dF_{\mu} \qquad \text{(J.-252)}$$

be the spectral decomposition of B. Since the transformation $I - B = 2C$ possesses an inverse (namely $(A + I)/2$, the value 1 is not a characteristic value of B, hence F_{μ} is a continuous function of $\mu = 1$, that is, $F_{1-0} = F_1 = I$. Consequently we have

$$A = (I+B)(I-B)^{-1} = \int_{-1-0}^{1} \frac{1+\mu}{1-\mu} dF_{\mu} = \in_{-0}^{\infty} \lambda dE_{\lambda}, \tag{J.-252}$$

where

$$E_{\lambda} = F_{\mu} \text{ for } \frac{1-\lambda}{1+\lambda}; \tag{J.-252}$$

J.18.2 Closure

Definition An operator is **closed** whenever a sequence of vectors φ_n in the domain of A converges to a limit vector and a sequence of vectors $A\varphi_n$ converges to a limit vector ϕ, then φ is in the domain of A and $\lim_{n\to\infty} A\varphi_n = \phi$.

□

Note that this property is different from the following property of bounded linear operator. If a linear operator A is bounded and thus continuous, and if $\{x_n\}$ is a sequence in $D(A)$ which converges in $D(A)$, then $\{Ax_n\}$ also converges. This need not hold for a closed operator.

Consider an infinite linear combination $\varphi = \sum_{k=1}^{\infty} x_k \phi_k$ where all the vectors are in the domain of A. Then each partial sum $\varphi_n = \sum_{k=1}^{n} x_k \phi_k$ is in the domain of A and

$$A\varphi_n = \sum_{k=1}^{n} x_k A\phi_k \tag{J.-252}$$

The sequence of vectors φ_n converges to a limit vector φ. Suppose the sequence of vectors $A\varphi_n$ converges to a limit vector

$$\phi = \sum_{k=1}^{n} x_k A\phi_k. \tag{J.-252}$$

If A is closed, then $\sum_{k=1}^{n} x_k \phi_k$ is in the domain of A and

$$A\sum_{k=1}^{\infty} x_k \phi_k = \sum_{k=1}^{\infty} x_k A\phi_k \tag{J.-252}$$

Theorem J.18.1 *Let A be an operator with dense domain. It's adjoint $A^{\dagger}$ is closed. (Which has the immediate corrollary that every self-adjoint operator is closed).*

Proof: Suppose a sequence of vectors ψ_n in the domain of $A^\dagger$ converges to a limit vector ψ, (i.e. $lim_{n\to\infty}\psi_n \to \psi$), and the sequence of vectors $A^\dagger\psi_n$ converges to a limit vector χ, (i.e. $lim_{n\to\infty}A^\dagger\psi_n \to \chi$). Because of the continuity of the inner product we obviously have

$$(\phi, A^\dagger\psi_n) \to (\phi, \chi) \tag{J.-252}$$

and

$$(A\phi, \psi_n) \to (A\phi, \psi). \tag{J.-252}$$

where ϕ is taken to be in the domain of A so that $A\phi$ is well defined.

Now, because

$$(\phi, A^\dagger\varphi_n) = (A\phi, \psi_n), \tag{J.-252}$$

equations (J.18.2) and (J.18.2) are equal, so that,

$$(\phi, \chi) = (A\phi, \psi). \tag{J.-252}$$

It follows that ψ is in the domain of $A^\dagger$ and $A^\dagger\psi = \chi$. This completes the proof.

$\square$

Definition We say the operator A is an **extension** of the operator B if $D(B) \subset D(A)$ and $A\psi = B\psi$ for $\psi \in D(B)$.

$\square$

Definition Let A and B be operators on $\mathcal{H}$. If $\Gamma(A) \supset \Gamma(B)$, then A is said to be an **extension** of B and we write $A \supset B$. This is equivalent to the earlier definition, $A \supset B$ if and only if $D(A) \supset D(B)$ and $A\varphi = B\varphi$ for all $\varphi \in D(B)$.

$\square$

Definition T is said to be **closable** if it has a closed extension to $(\overline{T})$, that is, $D(T) \subset D(\overline{T})$ and $\overline{T} = T$ in $D(T)$. The smallest closed extension is called the **closure** $\overline{T}$.

If A has a closed extension (i.e., an extension of its domain such that the associated graph is closed and such that the extended operator coincides with the original one on the original domain) it is called closable and the smallest such extension of A is called the closure $\overline{A}$ of A.

□

Definition Given a not neccessarily densely defined operator A with domain $D(A)$ consider the set

$$\Gamma(A) := \{[\psi, A\psi] : \psi \in D(A)\} \subset \mathcal{H} \oplus \mathcal{H},$$

called the **graph** of A.

$$< [\psi_1, \psi_2], [\psi_1', \psi_2'] > := < \psi_1, \psi_1' > + < \psi_2, \psi_2' > \qquad \text{(J.-252)}$$

The operator is closed if its graph is closed in the metric induced by the inner product. The relation to the first definition of a closed operator comes from the norm induced from the inner product (J.18.2),

$$\|\varphi - \varphi_n, \phi - A\varphi_n\|^2 \quad = \quad \|\varphi - \varphi_n\|^2 + \|\phi - A\varphi_n\|^2$$

We find $\delta > 0$ such that $\|\varphi - \varphi_n, \phi - A\varphi_n\|^2 \geq \delta$ if either or both do not converge. The graph $\Gamma(A)$ is closed if and only if

$$\lim_{n\to\infty} \|\varphi - \varphi_n, \phi - A\varphi_n\|^2 = 0$$

for φ in the domain $D(A)$ and at the same time, $A\varphi_n$ converges to the vector ϕ. The original awkard definition of a closed operator is replaced by the simple concept of a closed subspace.

Let us repeat the proof of theorem (J.18.1) in terms of graphs.

Theorem J.18.2 *If a is a densely defined operator on $\mathcal{H}$, then $a^\dagger$ is closed.*

Proof: Define $U : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ by

$$U[\phi, \psi] = [-\psi, \phi].$$

U is unitary, with $U^{-1} = U^\dagger = -U$. Now, let $[\phi, \psi] \in \Gamma(a)^\perp$, the orthogonal complement of the graph $\Gamma(a)$. Then for each $\chi \in D(a)$,

$$< \phi, \chi > + < \phi, a\chi > = \left\langle \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \begin{pmatrix} \chi \\ a\chi \end{pmatrix} \right\rangle = 0.$$

Hence,

$$< \phi, \chi > = - < \phi, a\chi >$$

for each $\chi \in D(a)$, so, by definition, $\psi \in D(a^\dagger)$ and $a^\dagger \psi = -\phi$. Hence,

$$[-a^\dagger \psi, \psi] \in \Gamma(a)^\perp.$$

But

$$U\Gamma(a^\dagger) = \{[-a^\dagger \psi, \psi] : \psi \in D(a^\dagger)\}$$

so

$$U\Gamma(a^\dagger) = \Gamma(a)^\perp. \tag{J.-253}$$

In a Hilbert space, the orthogonal complement of any subspace is closed. In particular, $\Gamma(a)^\perp$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. Unitary operators preserve closedness of subspaces, and

$$\Gamma(a^\dagger) = U^\dagger U \Gamma(a^\dagger) = U^\dagger[\Gamma(a)^\perp],$$

so $\Gamma(a^\dagger)$ is closed. Therefore $a^\dagger$ is a closed linear operator.

□

Theorem J.18.3 *If an operator T is closable, we can form its closure $\overline{T}$ via the following prescription:*

The domain $D(\overline{T})$ is taken to be the set of all $x \in \mathcal{H}$ for which there is a sequence $\{x_n\}$ in $D(T)$ and a $y \in \mathcal{H}$ such that

$$x_n \to x \qquad \textit{and} \qquad Tx_n \to y. \tag{J.-253}$$

It is not difficult to see that $D(\overline{T})$ is a vector space. Clearly, $D(T) \subset D(\overline{T})$. On $D(\overline{T})$ we define $\overline{T}$ by setting

$$y = \overline{T}x \tag{J.-253}$$

with y given by (J.18.3).

Proof: For $\overline{T}$ to be well defined there must correspond a unique y for every $x \in D(\overline{T})$. So in addition to $\{x_n\}$ let $\{\tilde{x}_n\}$ be another sequence in $D(T)$ such that

$$\tilde{x}_n \to x \qquad \text{and} \qquad T\tilde{x}_n \to \tilde{y},$$

we must have

$$T(x_n - \tilde{x}_n) \to y - \tilde{y} = 0.$$

Proof that, if it exists, $\overline{T}$ is a closed linear operator. Since T is linear, so is $\overline{T}$ by (J.18.3) and (J.18.3). Now consider any sequence $\{w_m\}$ in $D(\overline{T})$ such that

$$w_m \to x \qquad \text{and} \qquad \overline{T}w_m \to y, \tag{J.-253}$$

we will prove that $x \in D(\overline{T})$ and $\overline{T}x = y$.

Every w_m (for fixed m) is in $D(\overline{T})$. By definition of $D(\overline{T})$ there is a sequence in $D(T)$ which converges to w_m and whose image under T converges to $\overline{T}w_m$. Hence for every fixed m there is a $v_m \in D(T)$ such that

$$\|w_m - v_m\| < \frac{1}{m} \qquad \text{and} \qquad \|\overline{T}w_m - \overline{T}v_m\| < \frac{1}{m}.$$

From this and (J.18.2) we conclude that

$$v_m \to x \qquad \text{and} \qquad \overline{T}v_m \to y.$$

This shows that $x \in D(\overline{T})$ and $y = \overline{T}v_m$. Recall that an operator is closed if and only if

$$x_n \to x \quad [x_n \in D(T)] \qquad \text{and} \qquad Tx_n \to y.$$

together imply that $x \in D(T)$ and $Tx = y$. From the defintion, we see that every point of $D(\overline{T})$ must also belong to the domain of every closed linear extension of T. This shows that $\overline{T}$is the closure of T, moreover, it implies that the closure is unique.

□

The concept of a closable operator is helpful to view from the graph perspective.

Since a closed linear operator is determined by its closed graph, each closable linear operator a has a minimal closed extension $\overline{a}$: its graph $\Gamma(\overline{a})$ is in the intersection of the graphs of all closed extensions of a.

Proposition J.18.4 *If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

Proof: Suppose that S is a closed extension of T. Then $\overline{\Gamma(T)} \subseteq \Gamma(S)$ so if

Define an operator R with domain

$$D(R) = \{\psi : [\psi, \phi] \in \overline{\Gamma(T)}\}$$

by

$$R\psi = \phi$$

where $\phi \in \mathcal{H}$ is the unique vector so that $[\psi, \phi] \in \overline{\Gamma(T)}$.

□

The simplest way of extending an operator A is to take the closure of its graph $\Gamma(A) = \{[\psi, A\psi] : \psi \in D(A)\} \subset \mathcal{H} \oplus \mathcal{H}$. That is, if

$$[\psi_n, A\psi_n] \to [\psi, \phi]$$

we might try to define

$$A\psi = \phi.$$

For $A\psi$ to be well-defined, we need if

$$\psi_n \to \psi \qquad \text{and} \qquad \tilde{\psi}_n \to \psi$$

then we should have $\lim_{n\to\infty} A\psi_n = \lim_{n\to\infty} A\tilde{\psi}_n$, that is,

$$[\psi_n - \tilde{\psi}_n, A(\psi_n - \tilde{\psi}_n)] \to [0, 0].$$

Stateted otherwise, for $A\psi$ to be well-defined, we need that

$$[\psi_n, A\psi_n] \to [0, \phi] \qquad \text{implies} \qquad \phi = 0.$$

In this case A is called closable and the unique operator $\overline{A}$ which satisfies $\Gamma(\overline{A}) = \overline{\Gamma(A)}$ is called the closure of A. Clearly, A is called closed if $\overline{A} = A$, which is the case if and only if the graph of A is closed.

Give a first easily proven simple criterion fo an operator to be closable.

Theorem J.18.5 *If a is a densely defined linear operator on a Hilbert space $\mathcal{H}$, then a is closable if and only if $D(a^\dagger)$ is dense in $\mathcal{H}$. In such a case, $\overline{a} = a^{\dagger\dagger}$*

Proof: In a Hilbert space the closure of a subspace is equal to the orthogonal complement of its orthogonal complement:

$$\overline{M} = M^{\perp\perp}.$$

Since $\Gamma(a)$ is a subspace of $\mathcal{H} \oplus \mathcal{H}$,

$$\begin{aligned}
\overline{\Gamma(a)} &= \Gamma(a)^{\perp\perp} \\
&= (U^2\Gamma(a)^\perp)^\perp \\
&= (U(U\Gamma(a))^\perp)^\perp \quad (U[M^\perp] = (U[M])^\perp \text{ for any subspace } M) \\
&= (U\Gamma(a^\dagger))^\perp
\end{aligned}$$

Now, by the proof of (J.18.2) in theroem J.18.2, repeated here for the densely defined operator $a^\dagger$, we have $U\Gamma(a^{\dagger\dagger}) = \Gamma(a^\dagger)^\perp$. Equivalently, $\Gamma(a^{\dagger\dagger}) = U(\Gamma(a^\dagger)^\perp) = (U\Gamma(a^\dagger))^\perp$. Thus, if $a^\dagger$ is densely defined, then $\overline{\Gamma(a)}$ is the graph of $a^{\dagger\dagger}$. a is closable with closure $\overline{a} = a^{\dagger\dagger}$.

Now suppose that $D(a^\dagger)$ is not dense in $\mathcal{H}$, then there is a non-zero $\psi \in D(a^\dagger)^\perp$ (see corollary J.4.9). Let $\phi \in D(a^\dagger)$,and note that

$$\left\langle \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \begin{pmatrix} \phi \\ a^\dagger\phi \end{pmatrix} \right\rangle = <\psi, \phi> = 0,$$

so $[\psi, 0] \in \Gamma(a^\dagger)^\perp$. Thus

$$\begin{pmatrix} 0 \\ \psi \end{pmatrix} = U \begin{pmatrix} \psi \\ 0 \end{pmatrix} \in U[\Gamma(a^\dagger)^\perp].$$

is not the graph of a linear operator and so a is not closable.

□

Proposition J.18.6 *If a is closable, then $(\overline{a})^\dagger = a^\dagger$.*

Proof: Notice that if a is closable,

$$a^\dagger = \overline{(a^\dagger)} = a^{\dagger\dagger\dagger} = (a^{\dagger\dagger})^\dagger = (\overline{a})^\dagger.$$

□

Extending a symmetric operator

Every symmetric operator is closable.

Theorem J.18.7 *Let $T : D(T) \to \mathcal{H}$ be a linear operator, where $D(T)$ is dense in $\mathcal{H}$. Then if T is symmetric, its closure $\overline{T}$ exists and is unique. $\overline{T}$ is a symmetric linear extension of T.*

Proof. The domain $D(\overline{T})$ is taken to be the set of all $x \in \mathcal{H}$ for which there is a sequence $\{x_n\}$ in $D(T)$ and a $y \in \mathcal{H}$ such that $x_n \to x$ and $Tx_n \to y$. We define $\overline{T}$ by setting $y = \overline{T}x$. We must prove there corresponds a unique y for every $x \in D(\overline{T})$. In addition to $\{x_n\}$ let $\{\tilde{x}_n\}$ be another sequence in $D(T)$ such that

$$\tilde{x}_n \to x \qquad \text{and} \qquad T\tilde{x}_n \to \tilde{y}.$$

Since T is symmetric, we have for every $v \in D(T)$

$$(v, Tx_n - T\tilde{x}_n) = (Tv, x_n - \tilde{x}_n).$$

We let $n \to \infty$. By continuity of the inner product, we have

$$(v, y - \tilde{y}) = (Tv, x - x) = 0,$$

that is,

$$y - \tilde{y} \perp D(T).$$

Since $D(T)$ is dense in $\mathcal{H}$, we have $D(T)^\perp = \{0\}$ by lemma ??, and $y - \tilde{y} = 0$.

Proof that $\overline{T}$ is a symmetric linear extension of T. Since T is linear, so is $\overline{T}$ by (J.18.3) and (J.18.3). We show that the symmetry of $\overline{T}$ follows from the symmetry of T. By (J.18.3) and (J.18.3), for all $x, z \in D(\overline{T})$ there are sequences $\{x_n\}$ and $\{z_n\}$ in $D(T)$ such that

$$\begin{aligned} x_n &\to x, \qquad & Tx_n &\to y \\ z_n &\to z, \qquad & Tz_n &\to \overline{T}z. \end{aligned}$$

Since T is symmetric,

$$(z_n, Tx_n) = (Tz_n, x_n).$$

Letting $n \to \infty$, we obtain

$$(z, \overline{T}x) = (\overline{T}z, x)$$

by the continuity of the inner product. Since $x, z \in D(\overline{T})$ were arbitrary, this shows that $\overline{T}$ is symmetric.

□

The extension of a symmetric operator T can fail to be self-adjoint only if the domain of $A^\dagger$ is larger than that of T.

Definition If a symmetric operator cannot be extended to a symmetric operator on a larger domain, we call it a *maximal symmetric* operator.

□

Theorem J.18.8 *Every self-adjoint operator is a maximal symmetric operator.*

Proof. Let a be a self-adjoint operator and let b be a symmetric operator that is an extension of a. We proof the theorem by showing that b must be equal to a.

□

If A is a graph, then its null space Ker(A) consists of all $\psi \in \mathcal{H}$ such that $(\psi, 0)$ is in A.

If A is self-adjoint, then there is only one self-adjoint extension (if B is another one, we have $\overline{A} \subseteq B$ and hence $\overline{A} = B$ by theorem J.18.8. In this case A is called **essentially self-adjoint** and $D(A)$ is called a **core** for A. Otherwise there might be more than one self-adjoint extension or none at all.

Theorem J.18.9 *Let a be a symmetric operator on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

1) a is self-adjoint;

2) a is closed and Ker $(a^\dagger \pm i1_{\mathcal{H}}) = \{0\}$;

3) $[a \pm i1_{\mathcal{H}}]D(a) = \mathcal{H}$.

Proof. First we prove that 1) implies 2). Suppose that a is a self-adjoint operator and there is a $\varphi \in D(a^\dagger) = D(a)$ so that $a^\dagger\varphi = i\varphi$. Then $a\varphi = i\varphi$ and

$$-i(\varphi, \varphi) = (i\varphi, \varphi) = (a\varphi, \varphi) = (\varphi, a^\dagger\varphi) = i(\varphi, \varphi)$$

so $\varphi = 0$, that is, Ker $(a^\dagger - i1_{\mathcal{H}}) = \{0\}$. A similar proof shows that Ker $(a^\dagger + i1_{\mathcal{H}}) = \{0\}$.

We now prove how 2) implies 3). If $\psi \in$ Ran $(a - i)^\perp$

$$((a - i)\varphi, \psi) = 0$$

for all $\varphi \in D(a)$. Implying

$$(\varphi, (a^\dagger + i)\psi) = 0.$$

Recall the domain of $a^\dagger$ is the set of all vectors ψ for which there is a vector $a^\dagger\psi$ such that $(\varphi, a^\dagger\psi) = (a\varphi, \psi)$ for every vector φ in the domain of a. So $\psi \in D(a^\dagger)$ as $(a\varphi, \psi) = -i(\varphi, \psi) = (\varphi, a^\dagger\psi)$ for all $\varphi \in D(a)$. And $(a - i)^\dagger\psi = (a^\dagger + i)\psi = 0$ which is impossible since $a^\dagger\psi = -i\psi$ has no solutions.

Since Ran $(a-i)$ is dense, we need only prove it is closed to conclude that Ran $(a-i) = \mathcal{H}$. But for all $\varphi \in D(a)$

$$\begin{aligned}\|(a-i)\varphi\|^2 &= ((a-i)\varphi,(a-i)\varphi)\\ &= \|a\varphi\|^2 - i(a\varphi,\varphi) + i(\varphi,a\varphi) + \|\varphi\|^2\\ &= \|a\varphi\|^2 + \|\varphi\|^2.\end{aligned}$$

Thus if $\varphi_n \in D(a)$ and $(a-i)\varphi_n \to \psi_0$,

$$\infty > \|\psi_0\|^2 = \lim_{n\to\infty} \|a\varphi_n\|^2 + \lim_{n\to\infty} \|\varphi_n\|^2$$

we conclude that φ_n convergences to some vector φ_0, and $a\varphi_n$ converges too. Since a is closed, $\varphi_0 \in D(a)$ and $(a-i)\varphi_0 = \psi_0$. Thus $\mathrm{Ran}(a-i)$ is closed, so $\mathrm{Ran}(a-i) = \mathcal{H}$. Similarly, $\mathrm{Ran}(a+i) = \mathcal{H}$.

Finally, we will show that 3) implies 1). Let $\varphi \in D(a^\dagger)$ then $(a^\dagger - i)\varphi \in \mathcal{H}$. Since $\mathrm{Ran}(a-i) = \mathcal{H}$, there is $\eta \in D(a)$ so that $(a-i)\eta = (a^\dagger - i)\varphi$. As $D(a) \subset D(a^\dagger)$, $\varphi - \eta \in D(a^\dagger)$ (because $\|a^\dagger(\varphi-\eta)\| \le \|a^\dagger\varphi\|\ \|a^\dagger\eta\| = \|a^\dagger\varphi\|\ \|a\eta\|$) and

$$(a^\dagger - i)(\varphi - \eta) = (a - a^\dagger)\eta = 0.$$

Since $\mathrm{Ran}(a+i) = \mathcal{H}$, by lemma J.7.27, $\mathrm{Ker}(a^\dagger - i) = \{0\}$, so $\varphi = \eta \in D(a)$. This proves that $D(a^\dagger) = D(a)$, so that a is self-adjoint.

□

Computing $a^\dagger$ might be difficult, criterion 3) for doess not involve $a^\dagger$ and could be useful.

Lemma J.18.10 *If a is symmetric then*

$$Ran(a + i1_{\mathcal{H}})^\perp = Ker(a^\dagger - i1_{\mathcal{H}}), \qquad Ran(a - i1_{\mathcal{H}})^\perp = Ker(a^\dagger + i1_{\mathcal{H}})$$

Proof. In the proof of the previous theorem we established that if $\mathrm{Ker}(a^\dagger \pm i1_{\mathcal{H}}) = \{0\}$ then $\mathrm{Ran}(a \pm i1_{\mathcal{H}})$ is dense in $\mathcal{H}$.

□

Theorem J.18.11 *Let a be a symmetric operator on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

1) a is essentially self-adjoint

2) $Ker(a^\dagger \pm i1_{\mathcal{H}}) = \{0\}$;

3) $\overline{[a \pm i1_{\mathcal{H}}]D(a)} = \mathcal{H}$.

Proof. The equivalence of 2) and 3) follow from the lemma: 3) implies 2),

$$\begin{aligned}
\overline{\mathrm{Ran}(a+i1_{\mathcal{H}})} &= \mathrm{Ran}(a+i1_{\mathcal{H}})^{\perp\perp} \\
&= \mathrm{Ker}(a^{\dagger}-i1_{\mathcal{H}})^{\perp} \\
&= \{0\}^{\perp} = \mathcal{H}.
\end{aligned}$$

2) implies 3),

$$\begin{aligned}
\mathrm{Ker}(a^{\dagger}-i1_{\mathcal{H}}) &= \mathrm{Ran}(a+i1_{\mathcal{H}})^{\perp} \\
&= (\mathrm{Ran}(a+i1_{\mathcal{H}})^{\perp})^{\perp\perp} \\
&= (\mathrm{Ran}(a+i1_{\mathcal{H}})^{\perp\perp})^{\perp} \\
&= \mathcal{H}^{\perp} = \{0\}.
\end{aligned}$$

1) implies 2). If a is essentially self-adjoint then $\overline{a} = a^{\dagger}$ is self-adjoint.

Suppose that a is an essentially self-adjoint operator and there is a $\varphi \in D(a^{\dagger}) = D(\overline{a})$ so that $a^{\dagger}\varphi = i\varphi$. Then $\overline{a}\varphi = i\varphi$ and

$$-i(\varphi,\varphi) = (i\varphi,\varphi) = (\overline{a}\varphi,\varphi) = (\varphi, a^{\dagger}\varphi) = i(\varphi,\varphi)$$

so $\varphi = 0$, that is, Ker $(a^{\dagger} - i1_{\mathcal{H}}) = \{0\}$.

Hence $\mathrm{Ker}(a^{\dagger} \pm i1_{\mathcal{H}}) = \{0\}$.

A similar proof shows that Ker $(a^{\dagger} + i1_{\mathcal{H}}) = \{0\}$.

We prove that 2) implies 1).

....

$$\begin{aligned}
\|(a-i)\psi\|^2 &= \|a\psi\|^2 + \|\psi\|^2 \\
&\leq \|\psi\|^2
\end{aligned}$$

therefore $\mathrm{Ker}(a-i) = \{0\}$ and hence $(a-i)^{-1}$ exists. Moreover, setting $\psi = (a-i)^{-1}\varphi$ shows $\|(a-i)^{-1}\psi\| \leq |i|$. Hence $(a-i)^{-1}$ is bounded and closed. Since it is densely defined by assumption, its domain of dependence $\mathrm{Ran}(a+i)$ must be equal to $\mathcal{H}$. Replacing $(a-i)$ by $(a+i)$, we then apply theorem J.18.9.

□

J.18.3 Properties of Symmetric Operators

Theorem J.18.12 *Suppose an operator T has an inverse T^{-1} and that $D(T)$ and $D(T^{-1})$ are dense in $\mathcal{H}$, so that $T^\dagger$ and $(T^{-1})^\dagger$ exist. Then $T^\dagger$ also has an inverse, and*

$$(T^\dagger)^{-1} = (T^{-1})^\dagger \tag{J.-271}$$

Proof. We assume first that f transverses $D(T)$, and g transverses $D((T^{-1})^\dagger)$; then

$$(f, g) = (T^{-1}Tf, g) = (Tf, (T^{-1})^\dagger g).$$

This shows that

$$(T^{-1})^\dagger g \in D(T^\dagger),$$

and

$$T^\dagger (T^{-1})^\dagger g = g. \tag{J.-271}$$

On the other hand, if f transverses $D(T^{-1})$, and h transverses $D(T^\dagger)$, then

$$(f, h) = (TT^{-1}f, h) = (T^{-1}f, T^\dagger h).$$

$$(T^{-1})^\dagger T^\dagger h = h. \tag{J.-271}$$

The relations (J.18.3) and (J.18.3) show that

□

Theorem J.18.13 *If a self-adjoint operator A has an inverse A^{-1}, then A^{-1} is a self-adjoint operator.*

Proof. By () Ker(A) of an operator is

$$\text{Ker}(A) = \mathcal{H} \ominus \overline{\text{Ran}(A)}$$

but as the inverse exists, Ker= 0. Therefore $\overline{\mathrm{Ran}(A)} = \mathcal{H}$, i.e.,

$$\overline{D(A^{-1})} = \mathcal{H}.$$

The result to be proved then follows from theorem J.18.12.

□

J.18.4 Deficiency Indices and Self-adjoint Closed Extensions

For symmetric operators, there are always closed extensions. A smallest closed extension always exists (the double adjoint), but it is possible that none of these closed extensions is self-adjoint.

Definition The positive and negative deficiency indices of an operator a are defined respectively as

$$\begin{aligned} n_+(a) &= \dim(\mathrm{Ker}([a^\dagger + i1_{\mathcal{H}}])) \\ n_-(a) &= \dim(\mathrm{Ker}([a^\dagger - i1_{\mathcal{H}}])) \end{aligned}$$

□

Note a is essentially self-adjoint if and only if $n_+(a) = n_-(a) = 0$.

In general, symmetric operators may or may not have self-adjoint extension. One can show that this is possible if and only if the deficiency indices are equal to each other.

essentially self-adjoint operators their self-adjoint extension is unique and given by their closure.

Note if an operator a is closable then $\overline{a}^\dagger = a^\dagger$ so

$$n_+(a) = n_+(\overline{a}) \qquad \text{and} \qquad n_-(a) = n_-(\overline{a})$$

Definition A number λ (real or complex) is called a **point of regular type** for an operator T if there is a $k = k(\lambda) > 0$ such that, for all $f \in D(T)$,

$$\|(T - \lambda \mathbb{I})f\| \geq k\, \|f\|.$$

□

Therefore the eigenvalues of T are not points of regular type for T.

If λ is is a point of regular type for T, then the operator $(T - \lambda\mathbb{I})^{-1}$ exists...

If λ_0 is a point of regular type, then, for $|\lambda - \lambda_0| \leq \delta \leq \frac{1}{2}k(\lambda_0)$ and for any $f \in D(T)$,

$$\|(T - \lambda\mathbb{I})f\| \geq \|(T - \lambda\mathbb{I})f\| - |\lambda - \lambda_0| \, \|f\|. \tag{J.-272}$$

This shows that the set of points of regular type is open.

Definition The set of points of regular type is called the **field of regularity** of the operator.

□

If A is a symmetric operator, and if $z = x + iy$ (with $y \neq 0$), then () for any $f \in D(T)$,

$$\|(A - z\mathbb{I})f\| = \|(A - x\mathbb{I})f\| + y^2\|f\| \geq y^2\|f\|,$$

from which we see that each of the upper and the lower halves of the $z-$plane is a connected component of the field of regularity of any symmetric operator.

The field of regularity of any isometric operator V also consists of two connected components, the interior and exterior of the unit circle; because if $|\zeta| < 1$,

$$\|(V - \zeta\mathbb{I})f\| \geq \|Vf\| - |\zeta| \, \|f\| = (1 - |\zeta|)\|f\|,$$

and similarly, if $|\zeta| > 1$,

$$\|(V - \zeta\mathbb{I})f\| \geq |\zeta| - \|Vf\| \, \|f\| = (|\zeta| - 1)\|f\|.$$

Theorem J.18.14 *If Γ is a connected component of the field of regularity of a linear operator T, then the dimension of the subspace $\mathcal{H} \ominus Ran(T - \lambda\mathbb{I})$ is the same for all $\lambda \in \Gamma$.*

Proof: Let P_λ be the operator of orthogonal projection on to the subspace

$$\mathrm{Ker}(T - \lambda\mathbb{I})$$

If we can show that, for any $\lambda_0 \in \Gamma$, there is a $\delta = \delta(\lambda_0) > 0$ such that $|\lambda - \lambda_0| < \delta$ implies

$$\|P_\lambda - P_{\lambda_0}\| < 1, \tag{J.-272}$$

then by theorem J.4.16

$$\dim N_\lambda = \dim N_{\lambda_0}.$$

Any two points λ_1, λ_2 in Γ can be joined by a path which is covered by a finite number of these $\delta-$neighbourhoods. This leads to the result

$$\dim N_{\lambda_1} = \dim N_{\lambda_2}.$$

Let λ_0 be a fixed point of the domain Γ and let

$$\delta = \delta(\lambda_0) \leq \frac{1}{3} k(\lambda_0)$$

As

$$\begin{aligned} k(\lambda_0)\|f\| &= \|(T - \lambda_0)f\| \\ &\leq \|(T - \lambda_0)f\| + |\lambda - \lambda_0| \; \|f\| \end{aligned}$$

if we assume that $|\lambda - \lambda_0| \leq \delta$, then

$$\|(T - \lambda_0)f\| \geq \frac{2}{3} k(\lambda_0)\|f\|.$$

Then for any $h \in N_\lambda$ (so that $(h, (T - \lambda)f) = 0$) with $\|h\| = 1$,

$$\begin{aligned} \|(\mathbb{I} - P_{\lambda_0})h\| &= \sup \frac{|(h, (T - \lambda_0)f)|}{\|(T - \lambda_0 \mathbb{I})f\|} \\ &= \sup_{f \in D(T)} \frac{|(h, (T - \lambda)f + (\lambda - \lambda_0)f)|}{\|(T - \lambda_0 \mathbb{I})f\|} \\ &= \sup_{f \in D(T)} \frac{|\lambda - \lambda_0|(h, f)}{\|(T - \lambda_0 \mathbb{I})f\|} \\ &\leq \frac{1}{2} \end{aligned} \tag{J.-276}$$

and similarly for any $h \in N_{\lambda_0}$ with $\|h\| = 1$,

$$\|(\mathbb{I} - P_\lambda)h\| \leq \frac{1}{2} \tag{J.-276}$$

By the alternative defintion of the aperture (J.4.6), it follows from (J.-276) and (J.18.4) that

$$\|P_\lambda - P_{\lambda_0}\| \leq 1/2.$$

□

The deficiency number of a linear subspace M is

$$\dim(\mathcal{H} \ominus M)$$

The deficiency number of the linear subspace

$$M_\lambda = \mathrm{Ran}(a - \lambda\mathbb{I})$$

for points λ belonging to a given connected component of the field of regularity of the operator a. Note that for any operator in Hilbert space

$$(\mathrm{Ran}\ a)^\perp = \mathrm{Ker}\ a^\dagger,$$

therefore the deficiency number of the manifold is

$$M_\lambda = \mathrm{Ker}(a^\dagger - \overline{\lambda}\mathbb{I}).$$

As a consequence of this and theorem J.18.14, a symmetric operator's deficiency numbers are equivalent to

$$\dim(\mathcal{H} \ominus (A - z\mathbb{I})) = \begin{cases} n_+(A), & (Im(z) > 0), \\ n_-(A), & (Im(z) < 0). \end{cases}$$

We will define deficiency numbers of an isometric operator V as:

$$\dim(\mathcal{H} \ominus (V - \zeta\mathbb{I})) = \begin{cases} n_+(V), & (|\zeta| < 1), \\ n_-(V), & (|\zeta| > 1). \end{cases}$$

Lemma J.18.15 *The deficiency numbers of an isometric operator V can be defined by*

$$m = dim(\mathcal{H} \ominus D(V)), \qquad n = dim(\mathcal{H} \ominus Ran(V))$$

Proof: Only the first equality needs proof. For any $\zeta \neq 0$,

$$\begin{aligned}
\mathrm{Ran}(V - \zeta\mathbb{I}) &= (V - \zeta\mathbb{I})D(V) \\
&= \left(\frac{1}{\zeta}V - \mathbb{I}\right) D(V) \\
&= \left(\frac{1}{\zeta}\mathbb{I} - V^{-1}\right) V D(V) \\
&= \left(V^{-1} - \frac{1}{\zeta}\mathbb{I}\right) D(V^{-1}) \\
&= \mathrm{Ran}\left(V^{-1} - \frac{1}{\zeta}\mathbb{I}\right)
\end{aligned}$$

and therefore, for $|\zeta| > 1$,

$$\begin{aligned}
n_+(V) &= \dim(\mathcal{H} \ominus \mathrm{Ran}(V - \zeta\mathbb{I})) \\
&= \dim(\mathcal{H} \ominus \mathrm{Ran}\left(V^{-1} - \frac{1}{\zeta}\mathbb{I}\right)) \\
&= \dim(\mathcal{H} \ominus \mathrm{Ran}(V^{-1} - 0\mathbb{I}) \\
&= \dim(\mathcal{H} \ominus \mathrm{Ran}(V^{-1})) \\
&= \dim(\mathcal{H} \ominus D(V)).
\end{aligned}$$

□

Cayley transform

We know that every symmetric operator admits closure, so in what follows when discussing symmetric operators we shall assume them to be closed.

We have already introdued the Cayley transform V of a closed, symmetric operator A. The operator V is expressed in terms of the operator A by the formula

$$Vf = (A - z\mathbb{I})(A - \overline{z}\mathbb{I})^{-1} f$$

where the domain $D(V)$ is $\mathrm{Ran}(A - \overline{z}\mathbb{I})$.

In this section we assume $\mathrm{Im}(z) < 0$

Lemma J.18.16 *The deficiency index (m, n) of the operator A is the same as the deficiency index of the operator V.*

Proof: For, by definition,

$$m = \dim(\mathcal{H} \ominus \mathrm{Ran}(A - \overline{z}\mathbb{I})).$$

But $\mathrm{Ran}(A - \overline{z}\mathbb{I}) = D(V)$ and so

$$m = \dim(\mathcal{H} \ominus D(V))$$

Again, by definition,

$$n = \dim(\mathcal{H} \ominus \mathrm{Ran}(A - z\mathbb{I}))$$

and $\mathrm{Ran}(A - z\mathbb{I}) = \mathrm{Ran}(V)$ so

$$n = \dim(\mathcal{H} \ominus \mathrm{Ran}(V)).$$

□

Theorem J.18.17 *If V is an isometric operator, and if $Ran(V - \mathbb{I})$ is dense in $\mathcal{H}$, then A defined by*

$$Ah = (\overline{z}V - z\mathbb{I})(V - \mathbb{I})^{-1}h,$$

where $h \in D(A)$, is symmetric and V is its Cayley transform.

Proof: First we need to prove that since $\mathrm{Ran}(V - \mathbb{I})$ is dense in $\mathcal{H}$, the operator $(V - \mathbb{I})^{-1}$ exists. The operator $V - \mathbb{I}$ has an inverse if and only if

$$(V - \mathbb{I})g = 0$$

implies $g = 0$. Now for any $f \in D(V)$,

$$(Vf - f, g) = (Vf, g) - (f, g) = (Vf, Vg) - (f, g) = 0,$$

i.e., $g \perp \mathrm{Ran}(V - \mathbb{I})$. Since $\mathrm{Ran}(V - \mathbb{I})$ is dense, $g = 0$.

Since the operator $(V - \mathbb{I})^{-1}$ exists, the operator

$$A = (\overline{z}V - z\mathbb{I})(V - \mathbb{I})^{-1}$$

is well defined and its domain, $D(A) = \mathrm{Ran}(V - \mathbb{I})$, is dense in $\mathcal{H}$.

Let f, g be any elemnts of $D(A) = \mathrm{Ran}(V - \mathbb{I})$, then f, g are of the form

$$\begin{aligned} f &= V\phi - \phi \\ g &= V\psi - \psi \end{aligned}$$

for some $\phi, \psi \in D(V - \mathbb{I}) = D(V)$. Then

$$\begin{aligned} Af &= (\overline{z}V - z\mathbb{I})\phi = \overline{z}V\phi - z\phi \\ Ag &= (\overline{z}V - z\mathbb{I})\psi = \overline{z}V\psi - z\psi \end{aligned}$$

and so

$$\begin{aligned} (Af, g) &= (\overline{z}V\phi - z\phi, V\psi - \psi) \\ &= (z + \overline{z})(\phi, \psi) - \overline{z}(V\phi, \psi) - z(\phi, V\psi), \\[1em] (f, Ag) &= (V\phi - \phi, \overline{z}V\psi - z\psi) \\ &= (z + \overline{z})(\phi, \psi) - \overline{z}(V\phi, \psi) - z(\phi, V\psi). \end{aligned}$$

Hence

$$(Af, g) = (f, Ag).$$

It is easily shown that

$$V = \frac{A - z\mathbb{I}}{A - \overline{z}\mathbb{I}}$$

so V is the Cayley transform of A.

□

Theorem J.18.18 *If A_1, A_2 are symmetric operators and if V_1, V_2 are their Cayley transforms, then A_2 is an extension of A_1 if and only if V_2 is an extension of V_1.*

Proof: Suppose A_2 is an extension of A_1. Obviously

$$V_1\varphi = \frac{A_1 - z\mathbb{I}}{A_1 - \overline{z}\mathbb{I}}\varphi = \frac{A_2 - z\mathbb{I}}{A_2 - \overline{z}\mathbb{I}}\varphi = V_2\varphi$$

for all $\varphi \in D(V_1)$. Now $D(A_1) \subset D(A_2)$ implies $\mathrm{Ran}(V_1 - \mathbb{I}) \subset \mathrm{Ran}(V_2 - \mathbb{I})$ which in turn implies $D(V_1 - \mathbb{I}) \subset D(V_2 - \mathbb{I})$ or $D(V_1) \subset D(V_2)$.

Suppose V_2 is an extension of V_1. Obviously

$$A_1 f = (\overline{z}V_1 - z\mathbb{I})(V_1 - \mathbb{I})^{-1} f = (\overline{z}V_2 - z\mathbb{I})(V_2 - \mathbb{I})^{-1} f = A_2 f$$

for all $f \in D(A_1)$. Now $D(V_1) \subset D(V_2)$ implies $D(V_1 - \mathbb{I}) \subset D(V_2 - \mathbb{I})$ which, as $(V_2 - \mathbb{I})$ is an injection, implies $\mathrm{Ran}(V_1 - \mathbb{I}) \subset \mathrm{Ran}(V_2 - \mathbb{I})$ which means $D(A_1) \subset D(A_2)$.

□

This reduces the problem of symmetric extensions of a given operator A to the problem of isometric extensions of its Cayley transform V. This problem is much simplier than the original problem.

Two closed linear subspaces can be the domain and range of an isometric operator if and only if their dimensions are equal.

In the subspaces

$$\mathcal{H} \ominus D(V) \qquad \text{and} \qquad \mathcal{H} \ominus \mathrm{Ran}(V)$$

we choose two subspaces F and G of equal dimension, and construct an arbitrary isometric operator V_1 having F as its domain and G its range. We then define a linaer operator $\tilde{V}$, with

$$D(\tilde{V}) = D(V) \oplus F$$

as its domain and

$$\mathrm{Ran}(\tilde{V}) = \mathrm{Ran}(V) \oplus G$$

as its range, by

$$\tilde{V}f = \begin{cases} Vf & \text{for } f \in D(V) \\ V_1 f & \text{for } f \in F \end{cases}$$

$\tilde{V}$ is evidently an isometric extension of V_1, and by taking all possible choices of F, G, V_1, we obtain all possible isometric extensions $\tilde{V}$of the operator V and each extension once only.

To find some symmetric extension $\tilde{A}$ of an operator A, we have to take the Cayley transform V of A, find an isometric extension $\tilde{V}$ of V by the above procedure, and finally carry out a Cayley transform on $\tilde{V}$ to get $\tilde{A}$.

Recall an operator V from the subspace $D(V) \subset \mathcal{H}$ and mapping $D(V)$ to the subspace $\mathrm{Ran}(V) \subset \mathcal{H}$ is said to be isometric if, for all $f, g \in D(V)$,

$$(Vf, Vg) = (f, g).$$

A unitary operator is a particular case of an isometric operator where $D(V)$ and $\mathrm{Ran}(V)$ coincide. From the above considerations it follows that an operator A is maximal, symmetric (or self-adjoint) operator if and only if its Cayley transform V is a maximal, isometric (or unitary) operator.

Recall that the deficiency number of a not necessarily closed symmetric operator A is equal to the deficiency number of its closure $\overline{A}$.

Theorem J.18.19 *The closure $\overline{A}$ of a symmetric operator A is maximal if and only if one of its deficiency numbers is zero.*

The closure of a symmetric operator is self-adjoint if and only if both its deficiency numbers are zero, that is, a symmetric operator is essentially self-adjoint if and only if both its deficiency numbers are zero

Proof: An isometric operator is obviously maximal if and only if dimension of the deficiency subspace is zero.

□

Theorem J.18.20 *Any symmetric operator A can be extended into a maximal operator.*

If $n_+ \neq n_-$, then none of these extensions is self-adjoint;

if $n_+ = n_- < \infty$, then any maximal extension of A is self-adjoint;

if $n_+ = n_- = \infty$, then some of the maximal extensions of A are self-adjoint, and some are not.

Proof:

□

J.18.5 The Friedrichs Extension

Operators bounded below often occur in problems of mathematical physics. The importance of this class of operators is the remarkable porperty that each such operator has a self-adjoint extension (even if it is not essentially self-adjoint).

The Friedrichs extension is a self-adjoint extension of a non-negative densely defined symmetric operator.

characterising prooperty is it is the largest, self-adjoint extension with respect to the usual partial ordering of positive operators.

An operator T is said to be non-negative if and only if

$$(\xi, T\xi) \geq 0, \tag{J.-295}$$

for all elements ξ in the domain of T, and a symmetric operator if

$$(T\xi, \eta) = (\xi, T\eta) \tag{J.-295}$$

for all elements ξ and η in the domain of T.

Theorem J.18.21 *Every semi-bounded symmetric operator S can be extended to a semi-bounded self-adjoint operator A in such a way that A has the same bound as S.*

Proof: Let S be a symmetric operator such that

$$\|f\|_S^2 := (Sf, f) \geq \|f\|^2$$

for all $f \in D_S$. We introduce on D_S a new scalar product

If S is non-negative, then

$$q_S(\xi, \eta) = (\xi, S\eta) \tag{J.-295}$$

is a sesquilinear for on $D(S)$, i.e.,

$$\begin{aligned}
q_S(\xi + \zeta, \eta) &= q_S(\xi, \eta) + q_S(\zeta, \eta) \\
q_S(\alpha\xi, \eta) &= \alpha\, q_S(\xi, \eta) \\
q_S(\xi, \eta + \delta) &= q_S(\xi, \eta) + q_S(\xi, \delta) \\
q_S(\xi, \alpha\eta) &= \overline{\alpha}\, q_S(\xi, \eta).
\end{aligned} \tag{J.-297}$$

Also

$$q_S(\xi, \xi) = (\xi, S\xi) \;\geq \|\xi\|^2 \tag{J.-297}$$

Thus q_S defines an inner product on the domain of S. Let $\mathcal{H}_0$ be the completion of the domain of A with respect to q_S. $\mathcal{H}_0$ is an abstractly defined space; its elements can be represented as equivalence classes Cauchy sequences of elements of $D(S)$. It is not obvious that all elements in $\mathcal{H}_0$ can be identified with elements of $\mathcal{H}$. However, it turns out that the ideal elements can be identified of the space $\mathcal{H}$:

$$D_S \subseteq \mathcal{H}_0 \subseteq \mathcal{H}.$$

In fact, it follows from the inequality (J.18.5) that if a sequence of elements of D_S is a Cauchy sequence in the new norm, it is also a Cauchy sequence in in the original norm, and that two Cauchy sequences equivalent in the new norm are also equivalent in the original norm, so that they converge in the original metric to a well-determined element of the space $\mathcal{H}$. We can thus asign to each element $\tilde{g}$ ideal or not a definite element g of $\mathcal{H}$. Obviously it will be such that $\tilde{g} = g$ for the elements $\tilde{g} \in D_S$.

We must prove that two different elements of $\mathcal{H}_0$ can not be represented by the same element g of $\mathcal{H}$. Since the equation

$$q_S(f, \tilde{g}) = (Sf, g),$$

where g denotes the element of $\mathcal{H}$ which we have just identified with the element $\tilde{g}$ of $\mathcal{H}_0$, is valid by defintion if f and $\tilde{g}$ both belong to $D(S)$, extends to the case where $\tilde{g}$ is an arbitrary element of $\mathcal{H}_0$ by continuity. Now, let $\tilde{g}_1, \tilde{g}_2 \in \mathcal{H}_0$, then if they correspond to the same element $g \in \mathcal{H}$, then

$$q_S(f, \tilde{g}_1 - \tilde{g}_2) = (Sf, g - g) = 0$$

for $f \in D(S)$. But $D(S)$ is dense in $\mathcal{H}_0$ (in the new norm), hence $\tilde{g}_1 - \tilde{g}_2 = 0$.

We regard $\mathcal{H}_0$ as a subspace of $\mathcal{H}$.

Inequality (J.18.5) carries over by continuity to all elements $f \in \mathcal{H}_0$.

Let us fix an element an arbitrary element h of $\mathcal{H}$ and consider the functional

$$L_h(f) := (f, h).$$

If f is in $\mathcal{H}_0$, then

$$|L_h(f)| \leq \|f\| \, \|h\| \leq \|f\|_S \, \|h\|$$

On the Hilbert space $\mathcal{H}_0$, $L_h(f)$ is therefore a linear functional whose norm does not exceed $\|h\|$ ($|L_h(f)| \leq \|h\|$). By the Riesz lemma there exists an element g of $\mathcal{H}_0$ such that

$$L_h(f) = q_S(f, g)$$

and whose norm $\|g\|_S$ equals that of the functional, hence

$$\|g\| \leq \|g\|_S = |L_h(f)| \leq \|h\|. \tag{J.-297}$$

Moreover, this element g is uniquely determined by the functional, and hence by h, so that if we write

$$g = Bh$$

we define a unique aoperator whose domain is the entire space $\mathcal{H}$ and whose values are contained in $\mathcal{H}_0$. This oprator is obviously linear, and by defintion

$$(f, g) = q_S(f, Bh) \tag{J.-297}$$

for $f \in \mathcal{H}_0$, $h \in \mathcal{H}$; by (J.18.5),

$$\|Bh\| \leq \|h\|.$$

Setting $f = Bh'$ in (J.18.5), where h' is again an arbitrary element of $\mathcal{H}$, we obtain

$$(Bh', h) = q_S(Bh', Bh) = \overline{q_S(Bh, Bh')} = \overline{(Bh, h')} = (h', Bh),$$

and in particular, for $h = h'$, we have

$$(Bh, h) = q_S(Bh, Bh) \geq (Bh, Bh) \geq 0, \tag{J.-297}$$

hence B is a positive and symmetric operator in the space $\mathcal{H}$.

By (J.18.5) $Bh = 0$ implies that $(f, h) = 0$ for all $f \in \mathcal{H}_0$, and since $\mathcal{H}_0$ is dense in $\mathcal{H}$, this implies $h = 0$. By ?? B has an inverse.

Since B is bounded and symmetric, and therefore self-adjoint, its inverse

$$A = B^{-1}$$

will also be self-adjoint (see theorem J.18.13). A is semi-bounded below by 1; in fact if $g = Bh$ is an element of $D(A)$, then by (J.18.5)

$$(g, Ag) \geq \|g\|^2.$$

Moreover, by (J.18.5)

$$(f, Ag) = q_S(f, g)$$

for $f \in \mathcal{H}_0$ and $g \in D(A)$. Furthermore, the domain $D(A)$, which is a linear subset of $\mathcal{H}_0$, is dense in $\mathcal{H}_0$ in the sense of the new norm as well as the original norm. For if we assumed the contrary case there would be an element $f_0 \neq 0$ in $\mathcal{H}_0$ such that

$$(f_0, Ag) = q_S(f_0, g) = 0$$

for all elements $g \in D(A)$. But the $\mathrm{Ran}(A) = \mathcal{H}$, and consequently $f_0 = 0$, in contradiction to the hypothesis that $f_0 \neq 0$.

It remains to show that the self-adjoint operator A is an extension of the given symmetric operator S.

Let f and g be any two arbitrary elemets of $D(S)$. We have on the one hand, by (J.18.5),

$$(f, Sg) = q_S(f, BSg),$$

and on the other hand, by defintion of the new norm,

$$(f, Sg) = q_S(f, g).$$

Since $D(S)$ is dense in $\mathcal{H}_0$ (in the sense of the new norm), the equation

$$q_S(f, BSg) = q_S(f, g)$$

is possible for all elements f of $D(S)$ only if

$$BSg = g.$$

This shows that g belongs to the domain of A and that

$$Ag = Sg.$$

Hence $A \supseteq S$.

□

Another definition:

Let $D(S)$, $D(S^*)$ denote the domains of S and its adjoint S^*, respectively. The domain of $D_F(S)$ of the Friedrichs extension S_F of S consists of all y in $D(S^*)$ for which there exists a sequence $y_k \in D(S)$ such that

1. $y_k \to y$ in $\mathcal{H}$ as $k \to \infty$,
2. $(S(y_k - y_l), y_k - y_l) \to 0$ as $k, l \to \infty$,

and S_F is the restriction of S^* to $D_F(S)$.

J.18.6 Quadratic Forms

In the construction of operators a in physics one often starts from its matrix elements $Q_a(\psi, \psi')$, which should equal $< \psi, a\psi' >$. However, this is not enough to define an operator in inifite dimensions because given an ortho-normal basis (b_n) we must have

$$\|a\psi\| = \sum_n |Q_a(b_n, \psi)|^2 < \infty$$

in order that $\psi \in D(a)$. Hence it may happen that the quadratic form $Q_a(\psi, \psi')$ exists for ψ, ψ' in a dense subset of $\mathcal{H}$ but on the other hand it could be that $D(a) = \{0\}$.

A bilinear Hermitian form.

Definition A **quadrartic form** is called is a map

$$q : Q(q) \times Q(q) \to \mathbb{C},$$

where $Q(q)$ is a dense linear subset of $\mathcal{H}$ calfunctionled the **form domain**, such that

$$\begin{aligned} q(\lambda\psi_1 + \mu\psi_2, \phi) &= \lambda q(\psi_1, \phi) + \mu q(\psi_2, \phi) \\ q(\psi, \lambda\phi_1 + \mu\phi_2) &= \overline{\lambda} q(\psi, \phi_1) + \overline{\mu} q(\psi, \phi_2) \end{aligned} \qquad \text{(J.-297)}$$

If

$$q\{\psi, \phi\} = \overline{q\{\phi, \psi\}} \qquad \text{(J.-297)}$$

then we say that q is **symmetric**.

□

We use the notation

$$q(u) \equiv q(u, u). \qquad \text{(J.-297)}$$

Definition A quadrartic form is called **semibounded** provided that $q(u, u) \geq -c\|u\|^2$ for some $c \geq 0$ and **positive** if $c = 0$.

□

Definition The numerical range $W(a)$ of a quadratic form $a(u, v)$

Definition A quadratic form $q(u, v)$ will be **closed** if for $\{u_n\} \subset D(q)$, $u_n \to u$ in $\mathcal{H}$, and $q(u_n - u_m) \to 0$ as $m, n \to \infty$ imply that

$$u \in D(q) \text{ and } q(u_n - u) \to 0 \text{ as } n \to \infty.$$

□

We can give a defintion of a closed quadratic form analogous to the defintion for a closed operator, an operator A is closed if and only if its graph is closed which is the same as saying that $D(A)$ is complete under the norm $\|\psi\|_A = \|A\psi\| + \|\psi\|$.

Definition Let q be a semibounded quadratic form, $q(\psi, \psi) \geq -M\|\psi\|$. q is closed if $Q(q)$ is complete under the norm

$$\|\psi\|_{+1} = \sqrt{q(\psi, \psi) + (M+1)\|\psi\|^2}$$

□

Notice that $\|\psi\|_{+1}$ comes from the inner product

$$(\psi, \varphi)_{+1} = q(\psi, \varphi) + (M+1)(\psi, \varphi).$$

Proposition J.18.22 *The above two defintions for a closed form are equivalent.*

Proof: Proof of the equivalence. Say for $\varphi_m, \varphi_n \in Q(q)$, $\|\varphi_m - \varphi_n\|_{+1} \to 0$ as $n, m \to \infty$ implies $\|\psi - \psi_n\|_{+1} \to 0$.

$$\|\varphi_m - \varphi_n\|_{+1}^2 = (q(\varphi_m - \varphi_n, \varphi_m - \varphi_n) + M\|\varphi_m - \varphi_n\|^2) + \|\varphi_m - \varphi_n\|^2 \to 0$$

as $n, m \to \infty$ implies $q(\varphi_m - \varphi_n, \varphi_m - \varphi_n) \to 0$ and $\{\varphi_n\}$ is Cauchy in the norm of $\mathcal{H}$. Now

$$\|\varphi - \varphi_n\|_{+1}^2 = (q(\varphi - \varphi_n, \varphi - \varphi_n) + M\|\varphi_m - \varphi_n\|^2) + \|\varphi - \varphi_n\|^2 \to 0$$

implies that if $\varphi_n \to \varphi$ in the norm of $\mathcal{H}$ then $q(\varphi_n - \varphi, \varphi_n - \varphi) \to 0$ and

$$\|0\|_{+1}^2 = (M+1)\|0\|^2 = 0.$$

This implies the inner product $(\cdot, \cdot)_{+1}$ is continuous (and as a concequence $q(\cdot, \cdot)$ is continuous). Also, by the Schwarz inequality

$$|(\phi, 0)_{+1}| \leq \|\phi\|_{+1} \|0\|_{+1} = 0.$$

From this we have

$$q(\phi, 0) = (M+1)(\phi, 0) - (\phi, 0)_{+1} = 0.$$

For $\psi_1, \psi_2 \in Q(q)$ we can write

$$q(\lambda\psi_1 + \mu\psi_2, \varphi_n) = \lambda q(\psi_1, \varphi_n) + \mu q(\psi_2, \varphi_n)$$

By the continuity of the quadratic form, in the limit $n \to \infty$ we obatin

$$q(\lambda\psi_1 + \mu\psi_2, \varphi) = \lambda q(\psi_1, \varphi) + \mu q(\psi_2, \varphi).$$

This and the similarly proven result for $q(\varphi, \cdot)$ proves that $\varphi \in Q(q)$.

Now the other way around. Say whenever $\varphi_n \in Q(q)$, $\varphi_n \to \varphi$ in the norm of $\mathcal{H}$ and $q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0$, as $n, m \to \infty$, then

$$\|\varphi_m - \varphi_n\|_{+1} \to \sqrt{(M+1)}\, \|\varphi_m - \varphi_n\| \to 0$$

as $n, m \to \infty$. Now say that $\varphi \in Q(q)$ and $q(\varphi_n - \varphi, \varphi_n - \varphi) \to 0$, then

$$\|\varphi - \varphi_n\|_{+1} \to \sqrt{(M+1)}\, \|\varphi - \varphi_n\| \to 0,$$

that is, $Q(q)$ is complete in the norm $\| \cdot \|_{+1}$.

□

Definition A bilinear form $b(u, v)$ is called an *extension* of a bilinear form $a(u, v)$ if $D(b) \supseteq D(a)$ and $b(u, v) = a(u, v)$ for $u, v \in D(a)$.

□

Definition A set U will be called dense in $D(a)$ if for each $w \in D(a)$ and each $\epsilon > 0$ there is a $u \in U$ such that

$$a(w - u) < \epsilon \text{ and } \|w - u\| < \epsilon.$$

□

A symmetric operator can fail to be self-adjoint only if $A^\dagger$ is an extension of A to a larger domain.

Note on the distinction between symmetric operators and symmetric forms:

For symmetric operators, there are always closed extensions. A smallest closed extension always exists (the double adjoint), but it is possible that none of these closed extensions is self-adjoint. On the other hand, semibounded forms need not have any closed extensions (defintions for quadratic forms to be given below), but when such extensions exsit and are semibounded, they are the quadratic forms associated with self-adjoint operators.

Theorem J.18.23 *Let T be a semi-bounded quadratic form. Then T may not be closable, but if it is and the closure is semi-bounded, then T is the quadratic form of a unique self-adjoint operator T according to*

$$(\psi, T\phi) = (T\psi, \phi)??$$

Proof: When the quadratic form is real we have $T\{\psi, \psi\}$ is real for every vector ψ in the domain of T. In particular $T\{\psi + \phi, \psi + \phi\}$ and $T\{\psi + i\phi, \psi + i\phi\}$ are real. Therefore

$$(\psi + \phi, T(\psi + \phi)) = (T(\psi + \phi), \psi + \phi) \tag{J.-297}$$

$$(\psi + i\phi, T(\psi + i\phi)) = (T(\psi + i\phi), \psi + i\phi) \tag{J.-297}$$

Breaking up the inner product, we find that

$$\begin{aligned}(\psi, T\phi) + (\phi, T\psi) &= (T\psi, \phi) + (T\phi, \psi)\\ (\psi, T\phi) - (\phi, T\psi) &= (T\psi, \phi) - (T\phi, \psi)\end{aligned} \tag{J.-297}$$

and adding we have

$$(\psi, T\phi) = (T\psi, \phi) \tag{J.-297}$$

for arbitrary vectors ψ and ϕ in the domain of a_T, therefore T is self-adjoint.

Theorem J.18.24 *Let T be a positive, symmetric operator. Then the corresponding positive quadratic form a_T has a positive closure $\overline{a_T}$. The uniqe positive operator $\overline{T}$ corresponding to that closure via $a_{\overline{T}} = \overline{a_T}$ is called the* Friedrichs extension *of T. It may extend the closure $\overline{T}$ of T and is the only self-adjoint extension which contains $D(\overline{a_T})$.*

In an arbitrary orthonormal basis, we have

$$\langle x|H|x\rangle = x_i H^i_j x^j \equiv \sum_{i,j} H^i_j \bar{x}^i x^j \tag{J.-297}$$

where H^i_j are elements of a Hermitian matrix $\mathbf{H}$. The last expression is called a Hemitian quadratic form.

Theorem J.18.25 *The numerical range $W(a)$ of a bilinear form is convex. That is, if $u, v \in D(a)$ satisfying $\|u\| = \|v\| = 1$, then for each $0 < \tau < 1$, we can find a $w \in D(a)$ such that $\|w\| = 1$, and*

$$a(w) = (1-\tau)a(u) + \tau a(v). \tag{J.-297}$$

Proof: see details

J.19 Semianalytic Theory

J.19.1 Introduction

We introduce semianalytic functions, semianalytic manifolds,and semianalytic geometry. Used in proof of uniqueness theorem for the Kinematic Hilbert space of LQG.

Analytic difeomorphisms are non-local.

In particular, the group of anaytic diffeomorphisms has local degrees of freedom, in the sense that for every point $x \in \mathcal{M}$ and its neighbourhood $\mathcal{U}'$, there is a semianalytic diffeomorphism of $\mathbb{R}^n$ which moves x, but restricted to $\mathbb{R}^n/\mathcal{U}'$ is the identity map.

J.19.2 Semianalytic Functions in $\mathbb{R}^n$

Our non-analyticity surfaces will have an appropriate analytic structure. We introduce the idea of a semianalytic partition.

Let $\{h_1, \ldots, h_N\}$ be a set of analytic functions defined on a domain containing an open set $\mathcal{U}$. Consider the collection of all possible sequences of equalities/inequalities

$$h_1(x) > 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0;$$
$$h_1(x) < 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0;$$

$$h_1(x) > 0, \quad h_2(x) = 0, \ldots, h_N(x) < 0;$$

$$h_1(x) = 0, \quad h_2(x) = 0, \ldots, h_N(x) = 0. \tag{J.-301}$$

A particular member of this collection is identified by the $\{>, =, <\}$ associated to each h_I. Formally, there is a map

$$\sigma : h = \{h_1, \ldots, h_N\} \to \{>, =, <\} \tag{J.-301}$$

and we can write

$$\sigma_I := \sigma(h_I), \tag{J.-301}$$

where the integer I runs from 1 to N. A particular member of (J.-301) can then be written

$$\begin{array}{ccc} h_1(x) & \sigma_1 & 0, \\ & \cdots & \\ h_N(x) & \sigma_N & 0. \end{array} \tag{J.-302}$$

The set of conditions (J.-302) determines the following subset of $\mathcal{U}$.

$$\mathcal{U}_{h,\sigma} = \{x \in \mathcal{U} : (J. - 302)\}. \tag{J.-302}$$

Definition Given a finite set h of real valued analytic functions defined on a neighbourhood of an open subset $\mathcal{U}$ of $\mathbb{R}^n$, the corresponding semianalytic partition of $\mathcal{U}$ is the set of all the subsets $\mathcal{U}_{h,\sigma} \subset \mathcal{U}$ defined by (J.-302 , J.19.2) such that σ is an arbitrary map (J.19.2). Given $\mathcal{U}$ and h as above, the partition will be denoted by $\mathcal{P}(\mathcal{U}, h)$.

Obviously, every semianalytic partition covers $\mathcal{U}$.

$$\mathcal{U} = \bigcup_{\sigma} \mathcal{U}_{h,\sigma} \tag{J.-302}$$

where σ runs through all the maps (J.19.2). Also,

$$\sigma \neq \sigma' \quad \Rightarrow \quad \mathcal{U}_{h,\sigma} \cap \mathcal{U}_{h,\sigma'} = \emptyset, \tag{J.-302}$$

and a set $\mathcal{U}_{h,\sigma}$ may be empty itself. Another obvious property is that given a semianalytic covering $\mathcal{P}(\mathcal{U}, h)$ and an open suset $\mathcal{V} \subset \mathcal{U}$, the family of functions h defines a semianalytic covering $\mathcal{P}(\mathcal{V}, h)$.

Example

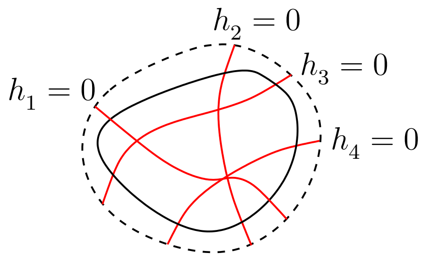


Figure J.20:

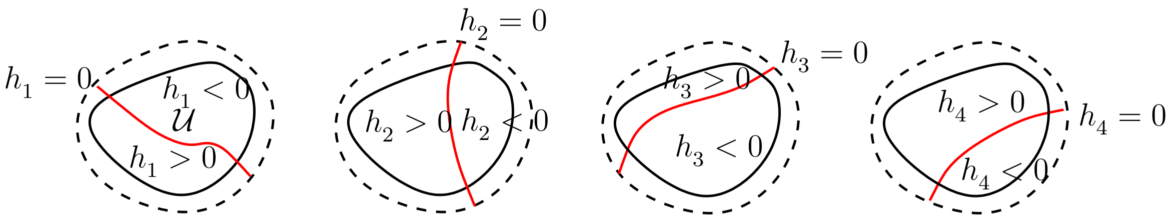


Figure J.21:

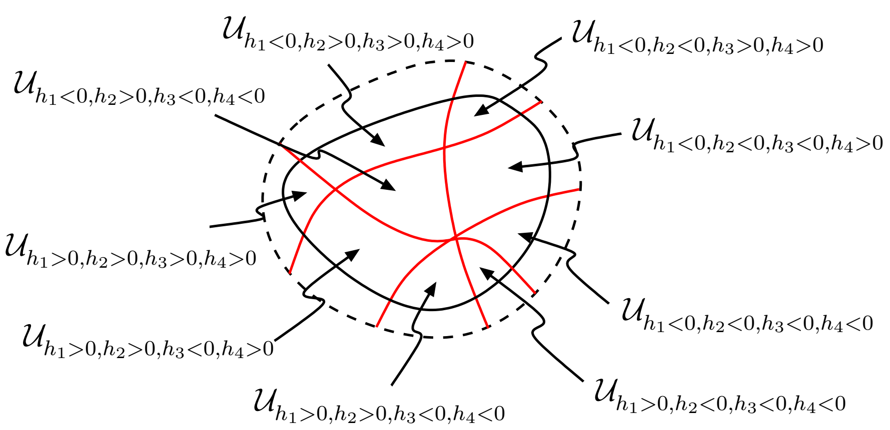


Figure J.22: Subsets, $\mathcal{U}_{h,\sigma}$, of dimension two.

The following subsets are empty sets:

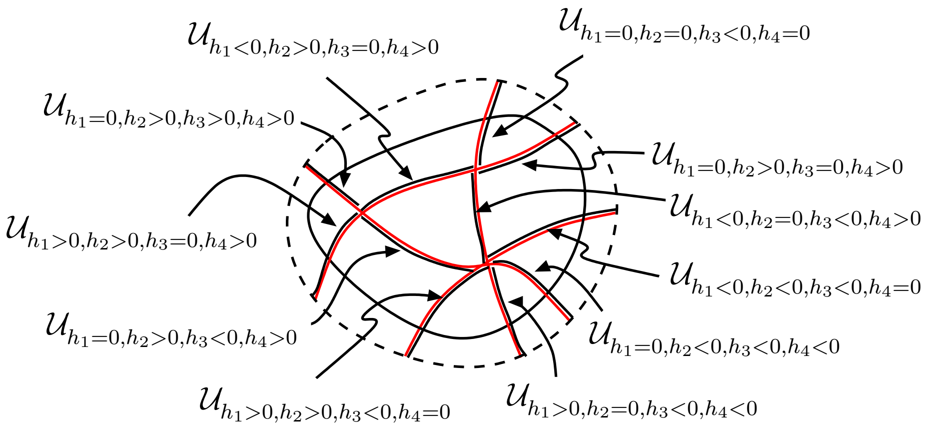


Figure J.23: Subsets, $\mathcal{U}_{h,\sigma}$, of dimension one.

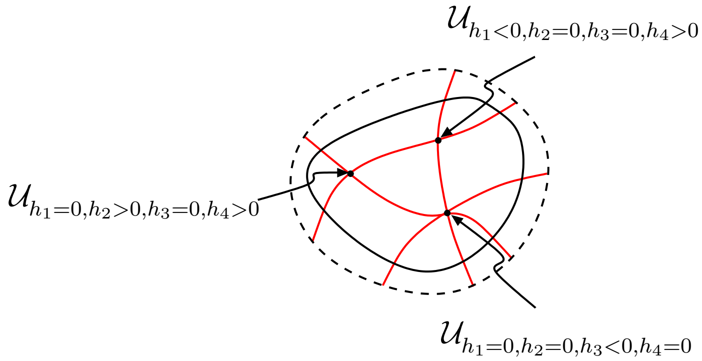


Figure J.24: Subsets, $\mathcal{U}_{h,\sigma}$, of dimension zero.

$$
\begin{array}{rcl}
\mathcal{U}_{h_1=0,\ h_2=0,\ h_3=0,\ h_4=0} & = & \emptyset \\
\mathcal{U}_{h_1>0,\ h_2=0,\ h_3=0,\ h_4=0} & = & \emptyset \\
\mathcal{U}_{h_1<0,\ h_2=0,\ h_3=0,\ h_4=0} & = & \emptyset \\
\mathcal{U}_{h_1=0,\ h_2=0,\ h_3=0,\ h_4>0} & = & \emptyset \\
\mathcal{U}_{h_1=0,\ h_2=0,\ h_3=0,\ h_4<0} & = & \emptyset \\
\mathcal{U}_{h_1=0,\ h_2=0,\ h_3=0,\ h_4<0} & = & \emptyset \\
etc\ldots & & .
\end{array}
\qquad \text{(J.-307)}
$$

Roughly speaking, a real valued function f defined on an open subset of $\mathcal{U} \subset \mathbb{R}^n$ will be called semianalytic if it is analytic on an open and dense subset of $\mathcal{U}$, and if the non-analyticity surfaces have also an appropriate analytic structure, and if the restrictions of f to the non-analyticity surfaces are again analytic in an appropriate sense.

Definition A function $f : \mathcal{U} \to \mathbb{R}^m$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$, is called semianalytic if every $x \in \mathcal{U}$ has an open neighbourhood $\tilde{\mathcal{U}}$ equipped with a semianalytic partition

$\mathcal{P}(\tilde{\mathcal{U}}, h)$, such that for every $\tilde{\mathcal{U}}_{h,\sigma} \in \mathcal{P}(\tilde{\mathcal{U}}, h)$ there is an analytic function $f_\sigma : \tilde{\mathcal{U}} \to \mathbb{R}^m$, such that

$$f\big|_{\tilde{\mathcal{U}}_{h,\sigma}} = f_\sigma\big|_{\tilde{\mathcal{U}}_{h,\sigma}}, \tag{J.-307}$$

that is, such that f_σ coincides with f on $\mathcal{U}_{h,\sigma}$.

Clearly, if $f : \mathcal{U} \to \mathbb{R}^n$ is semianalytic, and $\mathcal{V} \subset \mathcal{U}$ is open, then the restriction function $f\big|_{\mathcal{V}}$ is semianalytic.

Simple facts of semianalytic partitions

$$\mathcal{P}(\mathcal{U}, h^{(1)} \cup h^{(2)}) = \mathcal{P}(\mathcal{U}, h^{(1)}) \cap \mathcal{P}(\mathcal{U}, h^{(2)}).$$

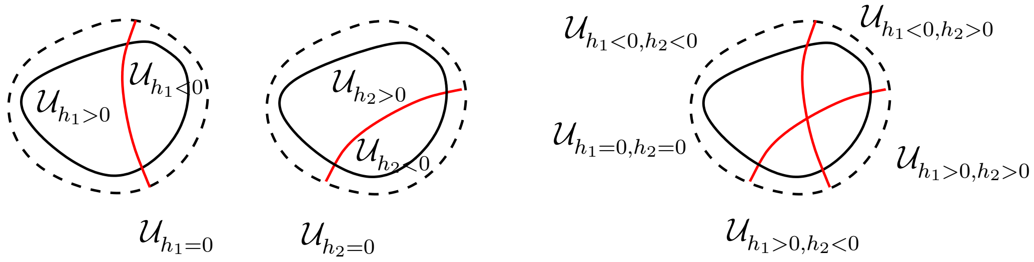


Figure J.25: An example of $\mathcal{P}(\mathcal{U}, h^{(1)} \cup h^{(2)}) = \mathcal{P}(\mathcal{U}, h^{(1)}) \cap \mathcal{P}(\mathcal{U}, h^{(2)})$. Also, for every $\mathcal{U}_{h,\sigma} \in \mathcal{P}(\mathcal{U}, h)$ there are some $\mathcal{U}^{(1)}_{h^{(1)},\sigma^{(1)}} \in \mathcal{P}(\mathcal{U}^{(1)}, h^{(1)})$ and $\mathcal{U}^{(2)}_{h^{(2)},\sigma^{(2)}} \in \mathcal{P}(\mathcal{U}^{(2)}, h^{(2)})$ such that $\mathcal{U}_{h^{(1)}\cup h^{(2)},\sigma} \subset \mathcal{U}^{(1)}_{h^{(1)},\sigma^{(1)}}$ and $\mathcal{U}_{h^{(1)}\cup h^{(2)},\sigma} \subset \mathcal{U}^{(2)}_{h^{(2)},\sigma^{(2)}}$.

Proposition J.19.1 *Let $f_1 : \mathcal{U} \to \mathbb{R}$, and $f_2 : \mathcal{U} \to \mathbb{R}^m$ be two semianalytic functions where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. Then the functions*

$$\mathcal{U} \ni x \mapsto f_1(x) f_2(x) \in \mathbb{R}^m, \qquad \mathcal{U} \ni x \mapsto (f_1(x), f_2(x)) \in \mathbb{R}^{m+1}, \tag{J.-307}$$

are also semianalytic.

Proof: Let $x \in \mathcal{U}$. Let $\mathcal{P}(\tilde{\mathcal{U}}^{(1)}, h^{(1)})$ be a semianalytic partition compatible with f_1 at x, and $\mathcal{P}(\tilde{\mathcal{U}}^{(2)}, h^{(2)})$ be a semianalytic partition compatible with f_2 at x. The proof becomes obvious if we construct a single semianalytic partition compatible at x with both functions. The natural choice is just the semianalytic partition of the intersection

$$\tilde{\mathcal{U}} := \tilde{\mathcal{U}}^{(1)} \cap \tilde{\mathcal{U}}^{(2)} \tag{J.-307}$$

defined by the set of functions

$$h := h^{(1)} \cup h^{(2)}. \tag{J.-307}$$

It is sufficient to notice, that for every $\tilde{\mathcal{U}}_{h,\sigma} \in \mathcal{P}(\mathcal{U}, h)$ there are some $\tilde{\mathcal{U}}^{(1)}_{h^{(1)},\sigma^{(1)}} \in \mathcal{P}(\tilde{\mathcal{U}}^{(1)}, h^{(1)})$ and $\tilde{\mathcal{U}}^{(2)}_{h^{(2)},\sigma^{(2)}} \in \mathcal{P}(\tilde{\mathcal{U}}^{(2)}, h^{(2)})$ such that

$$\tilde{\mathcal{U}}_{h,\sigma} \subset \tilde{\mathcal{U}}^{(1)}_{h^{(1)},\sigma^{(1)}} \qquad \text{and} \qquad \tilde{\mathcal{U}}_{h,\sigma} \subset \tilde{\mathcal{U}}^{(2)}_{h^{(2)},\sigma^{(2)}}. \tag{J.-307}$$

$\square$

J.19.3 Morphisms of Semianalytic Functions

It is obvious, that every analytic map $\phi : \mathcal{U} \to \mathcal{U}'$ between two open subsets $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{U}' \subset \mathbb{R}^{n'}$ pullbacks all the semianalytic functions defined on $\mathcal{U}'$ into semianalytic functions defined on $\mathcal{U}$.

Lemma J.19.2 *Let $\mathcal{P}(\tilde{\mathcal{U}}', h')$ be a semianalytic partition. Let $\phi : \mathcal{U} \to \tilde{\mathcal{U}}'$ be a semianalytic function, where the subset $\mathcal{U} \subset \mathbb{R}^n$ is open. For every $x_0 \in \mathcal{U}$ there exists an open neighbourhood $\tilde{\mathcal{U}}$ and a semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$ such that for every element of $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$, say $\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}}$, there is an element of $\mathcal{P}(\tilde{\mathcal{U}}', h')$, say $\tilde{\mathcal{U}}'_{\tilde{h}',\tilde{\sigma}'}$, such that*

$$\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}} \subset \phi^{-1}\big(\tilde{\mathcal{U}}'_{\tilde{h}',\tilde{\sigma}'}\big). \tag{J.-307}$$

Proof:

If ϕ is analytic, then the set of pullbacks of all the functions $h'_I \in h'$ defines a suitable partition of the whole of $\mathcal{U}$.

$$\tilde{h} := \{h'_I \circ \phi_\sigma\} \cup \{h_I\}$$

We now prove the corresponding semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$ satisfies the conclusion.

Denote the restriction of $\tilde{\sigma}$ to the set of functions h by σ

$$\sigma := \tilde{\sigma}\big|_h$$

so

$$
\begin{array}{ccc}
h_1 & \sigma_1 & 0 \\
 & \dots & \\
h_N & \sigma_N & 0
\end{array}
\tag{J.-308}
$$

Denote the restriction of $\tilde{\sigma}$ to the set of functions $\phi^*_\sigma(h')$ by σ'

$$
\sigma' := \tilde{\sigma}\big|_h
$$

so

$$
\begin{array}{ccc}
\phi^*_\sigma(h'_1) & \sigma'_1 & 0 \\
 & \dots & \\
\phi^*_\sigma(h'_M) & \sigma'_M & 0
\end{array}
\tag{J.-309}
$$

$$
\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}} \subset \tilde{\mathcal{U}}_{h,\sigma}
\tag{J.-309}
$$

$$
\phi_\sigma(\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}}) \subset \tilde{\mathcal{U}}'_{h',\sigma'}
\tag{J.-309}
$$

By the definition of semianalytic functions, we have

$$
\phi_\sigma\big|_{\tilde{\mathcal{U}}_{h,\sigma}} = \phi\big|_{\tilde{\mathcal{U}}_{h,\sigma}},
$$

then by (J.19.3) we see that ϕ is equal to every one of analytic functions ϕ_σ when restricted to the set $\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}}$. This combied with (J.19.3) allows us to define a suitable partition of the of $\tilde{\mathcal{U}}$ and concludes the proof.

□

Proposition J.19.3 *Let $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^{n'}$ be open subsets. Suppose the functions $f' : \mathcal{U}' \to \mathbb{R}^m$ and $\phi : \mathcal{U} \to \mathcal{U}'$ are semianalytic. Then, the composition function $f' \circ \phi : \mathcal{U} \to \mathbb{R}^m$ is semianalytic.*

Proof:

□

In general, the inverse of an invertible semianalytic function is not necessarilty semianalytic. However, a carefully formulated set of assumptions ensures the semianalyticity of the inverse.

Proposition J.19.4 *Let $\phi : \mathcal{U} \to \mathcal{U}'$ be a semianalytic and bijective function, where $\mathcal{U}, \mathcal{U}' \subset \mathbb{R}^n$ are open. Suppose, that for every $x_0 \in \mathcal{U}$ there exists a semianalytic partition $\mathcal{P}(\mathcal{U}, h)$ compatible with ϕ at x_0, and such that for every $\tilde{\mathcal{U}}_{h,\sigma} \in \mathcal{P}(\mathcal{U}, h)$ the restriction $\phi\big|_{\tilde{\mathcal{U}}_{h,\sigma}}$ is extendable to an analytic, injective function $\phi_\sigma : \tilde{\mathcal{U}} \to \mathcal{U}'$, such that:*

(i) $\phi(\tilde{\mathcal{U}})$ is an open subset of $\mathbb{R}^n$, and

(ii) the invrese $\phi_\sigma^{-1} : \phi_\sigma(\tilde{\mathcal{U}}) \to \tilde{\mathcal{U}}$ is analytic. Then, ϕ^{-1} is semianalytic.

Proof:

$\square$

Corollary J.19.5 *Suppose $\phi : \mathcal{U} \to \mathcal{U}'$ is a diffeomorphisms of the differentiability class C^m, where $\mathcal{U}, \mathcal{U}' \subset \mathbb{R}^n$ are open and $m > 0$. If ϕ is semianalytic, then so is $\phi^{-1} : \mathcal{U}' \to \mathcal{U}$*

Proof:

$\square$

Lemma J.19.6 *Suppose $\mathcal{P}(\mathcal{U}, h)$ is a semi-semianalytic partition of an open $\mathcal{U} \subset \mathbb{R}^n$. Then, every $x \in \mathcal{U}$ has a neighbourhood $\tilde{\mathcal{U}}$ which admits a semianalytic partition finer than $\mathcal{P}(\tilde{\mathcal{U}}, h)$.*

Proof:

$\square$

Proposition J.19.7 *For every semiaanlytic partition $\mathcal{P}(\mathcal{U}, h)$ of an open $\mathcal{U} \subset \mathbb{R}^n$, every point $x \in \mathcal{U}$ has a neighbourhood $\tilde{\mathcal{U}}$ which admits an analytic partitio finer than $\mathcal{P}(\tilde{\mathcal{U}}, h)$.*

Proof:

$\square$

J.19.4 Semianalytic Manifolds and Submanifolds

By analogy with the definitions of an analytic structure, analytic function, and analytic submanifolds, we introduce now natural semianalytic generalizations. This is possible by results of the previous two subsections. The main result of this subsection is the proof of the finite intersection property of semianalytic submanifolds, given in the form of proposition J.19.8. Then in the next subsection we prove the second important property of semianalytic structures - a local character.

In this subsection, Σ is an n-dimensional differential manifold. Henceforth we will be assuming that Σ and all considered functions are of a differentiable class C^m, where $m > 0$.

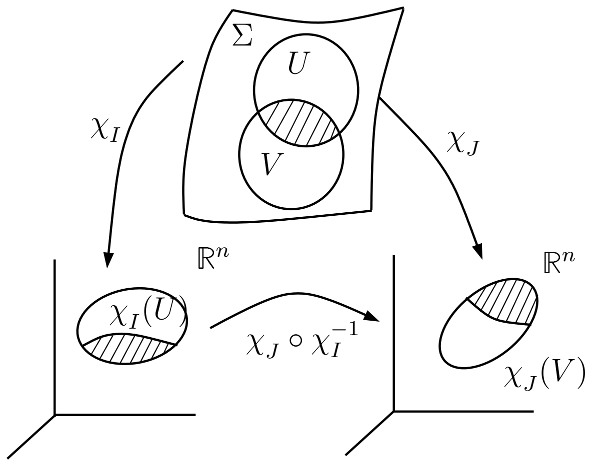


Figure J.26: semiana. The neighbourhoods U and V in Σ overlap. Their respective maps to $\mathbb{R}^n$, χ_I and χ_J, give two different coordinate systems to the overlap region. The differentiability class of the manifold is semianalytic when $\chi_J \circ \chi_I^{-1}$ is semianalytic.

Recall a collection of related charts such that every point of Σ lies in the domain of at least one chart forms an atlas.

We denote an atlas of Σ by $\{(\mathcal{U}_I, \chi_I)\}_{I\in\mathcal{I}}$, where $\mathcal{I}$ is some labeling set, $\{\mathcal{U}_I\}$ is an open covering of Σ, and $\{\chi_I\}_{I\in\mathcal{I}}$ is a family of diffeomorphisms $\chi_I : \mathcal{U}_I \to \mathcal{U}'_I \subset \mathbb{R}^n$.

Definition An atlas $\{(\mathcal{U}_I, \chi_I)\}$ of Σ is called semianalytic if for every pair $I, J \in \mathcal{I}$ the map

$$\chi_J \circ \chi_I^{-1} : \chi_I(\mathcal{U}_I \cap \mathcal{U}_J) \to \chi_J(\mathcal{U}_I \cap \mathcal{U}_J) \tag{J.-309}$$

The collection of all semianalytic charts forms a maximal semianalytic atlas.

□

Definition A semianalytic manifold is a differential manifold endowed with a maximal semianalytic atlas.

□

Definition Given two semianalytic manifolds Σ and Σ', a map $f : \Sigma \to \Sigma'$ is called semianalytic if fo every semianalytic chart χ_I of Σ, and every semianalytic chart $\chi'_{I'}$ of Σ' the function $\chi'_{I'} \circ \chi_I^{-1}$ (whenever the composition can be applied) is semianalytic. .2 In particular, if

$$\Sigma' = \mathbb{R}^{n'} \tag{J.-309}$$

and the semianalytic structure is the natural one defined by the atlas $\{(\mathbb{R}^{n'}, \mathrm{id})\}$, then the map f is a semianalytic function defined on Σ.

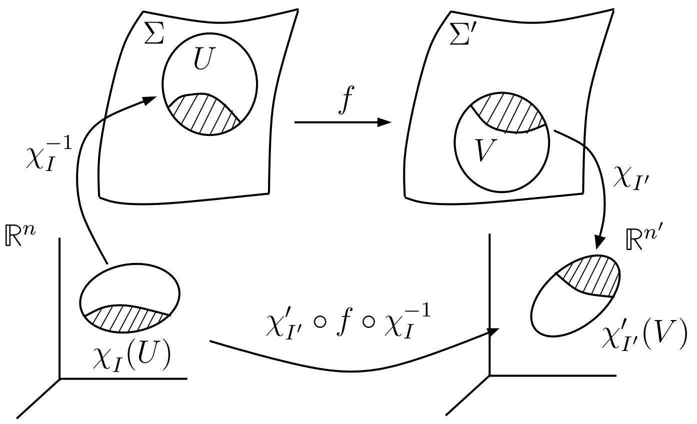


Figure J.27: Definition of a semianalytic function f between two semianalytic manifolds Σ and Σ'. Composition can be applied only when the range of $\chi_I(U) \subset \mathbb{R}^n$ under $\chi'_{I'} \circ f \circ \chi_I^{-1}$ overlaps with the subset $\chi'_{I'}(V) \subset \mathbb{R}^{n'}$ for subsets $U \subset \Sigma$ and $V \subset \Sigma'$.

□

Definition A semianalytic submanifold of a semianalytic manifold Σ is a subset $S \subset \Sigma$ such that for very $x \in S$, there is a semianalytic chart χ_I defined in a neighbourhood $\mathcal{U}_I$ of x, such that

$$\begin{aligned} \chi_I(S \cap \mathcal{U}_I) &= \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 = \cdots = x^{n-n'} = 0, \\ &\quad 0 < x^{n-n'+1} < 1, \ldots, 0 < x^n < 1\}, \end{aligned} \tag{J.-309}$$

where n' is a non-negative integer, $n' \leq n$, and n' is called the dimension of S.

□

Definition An n' dimensional semianalytic submanifold with boundary of Σ is a subset $S \subset \Sigma$ such that for every $x \in S$, there is a semianalytic chart χ_I defined in a neighbourhood $\mathcal{U}_I$ of x, such that either (J.-309) or

$$\begin{aligned} \chi_I(S \cap \mathcal{U}_I) &= \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 = \cdots = x^{n-n'} = 0, \\ &\quad 0 \leq x^{n-n'+1} < 1, 0 \leq x^{n-n'+2} < 1, \ldots, 0 < x^n < 1\}, \end{aligned} \quad \text{(J.-309)}$$

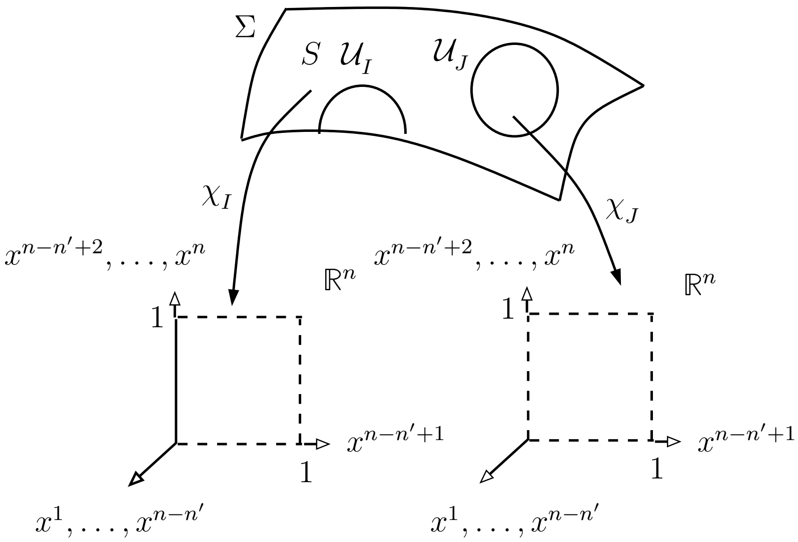


Figure J.28: .

□

We are using extensively two particular classes of submanifolds: edges and faces.

Definition A semianalytic edge is a connected, one-dimensional semianalytic submanifold of Σ with two-point boundary.

□

Definition A semianalytic face is a connected, codimensional one-dimensional semianalytic submanifold of Σ whose normal bundle is equipped with an orientation.

□

All these definitions and results were to prepare for the following key property of semianalytic submanifolds crucial to the proof of the LOST uniqueness theorem. An infinite set of disjoint, embedded intervals may also form a single submanifold, disconnected though. For example the intersection of two edges where one of them is allowed to "oscillate" arbitarily rapidly. In theorem L.5.1 it is proved that for semianalytic edges this cannot occur: either the intersection of two edges results in a finite number of isolated intersections or a common connected finite segment. What is important in the conclusion of the following proposition, a generalization of this result for edges, is the finiteness of the partition and the connectedness of its elements. The following proposition apllied to the edges and faces contained in compact sets shows the intersection of edges is a finite collection of edges and isolated points, the intersection of edges and faces is a finite collection of edges and isolated points and the intersection of faces is a finite collection of faces, s.a. submanifolds of co-dimension two,..., edges and isolated points.

Proposition J.19.8 *Let S_1 and S_2 be two semianalytic submanifolds of a semianalytic manifold Σ. Suppose $x \in S_1 \cap S_2$. Then, there is an open neighbourhood $\mathcal{W}$ of x in Σ, such that $\mathcal{W} \cap S_1 \cap S_2$ is finite, distjoint union of connected semianalytic submanifolds.*

Proof: For every point $x \in S_1 \cap S_2$, there is a neighbourhood $\mathcal{W}$ which can be mapped by a semianalytic chart into an open subset $\mathcal{U} \subset \mathbb{R}^n$.

□

J.19.5 Local Character of Semianalytic Structures

The property of semianalytic structures which distinguishes them from analytic ones is the local chararcter of the spaces of semianalytic functions and semianalytic diffeomorphisms. That feature is guaranteed by the existence of a partition of unity compatible with an arbitrary open covering.

Proposition J.19.9 *Suppose $\mathcal{W} \subset \Sigma$ is a compact subset. Let $\mathcal{U}_I \subset \Sigma$, $I = 1, \ldots, N$, be a family of open sets which covers $\mathcal{W}$. There exists a family of C^m semianalytic functions $\phi_I : \Sigma \to \mathbb{R}$, $I = 1, \ldots, N$ such that for every I,*

$$\mathit{supp}\, \phi_I \subset \mathcal{U}_I \tag{J.-309}$$

and

$$\sum_I \phi_I\big|_{\mathcal{W}} = 1. \tag{J.-309}$$

Proof: The proof relies on the following two properties of semianalytic functions:

(i) For every open ball in $\mathbb{R}^D$, there is a C^m semianalytic function greater than zero at every point inside the ball and identically zero everywhere else.

(ii) If f is a nowhere vanishing C^m semianalytic function then so is $1/f$.

One employs smooth functions of the form

$$\exp(-1/(x_1^2 + \cdots + x_n^2)).$$

These functions are actually analytic except at the point $x = 0$ where they are C^∞. First we prove that there exists a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ with $h \geq 0$ such that $h(x) = 1$ for $x_1^2 + \cdots + x_n^2 \leq 1$ and $h(x) = 0$ for $x_1^2 + \cdots + x_n^2 \geq 2$. Let

$$f(r) := \begin{cases} e^{-1/r^2} & r > 0 \\ 0 & r \leq 0. \end{cases}$$

Then $f : \mathbb{R} \to \mathbb{R}$ is smooth and the function

$$g(r) := \frac{f(2-r)}{f(2-r) + f(r-1)}$$

is C^∞ with values in $[0,1]$ such that $g(r) = 1$ for $r \leq 1$ and $g(r) = 0$ for $r \geq 2$. We then just have to set $h(x) = g(\sqrt{x_1^2 + \cdots + x_n^2})$.

Now $\mathcal{W}$ is covered by finitely many open sets of the form $\mathcal{U}_I = \varphi_{x_I}^{-1}(B_1(0))$.

we now define functions $\phi_I : \Sigma \to \mathbb{R}$ by

$$\phi_I(y) := \begin{cases} h(\varphi_J(y)) & y \in \mathcal{U}_J \\ 0 & y \notin \mathcal{U}_J. \end{cases}$$

It follows from property () and the properties of h that

we put

$$\phi_I = \sum \frac{g_n}{g}$$

to get the required partion of unity subordinate to $\{\mathcal{U}_I\}$.

□

J.20 Limits of Families of Measurable Spaces

The idea is actually quite simple.

Follows [410]

J.20.1 Hopf Theorem

$$
\begin{aligned}
E_1 \subset E_2 \quad &\Rightarrow \quad \mu(E_1) \le \mu(E_2) \quad \text{monotonic} && \text{(J.-308)}\\
\mu(\cup_i^\infty E_i) \quad &\le \quad \sum_i^\infty \mu(E_i) \quad \text{countably subadditive.} && \text{(J.-307)}
\end{aligned}
$$

A set E is said to be $\mu-$measurable if

$$
\mu(Y) = \mu(Y \cap E) + \mu(Y \cap E^c) \quad \text{for all } Y \subset X. \qquad \text{(J.-307)}
$$

Theorem J.20.1 *For a Caratheodory's outer measure μ, the family of all $\mu-$measurable sets is a $\sigma-$algebra and the restriction of μ on this $\sigma-$algebra is a measure.*

$\mathcal{B}$ is closed under complement; $E \in \mathcal{B} \Leftrightarrow E^c \in \mathcal{B}$. Then we have

$$
\begin{aligned}
\mu(Y) \quad &= \quad \mu(Y \cap E_1) + \mu(Y \cap E_1^c) = \mu(Y \cap E_1) + \mu(Y \cap E_1^c \cap E_2) + \mu(Y \cap E_1^c \cap E_2^c)\\
&= \quad \sum_{i=1}^n \mu(Y \cap E_1^c \cap \cdots \cap E_{i-1}^c \cap E_i) + \mu(Y \cap E_1^c \cap \cdots \cap E_n^c). && \text{(J.-307)}
\end{aligned}
$$

By monotonicity of the measure μ, that $Y \cap \left(\bigcup_{i=1}^\infty E_i\right)^c \subset Y \cap E_1^c \cap \cdots \cap E_n^c$ implies $\mu(Y \cap E_1^c \cap \cdots \cap E_n^c) > \mu\big(Y \cap \bigcup_{i=1}^\infty E_i)^c\big)$. Therefore by (J.-307) in the limit $n \to \infty$ we have

$$
\mu(Y) \ge \sum_{i=1}^\infty \mu(Y \cap E_1^c \cap \cdots \cap E_{i-1}^c \cap E_i) + \mu\big(Y \cap \bigcup_{i=1}^\infty E_i)^c\big) \qquad \text{(J.-307)}
$$

$$
\mu(Y) \ge \mu\big(Y \cap (\bigcup_{i=1}^\infty E_i)\big) + \mu\big(Y \cap (\bigcup_{i=1}^\infty E_i)^c\big) \qquad \text{(J.-307)}
$$

This implies $\bigcup_{i=1}^{\infty} E_i \in \mathcal{B}$, so the $\mu-$measurable sets form a $\sigma-$algebra. Now we establish that the measure μ is $\sigma-$additive. Assume $\{E_i\}$ are pairwise disjoint. If we choose as $Y = \cup_{i=1}^{\infty} E_i$ then E.q.(J.20.1) shows that

$$\begin{aligned}
\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) &\geq \sum_{i=1}^{\infty} \mu\Big(\bigcup_{n=1} E_n \cap E_1^c \cap \cdots \cap E_{i-1}^c \cap E_i\Big) + \mu(\emptyset) \\
&= \mu\Big(\bigcup_{n=1} E_n \cap E_1\Big) + \mu\Big(\bigcup_{n=1} E_n \cap E_1^c \cap E_2\Big) + \mu\Big(\bigcup_{n=1} E_n \cap E_1^c \cap E_2^c \cap E_3\Big) + \dots \\
&= \mu(E_1) + \mu\Big(\bigcup_{n=2}^{\infty} E_n \cap E_2\Big) + \mu\Big(\bigcup_{n=3}^{\infty} E_n \cap E_3\Big) + \dots \\
&= \sum_{i=1}^{\infty} \mu(E_i) \qquad \text{(J.-309)}
\end{aligned}$$

where we used pairwise disjointness in going from line 2 to line 3. This result means

$$\mu\Big(\bigcup_{i=1} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i) \qquad \text{(J.-309)}$$

because the converse inequality is true from (J.-307).

□

Theorem J.20.2 (Hopf Theorem)

Let $\mathcal{B}(\mathcal{F})$ is the smallest $\sigma-$algebra containing $\mathcal{F}$. A finitely additive measure μ on $\mathcal{F}$ can be extended to a $\sigma-$additive measure on $\mathcal{B}(\mathcal{F})$, if and only if for

$$\begin{aligned}
&F_i \in \mathcal{F} \;\; ; \;\; F_1 \supset F_2 \supset \cdots \supset F_n \supset \dots \\
&\lim_{n\to\infty} \mu(F_n) > 0 \implies \bigcap_{n=1}^{\infty} F_n \neq \emptyset \qquad \text{(J.-309)}
\end{aligned}$$

or as in [105] "if and only if for

$F_i \in \mathcal{F}$; $F_1 \supset F_2 \supset \cdots \supset F_n \supset \dots$ with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, we have

$$\lim_{n\to\infty} \mu(F_n) = 0. \qquad \text{(J.-309)}$$

Essentially, the condition (J.20.1) allows an extension $\tilde{\mu}$ to be consistently defined on elements of $\mathcal{B}(\mathcal{F})$ as limits of $\mu-$measures of sets in $\mathcal{F}$."

Proof of Theorem J.20.2:

The condition is necessary because if μ is extended to a $\sigma-$additive measure $\tilde{\mu}$ on $\mathcal{B}(\mathcal{F})$, we have

$$\lim_{n\to\infty} \mu(F_n) = \tilde{\mu}\big(\bigcap_{n=1}^{\infty} F_n\big). \tag{J.-309}$$

Conversely, assume the condition (J.20.2). Caratheodory's outer measure

$$\mu^*(E) = \inf\{\sum_{i=1}^{\infty} \mu(F_n) \mid E \subset \bigcup_{i=1}^{\infty} F_i\} \tag{J.-309}$$

Suppose $Y \subset X$ and $Y \subset \cup_{i=1}^{\infty} F_i$ are given. Since μ is finitely additive, we have

$$\sum_{i=1}^{\infty} \mu(F_i) = \sum_{i=1}^{\infty} \big(\mu(F_i \cap F) + \mu(F_i \cap F^c)\big) \geq \mu^*(Y \cap F) + \mu^*(Y \cap F^c) \tag{J.-309}$$

The infimum of a smaller set cannot be less than the infimum of a larger set, so $\mu(Y) \geq \inf \sum_{i=1}^{\infty} \mu(F_i)$. Taking the infimum of the left-hand side of (J.20.1) and noting that $\mu(Y) = \mu^*(Y)$, we have

$$\mu^*(Y) \geq \mu^*(Y \cap F) + \mu^*(Y \cap F^c) \tag{J.-309}$$

This means that F is μ^*-measurable.

Next, with $F_i = 0$ for $i > 1$ and setting $F_1 = F$ and $E = F$, it is clear that (J.20.1) implies $\mu^*(F) \leq \mu(F)$. We shall prove $\mu^*(F) \geq \mu(F)$, namely prove that

$$F \subset \bigcup_{i=1}^{\infty} F_i \implies \mu\,(F) \leq \sum_{i=1}^{\infty} \mu\,(F_i). \tag{J.-309}$$

since

$$\mu(F) = \inf \mu(F) \leq \inf \sum_{n=1}^{\infty} \mu(F_n) \tag{J.-309}$$

and (J.20.1) implies, (with $E = F$), $\mu(F) \leq \mu^*(F)$. Thus to this end we put

$$F'_n = \bigcap_{i=1}^{n} F_i^c \cap F.$$

Under the assumption of (J.20.1) then $\{F'_n\}$ is decreasing and $\bigcap_{n=1}^{\infty} F'_n = \emptyset$. From (J.20.2), we have $\mu(F'_n) \to 0$, therefore for all $\epsilon > 0$, there exists an n such that $\mu(\bigcap_{i=1}^{n} F_i^c \cap F) < \epsilon$. Since

$$\mu(F) = \mu(\bigcap_{i=1}^{n} F_i \cap F) + \mu(\bigcap_{i=1}^{n} F_i^c \cap F)$$

we get

$$\begin{aligned} \mu(F) - \epsilon &< \mu(\bigcup_{i=1}^{n} F_i \cap F) \\ &\leq \sum_{i=1}^{n} \mu(F_i \cap F) \leq \sum_{i=1}^{n} \mu(F_i) \\ &\leq \sum_{i=1}^{\infty} \mu(F_i) \end{aligned} \tag{J.-310}$$

This holds for every $\epsilon > 0$, so that we have (J.20.1).

□

Theorem J.20.3 *A $\sigma-$additive extension of a finitely additive measure μ on $\mathcal{F}$ is unique on $\mathcal{B}(\mathcal{F})$, if possible.*

Proof of Theorem J.20.3.

□

J.21 Regular Borel Measures on the Projective Limit: The Uniform Measure

Here we describe how we can, based on the Riesz representatioon theorem, how to construct $\sigma-$additive measures on the porjective limit $\overline{X}$ starting from a self-consistent family of (so-called cylindrical) maeusres μ_α on the various factor spaces X_α.

The spaces X_α are compact Hausdorff spaces and inparticular topological spaces and are therefore naturally equipped with the $\sigma-$algebra $\mathcal{B}_\alpha$ of Borel sets (recall, the smallest $\sigma-$algebra containing all open subsets of X_α).

J.22 Almost Periodic Functions and Bohr Compactification

$$\mathcal{N}_l(\overline{A}) = e^{il_jc} \tag{J.-310}$$

the space of square integrable fucntions on a suitable completion $\overline{\mathcal{A}}_S$ of the classical configuration space.

is given as the spectrum of the C$\star$-algebra of almost periodic functions on H.

Periodic Functions

Defintion A **trignometric polynomial** on T is an expression of the form

$$P \sim \sum_{n=-N}^{N} a_n e^{int}. \tag{J.-310}$$

The numbers n are called the frequencies of P. The largest integer n such that $|a_n| + |a_{-n}| \neq 0$ is called the **degree of** P.

$$P(t) = \sum_{n=-N}^{N} a_n e^{int}. \tag{J.-310}$$

Knowing the function P we can compute the coefficients a_n by the formula

$$a_n = \frac{1}{2\pi} \int P(t) e^{-int} dt \tag{J.-310}$$

which follows from

$$\frac{1}{2\pi} \int e^{i(m-n)t} dt = \delta_{mn} \tag{J.-310}$$

Definition Let f be a comlex-valued function on R and let $\epsilon > 0$. An almost periodic of f is a number τ such that

$$\sup_x |f(x-\tau) - f(x)| < \epsilon. \tag{J.-310}$$

Definition A function is almost periodic on $\mathbb{R}$ if it is continuous and if for every $\epsilon > 0$ there exists a number $\Sigma = \Sigma(\epsilon, f)$ such that every interval of length Σ on $\mathbb{R}$ is contains an $\epsilon-$almost period of f.

Examples

(1)

(2) $f = \cos x + \cos \pi x$

f is not continuous $f(x) = 2$ for $x = 0$ only.

(i) Every continuous a.p. function is bounded and uniformaly continuous on tha x-axis.

(ii) continuous periodic functions are also a.p. ,

(iii) The sums and products of a.p. functions and the limits of uniformaly convergent sequences of a.p. functions are also a.p.

J.22.1 Bohr Compactification

[157]

A function $f \in L^\infty(G)$ is, by definition, almost periodic if he set of all translates of f, $\{f_y\}_{y\in G}$ is precompact in the norm topology of $L^\infty(G)$.

As the dual of a discrete group, $\overline{G}$ it is compact; this is known as the *Borh compactification* of G.

A function f in sup norm L^∞ is, by definition, almost-periodic if the set of all translates of f, $\{f_y\}_{y\in G}$ precompact in the norm topology of L^∞. One proves that almost-periodic functions are uniformaly contiuous and are uniform limits of trignometric polynomials on G (i.e., of finite linear combinations of characters)

Bohr compacitification of the real line is the dual group of the discrete line and is usually called the Bohr group.

Assume $f \in AP(G)$; let $\{P_j\}$ be a sequence of trignometric polynomials which converges to f uniformally. Then, since G is dense in $\overline{G}$, $\{P_j\}$ converges uniformally on $\overline{G}$ (every character on G extends by continuity to a character on $\overline{G}$). It follows that f is the restriction to G of the $\lim P_i = F \in C(\overline{G})$. Conversely, since every continuous function F on $\overline{G}$ can be approximated uniformaly by trignometric polynomials, it follows that AP(G) is simply the restriction to G of $C(\overline{G})$.

J.22.2 Characters and the Dual Group

A *character* on a LCA group G is a continuous complex-valued function $\chi(x)$ on G with modulas 1 that preserves mulipilication, i.e.,

$$\chi(xy) = \chi(x)\chi(y) \text{ , with } |\chi(x)| = 1 \text{ for all } x, y \in G. \tag{J.-310}$$

That is, a character is a continuous homomorphism of G into the multiplicative group of complex numbers of modulus 1.

J.23 Introduction to Ergodic Theory

The original ergodic hypothesis concerned the long run average behavior of individual trajectories in the phase space Σ of a dynamical system. Such trajectories define a measure preserving flow φ

J.23.1 Motivation

For a measure $d\hat{\mu}$ to correspond to a physically interesting field theory, it has to satisfy (some version) of the Osterwalder-Schrader axioms []. These axioms guarantee that from the measure it is possible to construct the physical Hilbert space with a well defined Hamiltonian and Green's functions with appropriate proerties.

One of the most important of these axioms is the Euclidean invariance that requires that the measure $d\hat{\mu}$ be invariant under the action of theEuclidean group on $\mathbb{R}^{d+1}$.

J.23.2 Introduction

statistical mechanics concerning phase space. If a system of N particles is enclosed in a box, the state of the system is given by a point in 6-dimensional phase space with q_i representing coordinates and p_i reprsenting momenta.

An absractization of measure-preserving character of flow.; the basic problem of statistical mechanics is to study the asymptotic ptoperties of measure-preserving transformations.

The state of the system

a one parameter group of transformation, $T_{s+t} = T_s T_t$ Liouville's theorem the flow of phase space all volumes are invariant. In

Ergodic theory an absractization of such physical considerations.

So we would like to show that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(T_t x) dt \tag{J.-310}$$

exists, at least for continuous functions.

The map $f \mapsto f(x)$

(i) $\mu(1) = 1$

(ii) μ is linear

(iii) $\mu(f) \geq 0$ if $f \geq 0$.

Such a μ is always associated with a measure $\hat{\mu}$ on Ω_E with $\hat{\mu}(\Omega_E) = 1$, so that

$$\mu(f) = \int_{\Omega_E} f(w) d\hat{\mu}(w) \tag{J.-310}$$

Lioville's theorem in classical mechanics.

We have an isolated system of N particles whoes state is specified by N generalized coordinates and momentum $\{q_1, \cdots, q_N; p_1, \cdots, p_N\}$ in phase space.

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{J.-310}$$

$$\rho \dot{q}_1 \; dt dq_2 \cdots dp_N - \left[\rho \dot{q}_1 + \frac{\partial(\rho \dot{q}_1)}{\partial q_1} dq_1\right] dt dq_2 \cdots dp_N = -\frac{\partial}{\partial q_1}(\rho \dot{q}_1) dt \; dq_1 dq_2 \cdots \tag{J.-310}$$

$$\frac{\partial \rho}{\partial t} dt dq_1 \cdots dp_N = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial q_i}(\rho \dot{q}_i) - \sum_{i=1}^{N} \frac{\partial}{\partial p_i}(\rho \dot{p}_i)\right] dt \; dq_1 \cdots dp_N \tag{J.-310}$$

$$\mu_E(F) := \int_F \delta(\mathcal{H}(p,q) - E) d^{3N} p d^{3N} q \tag{J.-310}$$

If we pich a set of local coordinates at $x \in \Omega_E$, say $\mathcal{Q}_1, \ldots, \mathcal{Q}_{6N-1}$, which are orthogonal and normalized, then

$$d\mu_E = C d^{6N-1} \mathcal{Q} / |\text{grad } \mathcal{H}| \tag{J.-310}$$

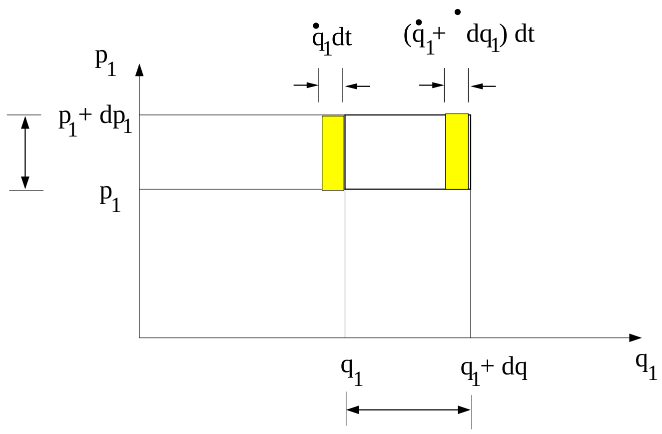


Figure J.29: Surface corresponding to first order term.

J.23.3 An Ergodic Theorem

we want to study

$$\frac{1}{T}\int_0^T (U_t f)(w) d\mu(w) \tag{J.-310}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} U^n f \tag{J.-310}$$

statistical ergodic theorom of J. Von Neumann concerning motions $P \rightarrow P_t$ of a measurably set Ω onto itself which preserve the measure. The "discrete" counterpart of this theorem concerns iterates

$\int_\Omega f(P)d\sigma = \int_\Omega f(P')d\sigma$

$$\psi_{m,n} = \frac{1}{n-m}\sum_{k=m}^{n-1} f(P^{(k)}) \tag{J.-310}$$

which tend in the mean, when $n - m \rightarrow \infty$.

Theorem (the mean ergodic theorem, or von Neumann's ergodic theorem) Let U be a unitary operator on a Hilbert space $\mathcal{H}$.

$$\lim_{N\rightarrow\infty}\frac{1}{N}\sum_{n=0}^{N-1} U^n f = Pf \tag{J.-310}$$

in justifying the statement that phase-space average and time averages are equal.

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where μ is a finite measure, and U is given by $Ux = x \circ f$ for some measure-preserving map f. A map f is said to be *measure-preserving* if it is measurable and $\mu(f^{-1}(B)) = \mu(B)$ for all measurable sets $B \subset \Omega$. In other words the transport μ_f of the measure μ via f is the same as μ. To prove that such a U is indeed unitary we calculate as follows:

$$(Ux, Uy) = \int_\Sigma x(f(\omega))y(f(\omega))\, d\mu(\omega) = \int_\Omega xy\, d\mu_f = \int_\Omega xy\, d\mu = (x, y). \tag{J.-310}$$

Theorem J.23.1 *(von Neumann mean ergodic theorem). Let $\mathbb{R} \to G$; $t \mapsto g_t$ be a one-parameter group and $\hat{U} : G \to \mathcal{B}(L_2(X, d\mu))$ be a unit representaation of G. Let $\hat{P}$ be the projection on the closure of the set of a.e. invariant vectors under $\hat{U}(g_t)$, $t \in \mathbb{R}$. Then*

$$(\hat{P}f)(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt(\hat{U}(g_t)f)(x) \quad \mu - a.e. \tag{J.-310}$$

J.24 The Infinite Tensor Product

We first consider the tensor product of a finite number of Hilbrt spaces. Say $f_k, g_k \in \mathcal{H}_k$ with inner product $(\cdot,\cdot)_k$ on $\mathcal{H}_k$. If an element of $\otimes_k \mathcal{H}_k$ is f, the inner product of the tensor product is defined as

$$(f, g) = \prod_{k=1}^{n} (f_k, g_k)_k$$

and the norm

$$\|f\| = \sqrt{\prod_{k=1}^{n} (f_k, f_k)_k} = \sqrt{\prod_{k=1}^{n} \|f_k\|_k^2} = \prod_{k=1}^{n} \|f_k\|_k.$$

We are lead to the consideration of the mathematics of products of arbitrary complex numbers.

Now when one forms the infinite tensor product of a collection of Hilbert spaces, a physical requirement is that this product must not depened on the order of the individual Hilbert spaces (whether the collection is countably or uncountably infinite).

Hence we are interested in the convergence properties of a countable or uncountable product of complex numbers which are independent of the ordering of the product. This was developed in the paper by von Neumann (availble at http://www.numdam.org/item?id=1939 0).

As we will see, convergence of products is related to convergence of corresponding summations. Now, it is a remarkable fact that whether an infinite series converges or not can depend on the ordering of the terms of that series. From which it follows that whether or not the coresponding product of complex numbers converges depends on the ordering of the terms of that product.

Absolutely and conditionally convergent series have completely different behaviours under rearrangement

Theorem J.24.1 *(***Riemann's Rearrangement Theorem***) Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. For each real number s, there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges and has sum s.*

Proof:

The nonnegative series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} (-q_n)$ diverge. In fact, if both were to converge, it would follow that $\sum_{n=1}^{\infty} |a_n|$ converges, that is, $\sum_{n=1}^{\infty} a_n$ would be absolutely convergent.

On the other hand, if one of these series converged and the other diverged, it would follow that the partial sums of $\sum_{n=1}^{\infty} a_n$ diverge to either $+\infty$ or to $-\infty$. The convergence of $\sum_{n=1}^{\infty} a_n$ itself implies that both $\{p_n\}$ and $\{q_n\}$ have limit zero.

Now we construct the rearrangement. Choose terms $p_1, p_2, \ldots$ up to the first index k_1 such that

$$p_1 + p_2 + \cdots + p_{k_1} > s.$$

This will occur , because $\sum_{n=1}^{\infty}$. Note

$$|p_1 + p_2 + \cdots + p_{k_1} - s| < p_{k_1}.$$

Next, we choose $q_1, q_2, \ldots$ up to the first index l_1

such that

$$(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) < s.$$

Note

$$|(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) - s| < \max\{p_{k_1}, |q_{l_1}|\}.$$

We then add just enough new p's to make the left hand side greater than s, followed by just enough q's to make it less than s, and continue. At each phase of the $2n-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_n}, |q_{l_n}|\}$, and at each phase of the $2n+1-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_{n+1}}, |q_{l_n}|\}$. As these have the limit 0, the rearranged series has sum s.

□

Corollary J.24.2 *Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. It has a rearrangement that diverges to $+\infty$.*

Proof:

□

Theorem J.24.3 *Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement. Then $\sum_{n=1}^{\infty} b_n$ converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.*

Proof: Let $\{s_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$, i.e,

$$s_n = \sum_{k=1}^{n} a_k,$$

and let s be the limit. Let $\{t_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} b_k$. Given $\epsilon > 0$, choose M so large that

$$\sum_{k=M+1}^{\infty} |a_k| < \epsilon/2. \tag{J.-310}$$

It follows from this that $|s - s_M| < \epsilon/2$,

$$|s - s_M| = |\sum_{k=M+1}^{\infty} a_k| \leq \sum_{k=M+1}^{\infty} |a_k| < \epsilon/2.$$

Choose N so large that every one of the first M terms of $\{a_k\}$ occurs among the first N terms of $\{b_k\}$. So that for any $n \geq N$

$$|t_n - s_M| = |t_n - \sum_{k=1}^{M} a_k| \leq \sum_{k=M+1}^{\infty} |a_k|.$$

Therefore

$$|t_n - s| \leq |t_n - s_M| + |s_M - s| < \epsilon/2 + \epsilon/2 = \epsilon.$$

$\square$

Theorem J.24.4 *Any bounded nondecreasing sequence of reals if convergent.*

Proof:

Let a be the least upper bound of the set of numbers $\{a_1, a_2, \dots\}$. If ϵ is positive, then the number $a - \epsilon$ is less than a and therefore not an upper bound for this set. As such, there is some N such that $a_N > a - \epsilon$. This, and the inequalities

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n, \qquad \text{all } n,$$

and the fact that a is an upper bound imply that for any integer $n \geq N$

$$a - \epsilon < a_N \leq a_n \leq a.$$

Therefore $n \geq N$ implies $|a_n - a| < \epsilon$, and a is the limit.

$\square$

Theorem J.24.5 *If $b_n \geq 0$ for all n, then $\sum_{n=1}^{\infty} b_n$ converges if and only if the sequence of partial sums is a bounded sequence.*

Proof:

$\square$

J.24.1 Infinite Products of Complex Numbers

A sequence of numbers $a_1, a_2, a_3, \ldots$ the infinite product

$$\prod_{n=1}^{\infty} a_n = a_1 \; a_2 \; a_3 \cdots$$

is defined to be the limit of the partial products $a_1 a_2 \ldots a_n$ as $n \to \infty$. That is, let P_n be the partial product

$$P_n = \prod_{k=1}^{n} a_k$$

then

$$\prod_{n=1}^{\infty} a_n := \lim_{n \to \infty} P_n.$$

The product is said to converge when the limit exists. If the product converges, then the limit of the sequence a_n as $n \to \infty$ must be 1. Proof is easy. Assume that an infinite product $\prod a_n$ is convergent. Then for each a_n of the series we have $a_n = P_n/P_{n-1}$. Since the product is convergent, there exists a such that as $n \to \infty$, $P_n = a$ and $P_{n-1} = a$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} P_n/P_{n-1} = a/a = 1.$$

The contrary to this is in general not true. Therefore, the logarithm a_n will be defined for all but a finite number of n, and for those we have ($a_n \neq 0$)

$$\ln \prod_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln a_n$$

with the product on the left converging if and only if the sum on the right converges. This allows the translation of convergence criteria for infinite sums into convergence criteria for infinite products.

Lemma J.24.6 *By the defintion of convergence , if $\prod_n^\infty a_n$, $\prod_n^\infty a'_n$ converge to a, a' respectively then $\prod_n^\infty a_n a'_n$ converges to aa'.*

Proof: Note that if $\{C_n\}$ and $\{C'_n\}$ are convergent sequences, converging to C, C' respectively, then

$$\lim_{n\to\infty} (C_n + C'_n) = \lim_{n\to\infty} C_n + \lim_{n\to\infty} C'_n$$

as

$$\begin{aligned} |(C_n + C'_n) - (C + C')| &= |(C_n - C) + (C'_n - C')| \\ &\leq |(C_n - C)| + |(C'_n - C')| \end{aligned}$$

and because we can always choose N large enough so that $|(C_n - C)| < \epsilon/2$ and $|(C'_n - C')| < \epsilon/2$ for all $n \geq N$.

Therefore, with $C_n = \sum_k^n \ln a_k$, $C'_n = \sum_k^n \ln a'_k$, we have

$$\begin{aligned} \lim_{n\to\infty} \sum_k^n (\ln a_k + \ln a'_k) &= \lim_{n\to\infty} (\sum_k^n \ln a_k + \sum_k^n \ln a'_k) \\ &= \lim_{n\to\infty} \sum_k^n \ln a_k + \lim_{n\to\infty} \sum_k^n \ln a'_k \end{aligned}$$

We can the use this to prove the lemma,

$$
\begin{aligned}
\ln \prod_{n}^{\infty}(a_n a_n') &= \sum_{n}^{\infty} \ln(a_n a_n') \\
&= \sum_{n}^{\infty}(\ln a_n + \ln a_n') \\
&= \sum_{n}^{\infty} \ln a_n + \sum_{n}^{\infty} \ln a_n' \\
&= \ln a + \ln a' \\
&= \ln(aa').
\end{aligned}
$$

□

A criterion for a product to converge.

Theorem J.24.7 *For $\rho_k > 0$, if $\prod_{n=1}^{\infty} |\rho_n - 1|$ converges then $\prod_{n=1}^{\infty} \rho_n$ converges.*

Proof: Set

$$
P_n = \prod_{k=1}^{n}(1 + a_k), \qquad \tilde{P}_n = \prod_{k=1}^{n}(1 + |a_k|)
$$

Note that

$$
1 + \sum_{k=1}^{n} |a_k| \leq \tilde{P}_n \leq \exp(\sum_{k=1}^{n} |a_k|).
$$

If the summation $\sum_{n=1}^{\infty} |a_n|$ converges then the sequence $\{B_n\}$ defined by $B_n := \exp(\sum_{k=1}^{n} |a_k|)$ is a bounded and a nondecreasing sequence and so converges. This implies that $\tilde{P}_n$ converges if the summation $\sum_{n=1}^{\infty} |a_n|$ convergences.

If $P_n - 1 < 0$ then, expanding the product, we see

$$
-(\tilde{P}_n - 1) < P_n - 1
$$

and if $P_n - 1 \geq 0$ then

$$
P_n - 1 \leq (\tilde{P}_n - 1),
$$

therefore

$$|P_n - 1| \leq \tilde{P}_n - 1$$

Hence if the product

$$\prod_{n=1}^{\infty}(1 + |a_n|)$$

converges, then so does

$$\prod_{n=1}^{\infty}(1 + a_n).$$

If $\sum_{n=1}^{\infty} |a_n| < \infty$ then

$$\prod_{n=1}^{\infty}(1 + a_n)$$

converges. Putting $\rho_n = 1 + a_n$ ($\rho_n > 0$) we have $\sum_{n=1}^{\infty} |\rho_n - 1|$ converges then $\prod_{n=1}^{\infty} \rho_n$ converges.

□

In fact the implication can be reversed. To prove this we need to first establish a couple of lemmas.

Lemma J.24.8 *For $r_n > 0$, $\sum_{n=0}^{\infty} r_n$ converges if and only if $\prod_{n=0}^{\infty}(1 + r_n)$ converges.*

Proof:

From part of the proof given in the previous theorem we have that $\prod_{n=0}^{\infty}(1+r_n)$ converges if $\sum_{n=0}^{\infty} r_n$ converges, noting that $r_n > 0$.

□

Lemma J.24.9 *If $0 < r_n < 1$, then $\sum_{n=0}^{\infty} r_n$ converges if and only if $\prod_{n=0}^{\infty}(1 - r_n)$ converges.*

Proof: Note the product obviously is finite, it only diverges if it equals zero. First say the product $\prod_n^\infty r_n$ converges. As $0 < x <$ then

$$\ln(1-x) = -x - x^2/2 - \cdots < -x,$$

so

$$x < -\ln(1-x).$$

This implies

$$\sum_n^\infty r_n < -\sum_n^\infty \ln(1-r_n) = \ln\left(1/\prod_{n=0}^\infty (1-r_n)\right) < \infty,$$

i.e., the summation converges.

Now say the summation $\sum_{n=0}^\infty r_n$ converges. Then there exists finite N such that $r_n < \delta$ for all $n \geq N$. Since $\ln(1-x)$ is convex, it follows that for $x < \delta$

$$\ln(1-x) > -kx$$

where $k = -(\ln(1-\delta))/\delta$. To see this more clearly draw the function $\ln(1-x)$. Replacing x by r_n, summing over n and then exponentiating gives

$$\prod_{n=N}^\infty (1-r_n) > \exp(-k\sum_{n=N}^\infty r_n) > 0,$$

i.e., the product is non-zero, and so by definition convergent.

□

Theorem J.24.10 *For $\rho_n > 0$, $\sum_{n=0}^\infty |\rho_n - 1|$ converges if and only if $\prod_{n=0}^\infty \rho_n$ converges.*

Proof: Consider the product

$$\prod_{n=0}^\infty \rho_n = \prod_{n=0}^\infty [(1 + (\rho_n - 1)]$$

for $\rho_n > 1$. It follows from lemma (J.24.8) that $\sum_{n=0}^{\infty} |\rho_n - 1|$ converges if and only if $\prod_{n=0}^{\infty}(1 + |\rho_n - 1|)$ converges. This proves the theorem for the case of $\rho_n > 1$ for all n.

Consider the product

$$\prod_{n=0}^{\infty} \rho_n = \prod_{n=0}^{\infty}[(1 - (1 - \rho_n)]$$

for $0 < \rho < 1$. It follows from lemma (J.24.9), by replacing r_n by $1 - \rho_n$, the theorem is proved for the case of $\rho_n < 1$ for all n.

Now we consider the general case. We factor the partial product

$$P_N = \prod_{n=1}^{N} \rho_n$$

into the product of terms for which $0 < \rho < 1$ and into the product of terms for which $\rho > 1$, which we denote respectively as Q_N and R_N (obviously we can ignore terms for which $\rho_n = 1$). Then

$$P_N = Q_N \cdot R_N.$$

We then use the fact that if $\prod_n^{\infty} \rho_n$, $\prod_n^{\infty} \rho'_n$ converge to ρ, ρ' respectively then $\prod_n^{\infty} \rho_n \, \rho'_n$ converges to $\rho\rho'$ to complete the proof of the theorem.

□

Now we consider products of complex numbers.

Definition The infinite product of complex numbers

$$\prod_{n=0}^{\infty} z_n \tag{J.-319}$$

is said to **converge** to the number z provided that for each positive number $\delta > 0$ there exists $N < \infty$ such that for all $n \geq N$

$$|z - \prod_{k=0}^{n} z_k| < \delta.$$

□

Lemma J.24.11 *If z_n are arbitrary complex numbers, then $\sum_n^\infty z_n$ converges if and only if $\sum_n^\infty |z_n|$ converges.*

Proof: Convergence of $\sum_n^\infty z_n$ is equivalent to the convegrence of both $\sum_n^\infty \mathrm{Re} z_n$ and $\sum_n^\infty \mathrm{Im} z_n$. Similarly, as

$$|\mathrm{Re} z_n|,\ |\mathrm{Im} z_n| \leq |z_n| \leq |\mathrm{Re} z_n| + |\mathrm{Im} z_n|.$$

the convergence of $\sum_n^\infty |z_n|$ is equivalent to the convegrence of both $\sum_n^\infty |\mathrm{Re} z_n|$ and $\sum_n^\infty |\mathrm{Im} z_n|$.

Sufficiency. As we have for real series

$$\left| \sum_k^n a_k \right| \leq \sum_k^n |a_k|,$$

$|\sum_k^n a_k|$ will be bounded as $\sum_k^n |a_k|$ is bounded by hypothesis, this proves sufficiency.

Now we prove necessity. Suppose $\sum_n^\infty z_n$ converges, and let a be its value.

□

in general it is not guatanteed we can write

$$\lim_{n\to\infty} (\prod_k^n |z_k|\ e^{i\varphi_k}) = \lim_{n\to\infty} \prod_k^n |z_k|\ \lim_{n\to\infty} \prod_k^n e^{i\varphi_k}$$

However we have: if $\{\xi_n\}$ and $\{\zeta_n\}$ are two complex convergent sequences with limits a and b, then

$$\lim_{n\to\infty} (\xi_n \zeta_n) = \xi\zeta$$

Proof:

$$\begin{aligned} |\xi_n\zeta_n - \xi\zeta| &= |(\xi_n\zeta_n - \xi\zeta_n) + (\xi\zeta_n - \xi\zeta)| \\ &\leq |(\xi_n\zeta_n - \xi\zeta_n)| + |(\xi\zeta_n - \xi\zeta)| \\ &= |\xi_n - \xi|\ |\zeta_n| + |\xi|\ |\zeta_n - \zeta| \end{aligned}$$

For $n \geq$ to some N, $|\zeta_n| = |\zeta + (\zeta_n - \zeta)| \leq |\zeta| + |(\zeta_n - \zeta)| \leq |\zeta| + 1$. Given $\epsilon > 0$ we can choose N so large that $n \geq N$ implies $|\zeta_n| \leq |\zeta| + 1$ and also

$$|\xi_n - \xi| < \frac{\epsilon}{2|\zeta| + 2}, \qquad |\zeta_n - \zeta| < \frac{\epsilon}{2|\xi|}.$$

It follows that $n \geq N$ implies $|\xi_n \zeta_n - \xi\zeta| < \epsilon$.

Theorem J.24.12 *Let* $z_n = \rho_n e^{i\varphi_n} \in \mathbb{C}$ *where* $\rho_n = |z_n|$, $\varphi_n \in [-\pi, \pi]$. *Then* $\prod_\alpha z_n$ *converges if and only if*

i) either $\prod_n \rho_n$ *converges to zero in which case* $\prod_n z_n = 0$, *prod ii) or* $\prod_n \rho_n$ *converges to* $\rho > 0$ *and* $\sum_n |\varphi_n|$ *converges in which case*

$$\prod_n z_n = \rho e^{i \sum_n \varphi_n}.$$

Proof: the partial product can be written

$$P_n = \prod_{k=1}^{n} a_k = (\prod_{k=1}^{n} \rho_k)\ e^{i\sum_{k=1}^{n} \varphi_k}$$

$$\lim_{n\to\infty} e^{i\sum_{k=1}^{n} \varphi_k}$$

$$|z - e^{i\sum_{k=1}^{n} \varphi_k}| < \delta.$$

We require

$$\lim_{n\to\infty} P_n / P_{n-1} = 1$$

that is

$$\varphi_n \to 0$$

We show that when $\prod_n^\infty |z_n| > 0$, then it implies the convergence of $\sum_n^\infty |\varphi_n|$ also.

As $\prod_n^\infty z_n$, $\prod_n^\infty |z_n|$ converge, with the latter > 0

$$\prod_n^\infty \frac{z_n}{|z_n|} = \prod_n^\infty e^{i\varphi_n}$$

converges also.

□

Corollary J.24.13 *Quasi-convergence of $\prod_n z_n$ to a non-vanishing value $\neq 0$ is equivalent with convergence with such a value (this is by definition of quasi-convergence). It holds if and only if all $z_n \neq 0$, and $\sum_n^\infty |z_n - 1|$*

Proof: As it is quasi-convergent we have that $\sum_n^\infty |\,|z_n| - 1|$

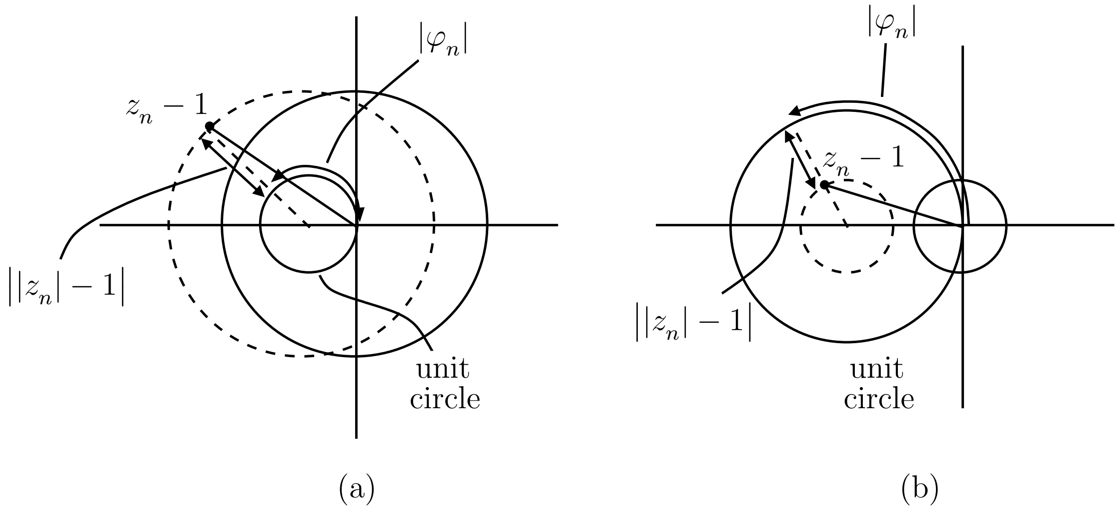

Figure J.30: Proof of the inequalities $\big||z_n| - 1\big| \leq |z_n - 1| \leq \big||z_n| - 1\big| + |\varphi_n|$. (a) Case where $|z_n| > 1$. (b) Case where $|z_n| < 1$.

$$\big||z_n| - 1\big|,\ \frac{1}{\pi}|\varphi_n| \leq |z_n - 1| \leq \big||z_n| - 1\big| + |\varphi_n|$$

Hence the convergence of the sum $\sum_n^\infty |z_n - 1|$ is equivalent to the convergence of both $\sum_n^\infty \big||z_n| - 1\big|$ and $\sum_n^\infty |\varphi_n|$.

Lemma J.24.14 *By the defintion of convergence, if*

1) $\prod_n^\infty z_n$, $\prod_n^\infty z'_n$ converge to z, z' respectively then $\prod_n z_n z'_n$ converges to zz',

2) if $\prod_n^\infty z_n$ converges to z, then $\prod_n^\infty z_n^*$ converges to z^*.

Proof: 1)

$$
\begin{aligned}
\ln\prod_n^{\infty}(z_n z_n') &= \sum_n^{\infty}(\ln\rho_n + \ln\rho_n' + i\varphi_n + i\varphi_n') \\
&= \sum_n^{\infty}(\ln\rho_n + i\varphi_n) + \sum_n^{\infty}(\ln\rho_n' + i\varphi_n') \\
&= \ln\prod_n^{\infty} z_n + \ln\prod_n^{\infty} z_n' \\
&= \ln(zz')
\end{aligned}
$$

□

Lemma J.24.15 *For any complex numbers, the convergence of one of* $\sum_{n=0}^{\infty} |\,|z_n| - 1|$, $\sum_{n=0}^{\infty} |\,|z_n|^2 - 1|$ *implies the convergence of the other.*

Proof: If one or other of the series converges, then $\lim_{n\to\infty} |z_n| = 1$. So

$$
\begin{aligned}
\lim_{n\to\infty} \frac{|\,|z_n|^2 - 1|}{|\,|z_n| - 1|} &= \lim_{n\to\infty} |\,|z_n| + 1| \\
&= 2 > 0.
\end{aligned}
$$

Hence there exists finite N such that

$$
1 < \frac{|\,|z_n|^2 - 1|}{|\,|z_n| - 1|} < 3 \qquad \text{for all } n \geq N.
$$

Equivalently

$$
|\,|z_n| - 1| < |\,|z_n|^2 - 1| < 3|\,|z_n| - 1| \qquad \text{for all } n \geq N.
$$

If the series $\sum_{n=0}^{\infty} |\,|z_n|^2 - 1|$ converges then $\sum_{n=0}^{\infty} |\,|z_n| - 1|$ also converges as

$$
\sum_{n=N}^{\infty} |\,|z_n| - 1| < \sum_{n=N}^{\infty} |\,|z_n|^2 - 1|.
$$

If the series $\sum_{n=0}^{\infty} |\,|z_n| - 1|$ converges then $\sum_{n=0}^{\infty} |\,|z_n|^2 - 1|$ also converges as

$$\sum_{n=N}^{\infty} |\, |z_n|^2 - 1| < \sum_{n=N}^{\infty} 3|\, |z_n| - 1|.$$

The result also follows by applying theorem J.24.10: $\sum_{n=0}^{\infty} |\, |z_n| - 1|$ converges if and only if $\prod_{n=0}^{\infty} |z_n|$ converges, which converges if and only if $\prod_{n=0}^{\infty} |z_n|^2$ converges, which in turn converges if and only if $\sum_{n=0}^{\infty} |\, |z_n|^2 - 1|$ converges.

□

It is understood that the sum is meaniful only if at most a countable number of terms are different from zero.

Recall that for a series $\sum_\alpha z_\alpha$ to converge absolutely it is necessary that $z_\alpha = 0$ for all but countably infinitely many $\alpha \in \mathcal{I}$.

Definition Let $\{z_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of complex numbers. The infinite product

$$\prod_{\alpha \in \mathcal{I}} z_\alpha \tag{J.-328}$$

is said to **converge** to the number z provided that for each positive number $\delta > 0$ there exists a finite set

$$\mathcal{I}_0 \subset \mathcal{I}$$

such that for any other finite $\mathcal{J}$ with

$$\mathcal{I}_0(\delta) \subset \mathcal{J} \subset \mathcal{I}$$

it holds that

$$|z - \prod_{\alpha \in \mathcal{J}} z_\alpha| < \delta.$$

□

In contrast to the case of an infinite series, absolute converge of an infinite product does not imply converge, the phases of the factors could fluctuate wildly. This motivates the following definition.

Definition We say that $\prod_{\alpha\in\mathcal{I}} z_\alpha$ is **quasi-convergent** if $\prod_{\alpha\in\mathcal{I}} |z_\alpha|$ converges. If $\prod_{\alpha\in\mathcal{I}} z_\alpha$ is quasi-convergent but not convergent we define $\prod_{\alpha\in\mathcal{I}} z_\alpha := 0$.

□

This definition assigns a value to the infinite product of numbers which converge absolutely but not necessarily non-absolutely.

Theorem J.24.16 *1) Let $\rho_\alpha \geq 0$.*

i) If for every $\alpha_0 \in \mathcal{I}$ it holds that $\rho_{\alpha_0} = 0$ then $\prod_\alpha \rho_\alpha = 0$.

ii) If for $\rho_\alpha > 0$ for all α then $\prod_\alpha \rho_\alpha$ converges if and only if

$$\sum_\alpha \max(\rho_\alpha - 1, 0)$$

converges.

iii) If for $\rho_\alpha > 0$ for all α then $\prod_\alpha \rho_\alpha$ converges if and only if

$$\sum_\alpha |\rho_\alpha - 1|$$

converges.

2) Let $z_\alpha = \rho_\alpha e^{i\varphi_\alpha} \in \mathbb{C}$ where $\rho_\alpha = |z_\alpha|$, $\varphi_\alpha \in [-\pi, \pi]$. Then $\prod_\alpha z_\alpha$ converges if and only if

i) either $\prod_\alpha \rho_\alpha$ converges to zero in which case $\prod_\alpha z_\alpha = 0$,

ii) or $\prod_\alpha \rho_\alpha$ converges to $\rho > 0$ and $\sum_\alpha |\varphi_\alpha|$ converges in which case

$$\prod_\alpha z_\alpha = \rho e^{i \sum_\alpha \varphi_\alpha}.$$

Proof:

□

Corollary J.24.17 *Quasi-convergence*

Proof:

$\square$

After having introduced convergence for infinite products of complex numbers we can now turn to ITP Hilbert spaces.

J.24.2 $C-$vectors of the ITP

Definition Let $\mathcal{H}_\alpha$, $\alpha \in \mathcal{I}$ be an arbitrary collection of Hilbert spaces. For a sequence $f := \{f_\alpha\}_{\alpha\in\mathcal{I}}$, $f_\alpha \in \mathcal{H}_\alpha$ the object

$$\otimes_f := \otimes_\alpha f_\alpha \tag{J.-328}$$

is called a $C-$**vector** provided that $\prod_\alpha \|f_\alpha\|_\alpha$ converges, where $\|\cdot\|_\alpha$ denotes the Hilbert norm of $\mathcal{H}_\alpha$. The set of vectors will be denoted V_C.

$\square$

Lemma J.24.18 *For two $C-$vectors $\otimes_f = \otimes_\alpha f_\alpha$, $\otimes_g = \otimes_\alpha g_\alpha$ the inner product*

$$< \otimes_f, \otimes_g > := \prod_\alpha < f_\alpha, g_\alpha >_\alpha \tag{J.-328}$$

is a quasi-convergent product of the individual inner products $< f_\alpha, g_\alpha >$ on $\mathcal{H}_\alpha$.

Proof: We prove that if $f_\alpha, g_\alpha \in \mathcal{H}_\alpha$ for all $\alpha \in \mathcal{I}$, and if $\prod_\alpha \|f_\alpha\|_\alpha$, $\prod_\alpha \|g_\alpha\|_\alpha$ are convergent, then so is $\prod_a |(f_\alpha, g_\alpha)_\alpha|$, that is, $\prod_a (f_\alpha, g_\alpha)_\alpha$ is quasi-convergent.

Since $\|f_\alpha\|_\alpha = 0$ or $\|g_\alpha\|_\alpha = 0$ implies $(f_\alpha, g_\alpha)_\alpha = 0$

by theorem J.24.16 we need only show that $\sum_\alpha \max(|(f_\alpha, g_\alpha)_\alpha| - 1, 0)$ converges as a consequence of the convergence of $\sum_\alpha \max(\|f_\alpha\|_\alpha^2 - 1, 0)$, $\sum_\alpha \max(\|g_\alpha\|_\alpha^2 - 1, 0)$.

Now as $|(f_\alpha, g_\alpha)_\alpha| \leq \frac{1}{2}\|f_\alpha\|_\alpha^2 + \frac{1}{2}\|g_\alpha\|_\alpha^2$,

$$|(f_\alpha, g_\alpha)_\alpha| - 1 \leq \frac{1}{2}(\|f_\alpha\|_\alpha^2 - 1) + \frac{1}{2}(\|g_\alpha\|_\alpha^2 - 1)$$

and hence

$$\max(|(f_\alpha, g_\alpha)_\alpha| - 1, 0) \leq \frac{1}{2}\max(\|f_\alpha\|^2_\alpha - 1, 0) + \frac{1}{2}\max(\|g_\alpha\|^2_\alpha - 1, 0).$$

$\square$

There are $C-$vectors $\otimes_f$ such that $\prod_\alpha \|f_\alpha\|_\alpha = 0$ although $\|f_\alpha\|_\alpha > 0$ for all $\alpha \in \mathcal{I}$. Thus, it is conceivable that it happens that $< \otimes_f, \otimes_g > \neq 0$ for some $C-$vector $\otimes_g$. If that were the case, the Schwarz inequality is violated for the inner product (J.24.18) for $C-$vectors.

Definition We say that for $f_\alpha \in \mathcal{H}_\alpha$ the ITP $\otimes_f := \otimes_\alpha f_\alpha$ is a C_0-**vector** (and $f = \{f_\alpha\}$ a C_0 sequence) if $\| \otimes_f \| := \prod_{\alpha\in\mathcal{I}} \|f_\alpha\|_\alpha$ converges to a non-vanishing number. The set of C_0-vectors will be denoted V_0.

$\square$

To distinguish trivial $C-$vectors from non-trivial ones we define

Definition A sequence $\{f_\alpha\}$ defines a C_0-**vector** $\otimes_f = \otimes_\alpha f_\alpha$ if and only if

$$\sum_\alpha | \; \|f_\alpha\|_\alpha - 1| \tag{J.-328}$$

converges. The set of C_0-vectors will be denoted V_0.

$\square$

Lemma J.24.19 *For any complex numbers,* $\sum_\alpha | \; |z_\alpha| - 1|$ *converges if and only if* $\sum_\alpha | \; |z_\alpha|^2 - 1|$.

Proof:

$\square$

For a C_0-vector $\|f_\alpha\|_\alpha \neq 0$.

The norm of a C_0-vector does not zanish. By the previous lemma C_0-vector can be defined equivalently as a $\otimes_f = \otimes_\alpha f_\alpha$ such that

$$\sum_\alpha | \; \|f_\alpha\|^2_\alpha - 1|$$

converges. By theorem J.24.16 this implies that

$$\prod_\alpha \|f_\alpha\|_\alpha^2 = \ (\otimes_f, \otimes_f) \ = \| \otimes_f \|^2$$

converges to a non-zero number, i.e. $\| \otimes_f \| > 0$.

J.24.3 Strong Equivalence Classes

Definition We will call two C_0-sequences $f = \{f_\alpha\}, g = \{g_\alpha\}$ **strongly equivalent**, denoted $f \approx g$, provided that

$$\sum_\alpha |(f_\alpha, g_\alpha)_\alpha - 1| \tag{J.-328}$$

converges.

□

Lemma J.24.20 *Strong equivalence of C_0-sequences is an equivalence relation, i.e., a relation such that*

1) (Reflexivity) for any $\otimes_f \in V_0$, we have $\otimes_f \approx \otimes_f$.

2) (Symmetry) for all $\otimes_f, \otimes_g \in V_0$, if $\otimes_f \approx \otimes_g$ then $\otimes_g \approx \otimes_f$.

3) (Transitivity) for all $\otimes_f, \otimes_g, \otimes_h \in V_0$, if $\otimes_f \approx \otimes_g$ and $\otimes_g \approx \otimes_h$ then $\otimes_g \approx \otimes_h$.

Proof: Condition 1) follows from the fact that a C_0-vector is defined by the condition that $\sum_\alpha | \ \|f_\alpha\|_\alpha^2 - 1|$ converges.

$$|(f_\alpha, g_\alpha)_\alpha - 1| = |\overline{(f_\alpha, g_\alpha)_\alpha} - 1| = |\overline{(f_\alpha, g_\alpha)_\alpha - 1}| = |(g_\alpha, f_\alpha)_\alpha - 1|.$$

Condition 3) We prove that, apart from a possible finite number of exceptions, the following

$$|(f_\alpha, h_\alpha)_\alpha - 1| \leq D \left(\left| \|f_\alpha\|_\alpha - 1 \right| + \left| \|g_\alpha\|_\alpha - 1 \right| + |(f_\alpha, g_\alpha)_\alpha - 1| + |(g_\alpha, h_\alpha)_\alpha - 1| \right)$$

holds for some constant D.

We put

$$\|f_\alpha\|_\alpha = 1+\lambda_f, \quad \|g_\alpha\|_\alpha = 1+\lambda_g, \quad \|h_\alpha\|_\alpha = 1+\lambda_h,$$
$$(f_\alpha, g_\alpha) = 1+\lambda_{fg}, \quad (g_\alpha, h_\alpha) = 1+\lambda_{gh}$$

So $|\lambda_f|$, $|\lambda_g|$, $|\lambda_h|$, $|\lambda_{fg}|$, $|\lambda_{gh}| \leq C$, and except for a finite number of α's, $|\lambda_g| \leq \frac{1}{2}$ (or $-1/2 \leq \|g_\alpha - 1\|_\alpha \leq 1/2$).

We express $g_\alpha, f_\alpha, h_\alpha \in \mathcal{H}_\alpha$ as:

$$\begin{aligned} g_\alpha &= a_{11}x_\alpha, \\ f_\alpha &= a_{21}x_\alpha + a_{22}x'_\alpha, \\ h_\alpha &= a_{31}x_\alpha + a_{32}x''_\alpha + a_{33}x'''_\alpha, \end{aligned}$$

where

$$\|x_\alpha\|_\alpha = \|x'_\alpha\|_\alpha = \|x''_\alpha\|_\alpha = 1, \qquad (x_\alpha, x'_\alpha) = (x_\alpha, x''_\alpha) = (x'_\alpha, x''_\alpha) = 0.$$

Then

$$\begin{aligned} \|g_\alpha\|_\alpha^2 &= |a_{11}|^2, \\ \|f_\alpha\|_\alpha^2 &= |a_{21}|^2 + |a_{22}|^2, \\ \|h_\alpha\|_\alpha^2 &= |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2, \\ \\ (f_\alpha, g_\alpha)_\alpha &= a_{21}\overline{a_{11}} \\ (g_\alpha, h_\alpha)_\alpha &= a_{11}\overline{a_{31}} \\ (f_\alpha, h_\alpha)_\alpha &= a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}} \end{aligned}$$

Now

$$\begin{aligned} |(f_\alpha, g_\alpha)_\alpha - 1| &= |a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}} - 1| \\ &= \left|(a_{21}\overline{a_{11}} \cdot a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1) + a_{22}\overline{a_{32}}\right| \\ &\leq \left|a_{21}\overline{a_{11}}a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1\right| + |a_{22}\overline{a_{32}}|. \end{aligned}$$

$$
\begin{aligned}
\left|a_{21}\overline{a_{11}}a_{11}\overline{a_{31}}|a_{11}|^{-2}-1\right| &\leq (1+|\lambda_{fg}|)\,(1+|\lambda_{gh}|)\,(1-|\lambda_g|)^{-2}-1 \\
&\leq (1+|\lambda_{fg}|+|\lambda_{gh}|+|\lambda_{fg}|\,|\lambda_{gh}|)\,(1+2|\lambda_g|)-1 \\
&\leq \big(1+|\lambda_{fg}|+|\lambda_{gh}|+\frac{C}{2}(|\lambda_{gh}|+|\lambda_{fg}|)\big)\,(1+2|\lambda_g|)-1 \\
&\leq D_1(|\lambda_g|+|\lambda_{fg}|+|\lambda_{gh}|)
\end{aligned}
$$

where in the second line we used that $1/(1-x)^2$ is convex.

$$
|a_{22}|^2 = (|a_{21}|^2+|a_{22}|^2) - \frac{|a_{21}\overline{a_{11}}|^2}{|a_{11}|^2} \leq (1+|\lambda_f|)^2 - \frac{(1-|\lambda_{fg}|)^2}{(1+|\lambda_g|)^2}
$$

$$
|a_{32}|^2 = (|a_{31}|^2+|a_{32}|^2+|a_{33}|^2) - \frac{|a_{11}\overline{a_{31}}|^2}{|a_{11}|^2} \leq (1+|\lambda_h|)^2 - \frac{(1-|\lambda_{gh}|)^2}{(1+|\lambda_g|)^2}
$$

$$
\begin{aligned}
(1+|\lambda_f|)^2 - \frac{(1-|\lambda_{fg}|)^2}{(1+|\lambda_g|)^2} &= 1+2|\lambda_f|+|\lambda_f|^2 - \frac{1-2|\lambda_{fg}|}{(1+|\lambda_g|)^2} - \frac{|\lambda_{fg}|^2}{(1+|\lambda_g|)^2} \\
&\leq (2+C)|\lambda_f| - (1-2|\lambda_{fg}|)[1-2|\lambda_g|] \\
&\leq (2+C)|\lambda_f| + 2|\lambda_{fg}| + 2|\lambda_g| - 4|\lambda_{fg}|\,|\lambda_g| \\
&\leq (2+C)|\lambda_f| + 2|\lambda_{fg}| + 2|\lambda_g| \\
&\leq (2+C)(|\lambda_f|+|\lambda_g|+|\lambda_{fg}|)
\end{aligned}
$$

where in the second line we used $1/(1+x)^2 \geq 1-2x$ for $x \geq 0$. Obviously we also have

$$
|a_{32}|^2 \leq (2+C)(|\lambda_g|+|\lambda_h|+|\lambda_{gh}|)
$$

Thus

$$
|a_{22}a_{32}| \leq D_2(|\lambda_f|+|\lambda_g|+|\lambda_h|+|\lambda_{fg}|+|\lambda_{gh}|).
$$

Combining all the inequalities, we obtain

$$
|(f_\alpha, h_\alpha)-1| \leq D(|\lambda_f|+|\lambda_g|+|\lambda_h|+|\lambda_{fg}|+|\lambda_{gh}|).
$$

with a finite number of exceptions.

□

An equivalence relation will split the set V_0 into disjoint subsets, the equivalence classes.

Definition The strong equivalence class of a C_0-sequence f is denoted by $[f]$. The set of strong equivalence classes of C_0-sequences will be called $\mathcal{S}$.

$\square$

Theorem J.24.21 *If two C_0-sequences f_α, g_α belong to two different equivalence classes, then*

$$(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = 0.$$

If they belong to the same equivalnce class, then $(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = 0$ if and only if some $(f_\alpha, g_\alpha) = 0$.

Proof: Clearly

$$(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = \prod_\alpha (f_\alpha, g_\alpha)$$

in the sence of quasi-convergence,

$\square$

C_0-vectors from different strong equivalence classes are always orthogonal and those from the same strong equivalence class are orthogonal if and only if they are orthogonal in at least one tensor product factor.

Theorem J.24.22 $[f] = [g]$ *if and only if both* $\sum_\alpha \|f^0_\alpha - g^0_\alpha\|^2_\alpha$ *and* $\sum_\alpha |Im(f^0_\alpha, g^0_\alpha)_\alpha|$ *converge*

Proof: In othere words, the combined convergence is equivalent to that of $\sum_\alpha |(f^0_\alpha, g^0_\alpha) - 1|$. Note that if $|z_{\alpha_1}| + |z_{\alpha_2}| + \cdots + |z_{\alpha_n}|$ is bounded then so is $|\zeta_{\alpha_1}| + |\zeta_{\alpha_2}| + \cdots + |\zeta_{\alpha_n}|$ if $|\zeta_{\alpha_1} - z_{\alpha_1}| + |\zeta_{\alpha_2} - z_{\alpha_2}| + \cdots + |\zeta_{\alpha_n} - z_{\alpha_n}|$ is bounded.

Now as $\sum_\alpha \big|\|f_\alpha\| - 1\big|$ converges if and only if $\sum_\alpha \big|\|f_\alpha\|^2 - 1\big|$, we can prove the theorem by demonstrating the convergence of

$$\sum_\alpha |(f^0_\alpha, g^0_\alpha) - \frac{1}{2}\|f^0_\alpha\|^2 + \frac{1}{2}\|g^0_\alpha\|^2|.$$

Now

$$
\begin{aligned}
\mathrm{Re}\big((f_\alpha^0, g_\alpha^0) - \frac{1}{2}\|f_\alpha^0\|^2 + \frac{1}{2}\|g_\alpha^0\|^2\big) &= -\frac{1}{2}\big(\|f_\alpha^0\|^2 + \|g_\alpha^0\|^2 - 2\mathrm{Re}(f_\alpha^0, g_\alpha^0)\big) \\
&= -\frac{1}{2}\|f_\alpha^0 - g_\alpha^0\|_\alpha^2
\end{aligned}
$$

and

$$
\mathrm{Im}\big((f_\alpha^0, g_\alpha^0) - \frac{1}{2}\|f_\alpha^0\|^2 + \frac{1}{2}\|g_\alpha^0\|^2\big) = \mathrm{Im}(f_\alpha^0, g_\alpha^0)
$$

Recall that the convergence of $\sum_\alpha z_\alpha$ is equivalent to the combined convergence of $\sum_\alpha |\mathrm{Re} z_\alpha|$, $\sum_\alpha |\mathrm{Im} z_\alpha|$. This completes the proof.

□

Lemma J.24.23 *for each $[f] \in \mathcal{S}$ there exists $f^0 \approx f$ such that $\|f_\alpha^0\|_\alpha = 1$ for all $\alpha \in \mathcal{I}$.*

Proof: Choose $f \in \mathcal{S}$. As $\sum_\alpha \big|\|f'_\alpha\|_\alpha - 1\big|$ converges, apart from a finite number of α's, we have

$$
\big|\|f_\alpha\|_\alpha - 1\big| \leq \frac{1}{2}, \qquad \|f_\alpha\|_\alpha \geq \frac{1}{2}.
$$

Recall that $f \approx f'$ if $f_\alpha \neq f'_\alpha$ for a finite number of α's. So we replace f_α by any f'_α such that $1/2 \leq \|f'_\alpha\|_\alpha \leq 3/2$ for the exceptional α's.

As $1/x$ is convex, for $1/2 \leq x \leq 1$ we have $(1/x) - 1 \leq 2(1-x)$ and for $1 \leq x \leq 3/2$ we have $1 - (1/x) \leq 2(x-1)$, that is,

$$
\left|\frac{1}{x} - 1\right| \leq 2|x-1|
$$

for $1/2 \leq x \leq 3/2$. So we have

$$
\left|\frac{1}{\|f'_\alpha\|_\alpha} - 1\right| \leq 2\big|\|f'_\alpha\|_\alpha - 1\big|,
$$

therefore

$$
\sum_\alpha \left|\frac{1}{\|f'_\alpha\|_\alpha} - 1\right|
$$

converges. By theorem J.24.16 ii), replace $|z_\alpha|$ by $1/\|f'_\alpha\|_\alpha$, as f'_α is a C_0−sequence so is

$$f^0_\alpha = \frac{1}{\|f'_\alpha\|_\alpha} f'_\alpha.$$

By theorem J.24.16 iv), replace z_α by $1/\|f'_\alpha\|_\alpha$, the C_0−sequences f'_α and f^0_α are equivalent. $\|f^0_\alpha\|_\alpha = 1$ is obvious, therefore the proof is complete.

□

By $\otimes_\alpha f_\alpha$ we denote the functional on X defined by

$$\otimes_\alpha f_\alpha(\{x_\alpha\}) = \prod_\alpha (f_\alpha, x_\alpha)_\alpha$$

We need to prove

$$\otimes_\alpha f_\alpha \neq 0 \quad \Leftrightarrow \quad \prod_\alpha \|f_\alpha\|_\alpha \neq 0$$

But $\otimes_\alpha f_\alpha \neq 0$ implies $\prod_\alpha \|f_\alpha\|_\alpha \neq 0$.

J.24.4 ITP Hilbert Space

1) All the $\mathcal{H}^\otimes_{[f]}$ are isomorphic and mutually orthogonal.

2) Every $\mathcal{H}^\otimes_{(f)}$ is the closed direct sum of all the $\mathcal{H}^\otimes_{[f']}$ with $[f'] \in \mathcal{S} \cap (f)$.

3) The ITP $\mathcal{H}^\otimes$ is the closed direct sum of all the $\mathcal{H}^\otimes_{(f)}$ with $(f) \in \mathcal{W}$.

4) Every $\mathcal{H}^\otimes_{[f]}$ has an explicitly known orthonormal von Neumann basis.

5) If s, s' are two different strong equivalence classes in the same weak equivalence class then there exists a unitary operator on $\mathcal{H}^\otimes$ that maps $\mathcal{H}^\otimes_s$ to $\mathcal{H}^\otimes_{s'}$, otherwise such an operator does not exist, the two Hilbert spaces are unitarily inequivalent subspaces of $\mathcal{H}^\otimes$.

Notice that two isomorphic Hilbert spaces can always be mapped into each other such that scalar products are preserved (just map some orthonormal bases) but here the question is whether this map can be extended unitarily to all of $\mathcal{H}^\otimes$. Intuitively then, strong classes within the same weak classes describe the same physics, those in different weak classes describe different physics such as an infinite difference in energy, magnetization, volume etc.

Definition By $\mathcal{H}_C$ we denote the completion of the complex vector space of finite linear combinations of elements from V_C equipped with the sesquilinear form $< \cdot, \cdot >$ obtained by extending (J.24.18) from V_C to $\mathcal{H}_C$ by sesquilinearity,

$$\begin{aligned} < \alpha\xi + \beta\chi, \zeta > &= \overline{\alpha} < \xi, \zeta > + \overline{\beta} < \chi, \zeta > \\ < \xi, \alpha\chi + \beta\zeta > &= \alpha < \xi, \chi > + \beta < \xi, \zeta > \end{aligned}$$

for all $\xi, \chi, \zeta \in \mathcal{H}_C$ and $\alpha, \beta \in \mathbb{C}$.

□

We can now give the defintion of the ITP.

Definition We denote by

$$\mathcal{H}^{\otimes} := \otimes_{\alpha} \mathcal{H}_{\alpha} \tag{J.-356}$$

the Cauchy completion of the pre-Hilbert space $\mathcal{H}_C$. It is called the ITP of the $\mathcal{H}_{\alpha}$.

□

The strong equivalence classes provide the basic tool to analyze the structure of $\mathcal{H}^{\otimes}$.

Definition For a strong equivalence class $[f] \in \mathcal{S}$ we define the closed subspace $\mathcal{H}_{[f]}$ of $\mathcal{H}^{\otimes}$ by the closure of the finite linear combinations of $\otimes_{f'}$'s with $f' \in [f]$, i.e., the closure of

$$\left\{ \sum_{k=1}^{N} z_k \otimes_{f^k} : z_k \in \mathbb{C},\ f^k \in [f],\ N < \infty \right\}. \tag{J.-356}$$

It is called the $[f]-$**adic incomplete ITP** of the $\mathcal{H}_{\alpha}$'s.

□

Theorem J.24.24 *The complete ITP decomposes as the direct sum over strong equivalence classes $[f]$ of $[f]-$adic ITP's,*

$$\mathcal{H}^{\otimes} = \overline{\otimes_{[f] \in \mathcal{S}} \mathcal{H}_{[f]}} \tag{J.-356}$$

Proof:

□

J.24.5 Non-Associativity of ITPs

The associative law of tensor products is false. By this we mean the following: Let us subdivide the index set $\mathcal{I}$ into mutually disjoint index sets

$$\mathcal{I} = \cup_\beta \mathcal{I}_\beta$$

where β runs over some other index set $\mathcal{L}$. One can now form the different ITP

$$\mathcal{H}^{\otimes\prime} = \otimes_\beta \mathcal{H}^\otimes_\beta, \qquad \mathcal{H}^\otimes_\beta = \otimes_{\gamma \in \mathcal{I}_\beta} \mathcal{H}_\gamma.$$

Unless the index set $\mathcal{L}$ is finite, a generic C_0-vector of $\mathcal{H}'^\otimes$ is orthogonal to all of $\mathcal{H}^\otimes$.

Scalar multiplication is not multi-linear. That is, if f and $z \cdot f$ are C_0-sequences where

$$(z \cdot f)_\alpha = z_\alpha f_\alpha$$

for some complex numbers z_α then $\otimes_{z \cdot f} = (\prod_\alpha z_\alpha) \otimes_f$.

Lemma J.24.25 *Let $\prod_\alpha z_\alpha$ be quasi-convergent. Then*

i) If f is a $C-$sequence, so is $z \cdot f$ with $(z \cdot f)_\alpha := z_\alpha f_\alpha$.

ii) If moreover $\sum_\alpha \big||z_\alpha| - 1\big|$ converges and f is a C_0-sequence, so is $z \cdot f$.

iii) The product formula

$$\otimes_{z \cdot f} = [\prod_\alpha z_\alpha] \otimes_f \tag{J.-356}$$

fails to hold only if

1) $\prod_\alpha z_\alpha$ is (quasi-convergent but) not convergent and

2) $(\otimes_f, \cdot\,) \neq 0$ considered as a linear functional over $C-$vectors.

In this case, $\{z_\alpha\}$, f satisfy the assumptions of ii), and all $z_\alpha \neq 0$.

iv) If $\{z_\alpha\}$, f satisfy the assumptions of ii), then $[z \cdot f] = [f]$ if and only if $\sum_\alpha |z_\alpha - 1|$ converges. If all $z_\alpha \neq 0$, then this is equivalent to the convergence of $\prod_\alpha z_\alpha$ (beyond mere quasi-convergence).

Proof: i): As both $\prod_\alpha |z_\alpha|$, $\prod_\alpha \|f_\alpha\|_\alpha$ converge, so does

$$\prod_\alpha \|z_\alpha f_\alpha\|_\alpha = \prod_\alpha |z_\alpha| \cdot \prod_\alpha \|f_\alpha\|_\alpha$$

ii) $|z_\alpha| \leq C$

$$\begin{aligned} \big|\|z_\alpha f_\alpha\|_\alpha - 1\big| &= \big||z_\alpha|\ \|f_\alpha\|_\alpha - 1\big| \\ &= \big|(|z_\alpha| - 1) + |z_\alpha|(\|f_\alpha\|_\alpha - 1)\big| \\ &\leq \big|(|z_\alpha| - 1) + C(\|f_\alpha\|_\alpha - 1)\big| \end{aligned}$$

and thus $\sum_\alpha \big|\|z_\alpha f_\alpha\|_\alpha - 1\big|$ converges.

□

J.24.6 Von-Neumann Algebras on ITPs

A von Neumann algebra over a Hilbert space is weakly (equivalently strongly) closed sub$-^*$ algebra of the algebra of bounded operators on that Hilbert space.

Given a bounded operator a_α on $\mathcal{H}_\alpha$ (notice that closed unbounded operators have a polar decomposition into a unitary and self-adjoint piece and that the self-adjoint operator is completely determined by its bounded spectral projections so that restriction to bounded operators is no loss of generality) we can extend it in the natural way to $\mathcal{H}^\otimes$ by defining $\hat{a}_\alpha$ densely on C_0-vectors through

$$\hat{a}_\alpha \otimes_f = \otimes_{f'}$$

with

$$f'_{\alpha'} = f_{\alpha'} \quad \text{for} \quad \alpha' \neq \alpha$$

and

$$f'_\alpha = a_\alpha f_\alpha$$

As we will see, it turns out that the algebra of these extended operators for a given label is automatically a von Neumann algebra for $\mathcal{H}^\otimes$ and we call the weak closure of all these algebras the von Neumann algebra $\mathcal{R}^\otimes$ of local operators.

Definition We denote by $\mathcal{B}(\mathcal{H}_\alpha)$ the set of bounded operators on $\mathcal{H}_\alpha$ and by $\mathcal{B}^\otimes := \mathcal{B}(\mathcal{H}^\otimes)$ the set of bounded operators on the ITP $\mathcal{H}^\otimes$.

□

Definition We denote by $\mathcal{B}_\alpha$ the extension of $\mathcal{B}(\mathcal{H}_\alpha)$ to the ITP, that is,

$$\mathcal{B}_\alpha = \{\hat{A}_\alpha : A_\alpha \in \mathcal{B}(\mathcal{H}_\alpha)\} \tag{J.-359}$$

□

Lemma J.24.26 *For all $\alpha \in \mathcal{I}$, the algebra $\mathcal{B}_\alpha$ is a von Neumann algebra over $\mathcal{H}^\otimes$.*

Proof: To prove the theorem one uses the von Neumann density theorem, proved in the next chapter. This states that a set $\mathcal{U}$ of the space of bounded operators of a Hilbert space $\mathcal{H}$ is a von Neuman algrba if it is equal to its double commutant, the commutant of a set $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ being

$$\mathcal{U}' = \{B \in \mathcal{B}(\mathcal{H}) : [A, B] = 0 \text{ for all } A \in \mathcal{U}\},$$

Let us write

$$\mathcal{B}^\otimes = \mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_{\overline{\alpha}})$$

where $\overline{\alpha} = \mathcal{I} - \alpha$.

$$\mathcal{B}'_{\overline{\alpha}} = \{\hat{B} \in \mathcal{B}^\otimes : [\hat{A}, \hat{B}] = 0 \text{ for all } \hat{A} \in \mathcal{B}_{\overline{\alpha}}\},$$

the commutant of $\mathcal{B}_{\overline{\alpha}}$.

□

Definition Two C_0-sequences f, g are said to be weakly equivalent, denoted $f \sim g$ provided that there are complex numbers z_α such that $z \cdot f$ and g are strongly equivalent, that is, $z \cdot f \approx g$.

□

Lemma J.24.27 *The definition of weak equivalenceis unchanged if we restrict to complex numbers with* $|z_\alpha| = 1$.

Proof: We prove that if f and $z \cdot f$ are C_0-sequences we can find z'_α with $|z'_\alpha| = 1$ such that $(z \cdot f) \approx (z' \cdot f)$.

As $\sum_\alpha \left| \|z_\alpha f_\alpha\|_\alpha - 1 \right|$ converges, $z_\alpha f_\alpha = 0$ can occur only for a finite number of α's. For these we may replace z_α by 1 and f_α by non-zero f^0_α. As this so constructed C_0-sequence differes from f by only a finite number of terms, it is srtongly equivalent to it (this follows from theorem J.24.22). So we may assume that $z_\alpha f_\alpha \neq 0$ for all α.

As $\sum_\alpha \left| \|f_\alpha\|_\alpha - 1 \right|$, $\sum_\alpha \left| \|z_\alpha f_\alpha\|_\alpha - 1 \right|$ converge, and all $\|f_\alpha\|_\alpha, \|z_\alpha f_\alpha\|_\alpha \neq 0$, by theorem J.24.16, $\prod_\alpha \|f_\alpha\|_\alpha$, $\prod_\alpha \|z_\alpha f_\alpha\|_\alpha$ converge and have non-zero values. Thus

$$\prod_\alpha \frac{1}{|z_\alpha|}$$

converges too, and has a non-zero value, as

$$\frac{\|f_\alpha\|_\alpha}{\|z_\alpha f_\alpha\|_\alpha} = \frac{1}{|z_\alpha|}.$$

Therefore

$$\sum_\alpha \left| \frac{1}{|z_\alpha|} - 1 \right|$$

converges by theorem J.24.16. Now it follows from lemma, (ii), (iv), that $(z \cdot f)$ and $(\frac{z}{|z|} \cdot f)$ are equivalent. Thus

$$z'_\alpha = \frac{z_\alpha}{|z_\alpha|}.$$

Lemma J.24.28 *Weak equivalence is an equivalence relation.*

Proof: Reflextivity: $f \sim f$ as $(1 \cdot f) \approx f$.

Symmetry: If $|z_\alpha| = 1$ we have

$$(f_\alpha, (1/z_\alpha) g_\alpha)_\alpha = z_\alpha (f_\alpha, g_\alpha)_\alpha = (z_\alpha f_\alpha, g_\alpha)_\alpha$$

and $\|(1/z_\alpha)g_\alpha\|_\alpha = \|g_\alpha\|_\alpha$. These together with the symmetry property of strong equivalence implies that if $(z \cdot f) \approx g$ with $|z_\alpha| = 1$ then $(1/z) \cdot g \approx f$ with $(1/z_\alpha)g_\alpha$ a C_0-sequence, that is, $g \sim f$.

Transitivity: If $z \cdot f \approx g$ and $z' \cdot g \approx h$, $|z_\alpha|, |z'_\alpha| = 1$, then $z \cdot f \approx (1/z') \cdot h$ and $z' \cdot z \cdot f \approx h$. So $f \sim g$.

□

Lemma J.24.29 *f and g are weakly equivalent if and only if*

$$\sum_\alpha \big||(f_\alpha, g_\alpha)_\alpha| - 1\big|$$

converges.

Proof: By definition, $f \sim g$ if we can find complex numbers z_α with $|z_\alpha| = 1$ such that

$$\sum_\alpha \big|(z_\alpha f_\alpha, g_\alpha)_\alpha - 1\big|$$

converges. If

$$\sum_\alpha \big||(f_\alpha, g_\alpha)_\alpha| - 1\big|$$

converges, then $f \sim g$ with $z_\alpha = (f_\alpha, g_\alpha)^*_\alpha / |(f_\alpha, g_\alpha)_\alpha|$.

Now, from fig (), it is easy to see that the minimum value of $|(z_\alpha f_\alpha, g_\alpha) - 1| = |z_\alpha (f_\alpha, g_\alpha)_\alpha - 1|$ is $\big||(f_\alpha, g_\alpha)_\alpha| - 1\big|$. Thus $\sum_\alpha \big|(z_\alpha f_\alpha, g_\alpha)_\alpha - 1\big|$ converges for some complex numbers z_α, when $f \sim g$, only if

$$\sum_\alpha \big||(f_\alpha, g_\alpha)_\alpha| - 1\big|$$

converges.

□

Lemma J.24.30 *Assume that a z_α with $|z_\alpha| = 1$ is given for each $\alpha \in \mathcal{I}$. Then there exists one and only one closed, linear operator U, such that*

$$U \otimes_f = \otimes_{z \cdot f}$$

for every C_0−vector f. The operator U is unitary.

Proof:

Proof of uniqueness. If U' is another closed, linear operator meeting the requirements, then

□

J.25 Biblioliographical notes

In this chapter I have relied on the following references:

Linear operators for Quantum Mechanics by Thomas F. Jordan.

Functional Analysis by Reed and Simon.

In proof of Tynchonov's Theorem I used J.L. Bell General topology [407].

An introduction to Hilbert space by N. Young.

To do with nets very good introduction that can be downloaded called "Topological spaces", King's College, London. http://www.mth.kcl.ac.uk/ iwilde/notes/fa2/fa2.pdf. [409].

T. Thiemann's notes "Introduction to Modern Canonical Quantum General Relativity" [32] and "Lectures of Loop Quantum Gravity" [33].

J.26 Worked Exercises and Details

Details: Numerical range of a form is convex.

Comming from Martin Schechter "princples of functional analysis".

Proof of Theorem J.18.25 Any linear combination of u and v is also in $D(a)$. We will show that for $w = \alpha e^{i\varphi}u + \beta$, $a(w)$ satisfies Eq.(J.18.25), where α, β, φ are real numbers to be determined.

$$\mathrm{Im}\gamma a(u) = \mathrm{Im}\gamma a(v) = d. \tag{J.-359}$$

$$\begin{aligned} g(\alpha,\beta) &= \gamma a(\alpha e^{i\varphi}u + \beta v) - id \, \|\alpha e^{i\varphi}u + \beta v\|^2 && \text{(J.-358)} \\ &= \alpha^2[\gamma a(u) - id] + \beta^2[\gamma a(v) - id] \\ &+ \alpha\beta e^{i\varphi}[\gamma a(u,v) - (u,v)id] + \alpha\beta e^{-i\varphi}[\gamma a(v,u) - (v,u)id] && \text{(J.-358)} \end{aligned}$$

Now given any two complex number z and z', we can find φ such that $\mathrm{Im}(ze^{i\varphi} + e^{-i\varphi}z') = 0$.[1] We will pick φ so that

$$\mathrm{Im}\{e^{i\varphi}[\gamma a(u,v) - (u,v)id] + e^{-i\varphi}[\gamma a(v,u) - (v,u)id]\} = 0$$

We see form $g(\alpha, \beta)$ is real. Now if

$$\alpha e^{i\varphi}u + \beta v = 0, \tag{J.-358}$$

then we have

$$\|\alpha e^{i\varphi}u\|^2 = \beta v\|^2$$

and

$$a(\alpha e^{i\varphi}u) = a(\beta v).$$

The first implies $\alpha^2 = \beta^2$, while the second gives

$$\alpha^2 a(u) = \beta^2 a(v).$$

Since we are assuming $a(u) \neq a(v)$, the only way (J.26) can hold is if $\alpha = \beta = 0$. From this, we see that the function

$$h(t) = \frac{g(t, 1-t)}{\|te^{i\varphi}u + (1-t)v\|^2} \tag{J.-358}$$

is contiuous and real valued in $0 \le t \le 1$. Moreover,

$$h(0) = \gamma a(v) - id, \qquad h(1) = \gamma a(u) - id.$$

Hence, there is a value t_1 satisfying $0 < t_1 < 1$ such that (see Fig(J.26))

$$\begin{aligned} h(t_1) &= \tau[\gamma a(v) - id] + (1-\tau)[\gamma a(u) - id] \\ &= \gamma[\tau a(v) + (1-\tau)a(u)] - id \end{aligned} \tag{J.-358}$$

Now equation (J.-358) tells us

[1] $\mathrm{Im}(|z|e^{i(\theta+\varphi)} + |z'|e^{i(\theta'-\varphi)}) = |z|\sin(\theta+\varphi) + |z'|\sin(\theta'-\varphi) = 0$ so that $-\frac{|z'|}{|z|} = \frac{\sin(\theta+\varphi)}{\sin(\theta'-\varphi)}$

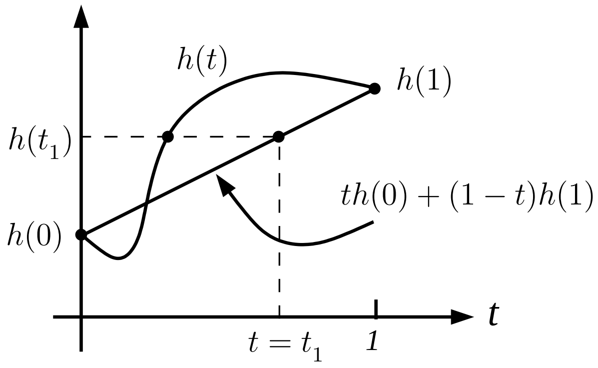

Figure J.31: NumConvex.

$$g(t_1, 1-t_1) = \gamma a(t_1 e^{i\varphi} u + (1-t_1)v) - id \, \|t_1 e^{i\varphi} u + (1-t_1)v\|^2$$

Dividing both sides by $\|t_1 e^{i\varphi} u + (1-t_1)v\|^2$

$$\begin{aligned} h(t_1) &= \frac{a(t_1 e^{i\varphi} u + (1-t_1)v)}{\|t_1 e^{i\varphi} u + (1-t_1)v\|^2} - id \\ &= \gamma a(w) - id. \end{aligned} \tag{J.-358}$$

where

$$w = \frac{t_1 e^{i\varphi} u + (1-t_1)v)}{\|t_1 e^{i\varphi} u + (1-t_1)v\|^2}. \tag{J.-358}$$

Equating (J.-358) with (J.-358) gives (J.18.25).

□

Details: Caratheodory's outer measure.

Prove that, under condition (N.-19), that $\mu(E) = \inf\{\sum_{j=1}^{\infty} \mu(F_j) \mid E \subset \bigcup_{j=1}^{\infty} F_j\}$ becomes Caratheodory's outer measure.

Proof:

A non-negative function on the power set. Monoticity: Let $E_1 \subset E_2$,

the infimum of a smaller set cannot be less than the infimum of a larger set $\Rightarrow \mu(E_1) \leq \mu(E_2)$.

subadditivity

$$\mu(\cup_i E_i) = \inf\{\sum_{j=1}^{\infty} \mu(F_j) \mid \cup_i E_i \subset \bigcup_{i=1}^{\infty} F_i\} \tag{J.-357}$$

$$\sum_i \mu(E_i) = \sum_i \inf\{\sum_{i=1}^{\infty} \mu(F_j) \mid E_i \subset \bigcup_{j=1}^{\infty} F_j\} \tag{J.-356}$$

Details: Axiom of choice and Zorn's Lemma.

Proof:

Appendix K

Quantization Schemes

K.1 Introduction

In view of these new physical settings we must first re-examine the quantization problem from a somewhat broader perspective. Standard methods?

One constructs a suitable algebra of functions on the classical phase space, promotes it to an operator algebra and then seeks its representations by operators on a Hilbert space.

[arXiv: quant-ph/ 0412015]. Quantisation is the problem of deriving the mathematical framework of a quantum mechanical system from the mathematical framework of the corresponding classical mechanical system. A method of quantisation must contain a map $\mathcal{A}$ from the set of classical observables to the set of quantum observables with the following properties:

Linear functionals on an observable's operator algebra. quantization as functionals on the set of observables. in other words expectation values of observables - these are called states ω

Operator Algebras Department of Mathematics:

a collection 'observables' and is capable of being in certain 'states'. We can define the state of a system as knowledge of the expected values of the observables. that is , a state is an assignment of an expected value of each observable.

K.2 Algebraic Quantum Theory

In quantum mechanics we often start by taking classical observables an write down Poisson brackets. We then promote these to operator equations that express that they don't commute. The first to spring to mind might well be

$$\hat{p}\hat{q} - \hat{q}\hat{p} = i\hbar \tag{K.0}$$

which came from

$$\{p, q\} = -1 \tag{K.0}$$

but there are many others.

When we do this, we are doing is defining an **"algebra of observables"** (or what mathematicians refer to as an algebra).

One way to get a hold of states is to take your algebra of observables and represent it as an algebra of operators on a Hilbert space. Then the unit vectors in your Hilbert space represent states. However, the same algebra of observables can have different representations as operators on a Hilbert space.

K.2.1 The Harmonic Oscillator

with eigenvector equation

$$\left(\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2\right)\psi = E\psi \tag{K.0}$$

where and are self-adjoint operators such that $[\hat{p}, \hat{x}] = -i\hbar I$.

The algebra $\mathcal{A}$ of observables for the one dimensional harmonic oscillator is generated by the operators the Hamiltonian $\mathcal{H}$, the operator P momentum, and Q. The defining algebraic relations are:

$$\mathcal{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2, \qquad [P, Q] = -i\hbar I. \tag{K.0}$$

The elements of $\mathcal{A}$ are assumed to be linear operators defined on a linear space Ψ. There is a scalar product $(\cdot, \cdot)$ defined on Ψ that provides probability amplitudes (but Ψ is not a Hilbert space).

There are many realisations of the vector space Ψ on which $\mathcal{A}$ is an algebra of operators. We have to make one further assumption to fully specify the realisation of $\mathcal{A}$

there exists at least one non-degenerate eigenvalue of $\mathcal{H}$ whose corresponding eigenvector

$$\Omega \text{ is an element of } \Psi. \tag{K.0}$$

the physics of the harmonic oscillator is described by an algebra of observables that satisfy the algebraic assumptions (K.2.1)-(K.2.1).

To construct the space Ψ, we make elements of $\mathcal{A}$ act on the eigenvector Ω.

$$\hat{a} = \hat{p} - im\omega\hat{x} \tag{K.0}$$

and its adjoint $\hat{a}^\dagger$

$$\hat{a}^\dagger = \hat{p} + im\omega\hat{x} \tag{K.0}$$

$$[\hat{a}, \hat{a}^\dagger] = \tag{K.0}$$

$$(\hat{p} \mp im\omega\hat{x})^*(\hat{p} - \pm im\omega\hat{x}) = \hat{p}^2 + m^2\omega^2\hat{x}^2 \mp m\omega\hbar I = 2m(\mathcal{H} \mp \frac{1}{2}\hbar\omega). \tag{K.0}$$

$$\begin{aligned}
(\varphi, a\psi) &= (a^\dagger\varphi, \psi), \qquad \text{for all } \varphi, \psi \in \Psi, && \text{(K.1)}\\
(\varphi, N\psi) &= (N\varphi, \psi), \qquad \text{for all } \varphi, \psi \in \Psi. && \text{(K.2)}
\end{aligned}$$

$$[a, a^\dagger] = I. \tag{K.2}$$

The assumption about the existence of Ω implies that there exists a $\varphi_\lambda \neq 0$ in Ψ such that

$$N\varphi_\lambda = \lambda\varphi_\lambda. \tag{K.2}$$

From (K.2) and (K.2.1) it follows that

$$\lambda(\varphi_\lambda, \varphi_\lambda) = (\varphi_\lambda, N\varphi_\lambda) = (N\varphi_\lambda, \varphi_\lambda) = \overline{\lambda}(\varphi_\lambda, \varphi_\lambda). \tag{K.2}$$

Therefore, $\lambda = \overline{\lambda}$, i.e., λ is real. From the commutation relation (K.2.1), it then follows that

$$\begin{aligned}
N(a\varphi_\lambda) &= a^\dagger a\varphi_\lambda = (aa^\dagger - I)a\varphi_\lambda\\
&= a(N - I)\varphi_\lambda = a(\lambda - 1)\varphi_\lambda\\
&= (\lambda - 1)a\varphi_\lambda && \text{(K.1)}
\end{aligned}$$

This implies either $a\varphi_\lambda$ is an eigenvector of N with eigenvalue $(\lambda - 1)$ or $a\varphi_\lambda = 0$. From (K.1) and from the commutation relation (K.2.1) it follows that

$$\|a^\dagger\varphi_\lambda\|^2 = (\varphi_\lambda, a^\dagger a\varphi_\lambda) + (\varphi_\lambda, I\varphi_\lambda) = \|a\varphi_\lambda\|^2 + \|\varphi_\lambda\|^2 \neq 0, \tag{K.1}$$

since φ_λ is different from the zero vector. This means that $a^\dagger\varphi_\lambda \neq 0$. equation (K.2.1) implies that

$$N(a^\dagger\varphi_\lambda) = (\lambda + 1)a^\dagger\varphi_\lambda, \tag{K.1}$$

i.e., $a^\dagger\varphi_\lambda$ is an eigenvector of N with eigenvalue $(\lambda + 1)$.

States as Expectation Values

In quantum mechanics one finding the expectation value of an operator. If the system is described by a pure state with wave function $\psi(\mathbf{r}, t)$ then the expectation value is given by

$$< \psi|\hat{A}|\psi >= \int \psi^*(\mathbf{r}, t)\hat{A}\psi(\mathbf{r}, t)d^3r \tag{K.1}$$

For every ψ in $\mathcal{H}$ the expectation value E_ψ has the following properties:

(i) E_ψ (I), where I denotes the identity operator on $\mathcal{H}$.

(ii) $E_\psi(A)$ is real for all self-adjoint operators A.

(iii) $E_\psi(A) \geq 0$ for all positive operators A.

(iv) $E_\psi(A)$ depends linearly on A, that is $E_\psi(\alpha A + \beta B) = \alpha E_\psi(A) + \beta E_\psi(B)$ for all complex numbers α and β and all linear operators A and B.

Proof:

(iii) When A is the identity operator

$$E_\psi(I) = \frac{< \psi|\psi >}{||\psi||^2} = 1. \tag{K.1}$$

(ii) Since A is self-adjoint we have

$$E_\psi(A) = \frac{< \psi|A\psi >}{||\psi||^2} = \frac{< A\psi|\psi >}{||\psi||^2} = \frac{\overline{< \psi|A\psi >}}{||\psi||^2} = \overline{E_\psi(A)}, \tag{K.1}$$

so that E_ψ is real.

(iii) A linear transformation is said to be positive if

$$< A\psi|\psi >\geq 0 \tag{K.1}$$

for all vectors ψ.

(iv)

$$\begin{aligned} E_\psi(\alpha A+\beta B) &= \frac{< \psi|\alpha A+\beta B\psi >}{||\psi||^2} \\ &= \alpha\frac{< \psi|A\psi >}{||\psi||^2}+\beta\frac{< \psi|B\psi >}{||\psi||^2} \\ &= \alpha E_\psi(A)+\beta E_\psi(B) \end{aligned} \tag{K.0}$$

Statistical Mechanical States

$$\mathrm{tr}(\mathcal{Q}_\psi A)=\frac{< \psi|\hat{A}\xi >}{\|\psi\|^2}=E_\psi \tag{K.0}$$

suppose we only know that there is a probability p_k that he state is described by the vector ψ_k. $< \psi|\hat{A}\ .\ > /\|\psi\|^2$ is sometimes referred to as a trace class operator. The expectation value should then be given by the weighted average

$$\sum_j p_j E_{\psi_j}(A)=\sum_j p_j \mathrm{tr}(\mathcal{Q}_\psi A)=\mathrm{tr}\left(\sum_j p_j \mathcal{Q}_\psi A)\right). \tag{K.0}$$

Definition In a quantum mechanical system whose state is described by a density operator ρ the expectation of the observable A is given by

$$E_\rho(A)=\mathrm{tr}(\rho A). \tag{K.0}$$

Definition definition of folium here??

A folium of a given state ω which may be defined to be the set of all states ω_σ which arise in the form $\mathrm{Tr}(\sigma\pi_\omega(\ \cdot\ ))$ where σ ranges over the density operators (trace-class operators with unit trace) on $\mathcal{H}_\omega$.

K.2.2 Abstract Formalism

An algebraic structure called a $C\star$ algebra. A concrete C^*-algebra is a linear space $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, that is, a bunch of operators closed under addition, multiplication, scalar multiplication, and taking adjoints which is also complete with respect to the operator norm.

A $C\star$ algebra can be defined abstractly without any reference to linear operators acting on a Hilbert space. An abstract C^*-algebra is given by a set on which addition, multiplication, adjoint conjugation, and a norm are defined, satisfying the same algebraic relations as their concrete counterparts.

Can we solve quantum mechanics problems without resorting to differential equations

we start by giving the main properties of an abstract $\star-$algebra. We then show a how a measure, ω, called a weight is introduced, which plays the role of the state. ω is a functional, mapping elements of the algebra onto the real fields. It is not difficult to show that this is equivalent to introducing the density operator in the usual approach (exercise)

Definition: Field

Examples of field:

Rings are the "number systems" in mathematics.

Definition: If $\mathcal{A}$ has both the structure

(i) of a vector space

(ii) and of a ring with identity such that for all $A \in \mathcal{A}$ and α,

$$(\alpha I)A = \alpha A \tag{K.0}$$

then $\mathcal{A}$ is said to be an *algebra*.

Definition: If in addition to the properties in Definition there exists a map $* : \mathcal{A} \to \mathcal{A}$ (meaning the map, $*$, sends any element of $\mathcal{A}$ into another element of $\mathcal{A}$), such that for all

(i) $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$

(ii) $(AB)^* = B^*A^*$

(iii) $A^{**} = A$

then $\mathcal{A}$ is said to be a $\star$-*algebra*.

Definition: A state on a $\star$-algebra $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \to C$ that satisfies:

(i) $\omega(A^*A)$ is real and non-negative for all $\mathcal{A}$

(ii) $\omega(I) = 1$.

Definition: A $\star$-algebra that contains all its conjugates is known as a C$\star$-algebra.

Proposition: For any state, ω, on a $\mathcal{A}$, and for any $A, B \in \mathcal{A}$,

$$\omega(A^*B) = \overline{\omega(B^*A)}. \tag{K.0}$$

An immediate corollary is $\omega(A^*) = \overline{\omega(A)}$, and so if A is self-adjoint, $A^* = A$, then $\omega(A)$ is real.

there is a Cauchy-Schwartz inequality:

$$\omega(A^*A)\omega(B^*B) \geq |\omega(A^*B)|^2. \tag{K.0}$$

Proof:

$$\omega((A + \lambda B)^* A(A + \lambda B)) = \omega(A^*A) + |\lambda|^2\omega(B^*B) + \lambda\omega(A^*B) + \overline{\lambda}\omega(B^*A). \tag{K.0}$$

The left hand side and the first two terms of the right-hand side are all positive and so real. This forces the sum of the remaining two terms to be real and gives

$$\lambda\omega(A^*B) + \overline{\lambda}\omega(B^*A) = \overline{\lambda\omega(A^*B)} + \lambda\overline{\omega(B^*A)}. \tag{K.0}$$

Rearranging this we obtain

$$\lambda(\omega(A^*B) - \overline{\omega(B^*A)}) = \overline{\lambda}(\overline{\omega(A^*B)} - \omega(B^*A)). \tag{K.0}$$

This is true for any value of λ. Let us denote the quantity inside the brackets on the left-hand side α and that of the right-hand side β. The then condition reads

$$\lambda\alpha = \overline{\lambda}\beta. \tag{K.0}$$

By taking $\lambda = 1$ we have $\alpha = \beta$ and by taking $\lambda = i$ we have $\alpha = -\beta$ hence both α and β are zero and so

$$\omega(A^*B) = \overline{\omega(B^*A)}. \tag{K.0}$$

Taking $B = I$ gives $\omega(A^*) = \overline{\omega(A)}$ and so, in particular, when $A^* = A$ we deduce that $\omega(A)$ is real.

It is straightforward to get the Cauchy-Schwartz inequality (exercise).

K.2.3 GNS (Gel'fand, Naimark, Segal)

If $(\mathcal{H}, \pi, \Omega)$ is a cyclic representation of a C^*-algebra $\mathcal{A}$, then $A \to \omega(A) := < \Omega|\pi(A)\Omega >$ defines a state on $\mathcal{A}$. The converse is also true, and is known as the GNS construction.

formulating things so that the $*-$algebra is a C^*-algebra, then the GNS representation is as every-where defined on $\mathcal{H}$ bounded operators and is irreducible if and only if the state is pure.

Theorem K.2.1 *Let ω be a state on a $\star$-algebra $\mathcal{A}$. There exists an inner product space $\mathcal{H}_\omega$, a unit vector $\Omega_\omega \in \mathcal{H}_\omega$, and a homomorphism $\pi : \mathcal{A} \to (\mathcal{H}_\omega)$, such that for all $A \in \mathcal{A}$*

$$\omega(A) =< \Omega_\omega|\pi(A)\Omega_\omega > . \tag{K.0}$$

Proof

The space $\mathcal{H}_\omega$ will be a subspace of the dual space $\mathcal{A}'$ of linear functionals on $\mathcal{A}$. We can define a homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{A}')$ by setting, for $f \in \mathcal{A}'$ and $X \in \mathcal{A}$,

$$(\pi(A)f) = f(XA). \tag{K.0}$$

To show that this defines a homomorphism we must establish $\pi(AB) = \pi(A)\pi(B)$ and $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$.

$$(\pi(AB)f) = f(XAB) = (\pi(B)f)(XA) = (\pi(A)\pi(B)f)(X), \tag{K.0}$$

so that $\pi(AB) = \pi(A)\pi(B)$.

$$(\pi(\alpha A+\beta B)f) = f(X(\alpha A+\beta B)) = \alpha f(XA)+\beta f(XB) = \alpha(\pi(A)f)(X)+\beta(\pi(B)f)(X) \tag{K.0}$$

$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$.

We have yet to define the inner product. That is a properties of an inner-product see (N.-19).

We now restrict attention to the subspace

$$\mathcal{H}_\omega = \{\pi(A)\omega : A \in \mathcal{A}\}, \tag{K.0}$$

which is invariant under the action of any $\pi(B)$, (that is $\pi(B)(\pi(A)\omega) \in \mathcal{H}_\omega$). To see that it is invariant take any member of $\mathcal{H}_\omega$, say $\pi(A)\omega$, and act on it with $\pi(B)$

$$\pi(B)(\pi(A)\omega) = (\pi(B)\pi(A))\omega = \pi(C)\omega \tag{K.0}$$

where $C \in \mathcal{A}$ therefore $\pi(B)(\pi(A)\omega) \in \mathcal{H}_\omega$. This is needed to ensure that the map $\pi : \mathcal{A} \to \mathcal{L}\mathcal{A}'$ is a homomorphism. On this subspace we define an inner product

$$< \pi(A)\omega|\pi(B)\omega >= \omega(A^*B). \tag{K.0}$$

This is well defined, since it can also be written as

$$\omega(A^*B) = (\pi(B)\omega)(A^*), \tag{K.0}$$

we need to check it has the properties of an inner:

(i) $< u|v >= \overline{< v|u >}$ for all $u, v \in V$,

(ii) $< u|\alpha v + \beta w >= \alpha < u|v > +\beta < u|w >$ for all $u, v, w \in V$, and for all $\alpha, \beta \in \mathbf{C}$,

(iii) $< u|u >> 0$ for all non-zero vectors u in V. $< 0|0 >= 0$

$$< \pi(A)\omega|\pi(B)\omega >= \omega(A^*B) = \overline{\omega(B^*A)} = \overline{< \pi(B)\omega|\pi(A)\omega >}, \tag{K.0}$$

giving the conjugate symmetry (condition (ii)). The linearity property follows from the linearity of ω

$$\begin{aligned} < \pi(A)\omega|\pi(\alpha B + \beta C)\omega > &= \omega(A^*(\alpha B + \beta C)) \\ &= \alpha\omega(A^*B) + \beta\omega(A^*C) \\ &= < \pi(A)\omega|\pi(B) > +\beta < \pi(A)\omega|\pi(C) > . \end{aligned} \tag{K.-1}$$

It is clear form the definition of ω that

$$< \pi(A)\omega|\pi(A)\omega >= \omega(A^*A) \geq 0, \tag{K.-1}$$

Hence to complete the proof we need to show that $< \pi(A)\omega | \pi(B)\omega >$ is strictly positive. To do this we use the Cauchy-Schwarz inequality; it tells us that

$$|(\pi(A)\omega)(B^*)|^2 = |\omega(B^* A)|^2 \leq \omega(B^* B)\omega(A^* A), \tag{K.-1}$$

so that $\omega(A^* A)$ can only vanish only if $(\pi(A)\omega)(B^*) = 0$ for all B, which forces $\pi(A)\omega = 0$ that is $< 0|0 >= 0$.

$$< \omega | \pi(A)\omega >= \omega(I A^*) = \omega(A), \tag{K.-1}$$

so Ω_ω.

Let us stop and sum up what has been proven here.

A reason for caution here is that a given physical system may lead to inequivalent spaces $\mathcal{H}_\omega$. It is the algebra that is associated with the physical system, not the inner product space. We give as an example a ferromagnetic material is described by different spaces, $\mathcal{H}_\omega$, according to whether the it is magnetized or not. The algebra, however, is the same.

This is the reason for concern whether we are working in an unphysical sector of the theory, that is, a sector that does not contain enough physical solutions for us to be able to recover the correct semi-classical limit.

If we have not only a unital $\star$-algebra but in fact a C$\star$-algebra one can show that the Hahn-Banach theorem that representations always exist, that ever non-degenerate representation is a direct sum of cyclic representations and that ever states is continuous so that the GNS representations are always bounded operators (see details).

Example: The Harmonic Oscillator

A second use of this term is *an expectation value functional.* This is to be chosen at some point in the quantization.

$$F_{\text{Sch}}(W(\zeta)) := e^{-\frac{1}{2}|\zeta|^2}. \tag{K.-1}$$

The expectation values of $\hat{U}$ and $\hat{V}$ are given by:

$$F_{\text{Sch}}(U(\lambda)) = e^{-\frac{1}{2}\lambda^2 d^2} \quad \text{and} \quad F_{\text{Sch}}(V(\mu)) = e^{-\frac{1}{2}\frac{\mu^2}{d^2}} \tag{K.-1}$$

The corresponding GNS "vacuum" - the cyclic state ψ_{Sch} is

$$\psi_{\mathrm{Sch}} = \frac{1}{(\pi d^2)^{\frac{1}{4}}} e^{-\frac{x^2}{2d^2}}, \tag{K.-1}$$

the ground state of the simple harmonic oscillator with fundamental length scale d.

observables form an algebra both in quantum and classical mechanics. Then states are defined as positive linear functionals on those algebras: in quantum mechanics pure states are labelled by certain vectors in a Hilbert spaces and classic pure states correspond to points of the phase space. Measurements are represented by evaluation of observables on particular states.

$$\hat{U}^{-1}\hat{q}\hat{U} = \hat{q} + a, \qquad \hat{U}^{-1}\hat{p}\hat{U} = \hat{p} \tag{K.-1}$$

K.2.4 Null Ideals of the Algebra

A property you want the inner product to have is that it be positive definite, i.e. $(A, A) = 0$ implies that A is the null vector??. If an inner product $(A, A) = 0$ does not imply that A is null it is said to be only positive semidefinite.

$$(A, B)_\omega = \omega(A^* B) \tag{K.-1}$$

$\omega(A^* A) = 0$ does not imply that A is null, means the algebra has a *null ideal*, usually denoted $\mathcal{N}$.

$$\mathcal{N} = \{A \in \mathcal{A} : \omega(A^* A) = 0\} \tag{K.-1}$$

Recall the definition of a two-sided ideal, (or simply an ideal): say that $\mathcal{I}$ is a linear subspace of $\mathcal{A}$. If $\mathcal{I}$ satisfies the conditions $ai \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ and if $ia \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ then $\mathcal{I}$ is a two-sided ideal.

The presence of a null ideal $\mathcal{N}$ requires to construct the Hilbert space by transferring our attention from the $\star-$algebra $\mathcal{A}$ and products of ... to consideration of equivalence classes ... and the induced multiplication between classes. This forces the positive semidefinite inner product to become positive definite.

System of semi-norms???

Let

$$\xi_\omega(\mathcal{A}) = \mathcal{A} = \mathcal{A}/\mathcal{N} \tag{K.-1}$$

We write $\xi_\omega(A)$ for the projection of $A \in \mathcal{A}$ to $[A] \in \mathcal{A}/\mathcal{N}$.

K.2.5 Algebra Automorphisms: Time evolution and Symmetries

Symmetries and Time evolutions: Algebra Automorphisms

Given a state (i.e. a positive, normalized, linear functional) on $\mathcal{U}$ that is invariant under the classical symmetry automorphisms of $\mathcal{U}$.

Most important is that when a classical theory has symmetries that act on $\mathcal{U}$ by a group of automorphisms in the GNS construction these automorphisms are unitarily implemented. the state is is invariant under some automorphism of $\mathcal{U}$, its action is automatically unitarily implemented in the representation.

$$\begin{aligned} U(t)\cdot f &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\{f,T\}_n \\ &:= f+t\{f,T\}+\frac{t^2}{2!}\{\{f,T\},T\}+\ldots \end{aligned} \tag{K.-1}$$

the Poisson structure of phase space is preserved, including an automorphisms on the algebra of observables. The corresponding quantum operator $\hat{U}$ should be an automorphism in the algebra of quantum operators, and therefore, a unitary operator.

It is desirable to have a cyclic and invariant representation of $\mathcal{A}$ satisfying

$$\omega\circ\alpha_{\varphi}=\omega \tag{K.-1}$$

for all φ in the symmetry group. Then the corresponding representation is then the GNS representation corresponding to ω. That the positive linear functional ω is invariant under a symmetry, general theorems from algebraic quantum mechanics tell us that we have a unitary representation of the symmetry on the GNS Hilbert space $\mathcal{H}_{\omega}$ defined by

$$U(\varphi)\pi_{\omega}(a)\Omega_{\omega}=\pi_{\omega}\Omega_{\omega}(\alpha_{\varphi}(a))\Omega_{\omega} \tag{K.-1}$$

Properties of GNS construction

K.2.6 Summary

mostly from [hep-th/0601035]

The formulation of quantum mechanics in terms of the algebra of observables. The starting point of this formulation is a unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ (the algebra of observables). One assumes that this algebra is equipped with antilinear involution $A \to A^*$. One says that a linear functional ω on A specifies a state if $\omega(1) = 1$ and $\omega \geq (AA^*)0$ (i.e. if the functional is normalized and positive). The probability distribution $\rho(\lambda)$ of real observable $A = A^*$ in the state ω is defined by the formula $\omega(A^n) = \int \lambda^n(\rho)d\lambda$.

In the textbooks on quantum mechanics the algebra of observables consists of operators acting on a (pre)Hilbert space. Every vector x having a unit norm specifies a state by the formula $\omega(A) = < Ax, x >$. (More generally, a density matrix K defines a state $\omega(A) = TrAK$). This situation is in some sense universal: for every state ω on $\mathcal{A}$ one construct a (pre)Hilbert space $\mathcal{H}$ and a representation of $\mathcal{A}$ by operators on this space in such a way that the state ω corresponds to a vector in this space. (To construct $\mathcal{H}$ one defines inner product on $\mathcal{A}$ by the formula $< A, B >= \omega(A^*B)$. The space $\mathcal{H}$ can be obtained from $\mathcal{A}$ by means of factorization with respect to zero vectors of this inner product (i.e. $< A, A >= 0$ when A is not the zero element of the vector space - the inner product is only positive semidefinite). The inner product on $\mathcal{A}$ descends to $\mathcal{H}$ providing it with a structure of preHilbert space. The state ω is represented by a vector of $\mathcal{H}$ that corresponds to the unit element of $\mathcal{A}$.)

It is important to notice that although every state of the algebra $\mathcal{A}$ can be represented by a vector in Hilbert space in general it is impossible to represent all states by vectors in the same Hilbert space.

Time evolution in algebraic formulation is specified by one-parameter group $\alpha(t)$ of automorphisms of the algebra $\mathcal{A}$ preserving the involution. This group acts in obvious way on the space of states. If ω is a stationary state (a state invariant with respect to time evolution) then the group $\alpha(t)$ descends to a group $U(t)$ of unitary transformations of corresponding space $\mathcal{H}$. The generator $\mathcal{H}$ of $U(t)$ plays the role of Hamiltonian. If the spectrum of $\mathcal{H}$ is non-negative one says that the stationary state ω is a ground state.

States

A state is an assignment of an expectation value to each member of a collection of 'observables' (the elements of a C^*-algebra). More precisely, a linear functional ω on a C^*-algebra $\mathcal{A}$ is a state if $\omega \geq 0$ (ie, for positive $A \in \mathcal{A}$, $\omega(A) \geq 0$), and $||\omega|| = 1$.

Representations

A representation of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space and π is a morphism of $\mathcal{A}$ to the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. The representation π is said to be faithful if, for $A \in \mathcal{A}$, $\pi(A) = 0 \Rightarrow A = 0$.

A cyclic representation of $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, \Omega)$, where $\Omega \in \mathcal{H}$ such that $||\Omega|| = 1$ and $\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$.

Automorphisms

A morphism g is a map from an algebra to itself that, that is $g : \mathcal{O} \to \mathcal{O}$. A morphism that has an inverse g^{-1} is called an automorphism. We say a state ω is invariant under a group of automorphisms G if

$$\omega(gA) = \omega(A)$$

for all $g \in G$ and $A \in \mathcal{O}$.

GNS construction

Given a state ω over an abstract C^*-algebra $\mathcal{A}$, the Gelfand-Naimark-Segal construction provides us with a Hilbert space $\mathcal{H}_\omega$ with a preferred state Ω_ω, and a representation π_ω of $\mathcal{A}$ as a concrete algebra of bounded operators on $\mathcal{H}_\omega$, such that

$$\omega(A) = < \Omega_\omega|\pi_\omega(A)|\Omega_\omega > . \tag{K.-1}$$

Definition A representation $(\mathcal{H}, \pi)$ of the C^*-algebra $\mathcal{A}$ is said to be non-degenerate if .

Proposition K.2.2 *Let $(\mathcal{H}, \pi)$ be a non-degenerate representation of the C^*-algebra $\mathcal{A}$. Then π is the direct sum of a family of cyclic sub-representations.*

Let $\{\Omega_\alpha\}_{\alpha\in I}$ denote a maximal family (not a proper subset of any other such family) of nonzero vectors in $\mathcal{H}$ such that

$$(\pi(A)\Omega_\alpha, \pi(B)\Omega_\beta) = 0$$

for all $A, B \in \mathcal{A}$, whenever $\alpha \neq \beta$. To prove the existence of such a family we need to resort to the Zorn's lemma: The collection of all families of such vectors in $\mathcal{H}$ can be partially ordered by inclusion. Moreover, any linearly ordered chain (a subfamily totally order by inclusion) has an upper bound (the union of sets of the chain). Hence, Zorn's lemma implies the existence of a maximal element.

Next define $\mathcal{H}_\alpha$ as the Hilbert subspace formed by closing the linear subspace $\{\pi(A)\Omega_\alpha; A \in \mathcal{A}\}$. This is an invariant subspace so we can introduce π_α by $\pi_\alpha = P_{\mathcal{H}_\alpha}\pi_\alpha P_{\mathcal{H}_\alpha}$ on $\mathcal{H}_\alpha$ where

$P_{\mathcal{H}_\alpha} : \mathcal{H} \to \mathcal{H}_\alpha$ is the orthogonal projection. Then $\mathcal{H} = \oplus_\alpha \pi_\alpha$ and $(\pi_\alpha, \mathcal{H}_\alpha, \Omega_\alpha)$ is a cyclic representation of the $\mathcal{A}$.

K.2.7 Modifications for $^*-$algebras

not necessarily bounded operators on the Hilbert space $\mathcal{H}$.

A representation of a $^*-$algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ consisting of a Hilbert space $\mathcal{H}$ and a morphism $\pi : \mathbb{U} \to \mathcal{L}(\mathcal{H})$ into the algebra of linear (not necessarily bounded) operators on $\mathcal{H}$ with common and invariant dense domain.

A representation is said to be cyclic if there exists a normed vector $\Omega \in \mathcal{H}$ in the common domain of all the $a \in \mathbb{U}$ such that $\pi(\mathbb{U})\Omega$ is dense in $\mathcal{H}$. Notice that the existence of a cyclic vector implies that the states $\pi(b)\Omega$, $b \in \mathbb{U}$ lie in the common dense and invariant domain for all $\pi(a)$, $a \in \mathbb{U}$. A representation is said to be irreducible if every vector in a common dense and invariant (for $\mathbb{U}$) domain is cyclic.

K.3 Stone-Von Neumann Theorem

It assures us that there is nothing in the detailed theory of wave functions that cannot be, in principle, also be achieved by the algebraic approach, since they are isomorphic spaces.

$$\hat{Q}\hat{P} - \hat{P}\hat{Q} = i\hbar\mathbf{1} \tag{K.-1}$$

Scrödinger found a representation of Eq(K.3) in the context of his wave mechanics:

$$(Q\Psi)(x) = x\Psi(x),\ x \in R, \tag{K.-1}$$

on $L_2(R)$ and P is the differential operator

$$(P\Psi)(x) = i\hbar\frac{d\Psi}{dx}(x),\ x \in R, \tag{K.-1}$$

on $L_2(R)$.

The relations in Eq(K.3) cannot be understood on the whole of the Hilbert space. To remedy this difficulty, Weyl introduced the *unitary* operators

$$U(a) = e^{-i2\pi aP/\hbar} \text{ and } V(a) = e^{-i2\pi aQ/\hbar}. \tag{K.-1}$$

As is well known, unitary operators preserve the norm of a state and hence are well defined on the whole Hilbert space.

The algebraic relations between Q and P expressed in Eq(K.3) are replaced by

$$U(a)V(a) = e^{i2\pi ab/\hbar}V(b)U(a). \tag{K.-1}$$

This is the Weyl form of the CCR for one degree of freedom. We can then ask *formally* what the algebra for "generated",

$$\begin{aligned} U(a)QU^{-1}(a) &= \exp(iaP)Q\exp(-iaP) \\ &= Q + ia[P,Q] + \frac{(ia)^2}{2!}[P,[P,Q]] + \dots \\ &= Q + a\hbar \quad (\text{as } [P,Q] = i\hbar \text{ a scalar}). \end{aligned} \tag{K.-2}$$

$$(U(a)\Psi)(x) = \Psi(xa) \ \text{ and } \ (V(b)\Psi)(x) = e^{-i2\pi bx/\hbar}\Psi(x), \tag{K.-2}$$

The formula involving bounded operators will typically imply the one for unbounded operators but not vice versa.

von Neumann uniqueness theorem in QM:

Theorem If $\{\tilde{U}(a) : a \in R\}$ and $\{\tilde{V}(a) : a \in R\}$ are (**weakly continuous**footnote here) families of unitary operators acting irreducibly on a (separable) Hilbert space $\mathcal{H}$ such that

$$\begin{aligned} \tilde{U}(a)\tilde{U}(b) = \tilde{U}(a+b) \quad &, \quad \tilde{V}(a)\tilde{V}(b) = \tilde{V}(a+b) \\ \tilde{U}(a)\tilde{V}(b) &= e^{i2\pi ab/\hbar}\tilde{V}(b)\tilde{U}(a), \end{aligned} \tag{K.-2}$$

then there exists a Hilbert space isomorphism $W : \mathcal{H} \to L^2(R)$ such that

$$W\tilde{U}(a)W^{-1} = U(a) \ \text{ and } \ W\tilde{V}(a)W^{-1} = V(a), \tag{K.-2}$$

for all $a \in R$, where $U(a)$ and $V(a)$ are the Weyl unitaries in the Schödinger representation in Eq(K.3).

Each Weyl system with a finite number M of degrees on freedom is unitarily equivalent to the Schrödinger representation. Each reducible Weyl system is with finite number M of degrees of freedom is the direct sum of irreducible representations; hence it is a multiple of the Schrödinger representation.

The Stone-von Neumann theorem ensures that every irreducible 1-parameter representation of **W** *which is weakly continuous in the parameter* ξ is unitarily equivalent to the standard Schrödinger representation, where the Hilbert space is the space of $L^2(R,\, dx)$ of square integrable functions on R. $W(\xi)$ are represented via:

$$\hat{W}(\xi)\psi(x) = e^{\frac{1}{2}\alpha\beta} e^{i\alpha x}\psi(x+\beta) \tag{K.-2}$$

$U(f+g) = U(f)U(g)$, $V(f+g) = V(f)V(g)$, and satisfy Weyl's form of the CCRs

$$V(f)U(g) = e^{i<f,g>}U(g)V(f). \tag{K.-2}$$

unlike the case of finite dimensions in which we have the Stone-von Neumann theorem which asserts that there is only one irreducible representation of the CCRs (up to unitary equivalence)[1], in infinite dimensional case the Stone-von Neumann theorem does not hold. There are an infinite number of irreducible representations of the CCRs .

K.3.1 Proof of the Stone-von Neumann Theorem

The Schödiinger representation

Instead of $U(a)$ and $V(b)$ one can consider the two parameter family

$$S(a,b) = \exp\left(-\frac{1}{2}iab\right)U(a)V(b). \tag{K.-2}$$

The Weyl form of the CCR entails commutation relation for $S(a,b)$:

$$S(a,b)S(c,d) = \exp\left(\frac{1}{2}i(ad-bc)\right)S(a+c,b+d). \tag{K.-2}$$

$$\begin{aligned}
S(a,b)S(c,d) &= \exp\left(-\frac{1}{2}iab\right)\exp\left(-\frac{1}{2}icd\right)U(a)V(b)U(c)V(d) \\
&= \exp\left(-\frac{1}{2}i(ab+cd+2cb)\right)U(a+c)V(b+d) \\
&= \exp\left(\frac{1}{2}i(ad-cb)\right)\exp\left(-\frac{1}{2}i(a+c)(b+d)\right)U(a+c)V(b+d)
\end{aligned}$$

We now define the representation of CCR in terms of bounded operators

[1]The theorem that guarantees the equivalence between Schrodinger's and Heisenberg's approaches to quantum mechanics.

Definition

$$(a,b) \in \mathbb{R} \mapsto S(a,b) \in \mathcal{B}(\mathcal{H})$$

is a representation of (the Weyl form) of CCR if

$$\begin{aligned} S(-a,-b) &= S(a,b)^{\dagger} \\ S(a,b)S(c,d) &= \exp\left(-\frac{1}{2}i(ad-bc)\right) S(a+c,b+d). \end{aligned} \tag{K.-5}$$

□

Two representations S and S' of the CCR on $\mathcal{H}$ are unitarily equivalent if there exists a unitary $U : \mathcal{H} \to \mathcal{H}$ such that

$$S(a,b) = US'(a,b)U^{\dagger} \quad \text{for all} \quad a,b. \tag{K.-5}$$

A representation S of CCR on $\mathcal{H}$ is unique if S is unitarily equivalent to every representation S' of CCR on $\mathcal{H}$.

These are the conditions ensuring uniqueness of representation:

The closed linear subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ is called invariant if

$$S(a,b)\xi \in \mathcal{H}_0$$

for all $\xi \in \mathcal{H}_0$ and for all a,b. The representation $(a,b) \mapsto S(a,b)$ is

(i) irreducible if there are no non-trivial invariant subspaces

(ii) (strongly) continuous if

$$(a_n,b_n) \to (a,b) \quad \text{entails} \quad S(a_n,b_n)\xi \to S(a,b)\xi \quad \text{for all} \quad \xi \in \mathcal{H}$$

Theorem K.3.1 *Stone-von Neumann's theorem. The Schrodinger representation of the CCR on a Hilbert space $\mathcal{H}$ is the unique irreducible, (strongly) continuous representation of the CCR.*

In detail: The theorem says that if S is any irreducible, continuous representation of CCR on $\mathcal{H}$ and S^{Sch} is the Schrodinger representation on $L^2(\mathbb{R},\mu)$, then there exists a unitary operator

$$U : L^2(\mathbb{R}, \mu) \to \mathcal{H}$$

such that

$$S(a, b) = US^{Sch}(a, b)U^\dagger \quad \textit{for all} \quad a, b.$$

Proof: Any proof of this theorem had to construct with aid of P, Q or

$$U(\alpha) = e^{i\alpha P} \qquad V(\beta) = e^{i\beta Q}$$

some operator, which has easily identifiable propeerties, determining it in a unique way - and which operator on the other hand can be used to determine some vectors in Hilbert space.

The general form of the operator determined by U and V is

$$A = \int\int \varphi(\alpha, \beta)U(\alpha)V(\beta)d\alpha d\beta \tag{K.-5}$$

with an integrable function $\mathbb{R}^2 \mapsto a(\alpha, \beta)$.

$A^\dagger = A$ implies by

$$\begin{aligned} (\int\int \varphi(\alpha, \beta)U(\alpha)V(\beta)d\alpha d\beta)^\dagger &= \int\int \overline{\varphi(\alpha, \beta)}V(\beta)^\dagger U(\alpha)^\dagger d\alpha d\beta \\ &= \int\int \overline{\varphi(\alpha, \beta)}V(-\beta)U(-\alpha)d\alpha d\beta \\ &= \int\int \left(\overline{\varphi(-\alpha, -\beta)}e^{i\alpha\beta}\right) U(\alpha)V(\beta)d\alpha d\beta \end{aligned}$$

that

$$\varphi(-\alpha, -\beta) = e^{-i\alpha\beta}\overline{\varphi(\alpha, \beta)}$$

and $A = A^2$ implies by

$$
\begin{aligned}
A^2 &= \int\int \varphi(\alpha,\beta)U(\alpha)V(\beta)\int\int \varphi(\gamma,\sigma)U(\gamma)V(\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \int\int\int\int \varphi(\alpha,\beta)\varphi(\gamma,\sigma)e^{-i\gamma\beta}U(\alpha)U(\gamma)V(\beta)V(\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \int\int\int\int \varphi(\alpha,\beta)\varphi(\gamma,\sigma)e^{-i\gamma\beta}U(\alpha+\gamma)V(\beta+\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \int\int\Big(\int\int \varphi(\alpha,\beta)\varphi(\gamma-\alpha,\sigma-\beta)e^{i\alpha-\gamma,\beta}d\alpha d\beta\Big)U(\gamma)V(\sigma)d\gamma d\sigma \\
&= \int\int \varphi(\gamma,\sigma)U(\gamma)V(\sigma)d\gamma d\sigma
\end{aligned}
$$

that

$$
\int\int \varphi(\alpha,\beta)\varphi(\gamma-\alpha,\sigma-\beta)e^{-i(\alpha-\gamma)\beta}d\alpha d\beta = \varphi(\gamma,\sigma).
$$

Furthermore, by Fourier analysis the operator A can vanish only if $\varphi\equiv 0$. As noted by von Neumann, if

$$
\varphi(\alpha,\beta) = \frac{1}{2\pi}e^{-i\alpha\beta}\exp\left(-\frac{1}{4}(|\alpha|^2+|\beta|^2)\right),
$$

then the conditions for $A = A^\dagger = A^2$ are satisfied. So A can be written in terms of $S(a,b)$ as

$$
A = \frac{1}{2\pi}\int\int \exp\left(-\frac{1}{4}(|\alpha|^2+|\beta|^2)\right)S(\alpha,\beta)d\alpha d\beta. \qquad \text{(K.-13)}
$$

$$
\begin{aligned}
A^2 &= \frac{1}{(2\pi)^2}\int\int\int\int e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)-\frac{1}{4}(|\gamma|^2+|\sigma|^2)}S(\alpha,\beta)S(\gamma,\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \frac{1}{(2\pi)^2}\int\int\int\int e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)-\frac{1}{4}(|\gamma|^2+|\sigma|^2)}e^{\frac{1}{2}i(\alpha\sigma-\beta\gamma)}S(\alpha+\gamma,\beta+\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \frac{1}{(2\pi)^2}\int\int\int\int e^{-\frac{1}{4}(|\alpha-\gamma|^2+|\beta-\sigma|^2+|\gamma|^2+|\sigma|^2)}e^{\frac{i}{2}[(\alpha-\gamma)\sigma-(\beta-\sigma)\gamma]}S(\alpha,\beta)d\alpha d\beta d\gamma d\sigma \\
&= \frac{1}{(2\pi)^2}\int\int\int e^{-\frac{1}{2}|\gamma|^2+\frac{1}{2}(\alpha+i\beta)\gamma}d\gamma\int e^{-\frac{1}{2}|\sigma|^2+\frac{1}{2}(\beta+i\alpha)\sigma}d\sigma\times e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)}S(\alpha,\beta)d\alpha d\beta \\
&= \frac{1}{2\pi}\int\int e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)}S(\alpha,\beta)d\alpha d\beta
\end{aligned}
$$

The crucial observation is that the operator

$$P = A$$

is a projection operator.

Crucial to what follow is the notion of an operator which relates two representations to one another

Definition Let U and W be two reperesentations of the same group G on spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. An interwining operator for U and W is a linear transformation, T, from $\mathcal{H}$ to $\mathcal{K}$ which satisfies

$$TU(x) = W(x)T$$

for all x in G. If there exists an invertible interwining operator then U and W are said to be equivalent.

If the representation $(a, b) \mapsto S(a, b)$ is irreducible then P is one dimensional, spanned by a unit vector $\xi \in \mathcal{H}$; hence if S and S' are two irreducible representations of the CCR's and ξ' is the analogously defined vector determined by S', then the map $U : \mathcal{H} \to \mathcal{H}$ defined by

$$US(a, b)\xi = S'(a, b)\xi'$$

extends lineaily to a unit operator of $\mathcal{H}$ that interwines between the two representations S and S'.

FOLLAND:

A matrix representation of the Heisenberg Lie algebra is

$$m(p, q, t) = \begin{pmatrix} 0 & p_1 & \cdots & p_n & t \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It is easily verified that

$$m(p, q, t)m(p', q', t') = m(0, 0, pq')$$

and so

$$[m(p,q,t), m(p',q',t')] = m(0,0,pq'-qp').$$

Using

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

we have

$$\exp m(p,q,t)\exp m(p',q',t') = \exp m\big(p+p', q+q', t+t'+\frac{1}{2}(pq'-qp')\big).$$

One identifies a point $X \in \mathbb{R}^{2n+1}$ with the matrix $e^{m(X)}$, and makes $\mathbb{R}^{2n+1}$ into a group with group law

$$(p,q,t)(p',q',t') = \big(p+p', q+q', t+t'+\frac{1}{2}(pq'-qp')\big).$$

This is called the Heisenberg group and is denoted $\mathbf{H_n}$. The element $(0,0,0)$ is the identity and the inverse of the element (p,q,t) is $(-p,-q,-t)$.

We regard the operators P_j, Q_j as continuous linear (bounded) operators on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. As such they satisfy the commutation relations

$$[P_j, P_k] = [Q_j, Q_k] = 0, \qquad [P_j, Q_k] = \frac{h\delta_{jk}}{2\pi i}1 \tag{K.-19}$$

and it follows that the map

$$d\rho_h(p,q,t) = 2\pi i(hpD + qX + tI)$$

Lie algebra homomorphism.

If $f \in L^2$, let

$$g(x,t) = [e^{2\pi i t(pD+qX)}f](x)$$

$$\frac{\partial g}{\partial t} - \sum_j p_j \frac{\partial g}{\partial x_j} = 2\pi qxg, \qquad g(x,0) = f(x)$$

$$\frac{dt}{d\tau}\frac{\partial g}{\partial t} + \sum_j \frac{dx_j}{d\tau}\frac{\partial g}{\partial x_j} - 2\pi qxg = 0$$

$\vec{a}$ is perpendicular to the normal to the solution surface. We parameterize the lines using the variable τ, then they satify

$$\frac{dt}{d\tau} = 1, \quad \frac{dx_j}{d\tau} = -p_j, \quad \frac{dg}{d\tau} = 2\pi qxg$$

$$\frac{dt}{d\tau} = 1 \quad \text{implies} \quad t = \tau + k(x)$$

But $t = 0$ when $\tau = 0$ implies $k = 0$. So

$$t = \tau.$$

$$\frac{dx_j}{dt} = -p_j \quad \text{implies} \quad x_j(t) = x_j - p_j t$$

and

$$\frac{dg}{dt} = 2\pi q_j(x_j - p_j t)g$$

says

$$g(x,t) = g_0(x)e^{2\pi itq_jx_j - \pi it^2p_jq_j}$$

or

$$g = f(x)e^{2\pi itq_jx_j - \pi it^2p_jq_j}$$

Setting $t = 1$ and replacing x by $x + p$, we obtain

$$e^{2\pi i(pD+qX)}f(x) = e^{2\pi iqx + \pi ipq}f(x+p)$$

Obviously

$$\|e^{2\pi i(pD+qX)}f\|_2^2 = \|f\|_2^2$$

so $e^{2\pi i(pD+qX)}$ is a unitary operator on L^2, and it is easily checked that

$$\begin{aligned} e^{2\pi i(pD+qX)}e^{2\pi i(rD+sX)} &= e^{2\pi i[(p+r)D+(q+s)X]+\frac{1}{2}(2\pi i)^2[pD+qX,rD+sX]} \\ &= e^{2\pi i[(p+r)D+(q+s)X]+\frac{1}{2}(2\pi i)^2(ps-qr)[D,X]} \\ &= e^{2\pi i[(p+r)D+(q+s)X]+\pi i(ps-qr)I} \end{aligned}$$

Therefore the map ρ from $\mathbf{H_n}$ to the group of unitary operators on L^2 defined by

$$\rho(p,q,t) = e^{2\pi i(pD+qX+tI)} = e^{2\pi it}e^{2\pi i(pD+qX)}$$

that is,

$$\rho(p,q,t)f(x) = e^{2\pi it+2\pi iqx+\pi ipq}f(x+p) \tag{K.-22}$$

is a unitary representation of $\mathbf{H_n}$.

□

$$\rho(F) = \int\int F(p,q)\rho(p,q)dpdq = \int\int F(p,q)e^{2\pi i(pD+qX)}dpdq \tag{K.-22}$$

$$\rho(F)f(x) = \int\int F(p,q)e^{2\pi iqx+\pi ipq}f(x+p)dpdq$$

Proposition K.3.2

$$\rho(a,b)\rho(F) = \rho(G) \quad \textit{and} \quad \rho(F)\rho(a,b) = \rho(H)$$

where

$$G(p,q) = e^{\pi i(bp-aq)}F(p-a,q-b) \quad \textit{and} \quad H(p,q) = e^{\pi i(aq-bp)}F(p-a,q-b).$$

Proof:

$$
\begin{aligned}
\rho(a,b)\rho(F) &= \int\int F(p,q)\rho(a,b)\rho(p,q)dpdq \\
&= \int\int F(p,q)e^{\pi i(aq-bp)}e^{2\pi i[(a+p)D+(b+q)X]}dpdq \\
&= \int\int F(p-a,q-b)e^{\pi i(a(q-b)-b(p-a))}e^{2\pi i(pD+qX)}dpdq \\
&= \int\int e^{\pi i(aq-bp)}F(p-a,q-b)\rho(p,q)dpdq
\end{aligned}
$$

□

We return to $L^1(\mathbf{H}_\mathbf{n}^{\mathbf{red}})$. This space is a Banach algebra under convolution,

Definition Twisted convolution.

$$
\begin{aligned}
F\natural G(p,q) &= \int\int F(p',q')G(p-p',q-q')e^{\pi i(p'q-q'p)}dp'dq' \\
&= \int\int F(p-p',q-q')G(p',q')e^{\pi i(pq'-qp')}dp'dq'
\end{aligned}
$$

We call $F\natural G$ the twisted convolution of F and G.

□

Its definition is set up so that

$$\rho(F\natural G) = \rho(F)\rho(G).$$

$$
\begin{aligned}
\rho(F\natural G) &= \int_0^1\int\int\int_0^1\int\int e^{2\pi i(t'+t)}F(p',q')G(p-p',q-q')e^{\pi i(p'q-q'p)}\rho(p,q)dp'dq'dt'dpdqdt \\
&= \int_0^1\int\int\int_0^1\int\int F(p,q)G(p',q')\rho(p'+p,q'+q,t'+t+\frac{1}{2}(p'q-q'p))dp'dq'dt'dpdqdt \\
&= \int_0^1\int\int\int_0^1\int\int F(p,q)G(p',q')\rho\big((p,q,t)(p',q',t')\big)dpdqdtdp'dq'dt' \\
&= \int\int F(X)G(Y)\rho(XY)dXdY \\
&= \rho(F)\rho(G)
\end{aligned}
$$

$$
\begin{aligned}
V(f,g)(p,q) &= (\rho(p,q)f,g) = (e^{2\pi i(pD+qX)}f,g) \\
&= \int e^{2\pi iqx+\pi ipq}f(x+p)\overline{g(x)}dx \\
&= \int e^{2\pi iqy}f(y+\frac{1}{2}p)\overline{g(y-\frac{1}{2}p)}dy
\end{aligned}
$$

By the Schwartz inequality

$$
\begin{aligned}
|V(f,g)| &\leq \left(\int |f(x+\frac{1}{2}p)|^2|g(x-\frac{1}{2}p)|^2dy\right)^{1/2} \\
&\leq \left(\int |f(y+\frac{1}{2}p)|^2dy\int |g(y-\frac{1}{2}p)|^2dy\right)^{1/2} \\
&= \left(\int |f(y)|^2dy\right)^{1/2}\left(\int |g(y)|^2dy\right)^{1/2} \\
&= \|f\|_2\|g\|_2
\end{aligned}
$$

then as

$$
\|V(f,g)\|_\infty := \sup|V(f,g)|
$$

we have

$$
\|V(f,g)\|_\infty \leq \|f\|_2\|g\|_2
$$

Proposition K.3.3 *The representation ρ_h is irreducible for any $h \in \mathbb{R}/\{0\}$.*

Proof: Suppose $M \subset L^2(\mathbb{R}^n)$ is a nonzero closed invariant subspace and $f \neq 0 \in M$. If $g \perp e^{2\pi i(hpD+qX)}f$ for all $p,q \in \mathbb{R}^n$, i.e. $V(f,g)=0$. But this implies that $\|f\|_2\|g\|_2=0$, whence $g=0$ and $M=L^2(\mathbb{R}^n)$.

□

Proposition K.3.4 *Let*

$$
\phi(x) = 2^{n/4}e^{-\pi x^2}, \qquad \Phi = V(\phi,\phi), \qquad \Phi^{ab} = V(\phi,\rho(a,b)\phi).
$$

Then

(a) $\Phi(p,q) = e^{-(\pi/2)(p^2+q^2)}$.

(b) $\Phi^{ab}(p,q) = e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]}$.

(c) $\rho(\Phi)\rho(a,b)\rho(\Phi) = e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)$.

(d) $\Phi\natural\Phi^{ab} = e^{-(\pi/2)(a^2+b^2)}\Phi$.

Proof:

$$\begin{aligned}
\Phi(p,q) &= 2^{n/2}\int e^{2\pi iqy}e^{-\pi[y+(p/2)]^2-\pi[y-(p/2)]^2}dy \\
&= 2^{n/2}e^{(-\pi/2)p^2}\int e^{2\pi iqy}e^{2\pi y^2}dy = e^{(-\pi/2)(p^2+q^2)}
\end{aligned}$$

$$(p,q,0)(p,q,0) = \big(p-c, q-d, \frac{1}{2}(dp-cq)\big).$$

using this

$$\begin{aligned}
V(f,\rho(c,d)g)(p,q) &= (\rho(p,q)f, \rho(c,d)g) \\
&= (\rho(-c,-d)\rho(p,q)f, g) \\
&= e^{\pi i(dp-cq)}(\rho(p-c,q-d)f,g) \\
&= e^{\pi i(dp-cq)}V(f,g)(p-c,q-d) \qquad \text{(K.-44)}
\end{aligned}$$

$$\begin{aligned}
\Phi^{ab}(p,q) &= V(\phi,\rho(a,b)\phi)(p,q) \\
&= e^{\pi i(bp-aq)}V(\phi,\phi)(p-a,q-b) \\
&= e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]} \qquad \text{(K.-45)}
\end{aligned}$$

$$\begin{aligned}
\rho(\Phi)\rho(p,q)\rho(\Phi)f &= \rho(\Phi)\rho(a,b)[(f,\phi)\phi] \\
&= \rho(\Phi)[(f,\phi)\rho(a,b)\phi] \\
&= (f,\phi)(\rho(a,b)\phi,\phi)\phi \\
&= (f,\phi)\Phi^{ab}(a,b)\phi \\
&= e^{-(\pi/2)(a^2+b^2)}(f,\phi)\phi \\
&= e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)f \qquad \text{(K.-49)}
\end{aligned}$$

Consider $\rho(a,b)\rho(\Phi)$, it follows from proposition N.-19 that

$$G(p,q) = e^{\pi i(bp-aq)}\Phi(p-a,q-b) = \Phi^{ab}(p,q)$$

and so $\rho(a,b)\rho(\Phi) = \rho(\Phi^{ab})$. Now by (c)

$$\begin{aligned}\rho(\Phi\natural\Phi^{ab}) &= \rho(\Phi)\rho(\Phi^{ab})\\ &= \rho(\Phi)\rho(a,b)\rho(\Phi)\\ &= e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)\end{aligned} \qquad \text{(K.-50)}$$

Since ρ is faithful (i.e. $\mathrm{Ker}\rho = \{0\}$), $\Phi\natural\Phi^{ab} = e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)$.

□

$$\phi(x) = 2^{n/4}e^{-\pi x^2}$$

by (K.3.1)

$$\phi^{ab}(x) := \rho(a,b)\phi(x) = 2^{n/4}e^{2\pi ibx+\pi iab}e^{-\pi(x+a)^2}$$

$$\begin{aligned}(\phi^{pq},\phi^{ab}) &= (\rho(p,q)\phi,\rho(a,b)\phi)\\ &= V(\phi,\rho(a,b)\phi)(p,q)\\ &= \Phi^{ab}(p,q) = e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]}\end{aligned} \qquad \text{(K.-51)}$$

$$\Phi\natural\Phi^{ab} = e^{-(\pi/2)(a^2+b^2)}\Phi \qquad \text{(K.-51)}$$

$$\pi(p,q)\pi(r,s) = \pi\big(p+r,q+s,\frac{1}{2}(ps-qr)\big) = e^{\pi i(ps-qr)}\pi(p+r,q+s)$$

We consider the integrated version of π,

$$\pi(F) = \int\int F(p,q)\pi(p,q)dpdq \qquad (F\in L^1(\mathbb{R}^{2n})),$$

and just as with ρ, we have

$$
\begin{aligned}
\pi(F)\pi(G) &= \pi(F\natural G) && && \text{(K.-50)}\\
\pi(F)\pi(a,b) &= \pi(G) &&\text{where} \quad G(p,q)=e^{\pi i(aq-bp)}F(p-a,q-b) && \text{(K.-49)}\\
\pi(a,b)\pi(F) &= \pi(H) &&\text{where} \quad H(p,q)=e^{\pi i(bp-aq)}F(p-a,q-b). && \text{(K.-48)}
\end{aligned}
$$

Proposition K.3.5 *We prove that π is faithful on $L^1(\mathbb{R}^{2n})$.*

Proof: If $\pi(F)=0$ then, by (K.-49) and (K.-48), for any $u,v\in\mathcal{H}$ and $a,b\in\mathbb{R}^n$,

$$
\begin{aligned}
0 &= (\pi(a,b)\pi(F)\pi(-a,-b)u,v)\\
&= (\pi(a,b)[\int\int e^{\pi i(bp-aq)}F(p+a,q+b)\pi(p,q)dpdq]\; u,v)\\
&= \int\int e^{2\pi i(bp-aq)}F(p,q)(\pi(p,q)u,v)dpdq
\end{aligned}
$$

Thus by the Fourier inverse theorem,

$$F(p,q)(\pi(p,q)u,v)=0 \quad \text{for a.e. } (p,q),$$

and sinse u and v are arbitary, $F=0$ a.e.

$\square$

Proposition K.3.6 *If we take F to be given by $\Phi=V(\phi,\phi)$ where $\phi(x)=2^{n/2}e^{-\pi x^2}$, $\pi(\Phi)$ is a projection.*

Proof: By (K.-48), (K.-51), (K.-50), and (K.3.1).

$$
\begin{aligned}
\pi(\Phi)\pi(a,b)\pi(\Phi) &= \pi(\Phi)\big(\pi(a,b)\pi(\Phi)\big)\\
&= \pi(\Phi)\int\int e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]}\pi(p,q)dpdq\\
&= \pi(\Phi)\pi(\Phi^{ab})\\
&= \pi(\Phi\natural\Phi^{ab})\\
&= e^{-(\pi/2)(a^2+b^2)}\pi(\Phi). && \text{(K.-54)}
\end{aligned}
$$

taking $a=b=0$ we obtain

$$\pi(\Phi)^2 = \pi(\Phi),$$

and since Φ is even and real it is easy to see that $\pi(\Phi)$ is self-adjoint:

$$\pi(\Phi)^\dagger = \pi(\Phi).$$

Thus $\pi(\Phi)$ is an orthogonal projection which is non-zero since $\Phi \neq 0$ and π is faithful.

$\square$

Proposition K.3.7 *Let $\{v_\alpha\}$ be an orthogonal basis for Ran$(\pi(\Phi))$, and let*

$$\mathcal{H}_\alpha := \overline{span\{\pi(p,q)v_\alpha\}}\ , \qquad p,q \in \mathbb{R}^n.$$

$\mathcal{H}_\alpha \perp \mathcal{H}_\beta$ and $(\oplus\mathcal{H}_\alpha)^\perp = \{0\}$.

Proof: By

we have

$$\begin{aligned}(\pi(p,q)u, \pi(r,s)v) &= (\pi(-r,-s)\pi(p,q)\pi(\Phi)u, \pi(\Phi)v) \\ &= e^{\pi i(ps-qr)}(\pi(\Phi)\pi(p-r,q-s)\pi(\Phi)u, v) \\ &= e^{\pi i(ps-qr)}e^{-(\pi/2)[(p-r)^2+(q-s)^2]}(u,v) \qquad \text{(K.-55)}\end{aligned}$$

For $\alpha \neq \beta$

$$(\pi(p,q)u_\alpha, \pi(r,s)u_\beta) = e^{\pi i(ps-qr)}e^{-(\pi/2)[(p-r)^2+(q-s)^2]}\ (u_\alpha, u_\beta) = 0$$

for all $p,q,r,s \in \mathbb{R}^n$. That is

$$\mathcal{H}_\alpha \perp \mathcal{H}_\beta.$$

$\square$

We claim that $\pi\big|\mathcal{H}_\alpha$ is equivalent to ρ for all α. Indeed, fix an α and let $v^{pq} = \pi(p,q)v_\alpha$. Then by () and (),

$$(v^{pq}, v^{rs}) = (\phi^{pq}, \phi^{rs})$$

so the correspondence $v^{pq} \to \phi^{pq}$ is an isometric map V,

$$T : v^{pq} \mapsto \phi^{pq}$$

The first obvious extension of elements which can be written as linear combinations of the v^{pq}'s.

$$T\big[\sum a_{jk} v^{p_j q_k}\big] = \sum a_{jk} T v^{p_j q_k} = \sum a_{jk} \phi^{p_j q_k}$$

It follows that if $u = \sum a_{jk} v^{p_j q_k}$ and $f = \sum a_{jk} \phi^{p_j q_k}$ then

$$\|u\|_{\mathcal{H}} = \sum a_{jk} \overline{a_{lm}} (v^{p_j q_k}, v^{p_l q_m}) = \sum a_{jk} \overline{a_{lm}} (\phi^{p_j q_k}, \phi^{p_l q_m}) = \|f\|_2$$

Therefore, if the v_n converge to some element, $\tilde{v}$, of $\mathcal{H}_\alpha$, so Tv_n must converge to some element of $L^2(\mathbb{R}^n)$.

In particular $u = 0$ if and only if $f = 0$. Therefore T is unitary.

$$T(\pi\big|\mathcal{H}_\alpha)(p, q) = \rho(p, q)T$$

for all $p, q \in \mathbb{R}^n$.

i.e. the map extends by linearity and continuity to a unitary map from $\mathcal{H}_\alpha$ to $L^2(\mathbb{R}^n)$ that interwines $\pi\big|\mathcal{H}_\alpha$ and ρ.

K.3.2 Superselection Sectors

Recall that a subspace $\mathcal{S}$ of $\mathcal{H}$ is said to be invariant under a set of operators $\mathcal{A}$ if

$$\mathcal{A}\mathcal{S} \subseteq \mathcal{S}. \tag{K.-55}$$

A set of operators is reducible if there is a subspace other than than the whole space or the zero vector, are invariant under the set of operators. If a set of operators is not reducible, we say it is irreducible. A representation $\mathcal{A}$ is irreducible if the Hilbert space does not split into orthogonal subspaces and so is preserved by the action of a set of operators $\mathcal{A}$.

$$\Psi = a\psi + b\phi$$

The existence of a restriction to the superposition principle for pure states, represented by vectors in the physical Hilbert space of quantum field theory, was discovered by Wightman and Wigner, showed that this Hilbert space is the direct sum of superselection sectors in such a way that the phase relations between vectors belonging to different sectors are unobservable. Haag and Kastler, this was recognized as an aspect of the representation problem in quantum field theory, i.e. the existence of several inequivalent irreducible representations of the algebra of observables for systems with an infinite number of degrees of freedom (in contrast to the situation for non-relativistic finite systems)

The representation should be irreducible on physical grounds, if there were superselection sectors this would imply that the physically relevant information is already contained in a closed subspace.

Irreducibility of a representation is equivalent to the condition that there must exist in the Hilbert space a cyclic vector whose image under the action of $\mathcal{A}$ is dense in the Hilbert space. We prove this below and also that superselection sectors can be identified with the irreducible sectors of a reducible representation.

Proposition K.3.8 *Let $\mathcal{A}$ be a self-adjoint set of bounded operators on the Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

1) $\mathcal{A}$ is irreducible;

2) the commutant $\mathcal{A}'$ of $\mathcal{A}$, i.e., the set of all bounded operators on $\mathcal{H}$ which commute with each $A \in \mathcal{A}$, consists of multiples of the identity operator;

3) every nonzero vector $\psi \in \mathcal{H}$ is cyclic for $\mathcal{A}$ in $\mathcal{H}$, or $\mathcal{A} = 0$ and $\mathcal{H} = \mathbb{C}$.

Let $\mathcal{S}$ be a subspace which is invariant under $\mathcal{A}$, then its orthogonal complement $\mathcal{S}^{\perp}$ is also invariant. The proof is as follows: let P be the projection operator onto $\mathcal{S}$

$$P : \mathcal{H} \to \mathcal{S}.$$

Let A be in $\mathcal{A}$. Then $AP\psi \in \mathcal{S}$, so $AP\psi = PAP\psi$ for all $\psi \in \mathcal{H}$, or

$$AP = PAP$$

As the set $\mathcal{A}$ is self-adjoint, if A is in $\mathcal{A}$ then so is its adjoint $A^{\dagger}$,

$$A^{\dagger}P = PA^{\dagger}P$$

By taking the adjoint we get

$$PA = APA$$

and we see that

$$PA = AP.$$

So that if $\varphi \in \mathcal{S}^\perp$ then $P(A\varphi) = A(P\varphi) = 0$ for all $A \in \mathcal{A}$, i.e., $\mathcal{A}\mathcal{S}^\perp \subseteq \mathcal{S}^\perp$.

Proof of proposition K.3.8.

(2) $\Rightarrow$ (1)

Let P be the projection operator onto $\mathcal{S}$, by the above, P commutes with $\mathcal{A}$. If multiples of the identity operator are the only bounded operators which commute with $\mathcal{A}$ then P is either the identity operator or zero, and hence $\mathcal{S}$ is either the whole space or the subspace containing only the zero vector, so the set of operators is irreducible.

(1) $\Rightarrow$ (3)

Suppose that $\mathcal{A}$ is irreducible. Assume there is a nonzero vector ψ such that the set $\{A\psi : A \in \mathcal{A}\}$ is not dense in $\mathcal{H}$. The orthogonal complement contains at least one nonzero vector and is invariant under $\mathcal{A}$ (unless $\mathcal{A} = 0$ and $\mathcal{H} = \mathbb{C}$), and this contradicts that $\mathcal{A}$ is irreducible.

(3) $\Rightarrow$ (2)

Let B be a bounded non-self-adjoint operator which commutes with every operator in $\mathcal{A}$. If A is in $\mathcal{A}$ then so is A^*. Taking the adjoint of $BA^* = A^*B$ gives $B^*A = AB^*$. It follows that the Hermitian operators

$$\mathrm{Re}B = \frac{B + B^\dagger}{2}, \qquad \mathrm{Im}B = \frac{B - B^\dagger}{2i}$$

are also in $\mathcal{A}'$. Thus there is a self-adjoint operator $H \in \mathcal{A}'$. Each of the projection operators in the spectral decomposition of H commutes with every operator in $\mathcal{A}$. But if P is any such projector and ψ is a vector in the range of P then $\psi = P\psi$ cannot be cyclic:

$$A\psi = AP\psi = P(A\psi)$$

for all $A \in \mathcal{A}$ so that $\mathcal{A}\psi$ is not dense in $\mathcal{H}$.

[157]

Proposition K.3.9 *Let $(\mathcal{H}, \pi)$ be a nondegenerate*[2] *representation of the C^*-algebra $\mathcal{A}$. It follows that π is the direct sum of a family of cyclic sub-representaions.*

reduces the discussion of general representations down to cyclic representations. This is important because there is a canonical manner of constructing cyclic representations.

For infinite systems there are not unique representations, however, uniqueness may be restored by imposition of physical requirements.

Poincare invariance and the ground state being the state having the lowest energy - these suffice to selection a unique vacuum state and a representation of the observable operators.

K.3.3 Existence of Representations

States

K.3.4 Construction of Representations

K.4 Gel'fand−Neumark Theorem

Since we require the analogue of a Hermitian conjugate we will be interested those algebras which are equipped with an involution (or $\star$ operation). An algebra $\mathcal{A}$ written

$$\star : \mathcal{A} \to \mathcal{A} \tag{K.-55}$$

A Banach algebra is a normed space which is *complete*.

A Banach $\star-$algebra is a complex Banach algebra endowed with an $\star-$ operation such that for all $a \in \mathcal{A}$, $\|a^*\| = \|a\|$. In particular for all $a \in \mathcal{A}$,

$$\|a^* a\| \leq \|a^*\| \, \|a\| = \|a\|^2. \tag{K.-55}$$

Definition A C^*-algebra is a Banach $\star-$algebra $\mathcal{A}$ such that for all $a \in \mathcal{A}$,

$$\|a^* a\| = \|a\|^2 \tag{K.-55}$$

[2] A representation $(\mathcal{H}, \pi)$ is said to be nondegenerate if $\{\psi; \psi \in \mathcal{H}, \pi(A)\psi = 0 \text{ for all } A \in \mathcal{A}\} = \{0\}$.

Example 1:

One of the most important Banach algebras is the set $\mathcal{C}(X)$ of all bounded continuous complex functions defined on a topological space X. The case in which X is a compact Hausdorff space will have particular significance for us in this report. If X has only one point, then $\mathcal{C}(X)$ can be identified with the simplest of all Banach algebras, the algebra of complex numbers.

Example 2:

(a)

If B is a non-trivial complex Banach space, then the set $\mathcal{B}(B)$ of all bounded operators on B is a Banach algebra.

$$\begin{aligned} \|AB\| &= \sup_{\|x\|=1} \|(AB)x\| \\ &= \sup_{\|x\|=1} (\|Bx\| \, \|A\Big(\frac{Bx}{\|Bx\|}\Big)\|) \\ &\leq \sup_{\|x\|=1} \|Bx\| \sup_{\|y\|=1} \|Ay\| \\ &= \|A\| \, \|B\| \end{aligned}$$

We assume that B is non-trivial in order to guarantee that the identity in the algebraic sense.

(b)

Example 3:

An L_p space essentially consists of all measurable functions f defined on a measure space X with measure μ which are such that $|f(x)|^p$ is integrable, with

$$\|f\|_p = (\int |f(x)|^p d\mu(x))^{1/p} \tag{K.-59}$$

taken as the norm.

Gel'fand-Neumark theorem asserts that that *every commutative C⋆-algebra with identity is isomorphic to the C⋆-algebra of all continuous, bounded complex functions on a compact, Hausdorff space.*

like this

$\boxed{\text{like this}}$

$$\left(2\sqrt{\frac{a}{b}}\right)^m$$

$$\left(a+b\right)$$

K.5 Gel'fand Theory for Abelean C* Algebras

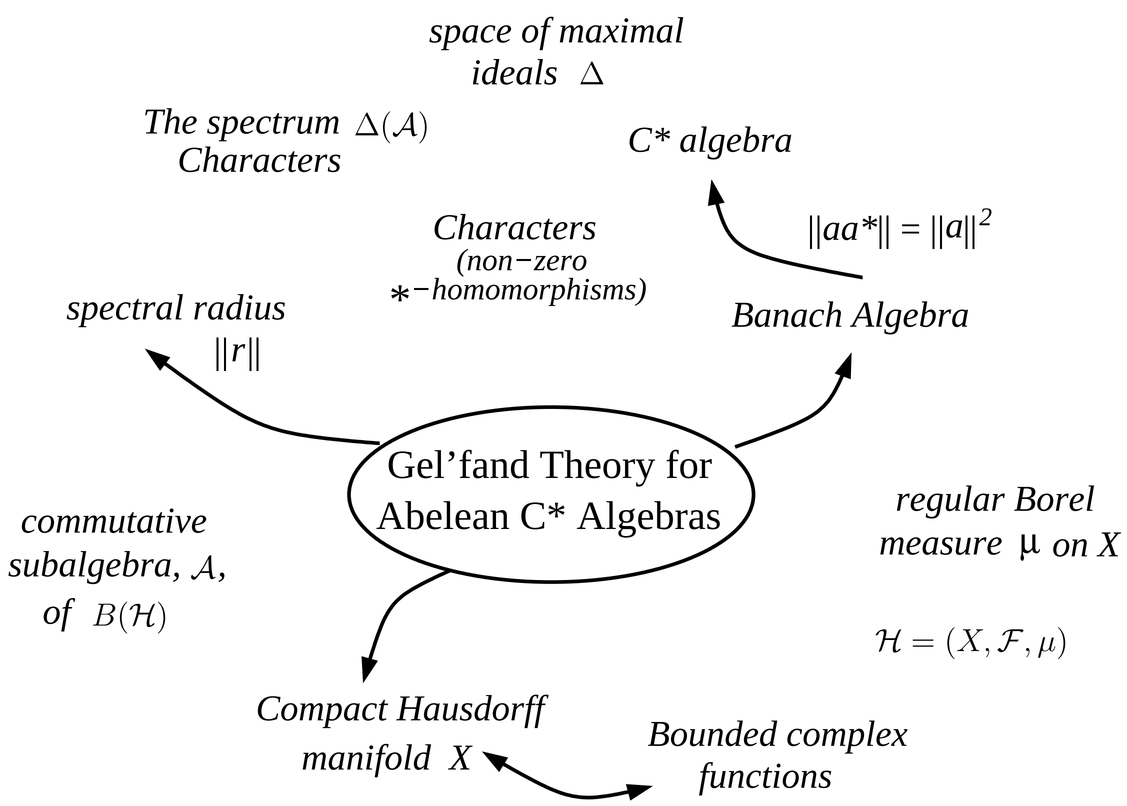

Figure K.1: Gelfand. Compact (and Hausdorff) in a natural topology. The algebra of the commutative C^*-algebra is isomorphic to $C(\Delta)$, the algebra of continuous functions on Δ.

K.6 Details of the Gel'fand−Neumark Theorem

If were only working with topologies of metric spaces (spaces that are equipped with a norm), we would need only consider sequence convergence. All the rest of the topological notions such as continuity, denseness, boundedness, closure, completeness, e.t.c. would be derived from the notion of sequence convergence. However, here we will also have to deal with topologies that are not defined by a norm. In these cases net convergence is sufficient to characterize closure of sets and that compactness can be characterized in terms of convergence of subnets.

K.6.1 Review of what we Wish to Establish

To help keep track of the spaces, maps and topologies that will be encountered in this section below we provide a list of definitions. Don't worry too much if you don't understand them straight away, their meaning will become more clear as you work through the section.

$\mathcal{A}$	an algebra
$\mathcal{A}'$	is the topological dual of $\mathcal{A}$
$\Delta(\mathcal{A})$	spectrum of the algebra $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ or space of maximal ideals
$\sigma(A)$	spectrum of the operator A
$\chi(a)$	commutative continuous homomorphism
Gel'fand map	a map from the commutative C^*-algebra $\mathcal{A}$ onto the space of continuous bounded complex functions on the spectrum of the algebra, $C(\Delta(\mathcal{A}))$
$\overset{\vee}{a}$	function on the maximal ideals
weak topology	weakest topology (the one with the least number of open sets) making all the maps $x \mapsto < \phi, x >$ continuous,
weak* topology	weakest topology making all the maps $\phi \mapsto < \phi, x >$ continuous,
Gel'fand topology	topology on the spectrum of an Abelean Banach algebra with unit is the weak* topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$

a)

b) Functional calculus

c) Isometric... commutative C^*-algebra is isomorphic with complex continuous bounded functions on a compact Hausdorff space.

K.6.2 Properties of the Spectrum of Operators

Let $\mathcal{A}$ be a Banach algebra with unit $\mathcal{I}$. An element A of $\mathcal{A}$ is invertible if there exists an element A^{-1} of $\mathcal{A}$ such that

$$A^{-1}A = AA^{-1} = \mathbb{I}.$$

Definition If $\mathcal{A}$ is a Banach algebra, then let $G(A)$ denote the group of invertible elements.

One calls the resolvent set of A the set

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda\mathcal{I} - A \text{ is invertible}\}. \tag{K.-59}$$

The partial sum which defines an element

$$S_n = \frac{1}{\lambda}\sum_{m=0}^{n}\left(\frac{A}{\lambda}\right)^m \tag{K.-59}$$

From the inequality

$$\|S_n\| \leq \frac{1}{|\lambda|}\sum_{m=0}^{n}\left(\frac{\|A\|}{|\lambda|}\right)^m \tag{K.-59}$$

we see that if $|\lambda| > \|A\|$ the partial sums $S_n := (1/\lambda)\sum_{m=1}^{n}(A/\lambda)^n$ form a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is complete, these partial sums converge to an element of $\mathcal{A}$, which is easily identified as the inverse of $\lambda\mathbb{I} - A$,

$$(\lambda\mathbb{I} - A)\frac{1}{\lambda}\left[\mathbb{I} + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\right] = \frac{1}{\lambda}\left[\mathbb{I} + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\right](\lambda\mathbb{I} - A) = \mathbb{I}. \tag{K.-59}$$

An immediate corollary of this is that an element A for which $\|A - \lambda\mathbb{I}\| < |\lambda|$ is invertible, and the inverse of such an element is given by the formula

$$A^{-1} = (\lambda\mathbb{I} - (\lambda\mathbb{I} - A))^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\mathbb{I} - A)^n}{\lambda^n} \tag{K.-59}$$

From the above, we see that $\rho(A)$ is included in $B(0, \|A\|)$.

Furthermore, if λ_0 is in $\rho(A)$ and if $|\lambda_0 - \lambda| < \|(\lambda_0\mathbb{I} - A)^{-1}\|$, then the (von Neumann) series

$$\sum_{m=0}^{\infty}(\lambda_0 - \lambda)^m(\lambda_0\mathbb{I} - A)^{-m-1} \tag{K.-59}$$

defines an element of $\mathcal{A}$ which is easily seen to be $(\lambda\mathbb{I} - A)^{-1}$:

$$
\begin{aligned}
&\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0\mathbb{I}-A)^{-m-1}(\lambda\mathbb{I}-A)\\
&=\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0\mathbb{I}-A)^{-m-1}[(\lambda_0\mathbb{I}-A)+(\lambda_0-\lambda)\mathbb{I}]\\
&=\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0\mathbb{I}-A)^{-m}-\sum_{m=0}^{\infty}(\lambda_0-\lambda)^{m+1}(\lambda_0\mathbb{I}-A)^{-(m+1)}\\
&=\mathbb{I}. \qquad \text{(K.-61)}
\end{aligned}
$$

Lemma K.6.1 *$G(A)$ is open.*

□

Definition The resolvent $\rho(A)$ of an element $A \in \mathcal{A}$ is the set of scalars such that λ such that $A - \lambda\mathbb{I}$ is regular. The spectrum $\sigma(A)$ is the set of all scalars not in $\rho(\mathcal{A})$.

Let $\mathcal{A}$ be a unital Banach algebra with unit $\mathbb{I}$. If $A \in \mathcal{A}$, then the spectrum of x is

$$\sigma(x) = \{\lambda \in \mathbb{C} : \ \lambda \cdot \mathbb{I} - x \notin G(A)\}, \qquad \text{(K.-61)}$$

and the spectral radius of x is

$$r(x) = \sup\{|\lambda| \in \mathbb{C} : \ \lambda \in \sigma(x)\}, \qquad \text{(K.-61)}$$

Theorem K.6.2 *If $\mathcal{A}$ is a Banach algebra with unit, then for each $x \in A$,*

$$
\begin{array}{ll}
(a) & \sigma(x) \textit{ is compact and nonempty.}\\
(b) & \rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}
\end{array} \qquad \text{(K.-61)}
$$

Now, if $\lambda \in \sigma(x)$, then

$$
\begin{aligned}
(\lambda^n\mathbb{I} - x^n) &= (\lambda\mathbb{I} - x)(\lambda^{n-1}\mathbb{I} + \lambda^{n-2}x + \cdots + x^{n-1})\\
&= (\lambda^{n-1}\mathbb{I} + \lambda^{n-2}x + \cdots + x^{n-1})(\lambda\mathbb{I} - x) \qquad \text{(K.-61)}
\end{aligned}
$$

implies that $\lambda^n \in \sigma(x^n)$ otherwise, $(\lambda^{n-1}\mathbb{I} + \lambda^{n-2}x + \cdots + x^{n-1})(\lambda^n\mathbb{I} - x^n)^{-1}$ is an inverse for $(\lambda\mathbb{I} - x)$. Then $|\lambda^n| \le \rho(x^n) \le \|x^n\|$. In particular, $|\lambda| \le \|x^n\|^{1/n}$. Combining with

$$\limsup_n \|x^n\|^{1/n} \;\le\; \rho(x) \;\le\; \inf_{n\ge 1} \|x^n\|^{1/n} \;\le\; \liminf_n \|x^n\|^{1/n}. \tag{K.-61}$$

This completes the proof.

□

Normal operators

Of particular interest will be operators that satisfy $x^*x = xx^*$. Such operators are called normal operators, self-adjoint and unitary operators are special cases. For normal operators we have $r(x) = \|x\|$. The proof uses the identity $\|y\|^2 = \|y^*y\|$.

$$\begin{aligned}
\|x^2\| &= (\|(x^2)\|^2)^{1/2} \\
&= \|(x^2)^*(x^2)\|^{1/2} \\
&= \|(x^*x)^*(x^*x)\|^{1/2} \qquad \text{from using } x^*x = xx^* \\
&= (\|x^*x\|^2)^{1/2} = (\|x^*x\|) \\
&= \|x\|^2
\end{aligned} \tag{K.-64}$$

From this we have $\|x\|^4 = \|x^2\|^2 = \|x^4\|$. By induction we get $\|x\|^{2n} = \|x^{2n}\|$. Thus,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \|x\| \tag{K.-64}$$

K.6.3 Functional Calculus

Theorem K.6.3 *Let A be a element of $\mathcal{A}$. The following are equivalent.*

i) A is positive

$t\cdot\mathbf{1} - A$ is a normal operator so $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} = \|A\|$.

$$\begin{aligned}
\|t\cdot\mathbf{1} - A\| &= \sup\{|\lambda| : \lambda \in \sigma(t\cdot\mathbf{1} - A)\} \\
&= \sup\{|\lambda - t| : \lambda \in \sigma(A)\} \le t \qquad \text{as } r(A) \le \|A\| \le t.
\end{aligned} \tag{K.-64}$$

□

$$\| \, \|A\| + \|B\| - A - B\| \leq \| \, \|A\| - A\| + \| \, \|B\| - B\| \leq \|A\| + \|B\| \tag{K.-64}$$

$\|t - (A + B)\| \leq t$ (with $t = \|A\| + \|B\|$), hence by Theorem N.-19 $A + B$ is positive.

Proposition K.6.4 *Let $(\mathcal{H}, \pi)$ be a representation of a C^*-algebra $\mathcal{A}$. Then the following are equivalent:*

(i) π is faithful (ker $\pi = \{0\}$).

(ii) $\|\pi(A)\| = \|A\|$ for all $A \in \mathcal{A}$

(iii) $\|\pi(A)\| > 0$ is $A > 0$.

Proposition K.6.5 *Let ω be a linear form on $\mathcal{A}$, a unital C^*-algebra. Then the following assertions are equivalent:*

(i) ω is positive;

(ii) ω is continuous with $\|\omega\| = \omega(\mathbf{1})$.

K.6.4 Null Ideals

Definition Say that $\mathcal{I}$ is a linear subspace of $\mathcal{A}$. If it satisfies the condition

(i) $i \in \mathcal{I} \Rightarrow xi \in \mathcal{I}$ for every element $x \in \mathcal{A}$, that is, $xi \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in \mathcal{A}$

then $\mathcal{I}$ is what is called a **left-sided ideal**, and if it satisfies the condition

(ii) $i \in \mathcal{I} \Rightarrow ix \in \mathcal{I}$ for every element $x \in \mathcal{A}$

it is called a **right-sided ideal**. If $\mathcal{I}$ satisfies both conditions then it is called a **two-sided ideal**, or simply an **ideal**. (Note that any Abelean algebra with a left or right ideal is automatically a two sided ideal).

Definition A maximal left ideal, MLI, in $\mathcal{A}$ is a proper left ideal not properly contained in any other proper left ideal, (proper meaning $\neq \mathcal{A}$ but only a subset of $\mathcal{A}$). A maximal right ideal, MRI, is defined similarly.

$[x] = [x_1]$ means that there is an equivalence relation $x \sim x_1$, that is, that $x - x_1$ is in $\mathcal{I}$. The elements x and x_1 are called **representatives** of the coset that contains them.

When we say a Banach algebra is **unital** this means it includes an element,

$$\mathbf{1} \cdot a = a \cdot \mathbf{1} = a, \quad \text{for all } a \in \mathcal{A}. \tag{K.-64}$$

Definition A **non-trivial** ideal is an ideal which does not contain invertible elements.

We explain the reason for the term non-trivial: say an ideal $\mathcal{I}$ contains an invertible element, denote it as a^{-1} with inverse a. The product $aa^{-1} = \mathbf{1}$ must also be in the ideal $\mathcal{I}$. Now, if $\mathbf{1}$ is in $\mathcal{I}$ the condition

$$x\mathbf{1} \in \mathcal{I} \text{ whenever } \mathbf{1} \in \mathcal{I} \text{ and } x \in \mathcal{A} \tag{K.-64}$$

on x is satisfied by every element of $\mathcal{A}$, i.e., the algebra $\mathcal{A}$ **itself is the ideal which contains 1**.

Definition Define its kernal as $ker(\chi) := \{a \in \mathcal{A};\ \chi(a) = 0\}$

Lemma K.6.6 *The kernal $ker(\chi)$ is a two-sided ideal of $\mathcal{A}$.*

Proof: ab is in the kernal $ker(\chi)$ for all $a \in \mathcal{A}$ and $b \in ker(\chi)$, making it a left-sided ideal.

$$\chi(ab) = \chi(a)\chi(b) = \chi(b) \times 0 = 0 \tag{K.-64}$$

It is obviously a right-sided ideal as well as the element ba is also in the kernel: $\chi(ba) = \chi(b)\chi(a) = 0$. The elements of $\mathcal{A}$ whose character vanishes forms an ideal of $\mathcal{A}$ for the multiplication. That is, the character of the product of an arbitrary element a with a element whose character vanishes b also vanishes.

$\square$

Lemma K.6.7 *The kernal of a character determines a maximal ideal in $\mathcal{A}$.*

Proof: Since χ is a linear functional on $\mathcal{A}$ considered as a vector space, that is, $\chi(\alpha a + \beta b) = \alpha\chi(a) + \beta\chi(b)$ for all $a, b \in \mathcal{A}$ and for all $\alpha, \beta \in \mathbb{C}$, it follows that $\ker(\chi)$ is a vector subspace of $\mathcal{A}$ (meaning for any $a, b \in \ker(\chi)$ we have $\alpha a + \beta b \in \ker(\chi)$ for all $\alpha, \beta \in \mathbb{C}$) of codimension one. To see this suppose the codimension is $N > 1$. Let $c_1, \ldots, c_N$ be a set of linearly independent elements not in $\ker(\chi)$, then

$$\chi(\alpha_1 c_1 + \cdots + \alpha_N c_N) = 0$$

can only be true if all the α_i's are zero. Now, χ being a linear functional this condition can be reexpressed

$$\alpha_1\chi(c_1) + \alpha_2\chi(c_2)\cdots + \alpha_N\chi(c_N) = 0. \tag{K.-63}$$

But this can be satisfied for non-zero α_i's, for example choose $\alpha_1/\alpha_2 = -\chi(c_2)/\chi(c_1)$ with $\alpha_3 = \cdots = \alpha_N = 0$. There is only no contradiction when the codimension is at most one. However, $\chi(a) = 0$ for all $a \in \mathcal{A}$ implies χ is identically zero which is ruled out by the defintion of the character, hence the codimension is one.

After taking the closure in $\mathcal{A}$ it is either still of codimension one or zero, the latter being impossible again since then χ would be identically zero. It follows that there exist elements $a \in \mathcal{A}-\ker(\chi)$ and that $\mathcal{A}$ is the closure of the span of a,$\ker(\chi)$. Thus, if there is an ideal $\mathcal{I}$ of $\mathcal{A}$ properly containg $\ker(\chi)$ then $\mathcal{I} = \mathcal{A}$. We conclude that the kernal of a character determines a maximal ideal in $\mathcal{A}$.

□

Definition The **quotient ring**, R with respect to the ideal $\mathcal{I}$:

Let $\mathcal{I}$ be an ideal in a ring, R, and let the coset of an element x in R be defined by $x + \mathcal{I} := \{x + i : i \in \mathcal{I}\}$. Then the distinct cosets form a partition of R. If we define addition and multiplication by

$$(x+\mathcal{I}) + (y+\mathcal{I}) = (x+y) + \mathcal{I} \quad \text{and} \tag{K.-62}$$
$$(x+\mathcal{I})(y+\mathcal{I}) = xy + \mathcal{I}, \tag{K.-61}$$

then these cosets form a ring denoted by $R/\mathcal{I}$ and called the quotient ring of R with respect to $\mathcal{I}$

Lemma K.6.8 *If $\mathcal{I}$ is a unital Banach algebra $\mathcal{A}$ then its closure $\overline{\mathcal{I}}$ is still an ideal in $\mathcal{A}$. Every maximal ideal is already closed.*

Proof: Recall that the closure of a subset Y in a topological is Y together with the limit points of

Recall that the subset Y in a topological space X is Y together with the limit points of convergent nets in Y. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$ and let (a^α) be a net in $\mathcal{I}$ converging to

$a \in \overline{\mathcal{I}}$. Then for any $b \in \mathcal{A}$ we have $ba^\alpha \in \mathcal{I}$ since $\mathcal{I}$ is an ideal and $ba^\alpha = ba$ in the limit $a^\alpha \to a$ since

$$\|b(a^\alpha - a)\| \leq \|b\| \, \|a^\alpha - a\| \to 0. \tag{K.-61}$$

Thus (ba^α) is a net in $\mathcal{I}$ converging to $ba \in \mathcal{A}$ and since all limit points of a converging net lie in $\overline{\mathcal{I}}$ we have $ba \in \overline{\mathcal{I}}$. Thus, $\overline{\mathcal{I}}$ is an ideal.

In the next step we prove that the set of non-invertible elements of $\mathcal{A}$ is closed subset. To show this, first recall a couple of facts: 1) every $a \in \mathcal{A}$ such that $\|a-1\| < 1$ is invertible; 2) that $b^\alpha \to b$ implies $\|b^\alpha\| \to \|b\|$ as follows from application of the triangle inequality Eq.(J.4.1):

$$\big| \|b^\alpha\| - \|b\| \big| \leq \|b^\alpha - b\|.$$

Now consider the set

$$\{c \in \mathcal{A};\ \|c - 1\| \geq 1\} \tag{K.-61}$$

and any convergent net (a^α) in it, converging to some element $a \in \mathcal{A}$. The net of real numbers $(\|a^\alpha - 1\|)$ belongs to the set

$$\{x \in \mathbb{R};\ x \geq 1\}$$

and since

$$a^\alpha - 1 \to a - 1 \quad \Rightarrow \quad \|a^\alpha - 1\| \to \|a - 1\|$$

it follows that $\|a-1\| \geq 1$ since $\{x \in \mathbb{R};\ x \geq 1\}$ is closed. Therefore every convergent net (a^α) in (K.6.4) converges to a point in (K.6.4) and we conclude it is a closed subset of $\mathcal{A}$.

We can now conclude that every non-trivial ideal $\mathcal{I}$, that is, those not containing invertible elements, must be contained in the closed set $\{c \in \mathcal{A};\ \|c-1\| \geq 1\}$ and so must its closure $\overline{\mathcal{I}}$. Obviously $1 \notin \{c \in \mathcal{A};\ \|c-1\| \geq 1\}$, hence, closures of non-trivial ideals are non-trivial.

Finally a maximal ideal must be closed as otherwise its closure would be a non-trivial ideal containing it.

□

K.6.5 The Gel'fand-Mazur Theorem

Theorem K.6.9 (Gel'fand)

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with $\mathbb{C}$.

$$[a] = \{a + \mathcal{I}\} \tag{K.-61}$$

By lemma K.6.8 $\mathcal{I}$ is closed in $\mathcal{A}$. The proof is split into three parts.

(a):

> If $\mathcal{I}$ is a maximal ideal in a unital Banach algebra $\mathcal{A}$ then $\mathcal{A}/\mathcal{I}$ is a Banach algebra.

The norm on $\mathcal{A}/\mathcal{I}$ is given by

$$\|[a]\| := \inf_{b\in[a]} \|b\| = \inf_{i\in\mathcal{I}} \|a + i\| \tag{K.-61}$$

(i)

$$\|[za]\| = \|z[a]\| = \inf_{b\in[a]} \|zb\| = |z| \; \|[a]\| \tag{K.-61}$$

(ii)

$$\begin{aligned} \|[a + a']\| &= \|[a] + [a']\| = \inf_{b\in[a]+[a']} \|b\| = \inf_{b\in[a]+[a']} \|b + b'\| \\ &\leq \inf_{b\in[a],b'\in[a']} (\|b\| + \|b'\|) = \|[a]\| + \|[a']\|. \end{aligned} \tag{K.-61}$$

(iii)

$$\|[a]\| = \inf_{b\in[a]} \|b\| = 0 \;\Rightarrow [a] = [0]. \tag{K.-61}$$

Suppose that $([a'_k])$ is a Cauchy sequence in $\mathcal{A}/\mathcal{I}$. Then for each n, there is a number $N(n) \geq N(n-1)$ such that

$$\|[a'_l] - [a'_k]\| < 2^{-n}, \quad l, k \geq N(n).$$

Set $a_n = a'_{N(n)}$. Then $\|[a_{n+1}] - [a_n]\| = \|[a_{n+1} - a_n]\| < 2^{-n}$. Since

$$\|[a_{n+1}] - [a_n]\| = \inf_{b_{n+1} \in [a_{n+1}], b_n \in [a_n]} \|b_{n+1} - b_n\| < 2^{-n} \tag{K.-61}$$

certainly find representatives $\|c_{n+k} - c_n\| < 2^{-n+1}$. Then

$$\|c_n - c_m\| = \|\sum_{k=m+1}^{n-1} c_{k+1} - c_k\| \leq \sum_{k=m+1}^{n-1} 2^{-k+1} = 2^{-m} \sum_{k=0}^{m-n-1} 2^k \leq 2^{-m+1} \tag{K.-61}$$

which shows that (c_n) is a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is complete this sequence converges to some $a \in \mathcal{A}$. But then

$$\|[a_n] - [a]\| = \inf_{b_n \in [a,], b \in [a]} \|b_n - b\| \leq \|c_n - a\| \tag{K.-61}$$

so $([a_n])$ converges to $[a]$. It follows that $\mathcal{A}/\mathcal{I}$ is complete, that is, a Banach space with unit $[1]$.

(b):

> For an Abelean, unital algebra $\mathcal{A}$ an ideal $\mathcal{I}$ is maximal in $\mathcal{A}$ if and only if $\mathcal{A}/\mathcal{I} - [0]$ consists of invertible elements only.

First say the ideal $\mathcal{I}$ is maximal and suppose $\mathcal{A}/\mathcal{I} - [0]$ does not consist of invertible elements only, that is, we find $[0] \neq [a] \in \mathcal{A}/\mathcal{I}$ but that $[a]^{-1}$ does not exist. This means that a^{-1} does not exist since $[a]^{-1} = [a^{-1}]$ as follows from $[a][a^{-1}] = [1]$. Consider now the ideal

$$\mathcal{A} \cdot a = \{ba : b \in \mathcal{A}\}$$

(this is a two-sided ideal because $\mathcal{A}$ is Abelean). Since $\mathcal{I} \subset \mathcal{A}$ we certainly have $\mathcal{I} \cdot a \subset \mathcal{A} \cdot a$ and since $\mathcal{I} \cdot a = \mathcal{I}$ we have

$$\mathcal{I} \subset \mathcal{A} \cdot a.$$

Now $a \in \mathcal{A} \cdot a$ since $1 \in \mathcal{A}$ and $a \notin \mathcal{I}$ because otherwise $[a] = \{i : i \in \mathcal{I}\} = \{0 + i : i \in \mathcal{I}\} = [0]$, which we excluded. It follows that $\mathcal{I}$ is a proper subideal of $\mathcal{A} \cdot a$. Finally, since

$a^{-1} \notin \mathcal{A}$, the unit element 1 can not be an element of $\mathcal{A} \cdot a$ and so $\mathcal{A} \cdot a$ cannot be all of $\mathcal{A}$. It follows

$$\mathcal{I} \subset \mathcal{A} \cdot a \subset \mathcal{A}.$$

and so $\mathcal{I}$ is not maximal, which is a contradiction.

Now say $\mathcal{A}/\mathcal{I} - [0]$ consists of invertible elements only and suppose $\mathcal{I}$ is not a maximal ideal. Then we find a proper subideal $\mathcal{J}$ of $\mathcal{A}$ of which $\mathcal{I}$ is a proper subideal. Since every non-zero element of $\mathcal{A}/\mathcal{I}$ is invertible so is every element $[a]$ of $\mathcal{J}/\mathcal{I}$ since $\mathcal{J} \subset \mathcal{A}$. But then $\mathcal{J}$ contains the invertible element $a \in \mathcal{A}$ and thus $\mathcal{J}$ coincides with $\mathcal{A}$ which is a contradiction.

(c)

> A unital Banach algebra $\mathcal{B}$ in which every non-zero element is invertible is isomorphic with $\mathbb{C}$.

Consider $b \in \mathcal{B}$ then we claim that $\sigma(b) \neq \emptyset$. Suppose that $\rho(b)$ is the whole complex plane. Let ϕ be a continuous linear functional on $\mathcal{A}$ considered as a vector space with metric $d(a, b) = \|a - b\|$.

Using linearity of ϕ and the expansion of $r_z(b)$ into an absolutely geometric series

$$r_z(b) = -\frac{1}{z}\sum_{n=0}^{\infty}(\frac{b}{z})^n$$

we see that $z \mapsto \phi(r_z(b))$

$$\phi(r_z(b)) = -\frac{1}{z}\sum_{n=0}^{\infty}(\frac{\phi(b)}{z})^n$$

is an entire analytic function - see sections K.15.1 and K.15.2 for more details. An entire function is a complex function analytic an arbitary distance from the origin. Since ϕ is linear and continuous, it is bounded with bound $\|\phi\|$. Thus $|\phi(r_z(b))| \leq \|\phi\| \; \|r_z(b)\|$. Consider

$$\begin{aligned}\Big|\sum_{n=1}^{N} z^{-n}\phi(b^{n-1})\Big| &\leq \sum_{n=1}^{N}\frac{\|\phi\|\ \|b\|^{n-1}}{k^n\|b\|^n}\\ &\leq \frac{\|\phi\|}{\|b\|}\sum_{n=1}^{\infty}\frac{1}{k^n}\\ &= \frac{\|\phi\|}{(k-1)\|b\|}\end{aligned}\qquad \text{(K.-62)}$$

Thus

$$\|r_z(b)\| \leq \frac{1}{(k-1)\|b\|}, \quad |z| \geq k\ \|b\|. \qquad \text{(K.-62)}$$

This shows that $\|r_z(b)\| \to 0$ as $|z| \to \infty$. In particular, $z \mapsto \phi(r_z(b))$ is an entire bounded function which therefore, by Liouville's theorem, is a constant.

Proof of Liouville's theorem: by Cauchy's integral formula,

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz.$$

If we take C to be a circle $|z - z_0| = r_0$, then

$$\begin{aligned}|f'(z_0)| &= \left|\frac{1}{2\pi i}\right| \oint_{C_0}\frac{|f(z)|}{|(z-z_0)^2|}|dz|\\ &< \frac{1}{2\pi r_0^2}M2\pi r_0 = \frac{M}{r_0},\end{aligned}\qquad \text{(K.-62)}$$

where $|f(z)| < M$ within and on C_0. As $f(z)$ is entire, we may take r_0 as large as we like. So given any $\epsilon > 0$ we may make $|f'(z_0)| < \epsilon$. That is, $|f'(z_0)| = 0$, which implies that $f'(z_0) = 0$ for all z_0, so $f(z_0) =$ constant. QED

Since $\phi(r_z(b)) \to 0$ as $|z| \to \infty$, we must have $\phi(r_z(b)) = 0$. Since ϕ was arbitrary it follows that

$$r_z(b) = (z_b 1 - b)^{-1} = 0$$

implying that $b - z_b \cdot 1$ does not exist for otherwise it would imply

$$1 = (z_b 1 - b)(z_b 1 - b)^{-1} = 0.$$

Thus we find $z_b \in \sigma(b)$, that is, $b - z_b \cdot 1$ is not invertible. By assumption, only zero elements are not invertible, hence $b = z_b \cdot 1$ for some $z_b \in \mathbb{C}$ for any $b \in \mathcal{B}$. The map $b \mapsto z_b$ is then the searched for isomorphism $\mathcal{B} \to \mathbb{C}$. Notice that $b = 0$ if and only if $z_b = 0$.

Let then $\mathcal{I}$ be a maximal ideal in a unital, Abelean Banach algebra $\mathcal{A}$. Then by a) $\mathcal{B} := \mathcal{A}/\mathcal{I}$ is a unital Banach algebra and by b) each of its non-zero elements is invertible. Thus by c) it is isomorphic with $\mathbb{C}$.

□

Corollary K.6.10 *In an Abelean, unital Banach algebra $\mathcal{A}$ there is a one-to-one correspondence between its spectrum $\Delta(\mathcal{A})$ and the set $I(\mathcal{A})$ of maximal ideals $\mathcal{A}$ via*

$$\Delta(\mathcal{A}) \to I(\mathcal{A});\ \chi \mapsto ker(\chi). \tag{K.-62}$$

Proof: We know by lemme K.6.7 that each character gives rise to a maximal ideal in $\mathcal{A}$.

Conversely, let $\mathcal{I}$ be a maximal ideal in a commutative unital Banach algebra then we can apply theorem K.6.9 and obtain a Banach algebra isomorphism

$$\chi : \mathcal{A}/\mathcal{I} \to \mathbb{C}; \quad [a] \mapsto \chi([a]).$$

We can extend this to a homomorphism

$$\chi : \mathcal{A} \to \mathbb{C}$$

by $\chi(a) := \chi([a])$ By construction $\chi(a) = 0$ if and only if $[a] = [0]$, that is, if and only if $a \in \mathcal{I}$.

□

Lemma K.6.11 *Let $\mathcal{A}$ be a unital, commutative Banach algebra and $a \in \mathcal{A}$. Then $z \in \sigma(a)$ if and only if there exists $\chi \in \Delta(\mathcal{A})$ such that $\chi(a) = z$.*

Proof:

The requirement $\chi(a) = z$ is equivalent to $\chi(a - z \cdot \mathbf{1}) = 0$ so that $a - z \cdot \mathbf{1} \in \ker(\chi)$. Since is a maximal ideal in $\mathcal{A}$ it cannot contain invertible elements, thus $(a - z \cdot \mathbf{1})^{-1}$ does not exist, hence $z \in \sigma(a)$.

□

Definition For a character χ in an Abelian, unital Banach algebra we define its norm by

$$\|\chi\| := \sup_{a\in\mathcal{A}} |\chi(a)| \tag{K.-62}$$

Lemma K.6.12 *The characters of an Abelian, unital Banach algebra form a subset of the unit sphere in $\mathcal{A}'$, the continuous linear functionals on $\mathcal{A}$ considered as a topological vector space.*

Proof:

$$\begin{aligned}
\|\chi\| &= \sup_{a\in\mathcal{A}} \frac{|\chi(a)|}{\|a\|} \le \sup_{a\in\mathcal{A}} \frac{\sup\{|\chi'(a)|;\ \chi' \in \Delta(\mathcal{A})\}}{\|a\|} \\
&= \sup_{a\in\mathcal{A}} \frac{\sup\{|z|;\ z \in \sigma(z)\}}{\|a\|} \qquad \text{by lemma K.6.11} \\
&= \sup_{a\in\mathcal{A}} \frac{r(a)}{\|a\|} \le 1
\end{aligned} \tag{K.-63}$$

since $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \le \|a\|$. On the other hand $\chi(1) = 1$, hence $\|\chi\| = 1$ for every character χ. This shows that every character is a bounded linear functional $\mathcal{A}$, that is, $\Delta(\mathcal{A}) \subset \mathcal{A}'$.

□

Definition Recall that the topological dual X' of the topological vector space X is the set of continuous (bounded) linear functionals. The weak * topology on the topological dual X' is defined by pointwise convergence, that is a net (ϕ^α) in X' converges to ϕ if and only if for any $x \in X$ the net of complex numbers $(\phi^\alpha(x))$ converges to $\phi(x)$).

More details here

This is equivalent to:

it is the weakest topology such that all the functions $x:\ X' \to \mathbb{C}; \phi \to \phi(x)$ are continuous.

K.6.6 Gel'fand Topology

weak* convergence of a sequence of functionals. Then weak* convergence of (ϕ_n) means that there is a $\phi \in X'$ such that $\phi_n(x) \to \phi(x)$ for all $x \in X$.

Every character is a bounded linear functional on $\mathcal{A}$, that is, $\Delta(\mathcal{A}) \subset \mathcal{A}'$. The Gel'fand topology on the spectrum of a unital, Abelean Banach algebra is the weak * topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$.

Proof

Product of topologies

Definition: Let $\{X_\alpha\}$ be a family of topological spaces, $\alpha \in A$. Define a subset

$$B(\beta, U_\beta) = \{f | f \in \times X_\alpha, f(\beta) \in U_\beta\} \tag{K.-63}$$

The Gel'fand topology on the spectrum of a unital, Abelian Banach algebra is the weak * topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$.

or put as (notes on the Spectral...):

the weak$-\star$ topology is the on A^* is the smallest topology making the maps $\phi \mapsto \phi(x)$ continuous from A^* to $\mathbb{C}$ for each $x \in A$. It follows that the Gel'fand topology is the restriction of the weak$-\star$ topology to $\Delta(A)$.

Lemma K.6.13 *Let X be a Banach space and X' its topological dual. Then the unit ball in X' is closed and compact in the weak topology.*

Proof: The unit ball B in X' is defined as the subset of elements ϕ with norm smaller than or equal to unity, that is,

$$\|\phi\| := \sup_{x \in X} \frac{|\phi(x)|}{\|x\|} \leq 1.$$

Let ϕ^α be a universal net in B and consider for any given $x \in X$ the net of complex numbers $(\phi^\alpha(x))$ which are bounded by $\|x\|$. Our x a function $X' \to \mathbb{C}; \phi \to \phi(x)$. If (x_α) is a universal net in a space X, and $f : X \to Y$ is a function, then $f(x_\alpha)$ is a universal net in Y, with no restriction on f. The net $(\phi^\alpha(x))$ is universal. It is contained in the set $\{z \in \mathbb{C} : |z| \leq \|x\|\}$ which is compact in $\mathbb{C}$ and therefore it converges. Define ϕ pointwise by the limit, that is, $\phi(x) := \lim_\alpha \phi^\alpha(x)$. Then

$$\|\phi\| = \sup_{x \in X} \lim_{\alpha} \frac{|\phi^\alpha(x)|}{\|x\|} \leq \|\phi^\alpha\| \leq 1. \tag{K.-63}$$

Thus ϕ^α converges pointwise to $\phi \in B$. Since a topological space is compact if and only if every universal net converges we conclude that B is compact.

A subset Y of a topological space X is closed if for every convergent net $(x_\alpha) \in X$ with $x_\alpha \in Y$ for all α the limit lies in Y. Say there is a convergent net (ϕ_α) which converges to an element ϕ not in B. Consider any open set U containing ϕ. Then for any $\alpha \in I$ such that $\phi^\alpha \in U$, $(\phi_{F(\beta)})_{\beta \in J} \in U$ for all $\beta \succeq \beta'$ for some β' by definition of a subnet. Therefore any subnet is eventually in U. But every net has a universal subnet. This contradicts that every universal net of B converges to a limit point in B. So in particular we have shown that B is closed.

□

Theorem K.6.14 *In Gel'fand topology, the spectrum $\Delta(\mathcal{A})$ of a unital, Abelian algebra is compact.*

Proof of Theorem K.6.14

By lemma K.6.12 $\Delta(\mathcal{A})$ is a subset of the unit ball B in $\mathcal{A}'$ and by lemma K.6.13 B is compact in the weak * topology. By () we know that closed subspaces of compact spaces are compact in the subspace topology. As the Gel'fand topology is the subspace topology induced from B, to prove the theorem we need only show that $\Delta(\mathcal{A})$ is closed in B

Let then (χ^α) be a net in $\Delta(\mathcal{A})$ converging to $\chi \in B$. We have, e.g., $\chi(ab) = \lim_\alpha \chi^\alpha(ab) = \lim_\alpha \chi^\alpha(a)\chi^\alpha(b)$ and similar for pointwise addition, scalar multiplication and involution in $\mathcal{A}$. It follows that the limit χ is a character as well, that is, $\chi \in \Delta(\mathcal{A})$.

□

K.6.7 The Gel'fand Transformation

Definition The Gel'fand transform is defined by

$$\bigvee : \mathcal{A} \to \Delta(\mathcal{A})'; \quad a \mapsto \overset{\vee}{a} \quad \text{where} \quad \overset{\vee}{a}(\chi) := \chi(a). \tag{K.-63}$$

Where $\Delta(\mathcal{A})'$ denotes the continuous linear functionals on $\Delta(\mathcal{A})$ considered as a topological vector space.

Theorem K.6.15 *The Gel'fand transform extends to a homomorphism*

$$\bigvee : \mathcal{A} \to C(\Delta(\mathcal{A})); \quad a \to \overset{\vee}{a} \tag{K.-63}$$

with the following additional properties:

i) range$(\overset{\vee}{a}) = \sigma(a)$.

ii) $\| \overset{\vee}{a} \| := \sup_{\chi \in \Delta(\mathcal{A})} | \overset{\vee}{a} (\chi)| = r(a)$.

iii) The image $\bigvee(\mathcal{A})$ *separates the points of* $\Delta(\mathcal{A})$.

Proof:

i) We have

$$\text{range}(\overset{\vee}{a}) = \{\overset{\vee}{a} (\chi) : \chi \in \Delta(\mathcal{A})\} = \sigma(a) \tag{K.-63}$$

as follows from lemma K.6.11.

ii) We have

$$\| \overset{\vee}{a} \| = \sup_{\chi \in \Delta(\mathcal{A})} | \overset{\vee}{a} (\chi)| = \sup_{\chi \in \Delta(\mathcal{A})} |\chi(a)| = \sup(\{|\chi(a)| : \chi \in \Delta(\mathcal{A}\}) = r(a) \tag{K.-63}$$

by definition of the spectral radius.

iii) Consider any $\chi_1, \chi_2 \in \Delta(\mathcal{A})$ with $\chi_1 \neq \chi_2$. By definition of $\Delta(\mathcal{A})$ there exists then $a \in \mathcal{A}$ such that $\chi_1(a) =\overset{\vee}{a} (\chi_1) \neq \chi_2(a) =\overset{\vee}{a} (\chi_2)$.

□

Lemma K.6.16 *The Gel'fand topology is on the spectrum of a unital Abelean Banach algebra is Hausdorff.*

Proof: The proof follows trivially from the fact that by theorem K.6.15 $\mathcal{C} := \{\overset{\vee}{a} \, ; a \in \mathcal{A}\}$ is a system of continuous functions separating the points of $\Delta(\mathcal{A})$ and applying lemma J.9.7.

□

Theorem K.6.17 *Let* $\mathcal{A}$ *be a unital commutative* C^*-*algebra (not only Banach algebra). Then the Gel'fand transform is an isometric isomorphism between* $\mathcal{A}$ *and the space of continuous functions on its spectrum.*

Proof:

For Abelean C^*-algebras we have

$$r(a) = \|a\|. \tag{K.-63}$$

By theorem K.6.15 we therefore have

$$\| \overset{\vee}{a} \| = \|a\| \tag{K.-63}$$

that is, isometry.

Consider now the system of complex valued functions on the spectrum given by $\mathcal{C} := \{\overset{\vee}{a}: a \in \mathcal{A}\}$. We claim that it has the following properties:

i) $\mathcal{C} \subset C(\Delta(\mathcal{A}))$

ii) $\mathcal{C}$ separates points of $\Delta(\mathcal{A})$

iii) $\mathcal{C}$ is a closed (in the sup-norm topology) $*$ subalgebra of $C(\Delta(\mathcal{A}), \mathbb{C})$

iv) The constant functions belong to $\mathcal{C}$.

We show iii) $\mathcal{C}$ is a closed * algebra in $C(\Delta(\mathcal{A}))$. Suppose that $(\overset{\vee}{a^\alpha})$ is a net in $\mathcal{C}$ converging to some $f \in C(\Delta(\mathcal{A}))$. Thus, $(\overset{\vee}{a}{}^\alpha)$ is in particular a Cauchy sequence, meaning that $\| \overset{\vee}{a}{}^\alpha - \overset{\vee}{a}{}^\beta\| = \|a^\alpha - a^\beta\|$ becomes arbitrarily small as α, β grow, where we have used isometry. It follows that (a^α) is a Cauchy sequence and therefore converges to some $a \in \mathcal{A}$ since $\mathcal{A}$ ia a banach algebra and therefore complete.

Now we can establish isomorphism. From theorem K.6.14 and corollary K.6.16 that $\Delta(\mathcal{A})$ is a compact Hausdorff space. Recall the Stone-Weierstrass theorem J.11.4 which states that if Y be a compact Hausforff space, and $\mathcal{C}$ be a closed subalgebra of $C(Y, \mathbb{C})$ such that $\mathcal{C}$ is unital, closed under complex conjugation and separates points of $C(Y, \mathbb{C})$ then $\mathcal{C} = C(Y)$. Properties i), ii), iii), iv) of $\mathcal{C}$ enable us to apply the theorem, Y being $\Delta(\mathcal{A})$, and conclude $\mathcal{C} = C(\Delta(\mathcal{A}))$. In other words, the Gel'fand tranform is a surjection. Finallly it is an injection since $\overset{\vee}{a} = \overset{\vee}{a}{}'$ implies $\| \overset{\vee}{a} - \overset{\vee}{a}{}' \| = \|a - a'\| = 0$ by isometry, hence $a = a'$.

□

Corollary K.6.18 *Every compact Hausdorff space X arises as the spectrum of an Abelean unital C^*-algebra $\mathcal{A}$, specifically $\mathcal{A} = C(X)$, $\Delta(\mathcal{A}) = X$.*

Let X be a compact Hausdorff space and define $\mathcal{A} := C(X)$ equipped with the sup-norm.

Next let (x^α) be a net in X which converges in $\Delta(C(X))$ then by theorem J.10.5 $\overset{\vee}{f}(x^\alpha)$ converges in $\mathbb{C}$ for any $\overset{\vee}{f} \in C(\Delta(C(X)))$, i.e., $f(x^\alpha)$ converges in $\mathbb{C}$ for any $f \in C(X)$. But by theorem J.10.5 $f : X \to \mathbb{C}$ is continuous only if whenever $f(x^\alpha)$ is a net convergent in $\mathbb{C}$ then the net (x^α) converges to $x \in X$. From theorem J.10.3 it follows that X is closed in $\Delta(C(X))$.

Suppose now that $\Delta(C(X)) - X \neq \emptyset$. Thus there exists $\chi_0 \in \Delta(C(X)) - X$. By lemma J.9.3 we know that one point sets in a Hausdorff space are closed. Therefore the sets $X, \{\chi_0\}$ are disjoint closed sets in the compact Hausdorff space $\Delta(C(X))$. Since compact Hausdorff spaces are normal spaces (theorem J.9.8) we may apply Urysohns's lemma (lemma J.9.10) to conclude that there is a continuous function $F : \Delta(C(X)) \to \mathbb{R}$ with range in $[0, 1]$ such that $F_{|X} = 0$ and $F|\{\chi_0\} = F(\chi_0) = 1$.

Consider then any $f \in C(X)$. Since $C(\Delta(C(X)))$ are all continuous functions on $\Delta(C(X))$, from the functions F just constructed we find different continuous extensions of f to $C(\Delta(C(X)))$: $\overset{\vee}{f} \to \overset{\vee}{f} + F$. However, this contradicts the fact that $\bigvee$ is an isomorphism since it would not be surjective.

□

Corollary tells us that a compact Hausdorff space can be reconstructed from its Abelean, unital C^*−algebra of continuous fnctions by constructing its spectrum.

K.7 Spectral Theorem and GNS-Construction

[408]:

"...C^*−algebras occupy a a very special place among all Banach algebras. This should be compared with the role played by Hilbert spaces among Banach spaces. In fact, as we will see there is an intimate relationship between Hilbert spaces and C^*−algebras thanks to the GNS construction and the Gel'fand-Neumark embedding theorem."

To each state, ω, the GNS-construction associates a representation, π_ω, of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ together with a cyclic vector $\Omega \in \mathcal{H}_\omega$ such that

$$\omega(A) = < \Omega | \pi_\omega(A) \Omega > \tag{K.-63}$$

A folium of a given state ω which may be defined to be the set of all states ω_σ which arise in the form $\mathrm{Tr}(\sigma \pi_\omega(\,\cdot\,))$ where σ ranges over the density operators (trace-class operators with unit trace) on $\mathcal{H}_\omega$.

Given a state, ω, and an automorphism, α, which preserves the state (i.e. $\omega \circ \alpha = \omega$) then there will be a unitary operator, U, on $\mathcal{H}_\omega$ which implements α in the sense that $\pi_\omega(\alpha(A)) = U^{-1}\pi_\omega(A)U$ and U is chosen uniquely by the condition $U\Omega = \Omega$.

Normal operators form a C^*-algebra

Let $\mathcal{A}$ be a unital abelean C^*-algebra generated by $1, x, x^*$. For normal operators $x^*x = xx^*$ the algebra generated by $C(\{1, x, x^*\})$ is a C^*-algebra.

$$\|x\psi\|^2 =< \psi, x^*x\psi >= < \psi, xx^*\psi >= \|x^*\psi\|^2$$

so that $\|x\| = \|x^*\|$ for any $x \in \mathcal{A}$. By the Schwartz inequality

$$\|x\|^2 = \|x^*\|^2 = | < \psi, x^*x\psi > | \leq \|\psi\| \; \|x^*x\psi\|$$

implying $\|x\|^2 = \|x^*\|^2 \leq \|x\|^2$. On the other hand, we always have $\|x^*x\| \leq \|x\| \; \|x^*\|$. Hence,

$$\|x^*x\| = \|x\|^2. \tag{K.-63}$$

Consider the spectrum $\Delta(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ and the map

$$z : \Delta(\mathcal{A}) \to \mathbb{C} \quad ; \quad \chi \mapsto \chi(a)$$

which is continuous by the definition of the Gel'fand topology on the spectrum. We have seen already that the range of this map coincides with $\sigma(a)$. Moreover, z is injective because $\chi(a) = \chi'(a)$ implies that χ, χ' coincide on all polynomials of a, a^* since they are homomorphisms,

$$\begin{aligned}
\chi(p(a, a^*)) &= \chi(c_0 + c_1 a + d_1 a^* + c_1 a^2 + \dots) \\
&= c_0 + c_1\chi(a) + d_1\overline{\chi(a)} + c_2\chi(a)^2 + \dots \\
&= c_0 + c_1\chi'(a) + d_1\overline{\chi'(a)} + c_2\chi'(a)^2 + \dots \\
&= \chi'(c_0 + c_1 a + d_1 a^* + c_2 a^2 + \dots) \\
&= \chi'(p(a, a^*))
\end{aligned}$$

and thus on all of $\mathcal{A}$ by continuity whence $\chi = \chi'$. Thus, z is a continuous bijection between the spectra $\Delta(\mathcal{A})$ and $\sigma(a)$. Since a is bounded, both spectra are compact Hausdorff spaces. Now a continuous bijection between compact Hausdorff spaces is automatically a homeomorphism.[3] We conclude that we can identify $\Delta(A)$ topologically with $\sigma(a)$.

By defintion the polynomials p in a, a^* lie dense in $\mathcal{A}$ and we have for $\chi \in \Delta(\mathcal{A})$ that

$$\chi(p(a, a^*)) = p(\chi(a), \overline{\chi(a)}) = p(z(\chi), \overline{z(\chi)}) = [p \circ (z, \overline{z})](\chi) = p(a, a^*) \vee (\chi) \tag{K.-68}$$

so that the Gel'fand isometric isomorphism can be thought of as a map

$$\bigvee : \mathcal{A} \to C(\sigma(a)) \quad ; \quad b \mapsto \overset{\vee}{b} \quad \text{with} \quad \overset{\vee}{b}(z) = \chi(b)_{z=\chi(a)}.$$

Now consider any state $\psi \in \mathcal{H}$ with $\|\psi\| = 1$. Then

$$\omega_\psi : \mathcal{A} \to \mathbb{C} \quad ; \quad b \mapsto <\psi, b\psi> \tag{K.-68}$$

is obviously a state on $\mathcal{A}$. Via the Gel'fand transform we obtain a positive linear functional on $C(\sigma(a))$ by

$$\Lambda_\psi : C(\sigma(a)) \to \mathbb{C} \quad ; \quad \overset{\vee}{b} \mapsto \omega_\psi(b) \tag{K.-68}$$

and since $\sigma(a)$ is a compact Hausdorff space we can apply the Riesz representation theorem in order to find a unique, regular Borel measure μ_ψ on $\sigma(a)$ such that

$$\omega_\psi(b) = \int_{\sigma(a)} d\mu_\psi \, b(z). \tag{K.-68}$$

The measure μ_ψ is caled a spectral measure. The meaning of this formula is explained by the following definition.

Extension to normal operators

Theorem K.7.1 *Let (a_I) be a self-adjoint collection of mutually commuting elements of a C^*-algebra $\mathcal{C}$. Then there exists a representation of the sub-C^*-algebra $\mathcal{A}$ generated by this collection on a Hilbert space $\mathcal{H}$ such that the $\pi(a_I)$ become multiplication operators.*

[3]Proof: Let $f : X \to Y$ be a continuous bijection and let X be compact and Y Hausdorff. We must show that $f(U)$ is open in Y for every open subset $U \subset X$, or by taking complements, that images of closed sets are closed. Now since X is compact, it follows that every closed set U is also compact. Since f is continuous, it follows that $f(U)$ is compact. Since Y is Hausdorff it follows that $f(U)$ is closed. See theorems N.-19 and N.-19.

K.7.1 Bits and Pieces

$$1 \in \mathcal{B} \subseteq \mathcal{A}.$$

Then clearly

$$\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x). \tag{K.-68}$$

Theorem K.7.2 *Suppose that $\mathcal{B}$ is a unital C^*-subalgebra of a C^*-algebra $\mathcal{A}$ (i.e., $1 \in \mathcal{B} \subseteq \mathcal{A}$ with $\star : \mathcal{B} \to \mathcal{B}$). Then for all $x \in \mathcal{B}$, $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$.*

Fix $x \in \mathcal{B}$. We need only show that if $x \notin \sigma_{\mathcal{A}}(x)$ then $x \notin \sigma_{\mathcal{B}}(x)$ (i.e., $x^{-1} \in \mathcal{B}$), then $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$ follows from Eq.(K.7.1).

$x^{-1} = (x^*x)^{-1}x^* \in \mathcal{B}$ if x is. As $x^* \in \mathcal{B}$ it suffices to show that $(x^*x)^{-1} \in \mathcal{B}$ if x is.

Let $\mathcal{C}$ be a C^*-subalgebra of $\mathcal{A}$ generated by x and x^{-1}, and let $\mathcal{D}$ be the $\star-$subalgebra generated by 1 and x.

Therefore, $\mathcal{D} = \mathcal{C}$ by the Stone-Weierstrass Theorem. In particular, $x^{-1} \in \mathcal{D} \subseteq \mathcal{B}$.

□

Theorem K.7.3 *J is a maximal ideal of $\mathcal{A}$ if and only if J is the kernal of some $h \in \Delta$.*

We establish the **first part** of K.7.3:

> If J is a maximal ideal of $\mathcal{A}$ then J is the kernal of some $h \in \Delta$.

Let J be the a maximal ideal of $\mathcal{A}$. Since J is closed, Now choose $x \in \mathcal{A}$ so that $\pi(x) \neq 0$. Thus, $x \notin J$, and

$$M = \{ax + y \ : \ a \in \mathcal{A} \text{ and } y \in J\} \tag{K.-68}$$

is an ideal in $\mathcal{A}$ so that $J \subset M$ but $J \neq M$. Therefore $M\mathcal{A}$; in particular, for some $a \in \mathcal{A}$ and $y \in JU$,

$$ax + y = e. \tag{K.-68}$$

...that π defines a complex homomorphism with kernal J.

We now move onto the **second part** of K.7.3:

If J is the kernal of some $h \in \Delta$ then J is a maximal ideal of $\mathcal{A}$.

If $h \in \Delta$, then we must show that $J = h^{-1}(0)$ is a maximal ideal.

$\square$

Notes on operator algebras (G. Jungman)

Theorem K.7.4 (Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative C^*-algebra, and equip Δ with the Gelfand topology as usual. Then the Gelfand transform is an isometric $*-$isomorphism of $\mathcal{A}$ onto the algebra of continuous complex-valued functions on Δ, $C(\Delta)$.*

The next theorem provides a continuous symbolic calculus for operators as long as they generate a commutative C^*-algebra. So, for example, if x is a normal operator then we apply the above theorem to the algebra generated by x and x^*, and we get the continuous functional calculus for normal operators.

Theorem K.7.5 (Inverse Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative C^*-algebra. Let $x \in \mathcal{A}$ be such that the polynomials in x and x^* are dense in $\mathcal{A}$. Then we can define an isometric isomorphism $\Phi : C(Spec(x)) \to \mathcal{A}$ by*

$$\widehat{(\Phi f)} = f \circ \hat{x},$$

and we have

$$\Phi f^* = (\Phi f)^*.$$

Proof. Let

$\square$

By the Gelfand-Naimark theorem, $f \circ \hat{x}$ is thus the Gelfand transform of a unique element in $\mathcal{A}$ which we denote Φf, and $\|\Phi f\| = \|f\|_\infty$. If $f(\lambda) = \lambda$, then $f \circ \hat{x} = \hat{x}$ and $\Phi f = x$.

Compare to section J.3.4

Theorem K.7.6 *Let (a_I) be a self-adjoint collection of mutually commuting elements of a C^*-algebra $\mathcal{C}$. Then there exists a representation of the sub$-C^*-$algebra $\mathcal{A}$ generated by this collection on a Hilbert space $\mathcal{H}$ such that the $\pi(a_I)$ become multiplication operators.*

K.7.2 Extension of Spectral Theorem to Unbounded Operators

The extension of the spectral theorem to unbounded self-adjoint operators on a Hilbert space can be traced back to the bounded case by using the following trick. (Recall that a densely defined operator a with domain $D(a)$ is called self-adjoint if $a^\dagger = a$ and $D(a^\dagger) = D(a)$ where

$$D(a^\dagger) = \{\psi \in \mathcal{H};\ \sup_{0 \neq \psi' \in D(a)} | < \psi, a\psi' > | \,/\, \|\psi'\| < \infty\}$$

K.8 *Further Details on Algebraic Quantization

If the set is over complete, we impose anti-commutation relations. For example if $FG = H$ where H is also a member of the algebra $\mathcal{S}$ then we impose the operator equation

$$\hat{F} \cdot \hat{G} + \hat{G} \cdot \hat{F} - 2\hat{FG} = 0, \tag{K.-68}$$

More generally, if $F_1, F_2, \cdots, F_n$, as well as thier product, $F_1 F_2 \cdots F_n$, belong to the space $\mathcal{S}$, we require that

$$F_{(1} F_2 \cdots F_{n)} - (F_1 \cdot \hat{\cdot} \cdot F_n) = 0. \tag{K.-68}$$

K.9 Examples: Shrödinger and Weyl-Representations

$$W(\xi_1) W(\xi_2) = e^{\frac{i}{2} \Im \xi_1 \bar{\xi}_2} W(\xi_1 + \xi_2), \tag{K.-68}$$

$$[W(\xi)]^* = W(-\xi) \tag{K.-68}$$

this property is called involution $*$

This is the Weyl-Heisenberg $\star$-algebra

$$W(\xi) = e^{\frac{i}{2} \lambda \mu} U(\lambda) V(\mu) \tag{K.-68}$$

$$U(\lambda_1) U(\lambda_2) = U(\lambda_1 + \lambda_2), \quad V(\mu_1) V(\mu_2) = V(\mu_1 + \mu_2),$$
$$U(\lambda) V(\mu) = e^{\frac{-i}{2} \lambda \mu} V(\mu) U(\lambda) \tag{K.-68}$$

K.10 Uniqueness Proof of the Ashtekar-Lewandowski-Isham Representation

K.10.1 Uniqueness

K.10.2 Irreducibility

Imcomplete

By the Stone-von Neumann theorem, every irreducible, weakly continuous representation of the Weyl algebra is unitarily equivalent to the Schrodinger representation.

.....

The proof was analogous to the original proof by von Neumann for the Schrodinger representation of the standard Weyl algebra.

K.11 Algebraic Quantum Field Theory

In AQFT one cleanly separates two parts of quantizing a field theory, namely first to define a suitable algebra $\mathcal{A}$ and then study its representations in a second step.

Reformulating QFT on an axiomatic basis: that is, starting from what seem to be physically necessary and mathematically precise principles which any QFT would have to satisfy, and then finding QFTs which actually satisfy them. This program is now generally referred to as algebraic quantum field theory, or AQFT; (see [114], and references therein, for extensive discussion of it).

Very few concrete theories have been found which satisfy the AQFT axioms. To be precise, the only known background dependent theories in four dimensions which do satisfy the axioms are interaction-free. The only known AQFT-compatible interacting field theories, and in particular the standard model are the background independent quantum field theories.

Operators of observables localized in an open region of spacetime $\mathcal{O}$ form an algebra $\mathcal{A}(\mathcal{O})$ of bounded operators. It is possible to encode all physically relevant properties in terms of these algebras and their transformation behaviour under Poincare group.

In AQFT one uses the mathematical framework of operators, which have been surveyed above and combines it with the physical concept of locality of nets of local algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$. for each open region $\mathcal{O}$ one assigns a C^*-algebra $\mathcal{A}(\mathcal{O})$.

(i) locality: operators localized in causally disjoint region commute,

(ii) covariance: the spacetime symetries act ....

(iii) stability??

hep-th/9901015 a QFT theory is cast into an inclusion preserving map

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) \tag{K.-68}$$

a unital C^*-algebra $\mathcal{A}(\mathcal{O})$. The Hermition elements of the abstract C^*-algebra $\mathcal{A}(\mathcal{O})$ are interpreted as the observables which can be measured at times and locations in $\mathcal{O}$. The physical states are described by positive, linear, and normalized functionals.

GNS-construction any state ω on $\mathcal{A}$ gives rise to a Hilbert space $\mathcal{H}_\omega$ and a representation π_ω together with a cyclic vector Ω_ω, such that

$$\omega(a) = (\Omega_\omega, \pi_\omega(a)\Omega_\omega) \tag{K.-68}$$

A measure μ_ω on K is induced by ω via the Riesz-Markov representation theorem.

The net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$

M. Rainer, *Is Loop Quantum Gravity a QFT?*, [gr-qc/9912011].

$$\mathcal{L}(\mathcal{H}) \supset \mathcal{A}(\mathcal{O}) := \{\phi(x)\ x\ \in \mathcal{O}\} \tag{K.-68}$$

Vice versa, the entity of point-like localized fields can be reconstructed from localized algebras as

$$\{\phi(x)\} = \cap_{x\in\mathcal{O}}\overline{\mathcal{A}(\mathcal{O})} \tag{K.-68}$$

K.11.1 Wightman Axioms

The Wightman axioms describe the action of the vacuum of the action of the group of inhomogeneous Lorentz transformations $\{a, \Lambda\}$, where Λ denotes a homogeneous Lorentz transformation and where a denotes a spacetime translation. they involve constructing the fields $\varphi(x, t)$, along with a unitary, positive-energy representation $U(a, \Lambda)$ of the inhomogeneous Lorentz group - they act on the Hilbert space $\mathcal{H}$ of the quantum theory. Furthermore, one requires that a zero-energy ground state $\Omega \in \mathcal{H}$ of the full group $U(a, \Lambda)$, and that is unique.

The unitary representation $U(a, \Lambda)$ of the Lorentz group of spacetime symmetries determines a $*-$automorphism group of transformations of fields,,

$$\alpha_{a,\Lambda} : \varphi \to U(a,\Lambda)\,\varphi\,U(a,\Lambda)^*. \tag{K.-68}$$

K.11.2 Haag-Kastler Axioms

Operator Algebras Department of Mathematics

Postulate 1

To each region $\mathcal{O}$ in Minkowski space, $\mathcal{M}$, there corresponds a sub$-C^*-$algebra $\mathcal{B}(\mathcal{O})$ of $\mathcal{B}$. Moreover, $\mathcal{B}$ generated by the algebras $\mathcal{B}(\mathcal{O})$ as $\mathcal{O}$ runs over regions of $\mathcal{M}$.

The next axiom expresses the notion that if one region is contained inside another, the bigger region will have as many or more observables associated with it.

Postulate 2 (Isotony)

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are regions in Minkowski space with $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then

$$\mathcal{B}(\mathcal{O}_1) \subseteq \mathcal{B}(\mathcal{O}_2). \tag{K.-68}$$

Observables associated with space-like separated regions should not affect each other and so should be simultaneously measurable. This means that elements of the C^*-algebra, they must commute.

Postulate 3 (Causality/locality)

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated regions, then the associated local algebras of observables $B \in \mathcal{B}(\mathcal{O}_1)$ and $B \in \mathcal{B}(\mathcal{O}_2)$ commute, i.e., for any $A \in \mathcal{B}(\mathcal{O}_1)$ and $B \in \mathcal{B}(\mathcal{O}_2)$, we have $AB = BA$.

Poincaré covariance of the theory is expressed in the following axiom.

Postulate 4 (Poincaré covariance)

There is a representation α of $\mathcal{P}_+^\dagger$, the restricted Poincaré group, in Aut $\mathcal{B}$ automorphisms group of $\mathcal{B}$, such that

$$\alpha(L)(\mathcal{B}(\mathcal{O})) = \mathcal{B}(\Lambda\mathcal{O} + a) \tag{K.-68}$$

for any region $\mathcal{O}$ and $L = (a,\Lambda) \in \mathcal{P}_+^\dagger$.

Postulate 5

The quasilocal algebra $\mathcal{B}$ is primitive, that is, $\mathcal{B}$ possesses a faithful, irreducible representation.

K.12 Constructive Quantum Field Theory

K.13 Algebraic Quantization

Algebraic Quantization

1. The set $\mathcal{S}$ should be a vector space large enough so that every function on Γ can be obtained by (possibly a limit of) sum of products of elements of $\mathcal{S}$. The purpose of this condition is that we have enough observables be unambiguously quantized.

2. The set $\mathcal{S}$ should be small enough so that it is closed under Poisson brackets from the vector space $\mathcal{S}$ as the free associative algebra generated by $\mathcal{S}$. It is this quantum algebra that we impose the Dirac quantization condition: Given A, B and $\{A, B\}$ in $\mathcal{S}$ we impose

$$[\hat{A}, \hat{B}] = i\hbar \{A, \hat{\,} B\} \tag{K.-68}$$

We must now find a vector space V and a representation of the elements of $\mathcal{A}$ as operators on V. Real observables must be represented by Hermition operators. One then completes V to get the Hilbert space $\mathcal{H}$ of the theory.

Refined Algebraic Quantization

RHS

K.14 von Neumann Algebras

K.14.1 Introduction

We tend to perceive time as 'flowing', as though it were in smooth and perpetual continuous motion,

'emergence of time'

has a thermodynamical origin.

observables living in a region R can be computed from observables living in region S if R is in the causal shadow of S. von Neumann algebras also enter here.

Also very important in non-commutative geometry.

K.14.2 The Emergence of Time

Generally covariant theories and The problem of time

see week 41 beaz

Rovelli wants to use thermodynamics to **define** what we call time as we usualy mean. does this as follows. Given a mixed state with density matrix D, find some operator H such that D is the Gibbs state $\exp(-H/kT)$. In lots of cases this isn't hard; it basically amounts to

$$H = -kT \ln D \tag{K.-68}$$

Of course, H will depend on T, but this is really is just saying that fixing your temperature fixes your units of time!

Operator theorists have pondered this notoin very carefully for a long time and generalised it into the Tomita-Takesaki theorem. This gives a very general way of finding a Hamiltonian (hence a notion of time evolution) from a state of a quantum system! For example, one can use this trick to start with a Robertson-Walkerr universe full of blackbody radiation, and recover a notion of "time".

Gibb's distribution

The quantum state is then given by the Gibbs density matrix

$$\omega = Ne^{\beta H} \tag{K.-68}$$

where H is the hamiltonian, defined on a Hilbert space $\mathcal{H}$, and $N = tr[e^{\beta H}]$.

The KMS condition

$$\omega() = \omega() \tag{K.-68}$$

Mathematics preliminaries of von Neumann Algebras

Given a state ω over an abstract C^*-algebra $\mathcal{A}$, the well known Gelfand-Naimark-Segal construction provides us with a Hilbert space $\mathcal{H}$ with a preferred state $|\Psi_0>$, and a representation π of $\mathcal{A}$ as a concrete algebra of operators on $\mathcal{H}$, such that

$$\omega(A) =< \Psi_0|\pi(A)|\Psi_0 > . \tag{K.-68}$$

Some definitions

The name trace class comes from its property that if A is trace class, then for any orthononormal basis $\{\varphi_n\}$

$$tr(A) = \sum_n < e_n, Ae_n > \tag{K.-68}$$

is finite and independent of the orthononormal basis.

A state's **folium** is the set of states ω_ρ on $\mathcal{U}$ defined by

$$\omega_\rho := \frac{\mathrm{tr}_{\mathcal{H}_\omega}(\rho\pi_\omega(a))}{\mathrm{tr}_{\mathcal{H}_\omega}(\rho)} \tag{K.-68}$$

where ρ is a positive trace class operator on the GNS Hilbert space $\mathcal{H}_\omega$.

The folium of states is the set of states determined by the density matrices on the Hilbert space of the given representation.

In the following, we denote $\pi(A)$ simply as A. Given ω and the corresponding GNS representation of $\mathcal{A}$ in $\mathcal{H}$, the set of all the states ρ over $\mathcal{A}$ that can be represented as

$$\rho(A) = Tr[A\rho] \tag{K.-68}$$

where ρ is a positive trace-class operator in $\mathcal{H}$, is denoted as the folium determined by ω. In the following, we shall consider an abstract C^*-algebra $\mathcal{A}$, and a preferred state ω. A von Neumann algebra $\mathcal{R}$ is then determined, as the closure of $\mathcal{A}$ under the weak topology determined by the folium of ω. (expand - make easier)

The KMS or modular condition is a mathematically rigorous requirement for a state on a C^*-algebra to be a thermal equilibrium states, [??]. Roughly this may be understood that for finite systems a thermal state is given by a (normal) trace-class opeerator in the GNS-Hilbert space. It is here that one is able to calculate the KMS condition as it is given in the below defition below. The important property of this relation now is, that it survives the thermodynamical limit, and it is a relation which can be formulated purely in terms of C^*-algebras without taking reference to any representation space.

Modular automorphisms

The modular flow of ω is

$$\alpha_t A = e^{i\beta t H} A e^{i\beta t H}, \tag{K.-68}$$

namely it is the time flow generated by the hamiltonian, with the time rescaled as $t \to \beta t$.

K.14.3 von Neumann Algebras

Let $\mathcal{R}$ be an algebra acting on a Hilbert space $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be a cyclic vector for $\mathcal{R}$ if the set $\{x\Omega : x \in \mathcal{R}\}$ is dense in $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be separating for $\mathcal{R}$ if $x\Omega = 0$ for $x \in \mathcal{R}$ implies that $x = 0$.

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on a complex Hilbert space $\mathcal{H}$ and 1 the identity operator on $\mathcal{H}$. The commutant of a set $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ is

$$\mathcal{U}' = \{A \in \mathcal{B}(H) : [A, B] = 0 \text{ for all } B \in \mathcal{U}\}.$$

If the commutant $\mathcal{U}'$ consists of multiples of the identity, then $\mathcal{U}$ is irreducible. Let $\mathcal{U}'' := (\mathcal{U}')'$, the double commutant of $\mathcal{U}$.

Lemma K.14.1 *(or theorem??) Ω is cyclic for $\mathcal{R}$ if and only if it is separating for the commutant $\mathcal{R}'$.*

Proof: Suppose Ω is cyclic for $\mathcal{R}$, we wish to show that Ω is separating for $\mathcal{R}'$. We suppose $y\Omega = 0$ for some y in $\mathcal{R}'$, then for any $x \in \mathcal{R}$ $yx\Omega = xy\Omega = 0$. So we have $y(x\Omega) = 0$ for all $x \in \mathcal{R}$. As $\mathcal{R}\Omega$ is dense in $\mathcal{H}$, it must be that $y = 0$.

Suppose Ω is separating for $\mathcal{R}'$, we wish to show that Ω is cyclic for $\mathcal{R}$. let p be the projection onto the closure of the subspace $\mathcal{R}\Omega$ in $\mathcal{H}$,

$$p : \mathcal{H} \to \overline{\mathcal{R}\Omega}$$

To prove the result we will show that in fact $p = \mathbf{1}$ as this implies that $\mathcal{R}\Omega$ is dense in $\mathcal{H}$. For any $f \in \mathcal{H}$, $pf \in \overline{\mathcal{R}\Omega}$ so that

$$pf = \lim_{n\to\infty} a_n \Omega$$

for some sequence $\{a_n\} \in \mathcal{R}$ (if $f \in \mathcal{H}$ but $f \notin \overline{\mathcal{R}\Omega}$ then $a_n = 0$??). For any $b \in \mathcal{R}\Omega$ we can write

$$bpf = \lim_{n\to\infty} ba_n\Omega = \lim_{n\to\infty} pba_n\Omega = pbpf$$

and so $bp = pbp$. For any $x \in \mathcal{R}$, setting $b = x$ and $b = x^*$ gives $xp = pxp$ and $x^*p = px^*p$ respectively. Taking adjoints of this last equality, we find $px = pxp$ and hence

$$px = xp,$$

i.e., $p \in \mathcal{R}'$. By the definition of p, $p\Omega = \Omega$, that is, $(\mathbf{1} - p)\Omega = 0$. It is obvious that $(\mathbf{1} - p) \in \mathcal{R}'$ and so as Ω is separating for $\mathcal{R}'$, we deduce $p = \mathbf{1}$.

□

It follows from the above that if Ω is both cyclic and separating for $\mathcal{R}$, it is also cyclic and separating for $\mathcal{R}'$.

Note that a weaker topology has less open sets so that if a set is closed in the weak topology it is necessarily closed in the strong and norm topologies.

Definition Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators containing the unit $\mathbf{1}$. $\mathcal{R}$ is said to be a von Neumann algebra if it is weakly closed in $\mathcal{B}(\mathcal{H})$.

$$\omega(a) = Tr[a\omega] \tag{K.-68}$$

Let $\mathcal{R}$ be a von Neumann algebra generated by $\pi_\omega(\mathcal{A})$, i.e., $\mathcal{R} = (\pi_\omega(\mathcal{A}))''$, where $\mathcal{A}'$ denotes the cummutant of $\mathcal{A}$

K.14.4 von Neumann's Density Theorem

for the other four topologies sequencies are not sufficient. the resulting topologies are not first countable, and so the closure of a subset N of $\mathcal{B}(\mathcal{H})$ is generally larger than the set of all limit points of sequencies in N. Rather, the closure of N is the set of all limit points of generalised sequencies (nets) in N.

$$
\begin{aligned}
\sum_{i=1}^{n} | < y_i, (C_\alpha - C)x_i > | &\leq \sum_{i=1} \|y_i\| \, \|(C_\alpha - C)\| \, \|x_i\| \\
&\leq \|(C_\alpha - C)\| \Big[\sum_{i=1}^{n} \|y_i\|^2 \Big]^{1/2} \Big[\sum_{i=1}^{n} \|x_i\|^2 \Big]^{1/2} \\
&< \infty \qquad \text{(K.-69)}
\end{aligned}
$$

Note that a weaker topology has less open sets so that if a set is closed in the weak topology it is necessarily closed in the strong and norm topologies.

A self-adjoint algebra $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if it satisfies one of the following equivalent conditions:

(1) $\mathbf{1} \in \mathcal{R}$ and $\mathcal{R}$ is closed in the strong operator topology.[4]

(2) $\mathbf{1} \in \mathcal{R}$ and $\mathcal{R}$ is closed in the weak operator topology.[5]

(3) $\mathcal{R} = \mathcal{R}''$.

Preliminary proposition

Proposition K.14.2 *For any subsets $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, we have that*

(1) $\mathcal{N}' \subseteq \mathcal{M}'$ and

(2) $\mathcal{M} \subseteq \mathcal{M}''$.

(3) For any subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, the set $\{\mathcal{M} \cup \mathcal{M}^*\}'$ is a von Neumann algebra.

Proof: (1) All the elements of $\mathcal{B}(\mathcal{H})$ which commute with $\mathcal{N}$ also commute with $\mathcal{M}$. As $\mathcal{N}$ is larger, it is a more demanding requirement that elements of $\mathcal{B}(\mathcal{H})$ commute with it. Hence, $\mathcal{N}' \subseteq \mathcal{M}'$.

(2) If B belongs to $\mathcal{M}'$ and A belongs to $\mathcal{M}$ then $AB = BA$, and thus $A \in (\mathcal{M}')' = \mathcal{M}''$. This proves the inclusion, but there may be elements of $\mathcal{B}(\mathcal{H})$ which commute with $\mathcal{M}'$ but are not in $\mathcal{M}$. Hence, $\mathcal{M} \subseteq \mathcal{M}''$.

(3) If $M \in \{\mathcal{M} \cup \mathcal{M}^*\}$ then the self-adjoint operators $(M + M^*)/2$ and $(M - M^*)/2i$ are in $\{\mathcal{M} \cup \mathcal{M}^*\}$. So we can take the elements of $\{\mathcal{M} \cup \mathcal{M}^*\}$ to all be self-adjoint. Now say $MD - DM = 0$ for all $M \in \{\mathcal{M} \cup \mathcal{M}^*\}$, by taking the adjoint of this, we see that if

[4]this requirement means that if $\{x_n\}$ is a sequence of operators in $\mathcal{R}$ such that for all $\Phi \in \mathcal{H}$ one has $x_n\Phi \to x\Phi$ for some $x \in \mathcal{B}(\mathcal{H})$.

[5]this requirement means that if $\{x_n\}$ is a sequence of operators in $\mathcal{M}$ such that for all $\Phi \in \mathcal{H}$ and $x' \in \mathcal{B}^*(\mathcal{H})$?? one has $(x', x_n\Phi) \to (x', x\Phi)$ for some $x \in \mathcal{B}(\mathcal{H})$.

$D \in \{\mathcal{M} \cup \mathcal{M}^*\}'$ then so is D^*. Again we can take the elements of $\{\mathcal{M} \cup \mathcal{M}^*\}'$ to all be self-adjoint.

It is straightforward show that, for every subset $\mathcal{U}$ of $\mathcal{B}(\mathcal{H})$, $\mathcal{U}'$ is weakly closed in $\mathcal{B}(\mathcal{H})$. From (2) we have $\mathcal{U} \subset \mathcal{U}''$, and applying (1) to this gives $\mathcal{U}''' \subset \mathcal{U}'$. On the other hand from (2) we also have $\mathcal{U}' \subset (\mathcal{U}')''$. So we have $\mathcal{U}''' \subseteq \mathcal{U}' \subseteq \mathcal{U}'''$, that is,

$$\mathcal{U}' = (\mathcal{U}')''$$

(This also implies $\mathcal{U}'' = (\mathcal{U}'')''$, $\mathcal{U}''' = (\mathcal{U}''')''$, and so on). That $\mathcal{U}'$ is weakly closed easily follows. If (A_n) is a sequence in $\mathcal{U}'$ which converges weakly to A in $\mathcal{B}(\mathcal{H})$ then for all $B \in \mathcal{U}$ and all $x, y \in \mathcal{H}$ we have

$$| < x, (AB - BA)y > | \leq | < x, (A - A_n)By > | + | < x, B(A - A_n)y > | \qquad \text{(K.-69)}$$

Thus A belongs to $\mathcal{U}'$. Putting $\mathcal{U}' = \{\mathcal{M} \cup \mathcal{M}^*\}'$ completes the proof of (3).

□

Theorem K.14.3 (von Neumann density theorem (Bicommutant theorem)). *Let $\mathcal{R}$ be a self-adjoint algebra in $\mathcal{B}(\mathcal{H})$ which contains the identity $\mathbf{1}$. Then the statements are equivalent:*

(1) $\mathcal{R}$ is weakly dense in $\mathcal{B}(\mathcal{H})$

(2) $\mathcal{R}$ is strongly dense in $\mathcal{B}(\mathcal{H})$

(3) $\mathcal{R}$ is ultra-strongly dense in $\mathcal{B}(\mathcal{H})$

(4) $\mathcal{R}$ is ultra-weakly dense in $\mathcal{B}(\mathcal{H})$

and their respective closures are all the same and equal to $\mathcal{R}''$.

(5) $\mathcal{R} = \mathcal{R}''$

Proof. of Theorem K.14.3.

$\mathcal{R}$ is a von Neumann algebra if and only if $\mathcal{R} = \mathcal{R}''$.

Firstly note that, by applying (ii) twice we see that $\mathcal{R}''$ is a von Neumann algebra and hence we have

$$\mathcal{R} \subseteq \mathcal{R}'' = \overline{\mathcal{R}''}^w.$$

We need to show that

$$\overline{\mathcal{R}}^{us} = \mathcal{R}''$$

that is, we need to show that if $B_\alpha \in \mathcal{R}$ and if A such that $\sum_{i=1}^{\infty} \|(B_\alpha - A)x_i\|^2 \to 0$ then A is in $\mathcal{R}''$.

for $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$

$$\sum_{i=1}^{\infty} \|(Ax_i - Bx_i)\|^2 < \epsilon \tag{K.-69}$$

$$x_i \in \mathcal{H}_i, \qquad \{x_1, x_2, \ldots, x_n, \ldots\} = x \in \bigoplus_i \mathcal{H}_i \tag{K.-69}$$

with $\tilde{A}$ defined by

$$\tilde{A}x = \{Ax_1, Ax_2, \ldots, Ax_n, \ldots\} \in \bigoplus_i \mathcal{H}_i \tag{K.-69}$$

We will say $\tilde{A} \in \tilde{\mathcal{R}}$. define

$$\|x\|^2 = \sum_{i=1}^{\infty} \|x_i\|^2 \quad \text{and} \quad \|(\tilde{A}x - \tilde{B}x)\|^2 = \sum_{i=1}^{\infty} \|(Ax_i - Bx_i)\|^2 \tag{K.-69}$$

Thus we wish to find for any given $\tilde{B} \in \tilde{\mathcal{R}}''$ any point $x \in \bigoplus_i \mathcal{H}_i$ such that $\|x\|^2 < \infty$ and any $\epsilon > 0$, there exists an $\tilde{A} \in \mathcal{R}$ such that $\|(\tilde{A}x - \tilde{B}x)\|^2 < \epsilon$.

It will be enough to show that there is some linear combinations of $\{Ax_1, Ax_2, \ldots, Ax_n, \ldots\}$, $y = (L_i A x_i)_1^{\infty}$ (L_i is a scalar), such that $\|(y - \tilde{B}x)\|^2 < \epsilon$. That is, it will be enough to show that $\tilde{B}x$ is in the closed linear span $\mathcal{V}$ of $\{\tilde{A}x : A \in \mathcal{R}\}$ in $\bigoplus_i \mathcal{H}_i$.

$$P : \bigoplus_i \mathcal{H}_i \to \mathcal{V} \tag{K.-69}$$

$C \in (\tilde{\mathcal{R}})'$

$$\sum_j AC_{ij} y_j = \sum_j C_{ij} A y_j \tag{K.-69}$$

for all $i \in \mathbb{N}$ and all $(y_j) \in \bigoplus_j \mathcal{H}_j$ for any $A \in \mathcal{R}$.

Take $y_1, y_2, \ldots, y_n \in \mathcal{H}$ and $y = y_1 \oplus + \cdots \oplus y_n \in \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. $y = (y_i)$

$\tilde{A}(x_1 \oplus \cdots \oplus x_n) = Ax_1 \oplus \cdots \oplus Ax_n$ or $\tilde{A}x = (Ax_i)$

let

$$C \in \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$$

$$E_i : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \to \mathcal{H}_i$$

$$C_{ij} = E_i C E_j$$

$C_{ij} \in \mathcal{B}(\mathcal{H})$

$C = \sum_{i,j} E_i C_{ij} E_j$

$$(\tilde{A}C - C\tilde{A})y = 0$$

for all $\tilde{A} \in \tilde{\mathcal{R}}$ then $C \in (\tilde{\mathcal{R}})'$

$$\sum_{i,j} (\tilde{A}C_{ij} - C_{ij}\tilde{A})y = 0$$

$$\sum_j A(C_{ij} y_i) = \sum_j C_{ij}(Ay_j)$$

In other words, $C \in (\tilde{\mathcal{R}})'$ if and only if $C_{ij} \in \mathcal{R}'$.

Find B such that

such that $\|(A \oplus \cdots \oplus A - B \oplus \cdots \oplus B)y\| < \epsilon$

□

Factors

Classification of von Neumann algebras.

If $\mathcal{P}(\mathcal{R})$ stands for the set of projection operator of a von Neumann algebra $\mathcal{R}$, then $\mathcal{R}$ is the smallest von Neumann algerba containing $\mathcal{P}(\mathcal{R})$. Infact von Neumann algebras are completely determine by the collection of projection operators

Type I. $0 < p < s$

Type II.

Type III.

$$\mathcal{R} \cap \mathcal{R}' \qquad \text{(K.-69)}$$

K.14.5 Tomita-Takesaki Theorem

$(\mathcal{R}, \omega)$ a von Neumann algebra acting on on some Hilbert space $\mathcal{H}$, with normal faithful state ω on $\mathcal{R}$.

The GNS representation:

i) π is a morphism form $\mathcal{R}$ to $\mathcal{B}(\mathcal{H})$.

ii) $\omega(A) =< \Omega, \pi(A)\Omega >$.

iii) $\omega(\mathcal{R})\Omega$ is dense in $\mathcal{H}$.

Proposition K.14.4 *The vector Ω is cyclic and separating for $\mathcal{R}$ and $\mathcal{R}'$.*

Proof. Ω is cyclic for $\mathcal{R}$ by iii) above. Let us see it is separating for $\mathcal{R}$. If $A \in \mathcal{R}$ is such that $A\Omega = 0$ then $\omega(A^*A) = 0$, but as ω is faithful (i.e. $\omega(A^*A) = 0$ only if $A = 0$) we have $A = 0$. By the comment after lemma K.14.1, Ω is cyclic and separating for $\mathcal{R}'$.

□

Let ω be of the form

$$\omega(A) = tr(\rho A)$$

Let (σ_t) be the following group of automorphisms

$$\sigma_t(A) = e^{itH} A e^{-itH}$$

for some self-adjoint operator H in...

(1)

$$\omega(A\sigma_t(B)) = \omega(\sigma_{t-\beta i}(B)A) \tag{K.-69}$$

(2)

$$\rho = \frac{1}{\mathcal{Z}} e^{-\beta H}, \tag{K.-69}$$

where $\mathcal{Z} = tr(\exp(-\beta H))$.

Proof. (1) $\Rightarrow$ (2)

$$\begin{aligned}
\omega(A\sigma_t(B)) &= \frac{1}{\mathcal{Z}} tr(e^{-\beta H} A e^{itH} B e^{-itH}) \\
&= \frac{1}{\mathcal{Z}} tr(A e^{itH} B e^{(-it-\beta)H}) \\
&= \frac{1}{\mathcal{Z}} tr(A e^{-\beta H} e^{(it+\beta)H} B e^{(-it-\beta)H}) \\
&= \frac{1}{\mathcal{Z}} tr(e^{-\beta H} e^{(it+\beta)H} B e^{(-it-\beta)H} A) \\
&= \omega(\sigma_{t-\beta i}(B)A)
\end{aligned} \tag{K.-72}$$

(2) $\Rightarrow$ (1)

$$\begin{aligned}
tr(AB\rho) &= tr(\rho AB) = \omega(AB) \\
&= \omega(\sigma_{-\beta i}(B)A) \\
&= tr(\rho e^{\beta H} B e^{-\beta H} A) \\
&= tr(A\rho e^{\beta H} B e^{-\beta H})
\end{aligned} \tag{K.-74}$$

this is valid for any A so

$$B\rho = \rho e^{\beta H} B e^{-\beta H} \tag{K.-74}$$

for all B.

$$B(\rho e^{\beta H}) = (\rho e^{\beta H})B.$$

we conclude that $\rho e^{\beta H}$ is a multiple of the identity.

The Tomita-Takesaki theorem asserts that the mappings $\alpha_t : \mathcal{R} \to \mathcal{R}$, $t \in \mathbb{R}$, given by

$$\alpha_t(b) = \Delta^{-it} b \Delta^{it} \tag{K.-74}$$

Theorem K.14.5 *Let $\mathcal{R}$ be a von Neumann algebra with a cyclic and separating vector Ω. Then $J\Omega = \Omega = \Delta\Omega$ and the following hold:*

$$J\,\mathcal{R}\,J = \mathcal{R}' \quad \textit{and} \quad \Delta^{-it}\,\mathcal{R}\,\Delta^{it} = \mathcal{R}, \qquad \textit{for all } t \in \mathbb{R}. \tag{K.-74}$$

two automorphisms of $\mathcal{R}$ are inner equivalent if the differ by an inner automorphism. are automorphisms which are inner equivalent

$$u\alpha''(a) = \alpha'(a)u \tag{K.-74}$$

for every $a \in \mathcal{R}$ and some $u \in \mathcal{U}$.

The set of all equivalence classes of this relation is the outer automorphisms of $\mathcal{R}$, denoted $\mathrm{Out}(\mathcal{R})$.

K.14.6 The Statistical State of the Universe

Rovelli "The Statistical State of the Universe" [290].

K.15 Fock-Bargmann Representation

In the conventional representation the Hilbert space of vectors is formed by the space of complex valued, square intergrable, coordinate or momentum functions $\psi(q)$ and $\tilde{\psi}(p)$. No analyticity conditions are placed on these complex functions. However, there exists a representation in which any state vector is described by an entire analytic function (a function that is analytic in every open set of the complex plane $\mathbb{C}$ is called an entire analytic function). This is called the Fock-Bargmann representation.

Applications are.... Ashtekar variables quantization of simple model. Where explicitly one can get the inner production from the requirement that real observables should correspond to self-adjoint operators.

An arbitrary state $|\psi>$ of the Hilbert space can be expanded in the Harmonic oscillator basis states $\{|n>\}$,

$$|\psi> = \sum_{n=0}^{\infty} c_n |n>, \qquad <\psi|\psi> = \sum_{n=0}^{\infty} |c_n|^2 = 1 \tag{K.-74}$$

For the coherent state

$$|\alpha> = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n> \tag{K.-74}$$

The projection of the state $|\psi>$ onto the coherent state $|z>$ is

$$<\alpha|\psi> = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} c_n \frac{\overline{\alpha}^n}{\sqrt{n!}} \tag{K.-74}$$

then

$$\begin{aligned} <\alpha|\psi> &= \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} c_n u_n(\overline{\alpha}) \\ &= \exp(-|\alpha|^2/2) \psi(\overline{\alpha}) \end{aligned} \tag{K.-74}$$

so we have

$$\psi(z) = \sum_{n=0}^{\infty} c_n u_n(z), \qquad u_n(z) = \frac{z^n}{\sqrt{n!}}. \tag{K.-74}$$

The series (K.15) converges uniformally in any compact domain of the complex plane $\mathbb{C}$ because of the condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$ (see next section on the Weierstrass M-test). Further as a consequence of this (see the section following the next section), $\psi(z)$ an entire analytic function.

normalized according to

$$\|\psi\|^2 = <\psi|\psi> = \int \exp(-|z|^2) |\psi(z)|^2 d\mu(z) \ < \infty \tag{K.-74}$$

Tha scalar product of two entire functions, satifying (K.15), is defined by

$$< \psi_1|\psi_2 > = \int \exp(-|z|^2)\overline{\psi}_1(z)\psi_2(z)d\mu(z) \tag{K.-74}$$

K.15.1 Weierstrass M-test

Theorem K.15.1 *Let $\sum_{k=1}^{\infty} f_k(z)$ be a series of functions, with each function defined on a subset U of $\mathbb{C}$. Suppose $\sum_{k=1}^{\infty} M_k$ is a series of real numbers such that:*

(i) $0 \leq |f_k(z)| \leq M_k$;

(ii) the series $\sum_{k=1}^{\infty} M_k$ converges

then $\sum_{k=1}^{\infty} f_k(z)$ converges uniformally.

Proof. For a series to be uniformally convergent, given any $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, we have

$$|\sum_{k=1}^{\infty} f_k(z)| < \epsilon,$$

for all $z \in U$. Since $\sum_{k=1}^{\infty} M_k$ converges, we know that we can find an N so that for all $n \geq N$, we have

$$\sum_{k=1}^{\infty} M_k < \epsilon.$$

Since $0 \geq |f_k(z)| \leq M_k$, for all $z \in U$, we have

$$|\sum_{k=1}^{\infty} f_k(z)| \leq \sum_{k=1}^{\infty} |f_k(z)| \leq \sum_{k=1}^{\infty} M_k < \epsilon,$$

$\square$

Example 1.

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

$$\left|\frac{z^{n+1}}{z^n}\right| = |z|, \tag{K.-74}$$

the Taylor series converges for $|z| < 1.$, but fails to converge for $|z| > 1$. $|z| = 1$ is said to be the *radius of convergence*

Consider the geometric series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k.$$

for $|z| < 1$.

we show that this series converges uniformally on any disc $|z| \leq a < 1$. Set

$$M_k = a^k.$$

For all z, we have $0 < |z|^n \leq a^n$. We know that the geometric series for $a < 1$, hence the geometric series converges uniformaly on any disc with radius less than 1 about the origin of the complex plane $\mathbb{C}$.

Example 2. Consider the series

$$\sum_{k=1}^{\infty} \frac{z^k}{k!}.$$

we show that this series converges uniformally on any disc $|z| \leq a$. Set

$$M_k = \frac{a^k}{k!}.$$

For all z , we have $0 < |z|^n/n! \leq a^n/n!$. Thus as the series $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} a^k/k!$ converges, we will have uniform convergence. Convergence comes from the ratio test

$$\lim_{k\to\infty} \frac{M_{k+1}}{M_k} = \lim_{k\to\infty} \frac{\frac{a^{k+1}}{(k+1)!}}{\frac{a^k}{k!}} = \lim_{k\to\infty} \frac{a}{k+1} = 0.$$

We find the Taylor series for e^z converges uniformaly on any disc about the origin of the complex plane $\mathbb{C}$.

Example 3. Consider the function

$$\frac{1}{1-(1/z)} = \sum_{k=0}^{\infty} \frac{1}{z^k}.$$

for $|z| > 1$.

Set

$$M_k = \frac{1}{a^k}.$$

For all z, we have $0 < |1/z|^n \leq (1/a)^n$. Hence the function converges uniformaly outside any disc of radius more than 1 about the origin of the complex plane $\mathbb{C}$.

Coherent state expansion

The same proof slightly modified will give the required result.

Consider the series

$$f(z) = \sum_k \frac{c_n}{\sqrt{k!}} z^k \tag{K.-74}$$

where $\sum_n |c_n|^2 = 1$.

Suppose, given ϵ, there exists an integer N such that for all $n \geq N$, we have

$$|\sum_{k=n}^{\infty} f_k(z)|^2 < \epsilon^2, \tag{K.-74}$$

for all $z \in U$, then for all $n \geq N$, we have

$$|\sum_{k=n}^{\infty} f_k(z)| < \epsilon,$$

for all $z \in U$, i.e., the series $\sum_{k=1}^{\infty} f_k(z)$ is uniformally convergent.

We will prove the series (K.15.1) is uniformally convergent by proving it satifies the statement regarding (K.15.1). As $\sum_k |c_k|^2 = 1$ we have

$$\sum_{k=n} \frac{|c_k|^2}{k!} < \infty$$

from which follows that we can find an N so that for all $n \geq N$, we have $\sum_{k=n} |c_k|^2/k! < \epsilon^2$. Therefore

$$|\sum_{k=n}^{\infty} f_k(z)|^2 \leq \sum_{k=n}^{\infty} |f_k(z)|^2 \leq \sum_{k=n}^{\infty} |c_k|^2/k! < \epsilon^2,$$

proving the inequality (K.15.1).

□

K.15.2 From the Space of Normalizable Functions to the Space of Entire Analytic Functions

Let U be an open set of the complex plane $\mathbb{C}$. A function $f(z)$ is analytic at z_0 if and only if in a neighbourhood of z_0, $f(z)$ is equal to a uniformaly convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{K.-74}$$

Recall a squence of functions $f_k(z)$ converges uniformaly to a function $f(z)$ if eventually all the functions $f_k(z)$ fall within any $\epsilon-$tube about the limit function $f(z)$.

If case.

Here we take the sequence functions to be *partial sums*. Let $h_1(z), h_2(z), \ldots$ be a sequence of functions. The series of functions

$$f(z) = \sum_{k=1}^{\infty} h_k(z)$$

converges uniformaly to a function $f(z)$ if the sequence of partial sums $f_1(z) = h_1(z)$, $f_2(z) = h_1(z) + h_2(z)$, $f_3(z) = h_1(z) + h_2(z) + h_3(z) \ldots$ converges uniformaly to $f(z)$.

An example of particular interest is when $h_k(z) = a_k(z - z_0)^k$ and

$$f_k(z) = \sum_{n=0}^{k} a_n (z - z_0)^n. \tag{K.-74}$$

This allows for a notion of uniform convergence for series. A series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformaly in an open set U of the complex plane $\mathbb{C}$ if the sequence of polynomials $\{\sum_{n=0}^{N} a_n (z - z_0)^n\}$ converges uniformaly in U.

Theorem K.15.2 *Let the sequence* $\{f_n(z)\}$ *of analytic functions converge uniformaly on an open set* U *to a function* $f : U \to \mathbb{C}$*. Then the function* $f(z)$ *is also analytic and the sequence of derivatives* $(f_n'(z))$ *converge pointwise to the derivative* $f'(z)$ *on the set* U*.*

By this theorem, since polynomials are analytic, we conclude that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{K.-74}$$

is a uniformaly convergent series, then the function $f(z)$ is analytic.

Only if case.

Suppose we have a function f which is analytic about a point z_0. Take a closed contour C around z_0. By the cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw, \tag{K.-74}$$

For any z inside C. Now we know for the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

for $|r| < 1$, it follows for all w and z with $|z - z_0| < |w - z_0|$.

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n . \end{aligned} \tag{K.-74}$$

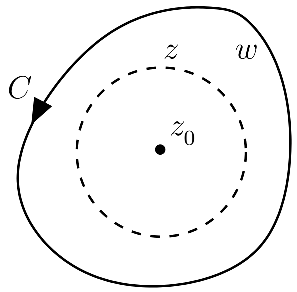

Figure K.2: contfornth. The contour used to evaluate the nth derivative, $|w-z_0| > |z-z_0|$.

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw \\
&= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n
\end{aligned}
\tag{K.-77}
$$

swaping the integral and summation follows from the fact that the series $\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$ converges uniformaly. The expansion (K.-77) is finite wherever the series does.

Where we used the Cauchy Integral Formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw.$$

Prove the first of these statements. From the triangle inequality

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)||dz| \leq |\max f(z)|L \tag{K.-77}$$

The series of partial sums

$$s_k(z) = \sum_{n=0}^{k} f_n(z) \tag{K.-77}$$

Uniform convergence implies that, given any $\epsilon > 0$, there exists some N such that $|f(z) - s_k(z)| < \epsilon$ everywhere on C. It follows

$$\left| \int_C f(z)dz - \int_C s_n(z)dz \right| \leq \epsilon L \tag{K.-76}$$

or as $s_n(z)$ is a finite sum we can interchange integration and summation,

$$\left| \int_C f(z)dz - \sum_{k=0}^{n} \int_C f_k(z)dz \right| \leq \epsilon L \tag{K.-75}$$

By choosing n large enough ϵL can be arbitary small, so

$$\lim_{n\to\infty} \sum_{k=0}^{n} \int_C f_k(z)dz = \int_C f(z)dz \tag{K.-74}$$

or

$$\sum_{n=0}^{\infty} \int_C f_n(z)dz = \int_C \sum_{n=0}^{\infty} f_n(z)dz. \tag{K.-73}$$

□

The proof

$$\begin{aligned} \frac{f(\xi) - f(z)}{\xi - z} &= \frac{1}{2\pi i} \int_C \left[\frac{f(w)}{(w-\xi)} - \frac{f(w)}{(w-z)} \right] \frac{dw}{w-z} \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-\xi)(w-z)} dw \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} \left(1 + \frac{\xi - z}{w - \xi} \right) dw. \end{aligned} \tag{K.-74}$$

writing $\xi - z = \epsilon e^{i\theta}$

$$\left| \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} - \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} dw \right| \leq \frac{1}{2\pi} \lim_{\epsilon \to 0} \epsilon \oint_C \frac{|f(w)||dw|}{|(w-z) - \epsilon e^{i\theta}||w-z|^2} \tag{K.-74}$$

We replace $|w - z|$ by its maximum value, m, and $|f(w)|$ by its maximum value, M, we obtain

$$\left| \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} - \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} dw \right| \leq \frac{1}{2\pi} \frac{ML}{m^2} \lim_{\epsilon \to 0} \frac{\epsilon}{m - \epsilon} = 0, \tag{K.-73}$$

where L is the length of the contour. Repeating the process, we obtain for the nth derivative

$$f^{(n)} = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw. \tag{K.-73}$$

Proof of theorem (K.15.2).

(a) We show term-by-term differentiation or integration of a power series yields a new power series with the same radius of curvature.

(b) The uniform-convergence property of a power series implies that term-by-term integration yields the integral of the sum function. We show that the integrated sum function is single-valued and analytic within the radius of convergence.

(c) We show that a power series converges to an analytic function within its circle of convergence.

□

(a) The convergence of a complex series is determined by the ratio test. The power series converges for $|z| < R$, but fails to converge for $|z| > R$.

$$\begin{aligned} \frac{d}{dz} s(z) &= \frac{d}{dz} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \frac{d}{dz} \left(a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots \right) \\ &= \sum_{n=0}^{\infty} a_{n+1} (n+1) (z - z_0)^n \end{aligned} \tag{K.-73}$$

The ratio test

$$\lim_{k\to\infty}\frac{M_{k+1}}{M_k}=\lim_{k\to\infty}\frac{(k+1)a_{k+1}}{ka_k}=\lim_{k\to\infty}\frac{a_{k+1}}{a_k} \tag{K.-73}$$

Hence the differentiation of a power series yields a new power series with the same radius of curvature. Similarly for integration:

$$\begin{aligned}\int s(z)dz &= \int\sum_{n=0}^{\infty}\frac{a_n}{n}(z-z_0)^n dz=\int\big(a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots\big)dz\\ &= a_{-1}+\sum_{n=1}^{\infty}\frac{a_{n-1}}{n}(z-z_0)^n,\end{aligned} \tag{K.-73}$$

the ratio test gives

$$\lim_{k\to\infty}\frac{M_{k+1}}{M_k}=\lim_{k\to\infty}\frac{ka_{k+1}}{(k+1)a_k}=\lim_{k\to\infty}\frac{a_{k+1}}{a_k} \tag{K.-73}$$

(b) The integral sum fucntion is $F(z)=\sum_{n=0}^{\infty}\int_C f_n(z)dz$. $F(z)$ is also

$$F(z)=\int_{z_0}^{z}\sum_{n=0}^{\infty}f_n(w)dw=\int_a^z f(w)dw$$

a is a fixed point and z an arbitrary point of the region, $F(z)$ only depends on a and z.

$$\begin{aligned}\frac{F(z+\Delta z)-F(z)}{\Delta z} &= \frac{1}{\Delta z}\int_a^{z+\Delta z}f(z')dz'-\frac{1}{\Delta z}\int_a^z f(z')dz'\\ &= \frac{1}{\Delta z}\int_z^{z+\Delta z}f(z')dz'\\ &= \frac{f(z)}{\Delta z}\int_z^{z+\Delta z}dz'+\frac{1}{\Delta z}\int_z^{z+\Delta z}[f(z')-f(z)]dz'\end{aligned} \tag{K.-74}$$

Take the inequality

$$\left|\frac{1}{\Delta z}\int_a^{z+\Delta z}[f(z')-f(z)]dz'\right|\le\max|[f(z')-f(z)]|. \tag{K.-74}$$

The RHS tends to zero as $\Delta z\to 0$ because $f(z)$ is continuous and therefore from (K.-74) we have

$$\frac{d}{dz}F(z) = f(z).$$

Hence the derivative exists and is single-valued, it is equal to $f(z)$. As its derivative is finite and unique, the integrated sum function $F(z)$ is analytic.

(c) If a fucntion $F(z)$ can be represented by

$$F(z) = \frac{1}{2\pi i}\int_C \frac{I(w)}{w-z}dw \tag{K.-74}$$

and $I(z)$ is contiuous on C, then, $F(z)$ is analytic at any point z which doesn't lie on C.

In particular, a function $F(z)$ that is analytic in some region can be expressed in this region by Cauchy's integral formula - C will be an arbitrary closed contour encircling the point z and $I(z) = F(z)$.

Proof. Consider the expression

$$\Delta := \left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{I(w)}{(w-z)^2}dw\right| \tag{K.-74}$$

Using the integral formula (K.15.2) for $F(z)$ and $F(z+\Delta z)$, one obtains

$$\Delta = \left|\frac{\Delta z}{2\pi}\int_C \frac{I(w)}{(w-z-\Delta z)(w-z)^2}dw\right| \tag{K.-74}$$

Since z and $z+\Delta z$ dont lie on C and $I(w)$ is continuous on C, the integrand is bounded, therefore $\Delta \to 0$ as $\Delta z \to 0$. This proves the differentiability of $F(z)$. An analogous result holds for the nth derivavtive of $F(z)$.

□

Thus, $F(z)$ also has a second derivative and hence $f(z)$ is differentiable throughout the region. And so the power series converges to an analytic function within its circle of convergence.

□

K.15.3 The Segal-Bargmann-Hall Transformation

Hall generalized the Segal-Bargmann transformation to the phase space of aribitrary compact guage group. The role of $\mathbb{C}$ is replaced by the complexification $G^{\mathbb{C}}$ of G.

Construction of coherent states in LQG.

For each $f \in L^2(G; dx)$, where dx is the normalized Haar measure on G, the image of f by the CST, $C_t f$, is the analytic continuation to $G^{\mathbb{C}}$ of the solution of the heat equation,

$$\frac{1}{\pi}\frac{\partial u}{\partial t} = \Delta_G u, \tag{K.-74}$$

in generalized coordinates the Laplacian

$$\triangle u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}g^{ij}\frac{\partial u}{\partial q_j}\right) \tag{K.-74}$$

with initial condition given by $u(0, x) = f(x)$.

K.15.4 The Ground-State Representation

K.16 Geometric Quantization

Not every sympletic manifold has the topology of a cotangent bundle over some configuration space.

Wave functions can be characterised as functions on phase space whic are annihilated by the Hamiltonian vector fields of maximally commuting set of observables. This way of looking at Schrodinger quantization is what will be generalised to other sympletic manifolds.

Geometric quantization provides a geometric, general framework for the quantization of a sympletic manifold M which is not neseccarily a cotangent bundle, for example when M it is compact.

Necessary background material in the quantization of black holes in LQG.

K.16.1 Pedestrian Overview

Let us begin by comparing [84]

Classical states are represented by on a symplectic manifold Γ, phase space. The space of observables consists of all the (smooth) real-valued functions on phase space. As a simple example: the classical state of a harmonic oscillator is represented by a point in a 2-d Euclieean manifold, coordinatized by the basic variables position, q and momentum

k. The total energy, E, is an example of an observable. It corresponds to the function $E = \frac{1}{2}\omega^2 q^2 + \frac{1}{2m}k^2$ on phase space. Any other function of q and k is also an observable, although it wont in general have as direct physical meaning as the total energy. Let us return to the general case. The (ideal) measurement of an observable f in a state $p \in \Gamma$ yields the simple value $f(p)$ at the point p and the state is left undisturbed. These outcomes occur with complete certainty. Then the dynamics of an evolving observable f_t is given by the differential equation

$$\frac{\partial f_t}{\partial t} = \{\mathcal{H}, f_t\}. \tag{K.-74}$$

where, as before, $\{,\}$ denotes the Poisson bracket. For the canonical choice of symplectic structure on T^*R^n, it is Hamilton's equations of motion as presented in Appendix E.

The arena for quantum mechanics, on the other hand, is a Hilbert space. States of the system correspond to normalized states, moduli a phase factor, (i.e. what are known as ***rays***), and observables are represented by self-adjoint linear operators on H. As in the classical description, the space of observables is a real vector space equip with

Taking the view in quantum mechanics that the observables evolve in time while the states remain fixed i.e. the Heisenberg picture. The fundamental equation describing the dynamical evolution of an observable $\mathcal{O}_t$ the equation,

$$\frac{d\mathcal{O}_t}{dt} = -\frac{i}{\hbar}[\hat{\mathcal{H}}, \mathcal{O}_t]. \tag{K.-74}$$

This direct analogy with the classical theory was first realized by Dirac.

The measurement theory is strikingly different.In the textbook description based on the Copenhagen interpretation, the (ideal) measurement of an observable $\hat{A}$ in a state $\psi \in H$ yields an eigenvalue of $\hat{A}$ and, immediately after the measurement, the state is thrown into corresponding eigenstate.

This lineararity the basis of the superposition of quantum states,

$$\phi = a\psi_1 + b\psi_2 \tag{K.-74}$$

however not ever linear combination of states is relevant because the coefficients must be chosen so that the new state is also normalized i.e. we must have $|a|^2 + |b|^2 = 1$.

Geometric quantization is a quantization scheme based on the above relation between Heisenberg's equation and Hamilton's equation.

fundamental objects

The fundamental objects are:

1) A set of observables $\mathcal{U}$

2) a set Ω of states

3) a probability interpretation map $\mathcal{U} \times \Omega \to \mathcal{P}$, where $\mathcal{P}$ denotes the set of all non-negative Lebesgue measurable functions $f : R \to R$ such that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \qquad \text{(K.-74)}$$

(i.e. probability distributions). For a state η and an observable $\mathcal{O}$, we write the associated probability distribution function as $\eta_{\mathcal{O}}(\lambda)$. Of course, there is a natural mean-value map from $\mathcal{P} \to R$, given by

$$\mapsto \int \lambda f(\lambda)\, d\lambda. \qquad \text{(K.-74)}$$

$\mathcal{U}$ and Ω are both form real vector spaces

The composition - observables $\times$ states

$$\mathcal{U} \times \Omega \to \mathcal{P} \to_{mean\ value} \to R, \quad \eta, \mathcal{O} \mapsto < \eta | \mathcal{O} > \qquad \text{(K.-74)}$$

there is a duality

It is evident that

In QM, the algebra is usually realized as an algebra of linear operators on a complex Hilbert space H, and the space $\mathcal{D}(H)$ of positive operators with unit trace (as we know from before *density matrices*) is taken as the space of states. In particular, this state space contains the *projective Hilbert space* (pure states)

$$P(\mathcal{H}) = \{\text{all projective operators onto } 1 - \text{dimensional subspaces}\} \simeq \mathcal{H}/ \qquad \text{(K.-74)}$$

where . Elements of $P(\mathcal{H})$ are known as pure states, while elements of $\mathcal{D}(\mathcal{H})$ which cannot be represented as one-dimensional projectors are the mixed states.

$$\eta_{\mathcal{O}}(\lambda) = \mathrm{tr}_{\mathcal{H}}(\eta P_{\mathcal{O}}(\lambda)) \qquad \text{(K.-74)}$$

Pre-quantization

complexification - We can tensor a real vector space with the complex numbers and get a complex vector space; this process is called complexification. For example, we can complexify the tangent space at some point of a manifold, which amounts to forming the space of complex linear combinations of tangent vectors at that point. (from Geometric Quantization John Baez August 11, 2000)

Consider a 2n-dimensional symplectic manifold $(\Gamma, \Omega_{\alpha\beta})$ and the space of symplectic potentials ω_α (recall append E, $\Omega_{\alpha\beta} = 2\partial_{[\alpha}\omega_{\beta]}$).

$$\Psi(\omega_\alpha + \partial_\alpha f) = e^{if/\hbar}\Psi(\omega_\alpha). \tag{K.-74}$$

More precisely, in the terminology of fibre bundles, we have the following. Pre-quantum wave functions are cross sections of a complex line bundle over Γ, associated with the principle $U(1)$ bundle which has $\Omega_{\alpha\beta}$ as the curvature tensor. The symplectic potentials are connection 1-forms whose curvature is given by $\Omega_{\alpha\beta}$. Equation (K.16.1) is just the cross sections under the $U(1)$ action.

$z = x + iy$ and $z^* = x - iy$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial \overline{z}}\right), \quad \frac{\partial}{\partial z^*} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial z^*}\right). \tag{K.-74}$$

$$\frac{\partial z}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z^*}\right)(x + iy) = 1, \quad \frac{\partial z^*}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z^*}\right)(x - iy) = 0. \tag{K.-74}$$

functions that satisfy the Cauchy-Riemann equations $f(z) = U + iV$ have the properties:

$$\frac{\partial f(z)}{\partial z^*} := \frac{1}{2}\left(\frac{\partial(U+iV)}{\partial x} + i\frac{\partial(U+iV)}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}\right) = 0 \tag{K.-74}$$

Consider the space of pre-quantum wave functions which have finite norm. The Cauchy completion of this space gives us the *pre-quantum Hilbert space*, $\mathcal{H}_p$.

Given the classical observable f, define an operator $\mathcal{O}_f$ on $\mathcal{H}_p$ by

$$\boxed{\mathcal{O}_f \circ \Psi := -i\hbar X_f^\alpha \nabla_\alpha \Psi + f\Psi \equiv -i\hbar X_f^\alpha\left(\partial_\alpha - \frac{i}{\hbar}\omega_\alpha\right)\Psi + f\Psi} \tag{K.-73}$$

(i) linear;
(ii) gauge-covariant;
(iii) symmetric (i.e. formally self-adjoint) with respect to the inner product.

Thus, at the pre-quantum level, the Dirac prescription of replacing Poisson brackets by $-i\hbar$ times the commutator is exact.

Moreover, this correspondence $f \mapsto \mathcal{O}_f$ is a linear, 1-1 mapping from the space of classical observables to the space of pre-quantum operators which preserves the natural Lie-algebra structure:

$$[\mathcal{O}_f, \mathcal{O}_g] = \frac{\hbar}{i}\mathcal{O}_{\{f,g\}}. \tag{K.-73}$$

that is, for any classical observables f and g we require that

$$\mathcal{O}_{\{f,g\}} \circ \Psi = -i\hbar X^{\alpha}_{\{f,g\}}\nabla_{\alpha}\Psi + f\Psi, \tag{K.-73}$$

where $X^{\alpha}_{\{f,g\}}$ is defined via,

$$X^{\alpha}_{\{f,g\}}\nabla_{\alpha}\Psi := [X^{\alpha}_f\nabla_{\alpha}, X^{\beta}_g\nabla_{\beta}]\Psi = [X_f, X_g]^{\alpha}\nabla_{\alpha}\Psi. \tag{K.-73}$$

This is indeed the case for the pre-quantum operator (K.-73), as is proven in box??.

Pre-quantum operators that preserve the natural Lie-algebra

We wish to show that the operator on $\mathcal{H}_p$ (K.-73) satisfies (K.16.1), or equivalently:

$$\mathcal{O}_f \circ \mathcal{O}_g \circ \Psi - \mathcal{O}_g \circ \mathcal{O}_f \circ \Psi = \frac{\hbar}{i}\mathcal{O}_{\{f,g\}} \circ \Psi \tag{K.-73}$$

First, we look at $\mathcal{O}_f \circ \mathcal{O}_g \circ \Psi$,

$$\begin{aligned}
\mathcal{O}_f \circ \mathcal{O}_g \circ \Psi &= \mathcal{O}_f \circ (-i\hbar X^{\alpha}_g\nabla_{\alpha} + g)\Psi \\
&= (\mathcal{O}_f \circ g)\Psi - i\hbar\mathcal{O}_f \circ (X^{\alpha}_g\nabla_{\alpha}\Psi) + f\mathcal{O}_g \circ \Psi + g\mathcal{O}_f \circ \Psi \\
&= \left[(-i\hbar X^{\alpha}_f\nabla_{\alpha}g)\Psi - i\hbar\mathcal{O}_f \circ (X^{\alpha}_g\nabla_{\alpha}\Psi)\right] + \underbrace{fg\Psi + f\mathcal{O}_g \circ \Psi + g\mathcal{O}_f \circ \Psi}_{\textit{symmetric in } f \textit{ and } g}
\end{aligned} \tag{K.-75}$$

$$[\mathcal{O}_f, \mathcal{O}_g] \circ \Psi = \frac{\hbar}{i}\left[(X^{\alpha}_f\nabla_{\alpha}g - X^{\alpha}_g\nabla_{\alpha}f)\Psi + \mathcal{O}_g \circ (X^{\alpha}_f\nabla_{\alpha}\Psi) - \mathcal{O}_f \circ (X^{\alpha}_g\nabla_{\alpha}\Psi)\right] \tag{K.-75}$$

Pre-quantum operators that preserve the natural Lie-algebra

$$
\begin{aligned}
[\mathcal{O}_f, \mathcal{O}_g] \circ \Psi &= (\mathcal{O}_f \mathcal{O}_g - \mathcal{O}_g \mathcal{O}_f)\Psi \\
&= (-i\hbar\nabla_{X_f} + f)(-i\hbar\nabla_{X_g} + g)\Psi - (-i\hbar\nabla_{X_g} + g)(-i\hbar\nabla_{X_f} + f)\Psi \\
&= -\hbar^2(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f})\Psi - i\hbar[\nabla_{X_f}(g\Psi) + f\nabla_{X_g}\Psi + fg\Psi \\
&\quad -\nabla_{X_g}(f\Psi) - g\nabla_{X_f}\Psi - gf\Psi] \\
&= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}]\Psi - i\hbar(\nabla_{X_f} g - \nabla_{X_g} f)\Psi
\end{aligned}
\tag{K.-78}
$$

Now

$$
(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})\Psi = 2\pi i\Omega(X_1, X_2)\Psi = 2\pi i\{f, g\}. \tag{K.-78}
$$

From section . Continuing where we left off,

$$
\begin{aligned}
[\mathcal{O}_f, \mathcal{O}_g] \circ \Psi &= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}]\Psi + 2\pi\hbar\{f, g\}\Psi \\
&= -\hbar^2(\nabla_{[X_f,X_g]} + \frac{i}{\hbar}\Omega(X_f, X_g))\Psi + 2\pi\hbar\{f, g\}\Psi \\
&= (\hbar^2\nabla_{[X_f,X_g]} - i\pi\hbar\{f, g\} + 2i\pi\hbar\{f, g\})\Psi \\
&= i\hbar(-i\hbar\nabla_{X_{\{f,g\}}} + \{f, g\})\Psi \\
&= i\hbar\mathcal{O}_{\{f,g\}}\Psi.
\end{aligned}
\tag{K.-81}
$$

$$
\delta^{\overset{\downarrow}{E}} \tag{K.-81}
$$

Quantization

To cut down the prequantum Hilbert space, we need to choose a POLARIZATION, say P. What's this? Well, for each point x in X, a polarization picks out a certain subspace Px of the complexified tangent space at x. We define the *quantum Hilbert space*, $\mathcal{H}$, to be the space of all square-integrable sections of L that give zero when we take their covariant derivative at any point x in the direction of any vector in Px. The quantum Hilbert space is a subspace of the prequantum Hilbert space.

To obtain quantum states, we need a new structure, called ***polarization.*** A polarization $\mathcal{P}$ is of the complexified tangent space such that:

(i) $v^\alpha \in \mathcal{P}|_\gamma$ for all $[v,w]^\alpha \in \mathcal{P}|_\gamma$ for all γ ;
(ii) Given any two vectors $v^\alpha, w^\alpha \in \mathcal{P}$, $\Omega_{\alpha\beta} v^\alpha w^\alpha = 0$ for all γ.

We devote this section to justify the main features related to the polarization, which are relevant to geometric quantization.

Given a polarization we define ***quantum wave functions*** as the cross-sections Ψ of (N.-19) satisfying:

$$v^\alpha \nabla_\alpha \Psi = 0, \quad \text{for all } v^\alpha \in \mathcal{P}. \tag{K.-81}$$

This condition on the wave function is called the ***polarization condition*** and effectively eliminates n degrees of freedom from the wave functions.

$$\left[v^\alpha \nabla_\alpha, w^\beta \nabla_\beta\right] \Psi = [v,w]^\alpha \nabla_\alpha \Psi - v^\alpha w^\beta \Omega_{\alpha\beta} \Psi. \tag{K.-81}$$

$$\mathcal{O}_{q^i} \Psi = q^i \Psi, \quad \mathcal{O}_{p_i} \Psi = -i\hbar \frac{\partial \Psi}{\partial q^i}, \tag{K.-81}$$

Which agree with the usual representation of the configuration and momentum operators.

Let us return to the general quantizatoin procedure.

Pre-quantum operators that preserve the natural Lie-algebra

$$[v,w]^\alpha \Psi - vw\Omega_{\alpha\beta} \tag{K.-81}$$

In usual "position representation" of a particle on the line, we start with the space of all (nice) functions on the phase space R^2, and then pick out the subspace of functions that depend only the position coordinate q. These are functions with

$$df/dp = 0 \tag{K.-81}$$

In the "momentum representation" we pick out the functions that depend only on the momentum coordinate p:

$$df/dq = 0 \tag{K.-81}$$

In the "Bargmann-Segal representation" we pick out the functions that depend only on the complex coordinate z:

$$df/dz* = 0z = q + ip, z* = q - ip \qquad \text{(K.-81)}$$

or in other words, functions that satisfy the Cauchy-Riemann equations.

The Kahler polarization

An almost complex structure on Γ is a tensor field $J^{\alpha}{}_{\beta}$ satisfying $J^{\alpha}{}_{\beta}J^{\beta}{}_{\mu} = -\delta^{\alpha}{}_{\mu}$.

$$g_{\alpha\beta} := \Omega_{\alpha\gamma}J^{\gamma}{}_{\beta} \qquad \text{(K.-81)}$$

$$\Omega = i\, dz \wedge\, d\overline{z}, \qquad \text{(K.-81)}$$

and the polarization condition (N.-19) we obtain:

$$\mathcal{O}_z\Psi = z\Psi, \quad \mathcal{O}_{\overline{z}}\Psi = \hbar\frac{\partial\Psi}{\partial z} + \frac{1}{2}\overline{z}\Psi \qquad \text{(K.-81)}$$

Then, the expressions of basic operators reduce to:

$$\mathcal{O}_z f(z) = zf(z), \quad \mathcal{O}_{\overline{z}} f(z) = \hbar\frac{\partial f(z)}{\partial z}. \qquad \text{(K.-81)}$$

(i) Construct the $\star$-algebra of quantum operators by incorporating the Poisson bracket relations and the reality conditions on the classical phase space.

(ii) Find an explicit represenation of the algebra by choosing a suitable polarization. At this stage, the $\star$-relations are ignored since one does not have accss to an inner product.

where this Hilbert space describes states of the black hole horizon.

This a field theory in 3 dimensions, and the reason it's called "topological" is that you don't need any metric or other geometrical structure on your 3d spacetime manifold for this theory to make sense.

Complex Manifolds

In two dimensions it is often very useful to combine the coordinates into complex coordinates $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$

A complex manifold is formally defined an a manner entirly similar to a real manifold. Instead of real coordinate neighborhoods $(U; x^a),\ x^a \in R$, we have complex coordinates neighborhoods $(U; z^a),\ z^a \in C,\ a = 1, \ldots, n$.

a complex manifold in which the vector space R^m is replaced with C^m and the overlap functions are required to be holomorphic. This latter requirement introduces profound changes in the manifold idea

K.16.2 Complex, Hermitian and Kähler Manifolds

To begin with, we define a holomorphic (or analytic) map on $\mathbb{C}^m$. This requires a simple higher-dimensional version of the Cauchy-Riemann relations.

Definition A complex-valued function $f : \mathbb{C}^m \to \mathbb{C}$ is holomorphic if $f(z) = u(x, y) + iv(x, y)$ satifies the Cauchy-Riemann relations for each $z^\mu = x^\mu + iy^\mu$,

$$\frac{\partial u}{\partial x^\mu} = \frac{\partial v}{\partial y^\mu}, \qquad \frac{\partial v}{\partial x^\mu} = -\frac{\partial u}{\partial y^\mu}. \tag{K.-81}$$

A map $f \equiv (f^1, \ldots, f^n) : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic if each function f^ν $(\nu = 1, \ldots, n)$ is holomorphic, that is

$$\begin{aligned} f^\nu(z) &= u^\nu(x, y) + iv^\nu(x, y), \qquad z^\mu = x^\mu + iy^\mu \\ &\Rightarrow \frac{\partial u^\nu}{\partial x^\mu} = \frac{\partial v^\nu}{\partial y^\mu}, \qquad \frac{\partial v^\nu}{\partial x^\mu} = -\frac{\partial u^\nu}{\partial y^\mu}, \quad \nu = 1, \ldots, n,\ \mu = 1, \ldots, m. \end{aligned} \tag{K.-81}$$

□

With an even dimensional manifold we can think of it being glued together by a number of coordinate patches, where each patch is an open region of the coordinate space $\mathbb{C}^m$ - the space of points are the $n-$tuples $(z^1, \ldots, z^m)$ of complex numbers. Given such a description, we then define a complex manifold to be one where the transition functions are given entirely by holomorphic functions. The formal definition is the following.

Definition $\mathcal{M}$ is a complex manifold if

(i) $\mathcal{M}$ is a topological space;

(ii) $\mathcal{M}$ is provided with a family of pairs $\{(U_I, z_I)\}$;

(iii) $\{U_I\}$ is a family of open sets which covers $\mathcal{M}$, z_I is a homomorphism from U_I to an open subset U'_I of $\mathbb{C}^m$;

(iv) Given U_I and U_J such that $U_I \cap U_J \neq \emptyset$, the map $\varphi_{IJ} = z_J \circ z_I^{-1}$ is holomorphic.

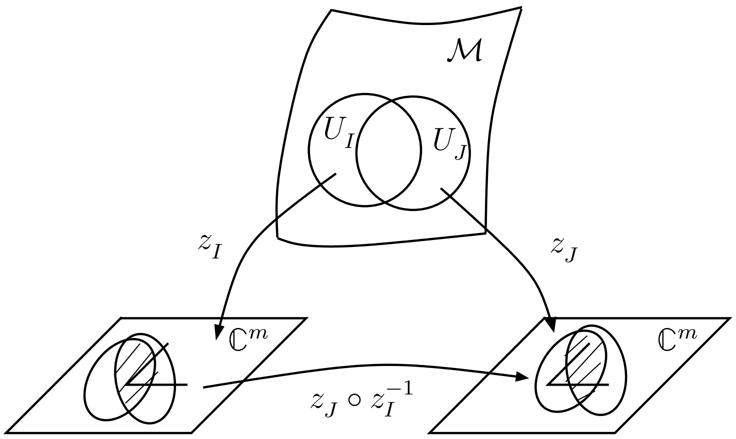

Figure K.3: Complex manifold.

□

The number m is called the complex dimension of $\mathcal{M}$. Each complex manifold is also a smooth real manifold of dimension $2m$. Any chart U of a complex manifold has coordinates $(z^1, \ldots, z^m)$ which may be regarded as real coordinates $(x^1, y^1, \ldots, x^m, y^m)$. The analytic property of the coordinate transformation functions ensures that they are differentiable when the manifold is regarded as a $2m$-dimensional differentiable manifold. However, holomorphic functions are much more rigid than smooth functions, so a complex structure (holomorphicity) is much stronger than a differentiable structure (smoothness) and hence most manifolds are not complex manifolds.

Complex forms

We define holomorphic vector fields, covectors, p-forms, tensors, etc., in just the same way as we did in the case of a real n-manifold. A complex manifold with local complex coordinates

$$z_I^\mu = x_I^\mu + i y_I^\mu$$

over U_I has the local coordinate vector fields

$$\frac{\partial}{\partial x_I^\mu}, \quad \frac{\partial}{\partial y_I^\mu}$$

and local coordinate one-forms

$$dx_I^\mu, \quad dy_I^\mu$$

as local real basis and co-basis respectively. Since we are allowed to take complex linear combinations in a complex manifold we may alternatively use complex basis and co-basis

$$\begin{aligned}
\frac{\partial}{\partial z_I^\mu} &= \frac{1}{2}(\frac{\partial}{\partial x_I^\mu} - i\frac{\partial}{\partial y_I^\mu}), \qquad \frac{\partial}{\partial \overline{z}_I^\mu} = \frac{1}{2}(\frac{\partial}{\partial x_I^\mu} + i\frac{\partial}{\partial y_I^\mu}) \\
dz_I^\mu &= dx_I^\mu + idy_I^\mu, \qquad d\overline{z}_I^\mu = dx_I^\mu - idy_I^\mu
\end{aligned} \tag{K.-81}$$

where $\overline{z} = x - iy$.

There is more than one way of formulating the notion of a complex structure. Form a complex vector field

$$\zeta_I^\mu = \partial/\partial x_I^\mu + i\partial/\partial y_I^\mu.$$

where $\partial/\partial x_I^\mu$ and $\partial/\partial y_I^\mu$ are ordinary real vector fields on the $2m$-manifold. Now consider the new complex vector field that arises when the complex vector field ζ_I^μ is multiplied by i:

$$i\zeta_I^\mu = -\partial/\partial y_I^\mu + i\partial/\partial x_I^\mu$$

The real vector field $\partial/\partial x_I^\mu$ is now replaced by $-\partial/\partial y_I^\mu$ and in turn $\partial/\partial y_I^\mu$ must be replaced by $\partial/\partial x_I^\mu$. The operation J which effects these replacements is a tensor field defined locally by

$$\begin{aligned}
(J_I[\partial/\partial x_I^\mu])\,(x_I(p)) &= \partial/\partial y_I^\mu \\
(J_I[\partial/\partial y_I^\mu])\,(x_I(p)) &= -\partial/\partial x_I^\mu.
\end{aligned} \tag{K.-81}$$

Actually J is globally defined. Assume that $U_I \cap U_K \neq \emptyset$ and denote $z_K = \varphi_{IK}(z_I)$, we need to show that $J_I = J_K$ on their overlap. The Cauchy-Riemann equations then read

$$\frac{\partial x_K^\mu}{\partial x_I^\nu} = \frac{\partial y_K^\mu}{\partial y_I^\nu}, \qquad \frac{\partial y_K^\mu}{\partial x_I^\nu} = -\frac{\partial x_K^\mu}{\partial y_I^\nu}$$

Then

$$\begin{aligned}
J_K[\partial/\partial x_I^\nu] &= J_K\left[\frac{\partial x_K^\mu}{\partial x_I^\nu}\frac{\partial}{\partial x_K^\mu}+\frac{\partial y_K^\mu}{\partial x_I^\nu}\frac{\partial}{\partial y_K^\mu}\right] \\
&= \frac{\partial x_K^\mu}{\partial x_I^\nu}J_K[\partial/\partial x_K^\mu]+\frac{\partial y_K^\mu}{\partial x_I^\nu}J_K[\partial/\partial y_K^\mu] \\
&= \frac{\partial x_K^\mu}{\partial x_I^\nu}\partial/\partial y_K^\mu-\frac{\partial y_K^\mu}{\partial x_I^\nu}\partial/\partial x_K^\mu \\
&= \frac{\partial y_K^\mu}{\partial y_I^\nu}\partial/\partial y_K^\mu+\frac{\partial x_K^\mu}{\partial y_I^\nu}\partial/\partial x_K^\mu \\
&= \partial/\partial y_I^\nu = J_I[\partial/\partial x_I^\nu] \qquad \text{(K.-84)}
\end{aligned}$$

We also find that $J_K[\partial/\partial y_I^\nu]=J_I[\partial/\partial y_I^\nu]$. J_I takes the form

$$J_I=\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \text{(K.-84)}$$

with respect to the basis

$$\{\partial/\partial x_I^1,\ldots,\partial/\partial x_I^m;\partial/\partial y_I^1,\ldots,\partial/\partial y_I^m\}.$$

Conversely, if $\mathcal{M}$ is a real $(2m)$-dimensional manifold which admits the globally defined tensor field J_0 then the diffeomorphisms between overlapping charts obey the Cauchy-Riemann equations: The requirement $J_K[\partial/\partial x_I^\nu]=J_I[\partial/\partial x_I^\nu]$ implies

$$\frac{\partial x_K^\mu}{\partial x_I^\nu}\partial/\partial y_K^\mu-\frac{\partial y_K^\mu}{\partial x_I^\nu}\partial/\partial x_K^\mu=\partial/\partial y_I^\nu$$

which becomes

$$\frac{\partial x_K^\mu}{\partial x_I^\nu}\partial/\partial y_K^\mu-\frac{\partial y_K^\mu}{\partial x_I^\nu}\partial/\partial x_K^\mu=\frac{\partial y_K^\mu}{\partial y_I^\nu}\partial/\partial y_K^\mu+\frac{\partial x_K^\mu}{\partial y_I^\nu}\partial/\partial x_K^\mu$$

upon writing $\partial/\partial y_I^\nu$ in terms of the basis $\{\partial/\partial x_K^\mu;\partial/\partial y_K^\mu\}$. We can simply read off the Cauchy-Riemann relations for the transition functions. The requirement $J_K[\partial/\partial x_I^\nu]=J_I[\partial/\partial x_I^\nu]$ leads to the same condition. Thus the existence of J_0 is equivalent to the existence of a complex structure.

Definition A $(2m)$-dimensional real manifold $\mathcal{M}$ admits a complex structure if and only if it admits a smooth tensor field $J_0\in T_1^1(\mathcal{M})$ with $J_0^2(p)=-id_{T_p(\mathcal{M})}$ which in suitable coordinates has canonical component matrix $\epsilon\otimes 1_m$. We then call $\mathcal{M}$ a complex m-dimensional manifold with complex structure J.

□

The condition

$$\mathbf{J}^2 = -id_{T_p(\mathcal{M})}$$

alone defines what is referred to as an almost complex structure.

Definition An m-dimensional real manifold $\mathcal{M}$ with smooth tensor field such that $J^2(p) = -id_{T_p(\mathcal{M})}$ is called an almost complex manifold with almost complex structure J.

□

Notice that $\det(J^2(p)) = (-1)^m = [\det(J(p))]^2 > 0$, hence almost complex manifolds have even-dimension. Not every almost complex manifold admits a complex structure. To have an actual complex structure, so that a consistent notion of holomorphic can arise, a certain differential equation in the quantity J must be satisfied. There is a deep theorem, the Newlander-Nirenberg theorem, which gives us a necessary and sufficient (the difficult part - Newlander A and Nirenberg L Ann. Math. **65** 391) condition for a $2m$-dimensional real manifold, with this J-structure, to qualify as a complex m-manifold.

Theorem K.16.1 (Newlander and Nirenberg). *Let the Nijenhuis tensor field $N \in T^1_2(\mathcal{M})$ on an almost complex manifold $\mathcal{M}$ be defined by*

$$N[u, v] := [u, v] + J[[J[u], v]] + J[[u, J[v]]] - [J[u], J[v]]. \tag{K.-84}$$

Then $\mathcal{M}$ admits a complex structure if and only if $N = 0$.

Proof omitted.

□

Definition Let $(\mathcal{M}, J)$ be a manifold of complex dimension m which at the same time is a Riemannian (2m)-dimensional real manifold with Riemannain structure g. Then $(\mathcal{M}, J, g)$ is called a **Hermitian manifold** provided that

$$g[J[u], J[v]] = g[u, v] \quad \text{for all } u, v \in T^1(\mathcal{M}). \tag{K.-84}$$

Then g is called a Hermitian structure and is said to be J-compatible.

□

Definition Let $(\mathcal{M}, J, g)$ be a Hermitian manifold. The so-called **Kähler two-form** is defined by

$$\omega[u,v] := g[J[u],v] \quad \text{for all } u,v \in T^1(\mathcal{M}). \tag{K.-84}$$

Notice that $\omega[u,v] = -\omega[v,u]$ due to $J^2(p) = -id_{T_p(\mathcal{M})}$. It also implies $\omega[J[u],v] = -g[u,v]$. So $g[J[u],J[v]] = \omega[u,J[v]]$ and $g[J[u],J[v]] = g[u,v]$ then becomes

$$\omega[u,J[v]] = -\omega[J[u],v]$$

or

$$\omega[J[u],J[v]] = \omega[u,v]. \tag{K.-84}$$

Thus ω is also J-compatible.

□

Definition A **Kähler manifold** is a Hermitian manifold $(\mathcal{M}, J, g)$ such that the corresponding Kahler two-form is closed. Equivalently, a Kähler manifold $(\mathcal{M}, J, g)$ is a complex manifold which also carries a J-compatible sympletic structure ω.

□

Lemma K.16.2 *With the notation $z^{\overline{\mu}} = \overline{z}^{\mu}$ and similarly for $dz^{\overline{\mu}}$, $\partial_{\overline{\mu}}$*

$$\omega = id \wedge \overline{d}K \tag{K.-84}$$

where K is called the local Kähler potential for ω. K is a real-valued function which is uniquely determined by ω up to $K(z,\overline{z}) \to K(z,\overline{z}) + f(z) + g(\overline{z})$ where f,g are holomorphic and antiholomorphic functions respectively.

Proof:

In local coordinates we have

$$\omega = \omega_{\mu\nu} dz^{\mu} \wedge dz^{\nu} + \omega_{\overline{\mu}\nu} d\overline{z}^{\mu} \wedge dz^{\nu} + \omega_{\mu\overline{\nu}} dz^{\mu} \wedge d\overline{z}^{\nu} + \omega_{\overline{\mu}\overline{\nu}} d\overline{z}^{\mu} \wedge d\overline{z}^{\nu} \tag{K.-84}$$

We have $J[\partial_{\mu}] = i\partial_{\mu}$, $J[\partial_{\overline{\mu}}] = -i\partial_{\overline{\mu}}$ and

$$\begin{aligned}\omega_{\mu\nu} &= \omega[\partial/\partial_\mu, \partial/\partial_\nu] \\ \omega_{\mu\overline{\nu}} &= \omega[\partial/\partial_\mu, \partial/\partial_{\overline{\nu}}] \\ \omega_{\overline{\mu}\nu} &= \omega[\partial/\partial_{\overline{\mu}}, \partial/\partial_\nu] \\ \omega_{\overline{\mu\nu}} &= \omega[\partial/\partial_{\overline{\mu}}, \partial/\partial_{\overline{\nu}}].\end{aligned} \tag{K.-86}$$

From the compatibility condition

$$\omega[\partial/\partial_\mu, \partial/\partial_\nu] = \omega[J[\partial/\partial_\mu], J[\partial/\partial_\nu]] = \omega[\partial/\partial_\nu, \partial/\partial_\mu]$$

implying $\omega_{\mu\nu} = 0$. Similarly we find $\omega_{\overline{\mu\nu}} = 0$. Hence (K.16.2) simplifies to

$$\omega = [\omega_{\mu\overline{\nu}} - \omega_{\overline{\nu}\mu}] dz^\mu \wedge d\overline{z}^\nu =: \Omega_{\mu\overline{\nu}} dz^\mu \wedge d\overline{z}^\nu \tag{K.-86}$$

Clearly $\Omega_{\overline{\nu}\mu} = -\Omega_{\mu\overline{\nu}}$. Reality $\omega = \overline{\omega}$ reads

$$\Omega_{\mu\overline{\nu}}\, dz^\mu \wedge d\overline{z}^\nu \equiv -\Omega_{\nu\overline{\mu}}\, d\overline{z}^\mu \wedge dz^\nu = \overline{\Omega_{\mu\overline{\nu}}\, dz^\mu \wedge d\overline{z}^\nu} = \overline{\Omega_{\mu\overline{\nu}}}\, d\overline{z}^\mu \wedge dz^\nu$$

implying

$$\overline{\Omega_{\mu\overline{\nu}}} = -\Omega_{\nu\overline{\mu}}. \tag{K.-86}$$

Closure $\partial_{[\alpha}\omega_{\beta\gamma]} = 0$ for $\alpha, \beta \in \{\mu, \nu, \overline{\mu}, \overline{\nu}\}$ implies

$$\partial_{[\mu}\Omega_{\nu]\overline{\rho}} = \partial_{[\overline{\mu}}\Omega_{\overline{\nu}]\rho} = 0. \tag{K.-86}$$

By Poincare's lemma this implies locally $\Omega_{\mu\overline{\nu}} = \partial_\mu f_{\overline{\nu}} = -\partial_{\overline{\nu}} g_\mu$.

$$= \tag{K.-86}$$

The reality condition reads $\overline{i\partial_\mu \partial_{\overline{\nu}} K(z, \overline{z})} = -i\partial_\nu \partial_{\overline{\mu}} K(z, \overline{z})$ or

$$\partial_\nu \partial_{\overline{\mu}} \overline{K(z, \overline{z})} = \partial_\nu \partial_{\overline{\mu}} K(z, \overline{z})$$

□

Notice that by definition

$$g_{\mu\overline{\nu}} = \tag{K.-86}$$

Any metric space defines a conformal or complex structure on S because one can always find isothermal coordinates, and the isothermal coordinates naturally define a complex structure.

n-dimensional Torus

The two-dimensional torus the points in R^2

$$(x, y) \ \ (x+1, y) \ \ (x, y+1) \tag{K.-86}$$

are identified as a single point in T^2.

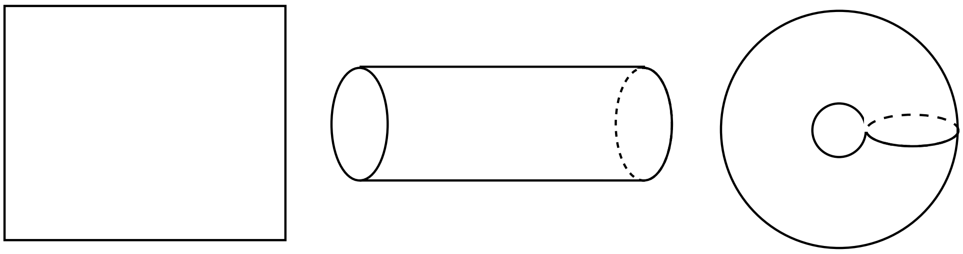

Figure K.4: Torus.

construct a holomorphic line bundle L over $\chi^{\mathcal{P}}$ with a connection whose curvature is $i\omega$.

$$\nabla_{x_i} = \partial_{x_i}, \quad \nabla_{y_i} = \partial_{y_i} + \frac{ik}{2\pi} z_i. \tag{K.-86}$$

Note that

$$[\nabla_{x_i}, \nabla_{x_j}] = 0, \quad [\nabla_{y_i}, \nabla_{y_j}] = 0, \quad [\nabla_{x_i}, \nabla_{y_j}] = \frac{ik}{2\pi}\delta_{ij}, \tag{K.-86}$$

using this curvature, we define parallel translation operators

$$U_i(t) = \exp(t\nabla_{x_i}), \quad V_i(t) = \exp(t\nabla_{y_i}) \tag{K.-86}$$

for all t in R.

and the above commutation relations imply

$$U_i(s)U_j(t) = U_j(t)U_i(s), \quad V_i(s)V_j(t) = V_j(t)V_i(s), \quad U_i(s)V_j(t) = e^{\frac{ik}{2\pi}st}V_j(t)U_i(s). \tag{K.-86}$$

the magnetic translation group quantum-Hall effect.

we obtain theta functions by the technique of group averaging. suppose we start with a holomorphic function on C^{n-1} that is invariant under $R(u)$ for real lattice vectors u, that is for $u \in (2\pi Z)^{n-1}$. Then try to average f with respect to the imaginary lattice directions, forming the function.

$$\psi = \sum_{v \in (2\pi Z)^{n-1}} R(iv)f. \tag{K.-86}$$

A basis of such functions is given by

$$f_a(z) = \exp(ia \cdot z) \tag{K.-86}$$

where $a \in (2\pi Z)^{n-1}$. If we apply group averaging to such a function f_a, we obtain a theta function

$$\psi_a(z) = \sum_{v \in (2\pi Z)^{n-1}} e^{\frac{k}{2\pi}(i\nu \cdot z - \frac{1}{2}v \cdot v)} e^{ia \cdot (z+iv)}, \tag{K.-86}$$

$$\psi = \sum_{v \in (2\pi Z)} R(iv) \exp(ia \cdot z). \tag{K.-86}$$

We define their inner product by

$$< f, g >= \int_{[0,2\pi]^{2(n-1)}} e^{-\frac{k}{2\pi}y \cdot y} \overline{f}(z) g(z) \, d^{n-1}x \, d^{n-1}y. \tag{K.-86}$$

arXiv:quant-ph/0006037 v1 7 Jun 2000 HARMONIC ANALYSIS WITH RESPECT TO HEAT KERNEL

Roughly a polarization is a choice of d coordinates on the $2d-$phase space $\mathcal{M}$, with the idea that the functions in our quantum Hilbert space will be independent of these d variables.

For example if $\mathcal{M} = \mathbb{R}^{2d}$, then we may take the usual position and momentum variables $x_1, \ldots, x_d, p_1, \ldots, p_d$ and then consider functions that depend only on $x_1, \ldots, x_d$ and are independent of $p_1, \ldots, p_d$. (This is called the vertical polarization.)

Alternatively, one may consider complex variables $z_1, \ldots z_d, \overline{z}_1, \ldots, \overline{z}_d$ and then consider the functions that are independent of z_k, that is, holomorphic. (This is a complex polarization.) In that case our Hilbert space is the Segal-Bargmann space. To be more precise, in geometric quantization the elements of the quantum Hilbert space are not functions, but rather sections of a certain complex line bundle with connection. The sections are required to be covariantly constant in the directions corresponding to the polarization.

Theta Functions

Riemann θ-function

$$\theta(u_1, u_2, \ldots, u_g; \tau_{ij}) := \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} e^{2\pi i \sum_{j=1}^{g} u_j m_j} e^{\pi i \sum_{j,k=1}^{g} m_i \tau_{ij} m_j} \tag{K.-86}$$

K.16.3 Mathematics Details of Geometric Quantization

The problem is that θ is neither necessarily globally defined nor unique. We can invoke fibre bundle theory to address this problem.

The integrality condition is closely related to the quantisation rule in old quantum theory.

1. A principal $(\mathbb{C} - \{0\})$ bundle B over M with globally defined connection A whose local sections θ have $\omega = d\theta$ as globally defined curvature.

2. A vector bundle E over M, associated with P under the defining representation of $\mathbb{C} - \{0\}$ with typical fibre $\mathbb{C}$ and local section ψ.

3. A $\nabla-$compatible fibre metric.

What is a necessary and sufficient criterion for the existence of these structures? To answer this we enter into the subject of Cech cohomology.

Introduction to Cech cohomology

We discussed de Rham cohomology previously. The point of cohomology came from the nilpotency of the exterior derivative: $d^2 = 0$.

the transition functions $g_{IJ} : U_I \cap U_J \to G$ satisfy

$$\begin{aligned} g_{IJ} g_{JI} &= 1, \\ g_{IJ} g_{JK} g_{KI} &= 1 \quad \text{(no summation)} \end{aligned}$$

or

$$g_{IJ} = (g_{JI})^{-1}, \quad \text{(K.-87)}$$
$$g_{JK}(g_{IK})^{-1}g_{IJ} = (\delta g)_{IJK} = 1 \quad \text{(no summation)} \quad \text{(K.-86)}$$

Such relations fall under the subject of Cech cohomology. There is an isomorphism between Cech cohomology and de Rham cohomology. This isomorphism allows us to establish the condition for the existence of a globally defined connection.

Definition Let M be a manifold with open cover $\mathcal{U} = (U_I)_{I\in\mathcal{I}}$ subordinate to an atlas of M.

(i) An $n-$**cochain** $\{g\} \in C^n(\mathcal{U})$ is a system of functions defined on $U_{I_1} \cap \cdots \cap U_{I_n}$:

$$g_{I_1 \ldots I_n} : U_{I_1} \cap \cdots \cap U_{I_n} \to \mathbb{C} - \{0\}$$

such that

$$g_{I_{\pi(1)} \ldots I_{\pi(n+1)}} = (g_{I_1 \ldots I_{n+1}})^{sgn(\pi)}. \quad \text{(K.-86)}$$

The n-cochains form an abelian group under pointwise multiplication for each muti-index.

(ii) We define a **cobounadry operator** (also called codifferential)

$$\delta_n : C^n(\mathcal{U}) \to C^{n+1}(\mathcal{U}),$$

$$(\delta g)(I_1, \ldots, I_{n+1}) := \prod_{k=1}^{n+1} [g(I_1, \ldots \hat{I}_k, \ldots, i_{n+1})]^{(-1)^{k-1}}$$

where the variable below the ˆ is omitted. The function δg is defined on $U_{I_1} \cap \cdots \cap U_{i_n} \cap U_{I_{n+1}}$ when it is non-empty. Some examples are

$$(\delta g)(I_1, I_2) = g(I_2)g(I_1)^{-1}$$
$$(\delta g)(I_1, I_2, I_3) = g(I_2, I_3)g(I_1, I_3)^{-1}g(I_1, I_2) \quad \text{(K.-86)}$$

Consider what happens when we take the operator δ again in these examples:

$$
\begin{aligned}
(\delta^2 g)(I_1, I_2, I_3) &= \delta(g(I_1)^{-1} g(I_2)) \\
&= (g(I_2)^{-1} g(I_3))\, (g(I_1)^{-1} g(I_3))^{-1}\, (g(I_1)^{-1} g(I_2)) \\
&= 1. && \text{(K.-87)}
\end{aligned}
$$

Also

$$
\begin{aligned}
(\delta^2 g)(I_1, I_2, I_3, I_4) &= \delta(g(I_2, I_3) g(I_1, I_3)^{-1} g(I_1, I_2)) \\
&= (g(I_3, I_4) g(I_2, I_4)^{-1} g(I_2, I_3))\, (g(I_3, I_4) g(I_1, I_4)^{-1} g(I_1, I_3))^{-1} \\
&\times\ (g(I_2, I_4) g(I_1, I_4)^{-1} g(I_1, I_2))\, (g(I_2, I_3) g(I_1, I_3)^{-1} g(I_1, I_2))^{-1} \\
&= 1. && \text{(K.-89)}
\end{aligned}
$$

We verify that $\delta^2(g) = \{1\}$ in general:

$$
\begin{aligned}
\delta^2(g) &= \delta(\prod_{k=1}^{n+1} [g(I_1, \ldots, \hat{I}_k, \ldots, I_{n+1})]^{(-1)^{k-1}}) \\
&= \prod_{l<k} g(I_1, \ldots, \hat{I}_l, \ldots, \hat{I}_k, \ldots, I_{n+2})^{(-1)^{k+l}} \\
&\times \prod_{l>k} g(I_1, \ldots, \hat{I}_k, \ldots, \hat{I}_l, \ldots, I_{n+2})^{(-1)^{k+l-1}} \\
&= 1. && \text{(K.-91)}
\end{aligned}
$$

We have the power $k+l-1$ in the second product as we have passed through an ommitted index.

(iii) Instead of a multiplicative notation we can use an additive one by writing

$$
g_{I_1 \ldots I_{n+1}} = \exp(f_{I_1 \ldots I_{n+1}}), \qquad f_{I_1 \ldots I_{n+1}} = sgn(\pi) f_{I_1 \ldots I_{n+1}}
$$

This is obviously in accord with (K.16.3). The coboundray operation can be expressed

$$
(\delta f)_{I_1 \ldots I_{n+2}} = (n+2) \chi_{[I_1} f_{I_2 \ldots I_{n+2}]} \tag{K.-91}
$$

where $\chi_I = \chi_{U_I}$ is the characteristic function of U_I. We verify the equivalence with the original definition:

$$\begin{aligned}
(\delta g)_{I_1 \ldots I_{n+1}} &= (\delta \exp(f))_{I_1 \ldots I_{n+1}} \\
&= \exp(\delta(f))_{I_1 \ldots I_{n+1}} \\
&= \exp\Big((n+1)\chi_{[I_1} f_{I_2 \ldots I_{n+1}]}\Big) \\
&= \exp(\chi_{I_1} f_{[I_2 \ldots I_{n+1}]} - \chi_{I_2} f_{[I_1 I_3 \ldots I_{n+1}]} + \ldots (-1)^n \chi_{I_{n+1}} f_{[I_1 \ldots I_n]}) \\
&= \exp(\chi_{I_1} f_{I_2 \ldots I_{n+1}} - \chi_{I_2} f_{I_1 I_3 \ldots I_{n+1}} + \ldots (-1)^{n+1} \chi_{I_{n+1}} f_{I_1 \ldots I_n}) \\
&= (g_{I_2 \ldots I_{n+1}})(g_{I_1 I_3 \ldots I_{n+1}})^{-1} \ldots (g_{I_1 \ldots I_n})^{(-1)^{n+1}} \qquad \text{(K.-95)}
\end{aligned}$$

It easy to verify that $(\delta^2 g) = \{1\}$. First we have

$$(\delta^2 f)_{I_1 \ldots I_{n+2}} = (n+2)(n+1)\chi_{[I_1} \chi_{I_2} f_{I_3 \ldots i_{n+2}]} = 0$$

So that

$$\begin{aligned}
(\delta^2 g)_{I_1 \ldots I_{n+3}} &= (\delta^2 \exp(f))_{I_1 \ldots I_{n+3}} \\
&= \exp(\delta^2 f))_{I_1 \ldots I_{n+3}} \\
&= 1. \qquad \text{(K.-96)}
\end{aligned}$$

(iv) A $n-$cochain is called an $n-$**cocycle** if it is in the kernel of δ_n. The cocycle group is

$$Z^n(\mathcal{U}) := \{g \in C^n(\mathcal{U}) : \delta_n g = \{1\}\}.$$

(v) An $n-$cochain is called an $n-$**coboundary** if it is in the image of δ_{n-1}. The coboundary group is

$$B^n(\mathcal{U}) := \{g \in C^n(\mathcal{U}) : g = \delta_{n-1} g', g' \in C^{n-1}(\mathcal{U})\}.$$

As $\delta^2 = \{1\}$ every coboundary is also a cocycle.

(vi) Two cococyles g, g' that differ by a coboundary are said to be **cohomologous**. Rexpressed in terms of the additive notation

$$f'_{I_1 \ldots I_n} = f_{I_1 \ldots I_n} + (\delta \tilde{f})_{I_1 \ldots I_n}.$$

(vii) The group

$$H^n(\mathcal{U}) = Z^n(\mathcal{U})/B^n(\mathcal{U})$$

is called the $n-th$**Cech cohomology group**. The equivalence class of f in $H^n(\mathcal{U})$ is denoted by $[f]$.

$\square$

The Cech cohomology seems to depend exlpicitly on the atlas $\mathcal{U}$. The dependence can be removed by taking an infinite refinement limit. However for our purposes this process is not required. Here we interested in the cases where M paracompact in which case one can choose a locally finite, contractible cover. We automatically have the so-called Leray cover for which the cohomology is already independent of the cover; the answer comes out the same whichever such covering is used. We have used the notation $H^n(\mathcal{U})$ in order to distinguish it from the de Rham cohomology $H^n(M)$ of forms.

In what follows we will only consider the Cech cohomology defined by locally constant functions. A function is locally constant, taking values for example in $\mathbb{R}, \mathbb{C}, \mathbb{Z}$, if it is constant in some neighbourhood of each point; it need not be globally constant if its domain is not connected. The resulting cohomology groups are said to have coefficients in $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and are denoted by $H^\cdot(\mathcal{U}, \mathbb{R}), H^\cdot(\mathcal{U}, \mathbb{C}), H^\cdot(\mathcal{U}, \mathbb{Z})$.

Definition A locally finite open cover of M is a cover such that any $p \in M$ is only in finitely many U_I. It is a contractible open cover of M if every U_I and every nonempty finite intersection $U_I \cap U_J \cap \dots$ is contractible to a point

$\square$

Definition Let M be paracompact. $\{f\}$ is a locally constant n-cochain if each $f_{I_1 \dots I_{n+1}}$ takes a constant value on each connected component of $U_{I_1} \cap \dots \cap U_{I_n}$, possibly a different one on each component.

$\square$

What is the relation between de Rham cohomology and Cech cohomology?

Let us define a map α from $C^n(\mathcal{U})$ to $C^n(M)$

$$\alpha : C^n(\mathcal{U}) \to C^n(M)$$

defined by

$$\alpha_{\{f\}}(p) := f_{II_1 \dots I_n}(p) e_I(p)\ de_{I_1}(p) \wedge \dots \wedge de_{I_n}(p) \tag{K.-96}$$

Theorem K.16.3 (de Rham isomorphism). *We have* $d\alpha_{\{f\}} = \alpha_{\{\delta f\}}$ *and* α *defines an isomorphism* $H^n(\mathcal{U}) \to H^n(M)$.

Proof: Notice that $df_{I_1 \dots I_{n+1}} = 0$ in the compact support of $e_{I_1}\, de_{I_2} \wedge \dots \wedge de_{I_{n+1}}$ due to local constancy. While it is not in general true that $df_{I_1 \dots I_{n+1}} = 0$ on $\partial U_{I_1} \cap \dots U_{I_n}$ this surface is not in the support of $e_{I_1}\, de_{I_2} \wedge \dots \wedge de_{I_{n+1}}$. Hence

$$d\alpha_{\{f\}} = f_{I_1 \dots I_{n+1}}\, de_{I_1} \wedge \dots \wedge de_{I_{n+1}}. \tag{K.-96}$$

Now next notice the relation $\chi_I e_I = \sum_I e_I = 1$ because $\mathrm{supp}(e_I) \subset U_I$. Hence $0 = de_I \chi_I + e_I d\chi_I = de_I \chi_I$ where we have used the fact that $d\chi_I$ is non-vanishing on ∂U_I only, which however is outside the support of e_I.

$$\begin{aligned}
\alpha_{\delta\{f\}} &= (n+2)\chi_{[I} f_{I_1 \dots I_{n+1}]} \\
&\quad + \sum_{k=1}^{n+1} (-1)^k \chi_{I_k}\, f_{I_1 \dots \hat{I}_k I \dots I_{n+1}}\, e_I\, de_{I_1} \wedge \dots \wedge de_{I_{n+1}} \\
&= [\chi_I e_I] f_{I_1 \dots I_{n+1}} de_{I_1} \wedge \dots \wedge de_{I_{n+1}} \\
&= d\alpha_{\{f\}} - \sum_{k=1}^{n+1} \chi_{I_k}\, f_{I I_1 \dots \hat{I}_k \dots I_{n+1}}\, e_I\, de_{I_1} \wedge \dots \wedge de_{I_{n+1}} \\
&= d\alpha_{\{f\}} + \sum_{k=1}^{n+1} (-1)^k \chi_{I_k}\, f_{I I_1 \dots \hat{I}_k \dots I_{n+1}}\, e_I\, de_{I_k} \wedge de_{I_1} \wedge \dots \wedge d\hat{e}_{I_k} \wedge \dots \wedge de_{I_{n+1}} \\
&= d\alpha_{\{f\}} + \left[\sum_{k=1}^{n+1} (-1)^k \right] [\chi_J de_J] \wedge \alpha_{\{f\}} = d\alpha_{\{f\}}
\end{aligned} \tag{K.-100}$$

Hence $\alpha_{\delta\{f\}} = d\alpha_{\{f\}}$ and so α maps coboundaries to closed forms. And if f and f' are cohomologous, then α_f and $\alpha_{f'}$ differ by an exact form.

We now define $\alpha : H^n(\mathcal{U}) \to H^n(M)$ by

$$\alpha_{[\{f\}]} := [\alpha_{\{f\}}]$$

where the brackets denote the respective cohomology classes.

Injectivity:

The map is one-to-one if for $f \neq f'$, $\alpha_{[\{f\}]} = \alpha_{[\{f'\}]}$ implies f and f' are Cech comologous. Now $\alpha_{[\{f\}]} = \alpha_{[\{f'\}]}$ means by definition $[\alpha_{\{f\}}] = [\alpha_{\{f'\}}]$. So $\alpha_{\{f'\}} = \alpha_{\{f\}} + d\alpha_{\{\tilde{f}\}}$

Surjectivity:

We now prove that the map is onto. We prove this only for $n = 2$, the general case is similar. Assume that $\tau \in Z^2(M)$, that is, τ is an arbitrary closed 2-form. We show that there exists an f_{IJK} such that $\tau - \alpha_{\{f\}}$ is exact, that is, $[\tau] = [\alpha_{\{f\}}] = \alpha_{[\{f\}]}$ proving surjectivity.

Since U_I is contractible, by Poincare's lemma we find $\beta_I \in C^1(U_I)$

$$\tau = d\beta_I$$

on U_I. If $U_I \cap U_J \neq \emptyset$ we have

$$d(\beta_I - \beta_J) = 0$$

on $U_I \cap U_J$. Since $U_I \cap U_J$ is conractible, again by Poincare's lemmaa, we find $\gamma_{IJ} \in C^0(U_I \cap U_J)$ (the space of smooth functions on $U_I \cap U_J$) such that

$$\beta_I - \beta_J = d\gamma_{IJ}$$

on $U_I \cap U_J$ and where γ_{IJ} satisfies $\gamma_{IJ} = -\gamma_{JI}$. If $U_I \cap U_J \cap U_K \neq \emptyset$ then

$$d(\gamma_{IJ} + \gamma_{JK} + \gamma_{KI}) = (\beta_I - \beta_J) + (\beta_J - \beta_K) + (\beta_K - \beta_I) = 0$$

on $U_I \cap U_J \cap U_K$, hence

$$f_{IJK} := \gamma_{IJ} + \gamma_{JK} + \gamma_{KI} = (\delta\gamma)_{IJK}$$

is locally constant. Notice that f_{IJK} is locally constant but not necessarily the γ_{IJ}. As we are only considering Cech cohomology of locally constant functions, f_{IJK} does not generally qualify as exact. However, the identity $\delta^2 = 0$ proved in () holds for any type of function not just locally constant ones, hence we have that $\{f\}$ is closed and on $U_I \cap U_J \cap U_K \cap U_L$

$$(\delta f)_{IJKL} = f_{JKL} - f_{IKL} + f_{IJL} - f_{IJK} = 0. \tag{K.-100}$$

Contracting with $e_I \, de_J \wedge de_K$ and using $\sum e_I = 1$ and $\sum de_J = 0$ we get

$$\begin{aligned}
\alpha_{\{f\}} &= f_{IJK} e_I de_J \wedge de_K \\
&= (f_{JKL} - f_{IKL} + f_{IJL})\, e_I de_J \wedge de_K \\
&= f_{LJK} de_J \wedge de_K \\
&= d(f_{LJK} e_J \wedge de_K) =: d\omega_I
\end{aligned} \tag{K.-102}$$

on the interior of U_I. Similarly, contracting with $e_J\, de_K$ we find

$$\begin{aligned}
f_{IJK} e_J\, de_K - f_{LJK} e_J\, de_K &= -f_{IKL} e_J\, de_K + f_{IJL} e_J\, de_K \\
&= f_{ILK} de_K \\
&= d(f_{ILK} e_K)
\end{aligned}$$

on the interior of $U_I \cap U_J$. This expressed in terms of the just defined ω_I reads

$$\omega_I - \omega_J = f_{IJK} de_K = d(f_{IJK} e_K) =: d\sigma_{IJ} \tag{K.-105}$$

on the interior of $U_I \cap U_J$. Noticing that

$$\sigma_{IJ} = f_{IJ} + f_{JK} e_K + f_{KI} e_K \tag{K.-105}$$

and defining on the interior of U_I

$$\lambda_I = \beta_I - \omega_I - d(f_{IJ} e_J) \tag{K.-105}$$

we have on the interior of $U_I \cap U_J$

$$\begin{aligned}
\lambda_I - \lambda_J &= \gamma_{IJ} - \omega_I + \omega_J - d(f_{IK} e_K - f_{JK} e_K) \\
&= -\omega_I + \omega_J + d(f_{IJ} + f_{JK} e_K + f_{KI} e_K) \\
&= -\omega_I + \omega_J + d\sigma_{IJ} = 0
\end{aligned} \tag{K.-106}$$

hence $\lambda = \lambda_I$ is globally defined. Hence on U_I

$$d\lambda = d\beta_I - d\omega_I = \tau - \alpha_{\{f\}} \tag{K.-106}$$

it follows that $\tau - \alpha_{\{f\}}$ is exact.

□

Theorem K.16.4 (Weil's integrality criterion). *A prequantisation of (M, ω), that is, a principal $\mathbb{C} - \{0\}$ bundle B with global connection ∇ and $\nabla-$compatible fibre metric ρ on an associated complex line bundle exists if and only if Weil's criterion holds: the Cech cohomology class of $\alpha^{-1}(\omega/(2\pi\hbar)$ is integral, that is, $[\alpha^{-1}(\omega/(2\pi\hbar)] \in \mathbb{Z}$ where $\alpha : H^2(\mathcal{U}) \to H^2(M)$ is the de Rham isomorphism*

Moreover, the inequivalent choices of (P, ∇, ρ) are parametrised by $H^1(\mathcal{U})$ with values in $U(1)$.

Proof:

Suppose first that Weil's criterion is satisfied and let $[\omega] = [\alpha_{\{f\}}]$. From the proof of the previous theorem we know that $f_{IJK} = \gamma_{IJ} + \gamma_{JK} + \gamma_{KI} = (\delta\gamma)_{IJK}$ on $U_I \cap U_J \cap U_K$ is locally constant with smooth functions $\gamma_{IJ} = -\gamma_{JI}$ on $U_I \cap U_J$. Moreover, by assumption

$$f_{IJK} = 2\pi\hbar n_{IJK}$$

where n_{IJK} takes locally constant integer values on $U_I \cap U_J \cap U_K$. Define

$$g_{IJ} = \exp(i\gamma_{IJ}/\hbar),$$

then

$$g_{IJ} g_{JI} = 1$$

on $U_I \cap U_J$ and because n_{IJK} is an integer

$$\begin{aligned} g_{IJ} g_{JK} g_{KI} &= \exp i(\gamma_{IJ} + \gamma_{JK} + \gamma_{KI})/\hbar \\ &= \exp(2\pi i n_{IJK}) = 1 \end{aligned} \tag{K.-106}$$

on $U_I \cap U_J \cap U_K$, hence g_{IJ} is a cocycle with values in $\mathbb{C} - \{0\}$ and therefore qualifies as the transition function of a principal $(\mathbb{C} - \{0\})$ bundle. If θ_I are the local potentials of ω then by definition

$$d\gamma_{IJ} = -i\hbar dg_{IJ} g_{IJ}^{-1} = \theta_I - \theta_J$$

or with $A_I = i\theta_I/\hbar$ we find

$$A_J = A_I - dg_{IJ} g_{IJ}^{-1} \tag{K.-106}$$

Here we use the results of section . We construct local one-forms $\nabla_U, \nabla_V, \ldots,$ on P from the local one-forms $A_U, A_V, \ldots,$ on $\mathcal{M}$. From the transformation (K.16.3) it follows that $\nabla_U = \nabla_V$ on $\pi^{-1}(U \cap V)$, thus the local one-forms $\nabla_U, \nabla_V, \ldots,$ collectively define a global connection one-form ∇ on P. The A_I qualify as the pull-backs by local sections of a globally defined $\mathbb{C} - \{0\}$ connection ∇.

Now we prove it the other way. Suppose that (P, ∇, ρ) exist and let g_{IJ} be the transition functions of the bundle P with values in $\mathbb{C} - \{0\}$. We wish to define a f_{IJK} in terms of the logarithm of g_{IJ}. However there is a subtlety in the definition of the logarithms. The complex logarithmic function has infinitely many branches. The function ln is called the fundamental or principal branch of the logarith.

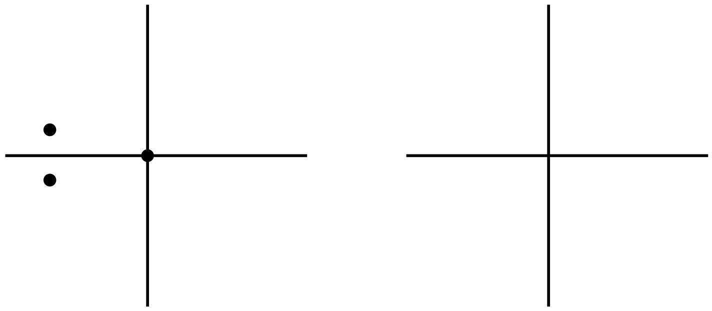

Figure K.5: .

We define

$$\frac{f_{IJK}}{2\pi\hbar} := \frac{1}{2\pi i}[\ln(g_{IJ}) + \ln(g_{JK}) + \ln(g_{KI})] \qquad \text{(K.-106)}$$

where we choose the fundamental branch of the logarithm over $U_I \cap U_J$ with cut at $\varphi = \pi$ so that $\ln(g_{IJ}) = -\ln(g_{IJ})$. Hence we have

$$\ln g_{IJ} = \ln|g_{IJ}| + i(\theta + 2\pi n_{IJ})$$

where $n_{IJ} \in \mathbb{Z}$. Since g_{IJ} satisfy the cocycle condition

$$\begin{aligned} \frac{1}{2\pi i}[\ln(g_{IJ}) + \ln(g_{JK}) + \ln(g_{KL})] &= \frac{1}{2\pi i}\ln(g_{IJ}g_{JK}g_{KL}) + n_{IJ} + n_{JK} + n_{KL} \\ &= n_{IJ} + n_{JK} + n_{KL}. \end{aligned}$$

Hence the right-hand side of (K.16.3) is integral $n_{IJK} \in \mathbb{Z}$ and as (relying on antisymmetry $\ln(g_{IJ}) = -\ln(g_{JI})$)

$$\frac{f_{IJK}}{2\pi\hbar} = \frac{1}{2\pi i}(\delta \ln g)_{IJK}$$

$\delta\{f\} = 0$. Since...

$$\begin{aligned}
\alpha_{\{f\}} &= -i\sum_{IJK}[\ln(g_{IJ}) + \ln(g_{JK}) + \ln(g_{KI})]\; e_I de_J \wedge de_K \\
&= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(K.-108)}
\end{aligned}$$

Finally we prove the last statement of the theorem. There is a freedom in the construction of P and ∇ from ω. If Weil's criterion is satisfied then $[\{f\}]$ is determined by γ_{IJ} only up to a coboundary $\delta\{x\}$

□

Recall that two bundles P, P' are equivalent if... . Now if M is simply connected (intuitively a connected space without any holes) then $H^1(M) = \{0\}$, hence by the de Rham isomorphism also $H^1(\mathcal{U}) = \{0\}$.

This is sinificant because if M is simply connected, then $H^1(M)$ is trivial and there is a unique choice of B and ∇.

Corollary K.16.5 *Weil's criterion is equivalent to the requirement that for any closed two-surface S in phase space*

$$\int_S \frac{\omega}{2\pi\hbar} = integer \qquad \text{(K.-108)}$$

Proof: Suppose first that Weil's criterion holds. We assume for simplicity that the contractible open cover $\mathcal{U}$ is such that the sets $D_I := S \cap U_I$ are open discs covering S such that no point of S lies in more than three different M_I.

$$\begin{aligned}
M_I &:= D_I - \cup_{J\neq I}(D_I \cap D_J) && \text{(K.-107)}\\
M_{IJ} &:= D_I \cap D_J - \cup_{K\neq I,J}(D_I \cap D_J \cap D_K) && \text{(K.-106)}\\
M_{IJK} &:= D_I \cap D_J \cap D_K && \text{(K.-105)}
\end{aligned}$$

$$\begin{aligned}
S_1 &= \cup_I M_I \\
S_2 &= \cup_{I<J} M_{IJ} \\
S_3 &= \cup_{I<J<K} M_{IJK} \qquad \text{(K.-106)}
\end{aligned}$$

$$S = S_1 \cup S_2 \cup S_3$$

$$M_I = D_I - \cup_{J\neq I} M_{IJ} - \cup_{J<K;J,K\neq I} M_{IJK} \tag{K.-106}$$

$$\begin{aligned}
\int_S \Omega &= \sum_I \int_{M_I} \Omega + \sum_{I<J} \int_{M_{IJ}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega \\
&= \left(\sum_I \int_{D_I} \Omega - \sum_{J\neq I} \int_{M_{IJ}} \Omega - \sum_{J<K,\ J,K\neq I} \int_{M_{IJK}} \Omega \right) + \sum_{I<J} \int_{M_{IJ}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega
\end{aligned} \tag{K.-107}$$

As $\sum_{J\neq I} = \sum_{I<J} + \sum_{I>J}$:

$$\int_S \Omega = \sum_I \int_{D_I} \Omega - \sum_{I>J} \int_{M_{IJ}} \Omega - \sum_{J<K,\ J,K\neq I} \int_{M_{IJK}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega$$

Rewritting the second term

$$\int_S \Omega = \sum_I \int_{D_I} \Omega - \left(\sum_{I>J} \int_{D_I\cap D_J} \Omega - \sum_{I>J,\ K\neq I,J} \int_{M_{IJK}} \Omega \right) - \sum_{J<K,\ J,K\neq I} \int_{M_{IJK}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega$$

Note

$$\begin{aligned}
\sum_{J<K,\ J,K\neq I} &= \sum_{I<J<K} + \sum_{J<I<K} + \sum_{J<K<I} \\
\sum_{I>J,\ J,K\neq I} &= \sum_{K<J<I} + \sum_{J<K<I} + \sum_{J<I<K} .
\end{aligned} \tag{K.-108}$$

Note the last two terms in the first and second lines are equal. We see

$$\begin{aligned}
\sum_{I<J<K} - \sum_{J<K,\ J,K\neq I} + \sum_{I>J,\ J,K\neq I} &= \sum_{I<J<K} - \sum_{I<J<K} + \sum_{K<J<I} \\
&= \sum_{I>J>K}
\end{aligned} \tag{K.-109}$$

So we finally have

$$\int_S \Omega = \sum_I \int_{D_I} \Omega - \sum_{I>J} \int_{D_I \cap D_J} \Omega + \sum_{I>J>K} \int_{D_I \cap D_J \cap D_K} \Omega$$
$$= \sum_I \int_{\partial D_I} \sigma_I - \sum_{I>J} \int_{\partial(D_I \cap D_J)} \sigma_J + \sum_{I>J>K} \int_{\partial(D_I \cap D_J \cap D_K)} \sigma_K \qquad \text{(K.-109)}$$

where we used $\Omega = d\sigma_I$ on (any subset of) D_I.

We take all D_I with orientation such that the loop ∂D_I is counterclockwise for definiteness.

$$\partial(D_I \cap D_J) = [(\partial D_I) \cap D_J] \cup [D_I \cap (\partial D_J)]$$
$$\partial(D_I \cap D_J \cap D_K) = [(\partial D_I) \cap D_J \cap D_K] \cup [D_I \cap (\partial D_J) \cap D_K]] \cup [D_I \cap D_J \cap (\partial D_K)] \qquad \text{(K.-110)}$$

$$\partial^2(D_I \cap D_J) = \partial[(\partial D_I) \cap D_J] \cup [D_I \cap (\partial D_J)] = 0 \qquad \text{(K.-110)}$$

$$\partial D_I = [\cup_{J \neq I} (\partial D_I) \cap M_{IJ}] \cup [\cup_{J<K;J,K\neq I} (\partial D_I) \cap M_{IJK}]. \qquad \text{(K.-110)}$$

$$\int_S \Omega = \sum_{I>J} \int_{(\partial D_I) \cap D_J} (\sigma_I - \sigma_J)$$
$$+ \sum_{I>J>K} \left\{ \int_{(\partial D_I) \cap D_J \cap D_K} (\sigma_K - \sigma_I) + \int_{D_I \cap (\partial D_J) \cap D_K} (\sigma_J - \sigma_K) \right\}$$

$$\int_S \Omega = \sum_{I>J} \int_{\partial[(\partial D_I) \cap D_J]} \gamma_{IJ}$$
$$+ \sum_{I>J>K} \left\{ \int_{\partial[(\partial D_I) \cap D_J \cap D_K]} \gamma_{KI} + \int_{\partial[D_I \cap (\partial D_J) \cap D_K]} \gamma_{KJ} \right\} \qquad \text{(K.-112)}$$

where $\sigma_I - \sigma_J = d(\gamma_{IJ})$ on (any subset of) $D_I \cap D_J$ was used.

$$\partial[(\partial D_I) \cap D_J] = \cup_{K \neq I,J} (\partial[(\partial D_I) \cap D_J)] \cap D_K \qquad \text{(K.-112)}$$

Next

$$\partial[D_I \cap (\partial D_J) \cap D_K] \;=\; (\partial[(\partial D_I) \cap D_J]) \cap D_K \cup (\partial[(\partial D_I) \cap D_K]) \cap D_J \tag{K.-112}$$

$$\begin{aligned}\sum_{I>J} \int_{\partial[(\partial D_I)\cap D_J]} \gamma_{IJ} &= \sum_{I>J} \sum_{K\neq I,J} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{IJ} \\ &= (\sum_{I>J>K} + \sum_{I>K>J} + \sum_{K>I>J}) \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{IJ}\end{aligned} \tag{K.-112}$$

$$\sum_{I>J>K} \int_{\partial[(\partial D_I)\cap D_J \cap D_K]} \gamma_{KI} \;=\; \sum_{I>J>K} \left\{ \int_{\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{KI} + \int_{(\partial[(\partial D_I)\cap D_K])\cap D_J} \gamma_{KI} \right\} \tag{K.-112}$$

$$\sum_{I>J>K} \int_{\partial[D_I\cap(\partial D_J)\cap D_K]} \gamma_{KJ} \;=\; \sum_{I>J>K} \left\{ \int_{\partial[(\partial D_J)\cap D_I])\cap D_K} \gamma_{KJ} + \int_{(\partial[(\partial D_J)\cap D_K])\cap D_I} \gamma_{KJ} \right\} \tag{K.-112}$$

Using $\gamma_{IJ} + \gamma_{JI} = 0$

$$\begin{aligned}\int_S \Omega &= \sum_{I>J>K} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} [\gamma_{IJ} + \gamma_{JK} + \gamma_{KI}] \\ &\quad +(\sum_{I>K>J} + \sum_{K>I>J}) \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{IJ} + 2 \sum_{I>J>K} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{KJ} \\ &\quad + \sum_{I>J>K} \left\{ \int_{(\partial[(\partial D_I)\cap D_K])\cap D_J} \gamma_{KI} + \int_{(\partial[(\partial D_J)\cap D_K])\cap D_I} \gamma_{KJ} \right\}\end{aligned} \tag{K.-113}$$

We arrive at

$$\begin{aligned}\int_S \Omega &= \sum_{I>J>K} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} [\gamma_{IJ} + \gamma_{JK} + \gamma_{KI}] \\ &= \sum_{I>J>K} n_{IJK} \in \mathbb{Z}\end{aligned} \tag{K.-113}$$

where we have used the fact that $\gamma_{IJ}+\gamma_{JK}+\gamma_{KI}=n_{IJK}$ is integral and constant on (any subset of) $D_I \cap D_J \cap D_K$.

We prove it the other way, that is, if $\int_S \omega/2\pi\hbar =$ integer then $[\alpha^{-1}(\omega/(2\pi\hbar))] \in \mathbb{Z}$. Fix some labels $I_0 > J_0 > K_0$ with $D_{I_0} \cap D_{J_0} \cap D_{K_0}$

□

Cotangent bundles T^*Q, equipped with the canonical sympletic strcuture $\omega = d\theta$, are always prequantisable as ω is exact.

Definition As sympletic manifold $(\mathcal{M},\omega)$ is said to be prequantisable if and only if Weil's integrality criterion is satisfied. The associated structure (P,∇,ρ) is called a prequantum bundle. The prequantum Hilbert space is $\mathcal{H}' = L_2(\mathcal{M},\Omega)$ with inner product between smooth sections of compact support of the associated line bundle E given by

$$< \psi, \psi' > = \int_{\mathcal{M}} \Omega\rho[\psi,\psi'], \qquad \rho[\psi,\psi'] = \rho\psi\psi' \tag{K.-113}$$

where $d\ln(\rho) = \frac{2}{\hbar} Im(\theta)$ and symmetric operators associated with real-valued functions $f \in C^\infty(\mathcal{M})$ are densely defined on E by

$$\hat{f} = i\hbar\nabla_{\chi_f} + f, \qquad \nabla = d + \frac{1}{i\hbar}\theta, \qquad \omega = d\theta. \tag{K.-113}$$

□

K.16.4 Integrability Condition

Consider the system of partial differential equations

$$\frac{\partial f}{\partial x} = g(x,y), \qquad \frac{\partial f}{\partial y} = h(x,y) \tag{K.-113}$$

This can be written as

$$f_{,i} = a_i,$$

where $a_x = g$ and $a_y = h$. This equation has the coordinate independent form

$$df = a, \tag{K.-113}$$

where a is a one-form with components g and h. If f is a solution to this equation then we must have

$$d(df) = da.$$

We have the necessary condition for the solution to exist is that

$$da = 0.$$

In components this is

$$a_{[i,j]} = 0,$$

which is

$$\frac{\partial g}{\partial y} - \frac{\partial h}{\partial x} = 0. \tag{K.-113}$$

These are sufficient conditions for the existence of a solution is guaranteed by Frobenius theorem in the differential form version.

K.16.5 Polarisation

The wave functions depend only on position. The wavefunctions of the associated prequantum Hilbert space would depend on momentum as well.

The prequantum Hilbert space H consists of functions ψ which depend on all the $2n$ coordinates of the sympletic manifold $(\mathcal{M}, \omega)$. Demand that the wavefunctions are constant along n vector fields in $\mathcal{M}$.

We choose some n-dimensional sub-bundle P of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$

As we see we proceed by choosing some k-dimensional subbundle P of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and consider only those wave functions that satisfy

$$D(X)\psi = 0 \qquad \text{for all } X \in P. \tag{K.-113}$$

Now there could be non-trivial integrability conditions for those equations which would form an obstruction to finding (or a sufficient number of) solutions to (K.16.5). From (K.16.5) it follows that $[D(X), D(Y)]\psi = 0$ for all $X, Y \in P$. Combined with $\Omega(X, Y) = i([[D(X), D(Y)]] - D([X, Y]))$ this leads to the integrability condition

$$D([X,Y])\psi - (i/\hbar)\omega(X,Y)\psi = 0 \qquad \text{for all } X, Y \in P. \tag{K.-113}$$

We see that this condition is automatically satisfied provided that

$$X, Y \in P \;\Rightarrow\; [X,Y] \in P \tag{K.-113}$$

and

$$X, Y \in P \;\Rightarrow\; \omega[X,Y] = 0. \tag{K.-113}$$

The first condition means that P is integrable. The second condition means that the integral manifolds are Lagrangian, or if you like, maximally isotropic:

$$P = P^{\perp} = \{X \in V : \omega[X,Y] = 0 \text{ for all } Y \in P\}.$$

Integrability condition

Given a smooth nonvanishing vector field, by solvong a system of ordinary differential equations one can always locally find a smooth family of integral curves, that is, nonintersecting curves that fill up a region and are always tangent to the vector field.

System of Mayer-Lie

Frobenius' theorem arose in the study of partial differential equations.

This is important in the section on geometric quantisation.

Complex structures

It turns out that one needs to complexify the tangent bundle of $\mathcal{M}$, $T\mathcal{M} \to T\mathcal{M}^{\mathbb{C}}$, and to then consider integrable Lagrangian sub-bundles of $T\mathcal{M}^{\mathbb{C}}$.

For example say the symplectic manifold were the sphere, it is well known that any vector field on this space must vanish at at least one point. In this case how you cannot define a polarisation. A resolution to this problem is to take two vector fields that vanish at different points and consider their complex linear combination.

Definition Let V be a vector space involving only real coefficients. The complexification $V_{\mathbb{C}}$ of V is a vector space consisting of vectors of the form $w = u + iv$, $u, v \in V$.

The linear operations on $V_{\mathbb{C}}$ are defined as follows:

$$w_1 + w_2 := (u_1 + u_2) + i(v_1 + v_2),$$

for $z = x + iy \in \mathbb{C}$ we have

$$zw := (xu - yv) + i(xv + yu)$$

and we define

$$\overline{w} := u - iv.$$

$\square$

Note we connot have $zw = \overline{w}$ as $(xu - yv) + i(yu + xv)$ does not have a solution (except for $u = 0$, $y = 0$, $x = -1$), i.e. the equations

$$\begin{array}{rcl} (xu - yv) & = & u \\ (yu + xv) & = & -v \end{array}$$

do not have a solution (except for $u = 0$, $y = 0$, $x = -1$).

Definition Let $(\mathcal{M}, \omega)$ be a sympletic manifold. A polarisation P of $(\mathcal{M}, \omega)$ is an integrable maximally isotropic (Lagrangian) sub-bundle of the complexified tangent bundle $T\mathcal{M}^{\mathbb{C}}$ of $\mathcal{M}$.

$\square$

A real distribution P on a manifold M is a sub-bundle of the tangent bundle.

Definition Let $(\mathcal{M}, \omega, J)$ be a sympletic vector space with ω-compatible complex structure J. The subspace $V^{\pm}$ of $V_{\mathbb{C}}$ consisting of vectors of the form

$$u^{\pm} := \frac{1}{2}(u \mp iJ[u])$$

is called the subspace of holomorphic (antiholomorphic) vectors since $J\left[u^{\pm}\right] = \pm iu^{\pm}$. We set $P_J := V^+$. Obviously u^- is the complex conjugate, $\overline{u^+}$, of u^+ in the sense of complex vectors and so $\overline{P_J} = \{\overline{w},\ w \in P_J\} = V^-$.

$\square$

Lemma K.16.6 *P_J is a Lagrangian subspace of $V_{\mathbb{C}}$ with the additional property that $P_J \cap \overline{P_J} = \{0\}$. Conversely, every Lagrangian subspace with this property determines a compatible complex structure on V.*

Proof: Recall that a Lagrangian subspace F of a sympletic vector space V is defined by the property $F = F^{\perp} := \{v \in V : \omega(u, v) = 0 \text{ for all } u \in F\}$. We have for any $u, v \in V$

$$\begin{aligned}
4\omega[u^+, v^{\pm}] &= 4\omega[\frac{1}{2}(u - iJ[u]), \frac{1}{2}(v \mp iJ[v])] \\
&= (\omega[u, v] \mp \omega[J[u], J[v]]) - i(\omega[J[u], v] \pm \omega[u, J[v]]) \\
&= [\omega[u, v] - i\omega[J[u], v](1 \mp 1) \qquad \text{(K.-116)}
\end{aligned}$$

by compatibility ($\omega[J[u], J[v]] = \omega[u, v]$, equivalently $\omega[J[u], v] = -\omega[u, J[v]]$). Hence, because ω nondegenerate (if $\omega[u, v] = 0$ for all $v \in V$ then $u = 0$), for $u^+, v^+ \in P_J$ we have $\omega[u^+, v^+] = 0$ so $P_J \subset (P_J)^{\perp}$ and when $u^+ \in P_J$, $v^- \in (P_J)^{\perp}$ we have $\omega[u^+, v^-] \neq 0$ so $\overline{P_J}$ is not in $(P_J)^{\perp}$, that is, $\overline{P_J} \cap (P_J)^{\perp} = \{0\}$. Since $V^+ = P_J$, $V^- = \overline{P_J}$ span $V_{\mathbb{C}}$ and satisfy $P_J \cap \overline{P_J} = \{0\}$ by $P_J \subset (P_J)^{\perp}$ for $w \in (P_J)^{\perp}$, $w \notin P_J$ we must have $w \in \overline{P_J}$ but this contradicts $\overline{P_J} \cap (P_J)^{\perp} = \{0\}$ so there is no such w, i.e. $P_J = (P_J)^{\perp}$. Thus we have proved that P_J is a Lagrangian subspace of $V_{\mathbb{C}}$.

Conversely, given a Lagrangian subspace $P \subset V_{\mathbb{C}}$ with $P \cap \overline{P} = \{0\}$ we know that $V_{\mathbb{C}} = P \oplus \overline{P}$ and can decompose any $w \in V_{\mathbb{C}}$ uniquely as $w = w^+ + w^-$ with $w^+ \in P$, $w \in \overline{P}$. Notice that since ω is real and P is Lagrangian, $\overline{P}$ is also Lagrangian since $\omega[\overline{w}, \overline{w}'] = \overline{\omega[w, w']} = 0$. We now define

$$J[w] := i(w^+ - w^-)$$

which determines J uniquely. Then

$$\begin{aligned}
\omega[J[w_1], J[w_2]] &= \omega[i(w_1^+ - w_1^-), i(w_2^+ - w_2^-)] \\
&= -\omega[w_1^+, w_2^+] + \omega[w_1^+, w_2^-] + \omega[w_1^-, w_2^+] - \omega[w_1^-, w_2^-] \\
&= \omega[w_1^+, w_2^-] + \omega[w_1^-, w_2^+] \qquad \text{(K.-117)}
\end{aligned}$$

where we have used the Lagrangian subspace property.

$\square$

Complex Lagrangian subspaces

Definition Let V be a vector space over the reals, or complex numbers, and let W be a subset of V. Then W is said to be a subspace of V. If W is itself a vector space, then W is said to be a vector subspace of V.

From the previous lemma we see that given a complex structure it is easy to construct Lagrangian subspaces of $V_{\mathbb{C}}$ with $P_J \cap \overline{P_J} = \{0\}$. The other extreme are Lagrangian subspaces of $V_{\mathbb{C}}$ with $P \cap \overline{P} = P$. They have the following property.

Lemma K.16.7 *Lagrangian subspaces P of $V_{\mathbb{C}}$ with the property $P \cap \overline{P} = P$ are the complexifications of Lagrangian subspaces of V.*

Proof:

□

A polarisation picks out for each point $x \in \mathcal{M}$ out a certain subspace P_x of the complexified tangent space at x.

The easiest kind of polarisation is a real polarisation. This is when the polarisation comes from the complexification of subspaces of the tangent spaces.

We now systematically study the intermediate cases of complex Lagrangian subspaces, in which the dimension of $P \cap \overline{P}$ is between zero and n, utilising the following results.

Lemma K.16.8 *Let*

$$F \subset V$$

be a subspace of the real vector space V and let

$$G \subset V_{\mathbb{C}}$$

be a subspace of its complexification. We define the real subspace of G by

$$G_{\mathbb{R}} := G \cap V.$$

The annihilator subspaces $F^{\perp}, G^{\perp}$ of V, $V_{\mathbb{C}}$ respectively are defined by

$$F^{\perp} = \{v \in V : \omega[u, v] = 0 \text{ for all } u \in F\}$$

and

$$G^\perp = \{v \in V_{\mathbb{C}} : \omega[u,v] = 0 \textit{ for all } u \in G\}$$

respectively. Then the following results hold:

(i) $(F^\perp)_{\mathbb{C}} = (F_{\mathbb{C}})^\perp$.

(ii) If $G = \overline{G}$ *then* $(G^\perp)_{\mathbb{R}} = (G_{\mathbb{R}})^\perp$.

(iii) If $G = \overline{G}$ *then* $(G_{\mathbb{R}})_{\mathbb{C}} = G$.

(iv) Define $\tilde{F} = F/(F \cap F^\perp) = \{(u) : u \in F\}$ *where the rest of classes are defined by* $[u] = \{u + v : v \in F \cap F^\perp\}$. *Then* $\tilde{\omega}[(u),(u')] := \omega[u,u']$ *is well defined and* $(\tilde{F}, \tilde{\omega})$ *is a sympletic vector space.*

(v) if F *is co-isotropic* ($F^\perp \subset F$), $\pi : F \to \tilde{F} = F/F^\perp$ *the canonical projection,* P *a Lagrangian subspace in* V *then* $\tilde{P} := \pi(P \cap F)$ *is a Lagrangian subspace in* $(\tilde{F}, \tilde{\omega})$.

Proof:

(i):

$$p \in F^\perp$$

$$p_{\mathbb{C}} \in (F^\perp)_{\mathbb{C}}, \quad p_{\mathbb{C}} = p_1 + ip_2, \quad p_1, p_2 \in F^\perp$$

now take

$$f_{\mathbb{C}} \in F_{\mathbb{C}}, \quad f_{\mathbb{C}} = f_1 + if_2, \quad f_1, f_2 \in F$$

$$\omega[p_{\mathbb{C}}, f_{\mathbb{C}}] = \omega[f_1 + if_2, p_1 + ip_2] = 0$$

so

$$p_{\mathbb{C}} \in (F_{\mathbb{C}})^\perp.$$

which implies

$$(F^\perp)_{\mathbb{C}} \subset (F_{\mathbb{C}})^\perp$$

The other way around

$$f_{\mathbb{C}} \in F_{\mathbb{C}}, \quad f_{\mathbb{C}} = f_1 + if_2, \quad f_1, f_2 \in F$$

$$q \in (F_{\mathbb{C}})^{\perp}$$

means

$$\omega[f_{\mathbb{C}}, q] = 0 \quad \text{for all } f_{\mathbb{C}} \in F_{\mathbb{C}}$$

$$q \in V_{\mathbb{C}}, \quad q_{\mathbb{C}} = q_1 + iq_2, \quad q_1, q_2 \in V$$

$$\omega[q, f_{\mathbb{C}}] = 0$$

implies

$$\omega[q_1 + iq_2, f_1 + if_2] = \omega[q_1, f_1] - \omega[q_2, f_2] + i\left(\omega[q_2, f_1] + \omega[q_1, f_2]\right) \tag{K.-116}$$

set $f_1 = f_2$, then the real and imaginary parts are

$$\begin{aligned} \omega[q_1 - q_2, f_1] &= 0 \\ \omega[q_1 + q_2, f_1] &= 0 \end{aligned} \tag{K.-117}$$

Adding gives

$$\omega[q_1, f_1] = 0$$

which implies

$$q_1 \in F^{\perp}.$$

Thus

$$(F^{\perp})_{\mathbb{C}} \subset (F_{\mathbb{C}})^{\perp}.$$

(ii): $G = \overline{G}$ implies if $u_1 + iu_2$ is in G then so is $u_1 - iu_2$.

$(G^{\perp})_{\mathbb{R}}$ means

$$\{v \in V_{\mathbb{C}} : \omega[v, w] = 0, \quad \text{for all } w \in G\} \cap V$$

and $(G_{\mathbb{R}})^{\perp}$ is defined by

$$\{v \in V : \omega[v, w] = 0, \quad \text{for all } w \in G \cap V\}.$$

We first prove $(G_{\mathbb{R}})^{\perp} \subset (G^{\perp})_{\mathbb{R}}$**:**

Let

$$g \in G$$

then

$$\overline{g} \in G$$

Let

$$f_{\mathbb{C}} \in F_{\mathbb{C}}, \quad f_{\mathbb{C}} = f_1 + if_2, \quad f_1, f_2 \in F$$

be in $G^{\perp}$, i.e.

$$f_{\mathbb{C}} \in G^{\perp}$$

then

$$f_1 \in (G^{\perp})_{\mathbb{R}}.$$

Now as $G = \overline{G}$

$$\omega[f, g] = \omega[f, \overline{g}] = 0$$

or

$$\begin{aligned} \omega[f, g_1 + ig_2] &= 0 \\ \omega[f, g_1 - ig_2] &= 0 \end{aligned} \tag{K.-118}$$

Adding gives

$$\omega[f, g_1] = 0$$

which implies

$$\omega[f_1, g_1] = 0$$

therefore

$$f_1 \in (G_{\mathbb{R}})^{\perp} \tag{K.-118}$$

Which implies

$$(G_{\mathbb{R}})^{\perp} \subset (G^{\perp})_{\mathbb{R}}. \tag{K.-118}$$

The other way around.

$$f_1 \in (G^{\perp})_{\mathbb{R}}$$

(iii):

$$(G_{\mathbb{R}})_{\mathbb{C}} = (G \cap V)_{\mathbb{C}} = \{v_1 + iv_2 : v_1, v_2 \in G \cap V\}$$

Let

$$g_1, g_2 \in G_{\mathbb{R}}$$

(iv): It is well defined if $\omega[u_1, u_1'] = \omega[u_2, u_2']$ where u_1 and u_2 are anny two representatives of the same class and similarly for u_1' and u_2'. It follows that we can put $u_1 = u_2 + v$ and $u_1' = u_2' + v'$.

$$
\begin{aligned}
\omega[u_1, u_1'] &= \omega[u_2 + v, u_2' + v'] \\
&= \omega[u_2, u_2'] + \omega[u_2, v'] + \omega[v, u_2'] + \omega[v, v'] \\
&= \omega[u_2, u_2'] + \omega[v, v'] \\
&= \omega[u_2, u_2']
\end{aligned}
$$

where $\omega[u_2, v'] = \omega[v, u_2'] = 0$ as $u_2, u_2' \in F$ and $v, v' \in F \cap F^\perp \subset F^\perp$. We have $\omega[v, v'] = 0$ as $v, v' \in F \cap F^\perp$. We now prove that $(\tilde{F}, \tilde{\omega})$ is a symplectic vector space. Must show that $\tilde{\omega}$ is non-degenerate, i.e.

$$
\tilde{\omega}[(u), (v)] = 0 \;\text{ for all }\; (v) \in \tilde{F} \;\text{ then }\; \in [u] = 0,
$$

and closed.

(v): Note that in this case $F \cap F^\perp = F^\perp$. If $P \subset V$ is a Lagrangian subspace, then $\tilde{P} = \pi(P \cap F)$ is an isotropic subspace of $\tilde{F}$ i.e. $\tilde{P} \subset \tilde{P}^\perp$. We have to show $\tilde{P}^\perp \subset \tilde{P}$. Let $\tilde{u} \in \tilde{P}^\perp$. Then $\tilde{u} = \pi(u)$ for some $u \in F$ such that

$$
\omega[u, v] = 0 \text{ for all } \; v \in F \cap P.
$$

That is,

$$
u \in (F \cap P)^\perp = F^\perp + P^\perp = F^\perp + P.
$$

As $\pi(F^\perp + P) = \pi(P) = \tilde{P}$, we have $\pi(u) \in \tilde{P}$.

$\square$

The following theorem is key to the classification of polarisations.

Theorem K.16.9 *Let $P \subset V_{\mathbb{C}}$ be a Lagrangian subspace and*

$$
E := (P + \overline{P})_{\mathbb{R}}, \qquad D := (P \cap \overline{P})_{\mathbb{R}} \tag{K.-122}
$$

(i) D is an isotropic ($D \subset D^\perp$) subspace of V.

(ii) E is a co-isotropic subspace of V and $E^\perp = D$.

(iii) $E_{\mathbb{C}} = P + \tilde{P}$, $D_{\mathbb{C}} = P \cap \tilde{P}$.

(iv) Let $\tilde{F} = E/D$. Then $\tilde{F}_{\mathbb{C}} = E_{\mathbb{C}}/D_{\mathbb{C}}$.

(v) If $\pi : E \to \tilde{F}$ is the canonical projection then $\tilde{P} := \pi(E_{\mathbb{C}} \cap P)$ is Lagrangian in $(\tilde{F}_{\mathbb{C}}, \omega)$.

(vi) $\tilde{P} \cap \overline{\tilde{P}} = \{0\}$.

Proof: We make use of the previous lemma, which is applicable since obviously $\overline{E} = E$, $\overline{D} = D$.

(i): $D^{\perp}$ is defined by

$$\{u \in V : \omega[u, v] = 0 \text{ for all } v \in D\}.$$

Consider

$$\omega[P \cap V, v] + \omega[\overline{P} \cap V, v] \text{ for } v \in D$$

As $D = (P \cap \overline{P}) \cap V$, $D \subset P$ and $D \subset \overline{P}$. It is easy to see that $D \subset D^{\perp}$ as $P = P^{\perp}$ and $\overline{P} = \overline{P}^{\perp}$.

(ii) We first prove $E^{\perp} = D$:

$$\begin{aligned} E &= (P + \overline{P})_{\mathbb{R}} \\ &= (P + \overline{P}) \cap V \\ &= P \cap V + \overline{P} \cap V \\ &= P^{\perp} \cap V + \overline{P}^{\perp} \cap V \\ &= P_{\mathbb{R}}^{\perp} + \overline{P}_{\mathbb{R}}^{\perp} \\ &= (P_{\mathbb{R}} \cap \overline{P}_{\mathbb{R}})^{\perp} \\ &= ((P \cap \overline{P})_{\mathbb{R}})^{\perp} = D^{\perp}. \end{aligned}$$

As $D \subset D^{\perp}$, the just proved equality implies $E^{\perp} \subset E$, or in other words, E is a co-isotropic subspace of V.

(iii) By (iii) of the previous lemma $E_{\mathbb{C}} = ((P + \tilde{P})_{\mathbb{R}})_{\mathbb{C}} = P + \tilde{P}$ and $D_{\mathbb{C}} = ((P \cap \tilde{P})_{\mathbb{R}})_{\mathbb{C}} = P \cap \tilde{P}$.

(iv) $\tilde{F}$ is defined by $F/F^{\perp} = \{[u] : u \in F\}$ where the equivalence classes are defined by $[u] = \{u + v : v \in F^{\perp}\}$. Consider an arbitrary element of $\tilde{F}_{\mathbb{C}}$: $[u_1] + i[u_2]$. Now $E_{\mathbb{C}}$ is $\{e_1 + ie_2 : e_1, e_2 \in E\}$ and $F_{\mathbb{C}}$ is $\{f_1 + if_2 : f_1, f_2 \in F\}$. It is fairly obvious that $\tilde{F}_{\mathbb{C}} = E_{\mathbb{C}}/F_{\mathbb{C}}$.

(v) E is a coisotropic subspace of V and

$$\pi : E \to \tilde{F} = E/D = E/E^{\perp}.$$

By (v) of the previous lemma, $\tilde{P}' = \pi(E \cap P)$ is Lagrangian subspace of $\tilde{F}$. Set $\tilde{P} = \pi(E_{\mathbb{C}} \cap P)$. Then $\tilde{P}$ is obviously a Lagrangian subspace of $\tilde{F}_{\mathbb{C}}$.

(vi) $\overline{\tilde{P}} = \overline{\pi(E_{\mathbb{C}} \cap P)} = \pi(\overline{E_{\mathbb{C}}} \cap \overline{P}) = \pi(E_{\mathbb{C}} \cap \overline{P})$.

$$\begin{aligned} \pi(E_{\mathbb{C}} \cap P) &= \pi((P + \overline{P}) \cap P) \\ &= \pi(P + P \cap \overline{P}) \\ &= \pi(P). \end{aligned}$$

So $\pi(E_{\mathbb{C}} \cap P) \cap \pi(E_{\mathbb{C}} \cap \overline{P}) = \pi(P) \cap \pi(\overline{P})$. As $\pi(D_{\mathbb{C}}) = \{0\}$, $\pi(P) \cap \pi(\overline{P}) = \{0\}$.

□

As $\tilde{P} \cap \overline{\tilde{P}} = \{0\}$, so $\tilde{P}$ determines a complex structure

Kahler polarisations are characterised by the condition $P \cap \overline{P} = \{0\}$. A Kahler manifold is a complex manifold with a compatible symplectic structure.

Definition The type of the Lagrangian subspace $P \subset V_{\mathbb{C}}$ is the pair of integers (r, s) just defined. Special types are:

(i) Kahler: $m = r + s$, that is $\dim(D) = 0$ so $P \cap \overline{P} = \{0\}$.

(ii) Positive: $m = r$, that is $s = 0$ and Kahler, hence $\omega = \tilde{\omega}$ and the associated Kahler metric $g[.,.] = \omega[.,.]$ is positive definite.

(iii) Non-negative: $s = 0$, that is, the Kahler metric $\tilde{g}$ on $\tilde{F}$ is positive definite, however, $\dim(D) > 0$ is possible in which case P contains a real subspace.

(iv) Real: $r = s = 0$, that is, $\dim(E) = \dim(D) = m$, hence $E = D$ and $P = \overline{P}$, so $P = L_{\mathbb{C}}$ where $L \subset V$ is Lagrangian.

□

After this preparation we can now generalise from sympletic vector spaces to sympletic manifolds $(\mathcal{M}, \omega)$. This will involve applying the previuos resluts to the indivdiual tangent spaces $T_p(\mathcal{M})$ of each $p \in \mathcal{M}$. In other words applying the results fibrewise.

Complex polarisations

We start by introducing a number of definitions.

Definition A complex distribution P on a real manifold $\mathcal{M}$ is an assignment of subspaces $p \mapsto P_p \subset (T_p(\mathcal{M}))_{\mathbb{C}}$ whose complex dimension k is constant and which is spanned by k complex vector fields in a neighbourhood of each point of $\mathcal{M}$.

$\square$

Definition A **complex polarisation** of a symplectic manifold $(\mathcal{M}, \omega)$ is a compex distribution P such that

(1) P_p is a Lagrangian subspace of $(T_p(\mathcal{M}))_{\mathbb{C}}$

and such that

(2) the type of P_p is constant (equivalently, the real dimension of $D_p = (P_p \cap \overline{P}_p) \cap T_p(\mathcal{M})$ is constant).

$\square$

Definition A complex distribution is called **integrable** provided that in a neighbourhood U of each point p of $\mathcal{M}$ there are smooth complex-valued functions $f_{k+1}, \ldots, f_{2m}$ with linearly independent differentials df_j, $j = k+1, \ldots, 2m$ such that $\overline{u}[f_j]$ for any vector field u tangential to P in U.

$\square$

Definition The complex distribution in the previous definition is said to be **strongly integrable** if in addition the real distribution $E_p = D_p^{\perp} = (P_p + \overline{P}_p) \cap T_p(\mathcal{M})$ is integrable.

$\square$

Definition A symplectic potential θ, $d\theta = \omega$ is said to be P-adapted provided that $i_{\overline{u}}\theta = 0$ for all u tangential to P. A polarisation is said to be **admissible** provided that local P-adapted symplectic potentials exist everywhere.

$\square$

Definition A polarisation is called **strongly admissible** if E is integrable and the spaces $\mathcal{M}/D$ and $\mathcal{M}/E$ are smooth Hausdorff manifolds.

□

Naively, one would like to construct the quantum Hilbert space from sections of the prequantum line bundle covariantly constant (parellel) along P. This leads to proposing the following definition.

Definition A complex-valued function $\psi \in C^{\infty}(\mathcal{M})$ is said to be P-polarised provided that $\overline{u}[\psi] = 0$ for all u tangental to P. We use the notation $C^{\infty}_{P}(\mathcal{M})$ for such functions.

□

Definition A polarisation P of a symplectic manifold $\mathcal{M}$ which satisfies $\overline{P} = P$ is the complexification of an integrable Lagrangian sub-bundle of $T(\mathcal{M})$ and is called a **real polarisation** of $\mathcal{M}$.

□

The importance of the notation of a strongly integrable polarisation is demostrated by the following theorem:

Theorem K.16.10 *Let P be a strongly integrable polarisation of a symplectic manifold $(\mathcal{M}, \omega)$ of dimension $dim(\mathcal{M}) = 2(m' + \tilde{m}) = 2m$. Then in the neighbourhood of each point we find a system of coordinates $\{q^a, p_a, z^\alpha\}_{a=1}^{m'}{}_{\alpha=1}^{\tilde{m}}$ with q, p real z complex such that $P = span_{\mathbb{C}}\{\partial/\partial p_a, \partial/\partial z^\alpha\}$ and their is a real valued function $K(q, z, \overline{z}$ such that $\omega = d\theta$ where*

$$\theta = p_a dq^a - \frac{i}{2}(\partial_\alpha K) dz^\alpha + \frac{i}{2}(\partial_{\overline{\alpha}} K) dz^{\overline{\alpha}}. \tag{K.-132}$$

Proof:

$$\tilde{\omega}_q = i\partial_\alpha \partial_{\overline{\beta}} K_q(z, \overline{z}) dz^\alpha \wedge dz^{\overline{\beta}} \tag{K.-132}$$

for some Kahler scalar K_q.

The general form of ω is now given by

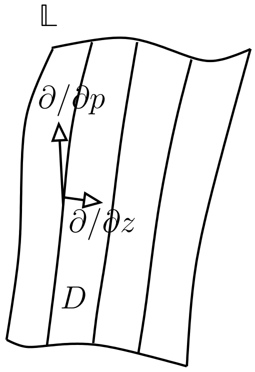

Figure K.6: Eleaf.

$$\begin{aligned}\omega &= dp_a \wedge dq^a + \sigma_{ab} dq^a \wedge dq^b + \sigma_{a\alpha} dq^a \wedge dz^\alpha + \overline{\sigma_{a\alpha}} dq^a \wedge d\overline{z}^\alpha \\ &\quad + i(\partial_\alpha \partial_\beta K) dz^\alpha \wedge d\overline{z}^\beta \end{aligned} \tag{K.-132}$$

We now compute

$$\begin{aligned} d\omega &= d\left(\sigma_{ab} dq^a \wedge dq^b + \sigma_{a\alpha} dq^a \wedge dz^\alpha + \overline{\sigma_{a\alpha}} dq^a \wedge d\overline{z}^\alpha + \ i(\partial_\alpha \partial_\beta K) dz^\alpha \wedge d\overline{z}^\beta\right) \\ &= \end{aligned} \tag{K.-132}$$

$$d(\sigma_{ab} dq^a \wedge dq^b) = \frac{1}{3!}(\partial_{q_c} \sigma_{ab} dq^c \wedge dq^a \wedge dq^b + \partial_{p_c} \sigma_{ab} dp^c \wedge dq^a \wedge dq^b + (\partial + \overline{\partial})\sigma_{ab} dq^a \wedge dq^b)$$

$$(\partial + \overline{\partial}) i(\partial_\alpha \partial_\beta K) dz^\alpha \wedge d\overline{z}^\beta = 0$$

$\partial_[$

□

K.16.6 Quantisation

Let P be a strongly integrable polarisation of a sympletic manifold $(\mathcal{M}, \omega)$ and let B be a prequantum bundle over $\mathcal{M}$.

Definition A smooth section $s : \mathcal{M} \to B$ is said to be polarised if $\nabla_{\overline{X}} s = 0$ for every $X \in V_P(\mathcal{M})$. The space of polarised sections is denoted by S_P.

$\square$

We would like to qunatise $\mathcal{M}$ by replacing the prequantisation Hilbert space $\mathcal{H}'$ by the subspace of square integrable polarised sections by restricting the prequantum Hilbert space $\mathcal{H}'$ to (the completion of) S_P, i.e. $\mathcal{H}_P := \overline{\mathcal{H}' \cap S_P}$. This is problematic for the following reasons:

(A) Normalisation

For most polarisations it is not true that S_P contains any square integrable element.

(B) Operators

Obviously the operator $\hat{f}$ corresponding to $f \in C^\infty(\mathcal{M})$ is only admissible if it maps local polarised sections to polarised sections. Hence we must have $\nabla_{\overline{X}}(\hat{f}\psi)$ for every polarised section ψ. It is easy to see from

$$\nabla_{\overline{u}}(\hat{f}\psi) = \hat{f}(\nabla_{\overline{u}}\psi) - i\hbar\nabla_{[\overline{u},\chi_f]}\psi$$

that this is the case if and only if $\nabla_{[\overline{u},\chi_f]}\psi = 0$. Hence not every function can be realised as a prequantum operator on the Hilbert space $\mathcal{H}_P$. For example in the real case f must be of the form $v^a(q)p_a + u(q)$, in the Kahler case, χ_f must be a Killing vector.

K.17 Non-Commutative Field Theories and Thier Relation to Quantum Geometry

K.18 Summary

K.18.1 Algebraic quantum mechanics

representation theory

GNS

K.18.2 Algebraic quantum field theory

Can we reconstruct in detail the hamiltonian Hilbert space, as well as kinematic and dynamical operator of loop theory, starting from the covariant spin foam definition of the theory? [20]: This amounts to an extension to the diffeomorphism invariant context of the Wightmann and Osterwalder-Schrader reconstruction theorems.

von Neumann algebras

K.18.3 Geometric quantization

Geometric quantisation provides a beautiful, geometric, general framework for the quantisation of a given sympletic manifold which is not necessarily of cotangent bundle type.

K.19 Biblioliographical notes

In this chapter I have relied on the following references:

Dana P. Williams notes on the SpectralTheorem for bounded normal operators

Introduction to topology and modern analysis G.F Simons.

K.20 Worked Exercises and Details

Cauchy-Schwartz inequality

To obtain the Cauchy-Schwartz inequality we first note that from (K.2.2) we have

$$0 \leq E(A^*A) + |\lambda|^2 E(B^*B) + \lambda E(A^*B) + \overline{\lambda} E(B^*A) \tag{K.-132}$$

and then set $\lambda = tE(B^*A) = t\overline{E(A^*B)}$ for real t. So we see that

$$0 \leq E(A^*A) + t^2 |E(A^*B)|^2 E(B^*B) + 2t|E(A^*B)|^2. \tag{K.-132}$$

If $0 < 2bt + at^2 + c$ for t real then the equation $at^2 + 2bt + c = 0$ can only be satisfied if the roots $t_\pm$ satisfying $t_\pm = a - b \pm \sqrt{b^2 - ca}$ are complex, that is $0 \leq b^2 - ac$. By setting

$$\begin{aligned} a &= E(A^*A) \\ b &= |E(A^*B)|^2 \\ c &= |E(A^*B)|^2 E(B^*B) \end{aligned} \tag{K.-133}$$

we obtain the Cauchy-Schwarz inequality.

Adjoining a unit to a C^*- algebra.

If a C^*-algebra does not contain a unit one can be added as follows. Consider the vector space $\mathcal{A}' = \mathcal{A} \oplus \mathbb{C}$ with addition $(\lambda, A) + (\mu, B) = (\lambda + \mu, A + B)$, product defined by

$$(\lambda, A)(\mu, B) = (\lambda\mu, AB + \lambda B + \mu A),$$

with the involution $(\lambda, A)^* = (\lambda, A^*)$, and norm

$$\|(\lambda, A)\| = \sup_{\|B\|=1} \|AB + \lambda B\|.$$

Equipped this way $\mathcal{A}'$ is a C^*-algebra with a unit $(1, 0)$. The algebra $\mathcal{A}$ defines an ideal of $\mathcal{A}'$:

$$(0, A)(0, B) = (0, AB).$$

The triangle and product inequalities are easily verified. The triangle inequality:

$$\begin{aligned}
\|(\lambda, A) + (\mu, B)\| &= \|(\lambda + \mu, A + B)\| \\
&= \sup_{\|C\|=1} \|(A + B)C + (\lambda + \mu)C\| \\
&= \sup_{\|C\|=1} \|(AC + \lambda C) + (BC + \mu C)\| \\
&\leq \sup_{\|C\|=1} \|AC + \lambda C\| + \sup_{\|C\|=1} \|BC + \mu C\| \\
&= \|(\lambda, A)\| + \|(\mu, B)\| \qquad \text{(K.-136)}
\end{aligned}$$

To prove the product inequality we note that for any C^*-algebra we need $\|A\| = \sup_{\|B\|=1} \|AB\|$.

$$\begin{aligned}
\|(\lambda, A)(\mu, B)\| &= \|(\lambda\mu, AB + \lambda B + \mu A)\| \\
&= \sup_{\|C\|=1} \|(AB + \lambda B + \mu A)C + \lambda\mu C\| \\
&= \sup_{\|C\|=1} \|(A + \lambda)(B + \mu)C\| \\
&\leq \sup_{\|C\|=1} \|A + \lambda\| \sup_{\|C\|=1} \|BC + \mu C\| \\
&= \sup_{\|C\|=1} \|AC + \lambda C\| \sup_{\|C\|=1} \|BC + \mu C\| \\
&= \|(\lambda, A)\| \, \|(\mu, B)\| \qquad \text{(K.-140)}
\end{aligned}$$

We now check that $\|(\lambda, A)\| = 0$ implies $A = 0$ and $\lambda = 0$.

$$\|B - AB\| \leq \|B\| \; \|(-A, 1)\|$$

Thus if $\|(-A,1)\| = 0$ then $B = AB$ for all $B \in \mathcal{A}$. Considering the involution gives $B = BA^*$ for all $B \in \mathcal{A}$. In particular $A^* = AA^* = A$ and thus

$$B = AB = BA,$$

This means that A is an identity, which contradics the assumption.

Poincare invariance and the unique representation

The requirement of Poincare invariance and stability of theory unquiley picks out a preferred ground state.

C^*-algebras.

For self-adjoint operator x we know $r(x) = |x\|$ (Eq (K.6.2)). By definition of $r(x)$, $\|x\| = r(x) = \sup\{|\alpha + \beta| : \alpha + i\beta \in \sigma(x)\}$. So that $\|x\|^2 \geq \alpha^2 + \beta^2$ for $\alpha + i\beta \in \sigma(x)$. From this we have $\|x + i\lambda\|^2 \geq \alpha^2 + (\beta + \lambda)^2$. We can write

$$\alpha^2 + (\beta + \lambda)^2 \leq \|x + i\lambda\|^2 = \|(x + i\lambda)(x - i\lambda)\| = \|x^2 + \lambda^2\| \leq \|x^2\| + \lambda^2$$

$\alpha^2 + \beta^2 + 2\beta\lambda \leq \|x\|$

Existence of representations of C^*-algebras.

If we have not only a unital $\star$-algebra but in fact a C$\star$-algebra one can show that by the Hahn-Banach theorem that representations always exist, that every non-degenerate representation is a direct sum of cyclic representations and that every state is continuous so that the GNS representations are always bounded operators

a subspace $\mathcal{B}$ defined by

$$\mathcal{B} = \{\alpha \mathbf{1} + \beta A^* A \; ; \; \alpha, \beta \in \mathbb{C}\} \tag{K.-140}$$

A linear functional defined on $\mathcal{B}$

$$f(\alpha\mathbf{1} + \beta A^*A) = \alpha + \beta\|A\|^2 \qquad \text{(K.-140)}$$

That $\lambda \cdot \mathbf{1} - A^*A$ non-invertable implies $(\alpha + \beta\lambda) \cdot \mathbf{1} - (\alpha \cdot \mathbf{1} + \beta A^*A)$ non-invertable. Now if T is normal ($TT^* = T^*T$) then by (K.6.2) the spectral radius of T is equal to $\|T\|$. One has by the spectral radius formula, (K.6.2), applied to the normal element $\alpha\mathbb{I} + \beta A^*A$,

$$\begin{aligned} |\alpha + \beta\|A\|^2| &\leq \sup\{|\alpha + \beta\lambda| : \lambda \in \sigma(A^*A)\} \\ &= \sup\{|\mu| : \mu \in \sigma(\alpha\mathbf{1} + \beta A^*A)\} \\ &= \|\alpha\mathbf{1} + \beta A^*A\| \end{aligned} \qquad \text{(K.-141)}$$

Thus $\|f\| \leq 1$. but $f(\mathbf{1}) = 1$ and hence $\|f\| = 1 = f(\mathbf{1})$

□

Proof of proposition K.6.5.

a self-adjoint operator element A of $\mathcal{A}$, with $\|A\| = 1$, positive if and only if $\|(t\mathbf{1} - A)\| \leq t$ with $\|A\| \leq t$. In particular, we have for any $A \in \mathcal{A}$, we have that $\|A^*A\|\mathbf{1} - A^*A$ is positive. We see this is so as the condition

$$\|(t\mathbf{1} - \|A^*A\|\mathbf{1} + A^*A)\| \leq t$$

and

$$\|\|A^*A\|\mathbf{1} - A^*A\| \leq t$$

are met for the choice $t = \|A^*A\|$.

(i) $\Rightarrow$ (ii). If (i) holds then ω applied to the element $\|A^*A\|\mathbf{1} - A^*A$ gives

$$\omega(A^*A) \leq \|A^*A\|\, \omega(\mathbf{1})$$

By the Cauchy-Schwarz (K.2.2) we have

$$\begin{aligned} |\omega(A)| &= |\omega(A^*\mathbf{1})| \\ &\leq |\omega(A^*A)|^{1/2}\, \omega(\mathbf{1})^{1/2} \\ &\leq \|A^*A\|^{1/2}\, \omega(\mathbf{1}) \\ &= \|A\|\, \omega(\mathbf{1}) \end{aligned} \qquad \text{(K.-143)}$$

So $|\omega(A)|$ is bounded and hence, because it is linear, is continuous. Also we have

$$\|\omega\| := \sup_{\|A\|=1} |\omega(A)| \leq \omega(\mathbf{1}),$$

from which this we easily see that $\|\omega\| = \omega(\mathbf{1})$.

(ii) $\Rightarrow$ (i). Supose ii) is satisfied. Assume $\omega(\mathbf{1}) = 1$. Let A be a self-adjoint element of $\mathcal{A}$. put $\omega(A) = \alpha + i\beta$ where α, β are real. For every $\lambda \in \mathbb{R}$ we have

$$\|A + i\lambda\mathbf{1}\|^2 = \|A^2 + \lambda^2\mathbb{I}\| = \|A\|^2 + \lambda^2.$$

Thus we have

$$\begin{aligned} \alpha^2 + \beta^2 + 2\lambda\beta + \lambda^2 &\leq |\alpha^2 + i(\beta + \lambda)|^2 \\ &= |\omega(A + i\lambda\mathbb{I})| \\ &\leq \|A\|^2 + \lambda^2 \end{aligned} \tag{K.-145}$$

If $|\lambda|$ is made large enough, we would have $\geq \|A\|$, so that that $\beta = 0$ and $\omega(A)$ is real. Consider now A positive. We have

$$\begin{aligned} |\ \|A\| - \omega(A)| &= |\omega(\|A\|\mathbf{1} - A)| \\ &\leq \|\ \|A\|\mathbf{1} - A\| \leq 1. \end{aligned} \tag{K.-146}$$

Thus $\omega(A)$ is positive.

$\square$

Representations of C^*-algebras.

Let A, B, C be elements of a C^*-algebra $\mathcal{A}$. The following is true:

(a) if $A \geq B \geq 0$ then $\|A\| \geq \|B\|$;

(b) if $A \geq 0$ then $A\|A\| \geq A^2$;

We first need the following lemma.

Lemma K.20.1 *If* $-M\mathbb{I} \leq A \leq M\mathbb{I}$ *then*

$$\|A\| \leq M. \tag{K.-146}$$

Proof. First we prove that if $|a(x)| \leq Mb(x)$ then

$$|a(x,y)| \leq Mb(x)b(y). \tag{K.-146}$$

If $a(x,y)$ is complex then $a(x,y) = e^{i\theta}|a(x,y)|$, which can be expressed as $|a(x,y)| = a(e^{-i\theta}x, y)$. Hence, it suffices to prove (K.20) for $a(x,y)$ real.

$$\begin{aligned} 4|a(x,y)| &\leq |a(x+y)| + |a(x-y)| \\ &\leq M(b(x+y) + b(x-y)) \\ &= 2M(b(x) + b(y)) \end{aligned} \tag{K.-147}$$

as $a(\alpha x, y/\alpha) = a(x,y)$ for real α we have

$$2|a(x,y)| \leq M\Big(\alpha^2 b(x) + \frac{b(y)}{\alpha^2}\Big) \tag{K.-147}$$

Assuming $b(x) \neq 0$ and $b(y) \neq 0$ then putting $\alpha^2 = b(y)/b(x)$ we get

$$|a(x,y)| \leq Mb(x)^{1/2}b(y)^{1/2} \tag{K.-147}$$

$$| < Ax, y > | \leq M\|x\|^2 \tag{K.-147}$$

Hence, by the above,

$$| < Ax, y > | \leq M\|x\| \; \|y\|$$

takeing $y = Ax$ gives $\|Ax\|^2 \leq M\|x\| \; \|Ax\|$ implying (K.20.1).

□

Proof of (a). (K.20.1) gives $A \leq \|A\|\mathbb{I}$ and hence $0 \geq B \geq \|A\|\mathbb{I}$. A second application of (K.20.1) gives $\|B\| \leq \|A\|$.

Proof of (b). As $\sigma(A) \subseteq [0, \|A\|]$ for a positive element, $\sigma(A - \|A\|\mathbb{I}/2) \subseteq [-\|A\|/2, \|A\|/2]$ and hence $\sigma((A - \|A\|\mathbb{I}/2)^2) \subseteq [0, \|A\|/4]$. Thus

$$0 \leq \left(A - \frac{\|A\|\mathbb{1}}{2}\right)^2 \leq \frac{\|A\|^2\mathbb{1}}{4},$$

multiplying out the bracket we see this is equivalent to $0 \leq A^2 \leq \|A\|A$.

Ireduciability of representations of C^*-algebras.

$$A \geq 0 \quad \Rightarrow \quad \pi(A) \geq 0 \tag{K.-147}$$

π is continuous

$$\|\pi(A)\| \leq \|A\| \tag{K.-147}$$

for all $A \in \mathcal{A}$.

Proof of proposition K.6.4

(i) $\Rightarrow$ (ii).

For a general A we have

$$\|\pi(A)\|^2 = \|\pi(A^*A)\| \leq \|(A^*A)\| = \|A\|^2.$$

(i) $\Rightarrow$ (ii): If ker= $\{0\}$ then the map π is invertible and so we can define a morphism π^{-1} from the range of π into $\mathcal{A}$ by $\pi^{-1}(\pi(A)) = A$,

$$\begin{gathered}
\pi^{-1}(\alpha\pi(A) + \beta\pi(B)) = \pi^{-1}(\pi(\alpha A + \beta B)) = \alpha A + \beta B, \\
\pi^{-1}(\pi(A)\pi(B)) = \pi^{-1}(\pi(AB)) = AB \\
\pi^{-1}(\pi(A)^*) = \pi^{-1}(\pi(A^*)) = A^*
\end{gathered}$$

For a morphism π, $\|\pi(A)\| \leq \|A\|$. This result also applies to π^{-1} now,

$$\|A\| = \|\pi^{-1}(\pi(A))\| \leq \|\pi(A)\|$$

hence $\|\pi(A)\| = \|A\|$.

(ii) $\Rightarrow$ (iii):

(iii) $\Rightarrow$ (i): We prove this by presuming (i) does not hold and then show this contradicts (iii). If (i) is false then there is a $B \in ker\pi$ with $B \neq 0$ and $\pi(B^*B) = 0$. But $\|B^*B\| \geq 0$ and as $\|B^*B\| = \|B\|^2$ one has $B^*B > 0$.

Decomposition into Sub-Representations

invariant subspace if

$\pi(\mathcal{A})\mathcal{H}_1 \subseteq \mathcal{H}_1$

$$P_{\mathcal{H}_1}\pi(\mathcal{A})P_{\mathcal{H}_1} = \pi(\mathcal{A})P_{\mathcal{H}_1}$$

$$\begin{aligned}
(P_{\mathcal{H}_1}\pi(\mathcal{A})P_{\mathcal{H}_1})^* &= P_{\mathcal{H}_1}^*\pi(\mathcal{A})^*P_{\mathcal{H}_1}^* = P_{\mathcal{H}_1}\pi(\mathcal{A})P_{\mathcal{H}_1} \\
&= (\pi(\mathcal{A})P_{\mathcal{H}_1})^* \\
&= P_{\mathcal{H}_1}\pi(\mathcal{A}) \qquad \text{(K.-148)}
\end{aligned}$$

$$\pi(\mathcal{A})P_{\mathcal{H}_1} = P_{\mathcal{H}_1}\pi(\mathcal{A})$$

$$\pi_1(A) = P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1} \qquad \text{(K.-148)}$$

$$\begin{aligned}
\pi_1(A)\pi_1(B) &= (P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1})(P_{\mathcal{H}_1}\pi(B)P_{\mathcal{H}_1}) \\
&= (P_{\mathcal{H}_1}\pi(A))(\pi(B)P_{\mathcal{H}_1}) \\
&= P_{\mathcal{H}_1}\pi(AB)P_{\mathcal{H}_1} \\
&= \pi_1(AB) \qquad \text{(K.-150)}
\end{aligned}$$

Every element of a C^*-algebra can be decomposed as a linear combination of four unitary elements.

Every element A of a C^*-algebra $\mathcal{A}$ can be written as a linear combination of unitaries in $\mathcal{A}$. As the conjugate of each element is also in the algebra, we can construct self-adjoint elements $A_+ = (A + A^*)/2$ and $A_- = (A - A^*)/2i$. But any self-adjoint element B can be written as $(U_+ + U_-)/2$ where

$$U_\pm = B \pm i\sqrt{\mathbb{I} - B^2}$$

Thus, for any $A \in \mathcal{A}$ we can write

$$
\begin{aligned}
A &= \frac{(A+A^*)}{2} + i\frac{(A-A^*)}{2i} = A^{(1)} + iA^{(2)} \\
&= \frac{(U_+^{(1)} + U_-^{(1)})}{2} + i\frac{(U_+^{(2)} + U_-^{(2)})}{2}.
\end{aligned}
\tag{K.-150}
$$

von Neumann algebras.

Check that $\tilde{A} = (Ax_i) \in \mathcal{H}(\mathcal{K})$ if $A \in \mathcal{H}(\mathcal{B})$

$$
\begin{aligned}
\|\tilde{A}\| &:= \sup_{\sum_i^\infty \|x_i\|=1} \sum_j^\infty \|A_j x_j\| \\
&\leq \sup_{\sum_i^\infty \|x_i\|=1} \|A\| \sum_j^\infty \|x_j\| \qquad \text{as } A = A_j \\
&= \|A\| < \infty.
\end{aligned}
\tag{K.-151}
$$

Disjoint representations of C^*-algebras.

$(\pi_1 \oplus \pi_2)(\mathcal{A})'$ is the commutant of $(\pi_1 \oplus \pi_2)(\mathcal{A})$ taken in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$

We may write B as

$$
B = \begin{pmatrix} X & S \\ T & Y \end{pmatrix} \in (\pi_1 \oplus \pi_2)(\mathcal{A})' \tag{K.-151}
$$

for suitable $X \in \mathcal{B}(\mathcal{H}_1)$, $Y \in \mathcal{B}(\mathcal{H}_2)$ and bounded linear operators $S : \mathcal{H}_2 \to \mathcal{H}_1$ and $T : \mathcal{H}_1 \to \mathcal{H}_2$.

As $\pi_1(\mathcal{A}) \oplus \pi_2(\mathcal{A}) \subseteq (\pi_1 \oplus \pi_2)(\mathcal{A})$ we must have

$$
\begin{pmatrix} X & S \\ T & Y \end{pmatrix} \begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix} = \begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix} \begin{pmatrix} X & S \\ T & Y \end{pmatrix} \tag{K.-151}
$$

so that

$$
\begin{pmatrix} X\pi_1(A) & S\pi_2(A) \\ T\pi_1(A) & Y\pi_2(A) \end{pmatrix} = \begin{pmatrix} \pi_1(A)X & \pi_2(A)S \\ \pi_1(A)T & \pi_2(A)Y \end{pmatrix} \tag{K.-151}
$$

We must have $X \in \pi_1(\mathcal{A})'$, $Y \in \pi_2(\mathcal{A})'$, $T\pi_1(A) = \pi_2(A)T$, and $S\pi_2(A) = \pi_1(A)S$ for all $A \in \mathcal{A}$.

Next.

Let $\tilde{S}\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H})$ and write π for $\pi_1 \oplus \pi_2$, we see

$$\begin{aligned}\tilde{S}\pi(A) &= \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix} = \begin{pmatrix} 0 & S\pi_2(A) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \pi_1(A)S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix}\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \\ &= \pi(A)\tilde{S}\end{aligned}$$

for all $A \in \mathcal{A}$. In particular, for any unitary element $U \in \mathcal{A}$, $\pi(U)$ is unitary and commutes with $\tilde{S}$. It follows that $\pi(U)$ also commutes with the polar isometry W, where $W|\tilde{S}|$ is the polar decomposition of $\tilde{S}$.

Operators and complex analysis.

(1)

From the "Taylor expansion" for $(z-A)^{-1}$ we have

$$\oint_C z^n(z-A)^{-1}dz = \sum_{k=1}^{\infty} A^{k-1} \oint_C z^n dz = A^n \oint_C \frac{dz}{z} = 2\pi i A^n. \tag{K.-154}$$

Thus

$$A^n = \frac{1}{2\pi i}\oint_C z^n(z-A)^{-1}dz. \tag{K.-154}$$

The partial fraction formula for constants $\mu, \lambda, \in \mathbb{C}$

$$\frac{1}{\mu - y} - \frac{1}{\nu - y} = (\mu - \nu)\frac{1}{\mu - y}\frac{1}{\nu - y}, \tag{K.-154}$$

where the variable $y \in \mathbb{C}$, has an operator (or matrix if you like) version. Let $x \in X$ and set

$$u = (\lambda - A)^{-1}x.$$

Thus $x = (\lambda - A)u$ and $(\mu - A)u = x + (\mu - \lambda)u$. Hence

$$u = (\mu - A)^{-1}x + (\mu - \lambda)(\mu - A)^{-1}u.$$

Substituting for u, we get

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}. \tag{K.-154}$$

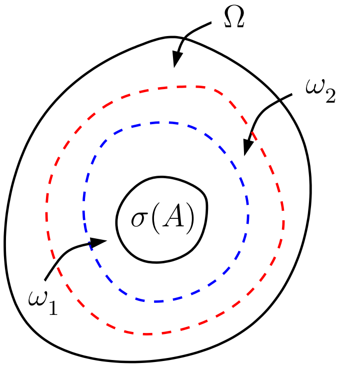

Figure K.7: OperProdsComp.

$$\begin{aligned} f(A)g(A) &= -\frac{1}{4\pi}\oint_{\partial\omega_1} f(z)(z-A)^{-1}\oint_{\partial\omega_2} g(\zeta)(\zeta - A)^{-1}dzd\zeta \\ &= -\frac{1}{4\pi}\oint_{\partial\omega_1} f(z)\oint_{\partial\omega_2} g(\zeta)\frac{(z-A)^{-1} - (\zeta - A)^{-1}}{\zeta - z}dzd\zeta \end{aligned} \tag{K.-154}$$

From fig (K.20) we easily see

$$\oint_{\partial\omega_2} \frac{g(\zeta)}{\zeta - z}d\zeta = 2\pi i g(z) \tag{K.-154}$$

for any $z \in \partial\omega_1$.

$$\oint_{\partial\omega_1} \frac{f(z)}{z - \zeta}dz = 0 \tag{K.-154}$$

for any $\zeta \in \partial\omega_2$. Hence from (K.-154)

$$f(A)g(A) = \frac{1}{2\pi i}\oint_{\partial\omega_1} f(z)g(z)(z-A)^{-1}dz = h(A). \tag{K.-154}$$

□

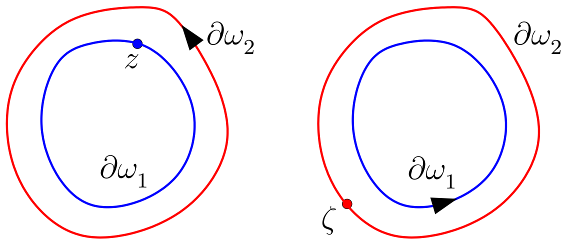

Figure K.8: OperProd2

(2)

the Taylor expansion around the point λ rather than μ,

$$\frac{1}{\lambda - x} = \frac{1}{\mu - x}\frac{1}{1 - \frac{\mu-\lambda}{\mu-x}} = \sum_{n=1}^{\infty} \frac{(\mu-\lambda)^{n-1}}{(\mu-x)^n} \tag{K.-153}$$

where $y \in \mathbb{C}$ (for $|\mu - \lambda|/|\mu - x| < 1$), has an operator (or matrix if you like) version: if $|\lambda - \mu| \cdot \|(\mu - A)^{-1}\| < 1$, then

$$(\lambda - A)^{-1} = \sum_{k=1}^{\infty} (\lambda - \mu)(\mu - A)^{-1} \tag{K.-153}$$

Note from (K.20) that $(\lambda - A)^{-1}$ and $(\mu - A)^{-1}$ commute. Substituting (K.20) it into itself one gets

$$\begin{aligned} (\lambda - A)^{-1} &= (\mu - A)^{-1} + (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} \\ &= (\mu - A)^{-1} + (\mu - \lambda)(\mu - A)^{-2} + (\mu - \lambda)^2(\lambda - A)^{-1}(\mu - A)^{-2} \end{aligned} \tag{K.-153}$$

Continuing to substitute (K.20) in this way one gets

$$(\lambda - A)^{-1} = \sum_{k=1}^{n} (\mu - \lambda)^{k-1}(\mu - A)^{-k} + (\mu - \lambda)^n(\lambda - A)^{-1}(\mu - A)^{-n} \tag{K.-153}$$

Since

$$\lim_{n\to\infty} \|(\mu - \lambda)^n(\lambda - A)^{-1}(\mu - A)^{-n}\| \le \lim_{n\to\infty} |\mu - \lambda|^n \|(\lambda - A)^{-1}\| \|(\mu - A)^{-1}\|^n = 0 \tag{K.-153}$$

completes the proof.

Appendix L

The Loop Representation

introducing heavy mathematical tools, often unfamiliar to the average physicist

to achieve certainty is to work at a high level of mathematical precision.

we search for a mathematical precision is that in quantum gravity

in the absence of any experimental observation at least for the moment by having a consistent theory

Ashtekar and Isham the representation of the loop algebra by using $C^\star-$algebra representation theory: $\overline{\mathcal{A}/\mathcal{G}}$ is the Gelfand spectrum (complex valued, bounded functions on a compact Hausdorff space) on the abelian part of the loop algebra.

$\mathcal{H}$ can be constructed as the projective limit of the projective family of the Hilbert spaces $\mathcal{H}_\gamma$, associated to a graph γ in $\mathcal{M}$.

L.1 Loop Representation

quantizing field theories requires one to smear fields, i.e. to integrate them over regions in order to obtain a well-defined algebra without $\delta-$functions. Usually this is done by integrating both configuration and momentum variables over three-dimensional regions, which requires an integration measure.

There is now a different smearing available which does not require a background metric. Instead of using three-dimensional regions we integrate the connection along one-dimensional curves e and exponentiate in a path-ordered manner, resulting in holonomies.

densitized vector fields can naturally be integrated over 2-dimensional surfaces, resulting in fluxes

$$F_S(E) = \int_S \tau^i E_i^a n_a d^2y \tag{L.0}$$

with the co-normal n_a to the surface.

The Poisson algebra of holonomies and fluxes is now well-defined and one can look for representations on a Hilbert space. We also require diffeomorphism group on the representation by moving edges and surfaces in space.

Spatial geometry can be obtained from fluxes representing the densitized triad. Since these are now momenta, they are represented by derivative operators with respect to values of connections on the flux surface. States as constructed above depend on the connection only along edges of graphs such that the flux operator is non-zero only if there are intersection points between its surface and the graph in the state it acts on

L.2 Algebraic Quantization of Loop Representation

$$W_k(x) = \exp(ikx) \tag{L.0}$$

$$\Psi(k) := \int dx W_k^*(x)\Psi(x) \tag{L.0}$$

$$\begin{array}{rcl} \{\mathcal{T}^0(k_1), \{\mathcal{T}^0(k_2)\} & = & 0, \\ \{\mathcal{T}^1(k_1), \{\mathcal{T}^0(k_2)\} & = & -ik_1\mathcal{T}^0(k_1+k_2), \\ \{\mathcal{T}^1(k_1), \{\mathcal{T}^1(k_2)\} & = & i(k_1-k_2)\mathcal{T}^0(k_1+k_2), \end{array} \tag{L.-2}$$

Its action on a wavefunction is to affect a translation

$$\hat{\mathcal{T}}^0\Psi(k) = \int dx e^{-ikx} e^{ik_1\hat{x}}\Psi(x) = \Psi(k-k_1) \tag{L.-2}$$

$$\boxed{\text{Simple example of a free algebra}}$$

for any finite order polynomial can be generated by multiplication and addition of the elementary variables

$$a \ \text{ and } \ x \tag{L.-2}$$

the associative algebra generated by finite sums and products of these elementary operators

$$a_0 + a_1 x + \cdots + a_k x^k + \cdots + a_N x^N \tag{L.-2}$$

L.2.1 Loop Algebra for $U(1)$

Let $\mathcal{A}$ be the space of smooth $U(1)$ connections whose cartesian components are functions of rapid decrease at infinity.

L.2.2 Mathematical Description

Let $\mathcal{L}_{x_0}$ denote the collection (or space) of oriented loops on R^3 with basepoint x_0. Them being oriented means there is a certain sense of flow around the loop. We can form a composition between loops as illustrated in fig(L.2.2) We denote the composition between two loops α and β as $\alpha \circ \beta$.

$$(a \cdot b)(t) := \begin{cases} a(2t), & 0 \leq t \leq 1/2 \\ b(2t-1) & 1/2 \leq t \leq 1 \end{cases} \tag{L.-2}$$

For any path α, we denote by α^{-1} the *inverse loop* formed by transversing α in the opposite direction.

can be found in section C.16.4 and in the maths glossary.

The set of all x_0-based loops in X is a semi-group with identity (a monoid?)

The collection of all equivalence classes of paths in a topological space X is called the *fundamental groupoid*, denoted $\Gamma(X)$.

The retracing identity

$$\mathcal{T} = \mathcal{T}[\alpha \cdot l \cdot l^{-1}]. \tag{L.-2}$$

Here l is a curve with one end on α, and $\alpha \cdot l \cdot l^{-1}$ is the loop obtained by going around α, along the curve and back along the curve α. And

$$\lim_{\gamma \to 0} \mathcal{T} = 1, \tag{L.-2}$$

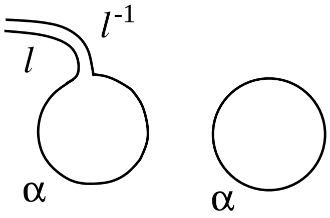

Figure L.1: Bridge.

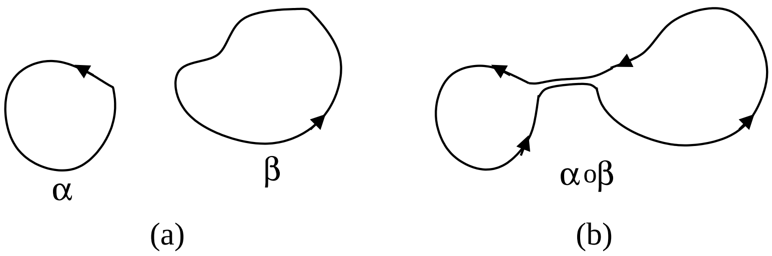

Figure L.2: Definition of the composition of two loop α and β (both in $\mathcal{L}_{x_0}$) - it is denoted as $\alpha \circ \beta$.

where $\gamma \to 0$ means γ shrinks to the loop to a point.

If we define multiplication of loops as the composition, the elements of $\mathcal{L}_{x_0}$ form a group under under this multiplication. The identity is just the loop contracted to a point at x_0. The inverse of the loop α is the same loop with opposite orientation, which we denote as α^{-1}. Associativity $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$ is demonstrated if fig().

i) The composition of parametrized curves is not associative, since the curves $(c_3 \circ c_2) \circ c_1$ and $c_3 \circ (c_2 \circ c_1)$ are related by a reparametrization:

$$c_3 \circ (c_2 \circ c_1) \;=\; c_3 \circ \left\{ \begin{array}{c} c_2(2t), \quad 0 \le t \le 1/2 \\ c_1(2t-1), \quad 1/2 \le t \le 1 \end{array} \right. = \left\{ \begin{array}{c} c_3(2t), \quad 0 \le t \le 1/2 \\ c_2(4t-2), \quad 1/2 \le t \le 3/4 \\ c_1(4t-3), \quad 3/4 \le t \le 1 \end{array} \right. \tag{L.-2}$$

$$(c_3 \circ c_2) \circ c_1 \;=\; \left\{ \begin{array}{c} c_3(2t), \quad 0 \le t \le 1/2 \\ c_2(2t-1), \quad 1/2 \le t \le 1 \end{array} \right. \circ c_1 = \left\{ \begin{array}{c} c_3(4t), \quad 0 \le t \le 1/4 \\ c_2(4t-1), \quad 1/4 \le t \le 1/2 \\ c_1(2t-1), \quad 1/2 \le t \le 1 \end{array} \right. \tag{L.-2}$$

Definition The set of equivalence classes of curves is denoted by $\mathcal{P}$. In order to distinguish the equivalence classes from their representative curves we will refer to them as **paths**.

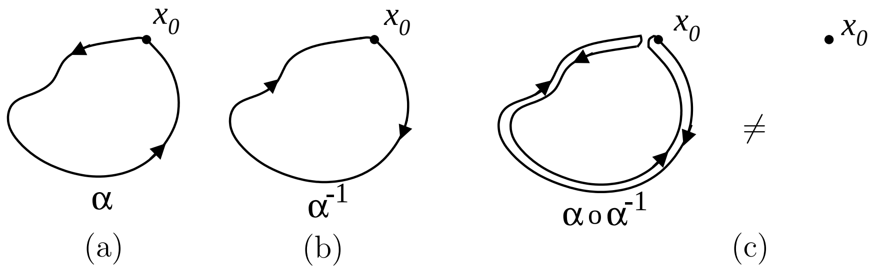

Figure L.3: inverse α^{-1} is a bit of a misnomer, $\alpha \circ \alpha^{-1} \neq o$.

Note that the definition of curves and paths is somewhat opposite to the defintions given in the appandix on the Hawking-Penrose singularity theorems.

The advantage of dealing with paths $\mathcal{P}$ rather than curves is that we now have almost a group structure since composition becomes associative and the path $p_c \circ p_c^{-1} = b(p_c)$ is trivial (stays at the beginning point). However, we still do not have a natural identity element in $\mathcal{P}$ and not all of its elements can be composed. The natural structure behind this is that of a **groupoid**.

L.2.3 Loops of Connections

Given a loop $\alpha \in \mathcal{L}_{x_0}$, the holonomy of $A_\mu(x)$ around α is $H_\alpha(A) := \exp(i \oint_\alpha A_\mu dx^\mu)$.

$$H_{\alpha\circ\beta}(A) = \qquad \text{(L.-2)}$$

If two loops α and β have the same holonomy

$$H_\alpha(A) = H_\beta(A) \qquad \text{(L.-2)}$$

for every $A_\mu(x)$ then we say they belong to the same holonomy loop class or just hoop. We denote such a class as $\tilde{\alpha}$.

$\mathcal{FA}$ free algebra

$$\sum_{i=1}^{N} a_i \alpha_i \in K \text{ if and only if } \sum_{i=1}^{N} a_i H_{\alpha_i}(A) = 0 \qquad \text{(L.-2)}$$

The function $H_{\alpha_i}(A)$ is a homorphism. Forms an ideal

$$H_{\alpha_i}(A)\left(\left(\sum_{i=1}^{N} b_i\beta_i\right)\left(\sum_{i=1}^{N} a_i\alpha_i\right)\right) = H_{\alpha_i}(A)\left(\sum_{i=1}^{N} b_i\beta_i\right) \times 0 = 0 \ \in \tag{L.-2}$$

α and β have a common point. Here # indicates joining of two loop at an intersection.

$$\mathcal{T}^0[\gamma] = U_\gamma(s) = \mathcal{P}e^{\oint_\gamma A}. \tag{L.-2}$$

$$\{\mathcal{T}^0[\alpha], \mathcal{T}^0[\beta]\} = 0 \tag{L.-2}$$

$$\mathcal{T}^a[\alpha](\alpha(s)) := \frac{1}{2}[U_\alpha(t,s)\tilde{E}^a(\alpha(s))U_\alpha(s,t)\tilde{E}^b(\alpha(t)) \tag{L.-2}$$

$$\{\mathcal{T}^a[\alpha], \mathcal{T}^0[\beta]\} = \frac{1}{2i}\int dt\delta^3(\gamma s, \eta(t))\dot{\eta}(t)[\mathcal{T}^0[\gamma\#\eta] - [\mathcal{T}^0[\gamma\#\eta^{-1}]]. \tag{L.-2}$$

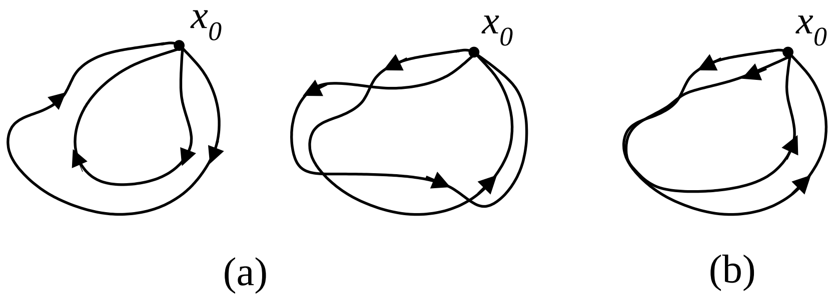

Figure L.4: (a) Two examples of strongly independent loops in $\mathcal{L}_{x_0}$. (b) An example of two strongly dependent loops in $\mathcal{L}_{x_0}$ - they have a segment in common.

$$\left(\hat{\mathcal{T}}^0[\alpha]\Psi\right)[\gamma] \equiv \Psi[\alpha \cup \gamma] \tag{L.-2}$$

$\cup$ stands for union in the set of loops.

$$(\hat{\mathcal{T}}^a[\alpha](s))[\gamma] \equiv \hbar c \int dt\delta^3(\alpha(s), \gamma(t))\dot{\gamma}^a(t)[\Psi[\gamma \circ \alpha] - [\gamma \circ \alpha^{-1}] \tag{L.-2}$$

can be found in section **??** and in the maths glossary.

The collection of all equivalence classes of paths in a topological space X is called the *fundamental groupoid*, denoted $\Gamma(X)$.

1. A finite set $\{e_1, \ldots, e_N\}$ of edges is said to be independent if the edges e_i can only intersect each other at their sources $s(e_i)$ or targets $t(e_i)$.

2. A finite graph is a collection of a finite set $\{e_1, \ldots, e_N\}$ of independent edges and their vertices. We denote by $E(\gamma)$ and $V(\gamma)$ respectively as sets of independent edges and vertices of a given finite graph γ. N_γ is the number of elements in $E(\gamma)$.

Details: loop algebra

$$T^a[\gamma](s) := \sqrt{2}\mathrm{tr}U_\gamma(s)^{AB}\tilde{\sigma}^a_{AB}(\gamma(s)) \tag{L.-2}$$

$$\begin{aligned} i\{T^a[\gamma](s), T^b[\eta](t)\} &= 2U_\gamma(s)^{AB}\{\tilde{\sigma}^a_{AB}(\gamma(s)), U_\eta(t)^{CD}\}\tilde{\sigma}^b_{CD}(\eta(t)) \\ &+ 2\tilde{\sigma}^a_{AB}(\gamma(s))\{U_\gamma(s)^{AB}, \tilde{\sigma}^b_{CD}(\eta(t))\}U_\eta(t)^{CD} \end{aligned} \tag{L.-2}$$

$$\begin{aligned} &= U_\gamma(s)^{AB} - i\sqrt{2}\int^t du\delta^3(\eta(u),\gamma(s))\dot{\eta}^a(u)U_\eta(0,u)^C{}_{(A}U_\eta(u,t)_{B)}^D\tilde{\sigma}^b_{CD}(\eta(t)) \\ &+ i\sqrt{2}\tilde{\sigma}^a_{AB}(\gamma(s))\int^s du\delta^3(\gamma(u),\eta(t)\dot{\gamma}^b(u)U_\gamma(0,u)^C{}_{(A}U_\gamma(u,s)_{B)}^D \\ &= \sqrt{2}\Delta^a[\gamma,\eta]U_\gamma(s)^{AB}U_\eta(0,u)^C{}_{(A}U_\eta(u,t)_{B)}^D\tilde{\sigma}^b_{CD}(\eta(t)) \\ &+ \sqrt{2}\Delta^b[\eta,\gamma]\tilde{\sigma}^a_{AB}(\gamma(s))U_\gamma(0,u)^A{}_{(C}U_\gamma(t,u)_{D)}^BU_\eta(t)^{CD} \end{aligned} \tag{L.-4}$$

$$\begin{aligned} \sqrt{2}\Delta[\gamma,\eta] \quad & \Big[U_\gamma(s)^{AB}U_\eta(u,t)_B{}^D\tilde{\sigma}^b_{CD}(\eta(t))U_\eta(t,u)^C{}_A \\ &+ U_\gamma(s)^{AB}U_\eta(u,t)_A{}^D\tilde{\sigma}^b_{CD}(\eta(t))U_\eta(t,u)^C{}_B\Big] \end{aligned} \tag{L.-4}$$

$$\{T^a[\gamma](s), T^b[\eta](t)\} = i\Delta^b[\eta,\gamma](t)\ T^a \tag{L.-4}$$

$$U_\gamma(s)_{AB} = -U_{\gamma^{-1}}(s)_{BA} \tag{L.-4}$$

$$U^A{}_B(0,u)U^B{}_C(u,s) = U^A{}_C(0,s) \tag{L.-4}$$

$$\Delta^a[\gamma,\eta](s) \equiv \frac{1}{2}\int dt\ \delta^3(\gamma(s),\eta(t))\ \dot{\eta}^a(t) \tag{L.-4}$$

$$\sqrt{2}\{\tilde{\sigma}^a_{AB}(x), U_\gamma^{CD}(0,s)\} = -i\int_0^s du\delta^3(\gamma(u),x)\ \dot{\gamma}^a(u)\ U_\gamma(0,u)^C{}_{(A}U_\gamma(u,s)_{B)}^D \tag{L.-4}$$

L.2.4 Differentiability Classes of Manifolds and Loops

See Chapter 3

L.2.5 Loop Space

$\mathcal{L}_\Sigma$ denotes the space of parametrize, differntiable, loops in Σ, which are the maps $\gamma : s \to \Sigma$. We also include in $\mathcal{L}_\Sigma$ those loops that with nowhere vanishing tangent vector, $\dot\gamma^a(s)$.

there is a subset of Diff(Σ) which leaves the curve γ invariant, and only reparamertizes it. The infinitesimal elements of this subset are the vector fields on Σ that are tangent to γ. Globally, one may show that the subset is thw diffeomorphism group of the complement of γ, that is, $\Sigma - \gamma$.

Exercise: Prove loop space is a differential manifold.

Proof:

L.2.6 Regularization of Holonomies

The Poison-brackets among the holonomies and the fluxes can be calculated by regularizing the edges and surfaces in three dimensions and then taking the limit of a family of functions, theat converge exactly to the holonomy along the particular edge and the flux through the particular surface.

$$H_\alpha(A) = \exp i \int_{R^3} X^\mu_\gamma(x) A_\mu(x) d^3x \tag{L.-4}$$

$$X^\mu_\gamma(x) := \oint_\gamma ds \delta^3(\vec\gamma(s), \vec x)\dot\gamma^\mu \tag{L.-4}$$

where s is a parametrization of the loop $\gamma, s \in [0, 2\pi]$. $X^\mu_\gamma(x)$ is called the form factor of γ. Its Fourier transform is

$$\begin{aligned} X^\mu_\gamma(k) &:= \frac{1}{(2\pi)^{3/2}} \int d^x X^\mu_\gamma(x) e^{-ik\cdot x} \\ &= \frac{1}{(2\pi)^{3/2}} \oint ds \dot\gamma^\mu e^{-i\vec k\cdot\vec\gamma(s)} \end{aligned} \tag{L.-4}$$

regularize by replacing the delta function $\delta^3(\vec{y}-\vec{x})$ with $f_r(\vec{y}-\vec{x})$ that approximates the delta function and such that $\lim_{r\to 0} f_r(\vec{y}-\vec{x}) \to \delta^3(\vec{y}-\vec{x})$

$$X^{\mu}_{\gamma(r)}(\vec{x}) := \int_{R^3} d^3y f_r(\vec{y}-\vec{x}) X^{\mu}_{\gamma}(\vec{y}) \tag{L.-4}$$

L.2.7 Classical Loop Algebra

L.2.8 Holonomy-Flux $*-$Algebra

The elementary classical observables in our representaion theory are the complex valued functions of holonomies $A(e)$ along paths e in Σ, and fluxes $E_i(S)$ of triad filed across 2-surfaces S, which are defined by

$$E_i(S) := \int_S \eta_{abc}\tilde{E}^c_i. \tag{L.-4}$$

$$\lim_{\epsilon\to 0}\int_S d^2y\ f_\epsilon(x^1,x^2;y^1,y^2)\ g(y^1,y^2) = g(x^1,x^2) \tag{L.-4}$$

$$[E_i]_f(x) := \int dy^a\wedge dy^b\ f_\epsilon(x,y)\eta_{abc}E^c_i(y) \tag{L.-4}$$

if the surface is given in local coordinates by $x^3 = const$, as ϵ tends to zero $[E_i]_f$ tends to $E^3_i(x)$.

$$\mathcal{T}[\alpha] \ := \ \frac{1}{2}\mathrm{trP}\exp\left[G\oint dt\dot{\alpha}^b A_b(\alpha(t))\right], \tag{L.-3}$$

$$\mathcal{T}^a[\alpha](\alpha(s)) \ := \ \frac{1}{2}\mathrm{trP}\left\{\exp\left[G\oint dt\dot{\alpha}^b A_b(\alpha(t))\right]\tilde{E}^a(\alpha(s))\right\} \tag{L.-2}$$

where $A_a(x) = A^i_a(x)\tau_i$ and $E^a(x) = 4E^{ai}(x)\tau_i$ are the Ashtekar connection and its conjugate frame field. (τ_i is the Paulli matrix divided by $2i$)

Invariance under inversion of loop is expressed as

$$\boxed{\mathcal{T}[\alpha]^{-1} = \mathcal{T}[\alpha^{-1}].} \tag{L.-2}$$

The spinor identity

$$\boxed{\mathcal{T}[\alpha]\mathcal{T}[\beta] = \frac{1}{2}\left(\mathcal{T}[\alpha\#\beta] + \mathcal{T}[\alpha\#\beta^{-1}]\right),} \tag{L.-2}$$

if α and β have a common point. Here # indicates joing of two loop at an intersection.

$$\mathcal{T}^0[\gamma] = \mathrm{tr}U_\gamma(s) = \mathrm{tr}\mathcal{P}e^{\oint_\gamma A}. \tag{L.-2}$$

$$\{\mathcal{T}^0[\alpha], \mathcal{T}^0[\beta]\} = 0 \tag{L.-2}$$

$$\mathcal{T}^a[\alpha](\alpha(s)) := \frac{1}{2}\mathrm{tr}[U_\alpha(t,s)\tilde{E}^a(\alpha(s))U_\alpha(s,t)\tilde{E}^b(\alpha(t)) \tag{L.-2}$$

$$\{\mathcal{T}^a[\alpha], \mathcal{T}^0[\beta]\} = \frac{1}{2i}\int dt\delta^3(\gamma s, \eta(t))\dot{\eta}(t)[\mathcal{T}^0[\gamma\#\eta] - [\mathcal{T}^0[\gamma\#\eta^{-1}]]. \tag{L.-2}$$

L.2.9 Quantization of Loop Algebra

The holonomy (corresponding to the configuration variable) opertor acts by multiplication as does (x in one particle quantum mechanics):

$$\hat{\mathcal{T}}[\alpha]\Psi_S(A) = -\mathrm{Tr}\left(\mathcal{P}\int_\alpha A\right)\Psi_S(A) \tag{L.-2}$$

$$[\hat{\mathcal{T}}^0[\alpha], \hat{\mathcal{T}}^0[\beta]] = 0 \tag{L.-2}$$

$$[\hat{\mathcal{T}}^0[\alpha], \hat{\mathcal{T}}^a[\beta](s)] \tag{L.-2}$$

$$\boxed{||T_\gamma|| := \sup_{[A]\in\mathcal{A}}|\ T_\gamma[A]\ |} \tag{L.-2}$$

and complete $\mathcal{HA}$ with respect to this norm we obtain a commutative C^*-algebra $\overline{\mathcal{HA}}$.

first key result we will use is the

> *Gel'fand-Naimark theorem, that every C^*-algebra with identity is isomorphic to the C^*-algebra of all continuous bounded functions on a compact Hausdorff space called the* spectrum *of the algebra.*

Completetion w.r.t. this norm gives us a commutative C∗-algebra with identity, $\overline{\mathcal{HA}}$. We will call the spectrum of $\overline{\mathcal{HA}}$ by:

$$\overline{\mathcal{A}/\mathcal{G}} \tag{L.-2}$$

The algebra structure allows us to construction of its representaions on Hilbert spaces. For every cyclic representation of $\overline{\mathcal{HA}}$ there is a Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ using which we get a Hilbert space:

$$\mathcal{H}_{aux} := L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu). \tag{L.-2}$$

(Exercise) the operator equation

$$e^{-\hat{B}}\hat{A}e^{\hat{B}} = \mathbb{I} + t\{\hat{A}, \hat{B}\} + \frac{t^2}{2!}\{\hat{A}, \{\hat{A}, \hat{B}\}\} + \cdots \tag{L.-2}$$

L.3 Spinor Network States

The basic canonical degrees of freedom are holonomies of a distributional $SU(2)$ connection and fluxes of the densitized triad conjugate to this connection. The Gauss law (local $SU(2)$ invariance) and momentum (spatial diffeomorphism) constraints are realized as self-adjoint operators constructed out of these variables. States annihilated by these constraint operators span the kinematical Hilbert space. Particularly convenient bases for this kinematical Hilbert space are the spin network bases. In any of these bases, a state is described in terms of links $l_1, \ldots, l_n$ carrying spins ($SU(2)$ irreducible representations) $j_1, \ldots, j_n$ and vertices carrying invariant $SU(2)$ tensors (intertwiners).

$$[\rho_{j_e}(H_e(A))]^{\alpha}_{\beta}, \quad \alpha, \beta = 1, \ldots, d_{\rho_{je}} \tag{L.-2}$$

where $d_{\rho_{je}} = 2j_e + 1$ is the dimension of the representation.

$$[\rho_{j_1}(H_{e_1}(A))]^{\alpha_1}_{\beta_1} \cdots [\rho_{j_n}(H_{e_1}(A))]^{\alpha_n}_{\beta_n}\, v^{\beta_1 \ldots \beta_n} = v^{\alpha_1 \ldots \alpha_n} \tag{L.-2}$$

$$\overline{v_i^{\alpha_1 \ldots \alpha_n}}\, v_{i' \alpha_1 \ldots \alpha_n} = \delta_{ii'} \tag{L.-2}$$

L.3.1 Spinor Network Decomposition of Kinematic Hilbert Space

Spinor Network Decomposition on Single edge

The decomposition of $\mathcal{H}_e = L^2(SU(2), d\mu_H)$ is provided by the Peter-Weyl theorem.

$$\int_{\overline{\mathcal{A}}_e} \overline{\rho^{j'}_{\alpha'\beta'}} \rho^{j}_{\alpha\beta}\, d\mu_e = \frac{1}{2j+1}\delta^{j'j}\delta_{\alpha'\alpha}\delta_{\beta'\beta} \tag{L.-2}$$

Spinor Network Decomposition on Finite Graph

L.4 Cylindrical Measure Theory

A key ingredient for discussing quantum physics is to have at hand an inner product to compute expectation values. It is not easy to develop functional measures in infinite dimensional spaces.

One wishes to compute: $(\psi_1, \psi_2) = \int_{\mathcal{A}/\mathcal{G}} d\mu([A])\overline{\psi_1([A])}\psi_2([A])$

To motivate the functional space we will consider let us start with a simpler example, that of a scalar field ϕ satisfying the Klein-gordon equation.

A configuration space C for such a theory would be given by the set of all smooth field configurations with appropriate fall off conditions at infinity, for instance C^2 functions. One therefore expects to have wavefunctions $\Psi(\phi)$, and wishes to compute,

$$(N_1; N_2) = N_1(p)N_2(p) \tag{L.-2}$$

And we therefore need a suitable measure and integration theory. To construct this, let us consider the set of test (or smearing) functions on R^3, that is, functions that fall off such that the integral,

$$F_f(\phi) = \langle f, \phi\rangle = \int_{\mathcal{R}^d} d^d x f(x)\phi(x). \tag{L.-2}$$

The functions f are called "Schwarz space" and define the simplest linear functionals on C.

A set of functions on C one can introduce are the "cylindrical" functions. Consider a finite dimensional subspace of the Schwarz space V_n, with a basis $(e_1, \ldots, e_n)$. We can define the projections, For any function $F : R^n \to C$

$$\pi_{e_1,\ldots,e_n}(\phi) = \{\langle e_1, \phi\rangle, \ldots, \langle e_n, \phi\rangle\} \tag{L.-2}$$

This represntation is not unique. In particular any function cylindrical with respect to V_n is cylindrical with respect to any V_m that contains V_n.

A cylindrical measure that allows to integrate cylindrical functions. Any measure in R^n would allow us to integrate cylindrical functions, but the tricky part is that there has to be consistency of these measures for different choices of V_n's.

$$\int_C d\mu(\phi) f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{\langle e_1,\phi\rangle,\ldots,\langle e_n,\phi\rangle} \tag{L.-2}$$

Suppose one has V_n and V_m which have non-vanishing intersection, and with $m > n$, and,

$$V_n^*(\eta_1, \ldots, \eta_n) \subset \tilde{V}_m^*(\tilde{\eta}_1, \ldots, \tilde{\eta}_n) \text{ with } e_i = \sum_{j=1}^{m} L_{ij} \tilde{e}_j \ ; \quad i = 1, \ldots, n \tag{L.-2}$$

Then for every cylindrical function f with respect to V_n defined by a function F on R^n one can make it cylindrical with respect to V_m via,

$$f(\phi) = F(\langle e_1, \phi\rangle, \ldots, \langle e_n, \phi\rangle) = F(\left\langle L_{1j}\tilde{e}_j, \phi\right\rangle, \ldots, \left\langle L_{nj}\tilde{e}_j, \phi\right\rangle = \tilde{F}(\left\langle \tilde{e}_j, \phi\right\rangle, \ldots, \left\langle \tilde{e}_j, \phi\right\rangle \tag{L.-2}$$

And therefore one has to have that,

$$\int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n) = \int_{R^m} \tilde{F}(\tilde{\eta}_1, \ldots, \tilde{\eta}_m) d\mu_{\tilde{e}_1,\ldots,\tilde{e}_m}(\tilde{\eta}_1, \ldots, \tilde{\eta}_n) \tag{L.-2}$$

Any set of measures one finite dimensional spaces satisfying these conditions for any cylindrical function F, defines a cylindrical measure via,

$$\int_C d\mu(\phi) f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n) \tag{L.-2}$$

And conversely, a cylindrical measure defines consistent sets of measures in finite dimensional settings.

$$f(\phi) = F\left(\langle e_1, \phi\rangle, \ldots, \langle e_n, \phi\rangle\right) \tag{L.-2}$$

A particularly simple example of this construction is to consider the normalized Gaussian measures in R^n. The resulting measure on C is the one used in textboks when quantizing the scalar field. The Fock space is obtained by completion of the sets of cylindrical measures with a certain weight.

However, in the Cauchy completion, we obtain states that which "genuinely" depend on an infinite number of degrees of freedom. these states can not be realized as functions on $\mathcal{C}$. appropriate measures are Gaussians and all quantun states can be realized on the space $\mathcal{S}'$ of tempered distributions, the topological dual of the space $\mathcal{S}$ of probes.

An obvious property of measures of integration involving a finite number of disjoint measure sets is that the measure of the union of the measure sets is equal to the sum of their measures, i.e.

$$\mu\left(\cup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu(A_i), \quad \text{for } N < \infty, \quad \text{where } A_i \cap A_j = \emptyset \text{ for all } i \neq j. \tag{L.-2}$$

However, this is not in general true for *countable* unions.

More precise account:

The possibility of extending a measure μ on $\mathcal{F}$ to a $\sigma-$additive measure $\tilde{\mu}$ on $\mathcal{B}(\mathcal{F})$ is in particular relevant to physical applications in quantum mechanics. Recall that quantum mechanical systems are often defined by first giving a linear pre-Hilbert space and then completing this space with respect to an inner product. In general, if μ is cylindrical but not $\sigma-$additive, the space $\mathcal{H}$ of -square integrable cylindrical functions on X (denoted through $\mathcal{C}L^2(X, \mathcal{F}, \mu)$) is only a pre-Hilbert space. Such spaces will be discussed in section ??. However, if μ is extendible to a $\sigma-$additive measure $\tilde{\mu}$ on $(X, \mathcal{B}(\mathcal{F}))$ then the Cauchy completion of $\mathcal{H}$ leads to the space $\tilde{\mathcal{H}} = L^2(X, \mathcal{B}(\mathcal{F}), \tilde{\mu})$ (see section 5).

On the other hand if μ is not extendible then the Cauchy completion of $\mathcal{C}L^2(X, \mathcal{F}, \mu)$ leads in general to a space with state-vectors which cannot be expressed as functions on the initial space X. This is the case in scalar field theory if one considers $X = \mathcal{S}(\mathbb{R}3)$ (the Schwarz space of rapidly decreasing smooth C^∞ functions on $\mathbb{R}^3$) and μ is a cylindrical measure defined with the help of a positive definite function on $\mathcal{S}(\mathbb{R}^3)$, continuous in the nuclear space topology (see []).

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As we shall see in Sect. 5 this is also the case in Yang-Mills theory if we take $\mathcal{H} = \mathcal{C}L^2(\mathcal{A}/\mathcal{G}, \mathcal{F} = \mathcal{C}, \hat{\mu}_{AL})$, where $\hat{\mu}_{AL}$ is the Ashtekar-Lewandowski measure on $\mathcal{A}/\mathcal{G}$. In the scalar field case the Cauchy completion of $\mathcal{C}L^2(\mathcal{S}(\mathbb{R}^3), \mathcal{F}, \mu)$ gives the space of square integrable functions on $\mathcal{S}'(\mathbb{R}^3)$ (the space of tempered distributions), while in the Yang-Mills case the completion of $\mathcal{C}L^2(\mathcal{A}/\mathcal{G}, \mathcal{C}, \hat{\mu}_{AL})$ gives the space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mathcal{B}(\overline{\mathcal{C}}), \mu_{AL})$ of square integrable functions on the Ashtekar-Isham space $\overline{\mathcal{A}/\mathcal{G}}$ of generalized distributional connections modulo gauge transformations.

L.4.1 Probability Densities

$$C(\lambda) \equiv < 0|e^{i\lambda Q}|0 > = e^{-\frac{1}{2}\lambda^2} = \int \rho(x) e^{i\lambda x} dx.$$

Inverting the Fourier transform one finds a Gaussian ground state ensity

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \tag{L.-2}$$

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,Ax)+(b,x)} d^n x = \prod_k \sqrt{\frac{(2\pi)^n}{det(A)}} e^{\frac{1}{2}(b,A^{-1}b)}. \tag{L.-2}$$

The “characteristic function”

$$C(\lambda) \;\equiv\; \sqrt{\frac{det(A)}{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,Ax)+i(\lambda,x)} d^n x = e^{-\frac{1}{2}(\lambda,A^{-1}\lambda)}. \tag{L.-2}$$

It is the Fourier transform of a probability measure

$$d\mu(x) = \rho(x) dx, \tag{L.-2}$$

we determined the probability density

$$\rho(x) = \sqrt{\frac{det(A)}{(2\pi)^n}} e^{-\frac{1}{2}(x,Ax)}$$

by doing an inverse Fourier transform on C.

How do we know whether a given function C is the Fourier transform of a probability measure?

The function

$$C(\lambda) \equiv \int_{\mathbb{R}^n} \rho(x) e^{(\lambda,x)} d^n x \tag{L.-2}$$

has the properties

(i) C is normalized

$$C(0) = \int_{\mathbb{R}^n} \rho(x) d^n x = 1 \qquad \text{(L.-2)}$$

(ii) C is continuous at zero, since

$$\begin{aligned} C(\lambda) - 1 &= \int_{\mathbb{R}^n} \left(e^{i(\lambda,x)} - 1\right) \rho(x) d^n x \\ &= \int_{\mathbb{R}^n} (\cos((\lambda, x)) - 1) \rho(x) d^n x + i \int_{\mathbb{R}^n} (\sin((\lambda, x)) \rho(x) d^n x \qquad \text{(L.-2)} \end{aligned}$$

the principle of dominated convergence permits us to take the limit $\lambda \to 0$ inside the integrals.

(iii) For any complex $a_1, \ldots, a_n$, and real $\lambda_1, \ldots, \lambda_n$

$$\int_{\mathbb{R}^n} \left| \sum_l a_l e^{i(\lambda_l, x)} \right|^2 \rho(x) d^n x = \sum_{k,l} a_k^* a_l C(\lambda_k - \lambda_l) \geq 0 \qquad \text{(L.-2)}$$

as a consequence of the positivity of ρ (positive definiteness of C).

Infinite dimensions

problems come from trying to extend

$$\rho(x) d^n x = \sqrt{\frac{1}{(2\pi)^n}} e^{\frac{1}{2}(x,x)} d^n x \qquad \text{(L.-2)}$$

to $n = \infty$. d^∞ doesn't make sense, $(x, x) = \sum_1^\infty x_n^2$ would require this infinite sum to be convergent. The factor goes to zero as $n \to \infty$. However,

$$\begin{aligned} \lim_{n\to\infty} \left(\Omega, e^{i(\lambda,Q)} \Omega\right) &= \left(\Omega, e^{i\varphi(f)} \Omega\right) = \left(\Omega, e^{i \sum \lambda_n \varphi e_n} \Omega\right) \\ &= e^{-\frac{1}{2} \sum_{k=1}^\infty \lambda_k^2} = e^{-\frac{1}{2} \int f^2(x) dx} \qquad \text{(L.-2)} \end{aligned}$$

could be well defined. In particular we note that for

$$C(f) = \left(\Omega, e^{i\varphi(f)}\Omega\right)$$

(i)

$$C(0) = 1$$

(ii) C is continuous in the test functions f.

(iii) For any complex $a_1, \ldots, a_n$ and real test functions $f_1, \dot{s}, f_n$

$$\sum_{k,l} a_k^* a_l C(f_k - f_l) \geq 0. \tag{L.-2}$$

As we will come to later, we have the following generalization of Bochner's theorerm

(*Bochner-Minlos)* Any normalized continuous positive defininte complex function on test function space $\mathcal{S}(\mathbb{R}^n)$ is the Fourier transform of a probability measure μ on distribution space $\mathcal{S}(\mathbb{R}^n)$.

now we have

$$C(f) = \left(\Omega, e^{i\varphi(f)}\Omega\right) = \int_{\mathcal{S}^*} e^{i<\omega,f>} d\mu(\omega) \tag{L.-2}$$

where $< \omega, f >$ is the application of the generalized function $\omega \in \mathcal{S}^*$ to the test function f.

recall the expansion of test functions in terms of a basis, where

$$f(x) = \sum \lambda_n e_n(x).$$

only admit rapidly decreasing sequences of coefficients (λ_n). we have

$$f \in L^2(\mathbb{R}) \quad \leftrightarrow \quad \sum \lambda_n^2 < \infty$$

whereas

$$f \in \mathcal{S}(\mathbb{R}) \quad \leftrightarrow \quad \sum n^k \lambda_n^2 < \infty \text{ for all } k.$$

$$\omega(x) = \sum \omega_n e_n(x).$$

the coeficients ω_n are not square summable.

$$< \omega, f > = \sum \omega_n \lambda_n = \int f(x)\omega(x)dx.$$

The $\omega(x)$ on the right may fail to exist pointwise, but the sum is well defined and finite.

L.4.2 Bochner-Minlos Theorems

This is "algebraic" part of the problem.

we consider projective limits of infinite families of finite dimensional and measurable spaces.

The appropriate space of histories turns out to be the space $\mathcal{S}'$ of (tempered) distributions on the Euclidean space-time and regular measures $d\mu$ on this space are in one to one correspondence with the socalled generating functionals, which are functionals on the Schwarz space $\mathcal{S}$ of test functions satisfying certain rather simple conditions. (Recall that the tempered distributions are continuous linear maps from the Schwarz space to complex numbers.)

In the characterization of typical configurations of measures on functional spaces the so called Bochner-Minlos theorems play a very important role. These theorems are infinite dimensional generalizations of the Bochner theorem for probability measures on $\mathbf{R}^N$. Let us, for the convenience of the reader, recall the latter result. Consider any (Borel) probability measure on $\mathbb{R}^N$, i.e. a finite measure μ, normalized so that $\mu(\mathbb{R}^N) = 1$. The generating functional χ_μ of this measure is its Fourier transform, given by the following function on $\mathbb{R}^N(\mathbb{R}^N)'$, the prime denotes the topological dual, see below)

$$\chi_\mu(\lambda) = \int_{\mathbf{R}^N} N d\mu(x) e^{i(\lambda,x)}, \tag{L.-2}$$

where $(\lambda, x) = \sum_{j=1}^N \lambda^j x_j$. Generating functionals of measures satisfy the following three basic conditions,

(i) Normalization: $\chi(0) = 1$;

(ii) Continuity: is continuous on $\mathbb{R}^N$;

(iii) Positivity: $\sum_{k,l=1}^m c_k \overline{c_l}(\lambda_k - \lambda_l) \geq 0$, for all $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{C}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}^N$.

The last condition comes from the fact that $\|f\|_\mu \geq 0$, for $f(x) = \sum_k^m c_k e^{i(\lambda,x)}$, where $\| \cdot \|_\mu$ denotes the $L^2(\mathbf{R}^N, d\mu)$ norm. The finite dimensional Bochner theorem states that the converse is also true. Namely, for any function χ on $\mathbf{R}^N$ satisfying (i), (ii) and (iii) there exists a unique probability measure on $\mathbf{R}^N$ such that χ is its generating functional.

$$\chi(\lambda) = \int_{S'} e^{\lambda \cdot x} d\mu(\lambda) \tag{L.-2}$$

(i) $\chi(0) = 1$
(ii) χ is continuous in every finite dimensional subspace of $\mathcal{S}$
(iii) For every $e_1, \ldots, e_N \in \mathcal{S}$ and $c_1, \ldots, c_N \in C$ we have

$$\sum_{i,j=1}^{N} \bar{c}_i c_j \chi(-e_i + e_j) \geq 0. \tag{L.-2}$$

$$\begin{aligned} \chi(-e) &= \overline{\chi(e)} && \text{(L.-1)} \\ |\chi(e)| &\leq \overline{\chi(0)} && \text{(L.0)} \end{aligned}$$

i.e. $\chi(e)$ is bounded

Proof: take $N = 2$, $c_1 = 1$ and $c_2 = z$

$$\begin{aligned} \sum_{i,j=1}^{2} \bar{c}_i c_j \chi(-e_i + e_j) &= \chi(0) + \chi(e_1 - e_2)\overline{z} + \chi(e_2 - e_1)z + \chi(0)|z|^2 \\ &= \chi(0)(1 + |z|^2) + \chi(e)\overline{z} + \chi(-e)z && \text{(L.0)} \end{aligned}$$

where $e = e_1 - e_2$. Set $z = 1$, then we have $2\chi(0) + \chi(e) + \chi(-e) \geq 0$ so that $\chi(e) + \chi(-e)$ is real and for $z = i$ we have $-i\chi(e) + i\chi(-e) \geq 0$ so that $-i(\chi(e) - \chi(-e))$ is real. As such we have:

$$\begin{aligned} \chi(e) + \chi(-e) &= \overline{\chi(e)} + \overline{\chi(-e)} \\ \chi(e) - \chi(-e) &= \overline{\chi(-e)} - \overline{\chi(e)} && \text{(L.0)} \end{aligned}$$

$\Longrightarrow$ $\chi(e) = \overline{\chi(-e)}$. Now choose z such that:

$$\overline{z}\chi(e) + |\chi(e)| = 0 \tag{L.0}$$

So that $z\overline{\chi(e)} + |\chi(e)| = 0$. Substituting this into (L.0)

$$\chi(0)(1+\left|\frac{|\chi(e)|}{\chi(e)}\right|^2)+2|\chi(e)|\geq 0,$$

so that we have

$$2\chi(0)-2|\chi(e)|\geq 0 \tag{L.0}$$

Putting $z=0$ in (L.0) we find

$$\chi(0)\geq 0 \tag{L.0}$$

$$\begin{vmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{vmatrix} \geq 0 \tag{L.0}$$

because

$$\chi(0)|c_1|^2+\chi(e)c_1\overline{c}_2+\chi(-e)\overline{c}_1c_2+\chi(0)|c_2|^2\geq 0,$$

by the assumption of (L.4.2). This can be rexpressed as

$$(\overline{c}_1,\overline{c}_2)\begin{pmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\geq 0, \tag{L.0}$$

As the matrix is hermitian, there exists eigenvectors $\tilde{c}_1$

$$\begin{pmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{pmatrix}\begin{pmatrix} \overline{c}_1 \\ \overline{c}_2 \end{pmatrix}=\lambda\begin{pmatrix} \overline{c}_1 \\ \overline{c}_2 \end{pmatrix} \tag{L.0}$$

$$\lambda_i\,(|\tilde{c}_1|^2+|\tilde{c}_2|^2)\geq 0 \quad i=1,2 \tag{L.0}$$

Hence both eigenvalues are real non-negative numbers

$$\lambda_1,\lambda_2\geq 0 \tag{L.0}$$

and so the determinte satisfies

$$\begin{vmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{vmatrix}=\lambda_1\lambda_2\geq 0. \tag{L.0}$$

Conversley???

Every positive-definite continuous function defines a generalized function on $\mathcal{S}$.

$$M_{jk} = \chi(x_k - x_j) \tag{L.0}$$

The conditions for a matrix to be positive-definiteness is that it be Hermitian $M^*_{jk} = M_{kj}$ and its eigenvalues to be non-negative. The condition for arbitary x_k and x_j

$$f(x_k - x_j) = f(x_j - x_k)^* \quad \text{setting} \;\; x = x_k - x_j \quad \Longrightarrow f(-x) = \overline{f(x)}, \tag{L.0}$$

$$(f, \varphi) = \int \overline{f(x)} \varphi dx \tag{L.0}$$

on $\mathcal{S}$

$$\begin{vmatrix} f(0) & f(x) \\ f(x) & f(0) \end{vmatrix} \leq 0, \tag{L.0}$$

$$|f(x)| \leq f(0), \tag{L.0}$$

i.e., $f(x)$ is bounded.

this generalized function is positive-definite:

$$\int \overline{f(x-y)} \varphi(y) \overline{\varphi(x)} dx dy. \tag{L.0}$$

$$\int_{-T}^{T} \int_{-T}^{T} \overline{f(x-y)} \varphi(y) \overline{\varphi(x)} dx dy \tag{L.0}$$

$\varphi(x)$ is summable ($\int \varphi(x) dx < \infty$?) and $f(x)$ is bounded ??? For each T the integral () is the limit of sums

$$\sum_{j,k=1}^{m} \overline{f(x_k - y_j)} \varphi(x_k) \overline{\varphi(x_j)} \Delta x_k \Delta x_j \tag{L.0}$$

the generalized function (f, φ) is positive-definite

Every continuous postive-definite function $\chi(\phi)$ is the Fourier transform of a finite positive measure $d\mu$.

$$(f,\varphi) = (2\pi)^{-n}\int \tilde{\varphi}(\lambda)d\mu(\lambda) \tag{L.0}$$

is $d\mu(\lambda) < \infty$ $\varphi_m(x) = \alpha_m \star \alpha_m^*(x)$, where $\{\alpha_m(x)\}$ is a $\delta-$sequence in S. we obtain

$$(f,\varphi_m) = (2\pi)^{-n}\int \tilde{\varphi}_m(\lambda)d\mu(\lambda) \tag{L.0}$$

$$\tilde{\varphi}_m(\lambda) = |\tilde{\alpha}_m(\lambda)|^2 \tag{L.0}$$

$$\begin{pmatrix} 1 & \chi(t-s) & \chi(t) \\ \overline{\chi(t-s)} & 1 & \chi(s) \\ \overline{\chi(t)} & \overline{\chi(s)} & 1 \end{pmatrix} \tag{L.0}$$

which is $\{\phi(t_i - t_j)\}$ with $t_1 = t, t_2 = s$ and $t_3 = 0$. In particular the determinant has to be non-negative.

$$\begin{aligned} 0 &\leq 1 + \chi(s)\chi(t-s)\overline{\chi(t)} + \overline{\chi(s)\chi(t-s)}\chi(t) - |\chi(s)|^2 \\ &\quad -|\chi(t)|^2 - |\chi(t-s)|^2 \\ &= 1 - |\chi(s)-\chi(t)|^2 - |\chi(t-s)|^2 - \chi(t)\overline{\chi(s)}(1-\chi(t-s)) \\ &\quad -\overline{\chi(t)}\chi(s)(1-\chi(t-s)) \\ &\leq 1 - |\chi(s)-\chi(t)|^2 - |\chi(t-s)|^2 + 2|1-\chi(t-s)| \end{aligned} \tag{L.-3}$$

or

$$\begin{aligned} |\chi(s)-\chi(t)|^2 &\leq 1 - |\chi(s)-\chi(t)|^2 + 2|1-\chi(t-s)| \\ &\leq 4|1-\chi(t-s)| \end{aligned} \tag{L.-3}$$

L.4.3 Proof of Bochner's Theorem

We use the dominated convergence theorem

$$\lim_{n\to\infty}\int f_n(t)d\mu = \int \lim_{n\to\infty} f_n(t)d\mu,$$

to prove

$$\begin{aligned} f(x) &= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} (1 - \frac{|t|}{T}) e^{-itx} \phi(t) dt, \\ &= \lim_{T\to\infty} \frac{1}{2\pi} \int_0^T \int_0^T e^{-i(t-s)x} \phi(t-s) dt ds \end{aligned} \qquad \text{(L.-3)}$$

a change of variables to show the second line

of χ to show the last

$$f(x) = \lim_{T\to\infty} \frac{1}{2\pi} \int_0^T \int_0^T e^{-itx} e^{isx} \phi(t-s) dt ds \qquad \text{(L.-3)}$$

and finally a Riemann sum approximation to the integral and the positive definitness of ϕ to show that (L.4.3) is non-negative,

$$\begin{aligned} f(x) &= \lim_{T\to\infty} \frac{1}{2\pi} \int_0^T \int_0^T e^{-itx} e^{isx} \phi(t-s) dt ds \\ &= \lim_{A\to 0} \frac{A^2}{4^n} \sum_{j=1}^{2^n} \sum_{k=1}^{2^n} \phi(\frac{A(j-k)}{2^n}) e(\frac{a_j}{2^n}) e(\frac{a_k}{2^n})^* \\ &\geq 0. \end{aligned} \qquad \text{(L.-4)}$$

L.4.4 Generalization to Infinite Dimensional Spaces: Bochner-Minlos Theorem

$\lambda(i) := \lambda^i$ is replaced by $f(x)$.

Then the simplest generalization of the Bochner theorem states that a function on $\mathcal{S}(\mathbb{R}^{d+1})$ satisfies the following conditions,

(i') Normalisation: $\chi(0) = 1$
(ii') Continuity: χ is continuous in every finite dimensional subspace of $\mathcal{S}(\mathbb{R}^{d+1})$
(iii') Positivity: $\sum_{k,l=1}^m c_k \overline{c}_l \chi(f_k - f_l) \geq 0$, for all $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{C}$ and $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^{d+1})$,

if and only of the Fourier transform of a probability measure μ on $\mathcal{S}(\mathbb{R}^{d+1})$, i.e.

$$\chi(f) = \int_{\mathcal{S}(\mathbb{R}^{d+1})} d\mu(\phi) e^{i\phi(f)}. \qquad \text{(L.-4)}$$

L.4.5 Background Independent Quantization of Linear Scalar Field Theory

[78]

$$\mathcal{L} = \frac{1}{2}\phi(x)\nabla_a\nabla^a\phi(x) + \frac{m}{2}\phi^2(x) \tag{L.-4}$$

The background independent quantization of the real scalar field - polymer representation of the scalar field [?] The classical configuration space, $\mathcal{Q}$, consists of all real-valued smooth functions ϕ on Σ. Instead of loops, given a set of finite number of points $X = \{x_1, \ldots, x_N\}$ in Σ, denote Cly_X the vector space generated by finite linear combinations of the following functions of ϕ:

$$\Pi_{X,\lambda}(\phi) := \prod_{x_j \in X} \exp[i\lambda_j \phi(x_j)],$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$ are arbitrary real numbers, which play a role of labelling of loops. It is obvious that Cly of all cylindrical functions on $\mathcal{Q}$ is defined by

$$Cyl := \cup_X Cly_X, \tag{L.-4}$$

(compare to (N.-19)). Completing Cyl with respect to the sup norm

$$\|\Pi_X\| := \sup_{\phi\in\mathcal{Q}} |\Pi_{X,\lambda}(\phi)|, \tag{L.-4}$$

(compare to (L.2.9)) one obtains an Abliean C^*-algebra with unit $\overline{Cyl}$. Thus one can use the GNS structure to construct its cyclic representations. A preferred positive linear functional ω_0 on $\overline{Cyl}$ is defined by

$$\omega_0(\Pi_{X,\lambda}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \text{ for all } j \\ 0 & \text{otherwise,} \end{cases} \tag{L.-4}$$

which defines a diffeomorphism invariant faithful Borel measure on $\mathcal{Q}$ as

$$\int_{\mathcal{Q}} d\mu(\Pi_{X,\lambda}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \text{ for all } j \\ 0 & \text{otherwise,} \end{cases} \tag{L.-4}$$

Thus one obtains the Hilbert space, $\mathcal{H}^{KG}_{Kin} \equiv L^2(\overline{\mathcal{Q}}, d\mu)$, of square integrable functions on a compact topological space $\overline{\mathcal{Q}}$ with respect to μ, where $\overline{Cly}$ acts by multiplication. The quantum configuration space $\overline{\mathcal{Q}}$ is the Gel'fand spectrum of $\overline{Cyl}$.

for a single point set $X \equiv \{x_0\}$, Cyl_{x_0} is the space of all almost periodic functions on the real line $\mathbb{R}$. The Gel'fand spectrum of the corresponding C^*-algebra $\overline{Cyl}_{x_0}$ is the Bohr completion $\overline{\mathbb{R}}_{x_0}$ of $\mathbb{R}$ (see section N.-19), which is a compact topological space topological space such that $\overline{Cyl}_{x_0}$ is the C^*-algebra of all continuous functions on $\overline{R}_{x_0}$. Since $\mathbb{R}$ is densely embedded in in $\overline{R}_{x_0}$, $\overline{R}_{x_0}$ can be regarded as a completion of $\mathbb{R}$.

Given a pair (x_0, λ_0), there is an elementary configuration for the scalar field, the so-called point holonomy,

$$U(x_0, \lambda_0) := \exp[i\lambda_0 \phi(x_0)]. \tag{L.-4}$$

It corresponds to a configuration operator $\hat{U}(x_0, \lambda_0)$, which acts on any cylindrical function $\psi(\phi) \in \mathcal{H}^{KG}_{Kin}$ by

$$\hat{U}(x_0, \lambda_0)\psi(\phi) = U(x_0, \lambda_0)\psi(\phi). \tag{L.-4}$$

All these operators are unitary. But since the family of operators $\hat{U}(x_0, \lambda_0)$ fails to be weakly continuous in λ, there is no operator $\hat{\phi}(x)$ on $\mathcal{H}^{KG}_{Kin}$ (in LQC this means the Stone-von Neumann theorem and so this quantization is not unitary equivalent to the usual Schrodinger representation). The momentum functional smeared on 3-dimensional region $R \subset \Sigma$ is expressed by

$$\pi(R) := \int_R d^3x \; \tilde{\pi}(x). \tag{L.-4}$$

The Poisson bracket between the momentum functional and a point holonomy can be easily calculated to be

$$\{\pi(R), U(x, \lambda)\} = -i\lambda \; \chi_R(x) U(x, \lambda), \tag{L.-4}$$

where $\chi_R(x)$ is the characteristic function for the region R. Recall from ordinary quantum mechanics: $\hat{p}\psi(q) := i\hbar\{p, \psi(q)\} = -i\hbar d\psi(q)/dq$. So the momentum operator is defined by the action on scalar network functions $\Pi_{c=(X,\lambda)}$ as

$$\hat{\pi}(R)\Pi_c(\phi) := i\hbar\{\pi(R), \Pi_c(\phi)\} = \hbar[\sum_{x_j \in X} \lambda_j \; \chi_R(x_j)]\Pi_c(\phi). \tag{L.-4}$$

so-called scalar network functions $\Pi_c(\phi)$ that are a orthonormal basis in $\mathcal{H}^{KG}_{Kin}$, where c denotes $(X(c), \lambda)$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$ are non-zero real numbers.

Tychonov Theorem

Then the direct product space $X_\infty = \prod_{l \in \mathcal{L}} X_l$ is a compact topological space in the Tychonov topology.

The Homotopy Group

We consider the set of paths with the same start and end points. Two such paths γ_1 and γ_2 are *homotopic* if one path may be continuously deformed into the other while holding the end points fixed. $\gamma_1 \sim \gamma_2$.

We introduce the group multiplication operator: we define the product of two such paths $\alpha \cdot \beta$ as a path that goes along α, then along β. That is, if $\gamma = \alpha \cdot \beta$, then

$$\begin{aligned} \gamma(t) &= \alpha(2t) \quad 0 \leq t \leq 1/2 \\ & \quad\ \beta(2t-1) \quad 1/2 \leq t \leq 1. \end{aligned} \tag{L.-4}$$

Notice that this product is compatible with the equivalence relation just defined. If α_1 and α_2 are homotopic, $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ are homotopic, and if $\alpha_1(1) = \alpha_2(1) = \beta_1(0) = \beta_2(0)$, then $\alpha_1 \cdot \beta_1$ is homotopic to $\alpha_2 \cdot \beta_2$

The inverse of α is d efined $\alpha^{-1}(t) = \alpha(1-t)$

$$\text{i)} \quad \bar{A}(\gamma^{-1}) = (\bar{A}(\gamma))^{-1} \tag{L.-3}$$

$$\text{ii)} \quad \bar{A}(\gamma_2 \cdot \gamma_1) = \bar{A}(\gamma_2) \cdot \bar{A}(\gamma_1) \tag{L.-2}$$

$$\boxed{\text{i)} \quad \bar{A}(\gamma^{-1}) = (\bar{A}(\gamma))^{-1} \qquad \text{ii)} \quad \bar{A}(\gamma_2 \cdot \gamma_1) = \bar{A}(\gamma_2) \cdot \bar{A}(\gamma_1)} \tag{L.-2}$$

$$\boxed{\int d\mu_0[\overline{A}]\Psi_{\Gamma,f}(\overline{A}) := \int_{SU(2)^n} dg_1 \ldots dg_n f(g_1, \ldots, g_n)} \tag{L.-2}$$

connections that cannot be expressed as continuous fields on $\mathcal{M}$ but which all the same asign well defined holonomies on $\mathcal{M}$. It is called the quantum configuration space.

projective limit of the projective family of Hilbert space $\mathcal{H}_\Gamma$

Unfortunately the projective family itself does not have a largest element from which one can project to any other. However, such an element can in fact be obtained by a standard procedure called the "projective limit".

The situation is strikingly similar to ordinary quantum mechanics, where the Hilbert space of physical states is obtained by suitable completions of square integrable functions on the configuration space. In many ways, Cly is analogous to the space $C_0^\infty(\mathbb{R})$ of smooth fucntions of compact support on $\mathbb{R}^3$ which is dense in the Hilbert space of quantum mechanics. In field theory the situation is more involved. Not every physical state is a function on just the configuration space, but distributions on the time=constant hypersurface are also generically involved.

The quotient $\overline{\mathcal{A}/\mathcal{G}} := \overline{\mathcal{A}}/\overline{\mathcal{G}}$ is the gauge invariant quantum configuration space.

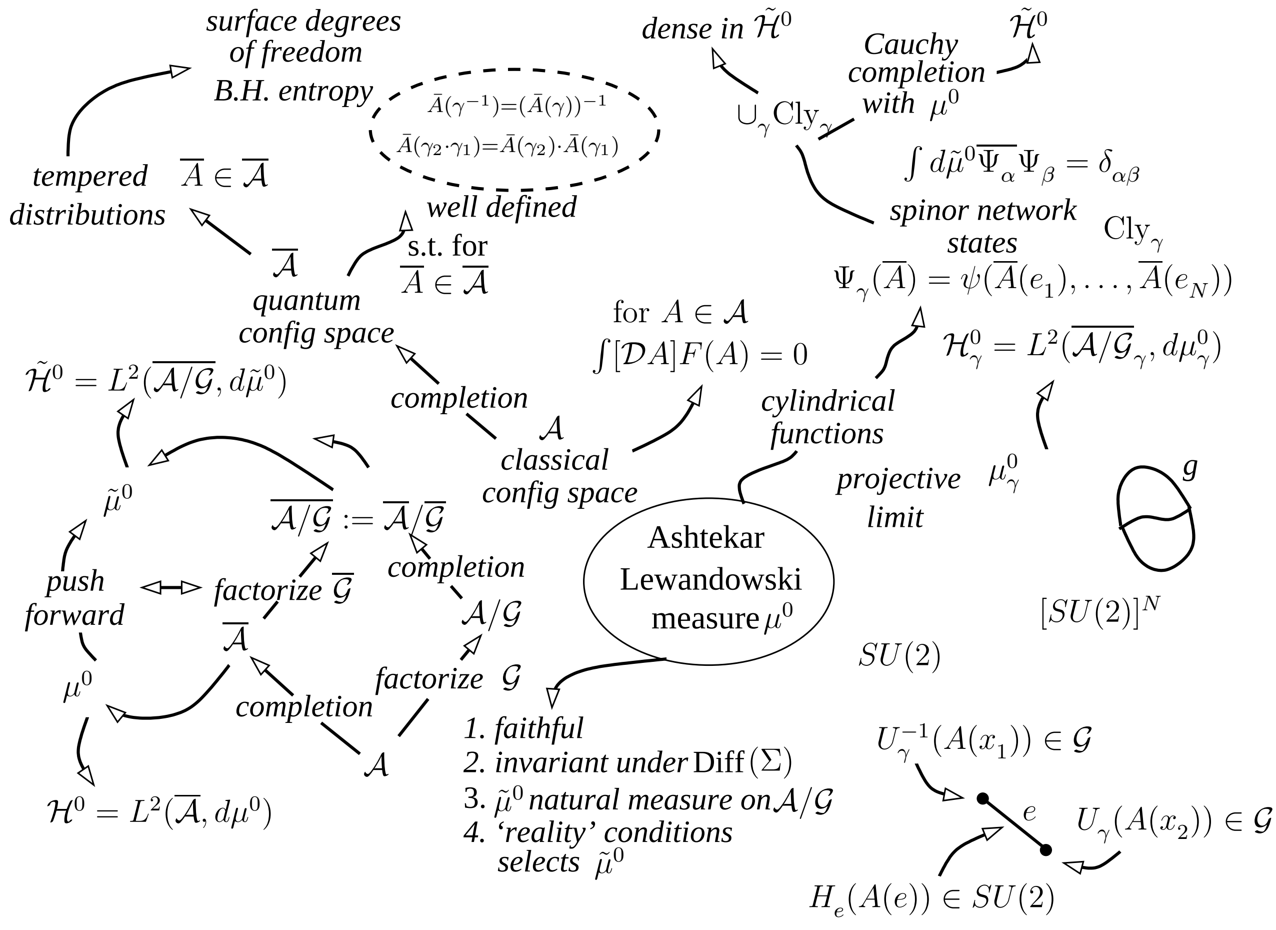

Figure L.5: Induce a measure on the projective limit. With the typical measures encountered in quantum field theory the set of smooth functions on the classical cofiguration space has measure zero. Tempered distributions not just needed but essential. Connections that cannot be expressed as continuous fields on $\mathcal{M}$ but which all the same asign well defined holonomies on $\mathcal{M}$. The quotient $\overline{\mathcal{A}/\mathcal{G}} := \overline{\mathcal{A}}/\overline{\mathcal{G}}$ is the gauge invariant quantum configuration space.

Classical configuration space $\mathcal{A}$ class of smooth functions of connections $A(x)$, such that they separate points in configuration space. When we say they separte points in configuration space, what we mean is that given the values of enough functions $F_1[A(x)], \ldots F_N[A(x)]$

we can determine uniquely $A(x)$ (??up to gauge transformations??).

Since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, it admits regular (Borel, normalized) measures and can construct Hilbert space of L^2-functions.

It turns out that $\overline{\mathcal{A}}$ admits a measure μ^0 that is preferred both mathematically and physically

as $\mathcal{A}_\gamma$ is isomorphic to $[SU(2)]^n$, the Haar measure on $SU(2)$ induces a measure μ_γ^0 on it

As we vary γ, we obtain a family of measures which turn out to be compatible and therefore induce a measure on the projective limit $\mathcal{A}$.

This measure has the nice properties (mathematically)

1. it is faithful; i.e., for any continuous, non-negative function f on $\overline{\mathcal{A}}$, $\int d\mu^0 f \geq 0$, equality holding if and only if f is identically zero.

2. it is invariant under the (induced) action of Diff[Σ], the diffeomorphism group of Σ

3. μ^0 induces a natural measure $\tilde{\mu}^0$ on $\mathcal{A}/\mathcal{G}$: $\tilde{\mu}^0$ simply the push-forward of μ^0 under the projection map that sends $\mathcal{A}$ to $\mathcal{A}/\mathcal{G}$.

(physically)

4. the classical phase space admits an (over)complete et of naturally defined configuration and momentum variables which are real, and the requirement that that the corresponding operators on the quantum Hilbert space be self-adjoint selects for us the measure $\tilde{\mu}^0$ [].

??the inner product was obtained on this set of states by requiring that the classical reality conditions be implemented as adjointness conditions on the corresponding quantum operators.??

L.5 The Space of Distributional Connections for Diffeomorphism Invariant Qunatum Gauge Theories

L.5.1 Introduction

This technical section involves category theory, covered in appendix X, and much of the maths on toplogy and measure theory needed is covered in appendix O.

Projective Limit

Ω_j is a topological Hausdorff space for every $j \in J$;

a **directed** set

J is a directed set of indexes, i.e. it is endowed with a partial order relationship $\preceq$ such that

if $i \preceq k$ $j \preceq k$ then $\pi_{ij} \circ \pi_{jk}$

if $i \preceq j$ then the maps for all $ij \in J$ there are are continuous surjective projections such that:

1. $\pi_{jj} = \mathrm{id}_{\Omega_j}$ for all $j \in J$

2. if $i \preceq j \preceq k$ then $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ (consistency relation).

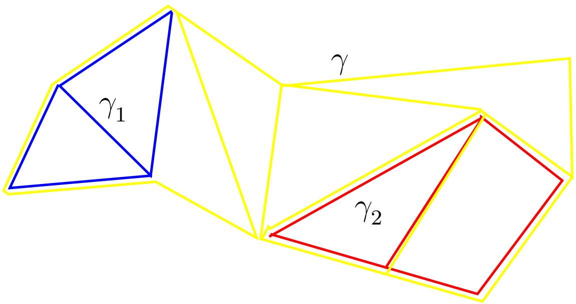

Figure L.6: direct set. The graphs $\gamma_1 \preceq \gamma_2$ if $\gamma_1 \subset \gamma_2$. Any two graphs γ_1 and γ_2 and there exists a graph γ such that $\gamma_1, \gamma_2 \subset \gamma$.

In the projective family there is, in general, no set $\overline{\chi}$ which can be regarded as the largest, from which we can project to any of the χ_S. Such a set emerges in an appropriate limit: The projective limit of Ω_j, π_{ij}, J is the subset of the cartesian product

$$\prod_{j \in J} \Omega_j \tag{L.-2}$$

given by all its wires, this space is indicated by

$$\Omega \overset{\leftarrow}{\lim} \Omega_j . \tag{L.-2}$$

The maps

$$\begin{aligned} \pi_j : \Omega &\rightarrow \Omega_j \\ \{\omega_i\}_{j\in J} &\mapsto \pi_j(\{\omega_i\}_{j\in J}) := \omega_j \end{aligned} \tag{L.-2}$$

are called the projections of Ω_j.

The projective limit Ω carries a natural topology,called initial topology , which is the smallest topology w.r.t.the projections j of are continuous.

A base of this topology is given by the sets $\prod_{j\in J} U_j$,where $U_j \in \Omega_j$ is an open set such that U j =j

L.5.2 The Label Set: Piecewise Analytic Paths

A groupoid is closed under binar operation, however, associativity, existence of identity, and inverse of each element is not required.

A groupoid is a special case of what is know as a **category** which is a general concept designed to encompass structures common in mathematics. The formal definintion of a category is the following:

Definition A category consists of a collection of **objects** $A, B, \ldots$ and **maps** between these objects. No restriction is placed on the objects, but the maps are required to satisfy the following conditions:

i) For any object A there is a map $\mathbf{1}_A : A \rightarrow A$, so that if $B \xrightarrow{f} A$ and $A \xrightarrow{g} C$ are maps in the category, the composite maps satisfy

$$g \cdot \mathbf{1}_A = g \quad \text{and} \quad \mathbf{1}_A \cdot f = f.$$

ii) If $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are maps in the category, we have associativity

$$(h \cdot g) \cdot f = h \cdot (g \cdot f).$$

One can define a map from one category to another as a pair of functions which takes objects to objects and maps to maps. Such a map $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ from category $\mathcal{C}_1$ to category $\mathcal{C}_2$ should satisfy

$$\begin{aligned} F(\mathbf{1}_A) &= \mathbf{1}_{F(A)} \\ F(g \cdot f) &= F(g) \cdot F(f). \end{aligned} \tag{L.-2}$$

Such a map from one category to another is called a **functor**.

Definition A morphism $f \in hom(x, y)$ is called an **isomorphism** provided there exists $g \in hom(y, x)$ such that

$$f \circ g = id_y$$

and

$$g \circ f = id_x$$

where id_x means the identity map from x to itself and ditto for id_y.

In other words, the maps f and g are inverses of one another. This leads to the categorical definition

Definition A category in which every morphism is an isomorphism is a **groupoid**.

Definition A **subcategory** is a category which contains a subclass of the class of objects and for each pair of objects (x, y) of the subcategory we have for the set of morphisms $hom'(x, y) \subset hom(x, y)$.

The definition of a category obviously applies to our situation with the following identifications:

Category: σ
Objects: points $x \in \sigma$.
Morphisms: paths between points $hom(x, y) := \{p \in \mathcal{P}; b(p) = x,\ f(p) = y\}$. Obviuously, every morphism is an isomorphism.
Collection of sets of morphisms: all paths $M(\sigma) = \mathcal{P}$.
Composition: composition of paths $p_{c_1} \circ p_{c_2} = p_{c_1 \circ c_2}$
Identities: $id_x = p \circ p^{-1}$ for any $p \in \mathcal{P}$ with $b(p) = x$.

L.5.3 The Topology: Tychonov Topology

For an element of $A \in \mathcal{A}$ its holonomy dependes only on p_c. To express this we use the notation

$$A(p_c) := h_c(A) \tag{L.-2}$$

We know that

$$A(p \circ p') = A(p)A(p'), \quad A(p^{-1}) = A(p)^{-1}, \tag{L.-2}$$

that is, every $A \in \mathcal{A}_p$ defines a **groupoid morphism**

(Or each A defines a functor between categories!)

Definition $Hom(\mathcal{P}, G)$ is the set of all groupoid morphisms from the set of paths in σ into the gauge group.

The set $Hom(\mathcal{P}, G)$ is larger than the classical space $\mathcal{A}$ as there are elements of $Hom(\mathcal{P}, G)$ that do not correspond to any smooth connection. We wish to equip $Hom(\mathcal{P}, G)$ with a topology, as measure theory becomes most powerful in the context of topology.

Definition The projective limit $\overline{X}$ of a projective family $(X_l, p_{ll'})_{l,l'\mathcal{L}}$ is the subset of the Cartesian product $\times_{l \in \mathcal{L}} X_l$ that satisfies certain consistency conditions:

$$\overline{X} := \{(x_l)_{l \in \mathcal{L}} \in \times_{l \in \mathcal{L}} X_l : \ l' \succeq l \Rightarrow p_{ll'} x_{l'} = x_l\}. \tag{L.-2}$$

The point of this definition is that in our application to gauge theory, this is the limit that gives us the continuum theory.

Definition Given a graph γ we denote by $l(\gamma)$ the subgroupoid generated by γ with $V(\gamma)$ as the set of objects and with the $e \in E(\gamma)$ together with their inverses and finite compositions as the set of homomorphisms.

The labels $\omega, 0$ in Γ_0^ω stand for "analytic" and "of compact support" respectively.

Definition The **tame subgroupoids** $l(\gamma)$ of $\mathcal{P}$ are those determined by graphs $\gamma \in \Gamma_0^\omega$.

Theorem L.5.1 *Let $\mathcal{L}$ be the set of all tame subgroupoids $l(\gamma)$ of $\mathcal{P}$. Then the relation $l \prec l'$ if and only if l is a subgroupoid of l' equips $\mathcal{L}$ with the structure of a partially ordered and directed set.*

Proof: Since l is a subgroupoid of l' if and only if all objects of l are objects of l' and all morphisms of l are morphisms of l' it is clear that $\prec$ defines a partial order. To see that $\mathcal{L}$ is a directed consider any two graphs $\gamma, \gamma' \in \Gamma_0^\omega$ and consider $\gamma'' := \gamma \cup \gamma'$. We must show that γ'' itself is an element of Γ_0^ω, that is, it has a finite number of edges.

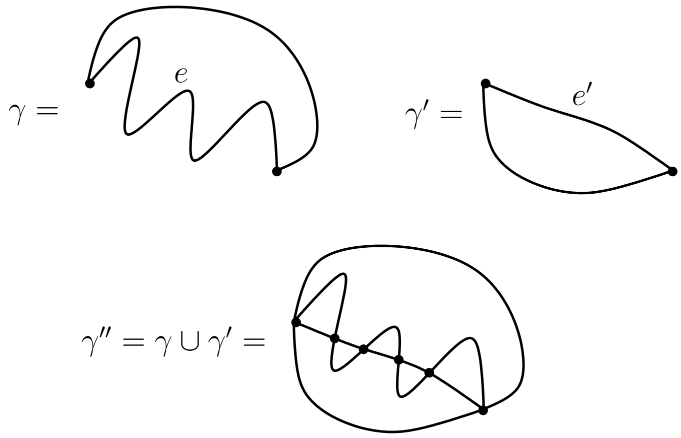

Figure L.7: The union of the two graphs γ and γ' has a finite number of edges. If the edge e were allowed to "oscillate" arbitarily rapidly the union of the two graphs would have an inifinte number of edges.

Although this seems intuiatively obvious, see fig. (Q.7), this is not so for paths of arbitrary differentiabilty; smooth curves can intersect in Cantor sets and thus define graphs which have an infinte number of edges. We prove, however, that it is not the case for peicewise analytic paths.

To prove this it is sufficient to show that any two edges $e, e' \in \mathcal{P}$ can only have a finite number of isolated intersections or $e \cap e'$ has a common finite segment. To prove this suppose then that $e \cap e'$ is an infinite discrete set of points. We may choose parameterizations of their representatives c, c' such that each of its component functions

$$f(t)^a := e'(t)^a - e(t)^a$$

vanishes in at least a countably infinite number of points t_m, $m = 1, 2, \ldots$. We show that for any function $f(t)$ which is real analytic in $[0, 1]$ implies $f = 0$. Since $[0, 1]$ is compact there is an accumulation point $t_0 \in [0, 1]$ of the t_m (here the compact support of the $c \in \mathcal{C}$ comes into play) and we assume that t_m converges to t_0. Since f is analytic we can write the absolutely convergent Taylor series

$$f(t) = \sum_{n=0}^{\infty} f_n (t - t_0)^n$$

(here the analyticity comes into play). We show that $f_n = 0$ by induction over $n = 0, 1, \ldots$. First we establish that $f_0 = 0$,

$$f_0 = f(t_0) = \lim_{m \to \infty} f(t_m) = \lim_{m \to \infty} 0 = 0.$$

Now suppose $f_1 = \cdots = f_n = 0$, we show that $f_{n+1} = 0$. Under this assumption we have

$$f(t) = f_{n+1}(t - t_0)^{n+1} + r_{n+1}(t)(t - t_0)^{n+2}$$

where $r_{n+1}(t)$ is uniformally bounded in $[0, 1]$, that is, there exists a number K such that $|r_{n+1}(t)| \leq K$ for all $x \in [0, 1]$. Thus

$$0 = f(t_m)/(t_m - t_0)^{n+1} = f_{n+1} + r_{n+1}(t_m)(t_m - t_0)$$

for all m, hence

$$f_{n+1} = \lim_{m\to\infty} [f_{n+1} + r_{n+1}(t_m)(t_m - t_0)] = 0.$$

□

Now that we have a partially ordered and directed index set $\mathcal{L}$ we must specify a projective family.

Definition For any $l \in \mathcal{L}$ define $X_l := Hom(l, G)$ the set of all homomorphisms from the subgroupoid l to G.

Definition For $l \prec l'$ define a projection by

$$p_{l'l} : X_{l'} \to X_l; \quad x_{l'} \mapsto (x_{l'})_l \tag{L.-2}$$

restriction of the homomorphism $x_{l'}$ defined on the groupoid l' to its subgroupid $l \prec l'$.

Lemma L.5.2 *The projections $p_{l'l}$, $l \prec l'$ are surjective, moreover, they are continuous.*

Proof:

□

The direct product X_∞ is compact by Tychonov's theorem. From section J.10.6 we have that the projective limit $\overline{X}$ is also a compact Hausdorff space in the subspace topology.

Let us collect these results in the following theorem.

Theorem L.5.3 *The projective limit $\overline{X}$ of the spaces $X_l = Hom(l, G)$, $l \in \mathcal{L}$ where $\mathcal{L}$ denotes the set of all tame subgroupoids of $\mathcal{P}$ is a compact Hausdorff space in the induced Tychonov topology whenever G is a compact Hausdorff space.*

Theorem L.5.4 *The map*

$$\Phi : Hom(\mathcal{P}, G) \to \overline{X}; \quad H \mapsto (H_l)_{l \in \mathcal{L}}$$

is a bijection.

Proof:

$\square$

L.6 The C^* Algebraic Viewpoint

A basic result in the Gel'fand-Naimark representation theory assures us that every Abelian C^*-algebra $\overline{\mathcal{C}}$ with identity is realized as the C^*-algebra of continuous functions on a compact Hausdorff space, called the spectrum of $\overline{\mathcal{C}}$.

the spaces can be seen as the Gel'fand spectra of certain C^*-algebras, as such, we make contact with so-called cylindrical functions on these spaces explicit which helps to construct measures on them.

Suppose that we are given a partially ordered and directed index set $\mathcal{L}$ which label compact Hausdorff spaces X_l and that we have surjective and continuous projections $p_{l'l} : X_{l'} \to X_l$ for $l \prec l'$ satisfying the consistency condition $p_{l'l} \circ p_{l''l'} = p_{l''l}$ for $l \prec l' \prec l''$. Let $X_\infty, \overline{X}$ be the corresponding direct product and projective limit respectively with Tychonov topology with respect to which we know that they are Hausdorff and compact from the previous sections.

Definition Let $C(X_l)$ be the continuous, complex valued functions on X_l and consider their union

$$Cyl'(\overline{X}) := \cup_{l \in \mathcal{L}} C(X_l). \tag{L.-2}$$

Let us define the following equivalence relation. Given $f_{l_1} \in C(X_{l_1})$ and $f_{l_2} \in C(X_{l_2})$ we will say f_{l_1} and f_{l_2} are equivalent, denoted $f_{l_1} \sim f_{l_2}$ if

$$p^*_{l_3 l_1} f_1 = p^*_{l_3 l_2} f_2 \tag{L.-2}$$

for all $l_1,\ l_2 \prec l_3$, where $p^*_{l_3 l_1}$ denotes the pull-back map from the space of functions on X_{l_1} to the space of functions on X_{l_3}.

The space of cylindrical functions on the projective limit $\overline{X}$ is defined to be the space of equivalence classes

$$Cyl(\overline{X}) := Cyl'(\overline{X})/\sim \tag{L.-2}$$

We will denote the equivalence class of $f \in Cyl'(\overline{X})$ by $[f]_\sim$. The quotient just gets rid of a redundancy: pull-backs of functions from a smaller set to a larger set are now identified with the functions on the smaller set.

Note to check condition (L.6) it is sufficient to find just one single l_3. For suppose that $f_{l_1} \in C(X_{l_1}), f_{l_2} \in C(X_{l_2})$ are given and that we find some $l_1, l_2 \prec l_3$ such that $p^*_{l_3l_1} f_1 = p^*_{l_3l_2} f_2$. Now let any $l_1, l_2 \prec l_4$ be given. Since $\mathcal{L}$ is a directed set we find l_5 such that $l_1, l_2, l_3, l_4 \prec l_5$ and due to the consistency condition among projections we have

$$\begin{array}{ll} i) & p_{l_4l_1} \circ p_{l_5l_4} = p_{l_5l_1} = p_{l_3l_1} \circ p_{l_5l_3} \\ ii) & p_{l_4l_2} \circ p_{l_5l_4} = p_{l_5l_2} = p_{l_3l_2} \circ p_{l_5l_3} \end{array} \tag{L.-2}$$

from which follows

$$\begin{array}{ll} i) & p^*_{l_5l_4} p^*_{l_4l_1} f_{l_1} = p^*_{l_5l_3} p^*_{l_3l_1} f_{l_1} \\ ii) & p^*_{l_5l_3} p^*_{l_3l_2} f_{l_2} = p^*_{l_5l_4} p^*_{l_4l_2} f_{l_2}. \end{array} \tag{L.-2}$$

Equality of i) and ii) in (L.-2) follows from using (L.6), we conclude

$$p^*_{l_5l_4}[p^*_{l_4l_1} f_{l_1} - p^*_{l_4l_2} f_{l_2}] = p^*_{l_5l_4} g_{l_4} = 0. \tag{L.-2}$$

where $g_{l_4} := p^*_{l_4l_1} f_{l_1} - p^*_{l_4l_2} f_{l_2}$. Now for any $f_{l_4} \in C(X_{l_4})$ the condition $f_{l_4}(p_{l_5l_4}(x_{l_5})) = 0$ for all $x_{l_5} \in X_{l_5}$ means that $f_{l_4} = 0$ because $p_{l_5l_4} : X_{l_5} \to X_{l_4}$ is surjective (onto), therfore

$$p^*_{l_4l_1} f_{l_1} = p^*_{l_4l_2} f_{l_2}.$$

Lemma L.6.1

Lemma L.6.2 *Let $f, f' \in Cly(\overline{X})$ then the following operations are well defined (independent of the representatives)*

i)

$$
\begin{aligned}
f+f' &:= [f_l+f'_l]_\sim && \text{(L.-1)}\\
ff' &:= [f_l f'_l]_\sim && \text{(L.0)}\\
zf &:= [zf_l]_\sim && \text{(L.1)}\\
\overline{f} &:= [\overline{f}_l]_\sim && \text{(L.2)}
\end{aligned}
$$

where l, f_l, f'_l are as in the previous lemma, $z \in \mathbb{C}$ and $\overline{f}_l$ denotes complex conjugation.

ii)

$Cyl(\overline{X})$ contains the constant functions.

iii)

The sup-norm for $f = [f_l]_\sim$

$$\|f\| := \sup_{x_1 \in X_1} |f_l(x_l)| \tag{L.2}$$

is well defined.

Proof:

i)

We condiser only pointwise multiplication, the other cases are similar. Let l, f_{l_1}, f'_{l_1} and l', f_{l_2}, f'_{l_2} as in lemma L.6.1. We find $l_1, l_2 \prec l_3$ and have $p^*_{l_3 l_1} f_{l_1} = p^*_{l_3 l_2} f_{l_2}$ and $p^*_{l_3 l_1} f'_{l_1} = p^*_{l_3 l_2} f'_{l_2}$. Thus

$$p^*_{l_3 l_1}(f_{l_1} f'_{l_1}) = p^*_{l_3 l_1}(f_{l_1}) p^*_{l_3 l_1}(f'_{l_1}) = p^*_{l_3 l_2}(f_{l_2}) p^*_{l_3 l_2}(f'_{l_2}) = p^*_{l_3 l_2}(f_{l_2} f'_{l_2}) \tag{L.2}$$

thus $f_{l_1} f_{l_1} \sim f_{l_2} f'_{l_2}$.

ii)

iii)

If $[f_{l_1}]_\sim = [f_{l_2}]_\sim$ is given, choose any $l_1, l_2 \prec l_3$ so that we know that $p^*_{l_3 l_1} f_{l_1} = p^*_{l_3 l_2} f_{l_2}$. Then from the surjectivity of $p^*_{l_3 l_1}, p^*_{l_3 l_2}$ we have

$$\sup_{x_{l_1} \in X_{l_1}} |f_{l_1}(x_{l_1})| = \sup_{x_{l_3} \in X_{l_3}} |(p^*_{l_3 l_1} f_{l_1})(x_{l_3})| = \sup_{x_{l_3} \in X_{l_3}} |(p^*_{l_3 l_2} f_{l_2})(x_{l_3})| = \sup_{x_{l'_1} \in X_{l'_1}} |f_{l'_1}(x_{l_1})| \tag{L.2}$$

$\square$

Recall that a norm induces a metric on a liner space via $d(f,f') := \|f-f'\|$ and that metric space is complete if all its Cauchy sequences converge to an element of the space. Any metric space can be uniquely (up to an distance preserving mapping carrying every point of the original metric space into itself, theorem J.4.4) embedded into a complete metric space extending it by its non-converging Cauchy sequences. The original metric space is dense in the extended one. We can complete $Cyl(\overline{X})$ in the norm $\|.\|$ and obtain an Abelean, unital Banach $^*-$algebra

$$\overline{Cyl(\overline{X})}$$

But notice that not only the submulitiplicativity of the norm ($\|ff'\| \leq \|f\|\|f'\|$) holds but in fact the C^* property $\|f\overline{f}\| = \|f\|^2$. Thus $\overline{Cyl(\overline{X})}$ **is in fact a unital, Abelean C$^*-$algebra**.

Denote by $\Delta(Cyl(\overline{X}))$ the **spectrum** of $Cyl(\overline{X})$, that is, the set of *all* (algebraic, i.e. not necessariliy continuous) homomorphisms from $Cyl(\overline{X})$ to the complex numbers and denote the Gel'fand isometric isomorphism by

$$\bigvee : \overline{Cyl(\overline{X})} \to C(\Delta(\overline{Cyl(\overline{X})})); \qquad f \mapsto \overset{\vee}{f} \quad \text{where} \quad \overset{\vee}{f}(\chi) := \chi(f) \tag{L.2}$$

where the space of continuous functions on the spectrum is equipped with the sup-norm.

....

It follows that $\chi(x)$ is a continuous linear (and therefore bounded) map from the normed linear space $Cyl(\overline{X})$ to the complete, normeed linear space $\mathbb{C}$. Hence, by the bounded linear transformation theorem (theorem J.4.6) each $\chi(x)$ can be uniquely extended to a bounded linear transformation (with the same bound) from the completetion $\overline{Cyl(\overline{X})}$ of $Cyl(\overline{X})$ to $\mathbb{C}$

....

Theorem L.6.3 *The map χ in ... is a homeomorphism.*

Proof:

Injectivity (one-to-oneness):

For χ to be one-to-one we must have $\chi(x) \neq \chi(x')$ whenever $x \neq x'$. Suppose then that $\chi(x) = \chi(x')$. In particular $[\chi(x)](f) = [\chi(x')](f)$ for any $f \in Cyl(\overline{X})$. Hence $f_l(x_l) = f_l(x'_l)$ for any $f_l \in C(X_l)$, $l \in \mathcal{L}$. Since X_l is a compact Hausdorff space, $C(X_l)$

separates points of X_l by the Stone-Weierstrass theorem (theorem J.11.4), hence $x_l = x'_l$ for all $l \in \mathcal{L}$. It follows that $x = x'$.

Surjectivity (onto):

Let $\chi \in Hom(\overline{Cyl(\overline{X})}, \mathbb{C})$ be given.

Continuity:

We have established that χ is a bijection. We must show that both, χ, χ^{-1} are continuous.

The topology on $\Delta(\overline{Cyl(\overline{X})})$ is the weakest topology such that the Gel'fand transforms $\overset{\vee}{f}, f \in \overline{Cyl(\overline{X})}$ are continuous while the toplogy on $\overline{X}$ is the weakest topology such that all projections p_l are continuous,

Recall a mapping f from one topological space X to another Y is continuous if and only if whenever $(x_\alpha)_I$ is a net convergent to x then the net $(f(x_\alpha))_I$ converges to $f(x)$.

Continuity of χ:

Let (x_α) be a net in $\overline{X}$ converging to x, that is, every net (x_l^α) converges to x_l.

hence $\chi(x^\alpha) \to \chi(x)$ in the Gel'fand topology.

Continuity of χ^{-1}:

Let (χ^α) be a net in $\Delta(\overline{Cyl(\overline{X})})$ converging to χ,

Hence $\chi^{-1}(\chi_\alpha) \to \chi^{-1}(\chi)$ in the Tychonov topology.

$\square$

....

Corollary L.6.4 *The closure of the space of cylindrical functions $\overline{Cyl(\overline{X})}$ may be identified with the space of continuous functions $C(\overline{X})$ on the projective limit $\overline{X}$.*

Proof:

$\square$

L.6.1 The spaces $\overline{X}/G$ and $\overline{X/G}$ are Homeomorphic

L.7 Regular Borel Measures on the Projective Limit: The Uniform Measure

Our spaces X_l are compact Hausdorff spaces and in particular topological spaces and are therefore naturally equipped with $\sigma-$algebra $\mathcal{B}_l$ of Borel sets (the smallest $\sigma-$algebra containing all open (equivalently closed) subsets of X_l).

regularity means that the measure of every measurable set can be approximately well by open and compact sets (hence closed since X_l is compact Hausdorff by lemma J.9.6)

Definition A family of measures $(\mu_l)_{l\in\mathcal{L}}$ on the projections X_l of a projection family $(X_l, p_{ll'})_{l\prec l'}$ where the $p_{l'l} : X_{l'} \to X_l$ are continuous and surjective projections is said to be consitent provided that

$$(p_{l'l})_* \mu_{l'} := \mu_{l'} \circ p_{l'l}^{-1} = \mu_l \tag{L.2}$$

for any $l \prec l'$. The measure $(p_{l'l})_* \mu_{l'}$ is called the push-forward of the measure $\mu_{l'}$.

....

Definition The Hilbert space $\mathcal{H}^0$ is defined as the space of square integrable functions over $\overline{\mathcal{A}}$ with respect to the uniform measure μ_0, that is

$$\mathcal{H}^0 := L_2(\overline{\mathcal{A}}, d\mu_0). \tag{L.2}$$

L.8 Operators

The specification of the topology in which the limit is taken is an integral part of the definition of the operator.

For limits in the involved in the regularization of quantum field theoretical operators, the limit cannot be taken in the Hilbert space topology where, in general, it does not exist. The limit must be taken in the topology that "remembers" the topology in which the corresponding limit is taken.

We say a sequence of qunatum states Ψ_n converges to a state Ψ if $\Psi_n[A]$ converges to $\Psi[A]$ for all smooth connections A. We define a domain M as the set for which $\{\Psi_n\} \subset M$, $\Psi_n \to \Psi$ implies that $\Psi \in M$. We use the corresponding operator topology: $O_n \to O$ if $O_n \Psi \to TO$ for all Ψ in the domain.

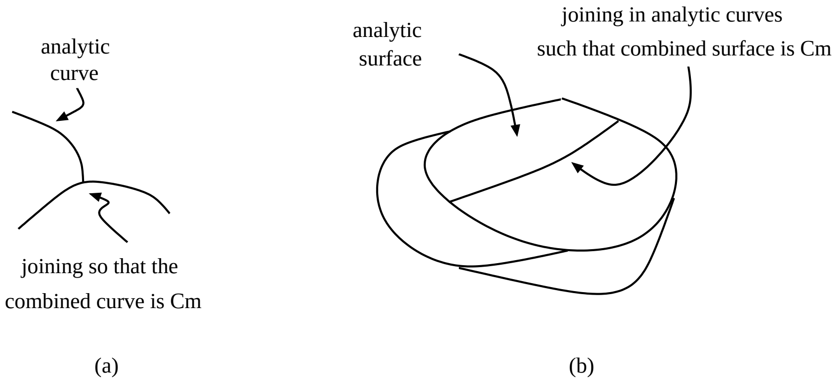

Figure L.8: (a) A semianalytic curve in $\mathbb{R}^3$. (b) A semianalytic surface in $\mathbb{R}^3$.

L.9 Functional Calculus on a Projective Limit

Functions

Differential Forms

Volume Forms

Vector Fields

Lie Brackets

Vector Field Divergences

L.10 Density and Support Properties of $\mathcal{A}, \mathcal{A}/\mathcal{G}$ with respect to $\overline{\mathcal{A}}, \overline{\mathcal{A}/\mathcal{G}}$

In this section we will see that $\mathcal{A}$ lies topologically dense, but measure theoretically thin in $\overline{\mathcal{A}}$ (similar results apply to $\mathcal{A}/\mathcal{G}$ with respest to $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$) with respect to the uniform measure μ_0.

We have seen that every element of $A \in \mathcal{A}$ defines an element of $Hom(\mathcal{P}, G)$ and that this space can be identified with the projective limit $\overline{X} \equiv \overline{\mathcal{A}}$. Now via the C^*-algebraic framework we know that $\overline{Cyl(\overline{X})}$ can be identified with $C(\overline{X})$ and the latter space of functions separates the points of $\overline{X}$ by the **Stone-Weierstrass theorem** (thereom J.11.4) since it is Hausdorff and compact.

Now does the set of functions $Cyl(\overline{X})$ separates the set of points $\mathcal{A}$. Let $A \neq A'$ be given then there exists a point $x \in \sigma$ such that $A(x) \neq A'(x)$.

We therefore we have that a collection $\mathcal{C} = Cyl(\overline{X})$ of bounded complex valued functions on a set $X = \mathcal{A}$ including the constants which separates the points of X. The following result is an abstract property of Abelean unit C^*-algebras (in our case, $\overline{\mathcal{C}} = \overline{Cyl(\overline{\mathcal{A}})}$ and $\overline{X} = \overline{\mathcal{A}}$).

Theorem L.10.1 *Let $\mathcal{C}$ be a collection of real-valued, bounded functions on a set X which contains the constants and separates points of X. Let $\overline{\mathcal{C}}$ be the Abelean, unital C^*-algebra generated from $\mathcal{C}$ by pointwise addition, multiplication, scalar multiplication and complex conjugation, completed in the sup norm. Then the image of X under its natural embedding into teh Gel'fand spectrum $\overline{X}$ of $\overline{\mathcal{C}}$ is dense with respect to the Gel'fand topology on the spectrum.*

Proof:

.....

Let $\overline{J(X)}$ be the closure of $J(X)$ in the Gel'fand topology on $\overline{X}$ of pointwise convergence on $\overline{\mathcal{C}}$. Suppose that $\overline{X} - \overline{J(X)} \neq \emptyset$ and take $\chi \in \overline{X} - \overline{J(X)}$. Since $\overline{X}$ is a compact Hausdorff space we find $a \in C(\overline{X})$ such that $1 = a(\chi) \neq a(J_x) = 0$ for any $x \in X$ by Urysohn's lemma, lemma J.9.10. (Urysohn's lemma applies to normal spaces. Compact Hausdorff spaces are normal spaces (theorem J.9.8). In Hausdorff spaces one point sets are closed, hence $\{x\}$ and $\overline{J(X)}$ are disjoint closed sets).

Since the Gel'fand map $\bigvee : \overline{\mathcal{C}} \to C(\overline{X})$ is an isomorphic we find $f \in \overline{\mathcal{C}}$ such that $\overset{\vee}{f} = a$. Hence

$$0 = a(J_x) = \overset{\vee}{f}(J_x) = J_x(f) = f(x)$$

for all $x \in X$, hence $f = 0$, thus $a \equiv 0$ contracdicting $a(\chi) = 1$. Therefore χ infact does not exist whence $\overline{X} = \overline{J(X)}$.

□

That is, $\mathcal{A}$ is topologically dense in $\overline{\mathcal{A}}$.

L.11 Uniqueness Theorem for the Ashtekar-Lewandowski Representation

Fleischhack

Every physical theory requires fundamental mathematical assumptions at the very beginning. It is highly desirable to justify them by even more fundamental axioms that are both mathematically and physically as plausible as possible.

This measure is "natural", since the Haar measure on a Lie group is "natural" as well. However, this is at most a mathematical statement or a statement of beauty. The deeper question behind is how one can justify this choice by mathematical physics arguments.

representation theory - diffeomorphism invariance.

physical selection, one case is, a unitary representation of the spacial diffeormorphism group (rather projective representation thereof as no representation of the infintesimal constraints cannot be well defined [96]). Remrkably it has been possible to show that such a representaion is unique '[6]' . More precisely, in general ..... taken from [214]

Quantum Geometry: Representation on $L_2(\overline{\mathcal{A}}/\overline{\mathcal{G}}, \mu_0)$ is unique if

1. diffeomorphism invariant;

2. semianalytic.;

A natural idea is to first look at irreducible or at least cyclic representations as the simple building blocks, out of which more complicated representations could eventually be built.

A simple formulation of these properties can be given by asking for a state (i.e. a positive, normalized, linear functional) on $\mathcal{U}$ that it is invariant under the classical symmetry automorphisms of $\mathcal{U}$. Given a state ω on $\mathcal{U}$ one can define a representation via the GNS construction. This representation will be cyclic by construction, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$,. If the state is invariant under some automorphism of $\mathcal{U}$, its action is automatically unitarily implemented in the representation.

Let G be a group of automorphisms of the $C*-$algebra $\mathcal{O}$ and ω a corresponding $G-$invariant state on $\mathcal{O}$. Then there is a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$

$$\pi_\omega(gA) = U_\omega(g)\pi_\omega(A)U_\omega(g)^{-1}, \quad U_\omega(g)\Omega_\omega = \Omega_\omega, \tag{L.2}$$

for all $g \in G$ and $A \in \mathcal{O}$.

Briefly, 'semianalytic' means 'piecewise analytic'. For example, a semianalytic sub-manifold would be analytic except for on some lower dimensional sub-manifolds, which in turn have to be piecewise analytic. We have already met the idea of semianalycity, see fig (L.9) (a). To convey the general idea, fig (L.9) (b) depicts a semi-analytic surface in $\mathbb{R}^3$.

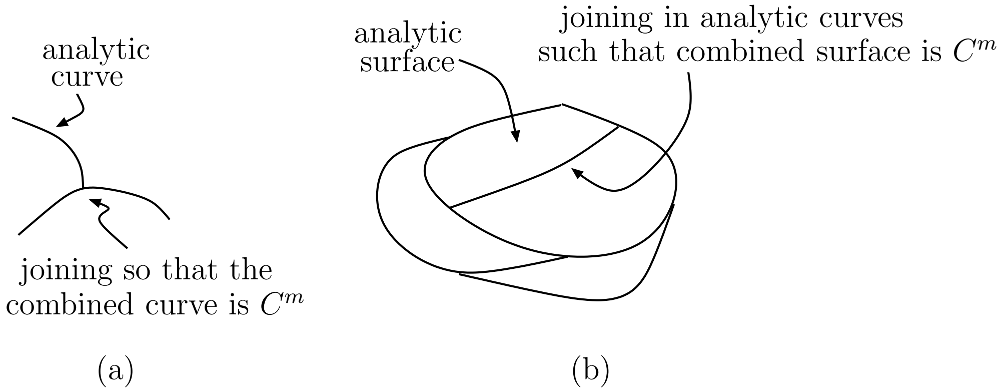

Figure L.9: (a) A semianalytic curve in $\mathbb{R}^3$. (b) A semianalytic surface in $\mathbb{R}^3$.

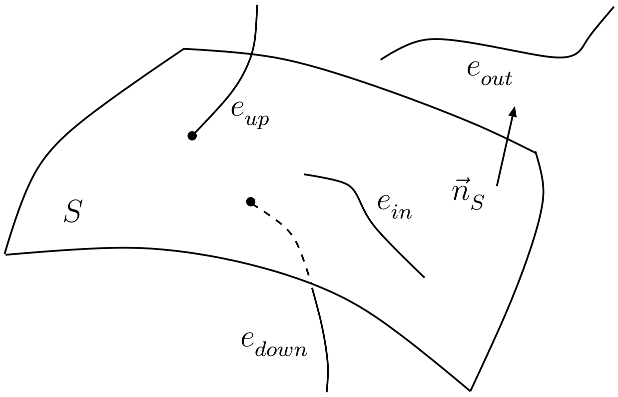

Figure L.10: Types of edges with respect to a face.

L.11.1 Flux Operators

The left invariant vector field in the $i-$th internal direction on the copy of G corresponds to the e-th edge

$$L_e^i \cdot \psi(h_{e_1}, \ldots, h_{e_N}) = (h_e \tau^i)_B^A \frac{\partial \psi}{\partial (h_e)_B^A} = \left(\frac{d}{dt}\right)_{t=0} \psi(h_{e_1}, \ldots, h_e e^{t\tau_i}, \ldots, h_{e_N})$$

$$R_e^i \cdot \psi(h_{e_1}, \ldots, h_{e_N}) = (\tau^i h_e)_B^A \frac{\partial \psi}{\partial (h_e)_B^A} = \left(\frac{d}{dt}\right)_{t=0} \psi(h_{e_1}, \ldots, e^{t\tau_i} h_e, \ldots, h_{e_N})$$

$$X_{S,n}[f] := \frac{1}{2} \sum_{p \in S \cap \gamma} \sum_{e_p} \sigma(e_p, S) n_i(p) X_{e_p}^i [f],$$

where the second sum is over the edges of γ adjacent to p,

$$\sigma(e_p, S) = \begin{cases} 1 & \text{if } e_p \text{ lies above } S \\ 0 & \text{if } e_p \cap S = \emptyset \text{ or } e_p \cap S = e_p \\ -1 & \text{if } e_p \text{ lies below } S \end{cases}$$

and $X^i_{e_p}$ is the *ith* left-invariant (right-invariant) vector field on $SU(2)$ acting on the argument of f corresponding to the holonomy h_{e_p} if e_p is pointing away from (towards) S.

Definition

$$W^n_t(S) := e^{t\beta\ell_p^2/2Y_n(S)} \tag{L.2}$$

$$e^L M e^{-L} = \sum_{n=0}^{\infty} \frac{1}{n!}[L, M]_n$$

$$\begin{aligned} W_\gamma(t_\gamma) f W_\gamma(t_\gamma)^{-1} &= e^{Y_\gamma(t_\gamma)} f e^{-Y_\gamma(t_\gamma)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}[Y_\gamma(t_\gamma), f]_{(n)} \\ &= f + [Y_\gamma(t_\gamma), f] + \frac{1}{2!}[Y_\gamma(t_\gamma), [Y_\gamma(t_\gamma), f]] + \end{aligned} \tag{L.0}$$

$$\begin{aligned} W^n_t(S) W^{n'}_{t'}(S') (W^n_t(S))^{-1} &= e^{t\beta\ell_p^2/2Y_n(S)} \big(\sum_{m'}^{\infty} \frac{(t'\beta\ell_p^2/2)^{m'}}{m'!} (Y^{n'}_{t'}(S'))^{m'} \big) e^{-t\beta\ell_p^2/2Y_n(S)} \\ &= \sum_{m'}^{\infty} \frac{(t'\beta\ell_p^2/2)^{m'}}{m'!} \sum_{m=0}^{\infty} \frac{(t\beta\ell_p^2/2)^m}{m!} [Y_n(S), (Y^{n'}_{t'}(S'))^{m'}]_{(m)} \\ &= \end{aligned} \tag{L.-2}$$

L.11.2 Algebra of Cylindrical Functions and Space of Generalised Connections

As we have seen there are several complementary characterisations of the Kinematic Hilbert space.

C^*-algebraic characterisation.

Cyl^∞ is an algebra: We know that if Ψ is compatible with γ and if $\gamma' \geq \gamma$, then Ψ is compatible with γ' as well. First note that every element Ψ of the space of finite linear coombination of smooth cylindrical functions $\{\Psi_i\}$ $(i = 1, \ldots, k)$ compatible, respectively, with graphs $\{\gamma_i\}$. There exists a graph γ such that $\gamma \geq \gamma_i$ for each i and so every function Ψ_i is compatible with γ. Hence Ψ is a smooth cylindrical function compatible with γ also. Now, given elements Ψ and Ψ' of Cyl^∞ we can find a graph γ' such that the two functions are compatible with it. Then $\Psi\Psi'$ is a smooth cylindrical function compatible with γ', thus an element of Cyl^∞.

The completion $\overline{\mathrm{Cyl}}$ of Cyl with respect to the sup norm $\|f\| := \sup_{A\in\mathcal{A}} |f(A)|$ defines an Abelean C^*-algebra. Define the space of generalsed connections $\overline{\mathcal{A}}$ as its Gel'fand spectrum $\Delta(\overline{\mathrm{Cyl}})$. By the Gel'fand isomorphism we can think of Cyl^∞ as the space $C(\mathcal{A})$ of continuous functions on the spectrum. The spectrum of an Abelean C^*-algebra is a compact Hausdorff space if equipped with the Gel'fand topology of pointwise convergence of nets. Hence, by the Riesz-Markov theorem the postive linear functional ω is in one to one correspondence with a regular Borel measure μ on $\overline{\mathcal{A}}$. The Hilbert space $\mathcal{H} := L_2(\overline{\mathcal{A}}, d\mu)$ is the space of square integrable functions on $\overline{\mathcal{A}}$ with respect to that measure.

L.11.3 Generalized Vector Fields Tangent to $\overline{\mathcal{A}}$

Definition The momentum variable space defined by a given space of smearing functions $\mathcal{F}$ is the real vector space spanned by the linear maps $\pi(f)$ such that $f \in \mathcal{F}$.

□

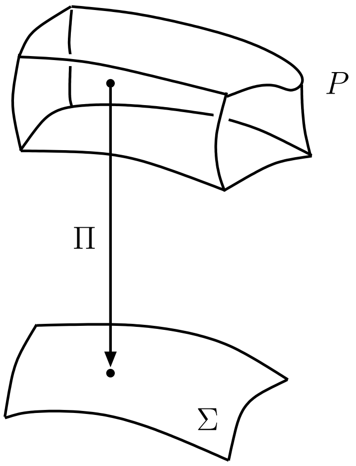

Figure L.11: Automorphisms of the bundle P implies action of the bundle automorphism group in the qunatum $*-$algebra .

Let S be a face. Consider the bundle

$$P_S := \Pi^{-1}(S) \subset P \tag{L.-2}$$

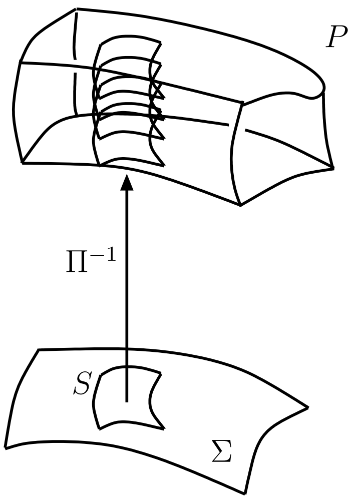

Figure L.12: .

Given a face S, a smearing

L.11.4 The Quantum $*-$algebra

$$
\begin{aligned}
(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_m) &= (a_1, \ldots, a_n, b_1, \ldots, b_m) && \text{(L.-1)} \\
(a_1, \ldots, a_n)^* &= (\overline{a_n}, \ldots, \overline{a_1}). && \text{(L.0)}
\end{aligned}
$$

L.11.5 Symmetries of $\mathfrak{A}$

The group of semianalytic automorphisms of the principal fiber bundle P act naturally in the space $\mathcal{A}$ of connections.

L.11.6 Implementation of Piecewise Analytic Diffeomorphisms on $\mathcal{H}_{kin}$

It is straightforward to implement the action of piecewise analytic diffeomorphisms on $\mathcal{H}_{kin}$: This Hilbert space consists of functions $f : \mathcal{A} \to \mathbb{C}$, which are cylindrical over some graph γ. The space of quantum configurations $\mathcal{A}$, i.e. the space of (distributional) connections on Σ carries a natural action of the diffeomorphism group Diff Σ. An element

$\phi \in \text{Diff}\,\Sigma$ simply acts by $A \to \phi^* A$ on a (distributional) connection A. With this, one can simplt define the action of Diff Σ on $\mathcal{H}_{kin}$ by

$$\alpha_\phi f(A) := f(\phi^* A)$$

where $\phi^* A$ is the pullback of the connection A under the diffeomorphism ϕ. Note that this defintion maps

$$\alpha_\phi \mathcal{H}_\gamma \to \mathcal{H}_{\phi(\gamma)}. \tag{L.0}$$

Let us discuss the action of the automorphisms/diffeomorphisms. One can build out of a map $\varphi : \Sigma \to \Sigma$ an induced map $B\varphi : P \to P$ of bundles by combining into one object all the maps of fibres induced by φ: that is, for each $v \in P$

$$B\varphi(v) = \varphi_{*\Pi(v)} v.$$

It cam be shown that

$$\Pi \circ B\varphi = \varphi \circ \Pi$$

For every bundle automorphism

$$\tilde{\varphi} : P \to P \tag{L.0}$$

there is a unique diffeomorphism

$$\varphi : \Sigma \to \Sigma \tag{L.0}$$

such that

$$\Pi \circ \tilde{\varphi} = \varphi \circ \Pi. \tag{L.0}$$

In our case both of them are semianalytic.

Definition Preservation of a fibre. The map $\tilde{\varphi}$ respects the bundle structure of P in the sense that if v and v' belong to the same fibre of P then their images $\tilde{\varphi}(v)$ and $\tilde{\varphi}(v')$ belong to the same fibre of P: this is the content of the property =.

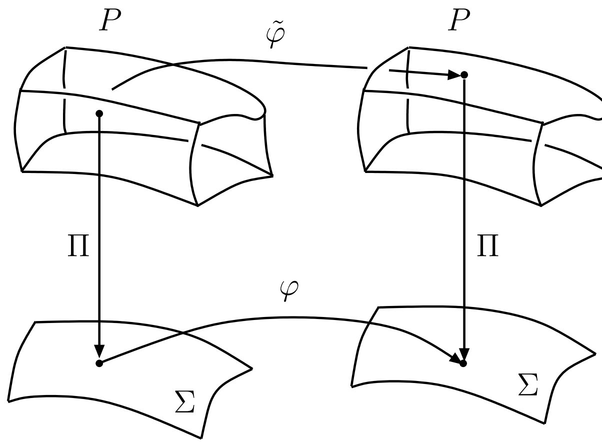

Figure L.13: A bundle map.

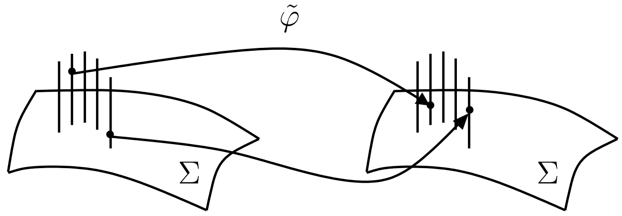

Figure L.14: Pictural illustraiton of a bundle map which preserves every fibre. The corresponding map on the base space is the identity - to be called a Yang-Mills gauge transformations.

A map $\tilde{\varphi} : P \to P$ which maps each fibre to itself is a particular case of a bundle map in which the corresponding map of the base is just the identity map of Σ.

In a sense the bundle automorphisms represent also the diffeomorphisms of Σ.

It is easy to check, that the new flow is the flow of the vector field $X_{\varphi(S),\tilde{\varphi}_* f}$.

L.11.7 Proof

In the Ashtekar-Lewandowski representation, ω_0, the quantum flux operator $X_{s,f}$ vanishes: $\pi_{\omega_0}[X_{s,f}] = 0$. The main part of the proof of the uniqueness theorem is to show that a consequence of diffeomorphism invaraince of any representation is the vanishing of the quantum flux operator, as it is fairly straightforward to show that the Ashtekar-Lewandowski representation is the only diffeomorphism invariant representation with this property.

Theorem L.11.1

$$[X_{S,f}] = 0 \tag{L.0}$$

Of course $[a] := \{a + b : b \in \mathfrak{A}$ such that $\omega(b^*b) = 0\}$ is the equivalence class of $a \in \mathfrak{A}$ with respect to the Gel'fand ideal of null vectors.

Proof:

For each point $p \in \mathrm{supp}(n)$, by the defnition of a face S we find a neighbourhood U_p of p and a chart x_p whose domain contains U_p such that

$$\chi_p(S \cap \mathcal{U}_p) = \{x \in \mathbb{R}^D : x^D = 0, 0 < x^1, \ldots, x^{D-1} < 1\}$$

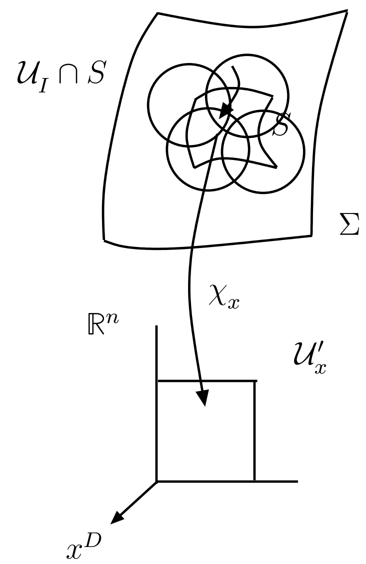

Figure L.15: x_I defined such tha the above is so.

Since the support of n is compact in P, choose a finite subcovering $\{\mathcal{U}_I\}_{I=1}^N$ of $\Pi(\mathrm{supp} f)$ with associated charts x_I. By the local character of semianalytic structures, there is a partition of unity subordinate to the covering $\{\mathcal{U}_I\}$, i.e., there exists a family of differentiable functions $\chi_I(x)$ such that

(i) $0 \leq \chi_I(x) \leq 1$

(ii) $\chi_I(x) = 0$ if $x \neq U_I$

(iii) $\sum_I^N \chi_1(x) = 1$ for any point $x \in \mathcal{M}$.

From (iii) it follows that

$$n(x) = \sum_I n(x)\chi_I(x) = \sum_I n_I(x)$$

where $n_I(x) \equiv n(x)\chi_I(x)$ vanishes outside of U_I by (ii).

Hence

$$n = \sum_{I-1}^{n} n_I$$

everwhere on Σ where

$$n_I = n \cdot \chi_I.$$

There doesn't exist global coordinates which ensure a basis for the collection of vector fields. The most that can be said is that for any point has a neighbourhood on which local vector fields are defined which form a basis of the tangent space at each point in the neighbourhood.

Furthermore, we may decompose

$$n_I = \sum_j n_I^j \tau_j \tag{L.0}$$

where τ_j is a basis in the Lie algebra of G and set $n_{Ij} = n_I^j \tau_j$ (no summation). It follows that

$$[X_n(S)] = \sum_{I=1}^{N} \sum_{j=1}^{\dim (G)} [X_{Ij}(S)]$$

and the result will follow from proving that $[X_{Ij}(S)] = 0$.

Consider for fixed I, j the following functional which assigns a number to any given pair of compactly supported functions $n_{Ij}, n'_{Ij} : S \cap U_I \to \mathbb{R}$,

$$(n_{Ij}, n'_{Ij})_S := < \left[X_{n_{Ij}}(S)^*\right], \left[X_{n'_{Ij}}(S)\right] > := \omega(X_{n_{Ij}}(S)^* X_{n'_{Ij}}(S)) \tag{L.0}$$

The product $(\cdot, \cdot)_S$ has the following properties:

(i) It is obviously bilinear and, due to the reality of the n, n', also symmetric.

(ii) It is invariant under semianalytic diffeomorphisms φ which preserve S and have support in U_I,

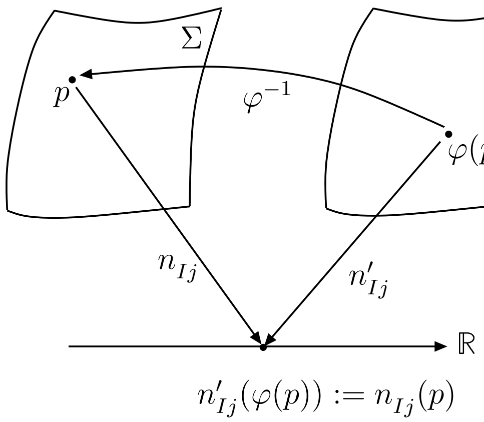

Figure L.16: The pull-back $(\varphi)^* n_{Ij}$ of the scalar function $n_{Ij}(x)$ is given by $n'_{Ij} = n_{Ij} \circ \varphi^{-1}$.

The point of the decomposition (L.11.7) is that, if additionally φ preserves S, then the action of $\tilde{\varphi}$ on $X_{n_{Ij}}(S)$ amounts to

$$\alpha_{\tilde{\varphi}} X_{n_{Ij}}(S) = X_{(\varphi)^* n_{Ij}}(S) = X_{n_{Ij} \circ \varphi^{-1}}(S).$$

It follows that the product $(\cdot,\cdot)_S$ is invariant under the specified $\tilde{\varphi}$

$$\begin{aligned} (n_{Ij}, n'_{Ij})_S &= \omega\big(\alpha_\varphi\big[X_{n_{Ij}}(S)^* X_{n'_{Ij}}(S)\big]\big) \\ &:= \omega\big(X_{n_{Ij}\circ\varphi}(\varphi(S))^* X_{n'_{Ij}\circ\varphi}(\varphi(S))\big) \\ &= (n_{Ij} \circ \varphi^{-1}, n'_{Ij} \circ \varphi^{-1})_S. \end{aligned} \tag{L.-1}$$

For $n = n'$

$$\|[X_{n_{Ij}}(S)]\|^2 = \big\langle \big[X_{n_{Ij}}(S)^* X_{n'_{Ij}}(S)\big] \big\rangle = \omega(X_{n_{Ij}}(S)^* X_{n_{Ij}}(S)). \tag{L.-1}$$

The trick to proving $\omega([X_{n_{Ij}}]^*[X_{n_{Ij}}]) = 0$ is to construct (which will be done in the next two lemmas) a semianalytic diffeomorphism φ_t which reduces to identity outside U_I, a semianalytic function N_{Ij}, and another semianalytic function f with $f|_S = 1$ such that

$$(\varphi_t)^* N_{Ij} = N_{Ij} + t f n_{Ij} \tag{L.-1}$$

for all $0 < t < t_0$, to which we can then apply (L.-1). This results in

$$\begin{aligned}(N_{Ij}, N_{Ij})_S &= ((\varphi_t)^* N_{Ij}, (\varphi_t)^* N_{Ij})_S \\ &= (N_{Ij} + tfn_{Ij}, N_{Ij} + tfn_{Ij})_S \\ &= (N_{Ij}, N_{Ij})_S + 2t(N_{Ij}, n_{Ij})_S + t^2(n_{Ij}, n_{Ij})_S \end{aligned} \tag{L.-2}$$

Since this holds for all $0 < t < t_0$ we may divide by $t > 0$ and find

$$2(N_{Ij}, n_{Ij})_S + t(n_{Ij}, n_{Ij})_S = 0 \tag{L.-2}$$

for all $0 < t < t_0$. Subtracting this equation evaluated at $0 < t_1 < t_2 < t_0$ one easily sees that $(n_{Ij}, n_{Ij})_S = 0$.

□

Lemma L.11.2 *There is $t_0 > 0$ such that for every $0 < t < t_0$, φ'_t is a semianalytic diffeomorphism of $\mathbb{R}^D$ equal to the identity outside of $\mathcal{U}'_I$ and preserving $\mathcal{U}'_I$.*

Proof: Using the coordinate system x_I associated with U_I we set $U'_I = x_I(U_I)$,

$$S'_I = x_I(S \cap U_I) = \{x \in \mathbb{R}^D : x^D = 0, 0 < x^1, \ldots, x^{D-1} < 1\}$$

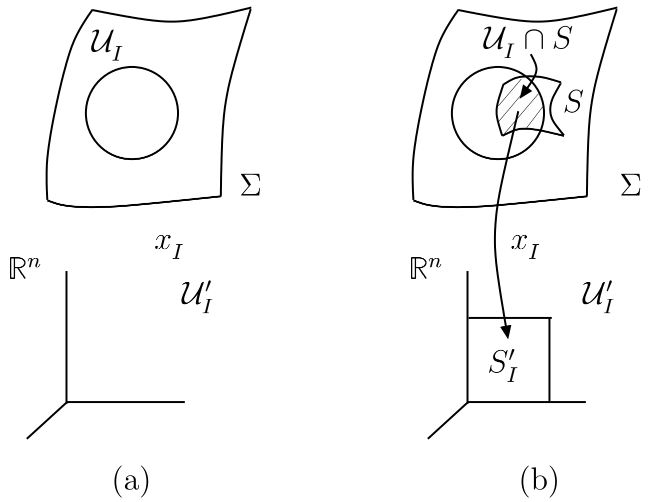

Figure L.17: x_I defined such tha the above is so.

and construct

$$n_{Ij} \circ x_I^{-1} : S'_I \to \mathbb{R}.$$

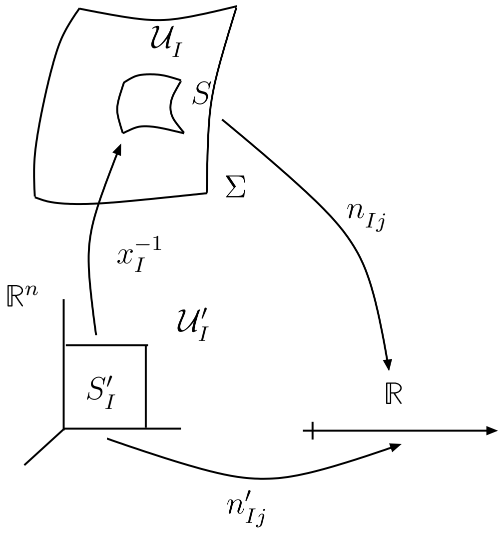

Figure L.18: .

To extend n'_{Ij} to U'_I, let $f' : \mathbb{R} \to \mathbb{R}$ be an arbitrary semianalytic function subject to $f'(0) := 1$ and such that

$$\tilde{n}'_{Ij}(x^1, \ldots, x^D) := n'_{Ij}(x^1, \ldots, x^{D-1}) f'(x^D)$$

From all this we can now define a map $\varphi'_t : \mathbb{R}^D \to \mathbb{R}^D$, where t is a real parameter by

$$\varphi'_t(x) = (x^1 + t\tilde{n}'_{Ij}(x^1, \ldots, x^D), x^2, \ldots, x^D). \qquad \text{(L.-2)}$$

$$\det\left(\frac{\partial \varphi'_t(x)}{\partial x}\right) = 1 + t\frac{\partial \tilde{n}'_{Ij}(x)}{\partial x^1} = 1 + tf'(x^D)\frac{\partial n'_{Ij}(x^1, \ldots, x^{D-1})}{\partial x^1} \qquad \text{(L.-2)}$$

The function $f' \partial n'_{Ij}/\partial x^1$ has compact support in U'_I and is at least continuous there. Thus, it is uniformally bounded whence there exists $t_0 > 0$ such that $1 + tf'\partial n'_{Ij}/\partial x^1 > 0$ for all $0 < t < t_0$.

Hence φ'_t is locally a semianalytic (since f', n'_{Ij}, x^k_I are semianalytic) diffeomorphism, provided $0 < t < t_0$. It is also a global diffeomorphism because outside of U'_I it acts as the identity.

A map which is identity in a subset will fail to preserve the complement only if it is not surjective and injective. That φ'_t preserves U'_I then follows from the fact that diffeomorphims are always bijective.

□

Lemma L.11.3

$$N_{Ij} \circ \varphi_t^{-1} = N_{Ij} + t f n_{Ij}.$$

Proof: We construct a semianalytic function N'_{Ij} with support in U'_I such that

$$N'_{Ij}(x^1, \ldots, x^D) = x^1 \qquad \text{whenever} \qquad (x^1, \ldots, x^D) \in \text{supp } n'_{Ij} f'. \tag{L.-2}$$

This is easily done by using an appropriate partition of unity.

We compute

$$\begin{aligned}
[(\varphi_t'^*) N'_{Ij}](x^1, \ldots, x^D) &= N'_{Ij}(x^1 + t\tilde{n}'_{Ij}(x^1, \ldots, x^D), x^2, \ldots, x^D) \\
&= \begin{cases} N'_{Ij}(x^1 + t\tilde{n}'_{Ij}, x^2, \ldots, x^D) & x \in \text{supp } (\tilde{n}'_{Ij}) \\ N'_{Ij}(x^1, x^2, \ldots, x^D) & x \notin \text{supp } (\tilde{n}'_{Ij}) \end{cases} \\
&= \begin{cases} x^1 + t\tilde{n}'_{Ij}(x) & x \in \text{supp } (\tilde{n}'_{Ij}) \\ N'_{Ij}(x^1, x^2, \ldots, x^D) & x \notin \text{supp } (\tilde{n}'_{Ij}) \end{cases} \\
&= \begin{cases} N'_{Ij}(x) + t\tilde{n}'_{Ij}(x) & x \in \text{supp } (\tilde{n}'_{Ij}) \\ N'_{Ij}(x^1, x^2, \ldots, x^D) & x \notin \text{supp } (\tilde{n}'_{Ij}) \end{cases} \\
&= N'_{Ij}(x) + t\tilde{n}'_{Ij}(x)
\end{aligned} \tag{L.-5}$$

Let us deonte by $N_{Ij}, n_{Ij}, f, \varphi_t$ the pull-back by x_I of $N'_{Ij}, n'_{Ij}, f', \varphi'_t$. Since x_I is a bijection and N'_{Ij}, n'_{Ij} have compact support in U'_I, it follows that $N_{Ij}, \tilde{n}_{Ij} = f n_{Ij}$ have compact support $U_I = x^{-1}(U'_I)$. We may thus extend them to all of Σ by setting them equal to zero outside of U_I. Likewise, φ_t equals the identity outside of U_I and preserves U_I for $0 < t < t_0$. Furthermore (L.-5) translates into

$$(\varphi_t)^* N_{Ij} = N_{Ij} + t f n_{Ij}.$$

Notice that

$$[\varphi'_t(x)]^D = x^D$$

preserves $x^D = 0$, hence it preserves S'_I and therefore φ_t preserves $S_I = U_I \cap S$. Since it is the identity outside of U_I, φ_t and its inverse are diffeomorphisms which preserve S. Also we see that $f = 1$ on S_I since $f' = 1$ when $x^D = 0$.

□

$$\int_{\mathcal{A}} X_{S,f}(\Psi) d\mu = 0 \tag{L.-5}$$

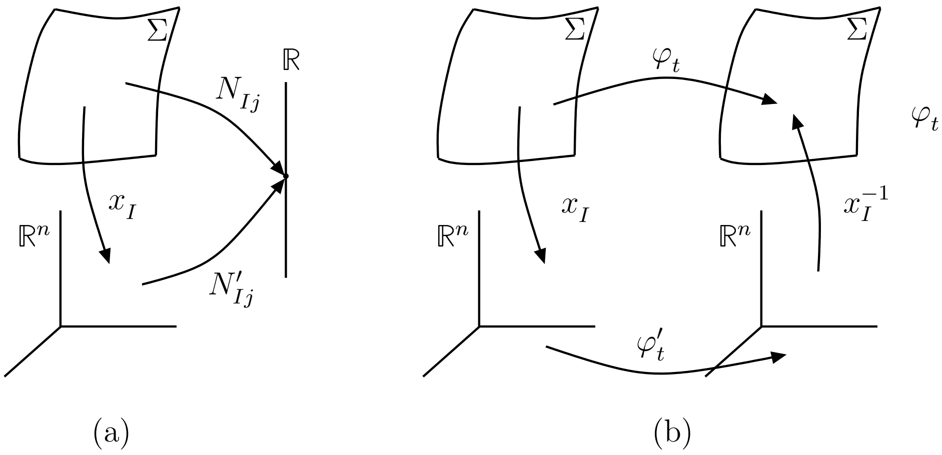

Figure L.19: (a) Pull-back N_{Ij} of N'_{Ij} under x_I is given by $N'_{Ij} \circ x_I$. (b) Pull-back φ_t of φ'_t under x_I is given by $x_I^{-1} \circ \varphi'_t \circ x_I$

Let us recall, that any regular, Borel, probability measure μ on the space $\overline{\mathcal{A}}$ is uniquely determined by its projections μ_γ on the spaces $\overline{\mathcal{A}}_\gamma$. Therefore, to prove the lemma it is enough to find out what restrictions are imposed by ()

Lemma L.11.4 *Every compact connected Lie group, G is isomorphic to a quotient $\tilde{G}/M$, where M is a central discrete subgroup of $\tilde{G}$, and $\tilde{G}$ is a simple product*

$$\tilde{G} = T \times P,$$

(that is, any $h \in \tilde{G}$ can be written as tp where $t \in T$ and $p \in P$), of an abelian group T and a semisimple group P.

Proof:

Recall that each Lie group possesses a Lie algebra g isomorphic to the tangent vector space at the identity element of the Lie group. An ideal in a Lie algebra is a Lie subalgebra $h \subset g$ such that $[X, Y] \in h$ for all $X \in h$, $Y \in g$. An ideal is said to be an invariant subalgebra.

An ideal is the Lie algebra equivalent of a closed, normal subgroup of a connected Lie group.

A connected Lie group can be defined to be simple if its Lie algebra is simple, or equivalently, if it contains no non-trivial, closed, connected normal subgroups. Under this definition, a simple connected Lie group can possess non-trivial, closed, normal subgroups, but if they exist they must be discrete.

A semisimple Lie algebra can be defined as a Lie algebra which has no non-trivial abelian ideals, but here we wish to characterise it as a Lie algebra which is the direct sum of simple Lie algebras. Semisimple Lie groups arae the direct products of simple Lie groups.

Clearly, a simple Lie algebra is semisimple.

Lemma L.11.5

Proof:

Let us consider an arbitrary graph γ consisting of edges $\{e_1, \ldots, e_N\}$. Divide each edge e_I

$$e_I = e_{I,1} \circ e_{I,2}$$

see fig (L.20)

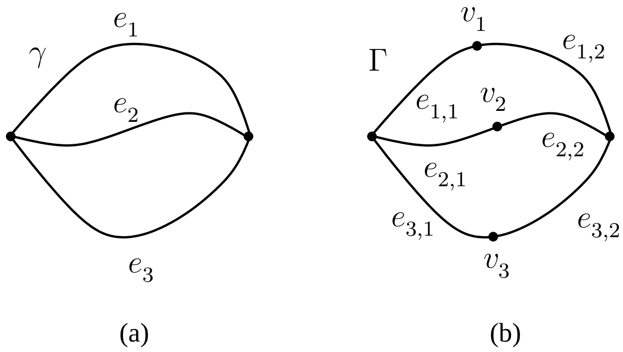

Figure L.20: $e_1 = e_{1,1} \circ e_{1,2}$

$$e_1 = e_{1,1} \circ e_{1,2}$$

where:

$$g_i(\overline{A}) := (\overline{A}(e_{1,i}), \ldots, \overline{A}(e_{1,N})) \in G^N$$

$$\Psi_\Gamma = \psi(g_1(), g_2)$$

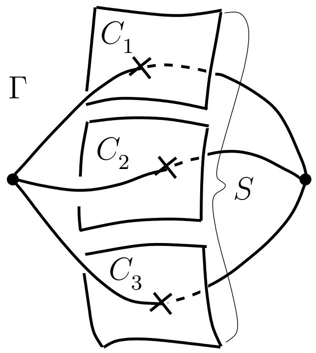

Figure L.21: We let S consist of 3 disjoint cubes C_1, C_2, C_3.

Semisimple case

Suppose the function f is defined on S in the following way

$$f\Big|_{C_I} := const_I f_I \in P'$$

Consider the operator

$$\hat{X}_{S,f} = \sum_{I=1}^{N} \hat{X}_{C_I,f_I}.$$

Assuming that Ψ_Γ is a smooth cylindrical function,

$$\begin{aligned}
0 &= \int_{\overline{A}} \hat{X}_{S,f} \Psi_\Gamma \, d\mu \\
&= \int_{\overline{A}_\Gamma} \hat{X}_{S,f} \psi \, d\mu_\Gamma \\
&= \sum_{I=1}^{N} \int_{\overline{A}_\Gamma} \hat{X}_{C_I,f_I} \psi \, d\mu_\Gamma \\
&= -\frac{i}{2} \int_{G^{2N}} \frac{d}{ds}\Big|_{s'=0} \psi(g_1 \exp(\vec{f}s'), \exp(\vec{f}s')g_2) \, d\mu_\Gamma \\
&= -\frac{i}{2} \frac{d}{ds'}\Big|_{s'=0} \int_{G^{2N}} \psi(g_1 \exp(\vec{f}s'), \exp(\vec{f}s')g_2) \, d\mu_\Gamma
\end{aligned} \tag{L.-8}$$

where

$$\vec{f} := (f_1, \ldots, f_N)$$

and

$$\exp(\vec{f}s) = (\exp(f_1 s), \ldots, \exp(f_N s)) \in P^N.$$

using this equality on the function

$$\tilde{\psi}(g_1, g_2) := \psi(g_1 \exp(\vec{f}s), \exp(\vec{f}s) g_2)$$

and that

$$\frac{d}{ds'}\Big|_{s'=0} \tilde{\psi}(s' + s) = \frac{d}{ds} \tilde{\psi}(s)$$

results in

$$\begin{aligned} 0 &= \frac{d}{ds'}\Big|_{s'=0} \int_{G^{2N}} \tilde{\psi}(g_1 \exp(\vec{f}s'), \exp(\vec{f}s') g_2)\, d\mu_\Gamma \\ &= \frac{d}{ds'}\Big|_{s'=0} \int_{G^{2N}} \psi(g_1 \exp(\vec{f}(s + s')), \exp(\vec{f}(s' + s)) g_2)\, d\mu_\Gamma \\ &= \frac{d}{ds} \int_{G^{2N}} \psi(g_1 \exp(\vec{f}s), \exp(\vec{f}s) g_2)\, d\mu_\Gamma \end{aligned} \tag{L.-9}$$

As the group is connected

$$\int_{G^{2N}} \psi(g_1 b, b g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma \tag{L.-9}$$

for every $b \in P^N$.

$$\zeta(a, a') := \int_{G^{2N}} \psi(g_1 a, a' g_2)\, d\mu_\Gamma$$

Equation (L.11.7) implies

$$\zeta(ba, a'b) = \zeta(a, a') \tag{L.-9}$$

for every $b \in P^N$. Let $b = a^{-1}$. Then

$$\zeta(a, a') = \zeta(\mathbb{I}_P, a'a^{-1}) =: \xi(a'a^{-1}),$$

where $\mathbb{I}_P$ is the identity element of P^N. Now, () reads

$$\xi(a'ba^{-1}b^{-1}) = \xi(a'a^{-1}).$$

The substitution $b_0 = a'a^{-1}$ gives an identity

$$\xi(b_0aba^{-1}b^{-1}) = \xi(b_0), \tag{L.-9}$$

which holds for every $a, b, b_0 \in P^N$. We now use this to prove that the function ξ, and consequently the function ζ, is constant. Let L_1, L_2 be arbitrary left invariant vector fields on P^N, then the vector $[L_1, L_2]_{b_0}$ tangent to P^N at the point b_0 can be generated by a curve of the form

$$b_0a(t)b(t)a^{-1}(t)b^{-1}(t)$$

for t small has the geometric interpretation of fig (L.22). Thus (L.11.7)

$$[L_1, L_2]_{b_0}\xi = 0$$

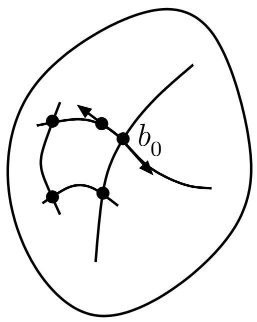

Figure L.22:

The commutator of two left invariant vector fields is also a left invariant vector field. This can easily be seen by fig (). The importance of this is that left invariant vector fields form a Lie algebra are isomorphic to the Lie algebra P'^N as the Lie algebra g is isomorphic to the tangent vector space at the identity element of the Lie group..

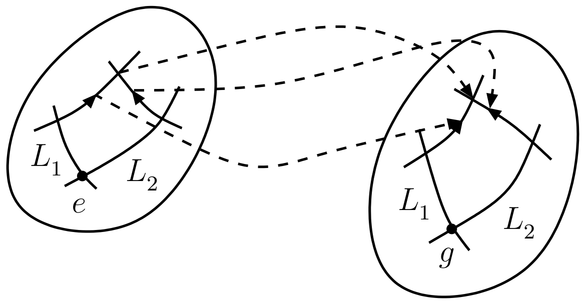

Figure L.23:

As the group P^N is semisimple, the algebra satisfies $[P'^N, P'^N] = P'^N$. So for arbitrary left invariant vector field L

$$L_{b_0}\xi = 0$$

Hence the function ξ and consequently the function ζ are both constant. Thus

$$\int_{G^{2N}} \psi(g_1 b_1, b_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma \tag{L.-9}$$

for every smooth function ψ on G^{2N} and for every $b_1, b_2 \in P^N$.

Abelian case

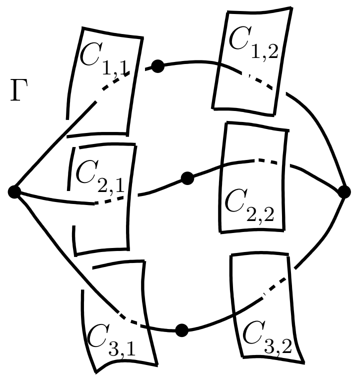

Figure L.24: We let S consist of 6 disjoint cubes $C_{1,1}, C_{2,1}, C_{3,1}, C_{1,2}, C_{2,2}, C_{3,2}$.

the operator

$$\hat{X}_{S,f} = \sum_{I=1}^{N} \hat{X}_{C_{I,i},f_{I,i}}.$$

$$\int_{G^{2N}} \psi(g_1 t_1, t_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma \tag{L.-9}$$

for every smooth function ψ on G^{2N} and for every $t_1, t_2 \in T^N$.

Combined

$$\int_{G^{2N}} \psi(g_1 t_1 b_1, t_2 b_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma$$

for every $t_i b_i \in T^N \times P^N$, that is

$$\int_{G^{2N}} \psi(g_1 h_1, h_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma \tag{L.-9}$$

for every $h_i \in \tilde{G}^N$

Obviously the space $C^\infty(G^{2N}, \mathbb{C})$ separates the points of G^{2N} and includes the constant functions, so the Stone-Weiestrass theorem applies, showing that the closure $C^\infty(G^{2N}, \mathbb{C})$ with respect to the sup-norm is $C^0(G^{2N}, \mathbb{C})$ and therefore equation (L.11.7) holds for every $\psi \in C^0(G^{2N}, \mathbb{C})$.

If we swap the function $\psi(g_1, g_2)$ by the function $\psi(g_1, g_2^{-1})$ then

$$\int_{G^{2N}} \psi(g_1 h_1, (h_2 g_2)^{-1})\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1 h_1, g_2^{-1} h_2^{-1})\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2^{-1})\, d\mu_\Gamma$$

consider the map

$$\omega(g_1, g_2) \mapsto \omega(g_1, g_2) := (g_1, g_2^{-1})$$

with the push forward measure

$$\mu_\Gamma^* := \omega^* \mu_\Gamma.$$

then

$$\int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma^* = \int_{G^{2N}} \psi(g_1 h_1, g_2 h_2^{-1})\, d\mu_\Gamma^* \tag{L.-8}$$

for every $(h_1, h_2) \in G^N \times G^N$.

The Haar measure μ_Γ^* on G^{2N} is a product of two copies of the Haar measure μ_H on G^N. Since G is compact, left and right invaraint measures on the group coincide. Therefore we can write

$$\begin{aligned} \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_H^{(1)} d\mu_H^{(2)} &= \\ &= \int_{G^{2N}} \psi(g_1, g_2^{-1})\, d\mu_H^{(1)} d\mu_H^{(2)} \end{aligned} \tag{L.-8}$$

and so

$$\mu_\Gamma^* = \mu_\Gamma.$$

Then gives for every $(h_1, h_2) \in G^N \times G^N$

$$\int_{G^{2N}} \psi(g_1, g_2) d\mu_\Gamma = \int_{G^{2N}} \psi(g_1 h_1, g_2 h_2) d\mu_\Gamma.$$

Every graph Γ is obtained by a subdivision of some graph γ, hence every cylindrical function Ψ compatible with γ is also compatible with Γ. Recall that if Ψ is compatible with two grpahs γ, γ', then

$$\int_{\overline{A}_\gamma} \psi d\mu_\gamma = \int_{\overline{A}_{\gamma'}} \psi' d\mu_{\gamma'},$$

and we conclude the push forward measure is the Haar measure on $G^{|E(\gamma)|}$. As the graph γ is arbitrary,

$$\mu = \mu_{AL}.$$

□

L.12 Irreduciblity of the Ashtekar-Lewandowski Representation

First let γ be a graph and split each edge $e \in E(\gamma)$ into two halves $e = e'_1 \circ (e'_2)^{-1}$ and replace the e's the e'_1, e'_2. We obtain a graph γ' which occupies the same points in Σ as γ but changes the set of edges of γ in such a way that each edge is outgoing from the vertex $b(e') = v \in V(\gamma)$. We call a graph refined in this way a standard graph. The reason for using this freedom is to simplify the following discussion. For notational simplicity we denote standard graphs as γ from now on.

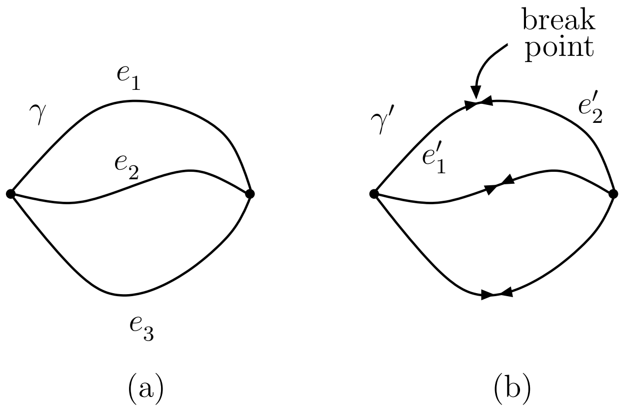

Figure L.25: The graph γ' is the standard graph associated with the original graph γ.

Lemma L.12.1 *Let γ be a standard graph. Assign to each $e \in E(\gamma)$ a vector $t_e = (t^j_e)_{j=1}^{dim(G)}$ and collect them into a label $t_\gamma = (t_e)_{e\in E(\gamma)}$.*

Then there exists a vector field $Y(t_\gamma, \gamma)$ in the Lie algebra of the flux fields $Y_{S,f}$ such that for any cylindrical function $f = p^*_\gamma f_\gamma$ over γ we have

$$Y_\gamma(t_\gamma) p^*_\gamma f_\gamma = p^*_\gamma \sum_{e\in E(\gamma)} t^j_e R^e_j f_\gamma . \tag{L.-8}$$

Any compact connected Lie group G has the structure $G/Z = A \times S$ where Z is a discrete subgroup, A is an abelean Lie group and S is a semisimple Lie group.

Abelean case:

Consider any $e \in E(\gamma)$ and take any surface S_e which intersects γ only

$$Y_j(S_e) p^*_\gamma f_\gamma = p^*_\gamma [R^j_{e_2} - R^j_{e_1}] f_\gamma \tag{L.-8}$$

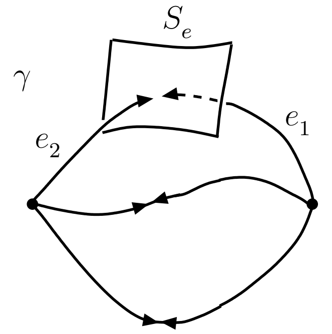

Figure L.26: Abelean case

Due to gauge invariance

$$(\tau^i h_{e_1})^A_B \frac{\partial f}{\partial (h_{e_1})^A_B} + (\tau^i h_{e_2})^A_B \frac{\partial f}{\partial (h_{e_2})^A_B} = 0$$

$$[R^j_{e_1} + R^j_{e_2}] f_\gamma = 0,$$

thus

$$Y^j_e p^*_\gamma f_\gamma = \frac{1}{2} Y_j(S_e) p^*_\gamma f_\gamma \tag{L.-8}$$

is an appropriate choice.

Non-Abelean case:

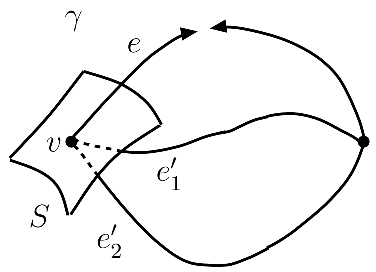

Figure L.27: Non-Abelean case

An analytic surface S is completely determined by its Taylor coefficients in the expansion of its parameterisation

$$S(u,v) = \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} S^{(m,n)}(0,0) \tag{L.-8}$$

e is determined

$$e(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{(n)}(0) \tag{L.-8}$$

In order that $s \subset S$ we just need to choose a parameterisation of S such that $S(t,0) = e(t)$ which fixes the Taylor coefficients

$$S^{(m,0)}(0,0) = e^{(m)}(0) \tag{L.-8}$$

for all m. Say that the other edges $e'_1, \ldots e'_n$ were to have a beginning segment s_k of e'_k in S then there would be an analytic function $v_k(t)$, such that

$$s_k(t) = S(t, v_k(t)).$$

Obviously, we can not have $v_k(t) = 0$ in an arbitrary small neighbourhood of $t = 0$ otherwise $s_k = s_e$. For each k let n_k be the first derivative such that $v_k^{(n_k)}(0) \neq 0$. By relabeling the edges we may arrange that $n_1 \leq n_2 \leq \cdots \leq n_N$. Consider $k = 1$ and take the n_1-th derivative at $t = 0$. We find

$$\frac{d^{n_1}}{dt^{n_1}} s_1(0) = S^{(n_1,0)}(0,0) + S^{(0,1)}(0,0) \frac{d^{n_1}}{dt^{n_1}} v_1(0)$$

Since $v_1^{(n_1)} \neq 0$ we can arrange the surface S, by using the freedom in $S^{(n_1,0)}(0,0)$, so that this equation does not hold and hence s_k is not in S.

$$\frac{d^{n_2+1}}{dt^{n_2+1}} s_1(0) = S^{(n_1,0)}(0,0) + 2S^{(1,1)}(0,0) \frac{d^{n_1}}{dt^{n_1}} v_2(0) + S^{(0,1)}(0,0) \frac{d^{n_2+1}}{dt^{n_2+1}} v_2(0)$$

Since $v_2^{(n_2)} \neq 0$ we can use the freedom in $S^{(1,1)}$ in order to violate this equation. Proceeding in this way we can use the coefficients $S^{(k-1,1)}$ in order for the edges to be transversal to S.

Having constructed the surfaces $S_{v,e}$ we can compute the associated vector field applied to a cylindrical function over γ

$$Y_j(s_{v,e})p_\gamma^* f_\gamma = p_\gamma^* \sum_{e' \in E(\gamma)-\{e\}, b(e')=v} \sigma(s_{v,e}, e') R^j_{e'} f_\gamma \tag{L.-8}$$

Taking the commutator

$$\begin{aligned}
[Y_j(s_{v,e}), Y_k(s_{v,e})]p_\gamma^* f_\gamma &= Y_j(s_{v,e})p_\gamma^* \sum_{e'} \sigma(s_{v,e}, e') R^k_{e'} f_\gamma - (j \leftrightarrow k) \\
&= p_\gamma^* \sum_{e'} \sigma(s_{v,e}, e') \sum_{e''} \sigma(s_{v,e}, e'') R^l_{e''} R^k_{e'} f_\gamma - (j \leftrightarrow k) \\
&= f_{jkl}\, p_\gamma^* \sum_{e'} R^l_{e'} f_\gamma
\end{aligned} \tag{L.-9}$$

where we used

$$[R^j_{e''}, R^k_{e'}] = \delta_{e'',e'} f_{jkl} R^l_{e'}.$$

$$R^j_v := \sum_{e' \in E(\gamma), b(e')=v} R^j_{e'}$$

we get

$$f_{jkl}[Y_k(s_{v,e}), Y_l(s_{v,e})]p_\gamma^* f_\gamma = p_\gamma^* [R^j_v - R^j_e] f_\gamma$$

This, if n_v is the valence of v

$$\begin{aligned}
Y^j_e p_\gamma^* f_\gamma &:= \Big(- f_{jkl}[Y_k(s_{v,e}), Y_l(s_{v,e})] + \frac{1}{n_v - 1} \sum_{e' \in E(\gamma)} f_{jkl}[Y_k(s_{v,e'}), Y_l(s_{v,e'})] \Big) p_\gamma^* f_\gamma \\
&= -p_\gamma^* [R^j_v - R^j_e] f_\gamma + \frac{1}{n_v - 1} \sum_{e' \in E(\gamma), b(e')=v} p_\gamma^* [R^j_v - R^j_{e'}] f_\gamma \\
&= -p_\gamma^* [R^j_v - R^j_e] f_\gamma + (\frac{n_v}{n_v - 1} - 1) R^j_v p_\gamma^* f_\gamma \\
&= p_\gamma^* R^j_e f_\gamma
\end{aligned} \tag{L.-11}$$

Collecting the vector fields Y^j_e for the Abelean and non-Abelean labels j respectively and contracting them with t^j_e and summing over $e \in E(\gamma)$ yields an appropriate vector field

$$Y_\gamma(t_\gamma) = \sum_{e\in E(\gamma)} t_e^j Y_e^j. \tag{L.-11}$$

Actually, here we have impiciyly assumed that where no $e \in E(v)$ is (a segment of) the analytic extension through v of another edge $e' \in E(v)$. We also need to consider the case where there is at least one pair of edges $e, \tilde{e} \in E(v)$ that are (segments of) analytic continuations of each other through v. See [31] for details.

$\Box$

Recall that the Hilbert space $\mathcal{H}_0$ has an orthonormal basis of particular cylindrical functions - the spin network functions - labeled by a spin network $s = (\gamma, \{\pi_e\}, \{m_e\}, \{n_e\})_{e\in E(\gamma)}$ defined by

$$T_s(A) = \prod_{e\in E(\gamma)} \{\sqrt{d_{\pi_e}}[\pi_e(h_e)]_{m_e n_e}\}$$

where π denotes an irreducible representation of G. Later we will need the right action R_e^j on T_s which is easily computed

$$\begin{aligned} R_e^j T_s &= [\tau^j \pi(h_e)]_{mn} \frac{\partial T_s}{\partial[\pi(h_e)]_{mn}} \\ &= \sqrt{d_{\pi_{e_1}}}[\pi_{e_1}(h_{e_1})]_{m_{e_1} n_{e_1}} \ldots\ [\tau_j]_{m_e l_e} \sqrt{d_{\pi_e}}[\pi_e(h_e)]_{l_e n_e} \cdots \sqrt{d_{\pi_{e_N}}}[\pi_{e_N}(h_{e_N})]_{m_{e_N} n_{e_N}} \end{aligned}$$

We now define for any two $\psi, \psi' \in \mathcal{H}_0$ the function

$$M_{\psi,\psi'}(t_\gamma, I_\gamma) := <\psi, T_{\gamma,I_\gamma}\ W_\gamma(t_\gamma)\psi' >_{\mathcal{H}_0} \tag{L.-13}$$

We exploit that for a compact connected Lie group the exponential map is onto.

Thus, there exists a region $D_G \subset \mathbb{R}^{\dim(G)}$ such that $\exp : D_G \to G;\ t \mapsto \exp(t^j\tau_j)$ is a bijection. Consider the measure μ on D_G defined by $d\mu(t) = d\mu_H(\exp(t^j\tau_j))$ where μ_H is the Haar measure on G. Finally, let $D_\gamma = \prod_{e\in E(\gamma)} D_G$ and let L_γ be the space of the I_γ.

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma := \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \overline{M_{\psi_1,\psi_1'}(t_\gamma, I_\gamma)}\ M_{\psi_2,\psi_2'}(t_\gamma, I_\gamma) \tag{L.-13}$$

where $d\mu(t_\gamma) = \prod_{e\in E(\gamma)} d\mu(t_e)$.

Lemma L.12.2 *i) For any* $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_0$ *we have*

$$|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \le \|\psi_1\|\, \|\psi_1'\|\, \|\psi_2\|\, \|\psi_2'\| \tag{L.-13}$$

ii) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_{0,\gamma}$ *we have*

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma = <\psi_2, \psi_1>_{\mathcal{H}_0} <\psi_1', \psi_2'>_{\mathcal{H}_0} \tag{L.-13}$$

where $\mathcal{H}_{0,\gamma}$ *denotes the closure of the cylindrical functions over* γ*.*

Proof:

$$\begin{aligned}
(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma &= \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') \overline{T_{\gamma,I_\gamma}(A)} T_{\gamma,I_\gamma}(A') \\
&\quad \psi_1(A) \overline{[W_\gamma(t_\gamma)\psi_1'](A) \psi_2(A')} [W_\gamma(t_\gamma)\psi_2'](A') \\
&= \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') [\sum_{I_\gamma} \overline{T_{\gamma,I_\gamma}(A)} T_{\gamma,I_\gamma}(A')] \ldots \\
&= \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') \int_{D_\gamma} d\mu(t_\gamma) \delta_\gamma(A, A') \ldots
\end{aligned} \tag{L.-15}$$

where we have defined the cylidrical $\delta-$distribution

$$\delta_\gamma(A, A') = \prod_{e \in E(\gamma)} \delta_{\mu_H}(h_e[A], h_e[A'])$$

which comes from the Plancherel formula

$$\delta_{\mu_H}(g, g') = \sum_{\pi,m,n} \overline{T_{\pi,m,n}(g)} T_{\pi,m,n}(g').$$

$$f(A) = F(A_{|\overline{\gamma}}, A_{|\gamma}) \tag{L.-15}$$

the (effective) measure on $\overline{A}_{\overline{\gamma}}$ by

$$\int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \left[\int_{\overline{A}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) F(A_{|\overline{\gamma}}, A_{|\gamma}) \right] = \int_{\overline{\mathcal{A}}} d\mu_0(A) f(A). \tag{L.-15}$$

all occuring f are countable linear combinations of spin network functions

$$T_s = \prod_{e' \in E(\gamma\cup\gamma(s))-E(\gamma)} \{\sqrt{d_{\pi_{e'}}}[\pi_{e'}(h_{e'})]_{m_{e'}n_{e'}}\} \prod_{e\in E(\gamma)} \{\sqrt{d_{\pi_e}}[\pi_e(h_e)]_{m_e n_e}\}.$$

Thus either integral can be written as a countable linear combination of integrals over spin-network functions T_s and then the prescription is to integrate only either over the degrees of freedom $A(e), e \in E(\gamma)$ or $A(e'), e' \in E(\gamma(s)\cup\gamma) - E(\gamma)$ for each individual integral with the corresponding product Haar measure.

$$\begin{aligned}(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma &= \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}}) \int_{\overline{A}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \\ &\times \Psi_1(A_{|\overline{\gamma}}, A_\gamma)\overline{[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}}, A_\gamma)\Psi_2(A'_{|\overline{\gamma}}, A_\gamma)} \\ &\times [W_\gamma(t_\gamma)\Psi_2'](A'_{|\overline{\gamma}}, A_\gamma)\end{aligned} \tag{L.-17}$$

In order to evaluate the Weyl operators, consider a spin network function T_s cylindrical over $\gamma(s)$ which we write in the form

$$T_s(A) = F(\{h_{e'}\}_{E(\gamma\cup\gamma(s))-E(\gamma)}, \{h_e\}_{e\in E(\gamma)}) \tag{L.-17}$$

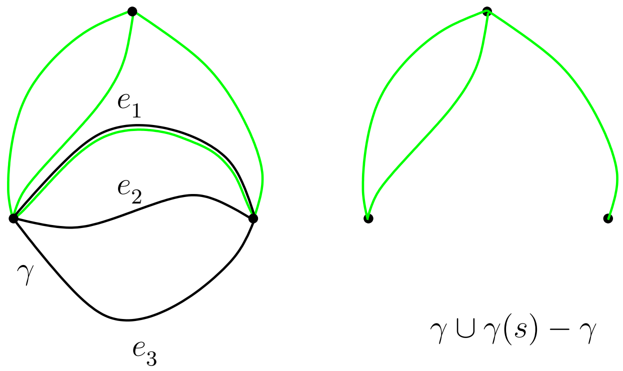

Figure L.28:

We know how the vector field $Y_\gamma(t_\gamma)$ acts on functions cylindrical over γ, (L.12.1), but how does it act on $\gamma(s)\cup\gamma-\gamma$?

it is easy to see that the action of $Y_\gamma(t_\gamma)$ on T_s is given by

$$Y_\gamma(t_\gamma)T_s = p^*_{\gamma(s)\cup\gamma}[\sum_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)} t_j^{e'}(t_\gamma)R^j_{e'} + \sum_{e\in E(\gamma)} t^e_j R^j_e]F \tag{L.-17}$$

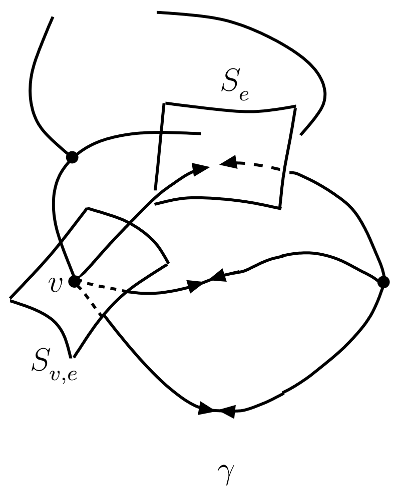

Figure L.29: Action of $Y_\gamma(t_\gamma)$ on T_s

where $t_j^{e'}(t_\gamma)$ is a certain linear combination of the t_j^e depnding on e' and the concrete surfaces S_e, $S_{v,e}$ used in the construction of $Y_\gamma(t_\gamma)$.

$$Y_\gamma(t_\gamma)\, T_s \;=\; F[\{t_j^{e'}(t_\gamma)\tau_j h_{e'}\}_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)} + \{t_j^e\}_{e\in E(\gamma)}\tau_j h_e] \tag{L.-16}$$

so that

$$\begin{aligned} W_\gamma(t_\gamma)T_s &= \sum_{m=0}^{\infty}\frac{1}{m!}Y_\gamma(t_\gamma)^m\, T_s \\ &= F(\{e^{t_j^{e'}(t_\gamma)\tau_j}h_{e'}\}_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)}, \{e^{t_j^e\tau_j}h_e\}_{e\in E(\gamma)}) \qquad \text{(L.-16)} \end{aligned}$$

the map $\alpha_{t_\gamma} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$; $A \mapsto W_\gamma(t_\gamma)AW_\gamma(t_\gamma)$ is just some right or left translation.

$$\begin{aligned} |(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \;\le\; & \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{A}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \\ & \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|\; |\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))| \\ & \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}})|\Psi_2(A'_{|\overline{\gamma}}, A_\gamma)|\; |\Psi_2'(\alpha_{t_\gamma}(A'_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|. \end{aligned} \tag{L.-18}$$

Consider the second line on the R.H.S., by the Cauchy-Schwarz inequality applied to functions $[\Psi_1(A_\gamma)](A_{|\overline{\gamma}}) = \Psi_1(A_{|\overline{\gamma}}, A_{|\gamma})$ in $L_2(\overline{\mathcal{A}}_{\overline{\gamma}}, d\mu_{0\overline{\gamma}})$ we can estimate

$$\int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|\,|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|$$
$$\leq \Big(\int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|^2\Big)^{1/2}\Big(\int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|^2\Big)^{1/2}. \tag{L.-19}$$

A similar result holds for functions $[\Psi_2(A_\gamma)](A'_{|\overline{\gamma}}) = \Psi_2(A'_{|\overline{\gamma}}, A_{|\gamma})$ in $L_2(\overline{\mathcal{A}}'_{\overline{\gamma}}, d\mu_{0\overline{\gamma}})$. From now on let us use the notation

$$\int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|^2 = \|\Psi_1(A_\gamma)\|^2_{|\overline{\gamma}}.$$

We can simplify the second factor on the R.H.S. of (L.-19) from the fact that

$$\int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|^2 = \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_\gamma))|^2 = \|\Psi_1'(\alpha_{t_\gamma}(A_\gamma))\|^2_{|\overline{\gamma}}$$

To prove this expand ψ_1' into spin-network functions

$$\psi_1'(A) = \Psi_1'(A_{|\overline{\gamma}}, A_{|\gamma}) = \sum_{n=1}^{\infty} z_n T_{s_n}(A).$$

Then the integral becomes

$$\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|^2$$
$$= \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\overline{T_{s_m}(\alpha_{t_\gamma}(A))}T_{s_n}(\alpha_{t_\gamma}(A)). \tag{L.-19}$$

The integration with measure $d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})$ over $\overline{\mathcal{A}}_{\overline{\gamma}}$ reduces to integration with the Haar measure over the space $G^{|E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma)-E(\gamma)|}$. Note first that by the bi-invariance of the Haar measure, for any $e' \in E(\gamma(s_m) \cup \gamma(s_n) \cup \gamma) - E(\gamma)$ it follows then, writing $E_{mn}(\gamma) \equiv E(\gamma(s_m) \cup \gamma(s_n) \cup \gamma) - E(\gamma)$, that

$$
\begin{aligned}
& \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \overline{T_{s_m}(\alpha_{t_\gamma}(A))}\, T_{s_n}(\alpha_{t_\gamma}(A)) \\
= & \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{G^{|E_{mn}(\gamma)|}} \prod_{e'\in E_{mn}(\gamma)} d\mu_H(h_{e'}) \overline{T_{s_m}(\{e^{t_j^{e'}(t_\gamma)\tau_j} h_{e'}\}, \{e^{t_j^e\tau_j} h_e\})}\, T_{s_n}\{e^{t_j^{e'}(t_\gamma)\tau_j}(h_{e'}\}, \{e^{t_j^e\tau_j} h_e\}) \\
= & \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{G^{|E_{mn}(\gamma)|}} \prod_{e'\in E_{mn}(\gamma)} d\mu_H(h_{e'}) \overline{T_{s_m}(\{h_{e'}\}, \{e^{t_j^e\tau_j} h_e\})}\, T_{s_n}(h_{e'}\}, \{e^{t_j^e\tau_j} h_e\}) \\
= & \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \overline{T_{s_m}(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_{|\gamma}))}\, T_{s_n}(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_{|\gamma})) \\
= & \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) |\Psi_1'(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_{|\gamma}))|^2 \qquad \text{(L.-22)}
\end{aligned}
$$

We now exploit that

$$\alpha_{t_\gamma}(A_{|\gamma}) = \{e^{t_j^e\tau_j} A(e)\}_{e\in E(\gamma)}$$

and introduce new integration variables $A'(e) := g(t_e)A(e)$ where $g(t_e) = \exp(t_j^e\tau_j)$. Since by definition

$$d\mu(t_\gamma) = \prod_{e\in E(\gamma)} d\mu(t_e) = \prod_{e\in E(\gamma)} d\mu_H(g(t_e))$$

we can estimate further

$$
\begin{aligned}
|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \;\leq\; & \int_{G^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(g_e) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \\
\times\; & \|\Psi_1(A_{|\gamma})\|_{|\overline{\gamma}}\, \|\Psi_1'(\{g_e A(e)\}_{e\in E(\gamma)})\|_{|\overline{\gamma}} \\
\times\; & \|\Psi_2(A_{|\gamma})\|_{|\overline{\gamma}}\, \|\Psi_2'(\{g_e A(e)\}_{e\in E(\gamma)})\|_{|\overline{\gamma}} \\
=\; & \left[\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\; \|\Psi_1(A_{|\gamma})\|_{\overline{\gamma}}\;\; \|\Psi_2(A_{|\gamma})\|_{\overline{\gamma}}\right] \\
\times\; & \left[\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma})\; \|\Psi_1'(A'_{|\gamma})\|_{\overline{\gamma}}\;\; \|\Psi_2'(A'_{|\gamma})\|_{\overline{\gamma}}\right] \\
\leq\; & \|\, \|\Psi_1\|_{\overline{\gamma}}\, \|_\gamma\, \|\, \|\Psi_1'\|_{\overline{\gamma}}\, \|_\gamma \|\, \|\Psi_2\|_{\overline{\gamma}}\, \|_\gamma\, \|\, \|\Psi_2'\|_{\overline{\gamma}}\, \|_\gamma \qquad \text{(L.-26)}
\end{aligned}
$$

where we have used Fubini's theorem and have again applied the Cauchy-Schwarz inequality to functions in $L_2(\overline{\mathcal{A}}_\gamma, d\mu_{0\gamma})$. But

$$
\begin{aligned}
\| \, \|\Psi_1\|_{\overline{\gamma}} \, \|^2_\gamma &= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \mid \|\Psi_1(A_{|\gamma})\|_{\overline{\gamma}} \mid^2 \\
&= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) |\Psi_1(A_{|\overline{\gamma}}, A_{|\gamma})|^2 \\
&= \int_{\overline{\mathcal{A}}} d\mu_0(A) |\psi_1(A)|^2 = \|\psi_1\|^2_{\mathcal{H}_0}
\end{aligned}
\tag{L.-28}
$$

ii)

If all functions in question are cylindrical L_2-functions over γ then the integrals over $\overline{\mathcal{A}}_{|\overline{\gamma}}$ are trivial and () simplifies to

$$
\begin{aligned}
(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma &= \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \Psi_1(A_\gamma) \overline{[W_\gamma(t_\gamma)\Psi_1'](A_\gamma)\Psi_2(A_\gamma)} [W_\gamma(t_\gamma)\Psi_2'](A_\gamma) \\
&= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma}) \Psi_1(A_\gamma) \overline{\Psi_1'(A'_\gamma)\Psi_2(A_\gamma)} \Psi_2'(A'_\gamma) \\
&= [\int_{\overline{\mathcal{A}}} d\mu_0(A) \overline{\psi_2(A)} \psi_1(A)] \, [\int_{\overline{\mathcal{A}}} d\mu_0(A') \overline{\psi_2'(A')} \psi_1'(A')] \\
&= < \psi_2, \psi_1 >_{\mathcal{H}_0} < \psi_1', \psi_2' >_{\mathcal{H}_0}
\end{aligned}
\tag{L.-30}
$$

□

Theorem L.12.3

Proof:

Proof was given in chapter 3.

□

L.12.1 Fleischhack

Regular: Weyl representation is weakly continuous - said to be regular.

Stone-von Neuman theorem says that if a representation is regular and irreducible then the representation is unique.

Quantum geometry:

1. diffeomorphism invariant;

2. regular;

3. irreducible;

4. semianalytic - stratified diffeomorphisms.

L.12.2 Properties of the Kinematic Hilbert Space

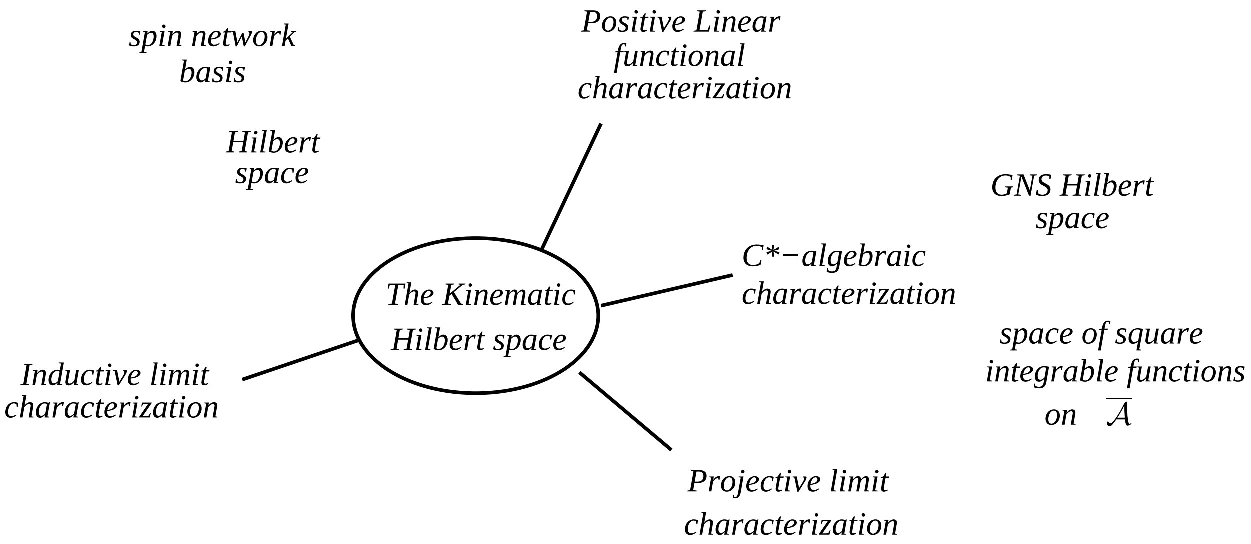

Figure L.30: KinHilbFig.

Spin networks states provide a natural decomposition of $\tilde{\mathcal{H}}^0$ into finite dimensional subspaces each of which can be identified with the space of states of a spin-system. This simplifies various constructions and calculations enermoursly.

L.13 Grassmann Integration

Grassmann Algebra

We consider a set of anticommutating Grassmann variables $\{\zeta_i\}_{i=1,\ldots,n}$, with complex linear coefficients, where n is the dimension of the algebra. The decisive relation defining the structure of the algebra is the anticommutation relation

$$\zeta_i\zeta_j + \zeta_j\zeta_i = 0 \tag{L.-30}$$

for all i and j. As a particular consequence of this condition the square and all higher powers of a variable vanish,

$$\zeta_i^2 = 0 \tag{L.-30}$$

The Grassmann algebra generate a Grassmann algebra of functions which have the form

$$f(\zeta) = f^{(0)} + \sum_i f_i^{(1)} + \sum_{i_1 < i_2} f^{(2)}_{i_1 i_2} \zeta_{i_1} \zeta_{i_2} + ... + f^{(n)} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_n} \tag{L.-30}$$

where the coefficients $f^{(k)}$ are ordinary complex numbers.

On this algebra we will need to define the derivative. We first consider a simple Grassmann algebra of order n=2 with the variables ζ_1 and ζ_2.

$$f(\zeta_1, \zeta_2) = f^{(0)} + f_1^{(1)} \zeta_1 + f_2^{(1)} \zeta_2 + f^{(2)} \zeta_1 \zeta_2$$

$$\frac{\partial f}{\partial \zeta_1} = f_1^{(1)} + f^{(2)} \zeta_2, \qquad \frac{\partial f}{\partial \zeta_1} = f_2^{(1)} - f^{(2)} \zeta_1. \tag{L.-30}$$

Note the minus sign in the last equation of (L.13). The general rule for differentiation of a product is given by

$$\frac{\partial}{\partial \zeta_j} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_m} = \delta_{j i_1} \zeta_{i_2} ... \zeta_{i_m} - \delta_{j i_2} \zeta_{i_1} \zeta_{i_3} ... \zeta_{i_m} + ... + (-1)^{m-1} \delta_{j i_2} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_{m-1}} \tag{L.-30}$$

The respective factor ζ_{i_k} is anticommuted to the left until the derivative operator can be directly applied. We may prove the following properties of the derivatives

$$\frac{\partial}{\partial \zeta_i} \frac{\partial}{\partial \zeta_j} + \frac{\partial}{\partial \zeta_j} \frac{\partial}{\partial \zeta_i} = 0 \tag{L.-30}$$

$$\frac{\partial}{\partial \zeta_i} \zeta_j + \zeta_j \frac{\partial}{\partial \zeta_i} = 0 \tag{L.-30}$$

Grassmann integration

An attempt to to introduce an indefinite integral as the inverse of differentiation is bound to fail. This illustrated by the fact that according to (L.13) the second derivative of any Grassmann function vanishes, so that the inverse operation does not exist.

We must be content with some formal definition. One way to arrive at it is to require that integration be translationally invariant. For an arbitrary function $g(\zeta) = g_1 + g_2\zeta$ we have

$$\begin{aligned} \int d\zeta g(\zeta+\eta) &= \int d\zeta\,[g_1 + g_2(\zeta+\eta)] = \int d\zeta\,[g_1 + g_2\zeta] + \int d\zeta g_2\eta \\ &= \int d\zeta g(\zeta) + \left[\int d\zeta 1\right] g_2\eta = \int d\zeta g(\zeta) \end{aligned} \tag{L.-30}$$

The translational invariance requires the integral of 1 is zero. The following postulates uniquely fix the value of any integral.

$$\int d\zeta 1 = 0, \tag{L.-30}$$

$$\int d\zeta\zeta = 1. \tag{L.-30}$$

Eq. (L.13) comes from the condition of translational invariance. The sole non-vanishing integral $\int d\zeta\zeta$ arbitrarily is assigned the value 1. This is a convenient normalisation condition.

We see that integration is equivalent to differentiation. Generalising integration rules to higher dimensions straightforward

$$\int d\zeta_i 1 = 0, \tag{L.-30}$$

$$\int d\zeta_i\zeta_j = \delta_{ij}. \tag{L.-30}$$

Note that the differentials $d\zeta_i$ must anticommute with all other Grassmann variables as integration is equivalent to differentiation. In order to obtain analog results of conventional integration we introduce complex Grassmann variables. Let us start with two disjoint sets of Grassmann variables $\zeta_1^*, ..., \zeta_n^*$ and $\zeta_1, ..., \zeta_n$, which are all mutually anticommutating

$$\{\zeta_i, \zeta_j\} = \{\zeta_i^*, \zeta_j^*\} = \{\zeta_i, \zeta_j^*\} = 0 \tag{L.-30}$$

The two sets are related, using complex cunjugation, according to

$$\begin{aligned}
(\zeta_i)^* &= \zeta_i^*, \\
(\zeta_i^*)^* &= -\zeta_i, \\
(\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m})^* &= \zeta_{i_m}^*...\zeta_{i_2}^*\zeta_{i_1}^* \\
(\lambda\zeta_i)^* &= \lambda^*\zeta_i^*
\end{aligned} \tag{L.-32}$$

where λ is a complex number.

In order to develop functional integral formalism for Grassmann fields we need to solve *Gaussian integrals.*

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp\left\{ - \sum_{k,l=1}^{N} \zeta_k^* M_{kl} \zeta_l \right\} \tag{L.-32}$$

To simplify the notation, let us write this as

$$I = \int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} \tag{L.-32}$$

The calculation in principle is very simply because grassmann functions can at worst be linear in each variable, causing the series expansion of the exponential function to terminate. On the other hand , according to the rules for Grassmann integration, the integrand must contain as many different Grassmann variables as there are integrals or else the overall integration vanishes. For the case of two pairs of variables one effectively has

$$\begin{aligned}
e^{\zeta^* M \zeta} &\rightarrow \frac{1}{2!}(\zeta^* M \zeta)^2 \\
&\rightarrow \frac{1}{2!}(\zeta_1^* M_{11}\zeta_1 \zeta_1^* M_{12}\zeta_2 \zeta_2^* M_{21}\zeta_1 \zeta_2^* M_{22}\zeta_2) \\
&\rightarrow (M_{11}M_{22} - M_{12}M_{21})\zeta_1^*\zeta_1\zeta_2^*\zeta_2
\end{aligned} \tag{L.-33}$$

where the last line follows from the anticommutating character of the Grassmann numbers. The integration of $\zeta_1^*\zeta_1\zeta_2^*\zeta_2$, gives unity, and so for this case

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} = detM \tag{L.-33}$$

One should suspect that this result holds in general. For the case of N pairs of variables, only the term of order $(\zeta^* M\zeta)^N$ survives in the expansion of the exponential. Moreover, only the terms which are multilinear in all the ζ_k^*, ζ_k can contribute and, in view of the anticommutativity of the Grassmann variables, these terms contain the appropriately signed products of matrix elements which define the determinant. But rather than go through this combinatorial exercise we will follow the method given in (Brown QFT).(page83) which is presented in Appendix(B). We do obtain the expected result:

$$\boxed{I = \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) e^{-\zeta^* M \zeta} = detM} \tag{L.-33}$$

This should be compared to the ordinary integration where the corresponding integral gives $\det M^{-1}$.

L.13.1 Grassmann generating Functional

It is not surprising that the Gaussian integral formula (L.13) can be generalised to the case of general bilinear forms in the exponent:

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp -\zeta^* M \zeta = detM e^{-\frac{1}{2}\rho^T A^{-1} \rho}. \tag{L.-33}$$

Here ρ is an n-component vector of Grassmann variables. Equation (L.13.1) is obtained by translating the integration variable, $\zeta' = \zeta + A^{-1}\rho$.

The construction of functional integration in section (4.1.2) did not make use of any special properties of the integration over field variables which might restrict the validity to ordinary c-numbers.

$$\boxed{\begin{array}{c} \int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\left[-\int d^dx'\, d^dx \bar{\chi}(x') A(x', x) \chi(x) + \int d^dx (\bar{\rho}(x)\chi(x) + \bar{\chi}(x)\rho(x))\right] \\ = detA \exp\left[\int d^dx'\, d^dx \bar{\rho}(x') A^{-1}(x', x) \rho(x)\right]. \end{array}} \tag{L.-33}$$

in which the measure is $\propto \prod_{\mathbf{r}} d\bar{\varphi}(r) d\varphi(r)$ and $Z(\rho = 0) = detA$. Note that to normalise the functional we divide by $detA$ as apposed to $det(A^{-1})$ in the bosonic case (**??**).

It is rather straightforward to extend the results of esction 4.1 to the fermionic case: The Grassmann functional derivative is defined

$$\frac{\delta G[\chi(y)]}{\delta \chi(x)} = \lim_{\Delta V_i \to 0} \frac{\partial G}{\partial \chi_i} \quad \text{where } \mathbf{x} \text{ is located in cell } \Delta V_i \tag{L.-33}$$

The (2n)-point correlators

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \langle \chi(y_n), \ldots, \chi(y_1); \bar{\chi}(x_1), \ldots, \bar{\chi}(x_n) \rangle \tag{L.-33}$$

can now be obtained by forming derivatives of the generating functional [1]

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \left. \frac{\delta^{2n} Z[\rho, \bar{\rho}]}{\delta\rho(x_n) \cdots \delta\rho(x_1) \delta\bar{\rho}(y_1) \cdots \delta\bar{\rho}(y_n)} \right|_{\rho=\bar{\rho}=0} . \tag{L.-33}$$

We could in fact use the Grassmann formalism instead of the Bosonic functional integral with the replica trick to do our calculations without too much adjustment. But we introduced the Grassmann function integral here to help form the supersymmetric formalism.

What about operators that act on this Hilbert space? All operators that are well defined on that Hilbert space arise from consistent family of operators. These operators on each of these individual finite Hilbert spaces fit together in a certain way. If it is well defined on here then it can be shown that that they come from something that fits together in this way.

L.14 Biblioliographical notes

In this chapter I have relied on the following refferences:

infinite product measures: Probability theory, S.R.S. Varadhan downloaded from www.math.nyu.edu/facult

L.15 Worked Exercises and Details

□

[1] The order of the derivatives was chosen in such that we get agreement with the bosionic case. This is not a trivial matter as the Grassmann derivatives $\delta/\delta\rho(x)$ and $\delta/\delta\rho(x)$ anticommute with the field variables $\chi(x)$ and $\bar{\chi}(x)$. One can show, however, that there is an even number of commutations when we carry out the differentiations of (L.13.1) and write the result in the form (L.13.1).

Curves (Thiemann)

a)

Despite the name, composition and inversion does not equip $\mathcal{C}$ with a group structure for many reasons.

i) Verify that composition is not associative and that the curve $c \circ c^{-1}$ is not simply $b(c)$ but rather a retracing.

ii) Moreover, contemplate that $\mathcal{C}$ does not have a unit and that not every two elements can be composed.

b)

Define composition and inversion of paths by taking the equivalence class of the compositions and inversions of any of their representatives and check that this definition is well defined.

Check that then composition of paths is associtive and that $p \circ p{-}1 = b(p)$. However, $\mathcal{P}$ still does not have a unit and still not every two elements can be composed.

c)

Let Obj:= σ and for each $x, y \in \Sigma$ let $\mathrm{Mor}(x, y) := \{p \in \mathcal{P} : b(p) = x, f(p) = y\}$. Recall the mathematical definition of a category (section **??**) and conclude that $\mathcal{P}$ is a category in which every morphism is invertible, that is, a groupoid.

d)

Define the relation $\prec$ on Γ by saying that $\gamma \prec \gamma'$ if and only if every $e \in E(\gamma)$ is a finite composition of the $e' \in E(\gamma')$ and their inverses.

Verify that $\prec$ equips Γ with the structure of a directed set, that is, for each $\gamma, \gamma' \in \Gamma$ we find $\gamma'' \in \Gamma$ such that $\gamma, \gamma' \prec \gamma''$.

For this to work, analyticity of the curve representatives is crucial. Smooth curves can intersect in Cantor sets and thus define graphs which are no longer finitely generated. Show first that this is not possible for analytic curves.

Answers:

a)

Bochner-Minlos

The characteristic function $\tilde{P}_m$ the Fourier transformation of the probability function:

$$\tilde{P}_m = \sum_l P_l e^{ilm} \tag{L.-33}$$

$$P_l = \sum_m \tilde{P}_m e^{-iml} \tag{L.-33}$$

say we also have $\tilde{P}'_m$

$$P_l = \sum_m \tilde{P}'_m e^{-iml} \tag{L.-33}$$

$$0 = P_l - P_l = \sum_m (\tilde{P}_m - \tilde{P}'_m) e^{-iml} = \sum_m a_m e^{-iml} \tag{L.-33}$$

We wish to show that the above condition can only hold if the coefficients $a_m = \tilde{P}_m - \tilde{P}'_m$ vanish. Multiply both sides by $e^{m'l}$ and sum over l,

$$0 = \sum_l \sum_m a_m e^{-i(m-m')l} = \sum_m a_m (\sum_{l'} e^{-i(m-m')} l) = \sum_m a_m \delta_{mn} = a_n \tag{L.-33}$$

Details: Operator identity

Prove the operator equation

$$e^{-\hat{B}} \hat{A} e^{\hat{B}} \tag{L.-33}$$

$$e^{-t\hat{B}} \hat{A} e^{t\hat{B}} = I + t\hat{C}_1 + \frac{t^2}{2!}\hat{C}_2 + \cdots \tag{L.-33}$$

$$e^{-(t+\delta t)\hat{B}} \hat{A} e^{(t+\delta t)\hat{B}} - e^{-t\hat{B}} \hat{A} e^{t\hat{B}} = \delta t (e^{-t\hat{B}} \hat{A}\hat{B} e^{t\hat{B}} - e^{-t\hat{B}} \hat{B}\hat{A} e^{t\hat{B}}) \tag{L.-33}$$

$$\hat{C}_1 = \frac{d}{dt}\left(e^{-t\hat{B}} \hat{A} e^{t\hat{B}}\right)|_{t=1} = \{\hat{A}, \hat{B}\} \tag{L.-33}$$

$$\frac{d^2}{dt^2}\left(e^{-t\hat{B}} \hat{A} e^{t\hat{B}}\right) = \frac{d}{dt}\left(e^{-t\hat{B}} \{\hat{A}, \hat{B}\} e^{t\hat{B}}\right) = \left(e^{-t\hat{B}} \{\{\hat{A}, \hat{B}\}, \hat{B}\} e^{t\hat{B}}\right) \tag{L.-33}$$

$$\frac{d^n}{dt^n}\left(e^{-t\hat{B}}\hat{A}e^{t\hat{B}}\right) = \left(e^{-t\hat{B}}\{\hat{A},\hat{B}\}_n e^{t\hat{B}}\right) \tag{L.-33}$$

$$\hat{C}_n = \{\hat{A},\hat{B}\}_n \tag{L.-33}$$

Details: The integral

$$I = \int_{-\infty}^{\infty} dx \left(\frac{\sin x}{x}\right)^2 \tag{L.-33}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} dx \left(\frac{\sin x}{x}\right)^2 &= \Big[-\sin^2 x\frac{1}{x}\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx \frac{2\sin x\cos x}{x} \\
&= \int_{-\infty}^{\infty} dx \frac{\sin 2x}{x} \\
&= \int_{-\infty}^{\infty} dy \frac{\sin y}{y}
\end{aligned} \tag{L.-34}$$

Can be evaluated using complex...

$$\begin{aligned}
I &= \left(\int_C dz\frac{e^{iz}}{2iz} - \int_C dz\frac{e^{-iz}}{2iz}\right) \\
&= \left(\int_{C_1} dz\frac{e^{iz}}{2iz} - \int_{C_2} dz\frac{e^{-iz}}{2iz}\right)
\end{aligned} \tag{L.-34}$$

Poisson's formula.

Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \tag{L.-34}$$

From Cauchy's integral formula, an explicit solution for the Dirichlet problem for a cirular region can be obtained. Without loss of generality, the circle can be taken to be of unit radius and centred at the origin. Let $z = e^{i\alpha}$, $\zeta = e^{it}$, $r < 1$.

$$f(z) = u(r,\alpha) + iv(r,\alpha)$$

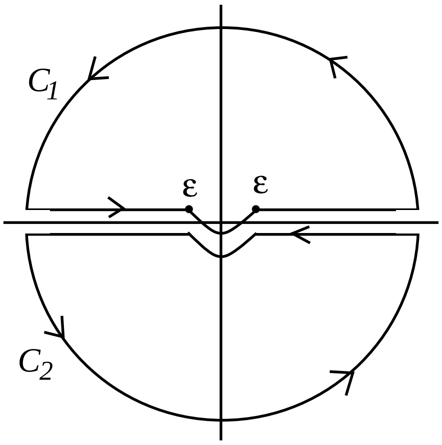

Figure L.31: $\sin x/x$.

Then since $d\zeta = i\zeta dt$, (L.15) gives

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z}dt \tag{L.-34}$$

But since z is inside the unit circle, $1/z^*$ is outside it, so that

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/z^*}dt \tag{L.-34}$$

or

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)}{\zeta - 1/z^*}d\zeta \tag{L.-34}$$

Consequently using $\eta = 1/\zeta^*$, (note $\eta = e^{it} = 1/(e^{it})^* = 1/\eta^*$)

$$\begin{aligned} \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/z^*}dt &= \frac{1}{2\pi i}\int_{2\pi}^{0} \frac{f(1/\eta^*)(1/\eta^*)z^*}{z^*/\eta^* - 1}dt \\ &= -\frac{1}{2\pi i}\int_{2\pi}^{0} \frac{f(1/\eta^*)z^*}{\eta^* - z^*}dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\eta)z^*}{\eta^* - z^*}dt \end{aligned} \tag{L.-35}$$

we arrive at

$$0 = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{z^*}{\zeta^* - z^*}dt \tag{L.-35}$$

adding or subtracting this to (L.15)

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{\zeta}{\zeta - z} \pm \frac{z^*}{\zeta^* - z^*}\right) dt \tag{L.-35}$$

Taking first the positive sign in (L.15),

$$\frac{\zeta}{\zeta - z} + \frac{z^*}{\zeta^* - z^*} = \frac{\zeta(\zeta^* - z^*) + z^*(\zeta - z)}{|\zeta - z|^2} = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2},$$

we obtain

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1 - |\zeta|^2}{|\zeta - z|^2} dt \tag{L.-35}$$

But the factor multiplying $f(\zeta)$ is purely real, so that the process of taking the real part gives

$$\begin{aligned} u(r,\alpha) &= \frac{1}{2\pi}\int_0^{2\pi} u(t)\ \frac{1 - r^2}{|e^{ti} - re^{i\alpha}|^2}\ dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(t)\ \frac{1 - r^2}{1 - 2r\cos(\alpha - t) + r^2} \end{aligned} \tag{L.-35}$$

where $u(t)$ is the value of the harmonic function u on the boundary. This result, which solves the Dirichlet problem for the circle, is known as Poison's formula.

If in (L.15) we take the negative sign,

$$\frac{\zeta}{\zeta - z} - \frac{z^*}{\zeta^* - z^*} = \frac{\zeta(\zeta^* - z^*) - z^*(\zeta - z)}{|\zeta - z|^2} = \frac{|\zeta|^2 + |z|^2 - 2\zeta z^*}{|\zeta - z|^2}$$

we shall obtain the conjugate function v in terms of $u(t)$,

$$\begin{aligned} v(r,\alpha) &= v(0) + \frac{1}{\pi}\int_0^{2\pi} f(\zeta)\frac{1 - 2\zeta z^* + |z|^2}{|\zeta - z|^2} dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(1 + \frac{2iIm(z\zeta^*)}{|\zeta - z|^2}\right) dt \end{aligned} \tag{L.-35}$$

Consequently,

$$v(r,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(t)\ \frac{r\sin(\alpha - t)}{1 - 2r\cos(\alpha - t) + r^2} dt \tag{L.-35}$$

Finally we obtain $f(z)$ in terms of $u(t)$, by combining the two results for u and v,

$$f(z) = iv(0) + \frac{1}{2\pi} \int_0^{2\pi} u(t) \, \frac{\zeta + z}{\zeta - z} \, dt \tag{L.-35}$$

Appendix M

Hamiltonian Constraint

M.1 Solving Quantum Constraint Equations

Rovelli

" The space of physical states must have the structure of a Hilbert space, namely a scalar product, in order to be able to compute expectation values. This Hilbert structure is determined by the requirement that real physical observables correspond to self-adjoint operators. In order to define a Hilbert space of physical states, it is convenient to define first a Hilbert space of unconstrained states. This is because we have a much better knowledge of the unconstrained observables than the physical ones. If we choose a scalar product on the unconstrained state space which is gauge invaiant then there exist standard techniques to "bring it down" to the space of the physical states. Thus, we need a gauge and diffeomorphism invariant scalar product, with respect to which real observable are self-adjoint operators. "

M.2 Projection Operator

We firstly quantized the theory ingnoring the constraints (here the spatial diff and Hamiltonian constraints) to get a kinematical Hilbert space H_{Kin}. The constraints are imposed as self-adjoint operators on H_{Kin} and the physical states are those that are annihilated by the by them (mathematicions call the set of such states that are annihilated by an operator the *kernel* of the operator).

The symmetry group is compact, (finite group volume), so the solutions to this constraint are honest members of the Hilbert space.

This follows Rovelli's

The space of square integrable functions i.e. $\int_\Lambda d^2x|\Phi(x)|^2$ $(H = L^2(M))$

Before going any further we introduce a simple toy model of a constrained quantum mechnical system.

$$\hat{J} := \hat{J}_z = i(x\partial_y - y\partial_x) = J_\varphi = i\partial_\varphi \tag{M.0}$$

Consider a simple

$$\hat{\mathcal{J}}\Psi = 0 \tag{M.0}$$

$$\hat{\mathcal{J}}\Psi = 0 \tag{M.0}$$

The constraint equation defines a subspace $\mathcal{H}_{Phys}$ of $\mathcal{H}_{aux}$ - the space of physical states.(the set of such solutions are known as the *kernel* of the operator.

$$\Pi : \quad H \to H_{Phys}. \tag{M.0}$$

Note that the (finite, see appendix d.2) action of the constraint on a general state functional $\Psi(r, \varphi)$ is given by

$$e^{i\alpha\hat{\mathcal{J}}}\Psi(r,\varphi) = \Psi(r, \varphi + \alpha) \tag{M.0}$$

Hence, the progection operator Π on the physical states is defined as

$$\begin{aligned} \Pi &= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\ \Psi(r,\varphi) \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\ \Psi(r,\varphi+\alpha) = \Psi\tilde(r) \end{aligned} \tag{M.0}$$

resulting in a new function $\Psi\tilde(r)$ which is independent of φ, just as one might have expected for physical states, $\hat{\mathcal{J}}\Psi_{Phys}(r,\varphi) = 0$.

Using a scalar product

$$\langle\Psi|\Phi\rangle = \int_0^{2\pi}\int_0^\infty d\varphi\overline{\Psi(r,\varphi+\alpha)}\Phi(r,\varphi+\alpha) \tag{M.0}$$

in H, one arrives at the important result

$$\langle\Psi|\Phi\rangle_{Phys} \equiv < \Psi|\Pi|\Phi > . \quad \text{(M.0)}$$

The quadratic form $\langle\rangle_{Phys}$ in H_{Phys} is indeed expressed as a scalar product over states which lie in H. Thus, knowing the matrix elements (P.12) of the progection operator in the unconstraind Hilbert space is equivalent to having solved the constraint!

A similar scheme can be applied to operators (we talk more about this when we come to the Master constraint). Suppose there exists a non-gauge invariant operator $O = O(r, \varphi)$ on H. Then a fully gaue invariant operator $R = R(r)$ in H_{Phys} can be constructed by defining

$$R := \Pi O \Pi. \quad \text{(M.0)}$$

The calculation of the physical operator R is then reduced to a calculation in the unconstrained Hilbert space, which gives

$$\langle\Psi|O|\Phi\rangle_{Phys} \equiv < \Psi|\Pi O \Pi|\Psi > . \quad \text{(M.0)}$$

Some of the Jargon

the physical state functionsals are only those solutions that have the zero eigenstate of the constraint operator $\mathcal{J}$, (the set of such solutions are known as the *kernel* of the operator).

Let us summarize the previous section in more techniqual language.

$$\Phi \in \mathcal{H} \in \Phi^* \quad \text{(M.0)}$$

where $\mathcal{H}$ is a Hilbert space, Φ is a dense subset of the Hilbert space, and Φ^* is the dual space of Φ, i.e Φ^* is the space of antilinear functionals over the space Φ.

M.3 Rigged Hilbert Space

M.3.1 Inadequancies of Hilbert Space.

We firstly quantized the theory ingnoring the constraints (here the spatial diff and Hamiltonian constraints) to get a kinematical Hilbert space H_{Kin}. The constraints are imposed as self-adjoint operators on H_{Kin} and the physical states are those that are annihilated by the by them (mathematicions call the set of such states that are annihilated by an operator the *kernel* of the operator). There is a problem here; such states are often not normalisible by the inner product of the unconstrained system, and hence do not belong to the kinematic Hilbert space! However, one should not be unduely alarmed as such "improper" solutions to quantum mechanical opretors are already evident in ordinary quantum mechanics. Infact, such issues come up whenever an operator has a continuous spectrum. For example,

Consider the space $\mathcal{K} = L^2[R^2, dxdy]$ and the self-adjoint operator

$H = -id/dx$. The solutions of $H\psi = 0$ or

$$-i\frac{d}{dx}\psi(x,y) = 0 \tag{M.0}$$

are functions $\psi(x,y)$ constants in the y and are non-normalizable in $\mathcal{K}$.

$$\tilde{f}(q_n) = \sum_{i=0}^{N} e^{ia_iq} f(a_i) \tag{M.1}$$

$$f_i = \sum_{qn2\pi/N} e^{-ia_iq_n} \tilde{f}(q_n) \tag{M.2}$$

finite intervals, say $-L/2$ to $L/2$

$$\tilde{f}(q_n) = \frac{1}{2\pi}\int_{-L/2}^{L/2} dx e^{ixq} f(x) \tag{M.3}$$

$$f(x) = \sum_{-\infty}^{\infty} e^{ixq_n} \tilde{f}(q_n) \tag{M.4}$$

$$\tilde{f}(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ixq} f(x) \tag{M.5}$$

$$f(x) = \int_{-\infty}^{\infty} dq e^{-ixq} \tilde{f}(q) \tag{M.6}$$

The "functions" $\psi(x) = e^{ixp}$ can only be normalized up to the delta function:

$$\int_{-\infty}^{\infty} dx\overline{\psi(p)}\psi(q) = \int_{-\infty}^{\infty} dx e^{ixp} e^{ixq} = 2\pi\delta(p-q) \tag{M.6}$$

We handle them according to the rules for using delta functions.

$$\hat{\mathcal{Q}}\psi(x) = x\psi(x). \tag{M.6}$$

The formal solution is the Dirac delta function

$$\phi(x) = \delta(x - \lambda). \tag{M.6}$$

$$\hat{\mathcal{Q}}|\lambda >= \lambda|\lambda >, \tag{M.6}$$

from which follows that

$$< \lambda'|\hat{\mathcal{Q}}|\lambda >= \lambda < \lambda'|\lambda > \tag{M.6}$$

On the other hand

$$< \lambda'|\hat{\mathcal{Q}}|\lambda >= \int_{-\infty}^{\infty} dx\delta(x - \lambda')\lambda\delta(x - \lambda) \tag{M.6}$$

Comparing we have

$$\boxed{< \lambda'|\lambda >= \delta(\lambda' - \lambda) \qquad \text{(M.7)}}$$

Making use of the delta function property (), we can calculate the inner product of the position eigenstate $|\lambda >$ with arbitrary $|\psi >$.

$$< \lambda|\psi >= \int_{-\infty}^{\infty} dx\delta(x - \lambda)\psi(x) = \psi(\lambda). \tag{M.7}$$

$$\boxed{\psi(x) =< x|\lambda >, \qquad \text{(M.8)}}$$

"projection operators" $|x >< x|$

$$\int_{-\infty}^{\infty} dx|x >< x|. \tag{M.8}$$

For any $|\phi >, |\psi >\in H$, we have

$$< \phi| \left(\int_{-\infty}^{\infty} dx|x >< x| \right) |\psi >= \int_{-\infty}^{\infty} dx < \phi|x >< x||\psi >= \int_{-\infty}^{\infty} dx\overline{\phi(x)}\psi(x) =< \phi|\psi > . \tag{M.8}$$

M.3.2 Introduction to Generalized Eigenfunctons and the RHS.

In undergaduate quantum mechanic courses we are told that wavefunctions are required to be square integrible in order to have a probabilistic interpretation. The Hilbert space is introduced as the space of squar integrable functions, $||\psi(x)||$, with respect to the usual inner product $\int dx\overline{\phi(x)}\psi(x)$. We are also told that plane waves are solutions to the Hamiltonian for free electrons and have energy $E = \hbar^2 k^2/2m$. However, as mentioned above, they cannot be normalized and so lie outside the Hilbert space.

In our undergraduate courses we circumvate this problem by considering electrons in a box of finite volume, V, with infinite barries. The wavefunctions that are eigenstates are obviously square integrable and so lie wihtin the Hilbert space. At the end we take the limit that the volume goes to infinity, $V \to \infty$. This is one way to avoid wavefunctions that are not normalizible. The justification given for taking this limit is that experiments are not going to be able to tell you the difference between a very large box of electrons and free electrons.

$$\begin{aligned} \mathcal{H}|E_n) &= E_n|E_n) \\ \mathcal{H}|E >&= E|E > \end{aligned} \tag{M.8}$$

These eigenstates are normalized according to:

$$(E_n|E_m) = \delta_{nm}, \quad < E|E' >= \delta(E - E'), \quad < E|E_n) = 0 \tag{M.8}$$

$$\varphi = \sum_n |E_n)(E_n|\varphi) + \int dE|E >< E||\varphi > . \tag{M.8}$$

kets $|\phi >$ are an astraction and must be understood to always be accompanated by a bra $< \psi|$.

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \tag{M.8}$$

$$< \phi(x)|\mathcal{H}\psi(x) >= \int_{-\infty}^{\infty} \phi^*(x)\mathcal{H}\psi(x) \tag{M.8}$$

In the case where $\psi(x) = e^{ikx}$ the vector $\mathcal{H}e^{ikx}$ is not defined. But we can define another Hamiltonian $\mathcal{H}^\times$ (that is a natural extension of $\mathcal{H}$) for which

$$< \phi(x)|\mathcal{H}^\times e^{ikx} >=< \mathcal{H}^\dagger \phi(x)|e^{ikx} > \tag{M.8}$$

where we integrated by parts in the third step

$$< \phi|\mathcal{H}\psi >=< \mathcal{H}^\times \phi|\psi > \tag{M.8}$$

where $\mathcal{H}^\times$ defines the dual action of $\mathcal{H}$.

$\mathcal{H}^\times$ is called the dual extension of $\mathcal{H}$.

experiments can never be complete

The inceptive motivation for introducing RHS in quantum mechanics was to provide Dirac's bra and ket formulism, already established calculational tool, with a proper mathematical content. Indicate how to make the above ideas more rigorous

$$(\varphi, H\varphi) \tag{M.8}$$

space of unconstrained observables

$$\Delta_\varphi H = \sqrt{(\varphi, H^2\varphi) - (\varphi, H\varphi)^2} \tag{M.8}$$

The expectation values cannot be computed for every element of the unconstrained Hilbert space, but only for those for which those $\varphi \in H$ that aslo belong to $\mathcal{D}(H)$

The solutions to the constraint are not square normalizable and so lie outside the Hilbert space. Therefore the unconstrained Hilbert space is not large enough to contain the non-normalizable physical states - a larger space is needed

$$\langle\varphi|\, H^\times \,|\rangle = \tag{M.8}$$

The dual space $\mathbf{\Phi}^\times$ contains the eigenkets associated with the solutions to the constraint operator.

Unitary representations of G on the rigged Hilbert space H_R instead on H. $U_R : G \to \mathcal{U}(H)$ for normalizable states $|\psi >$,

$$U_R(g) \equiv U(g)|\psi >, \tag{M.8}$$

and for non-normalizable states $|x>$

$$U_R(g) := |gx> \tag{M.8}$$

$$U(g)|\psi>=|\psi'> . \tag{M.8}$$

Then,

$$\begin{aligned} U(g)\psi(x) &= \psi'(x) =< x|\psi'>=< U(g)|\psi> \\ &= < x|U_R(g)|\psi>=< U_R^{-1}x|\psi>=< g^{-1}x|\psi>= \psi(g^{-1}x) \end{aligned} \tag{M.8}$$

satisfies the group homomorphism requirement $(g_1g_2)^{-1} = g_2^{-1}g_1^{-1}$

$$\begin{aligned} U(g_1)U(g_2)\psi(x) &= U(g_1)\psi(g_2^{-1}x) = \psi(g_1^{-1}g_2^{-1}x) \\ &= \psi((g_2g_1)^{-1}x) = U(g_1g_2)\psi(x) \end{aligned} \tag{M.8}$$

$$U(g_1)U(g_2)\psi(x) = U(g_1g_2)\psi(x) \tag{M.8}$$

M.3.3 Projector Technique with a Rigged Hilbert Space

We need to extend the progector (or group-averaging) technique to the caase where the symetry group is non-compact (has divergent group volume) where the solutions lie in the dual space.

An antilinear form $\eta : \Omega \to \Omega^*$ is provided by group averaging

$$(\psi| = \frac{1}{V}\int d\lambda \int dk \int d_{\mu_k} < \tag{M.8}$$

The physical inner product, denoted as $<,>$, is defined as

$$< \eta\psi', \eta\psi >_{phys}= (\psi|\psi'> . \tag{M.8}$$

M.4 Group-averaging

Projecting it onto the physical Hilbert space by gauge-orbit smearing.

Let us first suppose that the constraint operator $\hat{C}$ on the kinematical Hilbert space H_{kin} is self-adjoint, and that zero is a discrete point in its spectrum. Then, the kernel of $\hat{C}$ is a subspace of H_{kin}. Therefore, to extract physical states, one just has to project kinematical states to this subspace. In the case when the 1-parameter group $U(\lambda) = e^{-i\hat{C}}$ generated by $\hat{C}$ on H_{kin} provides a representation of $U(1)$, the projection procedure can be explicitly carried out through an integration: Given any $\Psi \in H_{kin}$, set

$$\Psi_{Phys} = \frac{1}{\Lambda}\int^{\Lambda} d\lambda e^{-i\hat{C}}\Psi \tag{M.8}$$

Since these physical states belong to H_{kin}, the scalar product between them is well-defined.

you have to always first of all you should define a subspace in the Hilbert space which should have stronger topology that the topology of the Hilbert space.

by Riesz' theorem, there is a unique vector $\eta'(\phi) \in \mathcal{H}_{Phys}$ which satisfies

$$\phi(f) := f[\phi] =< \eta'(\phi)|f >_{Phys} \tag{M.8}$$

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[269]:

The group averaging procedure [126], [127] need not result in a state which has finite and positive norm.

$$(\Psi_{Phy}| := \int d\lambda < e^{-i\lambda\hat{C}}\Psi|. \tag{M.8}$$

$$(\Psi_{Phy}|\Phi >= \int d\lambda < e^{-i\lambda\hat{C}}\Psi|\Phi > . \tag{M.8}$$

This procedure can be heuristically understood as follows. (M.4) extracts from Ψ a physical state $\Psi_{phy} \in \mathcal{S}^*$. This extractor $\hat{E}$ can be formally thought of as $\hat{E} = \delta(\hat{C})\Psi$. Therefore, the naive definition $< \Psi_{phy}|\Psi_{phy} >= (\delta(\hat{C})\Psi)|(\delta(\hat{C})\Psi)$ of the norm that one may first think of is divergent. In the correct definition, (M.4), one of the two delta-distributions is simply dropped, thereby removing the obvious infinity.

The norm of physical states is

$$\|\Psi_{Phy}\|^2 = \int d\lambda < e^{-i\lambda\hat{C}}\Psi|\Psi > . \tag{M.8}$$

For a general choice of the initial subspace $\mathcal{S}$, there is no guarantee that the norm would be finite and positive. The art in the group averaging procedure lies in selecting a dense subspace $\mathcal{S}$ of H_{kin} such that: i) the right side of (M.4) is well-defined for all $\Psi, \Phi \in \mathcal{S}$; i.e., Ψ_{Phy} is a well-defined distribution over S; and, ii) the norm (2.16) of each Ψ_{phy} is non-negative, vanishing if and only if Ψ_{phy} vanishes. The procedure succeeds in its goal of constructing the physical Hilbert space only if such a $\mathcal{S}$ can be located.

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M.4.1 The Rigging Map

Assume the constraints are Abelian

$$\{F_\mu, F_\nu\} = 0 \tag{M.8}$$

$$\eta : \mathcal{H}_{kin} \to \mathcal{H}_{phys}$$

$$\psi \mapsto [\eta(\psi)](\phi, Q) = \int [d\beta/(2\pi)]\, [e^{i\beta^\mu F_\mu}](\phi, Q) \tag{M.8}$$

We can write this formally as

$$\eta(\psi) = \prod_\mu \delta(F_\mu)\psi \tag{M.8}$$

To prove this we use

$$e^{\beta^\mu \pi_\mu}\psi(\phi) = \psi(\phi - \beta)$$

Denote

$$\beta(t) := \beta_1 + t(\beta_2 - \beta_1)$$

It follows that

$$
\begin{aligned}
V(\beta_2) - V(\beta_1) &= \int_0^1 dt_1 \frac{d}{dt_1} V(\beta(t_1)) \\
&= i\int_0^1 dt_1 V(\beta(t_1))\ \dot{\beta}(0) h'_\mu(\phi - \beta(t_1)) \qquad \text{(M.8)}
\end{aligned}
$$

We can find an iterative formula for $V(\beta(t))$. Note

$$
\beta_{t_1}(t_2) = \beta_1 + t_2(\beta(t_1) - \beta_1) = \beta_1 + t_1 t_2(\beta_2 - \beta_1) = \beta(t_1 t_2) \qquad \text{(M.8)}
$$

so that

$$
\begin{aligned}
V(\beta_2(t_1)) - V(\beta_1) &= \int_0^1 dt_2 \frac{d}{dt_2} V(\beta_{t_1}(t_2)) = i\int_0^1 dt_2 \frac{d}{dt_2} V(\beta(t_1 t_2)) t_1 \dot{\beta}(0) h'_\mu(\phi - \beta(t_1 t_2)) \\
&= i\int_0^{t_1} dt_2 V(\beta(t_2))\ \dot{\beta}(0) h'_\mu(\phi - \beta(t_2)) \qquad \text{(M.8)}
\end{aligned}
$$

$$
V(\beta(t_1)) = V(\beta_1) + i\int_0^{t_1} dt_2 V(\beta(t_2)) \dot{\beta}^\mu(0) h'_\mu(\phi - \beta(t_2))
$$

or

$$
V(\beta_1)^{-1} V(\beta(t_1)) = 1 + i\int_0^{t_1} dt_2 V(\beta_1)^{-1} V(\beta(t_2))\ \dot{\beta}^\mu(0) h'_\mu(\phi - \beta(t_2)) \qquad \text{(M.8)}
$$

Now set

$$
\begin{aligned}
U(t,0) &:= V(\beta_1)^{-1} V(\beta(t)) \\
\Phi(t) &:= \dot{\beta}^\mu(0) h'_\mu(\phi - \beta(t)) \\
&= [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta(t)) \qquad \text{(M.7)}
\end{aligned}
$$

Then () becomes

$$
U(t,0) = 1 + i\int_0^t dt_1 U(t_1, 0)\Phi(t_1) \qquad \text{(M.7)}
$$

From this equation we obtain successive approximations:

$$
\begin{aligned}
U_0(t,0) &= 1 \\
U_1(t,0) &= 1 + i\int_0^{t_1} dt_1 \Phi(t_1) \\
U_2(t,0) &= 1 + i\int_0^{t_1} dt_1 U_1(t_1,0)\Phi(t_1) \\
&= 1 + i\int_0^{t} dt_1 \Phi(t_1) + (i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \Phi(t_2)\Phi(t_1). \qquad \text{(M.5)}
\end{aligned}
$$

The $N-th$ order approximation being

$$
U_N(t,0) = 1 + \sum_{n=1}^{N} U^{(n)}(t,0)
$$

where

$$
U^{(n)}(t,0) = (i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \Phi(t_n)\ldots\Phi(t_1)
$$

We define the left-time-ordered product as

$$
T_l[\Phi(t_2)\Phi(t_1)] \equiv \begin{cases} \Phi(t_2)\Phi(t_1) & t_1 > t_2 \\ \Phi(t_1)\Phi(t_2) & t_2 > t_1 \end{cases}
$$

Thus

$$
U^{(2)}(t,0) = \frac{i^2}{2}\int_0^t dt_1 \int_0^t dt_2 T_l[\Phi(t_2)\Phi(t_1)]
$$

for n operators we obtain

$$
U^{(n)}(t,0) = \frac{i^n}{n!}\int_0^t dt_1 \int_0^t dt_2 \ldots \int_0^t dt_n T_l[\Phi(t_n)\ldots\Phi(t_1)] \qquad \text{(M.5)}
$$

$$
\begin{aligned}
U(t,0) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n T_l[\Phi(t_n) \ldots \Phi(t_1)] \\
&= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n \Phi(t_n) \ldots \Phi(t_1) \\
&= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int_0^t d\tau \Phi(\tau) \right)^n \qquad \text{(M.4)}
\end{aligned}
$$

We can formally write the expression for $U(t,0)$ as

$$
V(\beta_1)^{-1} V(\beta(t_1)) = T_l \exp(i \int_0^{t_1} dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))
$$

Setting $t_1 = 1$ we get

$$
V(\beta_1)^{-1} V(\beta_2) = T_l \exp(i \int_0^1 dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))
$$

Theorem M.4.1 *Equation () holds pointwise in ϕ spaceon a dense set of analytic vectors for the operator $h'(\phi) = \phi^\mu h'_\mu(\phi)$.*

Proof: Let $V_0(\phi) := 1$ for $N > 0$

$$
V_N(\phi) = 1 + \sum_{n=1}^{N} i^n \int_0^1 dt_1 \ \phi^{\nu_1} h'_{\nu_1}(t_1 \phi) \ldots \int_0^{t_{n-1}} dt_n \ \phi^{\nu_n} h'_{\nu_n}(t_n \phi) \qquad \text{(M.4)}
$$

We define for $N > 0$

$$
\begin{aligned}
R_N(\phi) &= -i^{N-1} \sum_{n=1}^{N} \int_0^1 dt_1 \ \phi^{\nu_1} h'_{\nu_1}(t_1 \phi) \ldots \int_0^{t_{n-1}} dt_n \ t_n \phi^{\nu_n} \{[h'_\mu, h'_{\nu_n}]\}(t_n \phi) \cdots \times \\
&\times \ldots \int_0^{t_{N-1}} dt_N \ \phi^{\nu_N} h'_{\nu_N}(t_N \phi) \qquad \text{(M.4)}
\end{aligned}
$$

and prove by induction for $N > 0$ that

$$
\partial_\mu V_N(\phi) = i h'_\mu(\phi) V_{N-1}(\phi) + R_N(\phi) \qquad \text{(M.4)}
$$

□

M.4.2 More on RAQ: Open Algebras

Say in the classical theory the constraints C_I are real and the structure functions ${f_{IJ}}^K$ are imaginary then

$$\hat{C}_I^\dagger = \hat{C}_I, \qquad ({\hat{f}_{IJ}}{}^K)^\dagger = -{\hat{f}_{IJ}}{}^K$$

Take the classical brackets

$$\{C_I, C_J\} = {f_{IJ}}^K C_K \tag{M.4}$$

and replace them by the arbitrary ordering (that depending on the constants λ_{IJ}),

$$[\hat{C}_I, \hat{C}_J] = \lambda_{IJ} {\hat{f}_{IJ}}{}^K \hat{C}_K + (1-\lambda_{IJ}) \hat{C}_K {\hat{f}_{IJ}}{}^K$$

as can easily be seen by taking the congujate of both sides

$$[\hat{C}_I, \hat{C}_J]^\dagger = \lambda_{IJ} \hat{C}_K^\dagger ({\hat{f}_{IJ}}{}^K)^\dagger + (1-\lambda_{IJ}) ({\hat{f}_{IJ}}{}^K)^\dagger \hat{C}_K^\dagger$$

which gives

$$[\hat{C}_I, \hat{C}_J] \quad = \quad (1-\lambda_{IJ}) {\hat{f}_{IJ}}{}^K \hat{C}_K + \lambda_{IJ} \hat{C}_K {\hat{f}_{IJ}}{}^K$$

therefore we must have $\lambda_{IJ} = 1/2$ and we would have to have

$$[\hat{C}_I, \hat{C}_J] = \frac{1}{2} ({\hat{f}_{IJ}}{}^K \hat{C}_K + \hat{C}_K {\hat{f}_{IJ}}{}^K) \tag{M.3}$$

It turns out that this is disasterous for solving the constraints. to see this impose

$$[(\hat{C}_I)' l](f) := l(\hat{C}_I^\dagger) = 0 \quad \text{for all } I \in \mathcal{I}, \ f \in \mathcal{D}_{kin}$$

on (M.4.2) we obtain

$$(({\hat{f}_{IJ}}{}^K)' \hat{C}_K)' + (\hat{C}_K)' ({\hat{f}_{IJ}}{}^K)') l = ([(\hat{C}_I K)', ({\hat{f}_{IJ}}{}^K)']) l = 0 \tag{M.3}$$

implying that l is not only annihilated by the dual constraint operators but also by $[(\hat{C}_I K)', (\hat{f}_{IJ}{}^K)']$, which is not necessarily proportional to a dual constraint, implying that the physical Hilbert space will be too small.

Therefore, in the case of an open constraint algebra the constraints should not be chosen to be self-adjoint operators. Notice that this is no contraction because self-adjointness is usually required to ensure that the spectrum (measurement values) of the operator lies in the real line, however, for constraint operators this is not a requirement because we are only intereseted in the kernel and the only requirement is that the point zero belongs to the spectrum at all.

In order to allow for non-self-adjoint constraints, in what follows we will assume that the set

$$\mathcal{C} := \{\hat{C}_I : I \in \mathcal{I}\}$$

is self-adjoint (i.e., contains with $\hat{C}_I$ also $\hat{C}_I^\dagger = \hat{C}_J$ for some J)

Consider the commutant of $\mathcal{C}$ within $\mathcal{O}_{kin}$, that is,

$$\mathcal{C}' := \{O \in \mathcal{O}_{kin} : [C, O] = \text{ for all } C \in \mathcal{C}\} \tag{M.3}$$

$I_{\mathcal{Z}}$ is a two-sided ideal in $\mathcal{C}'$ if

$$[I_{\mathcal{Z}}, \mathcal{C}'\ ] \in I_{\mathcal{Z}} \quad \text{and} \quad [\mathcal{C}'\ , I_{\mathcal{Z}}] \in I_{\mathcal{Z}}$$

The set of commutators $[\mathcal{C}', \mathcal{C}']$ is a subalgebra of $\mathcal{C}'$.

M.5 The Direct Integral Decomposition of the Hilbert Space

Consider the space $\mathcal{K} = L^2[\mathbb{R}^2, dxdy]$ and the self-adjoint operator $H = -id/dx$. The solutions of $H\psi = 0$ or

$$-i\frac{d}{dx}\psi(x, y) = 0 \tag{M.3}$$

are functions $\psi(x, y)$ constants in the y and are non-normalizable in $\mathcal{K}$. However, the decomposition

$$\mathcal{K} = \int_{\mathbb{R}} dy H_y. \tag{M.3}$$

where $H(y) = L^2[\mathbb{R}, dx]$.

$$(\psi, \phi)_{\mathcal{K}} = \int_{\mathbb{R}^2} dx \overline{\psi(x,y)} \phi(x,y) = \int_{\mathbb{R}} (\psi_y, \phi_y)_{H_y}, \tag{M.3}$$

where $\psi_y(x) = \psi(x,y)$ and

$$(\psi_y, \phi_y)_{H_y} = \int_{\mathbb{R}} dx \overline{\psi_y(x)} \phi_y(x). \tag{M.3}$$

The space of solutions of (M.5) is $\mathcal{H}(0)$ and has the natural Hilbert structure $\mathcal{H}(0) = L^2[\mathbb{R}, dx]$.

M.5.1 The Direct Integral Decomposition Theorem

Theorem M.5.1 *(Direct Integral Decomposition (DID)). Let a be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then there is a unitary operator U such that*

$$U\mathcal{H} = \mathcal{H}^{\oplus} = \int_{\mathbb{R}}^{\oplus} d\mu(\lambda) \mathcal{H}^{\oplus}(\lambda)$$

where μ is a probability measure and UaU^{-1} is represented on $\mathcal{H}^{\oplus}(\lambda)$ by multiplication by λ. Moreover, the measure class $< \mu >$ and the Hilbert spaces $\mathcal{H}^{\oplus}(\lambda)$ areuniquely determined.

Proof:

Let the projection-valued measure of a be denoted by $E(\lambda)$. Consider

$$W_t = \exp(ita) = \int_{\mathbb{R}} dt e^{it\lambda} E(\lambda)$$

this is bounded and weakly continuous

$$\psi_f := \int_{\mathbb{R}} dt \; f(t) W_t \psi \tag{M.3}$$

Stone's theorem regards one-parameter unitary groups which establishes a one-to-one correspondence between self-adjoint operators on a Hilbert space $\mathcal{H}$ and one parameter families of unitary operators

$$\{U_t\}_{t\in\mathbb{R}}$$

which are strongly continuous, that is

$$\lim_{t\to t_0} U_t\psi = U_{t_0}\psi \quad \text{for all} \quad t_0 \in \mathbb{R}, \psi \in \mathcal{H}$$

and are homomorphisms

$$U_{t+s} = U_t U_s.$$

Choose any vector ψ_1 and function $f_1 \in C_0^\infty(\mathbb{R})$ and set

$$\Omega_1 = \int_{\mathbb{R}} dt\ f_1(t) W_t \psi_1$$

Set

$$p(W) = \sum_{k=1}^{N} z_k W_{t_k}$$

Denote by $\mathcal{H}_1$ the closure of the finite linear span of the W_t, that is

$$\mathcal{H}_1 := \overline{\{p(W)(\int_{\mathbb{R}} dt\ f_1(t) W_t \psi_1) :\ \text{for } N < \infty, z_k \in \mathbb{C}\}}$$

If $\mathcal{H}_1 \neq \mathcal{H}$ choose $\psi_2 \in \mathcal{H}_1^\perp$ and $f_2 \in C_0^\infty(\mathbb{R})$. Then

$$W_{-t} \sum_{k=1}^{N} z_k W_{t_k} \Omega_1 = \sum_{k=1}^{N} z_k W_{t_k - t} \Omega_1 \in \mathcal{H}_1.$$

Hence

$$
\begin{aligned}
< \Omega_2, \Omega_1 > &= \int_{\mathbb{R}} dt\ \overline{f_2(t)} < W_t \psi_2, \Omega_1 > \\
&= \int_{\mathbb{R}} dt\ \overline{f_2(t)} < \psi_2, W_{-t} \Omega_1 > \\
&= 0
\end{aligned}
$$

so also $\Omega_2 \in \mathcal{H}_1^{\perp}$. Iterating, since $\mathcal{H}$ is separable we arrive at the direct sum

$$\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n \tag{M.0}$$

A dense set of vectors can be presented in the form

$$(p_n(W)\Omega_n)_{n=1}^{\infty}.$$

How $a^m \Omega_n$ are elements of $\mathcal{H}_n$

Thus, a dense set of vectors can be given in form

$$(p_n(a)\Omega_n)_{n \in \mathbb{N}}$$

where p_n is a polynomial in a and the Ω_n are C^{∞}-vectors for a.

Borel measures

Consider the probability Borel measure (all the μ_n are Borel measures)

$$\mu(\lambda) := \sum_{n=1}^{\infty} c_n\ \mu_n(\lambda) \tag{M.0}$$

Radon-Nikodym Derivative

$$\rho_n = \frac{d\mu_n}{d\mu}$$

□

Example:

Two commuting constraints

$$C_1 = p_1, \quad C_2 = p_2$$

for a particle moving in the plane.

$\square$

M.5.2 Comparing the RAQ and DID Programmes

M.6 Anomaly-freeness

Anomaly-freeness WAS HERE

M.7 Quantum Dirac Algebra

We may now compute the commutator $[\hat{C}(N), \hat{C}(N')]$ on Φ_{Kin} corresponding to the Poisson bracket $\{C(N), C(N')\}$ which is proportional to the spatial diffeomorphism constraint C_a. This commutator turns out to be non-vanishing on Φ_{Kin} as it should be, however,

$$\Psi([\hat{C}(N), \hat{C}(N')]f) = 0$$

for all $f \in \Phi_{Kin}, \Psi \in (\Phi^*_{Kin})_{Diff}$. This is precisely how we would expect it in the absence of an anomaly. Note that this is sometimes paraphrased....... in the strict sense, the commutator is defined on Φ_{Kin}, where it would not vanish, and not on $(\Phi^*_{Kin})_{Diff}$. On the other hand, the right hand side of the commutator on Φ_{Kin} does not obviously resemble the quantization of the classical expression $\int d^3x(NN'_{,a} - N_{,a}N')q^{ab}C_b$ so there are doubts, expressed in [], [] whether the quantization of $C(N)$ produces the correct quantum dynamics.

M.8 Quantization Ambiguity

source of ambiguity is associated with the choice of representation j of the new edge added by the action of the Hamiltonian constraint.

using again the expansion of the holonomy,

$$
\begin{aligned}
h_{s_k}^{(m)} &\approx I^{(m)} + A_a^j \tau_j^{(m)} s_k^a(\Delta) && \text{(M.1)} \\
h_{\alpha_{ij}}^{(m)} &\approx I^{(m)} + F_{ab}^j \tau_k^{(m)} s_i^a(\Delta) s_j^b(\Delta) && \text{(M.2)}
\end{aligned}
$$

$$
\operatorname{tr}\left(\tau_i^{(m)} \tau_j^{(m)}\right) = -\frac{1}{12} m(m+1)(m+2)\delta_{ij} \tag{M.2}
$$

$$
C(m) = \frac{1}{12} m(m+1)(m+2) \tag{M.2}
$$

then

$$
\mathcal{H}_T^m[N] := \sum_{\Delta \in T} \mathcal{H}_\Delta^m[N] \tag{M.2}
$$

Are the results on the spectrum of area and volume operators. Loop quantum cosmology.

However, starting with LQG in 2+1 dimensions, Perez [120] argues that any choice other than the fundamental representation leads to unphysical local degres of freedom also in 3+1 dimensions.

M.9 Constructing the Solution's

More precisely, the equation

$$
(\Psi | \hat{\mathcal{C}}(N) = 0, \tag{M.2}
$$

is equivalent to the following hierarchy of equations

$$
\begin{aligned}
(\Psi_{(1)[\alpha,j]} | \hat{\mathcal{T}}(N) &= 0 \\
(\Psi_{(2)[\alpha,j]} | \hat{\mathcal{T}}(N) &= (\Psi_{(1)[\alpha,j]} | \hat{\mathcal{C}}^{Eucl}(N) \\
&\cdots \\
(\Psi_{(n+1)[\alpha,j]} | \hat{\mathcal{T}}(N) &= (\Psi_{(n)[\alpha,j]} | \hat{\mathcal{C}}^{Eucl}(N) \\
&\cdots && \text{(M.-1)}
\end{aligned}
$$

M.10 The Generalized Wick Transformation

$$
\begin{aligned}
f &\rightarrow W(1)\cdot f \\
W(t)\cdot f &\equiv \sum_{n\geq 0}\frac{t^n}{n!}\{f,T\}_n \\
T &\equiv i\frac{\pi}{2}\int_\Sigma A_i^a E_a^i
\end{aligned}
\tag{M.-2}
$$

where $\{f,T\}_2=\{f,\{f,T\}\}$ and $\{f,T\}_3=\{f,\{f,T\}_2\},\ldots$ etc.

$$
\{E_a^i,T\}=\{E_a^i,\int_\Sigma dx^3 A_j^b E_b^j\}=iE_a^i
\tag{M.-2}
$$

$$
W(1)\circ E_a^i=\sum_{n\geq 0}\frac{t^n}{n!}\{E_a^i,T\}_n
\tag{M.-2}
$$

$$
W(1)\circ A_i^a=\sum_{n\geq 0}\frac{t^n}{n!}\{A_i^a,T\}_n
\tag{M.-2}
$$

$$
W(1)\circ S_E=\sum_{n\geq 0}\frac{t^n}{n!}\{S_E,T\}_n
\tag{M.-2}
$$

M.11 Inclusion of Matter

$$
H_{tot}=\Lambda^i G_i+N^a C_a+NC,
\tag{M.-2}
$$

where Λ^i, N^a and N are Lagrange multiplyers, and the three constraints in the Hamiltonian are

$$
\begin{aligned}
G_i &= \mathcal{D}_a\tilde{E}_i^a:=\partial_a\tilde{E}_i^a+\epsilon_{ij}{}^k A_a^i\tilde{E}_i^a, && \text{(M.-1)}\\
C_a &= \tilde{E}_i^b F_{ab}^i-A_a^i G_i+\tilde{\pi}\partial_a\phi, && \text{(M.0)}\\
C &= \frac{\kappa\gamma^2}{2\sqrt{|\det q|}}\tilde{E}_i^a\tilde{E}_j^b[\epsilon^{ij}{}_k F_{ab}^k-2(1+\gamma^2)K_{[a}^i K_{b]}^j] \\
&+ \frac{1}{\sqrt{|\det q|}}[\frac{\kappa^2\gamma^2\alpha_M}{2}\delta^{ij}\tilde{E}_i^a\tilde{E}_j^b(\partial_a\phi)\partial_b\phi+\frac{1}{2\alpha_M}\tilde{\pi}^2], && \text{(M.0)}
\end{aligned}
$$

$\tilde{\pi}$ is conjugate to ϕ

$$\tilde{\pi} := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\alpha_M}{N}\sqrt{|\det q|}(\dot{\phi} - N^a \partial_a \phi). \tag{M.0}$$

M.11.1 The "habitat" of functions

Another objection that was raised by Lewandowski and Marolf has to do with the fact that it is easy to construct a more general "habitat" of functions where Thiemann's Hamiltonian is well defined. Consider any function of a spin net with n vertices. Multiply i times a scalar function with n entries, evaluated at each vertex,

$$|s, f >= \int d^3x_1 \ldots d^3x_n f(x_1, \ldots, x_n) \tag{M.0}$$

These functions are invariant under diffeomorphisms that leave the vertices of the spin network fixed. Otherwise diffeomorphisms are correctly implemented geometrically: $C(N)|s, f >= |s, L_N f >$

It is obvious that on these states one can implement Thiemann's Hamiltonian. It is also obvious that on these states the Hamiltonian will have an abelian algebra too.

Is the theory inconsistent? For that we should evaluate the right hand side of the commutator, on these states.

$$\{H(N), H(M)\} = \int d^3x \omega_a(x) q^{ab}(x) \tilde{C}_b(x) \tag{M.0}$$

where $\omega_a(x) = (N\partial_a M - M\partial_a N)$.

Which in turn requires computing the doubly-contravariant metric. Remeber that we need to write it in terms of the fundamental variables. Using Thiemann's construction:

$$q^{ab}(x) = \frac{1}{4}\epsilon^{abc}\epsilon_{ijk}\epsilon^{bef}\epsilon_{ilm}\frac{e^j_c e^k_c}{\sqrt{det(q)}}\frac{e^l_e e^m_f}{\sqrt{det(q)}} \tag{M.0}$$

And it is not hard so see that this operator vanishes identically.

The theory is therefore consistent. But it apears (to some) that one is paying too high a price: the contravariant metric is a highly non-linear combinatin of the fundamental variables and its relation to operators that are well defined and non-vanishing like the area and volume.

So is it a problem or not? At the moment the issue is debated. It is interesting that the same problem arises in 2+1 dimensions and one nevertheless recovers Witten's correct quantization.

R. Gambini, J. Griego, C, Di Bartolo, JP (2000)

Habitats are unphysical and completely irrelevant in LQG

Habitats were introduced in [??, ??]. The idea was to take the limit $\epsilon \to 0$ for the duals of the Hamiltonian constraints on such a habitat in the sense of pointwise convergence. The habitat ambiguity is that there maybe zillions of habitats on which a limit of this kind can be performed. As was shown in those papers, there exists at least one such habitat and it has the property that the limit dual operators are Abelian.

We now show that this habitat ambiguity is actually absent: Namely, the habitat spaces must be genuine extensions of $\mathcal{D}^*_{Diff}$. Hence these spaces are not in the kernel of the spatial di eomorphism constraint and are therefore unphysical. Hence the only domain where to define the Hamiltonian constraints (rather than their duals) is on $\mathcal{D}$ i.e. on a dense subspace of the kinematical Hilbert space $\mathcal{H}$. This is the same domain as for the spatial di eomorphism constraints which thus treats both types of constraints democratically. This fact is widely appreciated in the LQG community and not a matter of debate, the habitat construction presented in [??] is outdated. Habitats are unphysical and completely irrelevant in LQG.

M.12 Analysis of the Volume Operator

$$V(\Omega) = \int_\Omega \sqrt{h} d^3x = \int_\Omega \sqrt{E} d^3x \tag{M.0}$$

E being

$$E = \frac{1}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a E^j_b E^k_c.$$

Given a graph α and a function $h_\alpha(A)$ of parallel transports along its edges e, the operator E acts as follows

$$\begin{aligned} E(x)h_\alpha(A) &= \frac{1}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a(x)E^j_b(x)E^k_c(x)h_\alpha(A) \\ &= \frac{8i\pi\gamma l_P^2}{3!}E^i_a(x)E^j_b(x)\left[\int_0^1 dt\frac{d\alpha^k}{dt}(U_\alpha)^t_0\tau_c(U_\alpha)^1_t\delta^3(x-\alpha(t))\right] \end{aligned}$$

And then by acting with E^j_b

$$
\begin{aligned}
&= \frac{(8i\pi\gamma l_P^2)^2}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a(x) \\
&= \frac{(8i\pi\gamma l_P^2)^2}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a(x)\left[\int_0^1 dt\int_0^t ds\frac{d\alpha^j}{ds}(U_\alpha)_0^s\tau_b(U_\alpha)_s^t\delta^3(x-\alpha(s))\ \frac{d\alpha^k}{dt}\times\right. \\
&\quad (U_\alpha)_0^t\tau_b(U_\alpha)_t^1\delta^3(x-\alpha(t))\ ] \\
&\quad \left.+\int_0^1 dt\frac{d\alpha^j}{dt}(U_\alpha)_0^t\tau_c(U_\alpha)_t^1\delta^3(x-\alpha(t))\ \ \int_t^1 ds\frac{d\alpha^j}{dt}(U_\alpha)_t^s\tau_b(U_\alpha)_s^1\delta^3(x-\alpha(s))\right]
\end{aligned}
$$

The third E^i_a operator...

The last condition produces a term

$$\epsilon_{ijk}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}\frac{d\alpha^k}{dt},$$

which is not vanishing if and only if there exists a point in the graph α where tangent vectors form a set of three independent vectors.

Let us consider the case in which only one vertex v with n out-going edges e_i , $i = 1, \ldots, n$ is contained in the region Ω we obtain for the volume operator

A consistency check [100] on the equivalence between a quantization based on triads and one based on fluxes implies that the latter is the correct one.

We consider the case in which only one vertex v with n out-going edges e_i, $i = 1, \ldots, n$ is contained in Ω.

$$V(\Omega)h_\alpha = (8\pi\gamma)^{3/2}l_P^3\sqrt{|q|}h_\alpha \tag{M.-6}$$

$$h_\alpha = \epsilon^{abc}\sum_{e,e',e''}o(v,e,e',e'')\tau_a^{j_e}\tau_b^{j_{e'}}\tau_c^{j_{e''}}h_\alpha, \tag{M.-6}$$

M.13 Non-Hermitian Hamiltonian in the Rovelli-Reisenberg Spin Foam

Is there a difference in using $\frac{1}{2}(C(N) + C^\dagger(N))$ or using $C(N)$?

M.13.1 Spin Foam

M.14 Spacially Diffeomorphism Invariant Hamiltonian Constraints

An unfortunate feature of the Hamiltonian constraint was that it could not be implemented on the spatially diffeomorpism invariant Hilbert space because it would map spatially diffeomorpism invariant states onto non-diffeomorpism invariant states - i.e. it did not close on the space.

Thus the inner product structure of H_{Diff} cannot be employed, via the same powerful technies used to construct the inner product H_{Diff} from the kinematic inner product structure ... , in the construction of the physical inner product.

In [301] it was shown that if one is given a constraint algebra of the form

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, C_k\} = f_{Jk}{}^l C_l, \quad \{C_j, C_k\} = f_{jk}{}^L C_L \tag{M.-6}$$

$$A_{lj} := \{C_l, T_j\}. \tag{M.-6}$$

$$\tilde{C}_j := \{M, T_j\} \approx \sum_{k,l} Q_{kl} C_k A_{lj} \tag{M.-6}$$

the constraint algebra can be simplified to

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, \tilde{C}_k\} = 0, \quad \{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l. \tag{M.-6}$$

Diffeomorphism Invariant Hilbert Space

M.15 Biblioliographical notes

In this chapter I have relied on the following references: .

M.16 Worked Exercies and Details

Proofs

Questions

1. Check that there are 8 of them.

Answers

1. A tetrahedal cannot have e_i and e_i^r edges.

The list reads:

$$
\begin{array}{lll}
e_1, e_2, e_3 & e_1, e_2, e_3^r & e_1^r, e_2^r, e_3 \\
e_1^r, e_2^r, e_3^r & e_1, e_3^r, e_2 & e_1^r, e_3^r, e_2 \\
 & e_2, e_3, e_1^r & e_2^r, e_3^r, e_1
\end{array} \tag{M.-7}
$$

Proofs. 3+1 decomposition and Legendre transformation

Questions

1. (Eq. M.11)

Appendix N

Spin Foams and Physical Observables.

N.1 Spin Foam

N.1.1 Spin Foam From Projector Technique Applied To the Hamiltonian Constraint

$$P\int[\mathcal{D}N]e^{i\hat{\mathcal{H}}[N]}=\int[\mathcal{D}N]e^{iN\hat{\mathcal{H}}}. \tag{N.0}$$

In the spin network basis, the matrix elements of P are

$$< s|P|s' >=< s|\int[\mathcal{D}N]e^{iN\hat{\mathcal{H}}}|s' > \tag{N.0}$$

It can be shown that a diffeomorphism invariant notion of integration exists for this functional integral.

$$< s|P|s' >\sim< s|s' > + \int[\mathcal{D}N]\left(N < s|\hat{\mathcal{H}}|s' > +NN < s|\hat{\mathcal{H}}\hat{\mathcal{H}}|s' > + \dots\right) \tag{N.0}$$

$$< s|P|s' >= \int_{Diff}\mathcal{D}[\phi]\int\mathcal{D}[N] < \mathcal{U}S|\left(\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}(H[N])^n\right)|s' > \tag{N.0}$$

N.1.2 State Sum

We change the signature of the metric by changing the gauge group.

Putting it all together we see it has the form

$$\mathcal{Z} = \sum_J \mathcal{N}(J) \sum_N \prod_{f \in J} \Delta_{N_f} \prod_{v \in J} A_v(N), \tag{N.0}$$

The first sum is over all spin foams Γ interpolating between a given initial spin network s_i and a final spin network s_f. Δ_{j_f} is the dimension of the G representation labelling the face f of Γ. A_v is the amplitude on the vertex v of Γ, a given function of the labels on the faces and edges adjacent to v.

N.2 Barrett-Crane Model

$$S(\omega, B, \phi) = \int_{\mathcal{M}} B^{IJ} \wedge F_{IJ}(\omega) \tag{N.0}$$

$$S(\omega, B, \phi) = \int_{\mathcal{M}} d^4x \epsilon_{abcd} B^{IJab} F^{cd}_{IJ}(\omega) \tag{N.0}$$

$$S(\omega, B, \phi) = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}(\omega) - \frac{1}{2}\phi_{IJKL} B^{IJ} \wedge B^{KL} - \frac{1}{2}\mu \epsilon^{IJKL} \phi_{IJKL} \right] \tag{N.0}$$

which is a BF theory with variables a 2-form B^{IJ}_{ab} and a 1-form connection (with curvature F^{IJ}_{ab}), all with values in $so(3,1)$, but with a constraint on the B field enforced by the Lagrangian multiplier ϕ_{IJKL}.

$$\begin{aligned}
DB &= dB + [\omega, B] = 0, \quad &&\text{(var. of } \omega) &(N.1)\\
F(\omega) &= \phi^{IJKL} B_{KL}, &&\text{(var. of } B) &(N.2)\\
B^{IJ} \wedge B^{KL} &= \mu \epsilon^{IJKL}, &&\text{(var. of } \phi) &(N.3)\\
\epsilon^{IJKL} \phi_{IJKL} &= 0, &&\text{(var. of } \mu) &(N.4)
\end{aligned}$$

contracting (N.3) with ϵ_{IJKL} we solve for μ:

$$\epsilon_{IJKL} B^{IJ} \wedge B^{KL} = \mu \epsilon_{IJKL} \epsilon^{IJKL}$$

implies

$$\mu = e := \frac{1}{4!}\epsilon_{IJKL}B^{IJ} \wedge B^{KL}. \tag{N.4}$$

For $e \neq 0$, equation (N.3) is equivalent to

$$\epsilon_{IJKL}B^{IJ}_{ab}B^{KL}_{cd} = e\epsilon_{abcd} \tag{N.4}$$

One can show that there are two types of solution to this equation and which are formed from co-triad fields e^I.

$$\begin{aligned} B^{IJ}_{ab} &= \pm\frac{1}{2}\epsilon^{IJ}{}_{KL}e^K_a e^L_b \quad \text{or} \quad B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL}e^K \wedge e^L && \text{(N.5)} \\ B^{IJ}_{ab} &= \pm e^I_{[a}e^J_{b]} \qquad\quad \text{or} \quad B^{IJ} = \pm e^I \wedge e^J && \text{(N.6)} \end{aligned}$$

substituting any of these solutions into one obtains the Palanti action,

$$S \rightarrow S_{EH} = \frac{1}{2}\int_{\mathcal{M}} d^4x\epsilon^{abcd}\epsilon_{IJKL}e^K_a e^L_b F^{IJ}_{cd} \tag{N.6}$$

$$S \rightarrow S_{EH} = \int_{\mathcal{M}} \epsilon_{IJKL}e^K \wedge e^L \wedge F^{IJ} \tag{N.6}$$

i.e. it reduces to pure 1st order Einstein gravity. Which as $e \neq 0$, is equivalent to the Einstein-Hilbert action.

choice (N.5) corresponds to the graviational sector and (N.6) to the topological sector - [287].

The geometric information is resides in the labels. This is an important difference from lattice gauge theories with a background metric, where the discretization itself determines the edge lengths and hence how refined the lattice is.

N.2.1 Latice BF-Theory

$\sigma_0, \ldots, \sigma_n$ triangulation of $\mathcal{M}$
C_0, C_1, C_2

dual 2-complex

$$c_0 \leftrightarrow \sigma_n \qquad c_1 \leftrightarrow \sigma_{n-1} \qquad c_0 \leftrightarrow \sigma_{n-2}$$

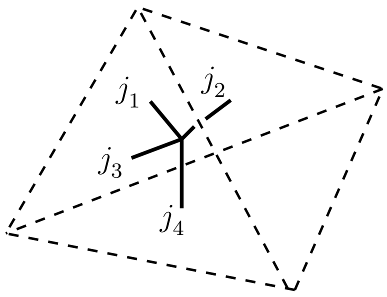


Figure N.1: The spins j_1, j_2, j_3, j_4 describe the areas of the triangles.

curvature associated with dual 2-cells

$$F(c_2) \int_{c_2} F \in g \tag{N.6}$$

B-field associated with $(n-2)-$simplicies

$$B(\sigma_{n-2}) \int_{c_2} B \in g \tag{N.6}$$

The discretized action is:

$$S = \sum_{\sigma_{n-2}} Tr\big(B(\sigma_{n-2})F(\sigma^*_{n-2})\big) \tag{N.6}$$

Lattice path integral quantization:

$$\begin{aligned} \mathcal{Z} &= \text{“} \int \mathcal{DA} \int \mathcal{DB} e^{iS} \text{”} \\ &= \text{“} \int \mathcal{DA} \prod_{\sigma_{n-2}} \underbrace{\delta(F(\sigma_{n-2})}_{\text{enforces flatness}} \quad)\text{”} \end{aligned} \tag{N.6}$$

where we are integrating over all connections $\mathcal{A}$.

$$\mathcal{Z} = \left(\prod_{c_1} \int_G dg(c_1) \right) \; \delta_G\big(g\partial_1(c_1)(g\partial_2)(c_2)\cdots\big) \tag{N.6}$$

Latice gauge theory of flat connections on the dual 2-complex of a generic triangulation.

N.2.2 Spin Foam BF Theory

N.3 State Sums in 3-d Gravity

N.3.1 Regge Calculus

denote the lengths of its edges by l_i $(I = 1, 2, \ldots, 6)$ (Fig N.-19). The Regge action for the tetrahedron is

$$S_{Regge} = \sum_{I=1} l_i \theta_i, \tag{N.6}$$

where θ_i is the angle between the outward normals of two faces sharing the I-th edge.

$$W(j) = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \tag{N.6}$$

Witten's $ISO(3)$ Chern-Simons theory [E. Witten, Nucl. Phys. **B311**, 46 (1988).]

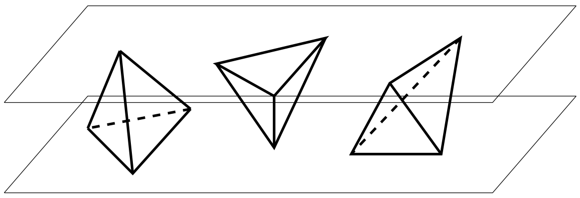


Figure N.2: The three types of tetrahedral building blocks used in 3d quantum gravity.

N.3.2 Ponzano-Regge-Turaev-Viro (PRTV)

Ponzano-Regge

$$\text{History amplitude} = \sum_{\{j\}} \prod_e (dimj_e) \prod_t (6j) \tag{N.6}$$

"triangulation independent"

It diverges. Turaev-Viromodel with regularization using q-deformed gauge group $SU(2)_q$, see ().

$$0 \leq j \leq \frac{k-1}{2} \tag{N.6}$$

k-level of $SU(2)_q$ is root of unitary i.e. $q = e^{2\pi/\kappa}$.

cellular decomposition $\triangle$ and associated dual 2-complex $\mathcal{F}_\triangle$.

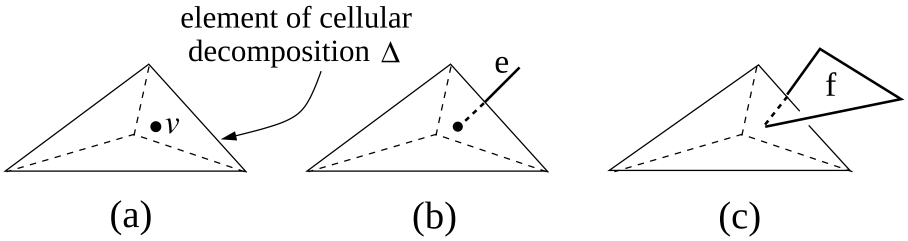


Figure N.3: (a) $v \in \mathcal{F}_\triangle$ - dual to 3-cells in $\triangle$; (b) $e \in \mathcal{F}_\triangle$ - dual to 2-cells in $\triangle$; (c) faces $f \in \mathcal{F}_\triangle$ - dual to 1-cells in $\triangle$

$su(2)$-valued 1-form field B represented by the assignment of a $B \in su(2)$ to each 1-cell. The connection field A is represented by the assignment of a group elements $g_e \in SU(2)$ to each edge in $\mathcal{F}_\triangle$.

$$B_a = B_a^i \tau_i, \quad [B_a, B_b] = f_{abc} B_c \tag{N.6}$$

$$S = \prod_{f \in \mathcal{F}_\triangle} B_{l_f} U_f \tag{N.6}$$

where $U_f = g_e^1 g_e^2 \dots g_e^N$

vertices $v \in \mathcal{F}_\triangle$ (dual to 3-cells in $\triangle$)
edges $e \in \mathcal{F}_\triangle$ (dual to 2-cells in $\triangle$)
and faces $f \in \mathcal{F}_\triangle$ (dual to 1-cells in $\triangle$)

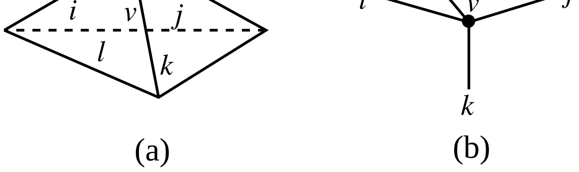


Figure N.4: $\triangle$dual3

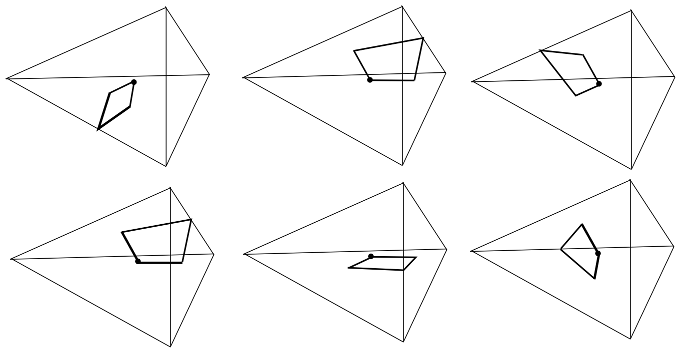


Figure N.5: atom2.

$$Z(\triangle) = \int \prod_{f\in\mathcal{F}} dB_f \prod_{e\in\mathcal{F}} dg_e e^{i\mathrm{tr}[B_f U_f]} \tag{N.6}$$

performing the integral over B_f

$$\int dB_f e^{i\mathrm{tr}[B_f U_f]} = \delta(U_f) = \delta(g_e^1 \dots g_e^N) \tag{N.6}$$

$$Z = \int \mathcal{D}A\delta[F], \tag{N.6}$$

namely an integral over flat $SU(2)$ connections.

$$\delta(g) = \sum_{j\in irrep(SU(2))} \Delta_j \mathrm{Tr}[\rho_j(g)] \tag{N.6}$$

$$Z(\triangle) = \sum_{C_f:\{f\}\to\{j\}} \int \prod_{e\in\mathcal{F}} dg_e \prod_{f\in\mathcal{F}} \Delta_{j_f} \mathrm{Tr}[\rho_{j_f}(g_e^1 \dots e_e^N)] \tag{N.6}$$

$$Z(\triangle) = \sum_{C_f:\{f\}\to\{j\}} \int \prod_{e\in\mathcal{F}} dg_e \prod_{f\in\mathcal{F}} \Delta_{j_f} \rho_{j_f}(g_e^1 \dots e_e^N)^\alpha{}_\alpha \tag{N.6}$$

$$Z(\triangle) = \sum_{C_f:\{f\}\to\{j\}} \sum_{C:\{e\}\to\{i\}} \prod_{f\in\mathcal{F}} \Delta_{j_f} \prod_{v\in\mathcal{F}} A_v(i_v, j_v) \tag{N.6}$$

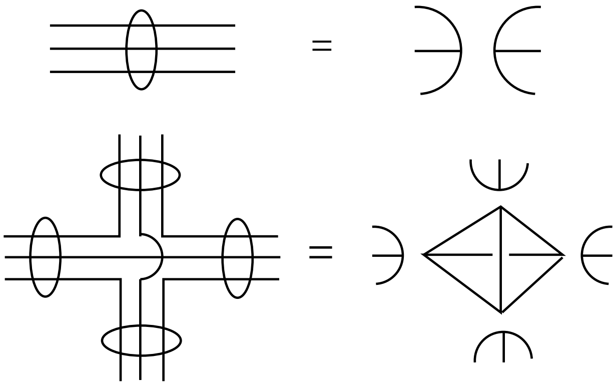


Figure N.6: How 6-j Symbols Appears.

How 6-j Symbols Appears:

N.4 10j symbols

$$\int_{(S^3)^5} \prod_{k<l} K^R_{2j_{kl}+1}(\phi_{kl}) \, \frac{dh_1}{2\pi^2} \cdots \frac{dh_5}{2\pi^2} \tag{N.6}$$

$$K^R_a(\phi) = \frac{\sinh a\phi}{\sin \phi} \tag{N.6}$$

Lorentzian $10j$ symbols - same sort of integral

$$\int_{(H^3)^5} \prod_{k<l} K^L_{a_{kl}}(\phi_{kl}) \, \frac{dh_1}{2\pi^2} \frac{\dots dh_5}{2\pi^2} \tag{N.6}$$

where

$$H^3 = \{t^2 - x^2 - y^2 - z^2 = 1, \ t > 0\} \tag{N.6}$$

$$K^L_a(\phi) = \frac{\sin a\phi}{\sinh \phi}. \tag{N.6}$$

N.5 4D State Sum

spin foams live on the 2-skeleton

each dual 0-cell has incident on it five endpoints of 1-cells and ten corners of 2-cells with each incident 2-cell corner bounded by two of the incident 1-cells. This corresponds to the fact that a 4-simplex (dual to a 0-cell) has in its boundary five 3-simplicies (dual to 1-cells) and ten 2-simplices (dual to 2-cells).

N.6 Group Field Theory for 2D Gravity

a model in two dimensions in which we perform a path integral quantization of a (interacting) group field theory defined on $SU(2) \times SU(2)$, and expand perturbatively the transition amplitudes defined in the theory in power of the coupling constant λ. We explicitly perform the main calculation to present in concrete terms the relationship between group field theory and spin foam models. The partition function obtained has the same form of the generic one that defines a spin foam model. Such identity makes clear the relation between field theory over a group and the spin foam formalism.

Consider the field theory defined by the following action

$$\begin{aligned} S[\Phi] \;&=\; \int_{SU(2)\times SU(2)} dg_1 dg_2 \Phi^2(g_1,g_2) + \\ &\frac{\lambda}{3!}\int_{SU(2)\times SU(2)\times SU(2)} dg_1 dg_2 dg_3 \Phi(g_1,g_2)\Phi(g_2,g_3)\Phi(g_3,g_1) \end{aligned} \tag{N.6}$$

for the real scalar field $\Phi(g_1, g_2)$ defined on the group manifold $SU(2) \times SU(2)$ and having the following properties

$$\Phi(g_1,g_2) = \Phi(g_1 g, g_2 g), \qquad \text{for all } g \in SU(2) \tag{N.6}$$

and

$$\Phi(g_1,g_2) = \Phi(g_2,g_1). \tag{N.6}$$

Using the Peter-Weyl theorem we expand the field in terms of irreducible representations of the group (repeated indices are summed over),

$$\Phi(g_1,g_2) = \phi_{j_1 j_2}^{a_1 b_1 a_2 b_2}\overline{R^{j_1}_{a_1 b_1}(g_1)}R^{j_2}_{a_2 b_2}(g_2) \tag{N.6}$$

where $R^{j_1}_{a_1 b_1}(g_1)$ are the matrix elements of the group element g_1 in the spin j representation of $SU(2)$. The symmetry property (N.-19) of the field can be used to simplify the expansion of $\Phi(g_1, g_2)$ in terms of R^j_{ab}. The symmetry property implies

$$\begin{aligned}\Phi(g_1,g_2) &= \int dg\ \Phi(g_1 g, g_2 g)\\ &= \int dg\ \phi^{a_1b_1a_2b_2}_{j_1j_2}\overline{R^{j_1}_{a_1b_1}(g_1g)}R^{j_2}_{a_2b_2}(g_2g)\\ &= \phi^{a_1b_1a_2b_2}_{j_1j_2}\overline{R^{j_1}_{a_1c_1}(g_1)}R^{j_2}_{a_2c_2}(g_2)\int dg\overline{R^{j_1}_{c_1b_1}(g)}R^{j_2}_{c_2b_2}(g). \qquad \text{(N.5)}\end{aligned}$$

Using the orthogonality relation

$$\int dU\overline{R^{(j)}_{ab}(U)}R^{(k)}_{cd}(U) = \frac{1}{2j+1}\delta^{jk}\delta_{ac}\delta_{bd}, \qquad \text{(N.5)}$$

we can write

$$\Phi(g_1,g_2) = \sqrt{2j+1}\ \phi^{a_1a_2}_j\ \overline{R^j_{a_1c}(g_1)}R^j_{a_2c}(g_2) \qquad \text{(N.5)}$$

where we have defined

$$\phi^{a_1a_2}_j := \frac{1}{\sqrt{2j+1}}\ \phi^{a_1b_1a_2b_2}_{jj_2}\ \delta^{jj_2}\delta_{b_1b_2}. \qquad \text{(N.5)}$$

The reality and the symmetry properties of the field Φ imply $\phi^{a_1a_2}_j = \overline{\phi^{a_2a_1}_j}$. This hermitian matrix has dimension $(2j+1)\times(2j+1)$ and represents the Fourier transform of the field Φ. Writing the action (N.6) in terms of these modes, we obtain for the kinetic term

$$\int \Phi^2(g_1,g_2)dg_1dg_2 = \phi^{a_1a_2}_j\phi^{a_2a_1}_j \qquad \text{(N.5)}$$

and for the potential term

$$\frac{\lambda}{3!}\int dg_1dg_2dg_3\Phi(g_1,g_2)\Phi(g_2,g_3)\Phi(g_3,g_1) = \frac{\lambda}{3!}\frac{1}{\sqrt{2j+1}}\phi^{ab}_j\phi^{bc}_j\phi^{ca}_j \qquad \text{(N.5)}$$

We can express the action in terms of the matrices ϕ_j of the Peter-Weyl expansion of the field

$$S[\Phi] = \sum_j\left(\frac{1}{2}Tr(\Phi^2_j) + \frac{\lambda}{3!}\frac{1}{\sqrt{2j+1}}Tr(\Phi^3_j)\right), \qquad \text{(N.5)}$$

where Φ_j is an hermitian matrix of dimension $N_j = 2j+1$, defined by $(\Phi_j)_{ab} = \phi_j^{ab}$. The action is a sum over j of terms which have the standard form for what are known as the matrix models action [196]. As such, the field theory on the groups leads to a generalization of the matrix models: the action (N.6) is the sum over all the values of the dimension N_j of the matrix Φ_j.

Counterparts with the $U(1)$ Case

for the real scalar field $\Phi(g_1, g_2)$ defined on the group manifold $U(1) \times U(1)$ and having the following properties

where $R^{j_1}(g_1)$ are the elements of the group element g_1 in the j representation of $U(1)$. Say $g_1 = e^{i\alpha_1}$

$$\Phi(g_1, g_2) = \Phi(g_1 g, g_2 g), \qquad \text{for all } g \in U(1) \tag{N.5}$$

and

$$\Phi(g_1, g_2) = \Phi(g_2, g_1). \tag{N.5}$$

Using the Peter-Weyl theorem we expand the field in terms of irreducible representations of the group (repeated indices are summed over),

$$\Phi(g_1, g_2) = \phi_{j_1 j_2} \overline{R^{j_1}(g_1)} R^{j_2}(g_2) \tag{N.5}$$

The Peter-Weyl theorem applied to $U(1)$ gives the Fourier series $f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}/\sqrt{2\pi}$

$$\Phi(g_1, g_2) = \sum_{j_1, j_2 = 0}^{\infty} \phi_{j_1 j_2} e^{-ij_1\alpha_1} e^{ij_2\alpha_2} \tag{N.5}$$

The symmetry property implies explicitly

$$\begin{aligned}
\Phi(g_1, g_2) &= \int_0^{2\pi} d\alpha \; \Phi(e^{i\alpha_1 + i\alpha}, e^{i\alpha_2 + i\alpha}) \\
&= \int_0^{2\pi} d\alpha \sum_{j_1, j_2 = 0}^{\infty} \phi_{j_1 j_2} e^{-ij_1(\alpha_1 + \alpha)} e^{ij_2(\alpha_2 + \alpha)} \\
&= \sum_{j_1, j_2 = 0}^{\infty} \phi_{j_1 j_2} e^{-ij_1\alpha_1} e^{ij_2\alpha_2} \int_0^{2\pi} d\alpha e^{-i(j_1 - j_2)\alpha} \qquad \text{(N.4)}
\end{aligned}$$

Using the orthogonality relation

$$\int dU \overline{R^{(j)}(U)} R^{(k)}(U) = \delta_{jk}, \tag{N.4}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta} e^{im\theta} = \delta_{nm}$$

$$\begin{aligned}
\Phi(g_1, g_2) &= \sum_{j_1, j_2=0}^{\infty} \phi_{j_1 j_2} e^{-ij_1\alpha_1} e^{ij_2\alpha_2}\ \delta_{j_1 - j_2} \\
&= \sum_{j_1=0}^{\infty} \phi_{j_1, -j_1} e^{-ij_1(\alpha_1-\alpha_2)} \\
&= \sum_{j=0}^{\infty} \phi_j\ e^{ij}\ \overline{R^j(g_1)} R^j(g_2)
\end{aligned} \tag{N.3}$$

$$\phi_j\ \overline{R^j(g_1)} R^j(g_2) = \int dk e^{-ik} \Phi(g_1, g_2)$$

$$\phi_j = e^i \int dk e^{-ik} \Phi(g_1, g_2)$$

$$\begin{aligned}
\int dk e^{-ik} \Phi(g_1, g_2) &= \sum_k e^{-ik} \sum_{j=0}^{\infty} \phi_j\ e^{ij}\ \overline{R^j(g_1)} R^j(g_2) \\
&= \sum_{j=0}^{\infty} \phi_j\ \int dk e^{i(j-k)}\ \overline{R^j(g_1)} R^j(g_2) \\
&= \sum_{j=0}^{\infty} \phi_j\ \int dk e^{i(j-k)}\ \overline{R^j(g_1)} R^j(g_2)
\end{aligned} \tag{N.2}$$

so that ϕ_j represents the Fourier transform of Φ.

N.7 GFT Dual to State Sums

N.7.1 TOCY model as a QFT over a group manifold

$$\begin{aligned}
S[\phi] &= \int dg_1 \ldots dg_4 \phi^2(g_1, g_2, g_3, g_4) + \frac{\lambda}{5!} \int dg_1 \ldots dg_4 \phi(g_1, g_2, g_3, g_4) \\
&\quad \phi(g_4, g_5, g_6, g_7)\phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1).
\end{aligned} \tag{N.2}$$

In fig(N.7) we give an mnemonic to remember the second term of (N.2), which also serves in perturbation theory we will develop later on. Here $g_i \in SO(4)$ and the field ϕ is a function of $SO(4)$. All the integrals are in the normalized Haar measure. The field ϕ is required to be invariant under any permutation of its arguments; that is, $\phi(g_1, g_2, g_3, g_4) = \phi(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}, g_{\sigma(4)})$, where; and under simultaneous right multiplication by any element g of $SO(4)$:

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_1 g, g_2 g, g_3 g, g_4 g) \tag{N.2}$$

if given any function $\tilde{\phi}$

$$\phi(g_1, g_2, g_3, g_4) = \int dg \tilde{\phi}(g_1 \gamma, g_2 \gamma, g_3 \gamma, g_4 \gamma) \tag{N.2}$$

(verify exercise) This condition is analogous to the "translational invariance" and leads to compatibility conditions on the representations, the analogy of "momentum conservation". Since vertices are 5-valent in our discretization the interaction term should contain the product of five field operators.

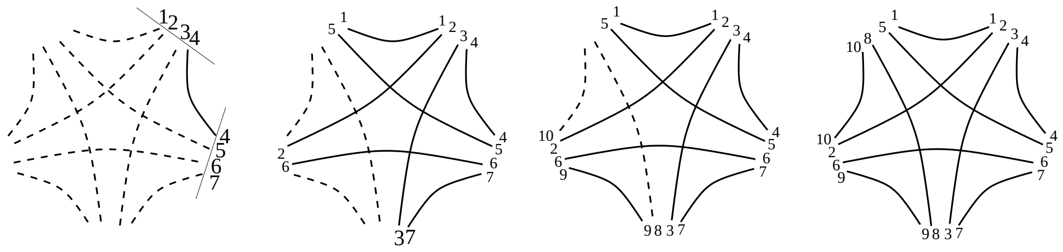


Figure N.7: Structure of interaction vertex. (a) $\phi(g_1, g_2, g_3, g_4)\phi(g_4, g_5, g_6, g_7)$. (d) Reading off the numbers clockwise gives $\phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1)$.

$$\phi(g) = \sum_{\Lambda} \Phi^{\Lambda}_{\alpha\beta} D^{\Lambda}_{\alpha\beta}(g) \tag{N.2}$$

$$\phi(g_1, g_2, g_3, g_4) = \sum_{(N_1 \ldots N_4)} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} \, D^{(N_1)\beta_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\beta_4}_{\alpha_4}(g_4) \tag{N.2}$$

$$\int_G dg D^{a}_{\alpha\beta}(g) D^{a'}_{\alpha'\beta'}(g) = \frac{1}{\dim a} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta^{aa'} \tag{N.2}$$

$$\int_G dg D^{(N_1)}_{\alpha_1\beta_1}(g) \ldots D^{(N_4)}_{\alpha_4\beta_4}(g) = \sum_{\Lambda} C^{N_1 \ldots N_4\,\Lambda}_{\alpha_1 \ldots \alpha_4} C^{N_1 \ldots N_4\,\Lambda}_{\beta_1 \ldots \beta_4} \tag{N.2}$$

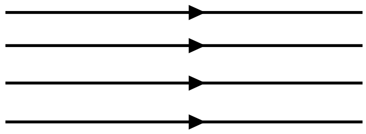


Figure N.8: Structure of interaction vertex.

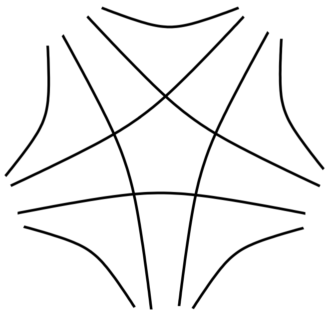


Figure N.9: Structure of interaction vertex.

defining

$$\phi^{\alpha_1 \ldots \alpha_4}_{N_1 \ldots N_4, \Lambda} := \frac{\Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4), \beta_1 \ldots \beta_4} C^{N_1 \ldots N_4, \Lambda}_{\beta_1 \ldots \beta_4}}{\Delta_{N_1} \Delta_{N_2} \Delta_{N_3} \Delta_{N_4}} \tag{N.2}$$

$$\begin{aligned} &\phi(g_1, g_2, g_3, g_4) = \\ &\phi^{\alpha_1 \ldots \alpha_4}_{N_1 \ldots N_4, \Lambda} \left((\Delta_{N_1} \ldots \Delta_{N_4})^2 D^{(N_1)\gamma_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\gamma_4}_{\alpha_4}(g_4) C^{N_1 \ldots N_4\ \Lambda}_{\gamma_1 \ldots \gamma_4} \right) \end{aligned} \tag{N.2}$$

A "Free" theory

Every n-point function of the field theory can be calculated as a functional derivatives of the generating function $\mathcal{W}(J)$

$$\mathcal{W}(J) = \int \mathcal{D}\phi \exp\left(iS[\phi] + J^{\alpha_1 \ldots \alpha_4}_{N_1 \ldots N_4, \Lambda} \phi^{N_1 \ldots N_4, \Lambda}_{\alpha_1 \ldots \alpha_4} \right). \tag{N.2}$$

The moments of the

$$< x^n >= \int_{-\infty}^{\infty} dx x^n \exp\left(i\alpha^2 x^2 \right) \tag{N.2}$$

by taking derivatives of

$$P(J)=\int_{-\infty}^{\infty}dx\exp\left(i\alpha^2x^2+Jx\right)=C\exp\left(\frac{1}{2}\frac{J^2}{\alpha}\right), \quad \text{(N.2)}$$

i.e.

$$< x^n >=\frac{d^n}{dJ^n}P(J)\bigg|_{J=0}. \quad \text{(N.2)}$$

$$\mathcal{W}(J)=\mathcal{C}\exp\left(\frac{1}{2}\frac{J^{\alpha_1\ldots\alpha_4}_{N_1\ldots N_4,\Lambda}J^{N_1\ldots N_4,\Lambda}_{\alpha_1\ldots\alpha_4}}{\Delta_{N_1}\ldots\Delta_{N_4}}\right) \quad \text{(N.2)}$$

$$W(s_1,s_2)=\left\{\frac{\delta}{\delta J^{\alpha_1\ldots\alpha_4}_{N_1\ldots N_4,\Lambda}}\frac{\delta}{\delta J^{\alpha_1\ldots\alpha_4}_{N_1\ldots N_4,\Lambda}}\frac{\delta}{\delta J^{\beta_1\ldots\beta_4}_{M_1\ldots M_4,\Lambda}}\frac{\delta}{\delta J^{\beta_1\ldots\beta_4}_{M_1\ldots M_4,\Lambda}}\mathcal{W}(J)\right\}_{J=0}, \quad \text{(N.2)}$$

N.7.2 Mode Expansion

The requirement of invariance under the right $SO(3)$ action can be written

$$\phi(g)=\int_{SO(3)}dh\phi(gh). \quad \text{(N.2)}$$

Expanding this into the modes, we have

$$\phi(g)=\sum_{\Lambda}\phi^{\Lambda}_{\alpha\beta}D^{(\Lambda)}_{\alpha\beta}(g)=\int_{SO(3)}dh\ \phi(gh)=\sum_{\Lambda}\int_{SO(3)}dh\ \phi^{\Lambda}_{\alpha\beta}D^{(\Lambda)}_{\alpha\gamma}(g)D^{(\Lambda)}_{\gamma\beta}(h). \quad \text{(N.2)}$$

$$\phi(g_1\ldots g_4)=\sum_{N}\phi^{N}_{\alpha}D^{(\Lambda)}_{\alpha\beta}(g)w_{\beta}. \quad \text{(N.2)}$$

$SO(4)/SO(3)$

$$g'x=ghx=gx \quad \text{(N.2)}$$

The coset $[g]$ then consists of all the elements of in $SO(3)$ which take the point $(0,0,1)$ to the point $(0,0,1)$. This point is specified by the polar coordinates (θ,ϕ). As such, each point on the unit circle corresponds to a coset and we have that

$$SO(3)/SO(2) \sim S^2 \tag{N.2}$$

Since $SO(2)$ is not a normal subgroup of $SO(3)$, S^2 does not admit a groups structure.

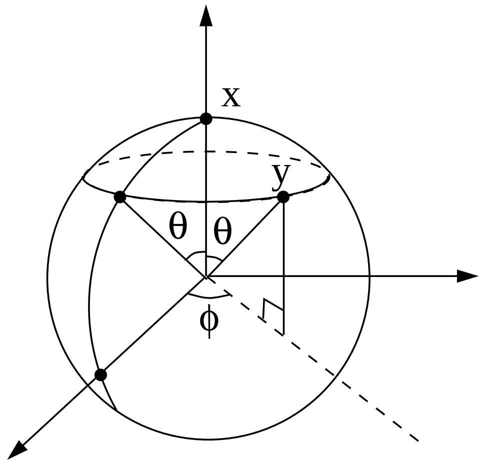


Figure N.10: $SO(3)/SO(2)$.

$SO(3)$ acts on S^2 transitively and we have $SO(3)/SO(2) \sim S^2$.

It is easy to generalise this result to higher dimensional rotation groups and we have the result

$$SO(n)/SO(n-1) \sim S^{n-1} \tag{N.2}$$

where S^{n-1} is the $(n-1)$-sphere.

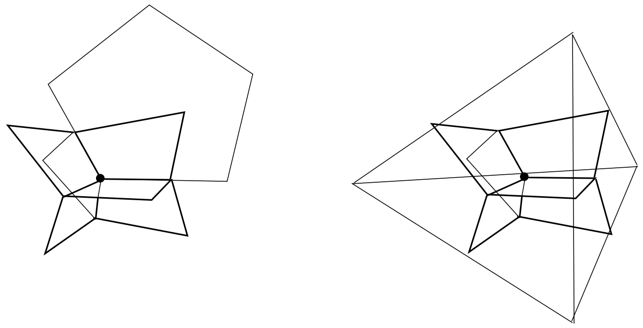


Figure N.11: wedge.

N.7.3 S^3 Spherical Functions

Often:

$$S^3 \cong SO(4)/SO(3) \cong Spin(4)/SU(2) \tag{N.2}$$

here:

$$S^3 \cong SU(2) \tag{N.2}$$

basis of the algebra $C_{alg}(S^3)$

$$?? \tag{N.2}$$

Transitive action of $SU(2) \times SU(2)$ on S^3:

$$(h_1, h_2) \cdot g := h_1 g h_2^{-1} \tag{N.2}$$

$$\text{stab}_{SU(2)\times SU(2)}(g) = \{(ghg^{-1}, h) : h \in SU(2)\} \tag{N.3}$$

Transitive action of $SO(4) \times SU(2)$ on S^3: $(SO(4) \cong SU(2) \times SU(2)/\mathbb{Z}_2)$

N.7.4 $SO(4)$ Barrett-Crane Model

$$\Sigma^{IJ} = \frac{1}{2}\Sigma^{IJ}_{ab}\; dx^a \wedge dx^b$$

defined

$$\Sigma^{IJ} = e^I \wedge e^J \tag{N.3}$$

$\Sigma^{IJ}_{ab} = e^I_a e^J_b - e^I_b e^J_a$

$$|\Sigma_{ab}|^2 = |B_{ab}|^2 \tag{N.3}$$

$$
\begin{aligned}
|B_{ab}|^2 &= \frac{1}{4}\epsilon^{IJ}{}_{KL}\Sigma_{ab}^{KL}\,\epsilon_{IJMN}\Sigma_{ab}^{MN} \\
&= \frac{1}{4}(\epsilon^{IJ}{}_{KL}\epsilon_{IJMN})\Sigma_{ab}^{KL}\Sigma_{ab}^{MN} \\
&= \frac{2}{4}(\delta_{KM}\delta_{LN}-\delta_{KN}\delta_{LM})\Sigma_{ab}^{KL}\Sigma_{ab}^{MN} \\
&= \frac{1}{2}(\Sigma_{ab}^{KL}\Sigma_{abKL}-\Sigma_{ab}^{KL}\Sigma_{abLK}) \\
&= \Sigma_{ab}^{IJ}\Sigma_{abIJ}
\end{aligned}
\tag{N.0}
$$

$$
|\Sigma_{ab}|^2 = g_{aa}g_{bb} - g_{ab}g_{ab} \equiv 2A_{ab}^2 \tag{N.0}
$$

$$
\begin{aligned}
|\Sigma_{ab}|^2 &= \frac{1}{2}(e_a^I e_b^J - e_a^J e_b^I)\frac{1}{2}(e_{aI}e_{bJ} - e_{aJ}e_{bI}) \\
&= \frac{1}{4}(e_a^I e_{aI}\, e_b^J e_{bJ} - e_a^I e_{bI}\, e_b^J e_{aJ} - e_a^J e_{bJ}\, e_b^I e_{aI} + e_a^J e_{aJ}\, e_b^I e_{bI}) \\
&= \frac{1}{2}(g_{aa}g_{bb} - g_{ab}g_{ab})
\end{aligned}
\tag{N.-1}
$$

and

$$
\Sigma_{ab}\cdot\Sigma_{ac} = B_{ab}\cdot B_{ac} \tag{N.-1}
$$

$$
\begin{aligned}
B_{ab}\cdot B_{ac} &= \frac{1}{4}\epsilon^{IJ}{}_{KL}\Sigma_{ab}^{KL}\,\epsilon_{IJMN}\Sigma_{ac}^{MN} \\
&= \frac{1}{4}(\epsilon^{IJ}{}_{KL}\epsilon_{IJMN})\Sigma_{ab}^{KL}\Sigma_{ac}^{MN} \\
&= \Sigma_{ab}^{IJ}\Sigma_{acIJ}
\end{aligned}
\tag{N.-2}
$$

$$
\Sigma_{ab}\cdot\Sigma_{ac} = g_{aa}g_{bc} - g_{ab}g_{ac} \equiv 2J_{aabc} \tag{N.-2}
$$

$$
\begin{aligned}
\Sigma_{ab}\cdot\Sigma_{ac} &= \frac{1}{2}(e_a^I e_b^J - e_a^J e_b^I)\frac{1}{2}(e_{aI}e_{cJ} - e_{aJ}e_{cI}) \\
&= \frac{1}{2}(g_{aa}g_{bc} - g_{ab}g_{ac})
\end{aligned}
\tag{N.-2}
$$

$$\int_S |\Sigma| = \int_S |B| = \int_S |\Sigma_{ab}| dx^a dy^b = \sqrt{2} Area(S) \tag{N.-2}$$

the 4-form

$$V \equiv \frac{1}{4!}\epsilon_{IJKL}\Sigma^{IJ} \wedge \Sigma^{KL} = \frac{1}{4!}\epsilon_{IJKL} B^{IJ} \wedge B^{KL} \tag{N.-2}$$

proof

$$\begin{aligned}
\epsilon_{IJKL} B^{IJ} \wedge B^{KL} &= \frac{1}{4}\epsilon^{IJ}{}_{I'J'}\Sigma^{I'J'}(\epsilon^{KL}{}_{K'L'}\Sigma^{K'L'}) \\
&= \frac{1}{4}(\epsilon_{IJKL}\epsilon^{IJ}{}_{I'J'}\epsilon^{KL}{}_{K'L'})\Sigma^{I'J'}\Sigma^{K'L'} \\
&= \frac{1}{2}(\delta_{KI'}\delta_{LJ'} - \delta_{KJ'}\delta_{LI'})\epsilon^{KL}{}_{K'L'}\Sigma^{I'J'}\Sigma^{K'L'} \\
&= \epsilon_{IJKL}\Sigma^{IJ} \wedge \Sigma^{KL}
\end{aligned} \tag{N.-4}$$

Using the Plebanski field, the action can be written in the BF-like form

$$\begin{aligned}
S[e,\omega] &= \frac{1}{2}\int \epsilon_{IJKL}\, \Sigma^{IJ}[e] \wedge F^{KL}[\omega] \\
&= \int B_{IJ}[e] \wedge F^{KL}[\omega].
\end{aligned} \tag{N.-4}$$

This condition can be expressed as a constraint equation for Σ:

$$\Sigma^{IJ} \wedge \Sigma^{KL} = V\ \epsilon^{IJKL} \tag{N.-4}$$

Say Σ^{IJ} is of the form $e^I \wedge e^J$, then

$$\begin{aligned}
\Sigma^{IJ} \wedge \Sigma^{KL} &= e^I \wedge e^J \wedge e^K \wedge e^L \\
&= e_a^{[I} e_b^J e_c^K e_d^{L]} dx^a \wedge \cdots \wedge dx^d \\
&= \alpha\epsilon^{IJKL} \\
&= \left(\frac{1}{4!}\epsilon_{I'J'K'L'} e^{I'} \wedge e^{J'} \wedge e^{K'} \wedge e^{L'}\right)\epsilon^{IJKL} \\
&= V\ \epsilon^{IJKL}
\end{aligned} \tag{N.-7}$$

Conversly if Σ^{IJ} satifies

$$\Sigma^{IJ} \wedge \Sigma^{KL} = \frac{1}{4!}(\epsilon_{I'J'K'L'}\Sigma^{I'J'} \wedge \Sigma^{K'L'})\epsilon^{IJKL} \tag{N.-7}$$

it must be of the form $e^I \wedge e^J$:

Condition (N.7.4) is equivalent to

$${}^*\Sigma_{ab} \cdot \Sigma_{cd} = \frac{1}{2}\tilde{V}\epsilon_{abcd} \tag{N.-7}$$

or

$$\frac{1}{2}\epsilon_{IJKL}\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd} = \frac{1}{2}\tilde{V}\epsilon_{abcd} \tag{N.-7}$$

where $V = \frac{1}{4!}\tilde{V}\epsilon_{abcd}dx^a \wedge \cdots \wedge dx^d$. To prove this, take this and use $\epsilon_{IJKL}\epsilon^{IJKL} = 4!$

$$\epsilon_{IJKL}(4!\Sigma^{IJ}_{[ab}\Sigma^{KL}_{cd]}) = \tilde{V}\epsilon_{abcd}\epsilon_{IJKL}\epsilon^{IJKL}$$

The summation on the left hand side is

$$\begin{aligned} &\epsilon_{1234}(4!\Sigma^{12}_{[ab}\Sigma^{34}_{cd]}) + \epsilon_{2134}(4!\Sigma^{21}_{[ab}\Sigma^{34}_{cd]}) + \epsilon_{1324}(4!\Sigma^{13}_{[ab}\Sigma^{24}_{cd]}) + \ldots \\ &= 4!(4!\Sigma^{12}_{[ab}\Sigma^{34}_{cd]}) \end{aligned}$$

so that

$$4!\Sigma^{12}_{[ab}\Sigma^{34}_{cd]} = \tilde{V}\epsilon_{abcd}$$

it is easy to see this implies

$$\Sigma^{IJ}_{[ab}\Sigma^{KL}_{cd]} = \frac{1}{4!}\tilde{V}\epsilon_{abcd}\epsilon^{IJKL}$$

which becomes upon applying $dx^a \wedge \cdots \wedge dx^d$

$$\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd}dx^a \wedge \cdots \wedge dx^d = \frac{1}{4!}\tilde{V}\epsilon_{abcd}dx^a \wedge \cdots \wedge dx^d\epsilon^{IJKL} \tag{N.-9}$$

giving (N.7.4)

Now contract (N.7.4) with ϵ_{IJKL} we find

$$\epsilon_{IJKL}\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd}(dx^a \wedge \cdots \wedge dx^d) = \frac{1}{4!}\tilde{V}\epsilon_{abcd}\,\epsilon_{IJKL}\epsilon^{IJKL}(dx^a \wedge \cdots \wedge dx^d)$$

which implies

$$\frac{1}{2}\epsilon_{IJKL}\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd} = \frac{1}{2}\tilde{V}\epsilon_{abcd} \tag{N.-9}$$

giving (N.7.4).

The system of constraints (N.7.4) can be decomposed into three parts:

$$\begin{array}{llcl} (a) & {}^*\Sigma_{ab}\cdot\Sigma_{ab} & = & 0, \\ (b) & {}^*\Sigma_{ab}\cdot\Sigma_{ac} & = & 0, \\ (c) & {}^*\Sigma_{ab}\cdot\Sigma_{cd} & = & \pm 2\tilde{V}. \end{array} \quad \begin{array}{r} \text{(N.-8)} \\ \text{(N.-7)} \\ \text{(N.-6)} \end{array}$$

where the indices *abcd* are all different, and the sign in the last equation is determined by the sign of their permutation. These are the simplicity constraints.

GR can be written as an $SO(4)$ BF theory whose B field satisfies the simplicity constraints (N.-8-N.-7-N.-6).

Proof:

$$\begin{aligned} B^{IJ} \wedge B^{KL} &= \frac{1}{4}(\epsilon^{IJ}{}_{I'J'}\Sigma^{I'J'})(\epsilon^{KL}{}_{K'L'}\Sigma^{K'L'}) \\ &= \frac{1}{4}(\epsilon^{IJ}{}_{I'J'}\epsilon^{KL}{}_{K'L'})\Sigma^{I'J'}\Sigma^{K'L'} \end{aligned} \tag{N.-6}$$

N.7.5 Simplicity Constraints for 4-Simplicies

N.7.6 Self-dual structure of $SO(4)$

$$\hat{X} = X^{IJ}J_{IJ}, \tag{N.-6}$$

$$[\hat{J}_{IJ}, \hat{J}_{KL}] = i\delta_{IK}\hat{J}_{JL} - i\delta_{JK}\hat{J}_{IL} - i\delta_{IL}\hat{J}_{JK} - i\delta_{JL}\hat{J}_{IK}. \tag{N.-6}$$

where $\hat{J}_{IJ}$ are generators of $SO(4)$

$X^{\pm i} = 1/2(X^i \pm X^{0i})$, where $X^i \equiv 1/2\epsilon^i{}_{jk}X^{jk}$ and

$$\hat{\mathbf{X}} = \hat{X}_+ + \hat{X}_-, \quad \hat{X}^{\pm} = X^{\pm i}\hat{J}_i^{\pm}. \tag{N.-6}$$

for Lie algebra

$$[\hat{J}_i^{\pm}, \hat{J}_j^{\pm}] = i\epsilon_{ij}{}^k \hat{J}_k^{\pm}, \quad [\hat{J}_i^{+}, \hat{J}_j^{-}] = 0. \tag{N.-6}$$

We can write each $U \in SO(4)$ can be written in the form

$$U = (g_+, g_-) \text{ where } g_+ \in SU(2) \quad \text{and} \quad g_- \in SU(2) \quad \text{and} \quad UU' = (g_+g'_+, g_-g'_-). \tag{N.-6}$$

Define selfdual and anti-selfdual generators

$$J_{\pm} := *J \pm J,$$

that satisfy

$$J_{\pm} = \pm * J_{\pm}.$$

then

$$[J_+, J_-] = 0.$$

J_+ span a three dimensional subalgebra $su(2)_+$ of $so(4)$, and the J_- span a three dimensional subalgebra $su(2)_-$ of $so(4)$, both isomorphic to $su(2)$.

N.8 Coherent State Formulation of New BC Models

N.8.1 New Geometric Criterion

N.8.2 Coherent States

$$1_j = \sum_m |j, m >< j, m|, \tag{N.-6}$$

$$\delta_{mm'} = d_j \int_{SU(2)} dg\ t^j_{mj}(g)\overline{t^j_{m'j}(g)} \tag{N.-6}$$

$t^j_{mj}(g)$ and $t^j_{mj}(gh)$ differ only by a phase for any group element h from the $U(1)$ subgroup of $SU(2)$. The $U(1)$ subgroup being of the form

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \tag{N.-6}$$

$$1_j = d_j \int_{G/H} dn|j, n >< j, n|. \tag{N.-6}$$

$$< j, n|\hat{J}^i|j, n > \sigma_i = jn\sigma_3 n^{-1} \equiv jn^i\sigma_i \tag{N.-6}$$

or

$$< j, \hat{n}|\hat{J}^i|j, \hat{n} > = < j, j|\hat{J}^{i\prime}|j, j > \tag{N.-6}$$

where $J^{i\prime} = g(\hat{n})^{-1}J^i g(\hat{n})$ is the rotated generator.

Thus the state $|j, n >$ describes a vector in $\mathbb{R}^3$ of length j and of direction...

$$\Delta J^2 = j + j^2 - m^2 \tag{N.-6}$$

$|j, \hat{n} > = g(\hat{n})|j, j >$, where $\hat{n}$ is a unit vector defining a direction on the sphere S^2 and $g(\hat{n})$ an $SU(2)$ group element rotating the direction $\hat{z} \equiv (0, 0, 1)$ into the direction $\hat{n}$.

Just as $|j, j >$ has direction z with minimal uncertainty, $|j, \hat{n} >$ has direction $\hat{n}$ with minimal uncertainty.

Thus, the highest and lowest states $m = \pm j$ minimize the uncertainty relation correspond to cohernt states.

$$|j, j > \quad \text{and} \quad |j, -j >$$

N.8.3 Partition Function

N.9 Perturbation Theory

N.9.1 Diagrammatic Perturbation Theory

In this section we investigate general rules for the perturbative calculation of correlation functions, rules designed to yield the result in the form of an expansion in powers of g,

$$G = G_0 + gG_1 + g^2 G_2 + g^3 G_3 + ... + g^n G_n + ... \qquad \text{(N.-6)}$$

Where G_0 is the correlation function of the Gaussian model, (non-interacting model). These rules are easily represented in diagrammatic form. These diagrams are the so-called *Feynman Diagrams*. As a simple example we examine the Ginzburg-Landau theory (see eq.(**??**)). It is impossible to find an exact closed formula for Z(0), but if g is small on can expand $\exp(-g\int d^d x\phi^4(x)/4!)$.

The calculation of $\mathbf{G^{(2)}}$ to order g

First we calculate the 2-point greens function to order g. One must evaluate the integral

$$I(x,y) = \int \mathcal{D}\phi\phi(x)\phi(y)e^{-H} = \int \mathcal{D}\phi\phi(x)\phi(y)e^{-H_0}\left[1 - \frac{g}{4!}\int d^d z\phi^4(z) + \cdots\right]. \qquad \text{(N.-6)}$$

The first term in the square brackets merely yields

$$\mathcal{N}\left\langle\phi(x)\phi(y)\right\rangle_0 = \mathcal{N}G_0(x-y) \quad \text{where } \mathcal{N} = Z_0(j=0). \qquad \text{(N.-6)}$$

To evaluate the integral in the second term,

$$\int \mathcal{D}\phi\phi(x)\phi(y)e^{-H_0}\int d^d z\phi^4(z), \qquad \text{(N.-6)}$$

we use Wick's theorem (**??**). There are two types of result from the contractions:

$$(a)\ \ \left\langle\phi(x)\phi(y)\right\rangle_0\left\langle\phi^4(z)\right\rangle \quad \text{and} \quad (b)\ \ \left\langle\phi(x)\phi(z)\right\rangle_0\left\langle\phi^2(z)\right\rangle_0\left\langle\phi(y)\phi(z)\right\rangle_0 \qquad \text{(N.-6)}$$

in the wick expansion there were $4\times = 12$ terms of type (a) and 3 terms of type (b). It is convenient to represent these contractions as diagrams, by drawing two "external" points x and y ("external" means that they refer to the arguments of the correlation function), and "internal" point or "vertex" z, which stems from the expansion of exp(-V), and over which we integrate. Every contraction is represented by a line joining arguments of ϕ. The two types of terms possible in (N.9.1) are drawn

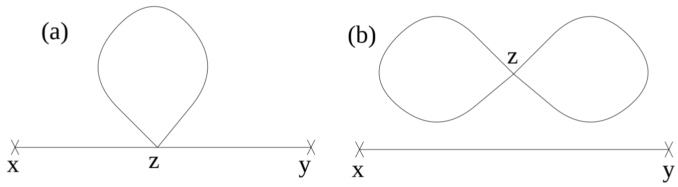


Figure N.12: The two diagrams of order g

These diagrams are called *Feynman diagrams (or graphs)*; one such diagram corresponds to every distinct group of terms of the perturbation expansion. The integral I reads

$$I(x,y) = \mathcal{N}\left[G_0(x-y) - \frac{1}{2}g\int d^dz G_0(x-z)G_0(0)G_0(z-y) - \frac{1}{8}gG_0(x-y)(G_0(0))^2\int d^dz\right] \tag{N.-6}$$

In order to obtain the correlation function, we must divide by Z(0):

$$Z(0) = \int \mathcal{D}\phi e^{-H_0}\left(1 - \frac{g}{4!}\int d^dz\phi^4(z) + \cdots\right) = \mathcal{N}\left[1 - \frac{g}{8}(G_0(0))^2\int d^dz + \cdots\right]. \tag{N.-6}$$

The second term in the square brackets is represented by the diagram.

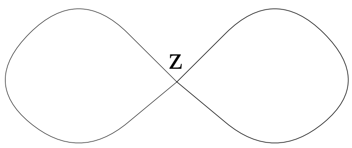


Figure N.13: The vacuum-fluctuation diagram

Dividing (N.9.1) by (N.9.1) we obtain the correlation function to order g

$$G^{(2)}(x-y) = \frac{I(x,y)}{Z(0)} = G_0(x-y) - \frac{1}{2}g\int d^dz G_0(x-z)G_0(0)G_0(z-y) + \mathcal{O}(g^2). \tag{N.-6}$$

The graph(b) from fig.(N.12) does not feature in the perturbation expansion of G. Diagrams of this type are called "vacuum-fluctuation" (sub)diagrams, meaning a subgraph that is completely disconnected from the "external" points x and y. The sum of all vacuum-fluctuation diagrams is equal to $Z(0) = \mathcal{D}\phi e^{-H}$. **Division by $Z(0)$ cancels all graphs containing "vacuum-fluctuations" parts disconnected from the rest of the diagram**. A proof is given in citeBellac (p 160).

On taking the Fourier transform, eq.(N.9.1) becomes

$$G^{(2)}(k) = G_0(k) - \frac{1}{2} g G_0(k) \left[\int \frac{d^d q}{(2\pi)^d} G_0(q) \right] G_0(k). \tag{N.-6}$$

The factor in front of the second term on the r.h.s. is called the *symmetry factor* of the diagram. To become familiar with the "Feynman rules", i.e. the rules for associating diagrams with the perturbation expansion, we move to the calculation of $G^{(2)}$ to order g^2.

The calculation of $\mathbf{G^{(2)}}$ to order $\mathbf{g^2}$

We use Wick's theorem to compute the expression

$$\left\langle \phi(x)\phi(y) \int d^d z d^d u \phi^4(z)\phi^4(u) \right\rangle_0 . \tag{N.-6}$$

Eliminating the terms that contain vacuum-fluctuation parts, one finds three types of graphs shown in fig.(N.14), with their symmetry factors given in brackets:

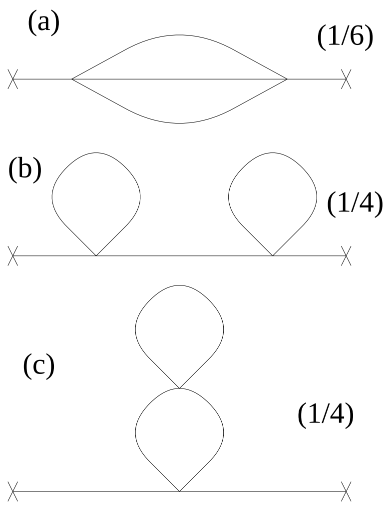


Figure N.14: The vacuum-fluctuation diagram

The vertices z and u may be permuted, which yields a multiplicative factor 2!; however this is exactly cancelled by the factor 1/2! from the expansion of the exponential. This is the same kind of cancellation happens in the nth order.

We shall settle for examining the contribution $\bar{G}(x-y)$ to the correlation function from graph (a) in fig.(N.14). Thus

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^dz d^du G_0(x-z)[G_0(z-u)]^3 G_0(u-y). \tag{N.-6}$$

Let us write $\bar{G}(x-y)$ as a Fourier transform, by replacing every factor G_0 by its Fourier representation

$$\begin{aligned} \bar{G}(x-y) = \frac{1}{6}g^2 \int d^dz d^du \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d} \prod_{l=1}^{3} \left\{ \frac{d^dq_l}{(2\pi)^d} e^{i\sum_{l=1}^{3} q_l\cdot(z-u)} \right\} \\ \times e^{ik\cdot(x-z)} e^{ik'\cdot(u-y)} G_0(k) G_0(k') \prod_{l=1}^{3} G_0(q_l). \end{aligned} \tag{N.-6}$$

The integration over z and u yield a product of two delta functions

$$2\pi)^d\delta^d(k-q_1-q_2-q_3) \times (2\pi)^d\delta^d(k'-q_1-q_2-q_3) \tag{N.-6}$$

which represent "momentum conservation" at the two vertices. Hence

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x-y)}[G_0(k)]^2 \times \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d} G_0(q_1)G_0(q_2)G_0(k-q_1-q_2). \tag{N.-6}$$

The last expression shows that $\bar{G}(x-y)$ is the Fourier transform of the function $\bar{G}(k)$,

$$\bar{G}(k) = \frac{1}{6}g^2 G_0(k) \left[\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d} G_0(q_1)G_0(q_2)G_0(k-q_1-q_2) \right] G_0(-k), \tag{N.-6}$$

(where we have used $G_0(k) = G_0(-k)$). (N.9.1) can be represented diagrammatically in fig.(N.15). The graph shown there has two external propagators $G_0(k)$ and $G_0(-k)$, and three internal propagators; because of the delta-functions $\delta^d(\ldots)$ ("momentum conservation"), only two of the three internal lines are independent. By following the internal propagators one can describe three different closed loops, but because of "momentum conservation" only two of these are independent; i.e. there are only two integration variables in (N.9.1).

Our experience with the previous examples suggest the following "Feynman rules" in k-space ("momentum space"):

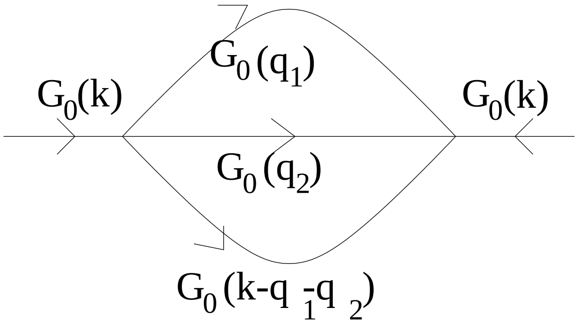


Figure N.15: Diagrammatic representation of (N.9.1)

1. We draw the Feynman diagram with a momentum assigned to each line. We must have overall momentum conservation and conservation at each vertex.

2. To every vertex we assign a factor -g

3. To every line we assign a factor $G_0(k)$

4. To every independent loop there corresponds an integration $\int d^d q/(2\pi)^d$.

5. Finally, every graph is multiplied by a symmetry factor.

N.9.2 The Generating Functional of Connected Diagrams

We start with an example, by investigating the correlation function $G^{(4)}$. It subdivides into one connected and three disconnected diagrams,

$$G^{(4)}(1,2,3,4) = G_c^{(4)}(1,2,3,4) + \{G_c^{(2)}(1,2)G_c^{(2)}(3,4) + permutations\}, \qquad \text{(N.-6)}$$

where G_c denotes a connected correlation function. (note $G_c^{(2)} = G^{(2)}$. In terms of graphs this is represented as in fig.(refbubble0)

The number of disconnected terms is $3 = 4!/[(2!)^2 \times (2!)]$. $4!$ is the number of permutations of the external points (1,2,3,4); but the result is unaffected by permuting (1,2), or (3,4), or the two bubbles (A) and (B), hence a factor $(2!)^2 \times 2!$.

We have been only considering theories where the n-point correlation functions with n odd vanish: $G^{(2k+1)} = 0$. For more generality, we shall assume that the interaction contains terms inφ^{2p+1}. Consider a disconnected diagram of $G^{(N)}$ corresponding to the subdivision into connected diagrams (fig N.17):

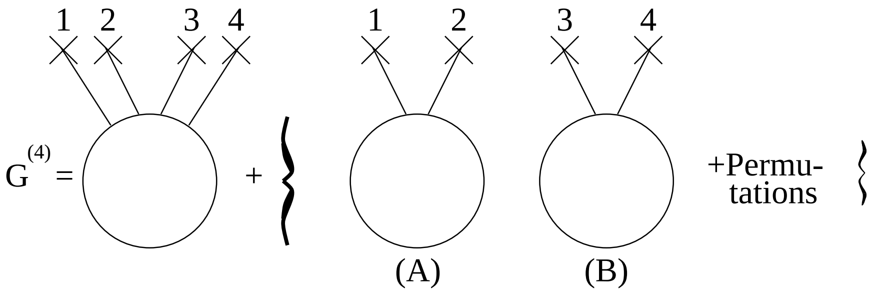


Figure N.16:

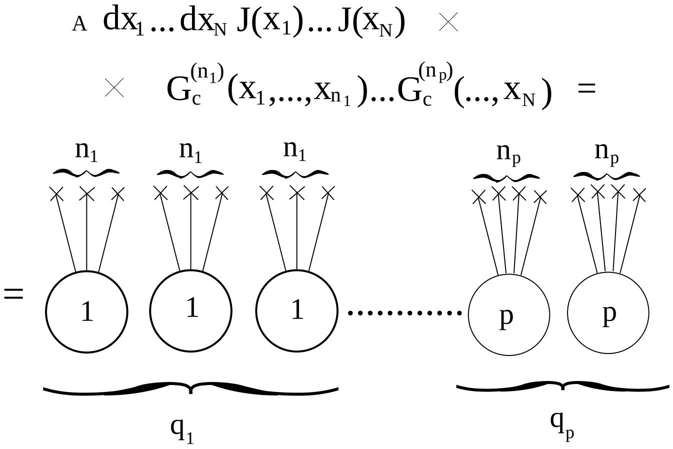


Figure N.17:

There are q_l bubbles connected to n_l external points,...., q_p bubbles connected to n_p external points, with

$$q_1 n_1 + \cdots + q_p n_p = N. \tag{N.-6}$$

The number of independent terms is

$$\frac{N!}{[(n_1!)^{q_1} q_1!] \ldots [(n_p!)^{q_p} q_p!]} \tag{N.-6}$$

It is found that the Functional that generates just connected diagrams is the logarithm of the normalised Generating functional. Hence, consider the exponential of the generating functional of connected diagrams:

$$\exp \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \ldots dx_N j(x_1) \ldots j(x_N) G_c^N (x_1 \ldots x_N) \tag{N.-6}$$

This should give the expansion for the generating function of all possible diagrams. When the exponential is expanded it is obvious that the amplitude for every possible disconnected diagram will be produced. To complete the proof that this is the correct Generating Functional we need to check each diagram comes with the correct prefactor.(i.e. equation (N.9.2). So expanding equation (N.9.2)

$$\sum_{q=0}^{\infty} \frac{1}{q!} \left(\sum_{n=1}^{\infty} \int dx_1 ....dx_n j(x_1)....j(x_n) G_c^{(n)}(x_1....x_n) \right)^q \tag{N.-6}$$

We convert this sum into a summation over N, the number of legs of the disconnected diagrams (figure).

$$\sum_{N=0}^{\infty} \sum_{q_1 n_1 + ... + q_p n_p = N} \prod_{i=1}^{p} \frac{1}{q_i!} \left[\frac{\int dx_1 ....dx_{n_i} j(x_1)....j(x_{n_i}) G_c^{n_i}(x_1....x_{n_i})}{n_i!} \right]^{q_i} \tag{N.-6}$$

Now we use (N.9.2) and the symmetry of G_c with respect to its arguments to rewrite the above equation as

$$\sum_{N=0}^{\infty} \frac{1}{N!} \int dx_1 ....dx_N j(x_1)....j(x_N) \sum_{q_1 n_1 + ... + q_p n_p = N} G_c^{n_i}(x_1....x_{n_i})....G_c^{n_p}(....x_N) \tag{N.-6}$$

Which is the correct form for the generating functional. Thus we have found that the generating functional of connected diagrams W(j) is indeed ln[Z(j)/Z(0)],

$$W(j) = ln \frac{Z(j)}{Z(0)} = \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 ... dx_N j(x_1)...j(x_N) G_c^N(x_1...x_N) \tag{N.-6}$$

N.9.3 1PI n-point functions

1PI $n-$point functions play a prominent role in the process of renormalization, as it is enough to renormalize 1PI $n-$point functions. This can be seen from their intmate relation to the effective action.

N.9.4 Rernormalization

since the observed magnitude of physical quantities (such as the charge of the electron) is finite, this number should arise from the addition of a “bare” (unobservable) value and the quantum corrections. Even though both of these quantities were divergent as only its (finite) sum that can be observed.

N.10 Coupling Matter to Spin Foams

group field theory

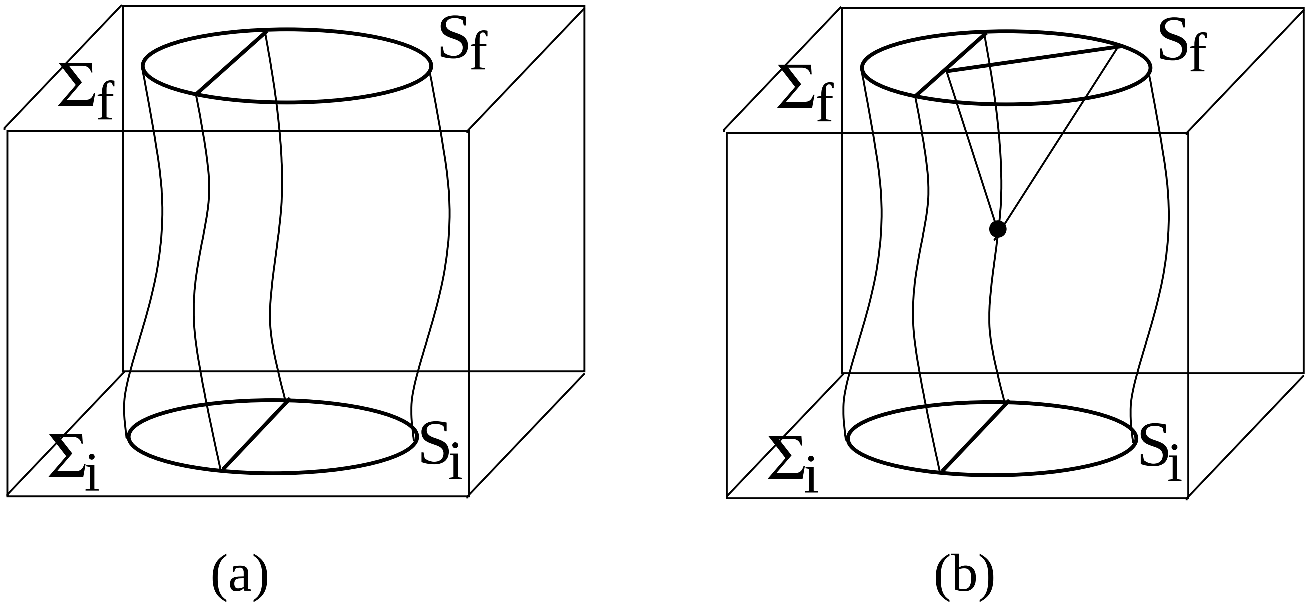


Figure N.18: Spin foam map.

N.11 Background-Independent Renormalization

Tentative ideas have been formulated by Markopoulou [122], [123] and Oeckl [124].

N.11.1 Background-Independent Renormalization a la Markopoulou

The partition function

$$Z_N[K] = Tre^{\mathcal{H}} \tag{N.-6}$$

An RG transformation

$$\exp(\mathcal{H}_N[K'], S'_I) = Tr' \exp(\mathcal{H}_N[K], S_i) \tag{N.-6}$$

This is accomplished by making a partial trace over the degrees of freedom $\{S_i\}$.

The lattice doesn't sit in a preexisting background geometry, the lattice itself represents the (quantized) geometry. The manifold the spinfoam is sitting in isn't equipped with a metric with respect to which scales can be defined; no lattice spacing is associated with the edges of the spin foam.

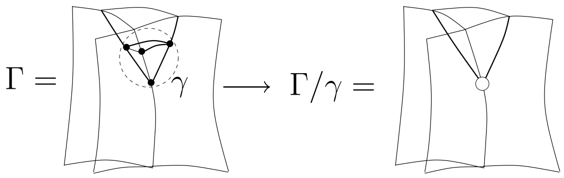


Figure N.19:

A nice example of a Hopf algebra.

background-independent course graining *a la* Fontini-Markopoulou [122], [?]

$$Z(s_i, s_f) = \sum_{\Gamma} N(\Gamma) \sum_{\text{labels on } \Gamma} \prod_{f \in \Gamma} \dim_{j_f} \prod_{v \in \Gamma} A_v(j) \tag{N.-6}$$

Two steps:

1. The calculation of a typical block transformation,

2. repeatedly apply it on the entire spin foam Γ to obtain a course grained one, Γ'.

If spin foams are highly irregular this makes the second step a non-trivial combinatorial problem.

Hopf Algebras

algebra: An algebra is simply a vector space over $\mathbb{C}$ (or over $\mathbb{R}$) in which there is defined a distributive and associative multiplication:

(i) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$;

(ii) $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$ for every scalar α, a complex (or real) number.

(iii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ associativity.

A multiplication associates to each pair of elements of A to an element from A. Mathematicians give this a formal description:

$$m \circ (a \otimes b) = a \cdot b \tag{N.-6}$$

Let's illustrate the notion

$$m \circ (m \otimes id_A)(a \otimes b \otimes c) = m \circ ((ab) \otimes c) = (ab)c \tag{N.-6}$$

(iii) a condition on the multiplication operation: $m \circ (m \otimes \mathrm{id}_A)(a \otimes b \otimes c) = m \circ (id_A \otimes m) \circ (a \otimes b \otimes c)$

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (id_A \otimes m) \tag{N.-6}$$

A unit operation ϵ

The comultiplication does the opposite: it associates a pair of elements from the set C with a single element from C - its coproduct, $\Delta(a) = b \otimes c, \quad b, c \in C$. a compatibility with m,

$$\Delta(a \cdot b) = \Delta(a)\Delta(a) \tag{N.-6}$$

There is a counit $\overline{\epsilon}$. This maps an element of C to a scalar $k : \overline{\epsilon}(a) = k$

A bialgebra is formed by combining an algebra and coalgebra for which $A = C$. There are conditions required of the multiplication and comultiplication so that they are compatible

$\gamma_1 \subset \gamma_2$ $\gamma_2 \subset \gamma_1$ $\gamma_1 \cap \gamma_2 = \emptyset$

let γ denote a proper sublattice of Γ, namely $\gamma \neq e$ and $\gamma \neq \Gamma$. We call the lattice that is left after we "cut out" γ the **remainder** and denote it Γ/γ.

$$\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p \tag{N.-6}$$

These are the **primitive elements** of the Hofp algebra.

The **counit** is an operation which annihilates every lattice except e.

$\gamma_1 \subset \gamma_2$ $\gamma_2 \subset \gamma_1$ $\gamma_1 \cap \gamma_2 = \emptyset$

let γ denote a proper sublattice of Γ, namely $\gamma \neq e$ and $\gamma \neq \Gamma$. We call the lattice that is left after we "cut out" γ the **remainder** and denote it Γ/γ.

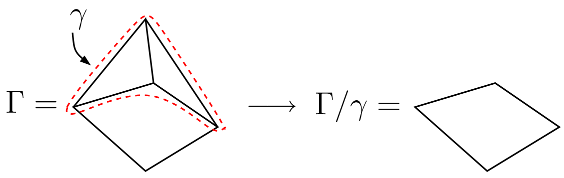


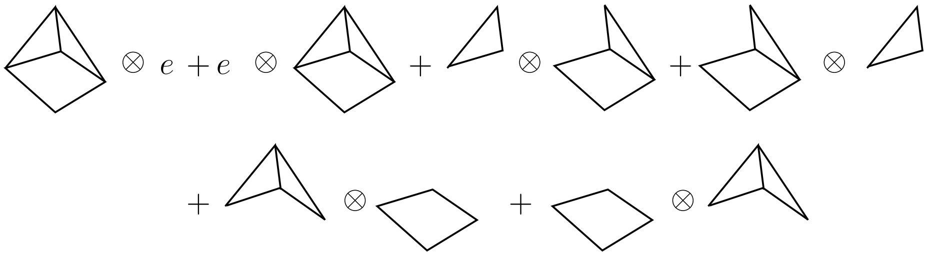
Figure N.20:

$$
\begin{aligned}
\Delta(\Gamma) &= \Gamma \otimes e + e \otimes \Gamma + \sum_{\gamma} \gamma \otimes \Gamma/\gamma && \text{(N.-5)}\\
\Delta(e) &= e \otimes e && \text{(N.-4)}\\
\Delta(\Gamma_1 \cdot \Gamma_2) &= \Delta(\Gamma_1)\Delta(\Gamma_2) && \text{(N.-3)}
\end{aligned}
$$

$$
\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p \qquad \text{(N.-3)}
$$

These are the **primitive elements** of the Hofp algebra.

$$\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \gamma_1\Gamma/\gamma_1 + \gamma_2\Gamma/\gamma_2 + \gamma_3\Gamma/\gamma_3 + \gamma_4\Gamma/\gamma_4$$



$$= \Gamma \otimes e + e \otimes \Gamma + \gamma_1\Gamma/\gamma_1 + \gamma_2\Gamma/\gamma_2 + \gamma_3 \otimes \gamma_4 + \gamma_4 \otimes \gamma_3.$$

Figure N.21: Markoprenorm2

The **counit** is an operation which annihilates every lattice except e.

$$
\bar{\epsilon}(\Gamma) = \begin{cases} 0 & \text{for } \Gamma \neq e, \\ 1 & \text{for } \Gamma = e. \end{cases} \qquad \text{(N.-3)}
$$

$$
\begin{aligned}
S(\Gamma) &= -\Gamma - \sum_{\gamma} S(\gamma)\Gamma/\gamma && \text{(N.-2)}\\
S(\gamma_p) &= -\gamma_p && \text{(N.-1)}\\
S(e) &= e && \text{(N.0)}\\
S(\Gamma_1 \cdot \Gamma_2) &= -S(\Gamma_2) \cdot -S(\Gamma_1) && \\
&= S(\Gamma_1)S(\Gamma_2) \qquad \text{as } \Gamma_1\Gamma_2 = \Gamma_2\Gamma_1. && \text{(N.0)}
\end{aligned}
$$

$S(\Gamma)$ is an iterative equation that stops when a primative lattice is reached.

(taken from hep-th/9805098) The full set of properties of a Hopf algebra canonly be guaranteed if equivalence works for products, in a certain sence

$$R\left(\prod_i R(\Gamma_1^i)\prod_j \Gamma_2^j\right) = \prod_i R(\Gamma_1^i)\prod_j R(\Gamma_2^j) \tag{N.0}$$

N.11.2 Background-Independent Renormalization a la Oeckl

renormalization *a la* R. Oeckl [124]

N.12 Reduced Phase Space Path-Integral

N.13 Operator Constraint Quantisation Path-Integral

Denote

$$\beta(t) := \beta_1 + t(\beta_2 - \beta_1)$$

It follows that

$$\begin{aligned} V(\beta_2) - V(\beta_1) &= \int_0^1 dt_1 \frac{d}{dt_1} V(\beta(t_1)) \\ &= i\int_0^1 dt_1 V(\beta(t_1))\ \dot\beta(0) h'_\mu(\phi - \beta(t_1)) \end{aligned} \tag{N.0}$$

We can find an iterative formula for $V(\beta(t))$. Note

$$\beta_{t_1}(t_2) = \beta_1 + t_2(\beta(t_1) - \beta_1) = \beta_1 + t_1 t_2(\beta_2 - \beta_1) = \beta(t_1 t_2) \tag{N.0}$$

so that

$$\begin{aligned} V(\beta_2(t_1)) - V(\beta_1) &= \int_0^1 dt_2 \frac{d}{dt_2} V(\beta_{t_1}(t_2)) = i\int_0^1 dt_2 \frac{d}{dt_2} V(\beta(t_1 t_2)) t_1 \dot\beta(0) h'_\mu(\phi - \beta(t_1 t_2)) \\ &= i\int_0^{t_1} dt_2 V(\beta(t_2))\ \dot\beta(0) h'_\mu(\phi - \beta(t_2)) \end{aligned} \tag{N.0}$$

$$V(\beta(t_1)) = V(\beta_1) + i\int_0^{t_1} dt_2 V(\beta(t_2))\dot{\beta}^\mu(0)h'_\mu(\phi - \beta(t_2))$$

or

$$V(\beta_1)^{-1}V(\beta(t_1)) = 1 + i\int_0^{t_1} dt_2 V(\beta_1)^{-1}V(\beta(t_2))\ \dot{\beta}^\mu(0)h'_\mu(\phi - \beta(t_2)) \tag{N.0}$$

Now set

$$\begin{aligned} U(t,0) &:= V(\beta_1)^{-1}V(\beta(t)) \\ \Phi(t) &:= \dot{\beta}^\mu(0)h'_\mu(\phi - \beta(t)) \\ &= [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta(t)) \end{aligned} \tag{N.-1}$$

Then () becomes

$$U(t,0) = 1 + i\int_0^t dt_1 U(t_1,0)\Phi(t_1) \tag{N.-1}$$

From this equation we obtain successive approximations:

$$\begin{aligned} U_0(t,0) &= 1 \\ U_1(t,0) &= 1 + i\int_0^{t_1} dt_1 \Phi(t_1) \\ U_2(t,0) &= 1 + i\int_0^{t_1} dt_1 U_1(t_1,0)\Phi(t_1) \\ &= 1 + i\int_0^t dt_1\Phi(t_1) + (i)^2\int_0^t dt_1\int_0^{t_1} dt_2\Phi(t_2)\Phi(t_1). \end{aligned} \tag{N.-3}$$

The $N-th$ order approximation being

$$U_N(t,0) = 1 + \sum_{n=1}^{N} U^{(n)}(t,0)$$

where

$$U^{(n)}(t,0) = (i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \Phi(t_n) \dots \Phi(t_1)$$

We define the left-time-ordered product as

$$T_l[\Phi(t_2)\Phi(t_1)] \equiv \begin{cases} \Phi(t_2)\Phi(t_1) & t_1 > t_2 \\ \Phi(t_1)\Phi(t_2) & t_2 > t_1 \end{cases}$$

Thus

$$U^{(2)}(t,0) = \frac{i^2}{2} \int_0^t dt_1 \int_0^t dt_2 T_l[\Phi(t_2)\Phi(t_1)]$$

for n operators we obtain

$$U^{(n)}(t,0) = \frac{i^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \dots \int_0^t dt_n T_l[\Phi(t_n) \dots \Phi(t_1)] \tag{N.-3}$$

$$\begin{aligned} U(t,0) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \dots \int_0^t dt_n T_l[\Phi(t_n) \dots \Phi(t_1)] \\ &= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \dots \int_0^t dt_n \Phi(t_n) \dots \Phi(t_1) \\ &= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int_0^t d\tau \Phi(\tau) \right)^n \end{aligned} \tag{N.-4}$$

We can formally write the expression for $U(t,0)$ as

$$V(\beta_1)^{-1} V(\beta(t_1)) = T_l \exp(i \int_0^{t_1} dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))$$

Setting $t_1 = 1$ we get

$$V(\beta_1)^{-1} V(\beta_2) = T_l \exp(i \int_0^1 dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))$$

N.14 Biblioliographical notes

In this chapter I have relied on the following refferences:

"On some aspects of canonical and covariant approaches to quantum gravity", D. Colosi.

N.15 Worked Exercies and Details

Details: Usefull formula.

Abelian subgroup - Cartan subgroup

$$h_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{N.-4}$$

Weyl integration formula

$$\int_G dg f(g) = \int_H \frac{\Delta(\theta)^2}{|W|} \left(\int_{G/H} f(x h_\theta x^{-1}) dx \right) d\theta \tag{N.-4}$$

where $\Delta(\theta) = \sin\theta$ and $|W|$ is the order of the Weyl group.

$$\int_G \delta_\theta(g) f(g) dg = \int_{G/H} \delta_\theta(g) f(x h_\theta x^{-1}) dx \tag{N.-4}$$

$$\delta_\theta(g) = \sum_j a_j \chi_j(g) = \sum_j (\delta_\theta \cdot \chi_j)\, \chi_j(g) \tag{N.-4}$$

where

$$\delta_\phi \cdot \chi_j \;=\; \int_G \delta_\phi(g)\chi_j(g) dg = \int_{G/H} \delta_\phi(g)\chi_j(x h_\phi x^{-1}) dx = \chi_j(h_\phi) \int_{G/H} dx = \chi_j(h_\phi) V_H \tag{N.-4}$$

$$\delta_\theta(g) = \sum_j \chi_j(h_\theta)\chi_j(g) = \sum_j \frac{\sin d_j\theta}{\sin\theta}(h_\theta)\chi_j(g) \tag{N.-4}$$

Details: 2D GFT.

(i)

$$
\begin{aligned}
\int \Phi^2(g_1,g_2)dg_1dg_2 &= \int \Phi(g_1,g_2)\Phi(g_2,g_1)dg_1dg_2 \\
& \int \left[\sqrt{2j+1}\phi_j^{a_1a_2}\overline{R^j_{a_1c}(g_1)}R^j_{a_2c}(g_2)\right] \times \\
& \qquad \left[\sqrt{2j'+1}\phi_{j'}^{b_1b_2}\overline{R^{j'}_{b_1d}(g_2)}R^{j'}_{b_2d}(g_1)\right] dg_1dg_2 \\
&= \sqrt{2j+1}\sqrt{2j'+1}\;\phi_j^{a_1a_2}\phi_{j'}^{b_1b_2}\left[\int \overline{R^j_{a_1c}(g_1)}R^{j'}_{b_2d}(g_1)dg_1\right] \times \\
& \qquad \left[\int \overline{R^{j'}_{b_1d}(g_2)}R^j_{a_2c}(g_2)dg_2\right] \\
&= \sqrt{2j+1}\sqrt{2j'+1}\phi_j^{a_1a_2}\phi_j^{b_1b_2}\delta_{jj'}\delta_{j'j}\left[\frac{\delta_{a_1b_2}\delta_{cd}}{2j+1}\right]\left[\frac{\delta_{b_1a_2}\delta_{dc}}{2j+1}\right] \\
&= \phi_j^{a_1a_2}\phi_j^{a_2a_1} \qquad \text{(N.-9)}
\end{aligned}
$$

where we used $\delta_{cd}\delta_{dc} = 2j+1$.

(ii)

$$
\begin{aligned}
& \int \Phi(g_1,g_2)\Phi(g_2,g_3)\Phi(g_3,g_1)dg_1dg_2dg_3 \\
= & \int \left[\sqrt{2j_1+1}\phi_{j_1}^{a_1a_2}\overline{R^{j_1}_{a_1d}(g_1)}R^{j_1}_{da_2}(g_2)\right] \times \left[\sqrt{2j_2+1}\phi_{j_2}^{b_1b_2}\overline{R^{j_2}_{b_1e}(g_2)}R^{j_2}_{eb_2}(g_3)\right] \times \\
& \left[\sqrt{2j_3+1}\phi_{j_3}^{c_1c_2}\overline{R^{j_3}_{c_1f}(g_3)}R^{j_3}_{fc_2}(g_1)\right]dg_1dg_2dg_3 \\
= & \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}\;\phi_{j_1}^{a_1a_2}\phi_{j_2}^{b_1b_2}\phi_{j_3}^{c_1c_2}\left[\int \overline{R^{j_1}_{a_1d}(g_1)}R^{j_3}_{fc_2}(g_1)dg_1\right] \times \\
& \left[\int \overline{R^{j_2}_{b_1e}(g_2)}R^{j_1}_{da_2}(g_2)dg_2\right] \times \left[\int \overline{R^{j_3}_{c_1f}(g_3)}R^{j_2}_{eb_2}(g_3)dg_3\right] \\
= & \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}\;\phi_{j_1}^{a_1a_2}\phi_{j_2}^{b_1b_2}\phi_{j_3}^{c_1c_2}\;\delta_{j_1j_3}\delta_{j_2j_1}\delta_{j_3j_2}\left[\frac{\delta_{a_1c_2}\delta_{df}}{2j_1+1}\right]\left[\frac{\delta_{b_1a_2}\delta_{ed}}{2j_2+1}\right]\left[\frac{\delta_{c_1b_2}\delta_{fe}}{2j_3+1}\right] \\
= & \frac{1}{\sqrt{2j+1}}\;\phi_j^{ab}\phi_j^{bc}\phi_j^{ca} \qquad \text{(N.-14)}
\end{aligned}
$$

Details:

(a):

$$
\begin{aligned}
\phi(g_1g,g_2g,g_3g,g_4g) &= \int dg\tilde{\phi}(g_1g\gamma,g_2g\gamma,g_3g\gamma,g_4g\gamma) \\
&= \qquad \text{(N.-14)}
\end{aligned}
$$

(b):

$$D^{(N)}_{\alpha\beta}(g_1 g) = D^{(N)\gamma}_{\alpha}(g_1) D^{(N)}_{\gamma\beta}(g) \tag{N.-14}$$

$$\begin{aligned}
\int \quad dg \quad & \phi(g_1 g, g_2 g, g_3 g, g_4 g) = \\
= \quad & \int dg \sum_{N_1 \ldots N_4} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} D^{(N_1)}_{\alpha_1 \beta_1}(g_1 g) \ldots D^{(N_4)}_{\alpha_4 \beta_4}(g_4 g) \\
= \quad & \sum_{N_1 \ldots N_4} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} D^{(N_1)\gamma_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\gamma_4}_{\alpha_4}(g_4) \underbrace{\int dg D^{(N_1)}_{\gamma_1 \beta_1}(g) \ldots D^{(N_4)}_{\gamma_4 \beta_4}(g)}_{Eq} \\
= \quad & \left[\sum_{N_1 \ldots N_4, \Lambda} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} C^{N_1 \ldots N_4 \ \Lambda}_{\beta_1 \ldots \beta_4} C^{N_1 \ldots N_4 \ \Lambda}_{\gamma_1 \ldots \gamma_4} \right] D^{(N_1)\gamma_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\gamma_4}_{\alpha_4}(g_4)
\end{aligned} \tag{N.-17}$$

$$\phi(x_1, x_2) = \phi(x_1 + y, x_2 + y) \quad \Longleftrightarrow \quad \Phi(g_1, g_2) = \Phi(g_1 g, g_2 g) \tag{N.-17}$$

and

$$\phi(x_1, x_2) = \phi(x_2, x_1) \tag{N.-17}$$

$$\begin{aligned}
\phi(x_1, x_2) \quad &= \quad \int dy \phi(x_1 + y, x_2 + y) \\
&= \quad \int dy \sum_{k_1, k_2} \phi_{k_1, k_2} e^{(x_1 k_1 + x_2 k_2) + iy(k_1 + k_2)} \\
&= \quad \sum_{k_1, k_2} \phi_{k_1, k_2} e^{(x_1 k_1 + x_2 k_2)} \int dy e^{iy(k_1 + k_2)} \\
&= \quad \sum_{k_1, k_2} \phi_{k_1, k_2} e^{(x_1 k_1 + x_2 k_2)} \delta(k_1 + k_2)
\end{aligned} \tag{N.-19}$$

so

the orthogonality relations

$$\int dy e^{ikx} e^{iky} = \delta(k_1 + k_2) \tag{N.-19}$$

$$\phi(x) = \sum_{n_1, n_2} \phi_{k_1, k_2} e^{ik_1 x_1 + ik_1 x_2} \quad \Longleftrightarrow \quad \Phi(g_1, g_2) = \sum_{ab}^{j} \phi^{ac}_{j} \ R^{j}_{ab}(g) R^{j}_{cd}(g) \tag{N.-19}$$

$$\phi_{k_1,k_2} \quad \Longleftrightarrow \quad \phi_j^{a_1 a_2} \tag{N.-19}$$

$$\begin{aligned}
\phi(x_1,x_2,x_3,x_4) &= \int dy \phi(x_1+y, x_2+y, x_3+y, x_4+y) \\
&= \int dy \sum_{k_1,\ldots,k_4} \phi_{k_1,\ldots,k_4} e^{i(x_1k_1+x_2k_2+k_3x_3+x_4k_4)+iy(k_1+k_2+k_3+k_4)} \\
&= \sum_{k_1,\ldots,k_4} \phi_{k_1,\ldots,k_4} e^{i(x_1k_1+x_2k_2+k_3x_3+x_4k_4)} \underbrace{\int dy e^{iy(k_1+k_2+k_3+k_4)}} \\
&= \sum_{k_1,\ldots,k_4} \phi_{k_1,\ldots,k_4} e^{i(x\cdot k)} \delta(k_1+k_2+k_3+k_4)
\end{aligned} \tag{N.-22}$$

$$\begin{aligned}
\phi(x_1,x_2,x_3,x_4) = \phi(x_1+a, x_2+a, x_3+a, x_4+a) \rightarrow \\
\phi(g_1,g_2,g_3,g_4) = \phi(g_1g, g_2g, g_3g, g_4g)
\end{aligned} \tag{N.-22}$$

$$\begin{aligned}
\phi(x_1,x_2,x_3,x_4) = \phi(x_1-x_4, x_2-x_4, x_3-x_4, 0) \Rightarrow \\
\phi(g_1,g_2,g_3,g_4) = \phi(g_1g_4^{-1}, g_2g_4^{-1}, g_3g_4^{-1}, I)
\end{aligned} \tag{N.-22}$$

$$\phi(x,y) = \phi(x-y) \Rightarrow \phi(g,\gamma) = \phi(g\gamma^{-1}) \tag{N.-22}$$

$$f(g) = \sum_{\Lambda}^{\alpha\beta} D_{\alpha\beta}^{(\Lambda)}(g) \Rightarrow f(x) = \sum_n f_n e^{inx} \tag{N.-22}$$

(c):

$$S[\phi] = \frac{i}{2}\int_G dg_1 \ldots dg_4 \phi^2(g_1,g_2,g_3,g_4) \tag{N.-22}$$

$$\begin{aligned}
S[\phi] &= \frac{i}{2} \sum_{N_1 \ldots N_4} \phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4),\Lambda} (\Delta_{N_1} \ldots \Delta_{N_4}) C^{N_1 \ldots N_4,\Lambda}_{\gamma_1 \ldots \gamma_4} \\
&\quad \sum_{M_1 \ldots M_4} \Phi^{\beta_1 \ldots \beta_4}_{(M_1 \ldots M_4),\delta_1 \ldots \delta_4} \int_G dg_1 D^{(N_1)\gamma_1}_{\alpha_1}(g_1) D^{(M_1)\delta_1}_{\beta_1}(g_1) \ldots \int_G dg_4 D^{(N_4)\gamma_4}_{\alpha_4}(g_4) D^{(M_4)\delta_4}_{\beta_4}(g_4) \\
&= \frac{i}{2} \sum_{N_1 \ldots N_4} \phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4),\Lambda} (\Delta_{N_1} \ldots \Delta_{N_4}) \left[\sum_{M_1 \ldots M_4} \Phi^{\beta_1 \ldots \beta_4}_{(M_1 \ldots M_4),\delta_1 \ldots \delta_4} C^{M_1 \ldots M_4,\Lambda}_{\gamma_1 \ldots \gamma_4} \delta^{\gamma_1 \delta_1} \ldots \delta^{\gamma_4 \delta_4} \right] \\
&\quad \delta_{\alpha_1 \beta_1} \ldots \delta_{\alpha_4 \beta_4} \delta^{N_1 M_1} \ldots \delta^{N_4 M_4} \delta^{\Lambda \tilde{\Lambda}}
\end{aligned} \tag{N.-24}$$

where we $C^{N_1 \ldots N_4, \Lambda}_{\gamma_1 \ldots \gamma_4} \rightarrow C^{M_1 \ldots M_4, \Lambda}_{\gamma_1 \ldots \gamma_4}$ because of the term $\delta^{N_1 M_1} \ldots \delta^{N_4 M_4}$.

$$S[\phi] = \frac{i}{2} \phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4), \Lambda} \phi^{\beta_1 \ldots \beta_4}_{(M_1 \ldots M_4), \tilde{\Lambda}} \left((\Delta_{N_1} \ldots \Delta_{N_4})^2 \delta_{\alpha_1 \beta_1} \ldots \delta_{\alpha_4 \beta_4} \delta^{N_1 M_1} \ldots \delta^{N_4 M_4} \delta^{\Lambda \tilde{\Lambda}} \right) \quad \text{(N.-24)}$$

SO(4)

$$U(g) = 1 - \frac{1}{2} J^j_i \epsilon^i_j + \mathcal{O}, \quad \text{(N.-24)}$$

where the J^j_i are $N \times N$ matrices

$$[J^j_i, J^l_k] = \delta^j_k J^l_i - \delta^i_l J^j_k + \delta^i_k J^l_j - \delta^j_l J^k_i \;, \quad i, j, kl = 1, \ldots, n. \quad \text{(N.-24)}$$

Hopf algebra of rooted trees.

$$\Delta(t_n) = t_n \otimes e + e \otimes t_n + \sum_{i=1}^{n-1} t_i \otimes t_{n-i} \quad \text{(N.-23)}$$

$$S(t_n) = -t_n - \sum_{i=1}^{n-1} S(t_i) t_{n-i} \quad \text{(N.-22)}$$

Proofs

Verify the vector space of partitioned lattices form a Hopf algebra under the operations (N.-19), (N.-19).

(a) Verify:

$$\Delta(\Gamma_1 \cdot \Gamma_2) = \Delta(\Gamma_1) \cdot \Delta(\Gamma_2). \quad \text{(N.-22)}$$

(b) Verify: Definition of the antipode

$$m(S \otimes \mathrm{id}) \Delta(\Gamma) = \bar{\epsilon}\epsilon(\Gamma) = \left\{ \begin{array}{l} e \;\; \text{for } \Gamma = e \\ 0 \;\; \text{for } \Gamma \neq e \end{array} \right. \quad \text{(N.-22)}$$

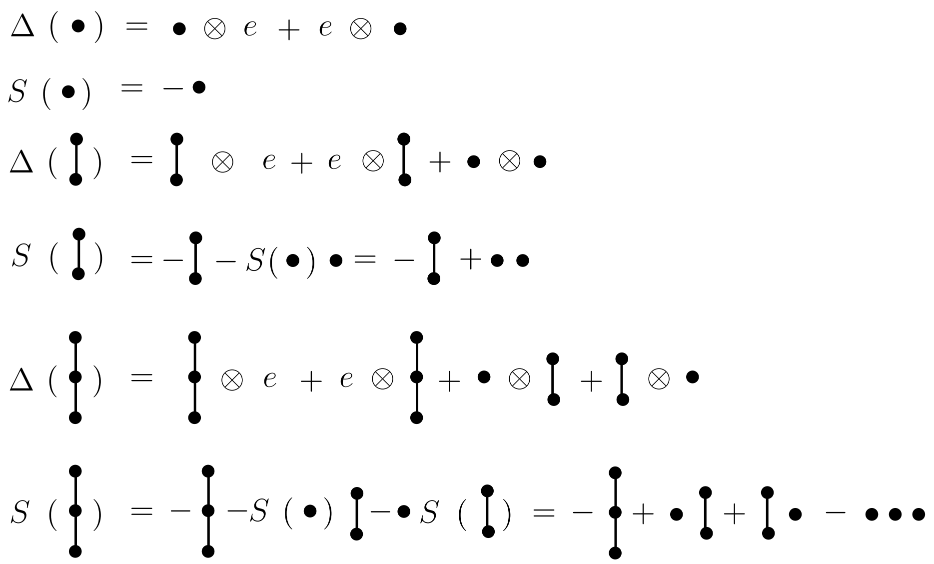


Figure N.22: Example of a Hopf algebra - rooted trees.

Verify: Antipode anti-homomorphism. Note that because the multiplication $\cdot$ is commutative ($\Gamma_1 \cdot \Gamma_2 = \Gamma_2 \cdot \Gamma_1$), the anti-homomorphism is equivalent to a homomorphism.

$$S(\Gamma_1 \cdot \Gamma_2) = S(\Gamma_1) \cdot S(\Gamma_2). \tag{N.-22}$$

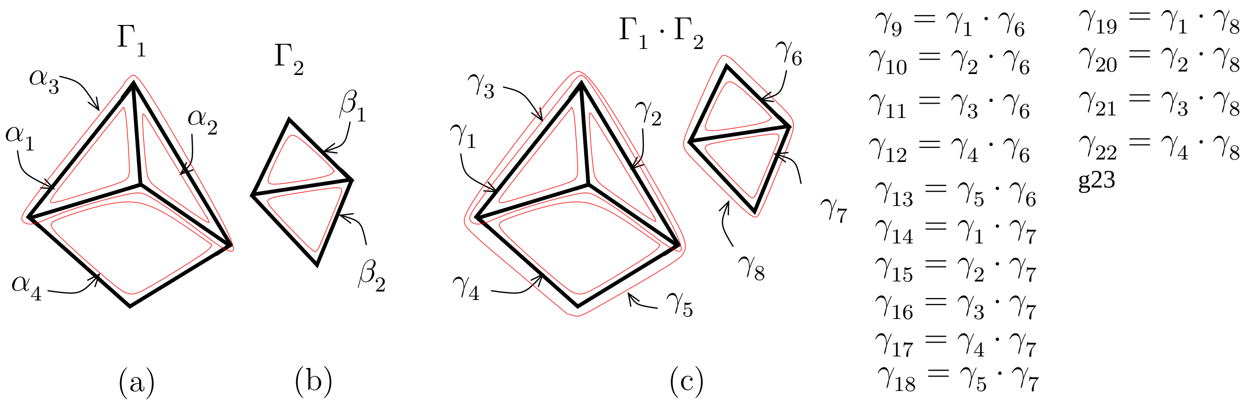


Figure N.23: Γ_1 and Γ_1 (c) all 22 of them.

$$\Delta(\Gamma_1 \cdot \Gamma_2) = (\Gamma_1 \cdot \Gamma_2 \otimes e + e \otimes \Gamma_1 \cdot \Gamma_2 + \textstyle\sum_\gamma \gamma \otimes \Gamma_1/\gamma)$$

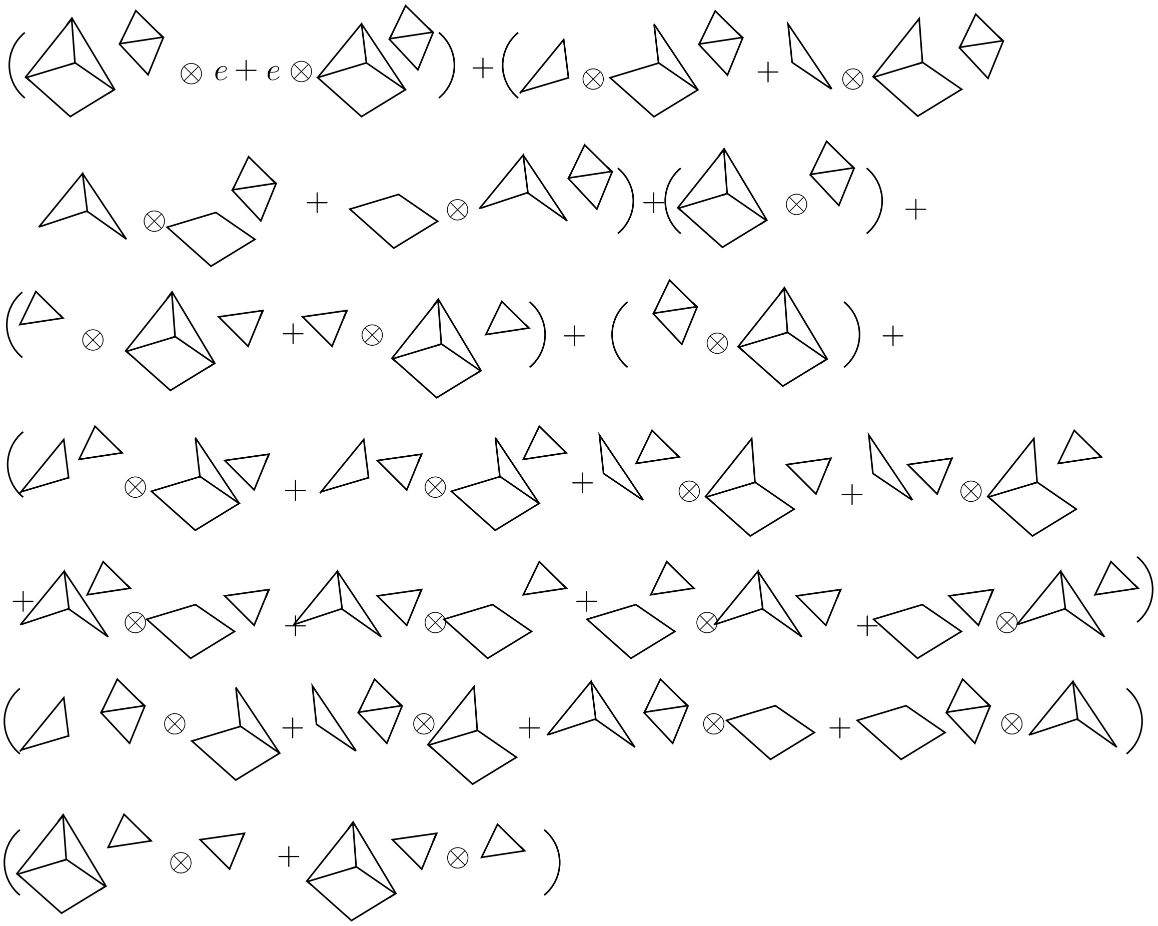


Figure N.24: Homomorphism of Δ.

$$\begin{aligned}
\Delta(\Gamma_1 \cdot \Gamma_2) &= \Gamma_1 \cdot \Gamma_2 \otimes e + e \otimes \Gamma_1 \cdot \Gamma_2 + \sum_\gamma \gamma \otimes \Gamma_1 \cdot \Gamma_2/\gamma \\
&= \Gamma_1 \cdot \Gamma_2 \otimes e + e \otimes \Gamma_1 \cdot \Gamma_2 + \\
&+ \sum_\alpha \alpha \otimes (\Gamma_1/\alpha) \cdot \Gamma_2 + \Gamma_1 \otimes \Gamma_2 \\
&+ \sum_\beta \beta \otimes (\Gamma_2/\beta) \cdot \Gamma_1 + \Gamma_2 \otimes \Gamma_1 \\
&+ \sum_{\alpha,\beta} \alpha \cdot \beta \otimes (\Gamma_1/\alpha) \cdot (\Gamma_2/\beta) + \\
&+ \sum_\alpha \alpha \cdot \Gamma_2 \otimes \Gamma_1/\alpha + \sum_\beta \Gamma_1 \cdot \beta \otimes \Gamma_2/\beta \\
&= (\Gamma_1 \otimes e + e \otimes \Gamma_1 + \sum_\alpha \alpha \otimes \Gamma_1/\alpha)\ (\Gamma_2 \otimes e + e \otimes \Gamma_2 + \sum_\beta \beta \otimes \Gamma_2/\beta) \\
&= \Delta(\Gamma_1)\Delta(\Gamma_2)
\end{aligned} \tag{N.-28}$$

(b) Anitipode definition.

$$(S \otimes \mathrm{id})\Delta(\gamma_p) = m(S \otimes \mathrm{id})(\gamma_p \otimes e + e \otimes \gamma_p) = S(\gamma_p) + \gamma_p = 0 \tag{N.-28}$$

$$\begin{aligned}(S \otimes \mathrm{id})\Delta(\Gamma) &= m(S \otimes \mathrm{id})(\Gamma \otimes e + e \otimes \Gamma + \sum_\gamma \gamma \otimes \Gamma/\gamma) \\ &= S(\Gamma) + \Gamma + \sum_\gamma S(\gamma) \cdot \Gamma/\gamma \end{aligned} \tag{N.-28}$$

Anitipode anti-homomorphism.

$$\begin{aligned} S(\gamma_p \cdot \gamma_{p'}) &= -\gamma_p \cdot \gamma_{p'} - [S(\gamma_p)\gamma_{p'} + S(\gamma_{p'})\gamma_p] \\ &= \gamma_p \gamma_{p'} \\ &= S(\gamma_p)S(\gamma_{p'}). \end{aligned} \tag{N.-29}$$

$$\begin{aligned} S(\Gamma_1 \cdot \Gamma_2) &= -\Gamma_1 \cdot \Gamma_2 - \sum_\gamma S(\gamma) \cdot (\Gamma_1 \cdot \Gamma_2)/\gamma \\ &= (-\Gamma_1 - \sum_\alpha S(\alpha) \cdot \Gamma_1/\alpha) \cdot (-\Gamma_2 - \sum_\beta S(\beta) \cdot \Gamma_2/\beta) \end{aligned} \tag{N.-29}$$

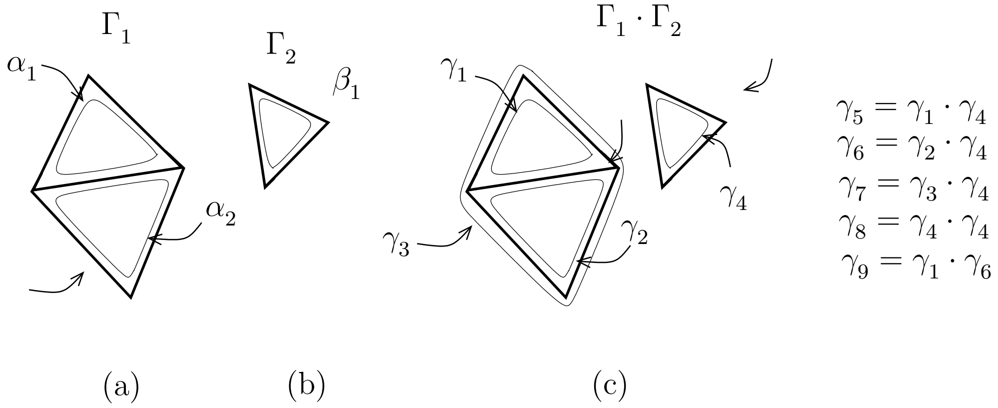


Figure N.25: Anti-Homomorphism of S.

$$\begin{aligned} S(\Gamma_1 \cdot \Gamma_2) &= -\Gamma_1 \cdot \Gamma_2 - \sum_\gamma S[\gamma](\Gamma_1 \cdot \Gamma_2)/\gamma \\ &= -\Gamma_1 \cdot \beta - S(\alpha_1) \cdot \alpha_2 \cdot \beta \\ &= [2\alpha_1 \cdot \alpha_2 \cdot -S(\Gamma_1)] \cdot \beta \\ &= S(\Gamma_1) \cdot \beta \end{aligned} \tag{N.-31}$$

where we have used

$$S(\Gamma_1) = -\Gamma_1 - S(\alpha_1)\cdot\alpha_2 - S(\alpha_2)\cdot\alpha_1 = -\Gamma_1 + 2\alpha_1\cdot\alpha_2. \qquad \text{(N.-31)}$$

Ising chain

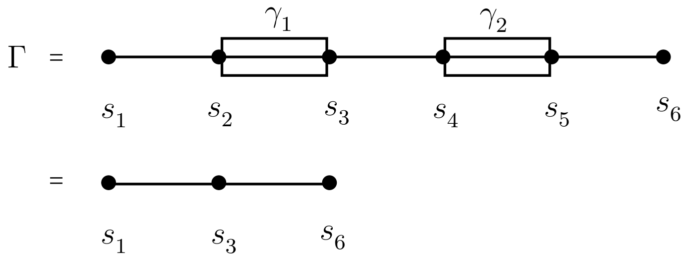


Figure N.26: Markoprenorm5.

$$\mathcal{H} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i \qquad \text{(N.-31)}$$

with $\sigma_i = \pm 1$ and $\sigma_{N+1} = \sigma_1$.

The partition function is given by

$$Z = Tr_{\sigma_i} e^{\mathcal{H}} = \sum_{\{\sigma_i\}=\pm 1}^{N} \exp\left\{\sum_{i=1}^{N}\left[K\sigma_i\sigma_{i+1} + \frac{1}{2}(\sigma_i + \sigma_{i+1})\right]\right\}. \qquad \text{(N.-31)}$$

Carry out sum over odd numbered degrees of freedom

$$\sum_{\sigma_1=\pm 1} e^{\mathcal{H}} = \sum_{\sigma_2=\pm 1}\sum_{\sigma_4=\pm 1}\cdots\sum_{\sigma_N=\pm 1}\left[\sum_{\sigma_1=\pm 1}\sum_{\sigma_3=\pm 1}\cdots\sum_{\sigma_{N-1}=\pm 1} e^{\mathcal{H}}\right] \qquad \text{(N.-31)}$$

$$\begin{aligned}
\sum_{\sigma_1=\pm 1} e^{K\sigma_1(\sigma_N+\sigma_2)+h\sigma_1} &= 2\cosh[K(\sigma_N+\sigma_2)+h] \\
\sum_{\sigma_3=\pm 1} e^{K\sigma_3(\sigma_N+\sigma_4)+h\sigma_3} &= 2\cosh[K(\sigma_N+\sigma_4)+h] \\
&\vdots
\end{aligned} \qquad \text{(N.-32)}$$

$$\sum_{\sigma_1=\pm1}\sum_{\sigma_3=\pm1}\cdots\sum_{\sigma_{N-1}=\pm1} e^{\mathcal{H}} = 2\cosh[K(\sigma_N+\sigma_2)+h]\times\cdots\times 2\cosh[K(\sigma_N+\sigma_2)+h] \times e^{K\sigma_2(\sigma_1+\sigma_3)+h\sigma_2}\times\cdots\times e^{K\sigma_{N-1}(\sigma_1+\sigma_3)+h\sigma_2} \quad \text{(N.-32)}$$

$$2e^{h(\sigma_N+\sigma_2)/2}\cosh[K(\sigma_N+\sigma_2)+h] \\ \exp\{2g+K'\sigma_N\sigma_2+\frac{1}{2}h'(\sigma_N+\sigma_2)\} \quad \text{(N.-32)}$$

where

$$K' = \frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h} \quad \text{(N.-32)}$$

$$h' = h+\frac{1}{2}\ln\frac{\cosh(2K+h)}{\cosh(2K-h)} \quad \text{(N.-32)}$$

and

$$g = \frac{1}{8}\ln[16\cosh(2K+h)\cosh(2K-h)\cosh^2 h]. \quad \text{(N.-32)}$$

$$\sum_{\sigma_1=\pm1}\sum_{\sigma_3=\pm1}\cdots\sum_{\sigma_{N-1}} e^{\mathcal{H}} = \exp\left\{Ng(K,h)+K'\sum_i\sigma_{2i}\sigma_{2i+2}+h'\sum_i\sigma_{2i}\right\} \quad \text{(N.-32)}$$

sum is over remaining even numbered sites.

Algebraic block transform

The coproduct on Γ is

$$\Delta[\Gamma] = \Gamma\otimes e + e\otimes\Gamma+\gamma_1\otimes\gamma_2+\gamma_2\otimes\gamma_1 \quad \text{(N.-32)}$$

$$\Delta[\Gamma] = \Gamma+\gamma_1\gamma_2 \quad \text{(N.-32)}$$

The shrinking antipode S'_R on w_Γ, gives

$$
\begin{aligned}
S_R'(w_\Gamma) &= -R(w_\Gamma) + R(w_{\gamma_1})w_{\gamma_2} + R(w_{\gamma_2})w_{\gamma_1} \\
&= w_\Gamma + 2w_{\gamma_1}w_{\gamma_2},
\end{aligned}
\tag{N.-32}
$$

where $w_{\Gamma'} = R(w_\Gamma)$

define the operation R

$$
R(\gamma) = \partial\gamma, \tag{N.-31}
$$

$$
R(w_\gamma) = \sum_{\text{Internal spins } \gamma=\pm} w_\gamma \tag{N.-30}
$$

Ising/Potts model in two dimensions.

$$
w_\Gamma = \exp\left(\sum_{<i,j>} \kappa_{ij} s_i s_j\right) \tag{N.-30}
$$

where means that i and j are adjacent sites in the lattice.

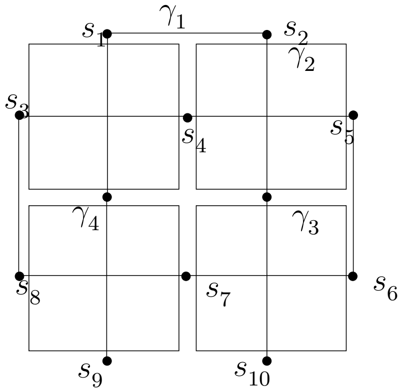


Figure N.27: Homomorphism of Δ.

The coproduct on Γ is

$$
\Delta[\Gamma] = \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1 \tag{N.-30}
$$

$$
\Delta[\Gamma] = \Gamma + \gamma_1\gamma_2 \tag{N.-30}
$$

The shrinking antipode S_R' on w_Γ, gives

$$
\begin{aligned}
S'_R(w_\Gamma) &= -R(w_\Gamma) + R(w_{\gamma_1})w_{\gamma_2} + R(w_{\gamma_2})w_{\gamma_1} \\
&= w_\Gamma + 2w_{\gamma_1}w_{\gamma_2}, \qquad \text{(N.-30)}
\end{aligned}
$$

where $w_{\Gamma'} = R(w_\Gamma)$

define the operation R

$$
R(\gamma) = \partial\gamma, \qquad \text{(N.-29)}
$$

$$
R(w_\gamma) = \sum_{\text{Internal spins } \gamma=\pm} w_\gamma \qquad \text{(N.-28)}
$$

Proofs

Appendix O

The Master Constraint

O.1 Weak Dirac Observables

A function O is a weak Dirac observable if and only if

$$\{O, \{O, \mathbf{M}\}\}_{\mathbf{M=0}} \qquad \text{Master Equation} \tag{O.0}$$

Proof:

It is a straight forward but tedious calculation, the end result is,

$$\begin{aligned}
\{O, \{O, \mathbf{M}\}\} &= \int_X d\mu(x) \Big[q^{jk}(x) \{O, C_j(x)\} \{O, C_k(x)\} + q^{jk}(x) \{O, \{O, C_j(x)\}\} C_k(x) \\
&+ \{O, q^{jk}(x)\} \{O, C_j(x)\} C_k(x) + \frac{1}{2} \{O, \{O, q^{jk}(x)\}\} C_j(x) C_k(x) \Big]
\end{aligned} \tag{O.0}$$

Restricting this expression to the constraint surface $\mathcal{C}$ is equivalent to setting $\mathbf{M} = 0$ hence

$$\{O, \{O, \mathbf{M}\}\}_{\mathbf{M=0}} = \int_X d\mu(x) q^{jk}(x) \{O, C_j(x)\}|_\rfloor \{O, C_k(x)\}|_\rfloor \tag{O.0}$$

Obviously, if $\{O, C_j(x)\}|_{\mathcal{C}=0} = 0$ then $\{O, \{O, \mathbf{M}\}\}_{\mathbf{M=0}} = 0$.

Now, since q is positive definite the condition $\{O, \{O, \mathbf{M}\}\}_{\mathbf{M=0}} = 0$ implies that

$$\{O, C_k(x)\}|_{\mathcal{C}=0} = 0 \quad \text{for } \forall\, x \in X. \tag{O.0}$$

Hence the conditions are equivalent. Eq.(O.1) can be reexpressed as

$$\{O, C_k(x)\}|_{\mathcal{C}=0} = 0 \tag{O.0}$$

for all smooth test functions of compact support.

Details: Use equation involving Poison brackets.

O.2 Regularization of the Master Constraint

rigging map

$$< \eta(T_s), \eta(T_{s'}) >= \eta(\eta(T_{s'}))[T_s]$$

$$\begin{aligned} < T_{[s]}, [\hat{\tilde{C}}'(\Delta)]T_{[s_2]} >_{Diff} &= < \hat{\tilde{C}}^\dagger(\Delta)T_{[s]}, T_{[s_2]} >_{Diff} \\ &= \eta(T_{s_2})[\hat{\tilde{C}}^\dagger(\Delta)T_{s_0(s)}] \\ &= T_{[s_2]}(\hat{\tilde{C}}^\dagger(\Delta)T_{s_0(s)}) \end{aligned} \tag{O.-1}$$

$$Q_M(T_{[s_1]}, T_{[s_2]}) = \lim_{\tau\Sigma} \sum_{\Delta\in\tau} \sum_{[s]} \overline{T_{[s_1]}(\hat{\tilde{C}}^\dagger(\Delta)T_{s_0([s])})}(T_{[s_2]}\hat{\tilde{C}}^\dagger(\Delta)T_{s_0([s])}), \tag{O.-1}$$

The part

$$\hat{\tilde{C}}^\dagger(\Delta)T_{s_0([s])}$$

has a finite number of terms so that

$$T_{[s_1]}\hat{\tilde{C}}^\dagger(\Delta)T_{s_0([s])}$$

obviously the number of $[s]$ contributing to (N.-19) is finite

O.3 Quantizing the Master Constraint

$$
\begin{aligned}
Q_{\mathbf{M}}(\Psi_{Diff}, \Psi'_{Diff}) \quad &:= \quad \lim_{\epsilon \to 0} \sum_{[s]} \frac{1}{2} \sum_{v \in V(\gamma)} \sum_{v' \in V(\gamma)} \frac{1}{C^3_{n(v)} C^3_{n(v')}} \sum_{v(\Delta)=v} \sum_{v(\Delta')=v'} \chi_\epsilon(v - v') \\
&\times \quad \Psi_{Diff} \hat{h}^\epsilon_v \; \Psi'_{Diff} \hat{h}^\epsilon_{v'}
\end{aligned} \tag{O.-1}
$$

O.4 Testing the Master Constraint on Toy Models

$$
\begin{aligned}
&p_1 = 0, \;\; p_2 = 0 \\
&M := c_1 p_1^2 + c_2 p_2^2, \quad \text{where } c_1, c_2 \text{ are positive numbers.}
\end{aligned} \tag{O.-1}
$$

Quantization: Schrödinger representation on $L_2(R^2)$

$$
\hat{M} := c_1 \hat{p}_1^2 + c_2 \hat{p}_2^2 \tag{O.-1}
$$

$$
\int dp_1 dp_2 \overline{\psi(p_1, p_2)} \phi(p_1, p_2) = \int d\lambda \left[dp_1 dp_2 \delta(\lambda - c_1 \hat{p}_1^2 - c_2 \hat{p}_2^2) \overline{\psi(p_1, p_2)} \phi(p_1, p_2) \right] \tag{O.-1}
$$

$$
\int d\lambda \int dp_1 dp_2 \delta(\lambda - c_1 p_1^2 - c_2 p_2^2) \psi(p_1, p_2) \tag{O.-1}
$$

$\lambda \int dx \delta(\lambda x) = 1 = \int_{-\infty}^{\infty} dx \delta(x)$ implies $\delta(\lambda x) \to \delta(x)/\lambda$

$$
\int d\lambda \frac{\lambda^{1-1}}{\sqrt{c_1 c_2}} \int d\tilde{p}_1 d\tilde{p}_2 \delta(1 - \tilde{p}_1^2 - \tilde{p}_2^2) \psi \left(\frac{\sqrt{\lambda} \tilde{p}_1}{\sqrt{c_1}}, \frac{\sqrt{\lambda} \tilde{p}_2}{\sqrt{c_2}} \right) \tag{O.-1}
$$

$$
d\Omega = \sin\theta d\theta \tag{O.-1}
$$

$$
\int d\lambda \frac{\lambda^{1-1}}{\sqrt{c_1 c_2}} \int_{S^1} d\Omega \mathcal{H}^{\oplus} \left(\frac{\sqrt{\lambda} \cos\theta}{\sqrt{c_1}} \frac{\sqrt{\lambda} \sin\theta}{\sqrt{c_2}} \right) \tag{O.-1}
$$

$\lambda = \mathbf{0}$

$$\int dp_1 dp_2 \delta(c_1 p_1^2 + c_2 p_2^2)\psi(p_1, p_2) \tag{O.-1}$$

$$u = c_2 p_2^2/c_1, \quad \text{so that} \quad p_2 = \sqrt{\frac{uc_1}{c_2}} \quad dp_2 = du\sqrt{\frac{uc_1}{uc_2}} \tag{O.-1}$$

$$\begin{aligned} &= \int dp_1 \frac{1}{c_2} \int \frac{du}{\sqrt{c_2}} \delta(1+u)\psi(p_1, \sqrt{\frac{c_1 u}{c_2}}) \\ &= \int dp_1 \frac{1}{c_2} \left[\int \frac{du}{\sqrt{c_2}} \psi(p_1, i\sqrt{\frac{c_1}{c_2}}) \right] \end{aligned} \tag{O.-1}$$

$$\mathcal{H}_{phys} = L_2(R). \tag{O.-1}$$

More general result:

$$\begin{aligned} &= \int_{R_+}^{\oplus} d\lambda \int_{R}^{\oplus} dp^{(n-k)} \delta(\lambda - \sum_{j>k} c_j p_j^2)\mathcal{H}^{\oplus}(k) \\ &= \frac{1}{\prod_{j>k}\sqrt{c_j}} \int_{R_+}^{\oplus} \lambda^{n-k-1} d\lambda \int_{S^{n-k-1}}^{\oplus} d\Omega \mathcal{H}^{\oplus}\left(\{ \frac{\sqrt{\lambda} n_j(\omega)}{\sqrt{c_j}} \} \right) \end{aligned} \tag{O.-1}$$

$$\mathcal{H}_{phys} = L_2(R^k) \tag{O.-1}$$

$$\{\tilde{\Omega}_{sl} := \exp(-\frac{1}{2}\sum_{j=1}^{m} p_j^2)\psi_{s,l} | s \in \mathbf{N}, l \in l_s\} \tag{O.-1}$$

A more direct approach:

$$\frac{1}{4}\nabla^2 = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{O.-1}$$

General solution is

$$g(z, \overline{z}) = f(z) + \tilde{f}(\overline{z}) \tag{O.-1}$$

Maxwell's theory in Minkowskian spacetime

canonical pair of fields (A_a, E^a)
sympletic structure $\{E^a(x), A_b(y)\} = e^2\delta^a_b\delta(x,y)$
fall off conditions $A \sim r^{-1}$, $E \sim r^{-2}$

$$G(\Lambda) = \int_{R^3} d^3x \Lambda(x)\partial_\mu E^\mu(x) \tag{O.-1}$$

The master constraint:

$$\mathbf{M} := \frac{1}{2} < \partial \cdot E, C \cdot \partial \cdot E >_{\mathcal{H}} \tag{O.-1}$$

where C is a positive definite operator on $\mathcal{H}$. $\mathcal{H} = L_2(R^3, d^3x)$

introduce

$$z^a = \frac{1}{\sqrt{2\hbar e^2}}[\sqrt{-\Delta}A_a - i\sqrt{\Delta}^{-1}E^a] \text{ and } \bar{z}^a \tag{O.-1}$$

$$\mathbf{M} = \frac{\hbar e^2}{4}\sum_{J,K} Q_{JK}(\bar{z}^{(3)}_J - z^{(3)}_J)(\bar{z}^{(3)}_K - z^{(3)}_K) \tag{O.-1}$$

$$Q_{JK} :=< b_J, \sqrt{-\Delta}^{3/2} C \sqrt{-\Delta}^{3/2} b_K >_{\mathcal{H}} \tag{O.-1}$$

-choose Q such that $\mathbf{M}$ convergent.

$$\hat{\mathbf{M}} = \sum_{J,K} Q_{JK}[(\hat{z}^{(3)}_J)^\dagger - \hat{z}^{(3)}_J][\hat{z}^{(3)}_K - (\hat{z}^{(3)}_K)^\dagger] \tag{O.-1}$$

Master Constraint

$$\mathbf{M} := \int_\sigma d^3x \frac{G_J G_K \delta^{JK}}{\sqrt{\det(q)}} \tag{O.-1}$$

O.5 Functional Analytic Issues of the Master Constraint

semibounded form

the domain of the form

it needs to be densely defined

Removal of regulator we end up with a quadratic for $Q_{\mathbf{M}}$ on H_{Diff} which is by inspection is *positive*, hence semibounded

If we can prove that the form is closable, then there is a unique positive, self-adjoint operator $\hat{\mathbf{M}}$ such that

$$Q_{\mathbf{M}}(\Psi_{Diff}, \Psi'_{Diff}) =< \Psi_{Diff}|\hat{\mathbf{M}}|\Psi'_{Diff} >_{Diff} \tag{O.-1}$$

O.5.1 Closure

Definition. An operator is *closed* whenever a sequence of vectors φ_n in the domain of A converges to a limit vector ψ and a sequence of vectors $A\varphi_n$ converges to a limit vector ϕ, then φ is in the domain of A and $A\varphi = \phi$.

Consider an infinite linear combination $\varphi = \sum_{k=1}^{\infty} x_k \phi_k$ where all the vectors are in the domain of A. Then each partial sum $\varphi_n = \sum_{k=1}^{n} x_k \phi_k$ is in the domain of A and

$$A\varphi_n = \sum_{k=1}^{n} x_k A\phi_k \tag{O.-1}$$

The sequence of vectors φ_n converges to a limit vector φ. Suppose the sequence of vectors $A\varphi_n$ converges to a limit vector

$$\phi = \sum_{k=1}^{n} x_k A\phi_k. \tag{O.-1}$$

If A is closed, then $\sum_{k=1}^{n} x_k \phi_k$ is in the domain of A and

$$A \sum_{k=1}^{\infty} x_k \phi_k = \sum_{k=1}^{\infty} x_k A\phi_k \tag{O.-1}$$

Quadratic form $Q_M\{\varphi, \phi\}$ is closed whenever a sequence of vectors ψ_n in the domain of $Q_M\{,\}$ converges to a limit vector ψ and a sequence of numbers $Q_M\{\varphi, \psi_n\}$ converges to a limit $Q_M\{\varphi, \phi\}$, then ϕ is in the domain of $Q_M\{,\}$ and $\phi = \psi$

$Q_M\{\psi, \varphi_n\}$

$$Q_M\{\psi, \varphi_n\} \tag{O.-1}$$

$$0 = (\varphi, \hat{M}\psi_n) - (\varphi, \hat{M}'\psi) = (\varphi, (\hat{M} - \hat{M}')\psi_n) \tag{O.-1}$$

O.6 The Associated Master Constraint Operator

is manifestly positive and sesquilinear. It remains to show that it is closable.

$- - --$

partly taken from Ouhabaz: Analysis of heat equations on domains

$u \in D(a)$ is in the domain $D(M)$ of M, if and only if there exists $v \in \mathcal{H}$ such that $a(u, \phi) = (v, \phi)$ for all $\phi \in D(a)$. Then we define the operator by

$$Mu := v. \tag{O.-1}$$

$D(M)$ is the set of vectors $u \in D(a)$ for which the mapping $\phi \mapsto a(u, v)$ is continuous on $D(a)$ with respect to the norm of $\mathcal{H}$.

$- - --1.$

O.6.1 Master Constraint from Quadratic form

$Q_M\{\psi, \varphi_n\}$

$$Q_M\{\psi, \varphi_n\} \tag{O.-1}$$

$$0 = (\varphi, \hat{M}\psi_n) - (\varphi, \hat{M}'\psi) = (\varphi, (\hat{M} - \hat{M}')\psi_n) \tag{O.-1}$$

as the quadratic form is real the operator $\hat{M}$ in $(\psi, \hat{M}\psi = Q_M\{\psi, \varphi\})$ will be self-adjoint (see section N.4.3). As it is self-adjoint it is automatically dense and hence is uniquely defined $\hat{\mathbf{M}}$. Suppose $\hat{\mathbf{M}}$ and $\hat{\mathbf{M}}'$ are operators such that

$$(\phi, \hat{\mathbf{M}}\chi) = (\phi, \hat{\mathbf{M}}\chi) \tag{O.-1}$$

for ever vector ϕ in the domain of $\hat{\mathbf{M}}$ (or $\hat{\mathbf{M}}'$). Then, because the domain of $\hat{\mathbf{M}}$ is dense, there is a sequence of vectors ϕ_n such that $\phi_n \to \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi$ and

$$(\phi_n, \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi) = 0 \tag{O.-1}$$

Therefore $\hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi = 0$, because

$$(\phi_n, \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi) \to (\hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi, \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi). \tag{O.-1}$$

Moreover $\hat{\mathbf{M}}$ is closed since $\hat{\mathbf{M}}^\dagger$ must be (see section).

In summary the operator would be densely defined and closed on H_{Diff}, so we really have pushed the constraint analysis one level up from H_{Kin}.

O.6.2 General Considerations

We assume that a judicious choice of ν, K has resulted in a positive, self-adjoint operaotr $\hat{\mathbf{M}}$ on some kinematic Hilbert space $\mathcal{H}_{Kin}$ which is assumed to be separable.

If zerois not in the spectrum of $\hat{\mathbf{M}}$ then compute the finite, positive number

$$\lambda_0 := \inf(\sigma(\hat{\mathbf{M}}))$$

and redifine $\hat{\mathbf{M}}$ by

$$\hat{\mathbf{M}} \to \hat{\mathbf{M}} - \lambda_0 \mathbf{1}_{\mathcal{H}_{Kin}}.$$

Here we assume that λ_0 vanishes in the limit $\hbar \to 0$ limit so that the modified operator still qualifies as a quantization of $\hat{\mathbf{M}}$. As we will see, this is justified in examples considered so far where λ_0 is usually related to some reordering of the operator. We will be assuming without loss of generality that $0 \in \sigma(\hat{\mathbf{M}})$.

Under these circumstances we can completely solve the Qunatum Master constraint equation

$$\hat{\mathbf{M}} = 0$$

and explicitly provide the phsical Hilbert space and its physical inner product.

O.6.3 Semi-Groups

Equations of the form

$$\frac{d}{dt}u(t) = Au(t), \quad t \geq 0, \qquad \text{with} \quad u(0) = f, \tag{O.-1}$$

where u is a function of a (time) variable $t \geq 0$, with values in a "state space" X, and A an operator on X, are a mathematical modeling of time dependent dynamical systems.

given a Banach space X

is strongly continuous if

$$\lim_{t \to 0^+} T(t)u = u. \tag{O.-1}$$

theory of strongly continuous semigroups is used in the study of existence and uniqueness of solutions to the evolution equations:

where

if B is the generator of the strongly continuoious semigroup $(T(t))_{t \geq 0}$, then for every $f \in D(B)$, the Cauchy problem has a unique solution, given by $u(t) = T(t)f$.

Definitions Let X be a Banach space.

Definition A one-parameter family $(S(t))_{t \geq 0}$ of bounded linear operators from X into X is called a semigroup of bounded linear operators on X if

$S(0) = I$,

$S(t+s) = S(t)S(s)$, for all $t, s \geq 0$.

The linear operator $A : D(A) \to X$, defined on the domain

$$D(A) = \left\{ y \in X : \lim_{t \to 0_+} \frac{S(t)y - y}{t} \text{ exists} \right\}$$

by

$$Ay = \lim_{t \to 0_+} \frac{S(t)y - y}{t}$$

for $y \in D(A)$, is called the infinitesimal generator of the semigroup $S(t)$.

Definition A semigroup S(t) of bounded linear operators is said

i) uniformly continuous if

$$\lim_{t \to 0_+} \|S(t) - I\| = 0;$$

ii) strongly continuous (or C_0 semigroup) if

$$\lim_{t \to 0_+} S(t)y = y,$$

for every $y \in X$.

Theorem O.6.1 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is bounded.*

For $\lambda \in \rho(A)$, let

$$R(\lambda, A) = (\lambda - IA)^{-1}$$

denote the resolvent of A....

Ergodic Semigroups

Let T be a bounded C_0−semigroup on a Banach space X. Denote by A the generator of T and by A^* the adjoint of A. We say that T is ergodic if

$$Px = \lim_{t \to \infty} \frac{1}{t} \int_0^t T(s)xds$$

exists for all $x \in X$.

O.7 Integral Decomposition and the Master Constraint.

$$(\phi, \hat{\mathbf{M}}\psi) = \int_{-\infty}^{\infty} dx\, d(\phi, E_x \psi) \tag{O.-1}$$

$$\hat{\mathbf{M}}\psi = \int_0^\infty d\lambda\lambda\, d(E_\lambda\psi) \tag{O.-1}$$

$$\mathcal{H}_{\mathbf{Diff}} = \int_0^\infty d\lambda\lambda\, d(E_\lambda\mathcal{H}_{Diff}) \tag{O.-1}$$

$$\mathcal{H}_{\mathbf{Diff}} = \int_0^\infty d\lambda\mathcal{H}_{Diff}(\lambda) \tag{O.-1}$$

$$\mathcal{H}_{\mathbf{Diff}} = \int_R^\oplus d\mu(\lambda)\mathcal{H}_{Diff}(\lambda) \tag{O.-1}$$

$$\begin{aligned} < \eta(f), \eta(f') >_{Phys} &= \int_R \frac{dt}{2\pi} < \hat{U}(t)f', f > \\ &= \int_R \frac{dt}{2\pi} \int_R d\mu(\lambda) < e^{i\lambda t} f'(\lambda), f(\lambda) >_{\mathcal{H}^\oplus_{Kin(\lambda)}} \\ &= \int_R \frac{dt}{2\pi} \int_R d\mu(\lambda) < \delta_R(\lambda) < f'(\lambda), f(\lambda) >_{\mathcal{H}^\oplus_{Kin(\lambda)}} \\ &= \mu(\delta) < f'(0), f(0) >_{\mathcal{H}^\oplus_{Kin(\lambda)}} \end{aligned} \tag{O.-3}$$

O.7.1 Dirac observables

Let $\hat{O}_{Diff}$ be a bounded self-adjoint operator on $\mathcal{H}_{Diff}$. Since the Master constraint operator is self-adjoint, we may construct the strongly continuous oneparameter family of unitarities $\hat{U}(t) := \exp(it\hat{\mathbf{M}})$. Then, if the uniform limit exists, the operator

$$[\hat{O}_{Diff}] := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\hat{U}(t)\hat{O}_{Diff}\hat{U}(t)^{-1} \tag{O.-3}$$

Find out for which diffeomorphism invariant, bounded self-adjoint operators $\hat{O}_{Diff}$ the corresponding ergodic mean $[\hat{O}_{Diff}]$ converges (in the topology induced by the topology of $\mathcal{H}_{Diff}$).

Then compute the induced operator $\hat{O}_{Phys}$ on $\mathcal{H}_{Phy}$ which is automatically self-adjoint.

answer if the integral diverges. use L'hopital's rule

$$[O] := \lim_{T\to\infty} [\alpha^{\mathbf{M}}_T(O) + \alpha^{\mathbf{M}}_{-T}(O)] \tag{O.-3}$$

Details:

uniform operator topology is:

Consider linear bounded operators from a Hilbert space $\mathcal{H}$ to itself, denoted as $\mathcal{L}(\mathcal{H},\mathcal{H})$. The norm of an operator is defined as

$$\|T\| := \sup_{\|x\|=1} \|Tx\| \tag{O.-3}$$

where x is an element of the Hilbert space, i.e. $x \in \mathcal{H}$ (the domain of T $\hat{T}$ which is the set of elements for which Tx exists, $\mathcal{D}(T)$). The induced topology on $\mathcal{H}$ is called the **uniform operator topology**.

Uniform convergence defined as

$$\|T_n - T\| \to 0, \text{ as } n \to 0. \tag{O.-3}$$

the norm $\|.\|_{Diff}$ provides a topology on the Hilbert space $\mathcal{H}_{Diff}$ by open sets

$$U_\epsilon(0) = \{\hat{O} \mid \|\hat{O}\|_{Diff} < \epsilon\} \tag{O.-3}$$

where $\|.\|_{Diff}$ is the norm induced by the scalar product on $< .,. >_{Diff}$.

This topology induces a topology on

uniform operator convergence is when in Banach spaces

$$\|\hat{O}_n - \hat{O}\| \to 0 \tag{O.-3}$$

(ii) It is automatically selfadjoint??

Proof:

O.7.2 Direct Integral Decomposition and Rigging Maps

Let Φ^*_{Kin} be the algebraic dual equipped with the topology of poinwise convergence (this is the space of all functionals on Φ_{kin}, not just the bounded functionals, and so no continuity assumptions are made and the definition does not involve a norm).

generalized eigenvector $l \in \Phi^*_{Kin}$ with eigenvalue λ with respect to the closable operator $\hat{\mathbf{M}}$ which together with its adjoin is is densely defined on the (invariant) domain Φ_{Kin} provided that

$$\hat{\mathbf{M}}' l = \lambda l \Longleftrightarrow l(\hat{\mathbf{M}} f) = \lambda l(f) \text{ for all} f \in \Phi_{kin} \tag{O.-3}$$

Here $\hat{\mathbf{M}}'$ is called the dual representation on Φ^*_{Kin}. The subspace of generalized eigenvectors with eigenvalue λ is denoted by $\Phi^*_{Kin}(\lambda) \subset \Phi^*_{Kin}$ and $(\Phi^*_{Kin})_{Phys} := \Phi^*_{Kin}(0)$ is the physical subspace.

A countably Hilbert space is always metrizable, i.e., we can always define a metric on it that yields the original topology. In terms of the norms, the meric is given by

$$d(\phi, \phi') := \sum_{n=1}^{\infty} 2^{-n} \frac{\|\phi - \phi'\|_n}{1 + \|\phi - \phi'\|_n} \tag{O.-3}$$

(Exercise show that (O.7.2) satisfies the conditions for a metric). Thus one can apply all the results for the well studied meric spaces to the countably Hilbert spaces.

It is easy to verify that $\Phi = \cap_{n=1}^{\infty} \Phi_n$ and the inclusion $\Phi_{n+1} \subset \Phi_n$ holds, (Exercise??).

Let Φ' be the topological dual of Φ (continuousl linear functionals) and Φ'_n the topological dual of Φ_n.

By Riez lemma Φ'_n is isometric isomorphic with Φ_n

and

$$\|F\|_{-n} := \sup_{0 \neq \phi \in \Phi_n} \frac{|F(\phi)|}{\|\phi\|_n} = \|\phi_F^{(n)}\|_n \tag{O.-3}$$

A rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi'$ is given by a Nuclear space Φ and a Hilbert space $\mathcal{H}$ which is the cauchy completion of Φ in yet another scalar product $< .,. > := < .,. >_0$

Given a *positive* self-adjoint operator $\hat{\mathbf{M}}$ on a Hilbert space $\mathcal{H}_{Kin}$, a corresponding Rigged Hilbert space as follows:

Let $\mathcal{D}$ be a dense, invariant domain for $\hat{\mathbf{M}}$, generically some space of smooth functions of compact support. Define positive sesquilinear forms $< \dot{,} >_n$ on $\mathcal{D}$ defined by

$$< \phi, \phi' >_n := \sum_{k=0}^{n} < \phi, (\hat{\mathbf{M}})^k \phi' > \tag{O.-3}$$

Theorem

A self-sdjoint operator $\hat{\mathbf{M}}$ on a separable Rigged Hilbert space $\Phi_{Kin} \subset \mathcal{H}_{Kin} \subset \Phi'_{Kin}$ has a complete set of generalized eigenvectors corresponding to real eigenvalues. More precisely:

Let

$$\mathcal{H}_{Kin} = \int_R^{\oplus} d\mu(\lambda)\mathcal{H}^{\oplus}_{Kin}(\lambda) \qquad \text{(O.-3)}$$

be the direct integral representation of $\mathcal{H}_{Kin}$. There is an integer n such that for $\mu - a.a.\ \lambda \in R$ there is a trace operator $T_\lambda : \Phi_n \to \mathcal{H}^{\oplus}_{Kin}(\lambda)$ which restricts to Φ_{Kin}

O.8 Toy Model III Free Field Theory: Maxwell Theory

O.8.1 Free Field Theories

O.8.2 Recap of Standard Canonical Quantization of Maxwell Theory

The canonical formulation of Maxwell theory on $\mathbb{R}^4$ consists of an infinite-dimensional phase space $\mathcal{M}$ with canonically conjugate coordinates (A_a, E^a) and symplectic structure

$$\{A_a(x), A_b(y)\} = \{E_a(x), E_b(y)\} = 0, \quad \{E_a(x), A_b(y)\} = e^2\delta^a_b\delta(x,y) \qquad \text{(O.-3)}$$

where e is the electric charge.

Maxwell's theory in Minkowskian spacetime

canonical pair of fields (A_a, E^a)
sympletic structure $\{E^a(x), A_b(y)\} = e^2\delta^a_b\delta(x,y)$
fall off conditions $A \sim r^{-1}$, $E \sim r^{-2}$

$$G(\Lambda) = \int_{R^3} d^3x\Lambda(x)\partial_\mu E^\mu(x) \qquad \text{(O.-3)}$$

In order to quantize the theory we want to introduce canonically conjugate coordinates (like x and p_x in non-relativistic quantum mechanics) for each degree of freedom and subject to these commutation relations.

We simplify the problem by considering radiation inside a large cubic box, of side L and volume $V = L^3$, and imposing periodic boundary conditions on the vector potential $\mathbf{A}$. The vector potential can then be represented as a Fourier series. The Fourier analysis corresponds to finding the normal modes of the radiation field.

One quantizes the radiation field by taking over the quantization of the harmonic oscillator from non-relativistic quantum mechanics.

With the boundary conditions

$$\mathbf{A}(0, y, z, t) = \mathbf{A}(L, y, z, t)$$

etc, the functions

$$\frac{1}{\sqrt{V}} \epsilon_r(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

form a complete set of transverse orthonormal vector fields. Here the wave vectors $\mathbf{k}$ must be of the form

$$\mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \qquad n_1, n_2, n_3 = 0, \pm 1, \ldots,$$

so the fields satisfy the periodicity conditions. $\epsilon_1(\mathbf{k})$ and $\epsilon_2(\mathbf{k})$ are two mutually perpendicular real unit vectors which are also orthogonal to $\mathbf{k}$:

$$\epsilon_r(\mathbf{k}) \cdot \epsilon_s(\mathbf{k}) = \delta_{rs}, \qquad \epsilon_r(\mathbf{k}) \cdot \mathbf{k} = 0, \quad r, s = 1, 2.$$

The last of these conditions ensures that the fields are transverse, satisfying the Coulomb gauge condition.

O.8.3 Master constraint

The master constraint:

$$\mathbf{M} := \frac{1}{2}(\partial \cdot E, K \cdot \partial \cdot E)_{\mathcal{H}} \tag{O.-3}$$

where K is a positive definite operator on $\mathcal{H}$. $\mathcal{H} = L_2(R^3, d^3x)$

Obviously $\mathbf{M} = 0$ if and only if $\partial \cdot E = 0$ a.e., that is, if and only if $G(\Lambda) = 0$ for all test functions of rapid decrease.

Recall that the Maxwell-Hamilonian is given by

$$H = \frac{1}{2e^2}\int d^3x\delta_{ab}(E^aE^b + B^aB^b) \simeq \hbar\int d^3x\delta_{ab}\overline{z^a}P^{\perp}_{ab}z^b \tag{O.-3}$$

where

$$z^a = \frac{1}{\sqrt{2\hbar e^2}}[\sqrt{-\Delta}A_a - i\sqrt{\Delta}^{-1}E^a] \text{ and } \bar{z}^a \tag{O.-3}$$

$$\mathbf{M} = \frac{\hbar e^2}{4}\sum_{J,K} Q_{JK}(\bar{z}_J^{(3)} - z_J^{(3)})(\bar{z}_K^{(3)} - z_K^{(3)}) \tag{O.-3}$$

$$Q_{JK} := (b_J, \sqrt{-\Delta}^{3/2}K\sqrt{-\Delta}^{3/2}b_K)_{\mathcal{H}} \tag{O.-3}$$

-choose Q such that $\mathbf{M}$ convergent.

$$\hat{\mathbf{M}} = \sum_{J,K} Q_{JK}[(\hat{z}_J^{(3)})^\dagger - \hat{z}_J^{(3)}][\hat{z}_K^{(3)} - (\hat{z}_K^{(3)})^\dagger] \tag{O.-3}$$

Master Constraint Quantization

$$\mathbf{M} := \int_\sigma d^3x\frac{G_JG_K\delta^{JK}}{\sqrt{\det(q)}} \tag{O.-3}$$

$$\begin{aligned} [\hat{z}_J^j, \hat{z}_K^k] = [(\hat{z}_J^j)^\dagger, (\hat{z}_K^k)^\dagger] &= 0 \\ [\hat{z}_J^j, (\hat{z}_K^k)^\dagger] &= \alpha\delta^{jk}\delta_{JK} \end{aligned} \tag{O.-3}$$

$$\hat{z}_J^j\Omega = 0. \tag{O.-3}$$

In terms of creation and annihilation operators, the Master constraint operator becomes

$$\hat{\mathbf{M}} = \frac{\alpha}{4}\sum_{J,K} Q^{JK}\left[\hat{z}_J^3 - (\hat{z}_J^3)^\dagger\right]^\dagger\left[\hat{z}_K^3 - (\hat{z}_K^3)^\dagger\right]. \tag{O.-3}$$

$$
\begin{aligned}
\hat{\mathbf{M}}(\hat{z}^3_J)^\dagger &= \frac{\alpha}{4}\sum_{J,K} Q^{JK}\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[\hat{z}^3_K - (\hat{z}^3_K)^\dagger\right](\hat{z}^3_L)^\dagger \\
&= \frac{\alpha}{4}\sum_{J,K} Q^{JK}(\alpha\delta_{KL} - \alpha\delta_{JL} + (\hat{z}^3_L)^\dagger)\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[\hat{z}^3_K - (\hat{z}^3_K)^\dagger\right] \\
&= \frac{\alpha^2}{4}\sum_{J,K} Q^{JK}(\delta_{KL} - \delta_{JL})\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[\hat{z}^3_K - (\hat{z}^3_K)^\dagger\right] + (\hat{z}^3_L)^\dagger\hat{\mathbf{M}}.
\end{aligned}
$$

As

$$
\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[(\hat{z}^3_K)^\dagger - \hat{z}^3_K\right] = \left[(\hat{z}^3_K)^\dagger - \hat{z}^3_K\right]\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]
$$

and Q_{IJ} is symmetric

$$
\hat{\mathbf{M}}(\hat{z}^3_J)^\dagger = (\hat{z}^3_J)^\dagger\hat{\mathbf{M}}.
$$

Therefore in general we have

$$
\hat{\mathbf{M}}(\hat{z}^j_J)^\dagger = (\hat{z}^j_J)^\dagger\hat{\mathbf{M}}. \tag{O.-6}
$$

The action of the Master constraint can be extended by linearity to the (dense) set of finite linear combinations of finite excitations of the vacuum Ω, thus it is densely defined operator on the Fock space, provided, that is, if $\hat{\mathbf{M}}\Omega$ has finite norm.

Theorem O.8.1 *The Master constraint operator is densely defined if and only if Q is a trace class operator.*

Proof: The Master constraint operator is densely defined if and only if $\hat{\mathbf{M}}\Omega$ has finite norm.

$$
\begin{aligned}
\|\hat{\mathbf{M}}\Omega\|^2 &= (\frac{\alpha}{2})^4\sum_{J,K,M,N} Q_{JK}Q_{MN}\left(\left[\hat{z}^3_J - (\hat{z}^3_J)^\dagger\right](\hat{z}^3_K)^\dagger\Omega, \left[\hat{z}^3_M - (\hat{z}^3_M)^\dagger\right](\hat{z}^3_N)^\dagger\Omega\right) \\
&= (\frac{\alpha}{2})^4\sum_{J,K,M,N} Q_{JK}Q_{MN}\left(\left[\alpha\delta_{JK} - (\hat{z}^3_J)^\dagger(\hat{z}^3_K)^\dagger\right]\Omega, \left[\alpha\delta_{MN} - (\hat{z}^3_M)^\dagger(\hat{z}^3_N)^\dagger\right]\Omega\right) \\
&= (\frac{\alpha}{2})^4\sum_{J,K,M,N} Q_{JK}Q_{MN}\left[\alpha^2\delta_{JK}\delta_{MN} + (\Omega, \hat{z}^3_K\hat{z}^3_J(\hat{z}^3_M)^\dagger(\hat{z}^3_N)^\dagger\Omega)\right] \\
&= (\frac{\alpha}{2})^4\left[2\mathrm{Tr}(Q^2) + (\mathrm{Tr}(Q))^2\right].
\end{aligned} \tag{O.-8}
$$

The first term in the last line is finite if Q is a Hilbert-Schmidt operator, the second if Q is trace-class. Since every trace class operator is Hilbert-Schmidt, it is necessary and sufficient that Q be trace class.

$\square$

O.8.4 Physical Hilbert Space

The Master Constraint operator acts as an identity on the Hilbert space of transversal modes so that we need only to consider the action of the Master Constraint operatoron the longitudinal Hilbert space.

O.9 Master Constraint for Gravity

The following theorem [215]

Theorem O.9.1 *The quadratic form $Q_{\mathbf{M}}(\ ,\ )$ is a closed quadratic form on $\mathcal{H}_{Diff}$. Hence there exists a unique densely deifned, positive self-adjoint operator $\hat{\mathbf{M}}$ on $\mathcal{H}_{Diff}$, leaving $\mathcal{H}_{Diff}$ invariant, such that:*

$$Q_{\mathbf{M}}(\Psi_{Diff}, \Psi'_{Diff}) = < \Psi_{Diff}|\hat{\mathbf{M}}|\Psi'_{Diff} >_{Diff} . \tag{O.-8}$$

Proof:

$$\hat{\mathcal{H}}^{\epsilon}_{C}\ f_{\gamma} = \sum_{v\in V(\gamma)} \frac{\chi_C(v)}{C^3_{n(v)}} \sum_{v(\Delta)=v} h^{\epsilon,\Delta}_{v}\ f_{\gamma} \tag{O.-8}$$

where $\chi_C(v)$ is the characteristic function of the cell C. The Master constraint operator, $\hat{\mathbf{M}}$

$$\boxed{\hat{\mathbf{M}} := \lim_{\mathcal{P}\to\Sigma} \sum_{C\in\mathcal{P}} \frac{1}{2} \hat{\mathcal{H}}'^{\dagger}_{C} \hat{\mathcal{H}}'_{C}} \tag{O.-8}$$

where

$$\begin{aligned} (\hat{\mathcal{H}}'_{C}\Psi)[f_{\gamma}] &:= \lim_{\epsilon\to 0} \Psi[\hat{\mathcal{H}}^{\epsilon}_{C}\ f_{\gamma}] \equiv \lim_{\epsilon\to 0}\ (\Psi|\hat{\mathcal{H}}^{\epsilon}_{C}|f_{\gamma} > \\ (\hat{\mathcal{H}}'^{\dagger}_{C}\Psi)[f_{\gamma}] &:= \lim_{\epsilon\to 0} \Psi[\hat{\mathcal{H}}^{\epsilon\dagger}_{C}\ f_{\gamma}] \end{aligned} \tag{O.-8}$$

$$(\hat{\mathbf{M}}\Psi_{Diff})[f_\gamma] := \lim_{\mathcal{P}\to\sigma,\epsilon,\epsilon'\to\sigma} \Psi_{Diff}[\sum \hat{H}^{\epsilon}_{C}\hat{H}^{\epsilon'\dagger}_{C} f_\gamma] \tag{O.-8}$$

Theorem O.9.2 (Closability of quadratic form $Q_{\mathbf{M}}(\ ,\ )$ - Thiemann's proof).
i. The positive quadratic form $Q_{\mathbf{M}}$ is closable and includes a unique, positive self-adjoint operator $\hat{\mathbf{M}}$ on $\mathcal{H}_{Diff}$.
ii. Moreover, the point zero is contained in the point spectrum of $\hat{\mathbf{M}}$.

Thiemann's proof

Thus, the heuristic idea is to define the quadratic form on $\mathcal{D}^*_{Diff}$ by

$$Q_M(l,l') := \lim_{\tau\to\Sigma}\sum_{\Delta\in\tau} < l, [\hat{\hat{C}}'(\Delta)]^*[\hat{\hat{C}}'(\Delta)]l' >_{Diff} \tag{O.-8}$$

where the prime denotes the operator dual as usual and * denotes the adjoint on $\mathcal{H}_{Diff}$. Unfortunately (N.-19) is ill defined as it stands because the operators $\hat{\hat{C}}'(\Delta)$ do not preserve $\mathcal{H}^*_{Diff}$. The cure is to extend $< .,. >_{Diff}$ to an inner product $< .,. >_*$ on all of $\mathcal{D}^*$. The final result turns out to be insensitive to the details of the extension because in the limit $\tau\to\Sigma$ the Riemann sum becomes well defined on $\mathcal{D}^*_{Diff}$.

$$\hat{\mathbf{M}}\, T_{[s_2]} := \sum_{T_{[s_1]}} Q_{\mathbf{M}}(T_{[s_1]}, T_{[s_2]})\; T_{[s_1]} \tag{O.-8}$$

$$\|\hat{\mathbf{M}}\, T_{[s_2]}\|^2 = \sum_{T_{[s_1]}} |Q_{\mathbf{M}}(T_{[s_1]}, T_{[s_2]})|^2 < \infty \tag{O.-8}$$

Let us fix $[s_1],[s_2]$ and consider theterm corresponding to $[s]$. In order that it does nto vanish

$$\sum_{v\in V(\gamma(s_0[s]))} \overline{T_{[s_1]}(\hat{C}^\dagger_v T_{s_0([s])})} T_{[s_2]}(\hat{C}^\dagger_v T_{s_0([s])})$$

must be non-zero. Hence the spin network decomposition of $\hat{C}^\dagger_v T_{s_0([s])}$ must contain a term diffeomorphic to T_{s_1} and a term diffeomorphic to T_{s_2} for at least one vertex $v\in V\big(\gamma(s_0[s])\big)$.

The first term adds an arc in between any possible pair of edges with two possible orientations and changes the spin of the two corresponding adjacent segments by $\pm 1/2$. Therefore it adds two more vertices.

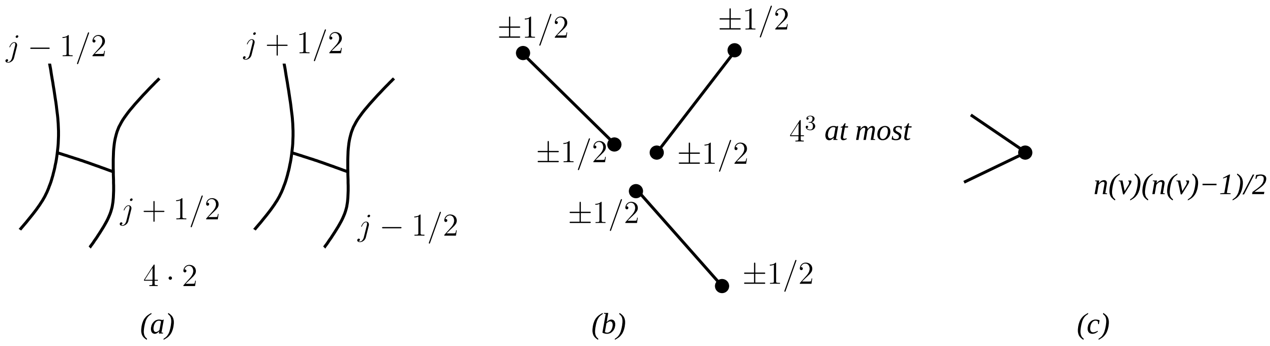


Figure O.1: MastatmostF. .

$$4 \cdot 2 \cdot 4^3 n(v)(n(v)-1)/2 = 4^4 n(v)(n(v)-1) \tag{O.-8}$$

The second term

$$4^8 n(v)^2 (n(v)-1)^2 \tag{O.-8}$$

O.10 Worked Exercises

O.10.1 Toy Models

Worked example:

(c) prove the spaces generated from the different $\tilde{\Omega}_{s,l}$ are mutually orthogonal.

Worked example: Differentiating series.

O.10.2 Functional Analysis Issues

Worked example: Absolute converge and interchanging series summation and differentiation

$$|s_i| \leq M. \tag{O.-8}$$

Worked example: Verify $\mathbf{\Phi} = \cap_{n=1}^{\infty} \mathbf{\Phi_n}$ and $\mathbf{\Phi_{n+1}} \subset \mathbf{\Phi_n}$.

A sequence (φ_k) is a Cauchy sequence in a norm $\|.\|_n$ if given $\epsilon > 0$ there exists N such that $j, k \geq N$ imply $\|\varphi_j - \varphi_k\|_n < \epsilon$. $\mathbf{\Phi}_n$ is the Cauchy completetion of Φ with respect to the norm $\|.\|_n$.

$\Phi = \cap_{n=1}^{\infty} \Phi_n$ and the inclusion $\Phi_{n+1} \subset \Phi_n$

there exists some $C > 0$ such that

$$\|\varphi\|_1 \leq C\|\varphi\|_2, \quad \text{for all } \varphi \in \mathbf{\Phi}. \tag{O.-8}$$

Every sequence (φ_j) that is Cauchy with respect to $\|\varphi\|_2$ is a Cauchy sequence with respect to $\|\varphi\|_1$. For $\epsilon > 0$ there exists N such that

$$\|\varphi_j - \varphi_k\|_2 < \epsilon/C \tag{O.-8}$$

$$\|\varphi_j - \varphi_k\|_1 < \epsilon \tag{O.-8}$$

The converse is not in general true: there may be sequencies Cauchy with respect to $\mathbf{\Phi_1}$ but which are not Cauchy with respect to $\mathbf{\Phi_2}$. The completetion with respect to $\|.\|_1$ so that $\mathbf{\Phi_2} \subset \mathbf{\Phi_1}$.

$$\mathbf{\Phi_1} \supset \mathbf{\Phi_2} \supset \mathbf{\Phi}. \tag{O.-8}$$

Worked example:

Given that the norms $\|\varphi\|_n$, $n = 1, 2, \ldots$ have the following proerties

$$\|\varphi\|_n \geq 0, \quad \text{for all } \varphi \in \Phi \tag{O.-8}$$

verify that $d(.,.)$ has satisfies the conditions for being a metric, i.e., show that

(*i*) $d(\varphi, \phi) \leq d(\varphi, \phi) + d(\phi, \psi)$,

(*ii*) $d(\varphi, \psi) = d(\psi, \varphi)$, (*iii*) $d(\varphi, \psi) \geq 0$, $d(\varphi, \psi) = 0$ implies $\varphi = \psi$.

Solution:

(*i*) Triangle inequality

$$f(x) = \frac{x}{1+x} \tag{O.-8}$$

Differentiation gives

$$f'(x) = \frac{1}{(1+x)^2} \geq 0. \tag{O.-8}$$

showing that it is monotonically increasing. Given that

$$\|a+b\|_n \leq \|a\|_n + \|b\|_n \tag{O.-8}$$

it means that

$$f(\|a+b\|_n) \leq f(\|a\|_n + \|b\|_n). \tag{O.-8}$$

$$\begin{aligned} \frac{\|a+b\|_n}{1+\|a+b\|_n} &\leq \frac{\|a\|_n + \|b\|_n}{1+\|a\|_n + \|b\|_n} \\ &= \frac{\|a\|_n}{1+\|a\|_n + \|b\|_n} + \frac{\|b\|_n}{1+\|a\|_n + \|b\|_n} \\ &\leq \frac{\|a\|_n}{1+\|a\|_n} + \frac{\|b\|_n}{1+\|b\|_n} \end{aligned} \tag{O.-9}$$

Making the substitution $a = \varphi - \phi$, $b = \phi - \psi$

$$\frac{\|\varphi-\psi\|_n}{1+\|\varphi-\psi\|_n} \leq \frac{\|\varphi-\phi\|_n}{1+\|\varphi-\phi\|_n} + \frac{\|\phi-\psi\|_n}{1+\|\phi-\psi\|_n} \tag{O.-9}$$

Multiplying both sides of this inequality by 2^{-n} and summing over n from zero to infinity, we have the inequality

$$d(\varphi, \psi) \leq d(\varphi, \phi) + d(\phi, \psi). \tag{O.-9}$$

(*ii*) $d(\varphi, \psi) = d(\psi, \varphi)$ easily follows from

$$\|\varphi - \psi\|_n = \|\psi - \varphi\|_n. \tag{O.-9}$$

(*iii*) $d(\varphi, \psi) \geq 0$: Since $\|\varphi - \psi\|_n \geq 0$ each term in the summation is ≥ 0.

$d(\varphi, \psi) = 0$

$$\sum_{n=1}^{\infty} 2^{-n} \frac{\|\varphi - \psi\|_n}{1 + \|\varphi - \psi\|_n} = 0, \tag{O.-9}$$

since $\|\varphi - \psi\|_n \geq 0$ for all $\varphi, \psi \in \Phi_n$????, this implies

$$\frac{\|\varphi - \psi\|_n}{1 + \|\varphi - \psi\|_n} = 0 \quad \Rightarrow \quad \|\varphi - \psi\|_n = 0. \tag{O.-9}$$

$$\|\varphi - \psi\|_n = 0, \quad \text{for all } n = 1, 2, \ldots \tag{O.-9}$$

This leads us to conclude $\varphi = \psi$.????

Worked example: Verify tha the collection of norms defined by the positive operator $\hat{\mathbf{M}}$ satisfy $\|.\|_n \leq \|.\|_{n+1}$.

positive sesquilinear forms $< ., . >_n$

or iteratively

$$< \phi, \phi' >_{n+1} := < \phi, \hat{\mathbf{M}} \phi' > + < \phi, \phi' >_n \tag{O.-9}$$

$$\|\phi\|_{n+1}^2 := < \phi, \hat{\mathbf{M}} \phi > + \|\phi\|_n^2 \tag{O.-9}$$

Since $\hat{\mathbf{M}}$ is positive $< \phi, \hat{\mathbf{M}} \phi > \geq 0$, $\|\phi\|_{n+1}^2 \geq \|\phi\|_n^2$ and so

$$\|\phi\|_{n+1} \geq \|\phi\|_n. \tag{O.-9}$$

Worked example:

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, C_k\} = f_{Jk}{}^l C_l, \quad \{C_j, C_k\} = f_{jk}{}^L C_L \tag{O.-9}$$

Master constraint

$$M := \frac{1}{2}\sum_{j,k} Q_{jk} C_j C_k \tag{O.-9}$$

We define new constraints

$$\tilde{C}_k := \{M, T_k\} \approx \sum_{k,l} Q_{kl} C_k A_{lj} \tag{O.-9}$$

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, \tilde{C}_k\} = 0, \quad \{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l \tag{O.-9}$$

Establish

(a) $\{C_J, \tilde{C}_k\} = 0$
(b) $\{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l$

Proof:

(a)

$$\begin{aligned}
\{C_J, M\} &= \{C_J, Q_{jk} C_j C_k\} \\
&= \{C_J, Q_{jk}\} C_j C_k + \{C_J, C_j\} Q_{jk} C_k + \{C_J, C_k\} Q_{jk} C_l \\
&= \{C_J, Q_{jk}\} C_j C_k + 2\{C_J, C_k\} Q_{jk} C_j \qquad \text{as } Q_{jk} = Q_{kj}.
\end{aligned} \tag{O.-10}$$

The condition that the Master constraint M be diffeomorphism invariant, i.e. $\{C_L, M\} = 0$, requires

$$\begin{aligned}
\{C_J, Q_{jk}\} C_k &= -2\{C_J, C_j\} Q_{jk} \\
&= -2 f_{Jj}{}^l C_l Q_{jk}
\end{aligned} \tag{O.-10}$$

As T_i is a strong Dirac observable with respect to the constraints C_J, i.e., $\{C_J, T_i\} = 0$. Using this and the Jacobi identity we find

$$\begin{aligned}
\{C_J, A_{lj}\} \equiv \{C_J, \{C_l, T_j\}\} &= -\{C_l, \{T_j, C_J\}\} - \{T_j, \{C_J, C_l\}\} \\
&= \{\{C_J, C_l\}, T_j\} \\
&= f_{Jl}{}^k \{C_k, T_j\} \qquad \text{remember } f_{Jl}{}^k \text{ are structure } \textbf{constants}, \\
&= f_{Jl}{}^k A_{kl}
\end{aligned} \tag{O.-12}$$

$$
\begin{aligned}
\{C_J, \tilde{C}_k\} &= \sum_{k,l} [\{C_J, Q_{jk}\} C_k A_{lj} + \{C_J, C_k\} Q_{kl} A_{lj} + \{C_J, A_{lj}\} Q_{jk} C_k] \\
&= \sum_{k,l} \left[-2 f_{Jj}{}^{l} C_l Q_{jk} A_{lj} + f_{Jk}{}^{l} C_l Q_{kl} A_{lj} + f_{Jl}{}^{k} A_{kl} Q_{jk} C_k \right] \\
&= 0. \qquad \text{(O.-13)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\{\tilde{C}_j, \tilde{C}_k\} &= \\
&= \tilde{f}_{jk}{}^{L} C_L + \tilde{f}_{jk}{}^{l} C_l \qquad \text{(O.-13)}
\end{aligned}
$$

Proof of the Mellin-Barnes integral.

$$
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n) n!} z^n \qquad \text{(O.-13)}
$$

Proof:

Definitions of the Gamma function $\Gamma(z)$

We first define the Gamma function $\Gamma(z)$ as the function of a complex variable by the integral

$$
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt
$$

with $\mathrm{Re} z > 0$. Using

$$
e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n
$$

we get

$$
\begin{aligned}
\Gamma(z) &= \lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt \\
&= \lim_{n \to \infty} n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau \qquad \text{(O.-13)}
\end{aligned}
$$

Integration by parts now gives

$$\begin{aligned}
\int_0^1 (1-\tau)^n \tau^{z-1} d\tau &= \left[\frac{1}{z}\tau^z (1-\tau)^n\right]_0^1 + \frac{n}{z}\int_0^1 (1-\tau)^{n-1}\tau^z d\tau \\
&= \frac{n(n-1)\ldots 2}{z(z+1)\ldots(z+n-1)}\int_0^1 \tau^{z+n} d\tau \\
&= \frac{n(n-1)\ldots 1}{z(z+1)\ldots(z+n)} \qquad \text{(O.-14)}
\end{aligned}$$

so that

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\ldots(z+n)} \\
&= \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^z \left(1+\frac{z}{n}\right)^{-1}\right] \qquad \text{(O.-14)}
\end{aligned}$$

We see that the poles of the Gamma function are at

We take this as our definition of $\Gamma(z)$.

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\ldots\left(1+\frac{z}{n}\right)\left(\frac{2}{1}\right)^{-z}\left(\frac{3}{2}\right)^{-z}\ldots\left(\frac{n}{n-1}\right)^{-z}\left(\frac{n+1}{n}\right)^{-z}\right] \\
&= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\ldots\left(1+\frac{z}{n}\right)e^{-z\ln n}\right] \\
&= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)e^{-z}\left(1+\frac{z}{2}\right)e^{-1/2z}\ldots\left(1+\frac{z}{n}\right)e^{-(1/n)z}\right. \\
&\qquad\left. e^{-(1/n)z}e^{[1+1/2+\cdots+(1/n)-\ln n]z}\right] \\
&= ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-(1/n)z}\right] \qquad \text{(O.-17)}
\end{aligned}$$

Using this we form the product

$$\frac{1}{\Gamma(z)}\frac{1}{\Gamma(-z)} = -z^2\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = -\frac{z}{\pi}\sin(\pi z)$$

Since by $\Gamma(z) = -(1/z)\Gamma(1+z)$ we have

$$\Gamma(-s)\Gamma(1+s) = -\pi \mathrm{cosec}(\pi s)$$

Take the logarithm of ()

$$\begin{aligned}
\frac{\Gamma'(a)}{\Gamma(a)} &= \frac{d}{da}[\ln\Gamma(a)] \\
&= \frac{d}{da}[-\ln a - \gamma a - \sum_{n=1}^{\infty}\ln\left(1+\frac{a}{n}\right) + \sum_{n=1}^{\infty}\frac{a}{n}] \\
&= -\gamma - \frac{1}{a} + \sum_{n=1}^{\infty}\frac{a}{n(a+n)}
\end{aligned}$$

The generalsed zeta function $\zeta(s,a)$

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(a+n)^s}$$

Since

$$(a+n)^{-s}\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-(n+a)x}dx$$

we have

$$\begin{aligned}
\Gamma(s)\zeta(s,a) &= \lim_{N\to\infty}\sum_{n=0}^{N}\int_0^{\infty} x^{s-1}e^{-(n+a)x}dx \\
&= \left\{\int_0^{\infty}\frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx - \int_0^{\infty}\frac{x^{s-1}}{1-e^{-x}}e^{-(N+1+a)x}dx\right\}
\end{aligned}$$

Now, when $x \geq 0$, $e^x \geq 1+x$ and then

$$\begin{aligned}
\int_0^{\infty}\frac{x^{\sigma-1}e^{-(N+1+a)x}}{1-e^{-x}}dx &= \int_0^{\infty}\frac{x^{\sigma-1}e^{-(N+a)x}}{e^x-1}dx \\
&\leq \int_0^{\infty} x^{\sigma-2}e^{-(N+1+a)x}dx \\
&= (N+a)^{1-\sigma}\Gamma(\sigma-1)
\end{aligned}$$

which (when $\sigma \geq 1+\delta$) tends to 0 as $N \to \infty$.

$$\zeta(s,a) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{\sigma-1}e^{-(N+1+a)x}}{1-e^{-x}}dx$$

Consider the contour integral

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz$$

where the contour $\mathcal{C}$ is as in fig (). On the real axis in the first part of this path we have $\arg(-z) = -\pi$, so that $(-z)^{s-1} = e^{-i\pi(s-1)}z^{s-1}$; and on the last part of the path $(-z)^{s-1} = e^{i\pi(s-1)}z^{s-1}$. On the circle write $-z = \delta e^{i\theta}$. We get

$$\begin{aligned}\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz &= \lim_{\rho\to\infty}\int_{\rho}^{\delta} \frac{e^{-i\pi(s-1)}z^{s-1}e^{-az}}{1-e^{-z}}dz + \lim_{\rho\to\infty}\int_{\delta}^{\rho} \frac{e^{i\pi(s-1)}z^{s-1}e^{-az}}{1-e^{-z}}dz + \\ &+ \int_{-\pi}^{\pi} \frac{(\delta e^{i\theta})^{s-1}e^{a\delta(\cos\theta+i\sin\theta)}}{1-e^{\delta(\cos\theta+i\sin\theta)}}\delta e^{i\theta}id\theta\end{aligned}$$

$$\int_{-\pi}^{\pi} \frac{(\delta e^{i\theta})^{s-1}e^{a\delta(\cos\theta+i\sin\theta)}}{1-e^{\delta(\cos\theta+i\sin\theta)}}\delta e^{i\theta}id\theta$$

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz = 2i\sin\pi(s-1)\int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx$$

Therefore

$$\zeta(s,a) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz$$

Take s to be zero or a negative integer $(= -m)$

$$\zeta(-m,a) = -\frac{\Gamma(1+m)}{2\pi i}\int_{\mathcal{C}} \frac{(-z)^{-m-1}e^{-az}}{1-e^{-z}}dz$$

By Cauchy's theorem, this is equal to the residue of the integrand at $z = 0$.

differentiate

$$-z\frac{e^{-az}-1}{e^{-z}-1} = \sum_{n=1}^{\infty} \frac{(-1)^n\phi_n(a)z^n}{n!}$$

a power series

$$\frac{z^2e^{-az}}{e^{-z}-1} = \sum_{n=1}^{\infty} \frac{(-1)^n\phi_n'(a)z^n}{n!}$$

so that

$$\frac{e^{-az}}{z^{m+1}(e^{-z}-1)} = \sum_{n=1}^{\infty} \frac{(-1)^n \phi'_n(a) z^{n-m-3}}{n!}$$

we want the coefficient of z^{-1}, that is we want the coefficient corresponding to the term $n = m+2$:

$$\frac{(-1)^{m+2}\phi'_{m+2}(a)}{(m+2)!}$$

Therefore

$$\zeta(-m,a) = \frac{-\phi'_{m+2}(a)}{(m+1)(m+2)} \tag{O.-28}$$

Writing $s = 0$ in , we see that

Hermite's formula for $\zeta(s,a)$

We need the formula by Plana: If x is an integer, and $\phi(z)$ is a function which is analytic and bounded for all values of z such that $x_1 \leq R(z) \leq x_2$, then

$$\sum_{n=x_1}^{x_2} \phi(n) = \frac{1}{2}\phi(x_1)+\frac{1}{2}\phi(N)+\int_{x_1}^{x_2} \phi(\xi)d\xi+\frac{1}{i}\int_0^{\infty} \frac{\phi(x_2+iy)-\phi(x_1+iy)-\phi(x_2-iy)+\phi(x_1-iy)}{e^{2\pi y}-1}dy \tag{O.-28}$$

integrate

$$\int \frac{\phi(z)dz}{e^{\pm 2\pi i z}-1}$$

round rectangles whose corners are $x_1, x_2, x_2 \pm i\infty, x_1 \pm i\infty$

Put

$$\phi(\zeta) = \frac{1}{(a+\zeta)^s}$$

$$
\begin{aligned}
\sum_{n=0}^{N}\frac{1}{(a+n)^s} &= \frac{1}{2}a^{-s}+\frac{1}{2}\frac{1}{(a+N)^s}+\int_0^N (a+y)^{-s}dy \\
&\quad +\frac{1}{i}\int_0^\infty \left[(a+N+iy)^{-s}-(a+iy)^{-s}-(a+N-iy)^{-s}+(a-iy)^{-s}\right]\frac{dy}{e^{2\pi y}-1} \\
&= \frac{1}{2}a^{-s}+\frac{1}{2}\frac{1}{(a+N)^s}+\int_0^N (a+y)^{-s}dy+2\int_0^\infty (q(N,y)-q(0,y))\frac{dy}{e^{2\pi y}-1}
\end{aligned}
\tag{O.-30}
$$

where the function $q(x,y)$ is defined by

$$
q(x,y)=\frac{1}{2i}[(a+x+iy)^{-s}-(a+x-iy)^{-s}]
$$

Is it legitimate to let $N\to\infty$ on the RHS? We will need the formula

$$
\sin\left\{s\ \tan^{-1}x\right\}=\frac{1}{2i}\frac{(1-ix)^s-(1+ix)^s}{(x^2+1)^{s/2}}.
$$

The proof is straightforward. Set $y=s\tan^{-1}x$ ($x=\tan(y/s)$), then

$$
\begin{aligned}
\sin\left\{s\ \tan^{-1}x\right\} &= \sin y \\
&= \frac{1}{2i}(e^{iy}-e^{-iy}) \\
&= \frac{1}{2i}\left[\left(\frac{e^{iy/s}}{e^{-iy/s}}\right)^{s/2}-\left(\frac{e^{-iy/s}}{e^{iy/s}}\right)^{s/2}\right] \\
&= \frac{1}{2i}\left[\left(\frac{\cos(y/s)+i\sin(y/s)}{\cos(y/s)-i\sin(y/s)}\right)^{s/2}-\left(\frac{\cos(y/s)-i\sin(y/s)}{\cos(y/s)+i\sin(y/s)}\right)^{s/2}\right] \\
&= \frac{1}{2i}\left[\left(\frac{1+ix}{1-ix}\right)^{s/2}-\left(\frac{1-ix}{1+ix}\right)^{s/2}\right] \\
&= \frac{1}{2i}\frac{(1+ix)^s-(1-ix)^s}{(x^2+1)^{s/2}}
\end{aligned}
$$

Using this we can rewrite $q(x,y)$ as

$$
\begin{aligned}
q(x,y) &= \frac{1}{2i}[(a+x+iy)^{-s}-(a+x-iy)^{-s}] \\
&= \frac{1}{2i}\frac{(a+x-iy)^{-s}-(a+x+iy)^{-s}}{[(a+x)^2+y^2]^s} \\
&= -[(a+x)^2+y^2]^{-s/2}\ \frac{1}{2i}\frac{(a+x+iy)^{-s}-(a+x-iy)^{-s}}{[(a+x)^2+y^2]^{s/2}} \\
&= -[(a+x)^2+y^2]^{-s/2}\ \sin\left\{s\ \tan^{-1}\frac{y}{x+a}\right\}
\end{aligned}
$$

$$
\int_0^\infty q(x,y)(e^{2\pi y}-1)^{-1}dy
$$

is convergent when $x \geq 0$ and tends to zero as $x \to \infty$

It is legitimate to let $N \to \infty$ and we have

$$
\zeta(s,a) = \frac{1}{2}a^{-s} + \int_0^\infty (a+x)^{-s}dx + 2\int_0^\infty (a^2+y^2)^{-\frac{1}{2}s}\left\{\sin\left(s\ \tan^{-1}\frac{y}{a}\right)\right\}\frac{dy}{e^{2\pi y}-1}
$$

We arrive at Hermite's formula

$$
\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty (a^2+y^2)^{-\frac{1}{2}s}\left\{\sin\left(s\ \tan^{-1}\frac{y}{a}\right)\right\}\frac{dy}{e^{2\pi y}-1} \tag{O.-40}
$$

Special values of $\zeta(s,a)$ and its derivative from Hermite's formula

Writing $s=0$ in Hermite's formula (O.10.2), we have

$$
\zeta(0,a) = \frac{1}{2} - a. \tag{O.-40}
$$

We will need the value of $\zeta'(0,a)$.

$$
\begin{aligned}
\left.\frac{d}{ds}\zeta(s,a)\right|_{s=0} &= \lim_{s\to 0}\left[-\frac{1}{2}a^{-s}\ln a - \frac{a^{-s}\ln a}{s-1} - \frac{a^{1-s}}{(s-1)^2}\right. \\
&\quad +2\int_0^\infty -\frac{1}{2}\ln(a^2+y^2)\ (a^2+y^2)^{-\frac{1}{2}s}\sin\left(s\ \tan^{-1}\frac{y}{a}\right) \\
&\quad \left. +(a^2+y^2)^{-\frac{1}{2}s}\tan^{-1}\frac{y}{a}\cos\left(s\ \tan^{-1}\frac{y}{a}\right)\frac{dy}{e^{2\pi y}-1}\right] \\
&= \left(a-\frac{1}{2}\right)\ln a - a + 2\int_0^\infty \frac{\tan^{-1}(y/a)}{e^{2\pi y}-1}dy
\end{aligned} \tag{O.-42}
$$

How to evaluate this integral?

$$\frac{d^2}{dz^2} \ln \Gamma(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

By ()

$$\sum_{n=0}^{N} = \frac{1}{2z^2} + \frac{1}{2} \frac{1}{(z+N)^2} + \int_0^N \frac{d\xi}{(z+\xi)^2} + 2 \int_0^\infty \frac{q(t, z+N) - q(t,z)}{e^{2\pi y} - 1}$$

where

$$q(t,z) = \frac{1}{2i} \left[\frac{1}{(z-it)^2} - \frac{1}{(z+it)^2} \right] = \frac{2tz}{(z^2+t^2)^2}$$

Hence

$$\frac{d^2}{dz^2} \ln \Gamma(z) = \frac{1}{z^2} + \frac{1}{z} + \int_0^\infty \frac{4yz}{(z^2+y^2)^2} \frac{dy}{e^{2\pi y} - 1}$$

Integrate from 1 to z

$$\frac{d}{dz} \ln \Gamma(z) = -\frac{1}{2z} + \ln z + C - \int_0^\infty \frac{ydy}{(z^2+y^2)(e^{2\pi y} - 1)}$$

where C is a constant. Integrating again,

$$\ln \Gamma(z) = \left(z - \frac{1}{2} \right) \ln z + (C-1)z + C' + 2 \int_0^\infty \frac{\tan^{-1}(y/z) dt}{(e^{2\pi y} - 1)}$$

where C' is a constant.

We see that $C = 0$ and $C' = \frac{1}{2} \ln(2\pi)$. Therefore

$$2 \int_0^\infty \frac{\tan^{-1}(y/z) dt}{(e^{2\pi y} - 1)} = \ln \Gamma(z) - \left(z - \frac{1}{2} \right) \ln z + z - \frac{1}{2} \ln(2\pi)$$

Substituting this into (O.-42) we finally obtain

$$\left. \frac{d}{ds} \zeta(s,a) \right|_{s=0} = \ln \Gamma(a) - \frac{1}{2} \ln(2\pi). \qquad \text{(O.-42)}$$

Now

$$\lim_{s\to 1}\left\{\zeta(s,a)-\frac{1}{s-1}\right\}=-\frac{\Gamma'(a)}{\Gamma(a)} \tag{O.-42}$$

Asymptotic expansion

$$\begin{aligned}
\frac{e^{-\gamma z}\Gamma(a)}{\Gamma(z+a)} &= e^{-\gamma z}\lim_{n\to\infty}\frac{n!n^a}{n!n^{z+a}}\frac{(z+a)(z+a+1)\dots(z+a+n)}{a(a+1)\dots(a+n)} \\
&= e^{-\gamma z}\lim_{n\to\infty} n^{-z}\left(1+\frac{z}{a}\right)\left(1+\frac{z}{a+1}\right)\left(1+\frac{z}{a+2}\right)\dots\left(1+\frac{z}{a+n}\right) \\
&= e^{-\gamma z}\left(1+\frac{z}{a}\right)\lim_{n\to\infty} e^{z(1+(1/2)+\dots+(1/n)-\ln n)}\left(1+\frac{z}{a+1}\right)e^{-z}\times \\
&\qquad\qquad\times\left(1+\frac{z}{a+2}\right)e^{-z/2}\dots\left(1+\frac{z}{a+n}\right)e^{-z/n} \\
&= \left(1+\frac{z}{a}\right)\prod_{n=1}^{\infty}\left(1+\frac{z}{a+n}\right)e^{-z/n}
\end{aligned}$$

the principal values of the logarithms

$$\begin{aligned}
&\ln\left(1+\frac{z}{a}\right)+\ln\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{a+n}\right)e^{-z/n}\right\} \\
=\;& \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{a^m}+\sum_{n=1}^{\infty}\left\{\ln\left(1+\frac{z}{a+n}\right)-\frac{z}{n}\right\} \\
=\;& \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{a^m}+\sum_{n=1}^{\infty}\left\{\left(\frac{-az}{n(a+n)}\right)+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{(a+n)^m}\right\}
\end{aligned}$$

Take the absolute value of the double sum

$$\begin{aligned}
\left|\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{(a+n)^m}\right| &= \sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{m}\frac{|z|^m}{(a+n)^m} \\
&= \sum_{n=1}^{\infty}\ln\left(1-\frac{|z|}{a+n}\right)-\frac{|z|}{a+n}
\end{aligned}$$

Consequently

$$
\begin{aligned}
\ln \frac{e^{-\gamma z}\Gamma(a)}{\Gamma(z+a)} &= \frac{z}{a} - \sum_{n=1}^{\infty} \frac{az}{n(a+n)} + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} z^m \left\{ \sum_{n=1}^{\infty} \frac{1}{(a+n)^m} + \frac{1}{a^m} \right\} \\
&= \frac{z}{a} - \sum_{n=1}^{\infty} \frac{az}{n(a+n)} + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} z^m \zeta(m,a)
\end{aligned}
$$

Now consider

$$
-\frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s,a)\, ds
$$

the contour of integration being that in fig (O.2) enclosing the points $s = 2, 3, 4, \ldots$ but not the points $1, 0, -1, -2, \ldots$. To find the residue we will need by L'hopital's rule

$$
\lim_{s \to m} \frac{s-m}{\sin(\pi s)} = \lim_{s \to m} \frac{\frac{d}{ds}(s-m)}{\frac{d}{ds}\sin(\pi s)} = \lim_{s \to m} \frac{1}{\pi \cos(\pi s)} = \frac{(-1)^m}{\pi}
$$

The residue of the integrand at $s = m (m \geq 2)$ is

$$
\frac{1}{2\pi i} \frac{(-1)^m}{m} z^m \zeta(m,a).
$$

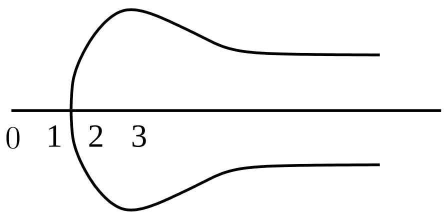


Figure O.2: .

Since as the real part of s tends to infinity,

$$
\begin{aligned}
\ln \frac{\Gamma(a)}{\Gamma(z+a)} &= -z[-\gamma - \frac{1}{a} + \sum_{n=1}^{\infty} \frac{a}{n(a+n)}] - \frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s,a)\, ds \\
&= -z \frac{\Gamma'(a)}{\Gamma(a)} - \frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s,a)\, ds
\end{aligned}
$$

$$
\begin{aligned}
|z^s| &= |z^{\sigma+it}| \\
&= |(|z|e^{iargz})^{\sigma+it}| \\
&= ||z|^{\sigma+it}|\,|e^{iargz(\sigma+it)}| \\
&= |z|^{\sigma} e^{-t\, argz}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\pi}{s\sin(\pi s)} &= \frac{\pi}{(\sigma+it)\sin(\pi(\sigma+it))} \\
&= \frac{2i\pi}{(\sigma+it)[e^{i(\pi(\sigma+it))}-e^{-i(\pi(\sigma+it))}]}
\end{aligned}
$$

$$
\int_{\mathcal{D}} \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds
$$

but see fig (O.3)

$$
\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} + \int_{\mathcal{D}} + \int_{\mathcal{C}} = 0
$$

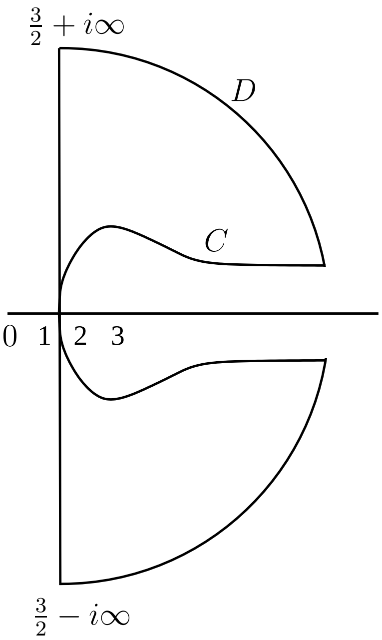


Figure O.3:

so

$$
\ln\frac{\Gamma(a)}{\Gamma(z+a)} = -z\frac{\Gamma'(a)}{\Gamma(a)} + \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds
$$

We wish to write the RHS in terms of the integral

$$
\frac{1}{2\pi i}\int_{-n-\frac{1}{2}-i\infty}^{-n-\frac{1}{2}+i\infty} \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds
$$

where n is a fixed integer. We have by Cauchy's theorem

$$\frac{1}{2\pi i}\left\{\int_{\frac{3}{2}-iR}^{\frac{3}{2}+iR}-\int_{-n-\frac{1}{2}+iR}^{\frac{3}{2}+iR}+\int_{-n-\frac{1}{2}-iR}^{\frac{3}{2}-iR}-\int_{-n-\frac{1}{2}-iR}^{-n-\frac{1}{2}+iR}\right\}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds=\sum_{m=-1}^{n}R_m$$

where R_m is the residue of the integrad at $s=-m$. So we must consider the integrals

$$\frac{1}{2\pi i}\int_{-n-\frac{1}{2}\pm iR}^{\frac{3}{2}\pm iR}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds.$$

Therefore

$$\ln\frac{\Gamma(a)}{\Gamma(z+a)}=-z\frac{\Gamma'(a)}{\Gamma(a)}+\frac{1}{2\pi i}\int_{-n-\frac{1}{2}-i\infty}^{-n-\frac{1}{2}+i\infty}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds+\sum_{m=-1}^{n}R_m$$

Consequently,

$$\ln\frac{\Gamma(a)}{\Gamma(z+a)}=-z\frac{\Gamma'(a)}{\Gamma(a)}+\sum_{m=-1}^{n}R_m+O(z^{-n-\frac{1}{2}})$$

when $|z|$ is large.

Now, when m is a positive integer,

$$R_m=\frac{(-1)^m z^{-m}\zeta(-m,a)}{-m}$$

and so by ()

$$R_m=\frac{(-1)^m z^{-m}\phi'_{m+2}(a)}{m(m+1)(m+2)}$$

R_0 is the residue at $s=0$

$$
\begin{aligned}
\frac{\pi}{\sin(\pi s)} &= \frac{2i}{e^{i\pi s} - e^{-i\pi s}} \\
&= \frac{2i}{\left((i\pi s) + \frac{(i\pi s)^2}{2!} + \frac{(i\pi s)^3}{3!}\right) - \left((-i\pi s) + \frac{(-i\pi s)^2}{2!} + \frac{(-i\pi s)^3}{3!}\right)} \\
&= \frac{1}{\pi s}\frac{1}{1 - \frac{(\pi s)^2}{6} + \ldots} \\
&= \frac{1}{\pi s}(1 + \frac{(\pi s)^2}{6} + \ldots) \\
z^s &= e^{s \ln z} = 1 + s\ln z + \ldots \\
\zeta(s,a) &= \zeta(0,a) + s\zeta'(0,a) + \ldots \\
&= \left(\frac{1}{2} - a\right) + s\zeta'(0,a) + \ldots
\end{aligned}
$$

The residue is the s^{-1} coefficient of

$$
\frac{1}{s} \times \frac{1}{s}\left(1 + \frac{(\pi s)^2}{6} + \ldots\right)(1 + s\ln z + \ldots)\left(\frac{1}{2} - a + s\zeta'(0,a)\right)
$$

and so

$$
\begin{aligned}
R_0 &= \left(\frac{1}{2} - a\right)\ln z + \zeta'(0,a) \\
&= \left(\frac{1}{2} - a\right)\ln z + \ln\Gamma(a) - \frac{1}{2}\ln(2\pi)
\end{aligned}
$$

Consequently if $|argz| \leq \pi - \delta$ and $|z|$ large is large,

$$
\ln\Gamma(z+a) = \left(z + a - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \sum_{m=1}^{\infty}\frac{(-1)^m z^{-m}\phi'_{m+2}(a)}{m(m+1)(m+2)} + \mathcal{O}(z^{-n-\frac{1}{2}})
$$

R_{-1} is the residue at $s = 1$. Set $s = S + 1$

$$
\begin{aligned}
\frac{1}{1+S} &= (1 - S + S^2 - \dots) \\
\frac{\pi}{\sin(\pi(1+S))} &= \frac{2i}{e^{i\pi(1+S)} - e^{-i\pi(1+S)}} \\
&= -\frac{2i}{e^{i\pi S} - e^{-i\pi S}} \\
&= -\frac{1}{\pi S}(1 + \frac{(\pi S)^2}{6} + \dots) \\
z^{1+S} &= e^{(S+1)\ln z} = z(1 + S\ln z + \dots) \\
\zeta(1+S, a) &= \zeta(1,a) + S\zeta'(1,a) + \dots \\
&= + \dots
\end{aligned}
$$

We finally have the asymptotic formula

$$
\ln\Gamma(z+a) = \left(z + a - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + o(1) \tag{O.-79}
$$

where the term $o(1)$ tends to zero as $|z| \to \infty$.

Proof of the formula for $|z| < 1$

With these preparations we can move onto the proof of the formula. Consider

$$
\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds
$$

The poles of $\Gamma(a+s)\Gamma(b+s)$ are $s = -a-n,\ -b-n\ (n = 1, 2, 3\dots)$

Now noting the relation $\Gamma(-s)\Gamma(1+s) = -\pi\text{cosec}(s\pi)$, consider

$$
\frac{1}{2\pi i}\int_C \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)}\frac{\pi(-z)^s}{\sin(s\pi)}ds
$$

the integrand tends to zero sufficiently rapidly to ensure

$$
\int_C \to 0 \quad \text{as} \quad N \to \infty.
$$

Now

$$
\int_{-i\infty}^{i\infty} - \left\{\int_{-i\infty}^{-i(N+\frac{1}{2})} + \int_C + \int_{i(N+\frac{1}{2})}^{i\infty}\right\}
$$

by Cauchy's theorem, is equal to minus $2\pi i$ times times the sum of the residues of the integrand at the points $s = 0, 1, 2, \ldots, N$.

We easily find the residue of the integrand of () for the pole at $s = n$

$$\begin{aligned} \lim_{s\to n} \quad & \frac{1}{2\pi i}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)} \cdot \frac{(s-n)}{\sin(s\pi)} \cdot \pi(-z)^s \\ = \quad & \frac{1}{2\pi i}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n \end{aligned} \tag{O.-79}$$

Let $N \to \infty$. In the case $|z| < 1$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n$$

We prove the integrand is an analytic function

$$\begin{aligned} \ln\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} &= \ln\Gamma(a+s)\ln\Gamma(b+s) - \ln\Gamma(c+s) \\ &= \left(s + (a+b-c) - \frac{1}{2}\right)\ln s - s + \frac{1}{2}\ln(2\pi) + o(1) \end{aligned} \tag{O.-79}$$

We prove the integrand tends to zero sufficiently rapidly

Proof of the formula for $|z| > 1$

$\square$

Appendix P

The Semi-Classical Limit

P.1 Introduction

For graph changing operators such as the Hamiltonian constraints it turns out to be extremely diffcult to define coherent (or semiclassical) states. That is, states labelled by points in the classical phase space with respect to which the operator assumes an expecion value which reproduces the value of the corresponding classical function at that point in phase space and with respect to which the (relative) fluctuations are small. The reason for why this happens is that the existing coherent states for LQG [] are defined over a finite collection of finite graphs and these suppress very e ectively the fluctuations of those degrees of freedom that are labelled by the given graph. However, the Hamiltonian constraints add degrees of freedom to the state on which they act and the fluctuations of those are therefore no longer suppressed. Indeed, the semiclassical behaviour of the Hamiltonian constraints with respect to these coherent states is rather bad.

P.2 Quantum Mechanics: Schrödinger vs. Polymer particle

P.2.1 Weyl representation as Apposed to the Schrödinger representation

The first cast the standard fock description into a different form that will more convenient for comparision with the background independent theory. We will demostrate this alternative representation with the simple example of the one-particle Schrödinger system.

Schrödinger

- Hilbert space: $\mathcal{H} = L_2(R, dx)$
-
- operators:

$$\hat{U}(\lambda)\psi(x) = e^{i\lambda x}\psi(x) \tag{P.1}$$
$$\hat{V}(\mu)\psi(x) = \psi(x+\mu) \tag{P.2}$$

Both are unitary, and as such are well defined operators as unitary transformations on wavefunctions preserve the norm.

$$W(\xi) = e^{\frac{i}{2}\lambda\mu}U(\lambda)V(\mu) \tag{P.2}$$

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1+\lambda_2), \quad V(\mu_1)V(\mu_2) = V(\mu_1+\mu_2),$$
$$U(\lambda)V(\mu) = e^{\frac{-i}{2}\lambda\mu}V(\mu)U(\lambda) \tag{P.2}$$

- Continuity in λ and μ: This means that have self-adjoint generators on H

$$\hat{U}(\lambda) = e^{i\lambda\hat{x}}, \quad \hat{V}(\mu) = e^{i\lambda\hat{k}} \tag{P.2}$$

where

$$\hat{x}\psi(x) = x\psi(x), \quad \hat{k}\psi(x) = \frac{1}{i}\frac{d}{dx}\psi(x) \tag{P.2}$$

$$[\hat{x}, \hat{k}] = i \tag{P.2}$$

Polymer Particle quantum mechanics

We still want a representation of the Weyl-Heisenberg algebra. We want to mimic LQG

- Graphs γ are countable sets of points on R satisfying certain restrictions.
- For a given graph , define cyl

$$f(k) = \sum_{x_j\in\gamma} f_j e^{-ix_jk} \tag{P.2}$$

- Put $CYL = \cup_\gamma CYL_\gamma$
- Introduce an inner product:

$$< e^{-ix_i k} | e^{-ix_j k} >= \delta_{x_i,x_j} \tag{P.2}$$

- Then $\mathcal{H}_{Poly} = L_2(R_d, d\mu_d)$ $f(k) \in \mathcal{H}_{Poly} \iff \sum_j |f_j|^2 < \infty$

Comparisons

Polymer particle		**Quantum Geometry**
• $CYL \subset \mathcal{H}_{Poly} \subset CYL^*$	$\iff$ CYL is based on "graphs" in each case	$CYL \subset \mathcal{H}_{Poly} \subset CYL^*$
• spin networks	$\iff$ Reflect the discrete nature	almost periodic functions.
• $\hat{x}$	$\iff$ Both are self-adjoint	$\hat{E}_S$
• $\hat{V}(\mu)$	$\iff$ Both are unitary	$\hat{h}_e$
• $\hat{k}$	$\iff$ Neither exists as an operator	$\hat{A}$

almost

Semi Classical States

A semiclassical state should have its expectation valuees for $\hat{x}$ and $\hat{p}$ peaked about $(0,0)$ with minimal uncertainty $(\Delta\hat{x})(\Delta\hat{p}) = \hbar/2$.

$$\hat{a}|\Psi_0 >= \frac{1}{\sqrt{2}}(\hat{x} + id^2\hbar\hat{p})|\Psi_0 >= 0 \tag{P.2}$$

$$(\Phi_0| e^{\sqrt{2}\alpha\hat{a}^\dagger} \hat{V}(-\alpha d) e^{\alpha^2/2} = (\Phi_0| \tag{P.2}$$

This has a unique solution in The dual space CYL^*

$$(\Phi_0| = \sum_{x \in R} e^{x^2/2d^2}(x| \tag{P.2}$$

Shadow States - Candidates for Semiclassical States?

$$(x_i|x_j >:= \delta_{x_i,x_j} \tag{P.2}$$

any state $(\Psi| \in CYL^*$ can be written

$$(\Psi| = \sum_{x \in R} \Psi(x)(x| \tag{P.2}$$

Let $(\Phi|$ be any state in CLY^*, and let γ be a graph. The shadow state $|\Phi_\gamma^{Shad}$ is the unique state in CYL_γ such that

$$< \Psi_\gamma^{Shad}|`\Phi >= (\Psi|\Phi > \tag{P.2}$$

What will we use for a momentum operator? formally

$$e^{-ik\mu} = 1 - ik\mu - \frac{k^2\mu^2}{2} + \dots \tag{P.2}$$

then

$$\frac{e^{-ik\mu} - e^{ik\mu}}{-2i\mu} = k + \mathcal{O}(k^2\mu) \tag{P.2}$$

$$\hat{k}_{\mu_0} := \frac{i}{2\mu_0}(\hat{V}(\mu_0) - \hat{V}(-\mu_0)) \tag{P.2}$$

Relation between polymer and Fock excitations [gr-qc/0107043]

Note: This stage of quantum geometry - Cyl* does not have a natural inner product with respect to which both eigenstates $< \mathcal{N}_{\alpha,\vec{n}}|$ of the electric flux-operators and the (images of the) Fock states are normalizable.

P.3 Varadarajan's Polymer Version of Maxwell's Theory

http://golem.ph.utexas.edu/ distler/blog/archives/000855.html:

As "traditional canonical" LQG(or LQG type quantizations) essentially relies on choosing a particular Poisson subalgebra(the so called holonomy-flux algebra) and quantizing it using a peculiar GNS functional (one that is spatially diffeomorphism invariant) one can also apply this procedure to field theories on flat spacetime. So for example take $U(1)$ theory on flat spacetime and quantize it this way. One will get a "loopy" Hilbert space with a spatially diffeomorphism invariant measure on which holonomies and smeared electric fields are well defined operators. Now one can ask the question, how is this representation (with holonomies as well defined operators) related to the usual Fock representation on which holonomies(without smearing) are not even well defined. This question was answered by Varadarajan in his 2000-2001 papers. The upshot is that, there is a so called r-Fock representation which is very closely tied to the usual Fock representation and whose states are distributions over the loopy Hilbert space. One can play the same game for scalar fields (Ashtekar et al. 2001).

Varadarajan's polymer states:

Graph changing polymer states:

the GNS Hilbert space $\mathcal{H}$. $\mathcal{D}$ is the linear subspace of $\mathcal{H}$ spanned by elements of the form

$$\hat{A}_a(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{k}} (e^{i\vec{k}\cdot\vec{x}} \hat{a}_a(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} \hat{a}^\dagger_a(\vec{k})) \tag{P.2}$$

where

The image of the Poincare condition in the r-Fock representation is

$$(\Gamma_F([\alpha, \{q\}]))^2 e^{-\int \frac{d^3x}{4q_0^2} X^a_{\alpha,\{q\}(r)}(x))E_{ra}(x)} \tag{P.2}$$

More General Version

The basic operators can be taken to be

$$U(f) \exp i \int d^3x \hat{\phi} f(x) \tag{P.3}$$

$$\hat{\pi} = \int d^3x \hat{\pi} g(x), \tag{P.4}$$

where f and g are test functions

$$[U(f)\Psi](\tilde{\phi}). \tag{P.4}$$

P.4 Coherent States

Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N := a^\dagger a$ (the number operator), then

$$[N, a^\dagger] = a^\dagger, \;\; [N, a] = -a, \;\; [a^\dagger, a] = -1. \tag{P.4}$$

Let H be a Fock space generated by a and $a^\dagger$. The actions of a and $a^\dagger$ on H are given by

$$\begin{aligned} a|n> &= \sqrt{n}|n-1> \\ a^\dagger|n> &= \sqrt{n+1}|n+1> \\ N|n> &= n|n>. \end{aligned} \tag{P.3}$$

$$|n>= \frac{(a^\dagger)^n}{\sqrt{n}}|0>. \tag{P.3}$$

These states satisfy the orthogonality and completeness conditions

$$< m|n >= \delta_{mn}, \quad \sum_{n=0}^{\infty} |n>< n| = I. \tag{P.3}$$

the following three conditions are equivalent

$$(i) \quad a|z>= z|z> \text{ and } <z|z>= 1 \tag{P.4}$$

$$(ii) \quad |z>= e^{-|z|^2/2} \sum_{-\infty}^{\infty} \frac{z^n}{\sqrt{n!}} |n>= e^{-|z|^2/2} e^{za^\dagger}|0> \tag{P.5}$$

$$(iii) \quad |z>= e^{za^\dagger - \bar{z}a}|0>. \tag{P.6}$$

In going from (P.6) to (P.6) we make use of the *Baker-Campbell-Hausdorff formula*

$$e^{A+B} = e^{\frac{1}{2}[A,B]} e^A e^B \tag{P.6}$$

which holds whenever $[A, [A, B]] = [B, [A, B]] = 0$.

$$|\Psi_\alpha >= e^{-|\alpha|/2} \sum_{n_1,\ldots,n_D} \frac{(\alpha_1)^{n_1} \cdots (\alpha_D)^{n_D}}{\sqrt{n_1!} \cdots \sqrt{n_D!}} |n_1, n_2, \ldots, n_D > . \tag{P.7}$$

in configuration space and momentum space representation:

$$\begin{aligned} \psi_z(x) &= \sqrt{\frac{\omega}{\pi\hbar}} \exp - \left\{ \frac{\omega}{2\hbar}(x - X_0) - \frac{i}{\hbar} x P_0 \right\}, & \text{(P.8)} \\ \psi_z(p) &= \sqrt{\frac{\hbar}{\pi\omega}} \exp - \left\{ \frac{\hbar}{2\omega}(p - P_0) - \frac{i}{\hbar} p X_0 \right\}. & \text{(P.9)} \end{aligned}$$

substituting $t = \hbar/\omega$ this is rewritten

$$\begin{aligned} \psi_z(x) &= \frac{1}{\sqrt{2t}} \exp - \left\{ \frac{1}{2t}(x - X_0) - \frac{i}{\hbar} x P_0 \right\}, & \text{(P.10)} \\ \psi_z(p) &= \frac{t}{\sqrt{\pi}} \exp - \left\{ \frac{t}{2}(p - P_0) - \frac{i}{\hbar} p X_0 \right\}. & \text{(P.11)} \end{aligned}$$

$$\begin{aligned} (\Delta \hat{q}_i)^2 &\equiv < \Psi_\alpha | \hat{q}_i^2 | \Psi_\alpha > -[< \Psi_\alpha | \hat{q}_i | \Psi_\alpha >]^2 = \frac{1}{2} \ell_i^2, \\ (\Delta \hat{p}_i)^2 &\equiv < \Psi_\alpha | \hat{p}_i^2 | \Psi_\alpha > -[< \Psi_\alpha | \hat{p}_i | \Psi_\alpha >]^2 = \frac{1}{2} \hbar^2 / \ell_i^2 & \text{(P.11)} \end{aligned}$$

$$< \Psi_\beta | : F(\hat{a}_i^\dagger, \hat{a}_i) : | \Psi_\alpha >= F(\overline{\beta}_i, \alpha_j) < \Psi_\beta | \Psi_\alpha > \tag{P.11}$$

read off the most important properties of coherent states

$$< \hat{X} >_{\Psi_z} = X_0, \quad < \hat{P} >_{\Psi_z} = P_0, \tag{P.11}$$

$$e^{-t\Delta} \delta_y(x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{1}{2t}(x-y)^2} \tag{P.11}$$

P.5 Coherent States of LQG

P.6 Coherent States of Compact Groups

In this section we describe the coherent state transforms (CST) for Lie groups introduced by Hall []. For simplicity, we will restrict ourselves in this section, to the case when K is simple. The general compact case can be treated in a similar way. In particular $K = U(1)^g$ will be considered in sections N.-19 and N.-19. Let K be a compact connected simple Lie group, $K_{\mathbb{C}}$ its complexification (see []) and Δ_K the Laplacian on K associated to an Ad-invariant inner product on its Lie algebra $Lie(K)$.

For each $f \in L^2(K; dx)$, where dx is the normalized Haar measure on K, the image of f by the CST, $C_t f$, is the analytic continuation to $K_{\mathbb{C}}$ of the solution of the heat equation,

$$\frac{1}{\pi}\frac{\partial u}{\partial t} = \Delta_K u, \tag{P.11}$$

in generalized coordinates the Laplacian

$$\triangle u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}g^{ij}\frac{\partial u}{\partial q_j}\right) \tag{P.11}$$

with initial condition given by $u(0, x) = f(x)$.

$$\psi_g^t(h) = \sum_\pi d_\pi e^{\frac{t}{2}\gamma_\pi}\chi_\pi(gh^{-1}) \tag{P.11}$$

To obtain a more explicit description of this CST, consider the expansion of $f \in L^2(K; dx)$ given by the Peter-Weyl theorem,

$$f(x) = \sum_R tr(R(x)A_R), \tag{P.11}$$

where the sum is taken over the set of (equivalence classes of) irreducible representations of K, and $A_R 2 EndVR$ is given by

$$A_R = (dimV_R)\int_K f(x)R(x)dx, \tag{P.11}$$

VR being the representation space for R. Let $X_i, i = 1, \ldots, dimK$ be an orthonormal basis for the Ad-invariant inner product on $Lie(K)$ for which the longest root has squared norm 2. Viewing $Lie(K)$ as the space of left-invariant vector fields on K, we have

$$\Delta_K = \sum_{i=1}^{dimK} X_i X_i \tag{P.11}$$

and one obtains

$$C_t f(g) = \sum_R e^{i\pi t\tau C_R} tr(R(g)A_R). \tag{P.11}$$

P.6.1 Coherent States of U(1)

Segal-Bargmann-Hall Transform

$$\Delta := \sum_{k=1}^{d} \frac{\partial^k f}{\partial x_k^2} \tag{P.11}$$

the heat equation

$$\frac{\partial u}{\partial s} = \frac{1}{2}\Delta u \tag{P.11}$$

$$B_t f(z) = \frac{1}{(2\imath t)^{-d/2}} \int_{\mathbf{R}^d} e^{-(z-x)^2/2t} f(x)dx, \quad z \in \mathbf{C}^d. \tag{P.11}$$

is just the convolution of f with a Gaussian.

Different bit

heat kernel measure:

$$\frac{d\mu}{dt} = \frac{1}{4}\Delta K_C \mu_t \tag{P.11}$$

let μ_t denote the associated heat kernel measure

$$d\mu_t(g) := \mu_t(g)dg. \tag{P.11}$$

In terms of a normalized left invariant vector field X^a on $U(1)$ as

$$\Delta = X^2. \tag{P.11}$$

$$\frac{dh(\theta,t)}{dt} = \frac{1}{2}\Delta h(\theta,t) \tag{P.11}$$

$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}h(\theta,t) = \frac{\partial}{\partial t}h(\theta,t) \tag{P.11}$$

$$\sum_m \left[-\frac{m^2}{2}a_m(t) - \dot{a}_m(t)\right] e^{im} = 0, \tag{P.11}$$

$$a_m(t) = A_m e^{-\frac{m^2}{2}t} \tag{P.11}$$

$$\begin{aligned} C_t(f)(z) \;&:=\; \frac{1}{2\pi}\int_{S^1}\left(\sum_m e^{-\frac{m^2}{2}t}x^{-m}z^m\right)\left(\sum_n a_n x^n\right)d\theta \\ &= \frac{1}{2\pi}\sum_m\sum_n a_n e^{-\frac{m^2}{2}t}z^m\int_{S^1} x^n x^{-m}d\theta \\ &= \sum_m a_m e^{-\frac{m^2}{2}t}z^m. \end{aligned} \tag{P.10}$$

$$\Delta_G = \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial p^2}. \tag{P.10}$$

$$\frac{dh}{dt} = \frac{1}{4}\Delta_G h. \tag{P.10}$$

$$\mu_t(z) = \frac{1}{\sqrt{t\pi}}e^{\frac{p^2}{t}}\left(\sum_m e^{-\frac{m^2}{2}t}e^{im\theta}\right). \tag{P.10}$$

Coherent States of U(1)

$$\begin{aligned} h &= e^{i\theta} \\ g &= e^{i(\phi-ip)} \\ \chi_\pi(gh^{-1}) &= e^{in(\phi-\theta)}e^{np} \;\text{ for } \pi = n \end{aligned} \tag{P.9}$$

$$\psi_g^t(h) = \sum_{n=-\infty}^{\infty} e^{\frac{t}{t}n^2} e^{in(\phi-\theta)} e^{np}. \tag{P.9}$$

Poisson for $SU(2)$

$m, n \in \{-j, -j+1, \dots, j-1, j\}$

$$\pi_j(g)_{mn} = \sum_\ell \frac{\sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}}{(j-m-\ell)!(j+n-\ell)!(m-n+\ell)!\ell!} a^{j+n-\ell} d^{j-m-\ell} b^{m-n+\ell} c^\ell \tag{P.9}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \tag{P.9}$$

$\det(g) = \lambda_1 \lambda_2$ and $tr(g) = a + d = \lambda_1 + \lambda_2$.

$$\lambda_1 + \frac{1}{\lambda_1} = a + d, \quad \text{and} \quad \lambda_2 + \frac{1}{\lambda_2} = a + d \tag{P.9}$$

implying the quadratic equation

$$\lambda^2 - \lambda(a+d) + 1 = 0$$

with solutions

$$\lambda_1 = x + \sqrt{x^2 - 1} \quad \text{and} \quad \lambda_2 = \lambda_1^{-1} = x - \sqrt{x^2 - 1} \quad \text{where } x = \frac{a+d}{2}.$$

If $\pi_j(g)_{mn}$ is diagonal ($b = c = 0$) then (with no summation implied),

$$\pi_j(g)_{mn} = \delta_{mn} \left(a^{j+m} d^{j-m} + \sum_{\ell \neq 0} \frac{(j+m)!(j-m)!}{(j-m-\ell)!(j+m-\ell)!\ell!} a^{j+m-\ell} d^{j-m-\ell} b^\ell c^\ell \right)$$

sum over ℓ reduces to one term $\ell = 0$ and

$$\chi_\ell(g) = \sum_{m=-\ell}^{\ell} a^{\ell+m} d^{\ell-m} \tag{P.9}$$

$$\chi_\ell(g) = \chi_\ell \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \tag{P.9}$$

as $ad = \alpha\overline{\alpha} = 1$ we have $\alpha = e^{i\theta}$

$$\begin{aligned} \chi_\ell(g) &= \sum_{n=-\ell}^{\ell} e^{i\theta(\ell+n)} e^{-i\theta(\ell-n)} = \sum_{n=-\ell}^{\ell} e^{2ni\theta} \\ &= \frac{e^{i(2\ell+1)\theta} - e^{-i(2\ell+1)\theta}}{e^{i\theta} - e^{-i\theta}} \\ &= \frac{\sin(2\ell+1)\theta}{\sin\theta} \end{aligned} \tag{P.8}$$

$$t_j^{mn}\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \sum_{s=-j}^{m} (-1)^{n-s} \frac{(j+n)!}{(n-s)!(j+s)!} \frac{(j-n)!}{(j-m-n+s)!(m-s)!} \alpha^{j+s} \beta^{m-s} \overline{\beta}^{n-s} \overline{\alpha}^{j-n-m+s} \tag{P.8}$$

P.6.2 Coherent States of $SU(2)$

$$\begin{aligned} J_+|J,m> &= \sqrt{(m+1)(2J-m)}|J,m+1>, \\ J_-|J,m> &= \sqrt{m(2J-m+1)}|J,m-1>, \\ J_3|J,m> &= (-J+m)|J,m>, \end{aligned} \tag{P.7}$$

where $|J,0>$ is a normalized vacuum $< J_-|J,0>= 0$ and $< J,0|J,0>= 1$). We denote by I_J the unit operator on on $\mathcal{H}_J$. states $|J,m>$ are given by

$$|J,m>= \frac{(J_+)^m}{\sqrt{m!_{2J}P_m}}, \tag{P.7}$$

where ${}_{2J}P_m = (2J)(2J-1)\cdots(2J-m+1)$.

These states satisfy the orthogonality and completeness conditions

$$< J,m|J,n>= \delta_{mn}, \quad \sum_{n=0}^{2J} |J,m>< J,m| = I_J. \tag{P.7}$$

We call a state

$$|v>:=e^{vJ_+-\bar{v}J_-}|J,0> \tag{P.7}$$

the generalized coherent state for $su(2)$.

$$|v>=\frac{1}{(1+|\eta|^2)^J}e^{\eta J_+}|J,0>=:|\eta> \tag{P.7}$$

P.6.3 Expectation Values and Variation Properties of $U(1)$ and $SU(2)$ Coherent States

$$\sum_n e^{-\epsilon(n-N)^2}f(n)=\sum_n e^{-\epsilon(y-N)^2}f(y)e^{2\pi iyn} \tag{P.7}$$

Let f be in $L_1(\mathbf{R},dx)$ function such that the series

$$\phi(y)=\sum_{n=-\infty}^{\infty}f(y+ns) \tag{P.7}$$

is absolutely and uniformly convergent for $y\in[0,s], s>0$. Then

$$\phi(y)=\sum_{n=-\infty}^{\infty}f(y+ns)=\frac{2\pi}{s}\sum_{n=-\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{s}\right) \tag{P.7}$$

where $\tilde{f}(k):=\int_{\mathbf{R}}\frac{dx}{2\pi}e^{-ikx}f(x)$.

$$\begin{aligned}
h_e^{\mathbf{C}} &:= g_e=\sum_{n=0}^{\infty}\frac{i^n}{n!}\{h_e,C\}_{(n)}\\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}(-{}^e_j\frac{\tau_j}{2})^n\ h_e\\
&= e^{-i\tau_j p_j^e/2}h_e
\end{aligned} \tag{P.6}$$

polar decomposition

P.7 Wigner Function

Quantum analog of the classical phase space probability distribution. For a particle travelling along one dimension, the Wigner function in terms of the wavefunction $\psi(x)$ through the formula:

$$W(q,p) = \frac{1}{2\pi\hbar}\int dx \psi^*(q + x/2)e^{ix/\hbar}\psi(q - x/2), \tag{P.6}$$

and it completely characterizes the quantum state. The integral of $W(q,p)$ over q and p is one, which follows from he normalization of the wavefunction. Expectation values of the quantum observables can be obtained from the Wigner function by integrating it with appropriate

Classical states statistics are described by the function $f(q,p)$, which is the probability function in the phase space, i.e.,

$$f(q,p) \geq 0, \quad \int f(q,p)\, dp = P(p), \quad \int f(q,p)\, dq = \tilde{P}(p), \tag{P.6}$$

with $P(p)$ and $\tilde{P}(p)$ probability distributions for position and momentum, respectively.

We consider an observable $X(q,p)$ which is a function on the phase space of the system under consideration. The characteristic function for the observable $X(q,p)$

$$\chi(k) =< e^{ikX} > \tag{P.6}$$

is given by the relation

$$\chi(k) = \int e^{ikX(q,p)} f(q,p)\, dq\, dp. \tag{P.6}$$

$$w(X) = \frac{1}{2\pi}\int \chi(k)e^{-ikX}\, dk \tag{P.6}$$

is a real nonnegative function which is normalized

$$\int w(X)\, dX = 1. \tag{P.6}$$

$$w(X) = \int f(q,p)\delta(X(q,p) - X)\, dq\, dp. \tag{P.6}$$

P.8 Noiseless Subsystems and Quantum Gravity

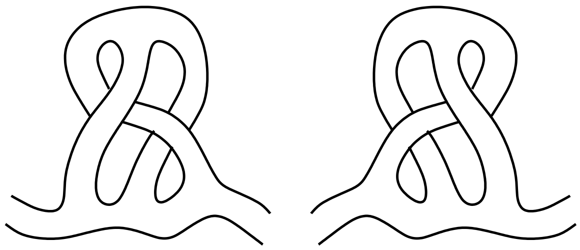


Figure P.1: ConElecNeutrF. The effect of charge conjugation on an electron-neutrino.

P.8.1 Propagating Modes vs Background Independence

"Normally when a particle is in interaction with an environment, information about its state dissipates into the environmnet - we say that it decoheres, It's difficult to prevent this decoherence from happening ... whihc depoendends on the efficacy on a particle's being in a pure quantum state. ... Markopoulou that their insights applied to teh problem of how a quantim particle could emerge from a quantum spacetime ... " not finished here

The NS method is background independent in the sense that it does **not** rely on a particular graph/state, there is a genuine superposition of spinfoams / states inside the boundary and is defined via the dynamics.

BUT:

The NS is a subsystem of a subsystem $= \mathcal{H}_A^{SBH}$ in ?? It finds the first if you give the second.

It does need the boundary / interaction dynamics:

P.8.2 NS and Locality - Microscopic vs Macroscopic Locality

One can try to assign geometric / local / causal properties to the subsystems / subgraphs. But they are not eigenstates of the dynamics.

We **have** to take into account the quantum sum over geometries.

Locality in an NS example:

$$\mathcal{H}^S \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

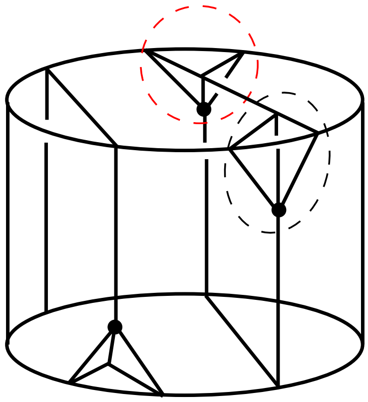


Figure P.2: NSlocalSpin.

with Z_3 .. in $A^{int'}$.

$$SOOO \otimes E$$

$\rightarrow \mathcal{H}^{NS} \simeq \mathbb{C}^2$, but it's **none** of the origin of Perez.

NS Suggests:

- give boundary (interaction) dynamics
- look at appropriate symmetries and extract NS
- assign locality / geometric / causal properties to the effective NS's.

P.8.3 Outlook

- Black hole case
  - excitations in $\mathcal{H}^{QSBH}$ that correspond classically to ..
  - Rotating black hole
  - black hole in spin foams / other models
  - $G_q \rightarrow SO(3)$
- Propagating modes in quantum gravity with boundary.
- Weaves (Dynamical) from translation symmetry in A^{int}.
- Separate scales: environment $\leftrightarrow$ small-scale ... system $\leftrightarrow$ course-grained weave.

- Theory of noiseless subsystems:

  - approx / emergent definition
  - Poincare.

P.9 The Standard Model from Loop Qunatum Gravity?

With the new decomposition, it is straightforward to check that operators in $\mathcal{A}_{evol}$ can only affect the $\mathcal{H}_\Gamma^T$ and that $\mathcal{H}_\Gamma^B$. Check this explicitly by showing that the actions of braiding and twisting of the edges of the graph and the evolution moves commute.

Proof:

$\square$

P.10 Summary

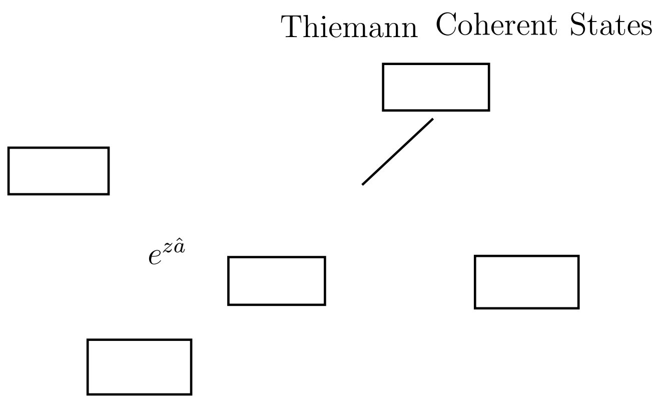


Figure P.3: u

P.11 Biblioliographical notes

In this chapter I have relied on the following reffferences:

P.12 Worked Exercises and Details

□

Poisson's formula and some inequalities.

Poisson's formula

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \varphi(2\pi n + x) &= \sum_{p=-\infty}^{\infty} a_n e^{ixp} \\
&= \sum_{p=-\infty}^{\infty} e^{ixp} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{-ipy} \sum_{n=-\infty}^{\infty} \varphi(2\pi n + y) dy \right\} \\
&= \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} e^{ixp} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \varphi(2\pi n + y) e^{-ipy} dy \qquad \text{(P.5)}
\end{aligned}$$

The sum over n in the last expression can be rewriten as follows:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \int_0^{2\pi} \varphi(2\pi n + y) e^{-ipy} dy &= \sum_{n=-\infty}^{\infty} \int_{2\pi(n+1)}^{2\pi n} \varphi(y) e^{-ipy} dy \\
&= \int_{-\infty}^{\infty} \varphi(y) e^{-ipy} dy. \qquad \text{(P.5)}
\end{aligned}$$

Therefore

$$\sum_{n=-\infty}^{\infty} \varphi(2\pi n + x) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} e^{-ipy} \int_{-\infty}^{\infty} \varphi(y) e^{-ipy} dy \qquad \text{(P.5)}$$

we set $x = 0$, we obtain

$$\sum_{n=-\infty}^{\infty} \varphi(2\pi n) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(y) e^{-ipy} dy. \qquad \text{(P.5)}$$

Coherent equations.

(b) $A = za^\dagger$ and $B = -\bar{z}a$

$$[A, B] = -|z|^2[a^\dagger, a] = |z|^2 \tag{P.5}$$

$$e^B|0>= e^{-\bar{z}a}|0>= (I - \bar{z}a + \frac{1}{2}a^2 - \dots)|0>= |0> \tag{P.5}$$

$$e^{za^\dagger - \bar{z}a}|0>= e^{-|z|^2/2}e^{za^\dagger}|0>= e^{-|z|^2/2}|z> . \tag{P.5}$$

$\hat{a}_i|\Psi_\alpha>= \alpha_i|\Psi_\alpha>$

$$< \Psi_\beta|\hat{a}_i|\Psi_\alpha>= \alpha_i < \Psi_\beta|\Psi_\alpha> \tag{P.5}$$

$< \Psi_\beta|\hat{a}_j^\dagger = (\hat{a}_j|\Psi_\beta>)^* = \overline{\beta}_j < \Psi_\beta|$

$$< \Psi_\beta|\hat{a}_i^\dagger|\Psi_\alpha>= \overline{\beta}_i < \Psi_\beta|\Psi_\alpha> \tag{P.5}$$

$$\begin{aligned} < \Psi_\beta|(\hat{a}_i^\dagger)^m(\hat{a}_j)^n|\Psi_\alpha> &= \\ &= (\overline{\beta_i})^m(\alpha_j)^n < \Psi_\beta|\Psi_\alpha> \end{aligned} \tag{P.5}$$

====================

$$< \Psi_\beta|\Psi_\alpha>= e^{-(|\alpha|^2+|\beta|^2)/2}e^{\overline{\beta}\alpha}$$

$$< \Psi_{phy}|\Phi_{phy}>= \frac{1}{\Lambda}\int_0^\Lambda d\lambda < e^{-i\lambda\hat{C}}\Psi|\Phi>$$

$$\hat{U}(\lambda)\Psi_\alpha := e^{-iC(\alpha)}\Psi_{\alpha(\lambda)}$$

where $\alpha_j(\lambda) = \alpha_j - i\kappa_j$

$$< e^{-iC(\alpha)}\Psi_{\alpha(\lambda)}|\Psi_\alpha>= e^{iC(\alpha)} < \Psi_{\alpha(\lambda)}|\Psi_\alpha>= e^{iC(\alpha)}e^{-(|\alpha|^2+|\alpha_j-i\kappa_j|^2)/2}e^{(\overline{\alpha}_j-i\overline{\kappa}_j)\alpha}$$

$$\|\Psi_\alpha^{phy}\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \tag{P.6}$$

======

$$
\begin{aligned}
&< \Psi_\alpha^{phy}|\Psi_\alpha^{phy} >= \frac{1}{2\pi}\int_0^{2\pi} d\lambda < \hat{U}(\lambda)\Psi_\alpha|\Psi_\alpha > \\
= &\frac{1}{2\pi}\int_0^{2\pi} d\lambda e^{-i\lambda\Delta}\left[e^{-|\alpha|^2/2}\sum_{n_1,\dots,n_D}\frac{(e^{-i\kappa_1}\alpha_1)^{n_1}\dots(e^{-i\kappa_D}\alpha_D)^{n_D}}{\sqrt{n_1!}\dots\sqrt{n_D!}}|n_1,\dots,n_D >\right]^\dagger \\
&\qquad e^{-|\alpha|^2/2}\sum_{\tilde{n}_1,\dots,\tilde{n}_D}\frac{\alpha_1^{n_1}\dots\alpha_D^{n_D}}{\sqrt{\tilde{n}_i!}\dots\sqrt{\tilde{n}_D!}}|\tilde{n}_1,\dots,\tilde{n}_D > \\
= &\frac{e^{-|\alpha|^2}}{2\pi}\int_0^{2\pi} d\lambda e^{-i\lambda\Delta}\sum_{n_1,\dots,n_D}\sum_{\tilde{n}_1,\dots,\tilde{n}_D} e^{i\sum_j n_j\kappa_j} < n_1,\dots,n_D|\frac{(\alpha_D^\dagger)^{n_D}\dots(\alpha_1^\dagger)^{n_1}\alpha_1^{\tilde{n}_1}\dots\alpha_D^{\tilde{n}_D}}{\sqrt{n_1!}\dots\sqrt{n_D!}\sqrt{\tilde{n}_i!}\dots\sqrt{\tilde{n}_D!}}|\tilde{n}_1,\dots,\tilde{n}_D > \\
= &e^{-|\alpha|^2}\sum_{n_1,\dots,n_D}\frac{|\alpha_1|^{2n_1}\dots|\alpha_D|^{2n_D}}{n_1!\dots n_D!}\frac{1}{2\pi}\int_0^{2\pi} d\lambda e^{-i\lambda\Delta}e^{i\sum_j n_j\kappa_j} \\
= &e^{-|\alpha|^2}\sum_{n_1,\dots,n_D}\frac{|\alpha_1|^{2n_1}\dots|\alpha_D|^{2n_D}}{n_1!\dots n_D!}\delta_{\sum_j n_j\kappa_j,\Delta}\ ,
\end{aligned}
\tag{P.1}
$$

Baker-Campbell-Hausdorff formula.

We are proving a reduced version of the Baker-Campbell-Hausdorff formula: the following

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-[\hat{A},\hat{B}]/2) \tag{P.1}$$

holds when $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$.

Proof:

□

First we prove

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{A}+\alpha[\hat{A},\hat{B}]+\frac{\alpha^2}{2!}[\hat{A},[\hat{A},\hat{B}]]+\frac{\alpha^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]+\cdots \tag{P.1}$$

To derive this we write $f(\alpha) = e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}}$, take derivatives of $f(\alpha)$ and then putting $\alpha = 0$. First

$$f(0) = \hat{B}.$$

Taking the first derivative

$$\frac{d}{d\alpha}e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = e^{\alpha\hat{A}}[\hat{A},\hat{B}]e^{-\alpha\hat{A}} \tag{P.1}$$

and setting $\alpha = 0$ gives $[\hat{A},\hat{B}]$. Taking the derivative of (P.12) we obtain

$$\frac{d^2}{d\alpha^2}e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = e^{\alpha\hat{A}}[\hat{A},[\hat{A},\hat{B}]]e^{-\alpha\hat{A}}$$

and setting $\alpha = 0$ gives $[\hat{A},[\hat{A},\hat{B}]]$ and so on.

As $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$ the RHS of (P.12) reduces to $\hat{B} + \alpha[\hat{A},\hat{B}]$ so that

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + \alpha[\hat{A},\hat{B}].$$

We now introduce the function $g(\alpha) = e^{\alpha\hat{A}}e^{\alpha\hat{B}}$, then

$$\begin{aligned} \frac{dg}{d\alpha} &= \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}} + e^{\alpha\hat{A}}\hat{B}e^{\alpha\hat{B}} \\ &= \left(\hat{A} + e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}}\right)e^{\alpha\hat{A}}e^{\alpha\hat{B}} \\ &= \left(\hat{A} + e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}}\right)g(\alpha). \end{aligned} \tag{P.0}$$

We have derived the differential equation

$$\frac{dg}{d\alpha} = \left(\hat{A} + \hat{B} + \alpha[\hat{A},\hat{B}]\right)g(\alpha) \tag{P.0}$$

whose solution is

$$g(\alpha) = e^{\alpha\hat{A}+\hat{B}+\frac{\alpha^2}{2}[\hat{A},\hat{B}]}. \tag{P.0}$$

So that

$$e^{\alpha\hat{A}}e^{\alpha\hat{B}} = e^{\alpha\hat{A}+\hat{B}+\frac{\alpha^2}{2}[\hat{A},\hat{B}]}. \tag{P.0}$$

Putting $\alpha = 1$ gives

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}. \tag{P.0}$$

As $[\hat{A},\hat{B}]$ commutes with both $\hat{A}$ and $\hat{B}$ we can write

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A}+\hat{B})\exp([\hat{A},\hat{B}]/2). \tag{P.0}$$

"Complexifier" form

$$\Psi_0(A) = \mathcal{N}\exp(-\hat{C})\delta(A) \tag{P.0}$$

where

$$\hat{C} := \frac{1}{4\hbar\kappa}\int d^3x \int d^3y\; W_\Lambda(\underline{x},\underline{y})(\hat{\tilde{g}}^{ab}_\Lambda(\underline{x}) - \delta^{ab}) \tag{P.0}$$

and

$$\hat{\tilde{g}}^{ab}_\Lambda(\underline{x}) := \int d^3x' \hat{E}^a_i(\underline{x})\delta_\Lambda(\underline{x}-\underline{x}')\hat{E}^b_i(\underline{x}') \tag{P.0}$$

Metric operator:

$$\begin{aligned} & \int d^3x' \hat{E}^a_i(\underline{x})\delta(\underline{x}-\underline{x}')\hat{E}^b_i(\underline{x}')\tilde{S} \\ = & \sum_v \sum_{edges e_1, e_2 of v} \int d^3x' F^a_{v,e_1}(\underline{x})\delta(\underline{x}-\underline{x}')F^a_{v,e_2}(\underline{x}) \end{aligned} \tag{P.0}$$

Irreducible representations of $SU(2)$.

We write a vector in $\mathbb{C}^2$ as a pair of complex numbers (z_1, z_2). We define the element of a Hilbert space $\mathcal{H}_\ell$ as an polynomial in z_1 and z_2 that is a linear combination of polynomials

$$P \in \mathbb{C}[z_1, z_2], \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad z = (z_1, z_2), \tag{P.0}$$

and

$$zg = (az_1 + cz_2, bz_1 + dz_2). \tag{P.0}$$

$$P_k(z_1, z_2) = z_1^k z_2^{n-k}, \quad 0 \geq k \geq n, \tag{P.0}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z_1^k z_2^{n-k} \;=\; (az_1 + cz_2)^r (bz_1 + dz_2)^{n-r} \tag{P.1}$$

Could take the basis of mononials of degree n in the order

$$z_1^n, z_1^{n-1}z_2, \ldots, z_1 z_2^{n-1}, z_2^n. \tag{P.1}$$

OR

$$P_\ell(z_1, z_2) = z_1^{\ell+j} z_2^{\ell-j}, \quad -\ell \geq j \geq \ell, \quad \ell = 0, \frac{1}{2}, 1, \ldots \tag{P.1}$$

where ℓ is an integer or half integer

$$\frac{z_1^{2\ell}}{\sqrt{(2\ell)!}}, \frac{z_1^{2\ell-1}z_2}{\sqrt{(2\ell-1)!(1)!}}, \frac{z_1^{2\ell-2}z_2^2}{\sqrt{(2\ell-2)!(2)!}}, \ldots, \frac{z_1 z_2^{2\ell-1}}{\sqrt{(1)!(2\ell-1)!}}, \frac{z_2^{2\ell}}{\sqrt{(2\ell)!}}. \tag{P.1}$$

These are the basis vector of a vector space we denote V_ℓ. We easily see that it is closed under the linear transformation

$$z_1^{\ell+j} z_2^{\ell-j} \to (az_1 + bz_2)^{\ell+j}(-cz_1 + dz_2)^{\ell-j}$$

as

$$\begin{aligned}
& (az_1 + bz_2)^{\ell+j}(-cz_1 + dz_2)^{\ell-j} \\
= \; & \sum_{r=0}^{\ell+j} \sum_{q=0}^{\ell-j} (az_1)^{\ell+j-r}(bz_2)^r(-cz_1)^q(dz_2)^{\ell-j-q} \\
= \; & \sum_{r=0}^{\ell+j} \sum_{q=0}^{\ell-j} \begin{pmatrix} \ell+j \\ r \end{pmatrix} \begin{pmatrix} \ell-j \\ q \end{pmatrix} \left(a^{\ell+j-r} b^r (-c)^q d^{\ell-j-q}\right) \cdot z_1^{\ell+(j-r+q)} z_2^{\ell-(j-r+q)}
\end{aligned} \tag{P.0}$$

Does $j - r + q$ vary between $-\ell$ and ℓ? The minimum value $j - r + q$ can take is when $r = \ell + j$ and $q = 0$, and hence is $-\ell$. The maximum value $j - r + q$ can take is when $r = 0$ and $q = \ell - j$, and hence is ℓ.

Since the monomial set is closed under the linear transformation g, it will provide a $(2\ell + 1) \times (2\ell + 1)$ matrix representation.

For any $g \in SU(2)$, let $U_\ell(g)$ be the linear transformation of $\mathcal{H}_\ell$ given by

$$(U_\ell(g)f)(v) = f(g^{-1}v) \tag{P.0}$$

for all $f \in \mathcal{H}_\ell$ and $v \in \mathbb{C}^2$. This is a representation: $U_\ell(1)$ is the identity, and for any $g, h \in SU(2)$ we have

$$
\begin{aligned}
(U_\ell(g)U_\ell(h)f)(v) &= (U_\ell(h)f)(g^{-1}v) \\
&= f(h^{-1}g^{-1}v) \\
&= f((gh)^{-1}v) \\
&= (U_\ell(gh)f)(v) \qquad \text{(P.-2)}
\end{aligned}
$$

for all $f \in \mathcal{H}_\ell$, $v \in \mathbb{C}^2$.

Matrix elements

We derive the symmetric version, based on the basis vectors (P.12),

$$\pi_\ell(g)_{mn} = \alpha^{m+n}(-\overline{\beta})^{\ell-m}\beta^{\ell-n}\sum_{s=0}^{\ell-n}\frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}{(m+n+s)!(\ell-m-s)!(\ell-n-s)!s!}\left(-\frac{|\alpha|}{|\beta|}\right)^s \tag{P.-2}$$

Setting

$$\phi_k^\ell(z_1,z_2) = \frac{z_1^{\ell+k}z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}} \tag{P.-2}$$

The matrix elements are defined by

$$U_\ell(g)\phi_n^\ell = \sum_{k=\ell}^{-\ell}\phi_k^\ell(z_1,z_2)\;\pi_\ell(g)_{kn} \tag{P.-2}$$

From (P.-2)

$$\begin{aligned}
U_\ell(g)U_\ell(h)\phi_n^\ell &= U_\ell(g)\sum_{j=-\ell}^{\ell}\phi_j^\ell\;\pi_\ell(h)_{jn} \\
&= \sum_{j=-\ell}^{\ell}\sum_{k=-\ell}^{\ell}\phi_k^\ell\;\pi_\ell(g)_{kj}\pi_\ell(h)_{jn} \\
&= U_\ell(gh)\phi_n^\ell \\
&= \sum_{k=-\ell}^{\ell}\phi_k^\ell\;\pi_\ell(h)_{kn}(gh)
\end{aligned} \tag{P.-4}$$

that is

$$\sum_{k=-\ell}^{\ell}\frac{z_1^{\ell+k}z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}}\left(\sum_{j=-\ell}^{\ell}\pi_\ell(g)_{kj}\pi_\ell(h)_{jn} - \pi_\ell(h)_{kn}\right) \tag{P.-4}$$

Setting $z_2 = 1$ and multiplying both sides by $z_1^{-\ell-m-1}$, integrating z_1 around the unit circle about the origin of the z_1-plane we obtain

$$\sum_{j=-\ell}^{\ell}\pi_\ell(g)_{mj}\pi_\ell(h)_{jn} = \pi_\ell(h)_{mn} \tag{P.-4}$$

i.e. $\pi_\ell(g)_{kj}$ is a matrix representation of $SU(2)$.

We defined the representation $\pi_\ell(g)_{mn}$ by

$$\begin{aligned}
U_\ell(g)\phi_n^\ell(z_1,z_2) &= \frac{1}{\sqrt{(\ell+n)!(\ell-n)!}}U_\ell(g)z_1^{\ell+n}z_2^{\ell-n} \\
&= \frac{(\alpha z_1-\overline{\beta}z_2)^{\ell+n}(\beta z_1+\overline{\alpha}z_2)^{\ell-n}}{\sqrt{(\ell+n)!(\ell-n)!}} \\
&= \sum_{m=\ell}^{-\ell}\phi_m^\ell(z_1,z_2)\ \pi_\ell(g)_{mn} \\
&= \sum_{m=\ell}^{-\ell}\frac{z_1^{\ell+m}z_2^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}\ \pi_\ell(g)_{mn} \qquad \text{(P.-6)}
\end{aligned}$$

To arrive at (P.12), first we expand into the binomial expansion

$$\begin{aligned}
&(\alpha z_1-\overline{\beta}z_2)^{\ell+n}(\beta z_1+\overline{\alpha}z_2)^{\ell-n} \\
= &\left(\sum_{t=0}^{\ell+n}(-1)^{\ell+n-t}\begin{pmatrix}\ell+n\\ t\end{pmatrix}\alpha^t\overline{\beta}^{\ell+n-t}z_1^tz_2^{\ell+n-t}\right)\left(\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\beta^{\ell-n-k}\overline{\alpha}^kz_1^{\ell-n-k}z_2^k\right) \\
= &\sum_{t=0}^{\ell+n}\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ t\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\alpha^t(-\overline{\beta})^{\ell+n-t}\beta^{\ell-n-k}\overline{\alpha}^k\cdot z_1^{\ell+t-n-k}z_1^{\ell-t+n+k}
\end{aligned} \qquad \text{(P.-8)}$$

We will show that this is equal to

$$\sum_{m=-\ell}^{\ell}\alpha^{m+n}(-\beta^*)^{\ell-m}\beta^{\ell-n}\left(\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ m+n+k\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\left(-\left|\frac{\alpha}{\beta}\right|\right)^k\right)z_1^{\ell+m}z_1^{\ell-m} \qquad \text{(P.-8)}$$

equivalently

$$\sum_{m=-\ell}^{\ell}\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ m+n+k\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}Q_{m,n,k} \qquad \text{(P.-8)}$$

where $Q_{m,n,k}=\alpha^{m+n+k}(-\beta^*)^{\ell-m-k}(\alpha^*)^k\beta^{\ell-n-k}\cdot z_1^{\ell+m}z_1^{\ell-m}$. We do this by working backwards. First write

$$
\begin{aligned}
B &:= \sum_{m=-\ell}^{\ell}\sum_{k=0}^{\ell-n}\left(\begin{array}{c}\ell+n\\ m+n+k\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{m,n,k}\\
&= \sum_{k=0}^{\ell-n}\sum_{m=-\ell}^{\ell}\left(\begin{array}{c}\ell+n\\ m+n+k\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{m,n,k}\\
&= \sum_{k=0}^{\ell-n}\sum_{t=n+k-\ell}^{\ell+n+k}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{t-n-k,n,k}
\end{aligned}
\tag{P.-10}
$$

but note that

$$
\left(\begin{array}{c}\ell+n\\ \ell+n+k\end{array}\right)=0 \quad \text{if} \quad k>0,
$$

because $1/(-N)! = 0$ for positive integer N. As such the upper limit on the t summation will always be $\ell + n$. We now explicitly write out each term in the k summation. In doing so we will use

$$
\left(\begin{array}{c}\ell+n\\ n+k-\ell\end{array}\right)=0 \quad \text{if} \quad k<\ell-n
$$

which again follows from $1/(-N)! = 0$ for positive integer N. Hence,

$$
\begin{aligned}
B &= \sum_{t=n-\ell}^{\ell+n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ 0\end{array}\right)Q_{t-n,n,0}+\sum_{t=n+1-\ell}^{\ell+n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ 1\end{array}\right)Q_{t-n-1,n,1}+\dots\\
&\quad \dots +\sum_{t=0}^{\ell+n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ \ell-n\end{array}\right)Q_{t-\ell,n,\ell-n}\\
&= \sum_{t=0}^{\ell+n}\sum_{k=0}^{\ell-n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{t-n-k,n,k}
\end{aligned}
\tag{P.-11}
$$

Now $Q_{t-n-k,n,k} = \alpha^t\beta^{\ell-n-k}\overline{\alpha}^k\overline{\beta}^{\ell+n-t}\cdot z_1^{\ell+t-n-k}z_1^{\ell-t+n+k}$ and we have proven (P.-8).

We divide the last line above by $[(\ell+n)!(\ell-n)!]^{1/2}$ and get $(\alpha z_1 - \overline{\beta}z_2)^{\ell+n}(\beta z_1+\overline{\alpha}z_2)^{\ell-n}/[(\ell+n)!(\ell-n)!]^{1/2}$. From (P.-6)

$$
\begin{aligned}
&\sum_{m=-\ell}^{\ell} \alpha^{m+n}(-\beta^*)^{\ell-m}\beta^{\ell-n}\left(\sum_{k=0}^{\ell-n}\left(\begin{array}{c}\ell+n\\ m+n+k\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)\left(-\left|\frac{\alpha}{\beta}\right|\right)^k\right)\frac{z_1^{\ell+m}z_1^{\ell-m}}{\sqrt{(\ell+n)!(\ell-n)!}}\\
= &\sum_{m=\ell}^{-\ell}\frac{z_1^{\ell+m}z_2^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}\,\pi_\ell(g)_{mn} \qquad \text{(P.-11)}
\end{aligned}
$$

We can then read off the matrix elements (P.12) .

□

Examples

$\ell = 0, 1/2, 1, 3/2, \ldots$

For $2\ell =$ to an even integer, the representatioon is the $2\ell+1$ dimensional tensorial representation of $SO(3)$. For $2\ell =$ to an odd integer, π_ℓ is a spinor representation. For matter, 1/2 describe elementary particles of half-integer spin.

(1) $\ell = 1/2$

$$
\begin{aligned}
\pi_{1/2}(g)_{\frac{1}{2}\frac{1}{2}} &= \alpha\sum_{k=0}^{0} = \alpha\\
\pi_{1/2}(g)_{-\frac{1}{2}-\frac{1}{2}} &= \alpha^{-1}(-\overline{\beta})\beta\sum_{k=0}^{1}\frac{1}{(k-1)!(1-k)!(1-k)!k!}\left(-\left|\frac{\alpha}{\beta}\right|\right)^k = \overline{\alpha}\\
\pi_{1/2}(g)_{-\frac{1}{2}\frac{1}{2}} &= \,= \beta \qquad \text{(P.-12)}
\end{aligned}
$$

$$
\pi_{1/2}\left(\begin{array}{cc}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{array}\right) = \left(\begin{array}{cc}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{array}\right) \qquad \text{(P.-12)}
$$

(2) $\ell = 1$:

$$
\pi_1\left(\begin{array}{cc}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{array}\right) = \left(\begin{array}{ccc}\alpha^2 & \sqrt{2}\alpha\beta & \beta^2\\ -\sqrt{2}\alpha\overline{\beta} & |\alpha|^2-|\beta|^2 & \sqrt{2}\overline{\alpha}\beta\\ \overline{\beta}^2 & -\sqrt{2}\overline{\alpha}\overline{\beta} & \overline{\alpha}^2\end{array}\right) \qquad \text{(P.-12)}
$$

Unitarity of the representation

the transpose $g \longrightarrow g^T$

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \to \pi_\ell(g^T)_{mn} = \pi_\ell(g)_{nm} = (\pi_\ell(g)^T)_{mn} \tag{P.-12}$$

the complex conjugate $g \longrightarrow \overline{g}$

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \to \pi_\ell(\overline{g})_{mn} = (\overline{\pi_\ell(g)})_{mn} \tag{P.-12}$$

Combining both operations $g \to g^\dagger$, this induces $\pi_\ell(g^\dagger) = \pi_\ell(g)^\dagger$ and so we can write

$$\pi_\ell(g)^\dagger = \pi_\ell(g^\dagger) = \pi_\ell(g^{-1}) = \pi_\ell(g)^{-1}. \tag{P.-12}$$

Therefore the representation is unitary.

□

According to Euler's theorem, every rotation R in $\mathbb{R}^3$ can be written as $R = R_3(\phi)R_2(\theta)R_3(\psi)$, see fig (P.4).

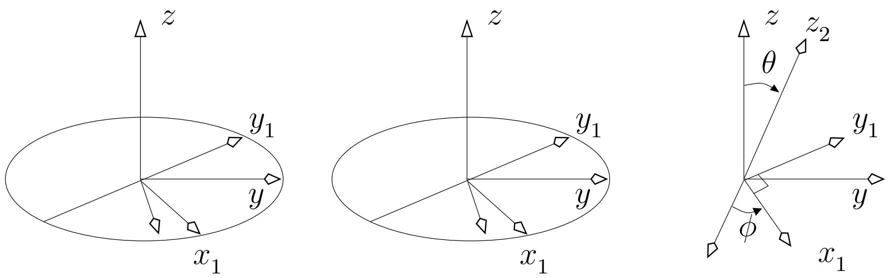


Figure P.4: EulerTherm.

$$\begin{aligned} \pi_{\frac{1}{2}}(\phi,\theta,\psi) &= \exp\left(-i\frac{\phi}{2}\sigma_3\right)\exp\left(-i\frac{\theta}{2}\sigma_2\right)\exp\left(-i\frac{\psi}{2}\sigma_3\right) \\ &= \begin{pmatrix} \exp(-i\frac{\phi}{2}) & 0 \\ 0 & \exp(i\frac{\phi}{2}) \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} \exp(-i\frac{\psi}{2}) & 0 \\ 0 & \exp(i\frac{\psi}{2}) \end{pmatrix} \end{aligned} \tag{P.-13}$$

Together

$$\begin{pmatrix} \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & -\exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \\ \exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \end{pmatrix} \tag{P.-13}$$

Irreducibility

According to Schur's lemma, the representation is irreducible if the matrix which commutes with all the elements of the representation is a constant matrix. This we will use to prove the representations (labelled by ℓ) are irreducible.

Consider special case in (P.12) $\alpha = e^{-im\theta/2}$ $\beta = 0$

$$\pi_\ell(0, 0, R_3(\phi))_{mn} = \delta_{mn} e^{-im\phi} \tag{P.-13}$$

where $m, n = \ell, \ell - 1, \cdots - \ell + 1, -\ell$,

$$(e^{-in\theta} - e^{-im\theta}) M_{nm} = 0. \tag{P.-13}$$

implying that M is diagonal

Thus any matrix M that commutes with the representation is a constant matrix.

□

Completeness

i.e. any function $f(g)_{mn}$ can be expanded in terms of elements $\pi_\ell(g)_{mn}$ of the irreducible elements with summation over labelled by ℓ,

$$f(g)_{mn} = \sum_\ell c_\ell \, \pi_\ell(g)_{mn} \tag{P.-13}$$

analagous to a Fourier transform corresponding to the abelean group $U(1)$ whose irreducible representations are $e^{i\ell\phi}$

$$f(\phi) = \sum_{\ell=-\infty}^{\infty} c_\ell \, e^{i\ell\phi} \tag{P.-13}$$

where ℓ is an integer labelling the irreducible representations.

character $\chi^{(\ell)}(\theta)$

$$\chi^{(\ell)}(\theta) = \sum_{m=-\ell}^{\ell} = \frac{\sin\left(j + \frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)} \tag{P.-13}$$

□

Clebsch-Gordon coefficients.

Vector addition

$$\chi^{(j_1)}\chi^{(j_2)} = \sum_{m_2=-j_2}^{j_2} e^{-im_2\theta} \sum_{m_1=-j_1}^{j_1} e^{-im_1\theta} \tag{P.-12}$$

Set $m = m_1 + m_2$, and assume $j_1 \geq j_2$ without loss of generality. Then

$$\begin{aligned}
\chi^{(j_1)}\chi^{(j_2)} &= (e^{+j_2\theta} + \cdots + e^{-j_2\theta})\left(\frac{e^{-i(j_1+1)\theta} - e^{ij_1\theta}}{e^{-i\theta} - 1}\right) \\
&= \frac{e^{-i(j_1+j_2+1)\theta} - e^{i(j_1+j_2)\theta}}{e^{-i\theta} - 1} + \cdots + \frac{e^{-i(j_2-j_1+1)\theta} - e^{i(j_2-j_1)\theta}}{e^{-i\theta} - 1} \\
&= \chi^{(j_1+j_2)} + \chi^{(j_1+j_2-1)} + \cdots + \chi^{(j_1-j_2)}
\end{aligned} \tag{P.-13}$$

Thus

$$\pi_{j_1} \otimes \pi_{j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \pi_j \tag{P.-13}$$

For example

$$\pi_{1/2} \otimes \pi_{1/2} = \pi_0 \oplus \pi_1 \tag{P.-12}$$

The composition of two spin haaalf particle is the direct sum of a scalar (singlet) and a spin one (doublet). Or two $j = 1/2$ edges of a spin network shares a tri-valent vertex with either a $j = 0$ or $j = 1$ edge.

$$\pi_1 \otimes \pi_1 = \pi_0 \oplus \pi_1 \oplus \pi_2 \tag{P.-11}$$

$$\begin{aligned}
\mathbf{J}^2\psi(j,m) &= j(j+1)\psi(j,m) \\
J_z\psi(j,m) &= m\psi(j,m)
\end{aligned} \tag{P.-11}$$

Appendix Q

Consistent Discrete Classical and Quantum General relativity

Q.1 Introduction

Given data on an initial time slice which fulfill the constaint equations, will the constraint equations hold on later time slices? That is, do the evolution equations preserve the constraint equations?

The usual starting point of numerical general relativity are: the six evolution equations for h_{pq} and K_{pq}

$$\dot{h}_{pq} = 2NK_{pq} + \mathcal{L}_{N^p} h_{pq} \tag{Q.1}$$

$$\dot{K}_{pq} = N_{|pq} - N(R_{pq} + KK_{pq} - 2K^r{}_p K_{qr}) + \mathcal{L}_{N^p} K_{pq}. \tag{Q.1}$$

and the four constraint equations which put conditions on the initial data:

$$\begin{aligned} C &= R + K^2 - K^{ab}K_{ab} = 0 \\ C_m &= \nabla^a (K_{am} - Kq_{am}) = 0. \end{aligned} \tag{Q.1}$$

The discrete formulations of constrained systems are often *inconsistent*. The discrete equations you get cannot be solved simultaneously. If you solve the constraint equations at the beginning they will fail to be solved when you evolve according to the discrete evolution equations, so the discretized evolution equations produce solutions in the future that do not satisfy the discretized constraints. This is a well known problem in numerical relativity. The discrete theory is also inconsistent with regards to the Poisson bracket algebra, the discrete versions of the constraints fail to close as an algebra.

Pullin et al developed a genral technique allowing to define consistent discrete theroies. One can define a consistent discrete theory for general relativity. What you do is you discretize the **action** of the theory and then you work out a canonical theory for the discrete action. Instead of just taking the EQMs and disretizing them discretize the action. Since derrived from an action, they are going to be consistent!

The lapse and shift are not free but are determined by imposing the preservation of constraints.

The resulting theory is different from GR, yet it will generically include solutions that approximate continuum general relativity very well.

For cosmological models the generic behaviour far from the big bang approximates very well the continuum.

$$L(n, n+1) \equiv L(q_n, q_{n+1}) \equiv \epsilon \hat{L}(q, \dot{q}) \tag{Q.1}$$

$$q + q_0 \quad \text{and} \quad \dot{q} \equiv \frac{q_{n+1} - q_n}{\epsilon}. \tag{Q.1}$$

$$S = \sum_{n=0}^{N} L(q_n, q_{n+1}) \tag{Q.1}$$

$$\frac{\partial S}{\partial q_n} = \frac{\partial L(q_{n-1}, q_n)}{\partial q_n} + \frac{\partial L(q_n, q_{n+!})}{\partial q_n} = 0. \tag{Q.1}$$

$$p_{n+1} = \frac{\partial L(q_n, q_{n+!})}{\partial q_{n+1}}, \quad p_n = -\frac{\partial L(q_n, q_{n+!})}{\partial q_n}. \tag{Q.1}$$

Relations that define a type I canonical transformation

$$L(Q_n, q_{n+1}) = m \frac{(q_{n+1} - q_n)^2}{2\epsilon} - V(q_n)\epsilon \tag{Q.1}$$

$$
\begin{aligned}
p_{n+1} &= m\frac{(q_{n+1}-q_n)}{\epsilon} \\
p_n &= m\frac{(q_{n+1}-q_n)}{\epsilon} + V'(q_n)\epsilon \\
q_{n+1} &= q_n + \frac{p_n}{m}\epsilon - V'(q_n)\frac{\epsilon}{m} \\
p_{n+1} &= p_n - V'(q_n)\epsilon. \qquad \text{(Q.-1)}
\end{aligned}
$$

$$
U = \exp\left(i\frac{V(q_{i-1})\epsilon}{\hbar}\right)\exp\left(i\frac{p_{i-1}^2\epsilon}{2m\hbar}\right) = \exp H. \qquad \text{(Q.-1)}
$$

$$
\mathcal{H} = \epsilon\left(\frac{p^2}{2m} + \frac{1}{2m\omega^2 q^2} - \frac{1}{2\epsilon pq\omega^2}\right) f(\epsilon) \qquad \text{(Q.-1)}
$$

Canonical formulation for constrained discrete dynamical systems

$$
L(n, n+1) = p_n(q_{n+1} - q_n) - \epsilon\mathcal{H}(q_n, p_n) - \lambda_{nB}\phi^B(q_n, p_n) \qquad \text{(Q.-1)}
$$

$$
\begin{aligned}
P^q_{n+1} &= \frac{\partial L(n,n+1)}{\partial q_{n+1}} = p_n, \;\; P^q_n = -\frac{\partial L(n,n+1)}{\partial q_n} = p_n + \epsilon\frac{\partial \mathcal{H}(q_n,p_n)}{\partial q_n} + \lambda_{nB}\frac{\partial \phi^B(q_n,p_n)}{\partial q_n} \\
P^p_{n+1} &= \frac{\partial L(n,n+1)}{\partial p_{n+1}} = 0, \;\; P^p_n = -\frac{\partial L(n,n+1)}{\partial p_n} = -(q_{n+1}-q_n) + \epsilon\frac{\partial \mathcal{H}(q_n,p_n)}{\partial p_n} + \lambda_{nB}\frac{\partial \phi^B(q_n,p_n)}{\partial p_n} \\
P^{\lambda_B}_{n+1} &= \frac{\partial L(n,n+1)}{\partial \lambda_{(n+1)}B} = 0, \;\; P^{\lambda_B}_n = \phi^B(q_n,p_n).
\end{aligned}
\qquad \text{(Q.-3)}
$$

$$
\phi^B(q_n, P^q_{n+1}) = 0. \qquad \text{(Q.-3)}
$$

$$
\phi^B(q_{n+1}, P^q_{n+1}, \lambda_{nB}) = 0, \qquad \text{(Q.-3)}
$$

$$
\lambda_{nB} = \lambda_{nB}(q_{n+1}, P^q_{n+1}, v^\alpha) \qquad \text{(Q.-3)}
$$

The final evolution equations are obtained by substituting the Lagrangian multipliers.

Notice that here the Lagrange multipliers were determined without imposing any gauge fixing. Notice that more precisely what has been determined is "$\lambda \times \epsilon$".

For a completely parametrized theory is no explicit dependence on ϵ, which may be fixed arbitrarily. Once the time interval (or the lattice spacing) is chosen, lapse is determined.

Q.1.1 "Dirac's" canonical approach to general discrete systems

They have recently developed a "Dirac's" canonical approach to general discrete systems.

$$\begin{aligned} L(n, n+1) &\equiv L(q_n, q_{n+1}) \\ P^q{}_{n+1}{}^a &= \frac{\partial L(n,n+1)}{\partial q^a{}_{n+1}} \\ P^q{}_n{}^a &= -\frac{\partial L(n,n+1)}{\partial q^a{}_n} \end{aligned} \tag{Q.-4}$$

$$\left| \frac{\partial^2 L(n,n+1)}{\partial q^b{}_{n+1} \partial q^a{}_n} \right| = 0. \tag{Q.-4}$$

Primary constraints

$$\Phi_A(q^a{}_n, P^q{}_n{}^a) = 0 \tag{Q.-4}$$

$$q^a_{n+1} = f^a(q^b{}_n, P^q{}_n{}^b, V^A, U^A) \tag{Q.-4}$$

V and U arbitrary functions. Consistency:

$$\Phi^A(q^a{}_{n+1}, P^q{}_{n+1}{}^a) = \Phi_A(f^a{}_n, \frac{\partial L(q_n, f^a)}{\partial q^a{}_{n+1}}) = 0 \tag{Q.-4}$$

$$C(q_n, P^q_n) \qquad V^A = V^A(q_n, P^q_n, v^\alpha, u^\rho) \tag{Q.-4}$$

$$q^a_{n+1} = f^a(q^b{}_n, P^q{}_n{}^b, V^A(q, P^q_n, v, u), U^A(q, P^q_n, v, u)) = \tilde{f}^a(q^b_n, P^{qb}_n, v^\alpha, u^\rho) \tag{Q.-4}$$

$$P^q{}_{n+1}{}^b = g^a(q^b{}_n, P^q{}_n{}^b, v^\alpha, u^\rho). \tag{Q.-4}$$

By using a Type II, II, or IV transformation one can show that this evolution is canonical, preserves the Poisson brackets and the canstraint surface. This is equivalent to what happens in the continuum, consistency may be achieved by determing the complete constraint surface and a total Hamiltonian that preserves the Poisson structure.

Finally one can recognise the second class constraints and impose them strongly. While some symmetries of the continuum are broken the discretization others are preserved.

The procedures may be extended to quantum field theories and reproduces, for standard gauge results obtained by using tranfer matrix techniques on a lattice.

$$S = -\frac{a_0 a^3}{2} \sum_P \frac{S_P}{A_P^2} \mathrm{Tr}(U_P) \tag{Q.-4}$$

It also provides a very simple descrition of BF theories on a lattice

$$\begin{aligned} L(n, n+1) \;&=\; \sum_v \mathrm{Tr}\left[B^0_{n,v} h^1_{n,v} h^2_{n,v+e_1} (h^1_{n,v+e_2})^\dagger (h^2_{n,v})^\dagger + B^2_{n,v} V_{n,v} h^1_{n+1,v} (V_{n,v+e_1})^\dagger \right. \\ &- \left. B^1_{n,v} V_{n,v} h^2_{n+1,v} (V_{n,v+e_2})^\dagger (h^2_{n,v})^\dagger + \mu_{n,v}(V_{n,v} V^\dagger_{n,v} - I) + \sum_{k=1}^{2} \lambda^k_{n,v} (h^k_{n,v} (h^k_{n,v})^\dagger - I) \right]. \end{aligned} \tag{Q.-5}$$

For standard gauge theories as Y-M and BF the symmeties may be preserved under the discretization and there is not any need of determining Lagrange multipliers.

Q.1.2 Canonical discrete quantization of general relativity

The Lagrangiean of Euclidean general relativity in terms of the Ashtekar variables.

$$L = \int_\Sigma E^{ai} F^i_{a0} + \epsilon_{abc} [E^{bi} E^{cj} \epsilon^{ijk} N + N^b E^{ck}] F^k_{de} \epsilon^{ade}, \tag{Q.-5}$$

This Lagrangian can easily be discretized as follows:

$$\begin{aligned} L(n, n+1) \;&=\; \sum_v \mathrm{Tr}\left[E^a_{n,v} h^a_{n,v} V_{n,v+e_1} (h^a_{n+1,v})^\dagger (V_{n,v})^\dagger \right. \\ &+ \; K_{1,n,v} h^2_{n,v} h^3_{n,v+e_2} (h^3_{n,v+e_3})^\dagger (h^3_{n,v})^\dagger + cyc. \\ &+ \left. \alpha_{a,n,v} \left(h^a_{n,v} (h^a_{n,v})^\dagger - 1 \right) + \beta_{n,v} (V_{n,v} V^\dagger_{n,v} - 1) \right] \end{aligned} \tag{Q.-6}$$

Where h and V are $SU(2)$ holonomies

$$K_{a,n,v} = \epsilon_{abc}[E^b_{n,v}E^c_{n,v}N_{n,v} + N^b_{n,v}E^c_{n,v}]. \tag{Q.-6}$$

The canonical quantization is now straightforward. Each of the action variables

$$E^a_{n,v},\ h^a_{n,v},\ V_{n,v},\ \alpha^a_{n,v},\ \beta^b_{n,v},\ N_{n,v},\ N^b_{n,v} \tag{Q.-6}$$

will have canonical momenta and evolution equations given by conanical transformations concerning level n and $n+1$.

The $SU(2)$ gauge symmetry is exactly preserved by the diffeomorhism and Hhhhamiltonian constraints are solved determining the multipliers N, N^a as we discussed above.

The final degrees of freedom are the electric field and the $SU(2)$ holonomy and as the Gauss invariance is exactly reserved the final quantum theory may be in principle treated in terms of loops.

Q.1.3 Discrete Quantum Gravity Applied to Cosmology

A very simple example

$$L = E\dot{A} + \pi\dot{\phi} - NE^2(-A^2 + (\Lambda + m^2\phi^2)|E|) \tag{Q.-6}$$

The system has four hase space variables and one constraint. Therefore it has two independent observables $(\{\mathcal{O}, C\})$:

$$\mathcal{O}_1 = \phi, \quad \mathcal{O}_2 = \pi + \frac{2}{3}\frac{m^2\phi}{\Lambda + m^2\phi^2}AE. \tag{Q.-6}$$

$$L(n, n+1) = E_n(A_{n+1} - A_n) + \pi_n(\phi_{n+1} - \phi_n) - N_nE^2_n(-A^2_n + (\Lambda + m^2\phi^2_n)|E_n|) \tag{Q.-6}$$

The evolution is given by the canonical transformation generated by the by L, the Lagrange multilier at each step is determined by the preservation of the constraint. The final discrete evolution equations are:

$$\Theta = \Lambda + m^2\phi^2, \tag{Q.-6}$$

$$
\begin{aligned}
N_n &= \frac{[-P^A_n\Theta + A^2_n]\Theta}{2A^5_n}.\\
P^A_{n+1} &= A^2_n\Theta^{-1}\\
A_{n+1} &= A_n + \frac{A^2_n - P^A_n\Theta}{2A_n}\\
\phi_{n+1} &= \phi_n\\
P^\phi_{n+1} &= P^\phi_n - (A^2_n - P^A_n\Theta)A_n m^2\phi_n\Theta^{-2}.
\end{aligned}
\tag{Q.-9}
$$

$A^2_n - P^A_n\Theta$ is a measure of the step of the discretization.

There are two constants of motion, that are the discrete counterpart of the observables

$$
O_n = \phi_n \qquad O'_n = P^\phi_n + \frac{2}{3}\frac{m^2\phi_n}{\Theta}A_n P^A_n, \quad O_{n+1} = O_n. \tag{Q.-9}
$$

Notice that the discrete theory has four phase space degrees of freedom instead of the two of the continuum theory. The additional degrees of freedom charactrize the step of the discretization and encode remnants of the gauge invariance in the discrete theory.

Although the graphs suggest that the triad goes to zero at $n = 0$ and therefore one has a singularity this is not the case.

We here show the approach to the singularity in the discrete and continuum case. The discrete theory has a small but non-vanishing triad a $n = 0$. R. Gambini and J. Pullin gr-qc/0212033

The rate of contraction/expansion changes when going through the big crunch/bang. Question: is that a remnent of the reparametrization invariance or does it have physical consequences?

The answer to this question is related with the existence of more degrees of freedom and therefore more constants of motion. In fact, the discrete canonical transformation is singuluar for $A = 0$. If one tries to introduce a generator of this evolution:

$$
A_{n+1} = A_n + \{A_n, \mathcal{H}_n\} + \frac{1}{2!}\{\{A_n, \mathcal{H}_n\}, \mathcal{H}_n\} + \cdots \tag{Q.-9}
$$

$$
\mathcal{H}_n = \frac{\mathcal{C}^2_n}{4\Theta A_n}\left[1 + \sum_{k=1}^{\infty} a_k \left(\frac{\mathcal{C}_n}{A^2_n}\right)^k\right] \tag{Q.-9}
$$

H_n diverges for $\left(\frac{\mathcal{C}_n}{A^2_n}\right)^k > 2$. This happens for $n = 0$ when the system goes through the singularity.

H_n, which is constant on each region characterizes the spacing of the discretization in an invariant way, and in that sense suggest a procedure for taking the continuum limit.

It may provide a mechanism for changing the values of the fundamental constants?

Quantization

$$\psi_{n+1}[A, \phi] \tag{Q.-9}$$

$$< A_1, \phi_1 | U | A_2, \phi_2 >= \sqrt{\frac{2|A_2|}{\Theta}} \exp(-iA_2^2(A_1 - A_2) sg A_2) \tag{Q.-9}$$

It is very easy to obtain the unitary evolution operator from the discrete action. The inner product is also trivially defined in this space. The elimiation of the constraints simplify all the quantization process.

The elimination of the constraints simplifies the quantization and allows to treat in a simlper way old conceptual problems as the issue of time.

Notice that the evolution variables n does not have any intrinsic meaning and it is not associate with any dynamical variable. n cannot be taken as a clock variable and it is not and observable quantity.

$$S = \int \left[\dot{q} + p_0 \dot{q}^0 - N(p_0 + \frac{p^2}{2m} + \lambda q) \right] d\tau, \tag{Q.-9}$$

$$L(n, n+1) = p^n(q_{n+1} - q_n) + p_0^n(q_{n+1} - q_n^0) - N_n(p_0^n + \frac{p_n^2}{2m} \lambda q_n). \tag{Q.-9}$$

the conditional robability to obtain $q = x$ given $q^0 = t$

$$(q = x | q^0 = t) = \frac{\sum_{n=-\infty}^{\infty} |\Psi(x, t, n)|^2}{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx |\Psi(x, t, n)|^2} \tag{Q.-9}$$

One can show that this relational description recovers usual quantum mechanics when the discrete approximation approaches the continuum limit and when the clock variable behaves sufficiently to a classical clock.

The procedure can be extended to situations where the simultaneity surfaces are not transverse to the classical orbits of the system.

The system has an unitary evolution in n. At t cannot br perfectly correlated with n, even in the semi-classical regime of the clock, the evolution in t is not perfectly unitary. In fact one can show that the density matrix evolves according to

$$\frac{\partial}{\partial t}\rho_2 = -i[\mathcal{H}_2, \rho_2(t)] - \sigma[\mathcal{H}_2, [\mathcal{H}_2, \rho_2(t)]]. \qquad \text{(Q.-9)}$$

This equation was first proposed by Milburn based on phenomenological arguments, and is a particular tpye of non-unitary evolutions considered by Lindblad.

Their derivation allows to estimate σ that is of order of the Planck time.

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t} \qquad \text{(Q.-9)}$$

This equation doesnot violate the conservation of energy like Hawking prosal for information loss. One could exect to confirm this type of equation by studying some mesoscopic quatum systems.

Information loss problem in Black Holes

It provides a new and very effective mechanism for treating the information loss problem in Black Holes. It eliminates the puzzle as a fundamental question.

They have shown that for any Black hole bigger than 600 Plank masses the information loss induced by our equation is enough to dissipate all the black hole information before to its evaporation.

For very small black holes, Hawking's semi-classical analysisis not valid.

Bose-Einstein Experiment of Quantum Gravity Decoherence

Plank-scale-induced deviations whose detection is a function of the number of particles.

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t} \qquad \text{(Q.-9)}$$

One could exect to confirm this type of equation by studying some mesoscopic quatum systems.

Q.2 Semi-Discrete Approach

Significant departure from what everyone else was doing in LQG. They could not make use of of kinematic tools of LQG, like spin-network states, Astekar-Lewendowski diff-invariant measure, ... Thiemann paper *"One way out could be to look at constraint quantization from an entirely new point of view which proves useful also in discrete formulations of classical GR, that is, numerical GR. While being a fascinating possibility, such a procedure would be a rather drastic step in the sense that it would render most results of LQG obtained so far obsolete."*, [77].

discrete time but keep space continuous in the classical action. But would be not be a discord with GR in which space and time on same footing?

Get coupled non-linear PDEs.

Same kinematics but deal with spacial constraint in the usual way.

$$\begin{aligned} S = & \int dtd^3x \left[Tr \left(\tilde{P}^a (A_a(x) - V(x) A_{n+1,a}(x) V^{-1}(x) + \partial_a (V(x)) V^{-1} \right) \right. \\ - & \left. N^a C_a - NC + \mu Tr(V(x)V^\dagger(x) - 1) \right] \end{aligned} \quad \text{(Q.-9)}$$

Q.3 Biblioliographical notes

In this chapter I have relied on the following refferences:

[449]

Q.4 Worked Exercies and Details

Proofs

Appendix R

Quntum Gravity Phenomenology

"An accelerator powerful enough to study ... Planckian objects would have to be as large as the entire galaxy."

- Leonard Susskind 2006.

phenomenology comes from small deviations away from classical effects. In LQG the techniques for obtaining the classical limit of the theory have not yet been developed.

R.1 Gamma ray bursts

Each burst is as powerful as a billion trillion suns

might see a lag in photon arrival times as they travel through the

Time delay Interferomtry [gr-qc/0409034].

cosmic rays and photons from gamma-ray bursts are used to probe the structure of spacetime at the Planck scale. This is because the discrete geometry is expected to modify energy-momentum relations at very high energies, thereby affecting the propagation of particles and photons over cosmological distances [349]

Below the Plank scale the classical picture of spacetime breaks down. Eisnsten's special theory of relativity is part of the classical picture, so we migh expect it to breakdown at this point.

R.2 Bose-Einstein Experiment of Quantum Gravity Decoherence

Plank-scale-induced deviations whose detection is a function of the number of particles.

[346]

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t} \tag{R.0}$$

One could exect to confirm this type of equation by studying some mesoscopic quatum systems.

R.3 Nonlinearities may be Observable in the Next Generation of Molecular Interferometry Experiments

While it is widely believed that gravity should ultimately be treated as a quantum theory, there remains a possibility that general relativity should not be quantized. If this is the case, the coupling of classical gravity to the expectation value of the quantum stress-energy tensor will naturally lead to nonlinearities in the Schrodinger equation. By numerically investigating time evolution in the nonrelativistic "Schrodinger-Newton" approximation, we show that such nonlinearities may be observable in the next generation of molecular interferometry experiments.

R.4 Spectrum of Fluctuations in Singularity-free Inflation

The control that loop quantum gravity has made it possible to compute predictions for real observations. They have been able to derive precise predictions for quatum gravity effects that may be seen in future observations of the cosmic microwave background, [348].

R.5 Formation and Evolution of Structure in Loop Cosmology

Abstract:

Inhomogeneous cosmological perturbation equations are derived in loop quantum gravity, taking into account corrections in particular in gravitational parts. This provides a framework for calcu- lating the evolution of modes in structure formation scenarios related to inflationary or bouncing models. Applications here are corrections to the Newton potential and to the evolution of large scale modes which imply non-conservation of curvature perturbations possibly noticeable in a run- ning spectral index. These e ects are sensitive to quantization procedures and test the characteristic behavior of correction terms derived from quantum gravity

Loop quantum gravity is one of the approaches where singularity resolution has been investigated using loop quantum cosmology [?] which results in the resolution of singularities in various situations including inhomogeneous ones [??]. Semiclassical bounce pictures in special models have been described in [??]. A key role is played by the underlying quantum nature of spatial geometry [??]. With such a discrete structure underlying classical space-time, effects not captured by low energy effective theory become possible. In particular, there are large dimensionless parameters, such as the number of spatial lattice sites in a discrete state, which can always spoil dimensional arguments. In such a context, orders of magnitude of quantum corrections can only be estimated with a detailed analysis of the effective equations arising from quantum gravity. Suitable techniques going beyond low energy effective theory are now available and are applied here.

R.6 Physical effects of the Immirzi parameter

[351]. Effects that are observable in principle, even independent of non-perturbative qunatum gravity.

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[412] State Sum Models for Quantum Gravity IGPG Seminar by Prof. John Barrett from University of Nottingham Thursday at 1:00 PM in (8/27/1998)

[413] Internal Logic of Casual Sets: What the Universe Looks Like from the Inside Relativity Seminar by Dr. Fotini Markopoulou from Penn State Thursday at 1:00 PM in (10/29/1998)

couldn't listen to this one - software problem

[414] Group Averaging: A Uniqueness Theorem IGPG Seminar by Dr. Donald Marolf from Syracuse University Monday at 1:00 PM in (11/2/1998)

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[426] Quantum Cosmology Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (11/8/2002)

haven't listened to this one yet

[427] Towards QFT on Quantum Geometry I: Ideas, Formalism, Conceptual Issues, Open Questions Gravity Seminar by Dr. Hanno Sahlmann from Penn State University Friday at 11:00 AM in 318 Osmond Laboratory (12/6/2002)

haven't listened to this one yet

[428] Towards QFT on Quantum Geometry II: Some Concrete Calculations Gravity Seminar by Dr. Hanno Sahlmann from Penn State University Friday at 11:00 AM in 318 Osmond Laboratory (12/13/2002)

[429] Black Hole Entropy: Inclusion of Distortion and Angular Momentum Gravity Seminar by Dr. Abhay Ashtekar from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/17/2003)

[430] Modeling the Low energy Limit of Loop Quantum Gravity: Shadow States and Quantum Mechanics Gravity Seminar by Josh Willis from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/24/2003)

[431] Quantum Cosmology: An Overview IGPG Seminar by Dr. Martin Bojowald from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (1/27/2003)

[432] Quantum Cosmology: Formalism Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/31/2003)

haven't listened to this one yet

[433] Quantum Gravity: Overview and Status IGPG Seminar by Dr. Abhay Ashtekar from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (2/3/2003)

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[434] Quantum Cosmology: Applications Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (2/7/2003)

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[435] Representations of the Sahlmann Star Algebra Gravity Seminar by Dr. Jerzy Lewandowski from University of Warsaw (Poland) Friday at 11:00 AM in 318 Osmond Laboratory (2/28/2003)

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[436] How Black Holes Grow Gravity Seminar by Dr. Badri Krishnan from Albert Einstein Institute Friday at 11:00 AM in 318 Osmond Laboratory (4/11/2003)

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[437] Diffeomorphism Invariance in the Smooth Category Gravity Seminar by Christian Fleischhack from Leipzig University and Penn State Monday at 3:00 PM in 318 Osmond Lab (4/28/2003)

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[438] An Update on Dynamical Horizons: Applications and Open Issues Gravity Seminar by Abhay Ashtekar from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (9/19/2003)

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[439] 2+1 Gravity and the Physical Scalar Product from Spin Foams Gravity Seminar by Alejandro Perez from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (10/24/2003)

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[440] Quadrupole Moments in Isolated Horizons Gravity Seminar by Jonathan Engle from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (10/31/2003)

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[441] Dynamics in Loop Quantum gravity as a Sum Over Histories of Quantum Geometry IGPG Seminar by Alejandro Perez from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (1/26/2004)

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[442] Discrete Quantum Gravity: Recent Developments and Applications Gravity Seminar by Rodolfo Gambini from University of Uruguay Friday at 11:00 AM in 318 Osmond Laboratory (2/6/2004)

[443] Master Constraint Programme for Loop Quantum Gravity Gravity Seminar by Thomas Thiemann from Perimeter Institute Monday at 12:00 PM in 318 Osmond Laboratory (2/9/2004) Audio, Presentation

[444] Testing the Master Constraint Programme for Loop Quantum Gravity Gravity Seminar by Bianca Dittrich from Perimeter Institute Friday at 11:00 AM in 318 Osmond Laboratory (2/13/2004)

[445] Goals and Achievements of Twistor Theory IGPG Seminar by Sir Roger Penrose from Oxford University and Penn State Monday at 3:00 PM in 318 Osmond (4/12/2004)

[446] Black Hole Entropy Gravity Seminar by Jerzy Lewandowski from University of Warsaw (Poland) Friday at 11:00 AM in 318 Osmond (9/3/2004)

[447] Black Hole Evaporation and Information Loss: Recent Advances IGPG Seminar by Abhay Ashtekar from Penn State Monday at 3:00 PM in 318 Osmond (9/20/2004)

[448] Free Vacuum for Loop Quantum Gravity Gravity Seminar by Florian Conrady from University of Marseille Friday at 9:00 AM in 318 Osmond (10/22/2004)

[449] Jorge Pullin, *Consistent discretization for Classical and Quantum General Relativity*, (17/01/05).

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