Appendix C

Basic Mathematics

C.1 Introduction

This appendix contains a miscellaneous assortment of mathematical ideas and results. those topics of relevance to the subject of the report.

C.1.1 Summary of Tensor Calculus

- Linear spaces.
- Curvilinear coordinates
- Curved spaces.
- Vector and tensors: covariant and contravriant.
- Affine and metric connections.
- Differential manifolds.

C.1.2 Linear operators and Matrices

A linear operator T maps a vector space V onto itself which obeys linearity

$$T(a\hat{x} + b\hat{y}) = aT(\hat{x}) + bT(\hat{y}). \tag{C.1}$$

If we have a basis $\{\hat{e}_i, i = 1, \ldots, n\}$ for V, then

$$\hat{x} = \sum_{i=1}^{n} a_i \hat{e}_i \tag{C.2}$$

$$\begin{aligned} T(\hat{x}) &= T\left(\sum_{i=1}^{n} a_i \hat{e}_i\right) = \sum_{i=1}^{n} a_i T(\hat{e}_i) \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j T_{ij} \hat{e}_j, \end{aligned} \tag{C.3}$$

where we have replaced each vector $T(\hat{e}_i)$ by its component form $\sum_{j=1}^{n} T_{ij}\hat{e}_j$.

Two successive linear transformations T and U acting on a space V produce the transformation UT:

$$\begin{aligned} UT(\hat{x}) &= U(T(\hat{x})) = U\left(\sum_{j=1}^{n} a_i T_{ij} \hat{e}_j\right) \\ &= \sum_{j=1}^{n} a_i T_{ij} U(\hat{e}_j) \\ &= \sum_{j=1}^{n} a_i T_{ij} U_{jk} \hat{e}_k \\ &= \sum_{j=1}^{n} a_i \left(T_{ij} U_{jk}\right) \hat{e}_k \end{aligned} \tag{C.4}$$

From which it follows that the components of UT are

$$\sum_{j=1}^{n} T_{ij} U_{jk}. \tag{C.5}$$

We see that this sum is just the matrix product of the respective matrices.

C.2 Group Theory

A group G is defined as a set of objects or operations (called elements) that may be combined or multiplied to form a well defined product and that satisfy the following four conditions. If we label the elements $a, b, c, \ldots,$ then the conditions are:

(1) If a and b are any two elements, then the product ab is an element.

(2) The defined multiplication is associative, $(ab)c = a(bc)$

(3) There is a unit element I, with $Ia = aI = a$ for all elements a.

(4) Each element has an inverse $b = a^{-1}$, with $aa^{-1} = a^{-1}a = I$ for all elements a.

In physics, these abstract conditions will take on a physical meaning, for example in terms of transformations of vectors and tensors.

C.2.1 Examples of Groups

Translation in time: $t' = t + a^0$
Translation in space: $\vec{r}' = \vec{r} + \vec{a}$
Rotation in space : $\vec{r}' = R\vec{r}$

Lie groups

Roughly, a Lie group is a group with an infinite number of elements but which can be parametrized by one or several real numbers (the total of which is refered to as the *dimension* of the Lie group).

The simplest example of a Lie group is $SO(2)$, the group of rotations in the plane. Each element $R(\theta)$ is labelled by rotation angel θ, with multiplication acting as $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$. Because the angle θ is defined only modulo 2π, the manifold of $SO(2)$ is a circle S^1.

interesting properties of Lie groups is that in neighbourhood of the identity element they can be expressed in terms of a set of generators T^a as

$$D(g) = \exp(-\alpha_a T^a) := \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \alpha_{a_1} \dots \alpha_{a_n} T^{a_1} \dots T^{a_n}, \tag{C.6}$$

where $\alpha_a \in \mathbb{C}$ are a set of coordinates of M in a neighbourhood of $\not\Vdash$.

Internal symmetries

for e.g. $SU(2)$.

finite-dimensional Lie groups of imporance relativity translations, rotations, Lorentz, Poincare and special unitary group $SU(2)$. Apart from these finite-dimentional Lie groups, the infinie-dimensional ones play an important role in GR. These are the Lie group of *diffeomorphisms* of the spacetime manifold.

two groups are isomorphic if there is a one-to-one correspondence between their elements and if they have exactly the same structure.

The neighbourhood of a group element is chararterized by the neighbourhood of the corresponding parameter set.

Cosets

Let G be a group and H be a subgroup of G. The set of the form

$$gH = \{gh : h \in H\} \tag{C.7}$$

for $g \in G$ is called the (left) coset of H in G. There is one coset for each $g \in G$.

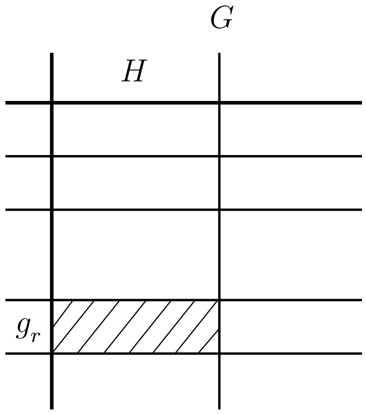


Figure C.1: coset. Suppose H is a subgroup of a finite group G, here the elements of H are listed first. The shaded box are the elements of the (left) coset of $g_r \in G$

As we will see now, cosets define an equivalence on G, where $g \sim g'$ if the set gH is equal to the set $g'H$, in other words there is an $h \in H$ with $gh = g'$.

Cosets have the property that if cosets overlap they are the same coset, i.e., that for $g_1, g_2 \in G$, $g_1 \neq g_2$

$$g_1 H = g_2 H \tag{C.8}$$

or that different cosets do not overlap

$$g_1 H \cap g_2 H = \emptyset. \tag{C.9}$$

Proof: Suppose there is an overlap between the two cosets corresponding to the elements g_1 and g_2, that is, for some $h_1, h_2 \in H$ we have

$$g_1 h_1 = g_2 h_2.$$

Therefore

$$g_1 = g_2 h_2 h_1^{-1}.$$

If h is *any* element of H then

$$g_1 h = g_2 h_2 h_1^{-1} h = g_2 h'$$

where $h' = h_2 h_1^{-1} h$. Since H is a subgroup h' is also an element of H. Now since no element of G appears more than once in each row, if cosets do overlap at all they are in fact the same coset.

proving (C.8).

□

Right cosets are defined similarly, they are the sets

$$Hg = \{hg : h \in H\}. \tag{C.10}$$

These also define an equivalence realtion on the group G.

The equivalence class of a particular representative g is denoted

$$[g]. \tag{C.11}$$

Now the set of cosets gH, under the binary operator

$$[g] \cdot [g'] = [gg']. \tag{C.12}$$

forms a group if and only if H is what's known as a normal subgroup.

Definition Normal subgroups.

A normal N subgroup of a group G is

$$h \in H \text{ if } ghg^{-1} \in H \text{ for all } g \in G. \tag{C.13}$$

We write $N \lhd G$ to indicate that N is a normal subgroup of G.

We need to establish that induced multiplication between equivalence classes is independent of the representative chosen, i.e., that $[g_1'][g_2'] = [g_1][g_2]$, where g_1 is any element of $[g_1']$ and similarly for g_2.

$$[g_1'][g_2'] = [g_1h_1][g_2h_2] = [g_1h_1g_2h_2] = [g_1g_2hh_2] = [g_1g_2h] = [g_1g_2] = [g_1][g_2]. \tag{C.14}$$

$$[hg] = [h][g] = [e][g] = [eg] = [g] = [ge] = [g][e] \tag{C.15}$$

$[e] = [h]$

$$[e] = [gg^{-1}] = [g][g^{-1}] = [g][g]^{-1} \tag{C.16}$$

$[g]^{-1} = [g^{-1}]$

This makes G/N into a group, called the quotient group.

Definition The **Quotient group** is the subgroup is the collection of cosets, each being considered an element. The quotient group is denoted G/H.

We will be interested in other mathematical things such as algebras, rings, categories amongt others. There are analogous notions of the normal subgroups for each of these goining by the names of ideals, , quotient categories.

Simple and Semi-simple Lie Groups

A Lie group is called simple if it does not possess an invariant subgroup.

A Lie group is called semi-simple if it does not possess an abelian invariant subgroup. Note that a semi-simple subgroup can possess a non-abelian invariant sugroup.

An invariant subset is an ideal if

$$[A_i, G_j] = \sum a_{ijk} A_k \tag{C.17}$$

Cartan killing form k_{ij} built from the structure constants

$$k_{ij} = C_{iab} C_{jba} \tag{C.18}$$

Cartan killing form

The killing form k_{ab} is nondegenerate if the Lie algebra is semisimple

$$\det(k_{ij}) \neq 0. \tag{C.19}$$

Proof:

we have to demonstrate that $\det(k_{ij}) = 0$ if there is an abelian ideal. suppose that the Lie algebra has an abelian ideal. generators belonging to the ideal will be labelled marked indices. Then for the b' column of the killing form,

$$\det(k_{ij'}) = C_{iab} C_{j'ba} = C_{ia'b} C_{j'ba'} \tag{C.20}$$

as by (C.17) $C_{j'ba} = 0$ for those values of a which do not belong to the ideal. it follows from (C.20) that

$$\det(k_{ij'}) = -C_{a'ib} C_{j'ba'} = -C_{a'ib'} C_{j'b'a'} \tag{C.21}$$

since $C_{a'ib} = 0$ for all $b \neq b'$ the same reason. $[A_i, A_j] = 0$ for an abelian ideal so

$$C_{j'b'a'} = 0$$

holds. As the j' column of the killing form vanishes it follows that

$$\det(k_{ij}) = 0. \tag{C.22}$$

Cartan's condition can be stated that if a Lie algebra is semisimple its killing form is invertible, that is, there is an inverse $k^{ij} := (k_{ij})^{-1}$ for which

$$k^{ik} k_{kj} = \delta^i_j$$

holds.

C.2.2 Unitary Representations of Groups

R	Group elements.
$[G]$	The collection of group elements
R_a	An indexed group element.
$\alpha, \beta, \ldots$	Labels irreducible representations.
D	Matrix representation.
$D^{(\alpha)}$	Indexed irriduceable matrix representation.
$i, j, \ldots$	Labels components of matrix representations
h	Dimension of group.
χ	Character

Proof Let $\{A_a\}$ be a representation of the group G. Construct the Hermitian matrix $H = \sum_{a=1}^{h} A_a A_a^\dagger$

$$H^\dagger = [\sum_a^h (A_a A_a^\dagger)]^\dagger = \sum_a^h (A_a A_a^\dagger)^\dagger = \sum_a^h A_a A_a^\dagger = H \tag{C.23}$$

From the earlier discussion of matrix algebra, any Hermitian matrix can be diagonalized by a similarity transformation. Let

$$d = \sum_a^h U^\dagger A_a A_a^\dagger U = \sum_\alpha (U^\dagger A_a U)(U^\dagger A_a^\dagger U) = \sum_\alpha (U^\dagger A_a U)(U^\dagger A_a U)^\dagger \tag{C.24}$$

Hence, $d = \sum_a \tilde{A}_a \tilde{A}_a^\dagger$ where $\tilde{A}_a = U^\dagger A_a U = U^{-1} A_a U$. The elements of the diagonal d matrix are real and positive.

$$d_{jj} = \sum_a^h \sum_k (\tilde{A}_a)_{jk} (\tilde{A}_a^\dagger)_{kj} = \sum_a^h \sum_k (\tilde{A}_a)_{jk} (\tilde{A}_a)^*_{jk} = \sum_a^h \sum_k |(\tilde{A})_{jk}|^2 \tag{C.25}$$

for all $j = 1, \ldots, h$. Since d is diagonal, we can define its square-root $d^{1/2}$

$$d^{1/2} = \begin{pmatrix} d_{11}^{1/2} & 0 & \cdots & 0 \\ 0 & d_{11}^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{hh}^{1/2} \end{pmatrix} \tag{C.26}$$

and $d^{-1/2}$, which is given by an analogous expression. Evidently, $(d^{1/2})^2$ and $d^{1/2} d^{-1/2} = 1$, where I is the identity matrix. Diagonal matrices commute with each other, and we can write for the identity I

$$I = d^{-1/2} d d^{-1/2} = d^{-1/2} \sum_a^h \tilde{A}_a \tilde{A}_a^\dagger \, d^{-1/2}. \tag{C.27}$$

By the rearangement theorem, $\{\tilde{A}_b \tilde{A}_a\}$ all a, and any one b is equal to $\{\tilde{A}\}$ for all a. Hence,

$$\begin{aligned} I &= d^{-1/2} \sum_a (\tilde{A}_b \tilde{A}_a) \, (\tilde{A}_b \tilde{A}_a)^\dagger d^{-1/2} \\ &= d^{-1/2} \sum_a (\tilde{A}_b \tilde{A}_a \; \tilde{A}_a^\dagger \tilde{A}_b^\dagger) d^{-1/2} \\ &= d^{-1/2} \tilde{A}_b d^{1/2} \left(d^{-1/2} \sum_a \tilde{A}_a \tilde{A}_a^\dagger d^{-1/2} \right) d^{1/2} \tilde{A}_b^\dagger d^{-1/2} \end{aligned} \tag{C.28}$$

using (C.27) in this gives

$$\begin{aligned} I &= (d^{-1/2} \tilde{A}_b d^{1/2})(d^{-1/2} \tilde{A}_b d^{1/2})^\dagger \\ &= (d^{-1/2} U^{-1} A_b U d^{1/2})(d^{-1/2} U^{-1} A_b U d^{1/2})^\dagger. \end{aligned} \tag{C.29}$$

Let us define the matrix $B_a := d^{-1/2} U^{-1} A_a U d^{1/2}$. It is easy to see that B_a has the same multiplication table as A_α. If we have $A_a A_b = A_c$, then

$$\begin{aligned} B_a B_b &= (d^{-1/2} U^{-1} A_\alpha U d^{1/2})(d^{-1/2} U^{-1} A_b U d^{1/2}) \\ &= (d^{-1/2} U^{-1} A_a) \, (U d^{1/2} d^{-1/2} U^{-1}) \, (A_b d^{1/2} U) \\ &= (d^{-1/2} U^{-1} \; A_a A_b \; U d^{1/2}) \\ &= (d^{-1/2} U^{-1} A_c U d^{1/2}) = \; B_c \end{aligned} \tag{C.30}$$

Therefore $\{B_a\}$ is a unitary reprsentation of the group G.

Representation of finite groups can always be taken to be unitary. It is essential that the sum over $g \in G$ converge. This is guaranteed for a finite group, but may not work for infinite groups. In particular, non-compact Lie groups, such as the Lorentz group, have no finite dimensional unitary representations.

C.2.3 Schur's First Lemma

Theorem C.2.1 *A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix.*

Proof:

We can take these matrices to be unitary without loss of generality. Suppose there is a matrix M that commutes with all of the A_a but which is not a constant multiple of the unit matrix:

$$MA_a = A_a M \tag{C.31}$$

for all $a = 1, 2, \ldots, |G|$. By taking the edjoint of each of these equations, we obtain

$$A_a^\dagger M^\dagger = M^\dagger A_a^\dagger. \tag{C.32}$$

Since A_a are unitary

$$A_a^{-1} M^\dagger = M^\dagger A_a^{-1}. \tag{C.33}$$

Multiplying both this from left and right we obtain

$$M^\dagger A_a = A_a M^\dagger \tag{C.34}$$

So that, if M commutes with every matrix of the representation, then so does $M^\dagger$. As such the Hermitian matries

$$H_1 = M + M^\dagger, \quad H_2 = i(M - M^\dagger) \tag{C.35}$$

also commute with every matrix of the representation. We prove the statement of the theorem for Hermitician matrices. We start with

$$HA_a = A_a H. \tag{C.36}$$

Since a hermitian matrix H can be diagonalized by some U, $D = U^{-1}HU$.

$$U^{-1}HUU^{-1}A_aU = U^{-1}A_aUU^{-1}HU, \tag{C.37}$$

Let $\tilde{A}_a = U^{-1} A_a U$

$$D\tilde{A}_a = \tilde{A}_a D. \tag{C.38}$$

A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix.

$$(\tilde{A}_a)_{ij}(D_{ii} - D_{jj}) = 0. \tag{C.39}$$

(i) Suppose all the diagonal elements of D are distinct: $D_{ii} \neq D_{jj}$ if $i \neq j$. Then (C.39) implies that

$$(\tilde{A}_a)_{ij} = 0, \qquad i \neq j,$$

i.e., the off-diagonal elements of $\tilde{A}_\alpha$ must vanish. They form a reducible representatoin composed of d one-dimentional representations. Since the $\tilde{A}_a$ are obtained from the A_a by a similarity transformation, the A_a are a reducible representaion.

(ii) Suppose p of the diagonal elements of D are equal, but the remaining entries are distinct from these and from each other. Using similarity transfromations, (for example (C.40)), we can arrange the diagonal elements so that the first p are equal.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{C.40}$$

Hence, we can assume this is the case without loss of generality, i.e. $D_{11} = D_{22} = \cdots = D_{pp}$, $D_{mm} \neq D_{nn}$, otherwise. The $(\tilde{A}_i)_{mn}$ must vanish for any pair of unequal diagonal entries.

$$\tilde{A}_i = \begin{pmatrix} B_1 & 0 \\ 0 & B_2, \end{pmatrix} \tag{C.41}$$

where B_1 is a $p \times p$ matrix and B_2 is a $(p-d) \times (p-d)$ diagonal matrix.

We have shown that if a Hermtian matrix is not a multiple of the unit matrix and commutes with all the matrices of a representation, then that representation is necessarily reducible. Thus, if a non-zero Hermtian matrix which commutes with all the matrices of an irreducible representation that matrix must be a multiple of the unit matrix.

Given that both $M + M^\dagger$ and $i(M - M^\dagger)$ are both Hermitician matrices that commute with all of the matrices of an irreducible representation is a constant multiple of the unit matrix. we have $H_1 = M + M^\dagger = c_1 I$ and $H_2 = i(M - M^\dagger) = c_2 I$. We have that

$$M = H_1 - iH_2 = (c_1 - ic_2)I$$

and hence must be proportional to the unit marix.

□

C.2.4 Schur's Second Lemma

Theorem C.2.2 *Let $\{A_1, A_2, \ldots, A_{|G|}\}$ and $\{B_1, B_2, \ldots, B_{|G|}\}$ be two irreducible representations of a group G of dimensionalities d an d' respectively. If there is a matrix M such that*

$$MA_a = B_a M$$

for $a = 1, 2, \ldots, |G|$, then if $d = d'$, either $M = 0$ or the two representations are related by a similarity transformation. If $d \neq d'$, then $M = 0$.

As

$$MA_a = B_a M$$

then

$$A_a^\dagger M^\dagger = M^\dagger B_a^\dagger$$

using unitary

$$A_a^{-1} M^\dagger = M^\dagger B_a^{-1}. \tag{C.42}$$

Multiplying on the right by M, we get

$$A_a^{-1} M^\dagger M = M^\dagger B_a^{-1} M. \tag{C.43}$$

By the group properties B_a^{-1} is some B_b and A_a^{-1} is the corresponding A_b, and so by the posulate of the theorem, $MA_b = B_b M$, or

$$MA_b^{-1} = B_b^{-1}M$$

substituting this into (C.43), gives

$$A_a^{-1}M^\dagger M = M^\dagger M A_a^{-1}. \tag{C.44}$$

Thus, the $d' \times d'$ matrix $MM^\dagger$ commutes with all the matrices of an irreducible representation. By Schur's first lemma, $MM^\dagger$ must therefore be a constant multiple of the unit matrix,

$$MM^\dagger = cI. \tag{C.45}$$

(i) Say $d = d'$. If $c \neq 0$, equation (C.45) implies that

$$M^{-1} = \frac{1}{c}M^\dagger.$$

plus (C.42) can be rearranged

$$A_a = M^{-1}B_a M,$$

so the two representations are related by a similarity transfromation and are, therefore, equivalent and since $M^\dagger M = c^d$,it follows that $c^{-d'/2}M$ is a unitary matrix. If $c = 0$ then

$$(MM^\dagger)_{ij} = \sum_k M_{ik}(M^\dagger)_{kj} = \sum_k M_{ik}M^*_{jk} = 0. \tag{C.46}$$

By setting $i = j$, we obtain

$$\sum_k M_{ik}{M^*}_{ik} = \sum_k |M_{ik}|^2 = 0 \tag{C.47}$$

which implies that $M_{ik} = 0$ for all i and k, i.e., that M is the zero matrix.

(ii) Say $d \neq d'$. We take $d < d'$. Them M is a rectangular matrix with d columns and d' rows,

$$\begin{pmatrix} M_{11} & \cdots & M_{1d} \\ M_{21} & \cdots & M_{2d} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ M_{d'1} & \cdots & M_{d'd}. \end{pmatrix} \tag{C.48}$$

We add $d' - d$ columns of zeros

$$N = \begin{pmatrix} M_{11} & \cdots & M_{1d} & 0 & \cdots & 0 \\ M_{21} & \cdots & M_{2d} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & & \vdots & & \vdots \\ M_{d'1} & \cdots & M_{d'd} & 0 & \cdots & 0 \end{pmatrix} \tag{C.49}$$

Taking the adjoint of this matrix yields

$$N^\dagger = \begin{pmatrix} M_{11}^* & M_{21}^* & \cdots & M_{d'1} \\ M_{12}^* & M_{22}^* & \cdots & M_{d'2} \\ \vdots & & \cdots & \vdots \\ M_{d1}^* & M_{2d}^* & \cdots & M_{d'd}^* \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{C.50}$$

We have

$$NN^\dagger = MM^\dagger = cI.$$

Taking the determinant

$$\det(NN^\dagger) = \det(N)\det(N^\dagger) = c^{d'} = 0.$$

so $c = 0$, so that, as before we arrive at

$$\sum_k |N_{ik}|^2 = 0. \tag{C.51}$$

which implies that $N_{ik} = 0$ for all i and k, i.e. N is the zero matrix. Hence, $M = 0$.

$\square$

C.2.5 Orthogonality relations

Let $D^{(\alpha)}(R)_{ij}$ be the (i,j) matrix elements of the α^{th} irreducible unitary representation of the element R of G. The range of R will be the h elements of G. The range of α will be the number of inequivalent irreducible representations. The range of i and j will be the dimension d_α of the α^{th} representation.

Theorem:

$$\sum_R D^{(\alpha)}(R)^*_{i'j'} D^{(\beta)}(R)_{ij} = \frac{h}{d_\alpha}\delta_{\alpha\beta}\delta_{ii'}\delta_{jj'} \tag{C.52}$$

Proof:

Let

$$M = \sum_R D^{(2)}(R) X D^{(1)}(R^{-1}). \tag{C.53}$$

$$\begin{aligned} D^{(2)}(S)M &= \sum_R D^{(2)}(S)\left[D^{(2)}(R)XD^{(1)}(R^{-1})\right] D^{(1)}(S^{-1})D^{(1)}(S) \\ (S)M &= \sum_R \left[D^{(2)}(S)D^{(2)}(R)\right] X \left[D^{(1)}(R^{-1})D^{(1)}(S^{-1})\right] D^{(1)}(S) \end{aligned} \tag{C.54}$$

If $SR = T$, then

$$\begin{aligned} D^{(\alpha)}(S)D^{(\alpha)}(R) = D^{(\alpha)}(SR) = D^{(\alpha)}(T) \;\; \text{and} \\ D^{(\alpha)}(R^{-1})D^{(\alpha)}(S^{-1}) = D^{(\alpha)}(R^{-1}S^{-1}) = D^{(\alpha)}(T^{-1}) \end{aligned} \tag{C.55}$$

$$\begin{aligned} D^{(2)}(S)M &= \sum_T D^{(2)}(T)XD^{(1)}(T^{-1})D^{(1)}(S) \\ &= MD^{(1)}(S) \;\; \text{for all S.} \end{aligned} \tag{C.56}$$

Hence by theorem ands its corollary, $M = 0$. Now choose X to be a matrix every element of which is zero except one, which we shall call $X_{j'i}$, $X_{j'i} = 1$.

$$\begin{aligned}
M = 0 &= \sum_R D^{(2)}(R) X D^{(\alpha)}(R^{-1}) \\
0 &= \sum \sum_R D^{(2)}(R)_{i'l} X_{kl} D^{(2)}(R^{-1})_{lj} \\
0 &= \sum_R D^{(2)}(R)_{i'j'} D^{(2)}(R^{-1})_{ij} \\
0 &= \sum_R D^{(2)}(R)^*_{ji} D^{(2)}(R^{-1})_{i'j'}
\end{aligned} \tag{C.57}$$

The last step follows from the unitarity of the representation.

$$D^{(\alpha)}(R^{-1})_{ij} = [D^{(\alpha)}(R^{-1})]^{-1}_{ij} = D^{(\alpha)}(R)^*_{ji} \tag{C.58}$$

This proves the theorem for $i \neq j$

$$\sum_i D^{(\alpha)}_{ii}(R) \sum_j D^{(\beta)}_{jj}(R)^* = g\delta_{\alpha\beta} \tag{C.59}$$

case(b): $i = j = 1$

We define $M = cI = \sum_R D^{(1)}(R) D^{(1)}(R^{-1})$. Then just as in case (a)

□

C.2.6 The Characters of a Representation

First Orthogonal Relation

If α and β are irreducible and N_k is the number of elements in c_k,

then

$$\sum_k \chi^{(\alpha)}(c_k)^* \chi^{(\beta)}(c_k) N_k = h\delta_{\alpha\beta} \tag{C.60}$$

Proof:

Starting with the orthogonality relatioins, theorem and setting $i = i'$ and summing over i and j, we get

$$\sum_R \chi^{(\alpha)*}(R)\chi^{(\beta)}(R) - \frac{h}{l_\alpha}\delta_{\alpha\beta}\underbrace{\sum_{i,j}\delta_{ij}\delta_{ij}}_{=l_\alpha} \tag{C.61}$$

$$\sum_R \chi^{(\alpha)}(c_k)^*\chi^{(\beta)}(c_k)N_k = h\delta_{\alpha\beta}. \tag{C.62}$$

The Second Orthogonality Relation If $D^{(\alpha)}$ is irreducible, then

$$\sum_\alpha \chi^{(\alpha)}(c_k)^*\chi^{(\alpha)}(c_l) = \frac{h}{N_l}\delta_{kl} \tag{C.63}$$

$$\chi = \sum_\alpha a_\alpha \chi^{(\alpha)} \tag{C.64}$$

$$\begin{aligned} \chi \cdot \lambda &= \sum_\alpha a_\alpha \chi(c_k)\cdot\chi(c_\beta) \\ &= \sum_\alpha a_\alpha h\delta_{ij} = ha_j \end{aligned} \tag{C.65}$$

Hence,

$$\begin{aligned} \chi &= \sum_\alpha \frac{\chi\cdot\chi^{(\alpha)}}{h}\chi^{(\alpha)} \\ &= \chi\cdot\sum_\alpha \frac{\chi^{(\alpha)}\chi^{(\alpha)}}{h} \end{aligned} \tag{C.66}$$

$$\chi(c_l) = \sum_{k,\alpha} N_k\chi(c_k)\chi^{(\alpha)}(c_k)\chi^{(\alpha)}(c_l)\frac{1}{h} \tag{C.67}$$

$$0 = \sum_k \chi(c_k)\left[\frac{N_k}{h}\sum_\alpha \chi^{(\alpha)}(c_k)\chi^{(\alpha)}(c_l) - \delta_{kl}\right] \tag{C.68}$$

C.2.7 Direct Products

Details Direct Product of matrices and vectors.

By definition the vector product of two 2-by-2 matrices is

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \tag{C.69}$$

A more explicit form

$$A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{12}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \tag{C.70}$$

$$C = \begin{pmatrix} c_{11;11} & c_{11;12} & c_{12;11} & c_{12;12} \\ c_{11;21} & c_{11;22} & c_{12;21} & c_{12;22} \\ c_{21;11} & c_{21;12} & c_{12;11} & c_{22;12} \\ c_{21;21} & c_{21;22} & c_{22;21} & c_{22;22} \end{pmatrix} \tag{C.71}$$

In general the direct product C of two matrices A and B is defined in terms of matrix elements by

$$a_{ij}b_{kl} = c_{ik;jl}. \tag{C.72}$$

The row and column labels of C are ***composite*** labels:

$$M_{i1}, M_{i,2}, \dots, M_{1,n} \tag{C.73}$$

whereas the as one goes along a row the read

$$c_{ik;1,1},\ c_{ik;1,2},\ \dots,\ c_{ik;1,n},\ c_{ik;2,1},\ c_{ik;2,2},\ \dots,\ c_{ik;2,n,},\ \dots,\ c_{ik;n,1},\ c_{ik;n,2},\ \dots,\ c_{ik;n,n,}. \tag{C.74}$$

the row label, is obtained from the

We prove these direct products have the same operations as matries.

Products

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{C.75}$$

$$\begin{aligned}
(A \otimes B)(C \otimes D) &= \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \begin{pmatrix} c_{11}D & c_{12}D \\ c_{21}D & c_{22}D \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}c_{11} + a_{12}c_{21})BD & (a_{11}c_{12} + a_{12}c_{22})BD \\ (a_{21}c_{11} + a_{22}c_{21})BD & (a_{21}c_{12} + a_{22}c_{22})BD \end{pmatrix} \\
&= AC \otimes BD
\end{aligned} \tag{C.76}$$

$$D \times D \tag{C.77}$$

$$\chi^{(\alpha\times\beta)} = \sum_{ij} D^{(\alpha\times\beta)}_{(ii)(jj)}(G_a) = \sum_i D^{(\alpha)}_{ii}(G_a) \sum_j D^{(\beta)}_{jj}(G_a) \tag{C.78}$$

The character of the representation of the direct product is equal to the product of the characters of the original representations α and β, which implies that

$$\chi^{(\alpha\times\beta)} = \chi^{(\alpha)} \cdot \chi^{(\beta)}. \tag{C.79}$$

Thr representations resulting from (C.77) is in general reducible, that is

$$D^{(\alpha\times\beta)}(G_a) = \oplus_\gamma m_\gamma D^\gamma(G_a) \tag{C.80}$$

$$m_\gamma = \frac{1}{g} \sum_p c_p \chi^{(\gamma)*} \chi^{(\alpha)}_p \chi^{(\beta)}_p. \tag{C.81}$$

C.3 Continuous Groups, Lie Groups and Lie algebras

We are only interested in symmetery transformations are all based on *contiuous* quantities determing how summations over group elements are carried out.

The neighbourhood of a group element is characterized by the neighbourhood of the corresponding parameter set.

symetries e.g. rotational each point in three spacial manifold).

Instead of having to consider the group as a whole, for many purposes it is sufficient to consider the an infinitesimal transformation around the identity. Any finite transformation can then be constructed by the reapeted application of this infinitesimal transformation.

We will the properties of Lie groups and algebras in terms of specific examples, especially the two and three dimensional rotation groups and the Lorentz group.

C.3.1 Infinitesmal Generating Technique

A function

$$f(x+\delta x) = f(x) + \delta x \frac{df(x)}{dx} \tag{C.82}$$

$$f(x+2\delta x) = f(x+\delta x) + \delta x \frac{df(x+\delta x)}{dx} = f(x) + 2\delta x \frac{df(x)}{dx} + \delta x \frac{d^2 f(x)}{dx^2} \tag{C.83}$$

Binomial type expansion

$$f(x+N\delta x) = \sum_{r=0}^{N} \frac{N!}{r!(N-r)!} \delta x^r \frac{d^r f(x)}{dx^r} \tag{C.84}$$

In the limit $N \to \infty$ the factor $N!/(N-r)$ can be replaced by N^r. We put $N\delta x = a$ where, in the limit $N \to \infty$ a is a finite number.

$$f(x+a) = \sum_{r=0}^{\infty} \frac{a^r}{r!} \frac{d^r f(x)}{dx^r} = f(x) + a\frac{df(x)}{dx} + \frac{a^2}{2!}\frac{d^2 f(x)}{dx^2} + \ldots \tag{C.85}$$

which we can formally write

$$f(x+a) = \exp\left(a\frac{d}{dx}\right) f(x) \tag{C.86}$$

Rotation operators

$$R(\delta\theta) = \begin{pmatrix} 1 & d\theta \\ -d\theta & 1 \end{pmatrix} = 1 + id\theta J \quad \text{where } \; J = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{C.87}$$

$$J^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{C.88}$$

$$
\begin{aligned}
R(\theta) &= \left(1+\frac{i\theta}{N}J\right)^N = \exp\left(i\theta J\right) = I + iJ + \frac{(i\theta)^2}{2!}J^2 + \frac{(i\theta)^3}{3!}J^3 + \\
&= I\left[1-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\ldots\right] + iJ\left[1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\ldots\right] = I\cos\theta + iJ\sin\theta \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
\end{aligned} \tag{C.89}
$$

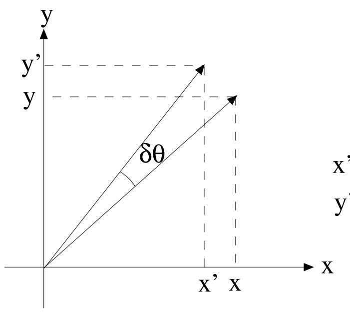


Figure C.2: infintesmal rotation.

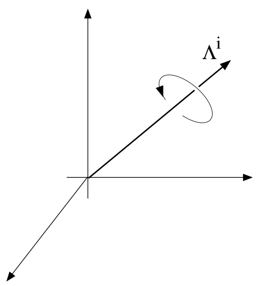


Figure C.3: infintesmal rotation.

$$
\mathbf{r}' = \mathbf{r} + \delta\phi \times \mathbf{r} \tag{C.90}
$$

expressed in component form

$$
x_I' = x_i + \epsilon_{ijk}x_k = (\delta_{ik} + \epsilon_{ijk})x_k \tag{C.91}
$$

$$
\hat{R} = \begin{pmatrix} 1 & \delta\phi_z & \delta\phi_y \\ -\delta\phi_z & 1 & \delta\phi_x \\ \delta\phi_y & -\delta\phi_x & 1 \end{pmatrix} \tag{C.92}
$$

$$\left(\mathbf{I} - \frac{i}{\hbar}\delta\phi \cdot \hat{\mathbf{L}}\right) \tag{C.93}$$

where

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{C.94}$$

Thus, given two vector angles α and β, there must exist a third one, γ, such that

$$\exp(i\alpha \cdot \mathbf{J}) \exp(i\beta \cdot \mathbf{J}) = \exp(i\gamma \cdot \mathbf{J}). \tag{C.95}$$

For this to be so the matrices must have to satisfy some condition, which can be found by considering the angles α^i and β^i as small.

$$U(\delta\alpha_\mu) = I + i\alpha^i J_i - \frac{1}{2}\sum_{i,j} \alpha^i\alpha^j J_i J_j + \dots \quad , \tag{C.96}$$

as J_i are *unitary*, i.e. $U^\dagger(\alpha^i) = U^{-1}(\alpha^i)$, then from() we obtain

$$U^\dagger(\alpha^i) I - i\alpha^i J_i^\dagger = U^{-1}(\alpha^i) = I - i\alpha^i J_i. \tag{C.97}$$

from this follows the *Hermiticity of the generators*

$$U_i^\dagger = U_i \tag{C.98}$$

Next we calculate the inverse operator to second order in α^i

$$U^{-1}(\alpha^i) = I - i\alpha^i J_i - \frac{1}{2} \tag{C.99}$$

$$I + i(\alpha^i J_i + \beta^i J_i) - \frac{1}{2}(\alpha^i J_i + \beta^i J_i)^2 = I + i(\alpha^i J_i + \beta^i J_i) - \frac{1}{2}(\alpha^i\alpha^j + \alpha^i\beta^j + \alpha^j\beta^i + \beta^i\beta^j) J_i J_j \alpha^i\beta^i [J_i, J_j] + \dots \tag{C.100}$$

$$+ \dots = i(\alpha^i + \beta^i) J_i - \frac{1}{2} \tag{C.101}$$

the commutator must be a linear combination of the J_i, that is

$$[J_i, J_j] = C_{ij}^k J_k. \tag{C.102}$$

The generators of *any* Lie group must have such commutation relations. The coefficients C_{ij}^k are called the *structure constants*

The product of two rotations $\exp(-iT_y)\exp(-iT_z)$ can always be written as a single exponential, say $\exp(-i\alpha\cdot T)$ where $\alpha\cdot T := \alpha_x T_x + \alpha_y T_y + \alpha_z T_z$. Suppose we set $\exp(-i\alpha\cdot T)\exp(-i\beta\cdot T) = \exp(-i\gamma\cdot T)$ and try to calculate γ in terms of α and β. If we expand the exponentials we find

$$\begin{aligned}[1 - i\alpha\cdot t - \frac{1}{2}(\alpha\cdot t)^2 \quad &+ \quad \dots][1 - i\beta\cdot t - \frac{1}{2}(\beta\cdot t)^2 + \dots] \\ &= \quad \exp(-i(\alpha+\beta)\cdot t - \frac{1}{2}[\alpha\cdot t, \beta\cdot t] + \dots)\end{aligned} \tag{C.103}$$

(and this is known as the Campbell-Baker-Hausdorff theorem - see Appendix S). It is for this reason that we can learn all we need to know about Lie groups by studying the commutation algebras of the generators

C.3.2 General Structure of Lie Groups

An infinite group is a group that contains an infinite nuber of elements. The rotation group is an example of such a group.

$$SO(3) = \exp(-i\hat{\phi}\cdot J). \tag{C.104}$$

The fact that these matrices are functions of only three fundamental matrices $\{\hat{J}_k\} = \{\hat{J}_1\hat{J}_2\hat{J}_3\}$ allows us to represent them in a simple way.

The set of group elements are characterized by a set of continuous real parameters.

A continuous group G is said to be *compact* if the parameter space is finite and non-compact if it is infinite. The rotation group $SO(3)$ is an example of a compact group with the Lorentz group $SO(3,1)$ as an example of a non-compact group.

$$\begin{aligned}[A,[B,C]] &= A[B,C] - [B,C]A \\ &= ABC - ACB - BCA + CBA\end{aligned} \tag{C.105}$$

We now add and subtract the quantities BAC and CAB on the right-hand side of this equation and rearange the resulting expression

$$\begin{aligned}
[A,[B,C]] &= ABC - ACB - BCA + CBA \\
&\quad -BAC + BAC + CAB - CAB \\
&= -C(AB - BA) + (AB - BA)C \\
&\quad +B(AC - CA) - (AC - CA)B \\
&= -[C,[A,B]] - [B,[C,A]]
\end{aligned} \tag{C.106}$$

A simple rearrangement yields the *Jacobi identity*:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \tag{C.107}$$

The important properties of the structure constants are the following

(i) They are antisymmetric in their lower indices

$$C^k_{ij} = -C^k_{ji} \tag{C.108}$$

(ii) The Jacobi identity defined by the infinitesimal generators (e.g. for the rotation group: $A = J_k,\ B = J_l,\ C = J_m$) leads to the condition on the scructure constants

$$C^n_{kl}C^p_{mn} + C^n_{lm}C^p_{kn} + C^n_{mk}C^p_{ln} = 0 \tag{C.109}$$

C.3.3 Rotations SO(3) and SU(2)

By a *representation* we mean a set of matrices T_x, T_y , and T_z with the same commutation relations as the t's. The T's of Eqs() and () are an examples in which the matrices are 3×3 and the representation is said to be of dimension three.

We recall the construction in standard quantum mechanics lectures.

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2 \tag{C.110}$$

$$\hat{J}^2\psi_{jm} = j(j+1)\psi_{jm}, \quad \hat{J}_3\psi_{jm} = m\psi_{jm} \tag{C.111}$$

It is convenient to define

$$J_+ = iJ_1 - J_2, \text{ and its complex conjugate } J_- = -iJ_1 - J_2 \tag{C.112}$$

the commutation relations become

$$[J_z, J_+] = J_+, \quad [J_z, J_-] = -J_-, \quad [J_+, J_-] = 2J_z. \tag{C.113}$$

$$J_3\psi_1 = J_3J_+\psi = [J_3, J_+]\psi + J_+J_3\psi \tag{C.114}$$

$$J_+J_- = J^2 - J_3(J_3 - 1) \tag{C.115}$$

$$J_-J_+ = J^2 - J_3(J_3 + 1) \tag{C.116}$$

$$J_+J_-\psi_{jm} = [j(j+1) - m(m-1)] \tag{C.117}$$

$$J_-J_+\psi_{jm} = [j(j+1) - m(m+1)] \tag{C.118}$$

$$U = e^{-i\theta_i\sigma_i} \tag{C.119}$$

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.120}$$

$$\tau^{\mathbf{i}}\tau^{\mathbf{j}} - \tau^{\mathbf{j}}\tau^{\mathbf{i}} = \epsilon_{ijk}\tau^{\mathbf{k}} \tag{C.121}$$

$SU(2)$ is the universal cover of $SO(3)$

By definition $SU(2)$ is the group of 2×2 special unitary matricies with determinant one. If

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$ then the requirements $P{-}1 = P^*$ and $\det P = 1$ translates to

$$\overline{a} = d \text{ and } \overline{b} = -c.$$

That is

$$P = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

and the condition that $\det P = 1$ gives the condition

$$a\overline{a} + b\overline{b} = 1$$

This says that P is fully determined by a vector $(a, b) \in \mathbb{C}^2$ of length one. If we write a, b in terms of their real and imaginary parts, then the above condition becomes

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

that is the unit sphere S^3 in $\mathbb{R}^4$, which is simple connected as the unit sphere S^2 in $\mathbb{R}^3$ is.

Details The Pauli matrices form a vector space

$$\tau_i = \frac{\hat{\sigma}_i}{2} \tag{C.122}$$

They form linear independent complete vector space for 2×2 matrices

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \hat{\sigma}^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{C.123}$$

$$\begin{aligned} & a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + bI = \mathbf{a} \cdot \sigma \\ & = \begin{pmatrix} b + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & b - a_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \mathbf{A} \end{aligned} \tag{C.124}$$

$$TrI = 2, \qquad Tr\sigma_i = 0 \tag{C.125}$$

Hence,

$$A = \sum_i \hat{\sigma}_i \frac{1}{2} Tr(A\hat{\sigma}_i) + (\frac{1}{2} TrA)I \tag{C.126}$$

C.3.4 Spin Direct Products

A subspace V is said to be **invariant** if it is mapped into itself by application of each matrix element of the group.(refine definition) A less trival example of a reduciable representation is the "addition of angular momentum" in quantum mechanics

We combine two irreduciable representations by forming the product space $V = V_1 \otimes V_2$.

$$\begin{aligned}
\hat\sigma^x_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \\
\hat\sigma^y_1 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & & -i & \\ & & & -i \\ i & & & \\ & i & & \end{pmatrix}, \\
\hat\sigma^z_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},
\end{aligned} \tag{C.127}$$

and

$$\begin{aligned}
\hat\sigma^x_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \\
\hat\sigma^y_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{pmatrix}, \\
\hat\sigma^z_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.
\end{aligned} \tag{C.128}$$

Note that

$$\left[\hat\sigma^i_1, \hat\sigma^j_2\right] = 0 \tag{C.129}$$

for any $i, j \in \{x, y, z\}$ as can be shown directly or better still using the rule for multiplying direct products of matrices :$(A \otimes B)(C \otimes D) = (AC \otimes BD)$,

$$\left[\hat{\sigma}_1^i, \hat{\sigma}_2^j\right] = (\hat{\sigma}^i \otimes I)(I \otimes \hat{\sigma}^j) - (I \otimes \hat{\sigma}^j)(\hat{\sigma}^i \otimes I) = (\hat{\sigma}^i \otimes \hat{\sigma}^j) - (\hat{\sigma}^i \otimes \hat{\sigma}^j) = 0 \tag{C.130}$$

These operators act on the direct product space $V \otimes V$, with, elements

$$\eta \otimes \omega = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \otimes \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \eta_1\omega_1 \\ \eta_1\omega_2 \\ \eta_2\omega_1 \\ \eta_2\omega_2 \end{pmatrix} \tag{C.131}$$

$$\begin{aligned}
|\psi_{(1,1)} > &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
|\psi_{(1,0)} > &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
|\psi_{(1,-1)} > &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
|\psi_{(0,0)} >= &= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned} \tag{C.132}$$

This result

$$S := \frac{1}{2}(\hat{\sigma} \otimes I + I \otimes \hat{\sigma}) \tag{C.133}$$

are generators of a *reducible* 4-dimensional reperentation of $\int\sqcap(2)$. There exists a unitary matrix U such that

$$USU^\dagger = \begin{pmatrix} S_{(S=1)} & \\ & S_{(S=0)} \end{pmatrix}, \tag{C.134}$$

where $S_{(S=1)}$ and $S_{(S=0)}$ representaions of dimension 3 and 1, respectively.

$$\begin{aligned} S^2 &= \frac{1}{4}(\hat{\sigma} \otimes I + I \otimes \hat{\sigma})^2 \\ &= \frac{1}{4}(\hat{\sigma}^2 \otimes I + 2\hat{\sigma} \otimes \hat{\sigma} + I \otimes \hat{\sigma}^2) \\ &= \frac{1}{2}\hat{\sigma} \otimes \hat{\sigma} + \frac{3}{2} I \otimes I, \end{aligned} \tag{C.135}$$

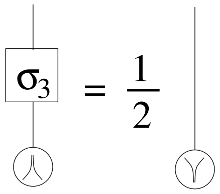
since $\hat{\sigma}^2 = 3I$.

$$\vec{S}^z = \hat{\sigma}^z \otimes I + I \otimes \hat{\sigma}^z \tag{C.136}$$

$$\vec{S}^z | S, S^z >= S^z | S, S^z > \tag{C.137}$$

$$\vec{S}^2 | S, S^z >= S(S+1) | S, S^z >, \quad S^z = -S, \ldots, S; \;\; S = 0, 1. \tag{C.138}$$

I will first go through the explicit calculation of the action of the z-component of the angular momentum, $\hat{J}_3$, and the total angular momentum, $\hat{J}^2$, so the reader may have a better feel for what is going on in the abstract diagrammatic calculation (and to appreciate the simplicity of the diagrammatic version). Moreover, I wish to go through the calculation for the total angular momentum operator both ways because of its importance in finding the (main sequence of) eigenvalues of the area operator (see appendix on geometric operators).



Details

$$\begin{aligned}
\omega_{AB\ldots F}(i=0) &= \delta_{A0}\delta_{B0}\ldots\delta_{E0}\delta_{F0} && \text{(C.139)}\\
\omega_{AB\ldots F}(i=1) &= \delta_{A1}\delta_{B0}\delta_{C0}\ldots\delta_{E0}\delta_{F0}\\
&\quad +\delta_{A0}\delta_{B1}\delta_{C0}\ldots\delta_{E0}\delta_{F0}\\
&\quad +\delta_{A0}\delta_{B0}\delta_{C1}\ldots\delta_{E0}\delta_{F0}+\ldots\\
&\vdots\\
\omega_{AB\ldots F}(i=2s) &= \delta_{A1}\delta_{B1}\ldots\delta_{E1}\delta_{F1} && \text{(C.140)}
\end{aligned}$$

$$\hat{\sigma}_z = \sum_{k=1}^{2s+1} 1\otimes\cdots\otimes 1\left(\frac{\hat{\sigma}^3}{2}\right)\otimes\cdots\otimes 1 \tag{C.141}$$

which written in component form reads

$$\begin{aligned}
\frac{1}{2}\hbar\hat{\sigma}^3_{AA'BB'\ldots EE'FF'} &:= \frac{1}{2}\hbar\hat{\sigma}^3_{AA'}\delta_{BB'}\ldots\delta_{FF'}\\
&\vdots\\
&\quad\ \frac{1}{2}\hbar\hat{\sigma}^3_{FF'}\delta_{AA'}\delta_{BB'}\ldots\delta_{EE'}\\
&\quad\ (AA';BB';\ldots;EE';FF'=0,1). && \text{(C.142)}
\end{aligned}$$

We verify that the "states" $\omega(i)$ fulfill the eigenvalue equation

$$\begin{aligned}
\hbar\frac{1}{2}\hat{\sigma}^3\omega(i) &= \hbar(s-i)\omega(i)\\
i &= 0,1,\ldots,2s. && \text{(C.143)}
\end{aligned}$$

Using

$$\hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.144}$$

and (C.142) we get

$$\begin{aligned}
\left(\frac{1}{2}\hbar\,\omega(i=0)\right)_{AB\ldots F} &= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'BB'\ldots EE'FF'}\ \omega_{A'B'\ldots E'F'}(i=0) \\
&= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'}\delta_{A'0}\delta_{B0}\ldots\delta_{E0}\delta_{F0} + \\
&\quad + \delta_{A0}\hbar\frac{1}{2}\hat{\sigma}^3_{BB'}\delta_{B'0}\ldots\delta_{E0}\delta_{F0} + \ldots \\
&\quad + \delta_{A0}\delta_{B0}\ldots\delta_{E0}\hbar\frac{1}{2}\hat{\sigma}^3_{FF'}\delta_{F'0} \\
&= \frac{\hbar}{2}2s(\delta_{A0}\delta_{B0}\ldots\delta_{F0}) = \hbar s\omega_{AB\ldots F}(i=0)
\end{aligned} \tag{C.145}$$

$$\begin{aligned}
\left(\frac{1}{2}\hbar\,\omega(i=1)\right)_{AB\ldots F} &= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'}\delta_{A'1}\delta_{B0}\ldots\delta_{E0}\delta_{F0} + \delta_{A1}\frac{\hbar}{2}\hat{\sigma}^3_{BB'}\delta_{B'0}\ldots\delta_{F0} + \\
&+ \cdots + \ \delta_{A1}\delta_{B0}\ldots\frac{\hbar}{2}\hat{\sigma}^3_{FF'}\delta_{F'0} + \frac{\hbar}{2}\hat{\sigma}^3_{AA'}\delta_{A'0}\delta_{B1}\ldots\delta_{F0} \\
&+ \delta_{A0}\frac{\hbar}{2}\hat{\sigma}^3_{BB'}\delta_{B'1}\ldots\delta_{F0} + \ldots \\
&+ \delta_{A0}\delta_{B1}\ldots\frac{\hbar}{2}\hat{\sigma}^3_{FF'}\delta_{F'1} + \ldots \\
&= \frac{\hbar}{2}\left[-\delta_{A1}\delta_{B0}\ldots\delta_{F0} + (2s-1)\times\delta_{A1}\delta_{B0}\ldots\delta_{F0} + \right. \\
&+ \delta_{A0}\delta_{B1}\ldots\delta_{F0} - \delta_{A0}\delta_{B1}\ldots\delta_{F0} + \\
&+ \left.(2s-3)\times\delta_{A0}\delta_{B1}\ldots\delta_{F0} + \ldots\right] \\
&= \hbar(s-1)\omega_{AB\ldots F}(i=1),
\end{aligned} \tag{C.146}$$

$$\begin{aligned}
&\vdots \\
\left(\frac{1}{2}\hbar\,\omega(i=2s)\right)_{AB\ldots F} &= \\
&= \hbar\frac{1}{2}\hat{\sigma}^3_{AA'}\delta_{A'1}\delta_{B1}\ldots\delta_{F1} + \delta_{A1}\frac{\hbar}{2}\hat{\sigma}^3_{BB'}\delta_{B'1}\ldots\delta_{E1}\delta_{F1} + \ldots \\
&+ \cdots + \delta_{A1}\delta_{B1}\ldots\frac{\hbar}{2}\hat{\sigma}^3_{FF'}\delta_{F'1} \\
&= -\hbar s\omega_{AB\ldots F}(i=2s).
\end{aligned} \tag{C.147}$$

Details K1.3

$$\hat{\sigma}^2 = \sum_{k=1}^{2s+1}\sum_{k'=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes \left(\frac{\hat{\sigma}^{k'}}{2}\right) \otimes \cdots \otimes 1 \tag{C.148}$$

$$(\hbar^2/4)\hat{\sigma}^2_{AA'BB'\ldots FF'} \tag{C.149}$$

has eigenvalue $\hbar^2 s(s+1)$

$$\left(\frac{1}{4}\hat{\sigma}^2\right)_{AA'BB'\ldots FF'} = \frac{1}{4} \tag{C.150}$$

$$\hat{\sigma}^2 = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k\hat{\sigma}^k}{4}\right) \otimes \cdots \otimes 1 \tag{C.151}$$

$$\hat{\sigma}^2_{AA'} = (\hat{\sigma}^1)^2_{AA'} + (\hat{\sigma}^2)^2_{AA'} + (\hat{\sigma}^3)^2_{AA'} = 3\delta_{AA'} \tag{C.152}$$

$$\begin{aligned}\left(\frac{1}{4}\hat{\sigma}^2\right)\omega(i=0) &= \left[\frac{3}{2}s + \frac{1}{4}(2s-1)2s\right]\omega(i=0) \\ &= s(s+1)\omega(i=0).\end{aligned} \tag{C.153}$$

$$\begin{aligned}\left(\frac{1}{4}\hat{\sigma}_{AA'}\hat{\sigma}_{BB'}\right)\delta_{A'1}\delta_{B'0} &= \\ &= \left(\hat{\sigma}^1_{AA'}\hat{\sigma}^1_{BB'} + \hat{\sigma}^2_{AA'}\hat{\sigma}^2_{BB'} + \hat{\sigma}^3_{AA'}\hat{\sigma}^3_{BB'}\right)\delta_{A'1}\delta_{B'0} \\ &= \frac{1}{4}\left((+1)\delta_{A0}\delta_{B1} + (+1)\delta_{A0}\delta_{B1} + (-1)\delta_{A1}\delta_{B0}\right)\end{aligned} \tag{C.154}$$

$$\begin{aligned}\left(\frac{1}{4}\hat{\sigma}_{AA'}\hat{\sigma}_{BB'}\right)(\delta_{A'1}\delta_{B'0} + \delta_{A'0}\delta_{B'1}) &= \\ \frac{1}{4}(\delta_{A0}\delta_{B1} + \delta_{A1}\delta_{B0})\end{aligned} \tag{C.155}$$

C.3.5 Direct Products and Clebsch-Gordan Coefficients

$$\tau^{\mathbf{i}}_{(\mathbf{j})} = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes 1 \tag{C.156}$$

We wish to calculate $\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})}$. The terms of (M.-19) for $k \neq k$ wont contribute to the commutator as the order of multiplication irrelevant,

$$\sum_{k\neq k'} \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes 1\right) \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^{k'}}{2}\right) \otimes \cdots \otimes 1\right) - (k \leftrightarrow k') = 0 \tag{C.157}$$

$$\begin{aligned}
\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})} &= \\
&= \sum_{k=1}^{2s+1} \left[1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i}{2}\right)\left(\frac{\hat{\sigma}^j}{2}\right) \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^j}{2}\right)\left(\frac{\hat{\sigma}^i}{2}\right) \otimes \cdots \otimes 1\right] \\
&= \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i\hat{\sigma}^j}{4} - \frac{\hat{\sigma}^j\hat{\sigma}^i}{4}\right) \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk} \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \frac{\hat{\sigma}^k}{2} \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk}\tau^{\mathbf{k}}_{(\mathbf{j})}
\end{aligned} \tag{C.158}$$

$A \otimes B$

$$a_{ij}b_{kl} = c_{ik;jl}. \tag{C.159}$$

the row and column labels of the matrix elements of C are *composite* labels: the row label ik, is obtained from the row labels of the matrix elements of A and B and the column label, jl is obtained from the corresponding column labels.

$$\begin{aligned}
\Delta_j(\theta) &= \sum_{m=-j}^{j} e^{-im\theta} \\
&= e^{ij\theta}\frac{e^{-i(2j+1)\theta} -}{e^{-i\theta} - 1} \\
&= \frac{e^{i(j+\frac{1}{2})\theta} - e^{-i(j+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}} \\
&= \frac{\sin\left[(j+\frac{1}{2})\theta\right]}{\sin(\frac{1}{2}\theta)}
\end{aligned} \tag{C.160}$$

$$\int_0^{2\pi} \overline{\Delta_k(\theta)} \Delta_j(\theta) \frac{1-\cos\theta}{2\pi}\, d\theta = \delta_{kj}. \tag{C.161}$$

Since the spinor indices take only one of two values an antisymmetrizable over three indices of a multivalent spinor $\tau_{\ldots ABC \ldots}$ is zero. In particular we have the *Jacobi identity*

$$\epsilon_{A[B}\epsilon_{CD]} = 0 = \epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC} \tag{C.162}$$

$$\tau_{\ldots AB \ldots} = \tau_{\ldots(AB)\ldots} + \frac{1}{2}\epsilon_{AB}\tau_{\ldots C}{}^{C}{}_{\ldots}. \tag{C.163}$$

The proof

$$\epsilon_{AB}\tau_{C}{}^{C} - \tau_{AB} + \tau_{BA} = 0 \tag{C.164}$$

$$\tau_{[AB]} = \frac{1}{2}\epsilon_{AB}\tau_{C}{}^{C}. \tag{C.165}$$

Since $\tau_{AB} = \tau_{(AB)} + \tau_{[AB]}$ the result follows

$$\tau_{AB} = \tau_{(AB)} + \frac{1}{2}\epsilon_{AB}\tau_{C}{}^{C} \tag{C.166}$$

It should be evident that this result will also apply to the more general case where we consider only two particular indices of a multivalent spinor $\tau_{\ldots ABC \ldots}$ of valence > 2, hence we have the result (C.163).

The same is true for the followings cases.

(a) Case τ_{AB} we already have

(b) Case τ_{ABC}

$$\begin{aligned} 3\tau_{(ABC)} &= \tau_{A(BC)} + \tau_{B(AC)} + \tau_{C(AB)} \\ &= 3\tau_{A(BC)} - (\tau_{A(BC)} - \tau_{B(AC)}) - (\tau_{A(BC)} - \tau_{C(AB)}) \\ &= 3\tau_{A(BC)} - \epsilon_{AB}\sigma_C - \epsilon_{AC}\sigma_B \end{aligned} \tag{C.167}$$

where $\sigma_C = \epsilon^{AB}(\tau_{A(BC)} - \tau_{B(AC)})$.

This rearanged gives

$$\tau_{A(BC)} = \tau_{(ABC)} + \frac{1}{3}\epsilon_{AB}\sigma_C + \frac{1}{3}\epsilon_{AC}\sigma_B. \tag{C.168}$$

$$\tau_{A(BC)} = \tau_{ABC} - \frac{1}{2}\epsilon_{BC}\tau_{AD}{}^{D} \tag{C.169}$$

Using (C.169) in (C.168) we have the desired expansion for τ_{ABC}

$$\tau_{ABC} = \tau_{(ABC)} + \frac{1}{2}\epsilon_{BC}\tau_{AD}{}^{D} + \frac{1}{3}\epsilon_{AB}\ \sigma_C + \frac{1}{3}\epsilon_{AC}\ \sigma_B. \tag{C.170}$$

or more generally

$$\tau_{...ABC...} = \tau_{...(ABC)...} + \frac{1}{2}\epsilon_{BC}\tau_{...AD}{}^{D}{}_{...} + \frac{1}{3}\epsilon_{AB}\ \sigma_{...C...} + \frac{1}{3}\epsilon_{AC}\ \sigma_{...B...}. \tag{C.171}$$

where $\sigma_{...C...} = \epsilon^{AB}(\tau_{...A(BC)...} - \tau_{...B(AC)...})$

(c) Case $\tau_{A...F}$:

Proof is by induction

Any spinor $\tau_{(A...F)}$

$$n\tau_{(ABC...F)} = \tau_{A(BC...F)} + \tau_{B(AC...F)} + \cdots + \tau_{F(BC...A)}. \tag{C.172}$$

$$\tau_{A(BCD...F)} - \tau_{B(ACD...F)} = \frac{1}{2}\epsilon_{AB}\sigma_{(CD...F)} \tag{C.173}$$

where $\sigma_{(CD...F)} = \epsilon^{AB}(\tau_{A(BCD...F)} - \tau_{B(ACD...F)})$

$$n\tau_{(ABC...F)} = n\tau_{A(BC...F)} - (\tau_{A(BC...F)} - \tau_{B(AC...F)}) - \cdots - (\tau_{A(BC...F)} - \tau_{F(BC...A)}) \tag{C.174}$$

$$\tau_{A(BC...F)} = \tau_{(ABC...F)} + \frac{1}{n}\epsilon_{AB}\rho_{(C...F)} + \cdots + \frac{1}{n}\epsilon_{AF}\rho_{(B...F)} \tag{C.175}$$

where

$$\rho_{(C...F)} = \epsilon^{AB}(\tau_{A(BC...F)} - \tau_{B(AC...F)}) \tag{C.176}$$

C.3.6 Recoupling Theory

Recoupling Theory of Three Angular Momenta - 6-j-Symbols

$$|j_{12}(j_1, j_2), j(j_{12}, j_3) > \;\; = \;\; | \tag{C.177}$$

C.3.7 SO(3,1) and SL(2,C)

$$x' = \frac{x + vt}{\sqrt{1 - c^2 t^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t + vx/c^2}{\sqrt{1 - c^2 t^2}} \tag{C.178}$$

$$\gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi, \tag{C.179}$$

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{C.180}$$

$$K_x = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.181}$$

$$K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.182}$$

$$J_x = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_y = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.183}$$

$$
\begin{aligned}
\left[K_x, K_y\right] &= -iJ_z \\
[J_x, K_x] &= 0 \\
\left[J_x, K_y\right] &= iK_z
\end{aligned}
\tag{C.184}
$$

$$
\begin{aligned}
A &= \frac{1}{2}(J + iK), \\
B &= \frac{1}{2}(J - iK).
\end{aligned}
\tag{C.185}
$$

$$
\begin{aligned}
\left[A_x, A_y\right] &= iA_z \text{and cyclic perms,} \\
[B_x, B_x] &= iB_z \text{and cyclic perms,} \\
\left[A_x, B_y\right] &= 0 \;\; (i, j = x, y, z).
\end{aligned}
\tag{C.186}
$$

This shows that A and B each generate a group $SU(2)$ $SU(2) \otimes SU(2)$

$$
X(x) := \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}
\tag{C.187}
$$

unique vector x^μ

$$
x^\mu = \frac{1}{2}\mathrm{tr}(\sigma_\mu X)
\tag{C.188}
$$

$$
\begin{aligned}
\det X(x) &> 0 \quad \text{if } x \text{ is a timelike vector} \\
&= 0 \quad \text{if } x \text{ is on the light cone} \\
&< 0 \quad \text{if } x \text{ is a spacelikevector}
\end{aligned}
\tag{C.189}
$$

$$
A = \exp\left(-i\frac{\theta}{2}(\vec{n}\cdot\sigma)\right) = \cos\frac{\theta}{2}I - i(\vec{n}\cdot\sigma)\sin\frac{\theta}{2}
\tag{C.190}
$$

$$
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C)
\tag{C.191}
$$

$$
A = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta e^{i\gamma} \\ -\sin\beta e^{-i\gamma} & \cos\beta \end{pmatrix} B
\tag{C.192}
$$

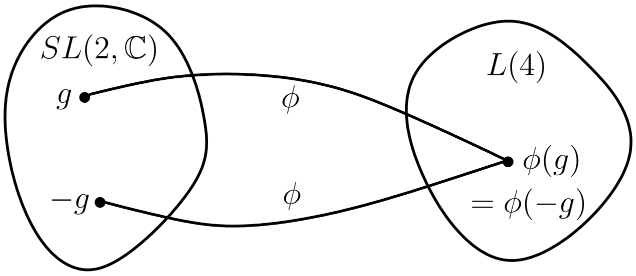


Figure C.4:

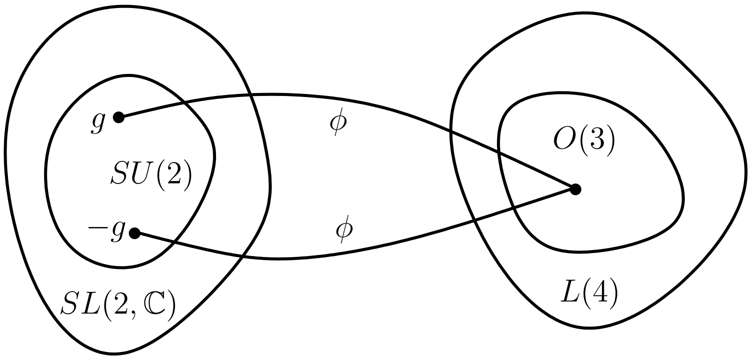


Figure C.5:

C.3.8 SO(4)

$$U(g) = 1 - \frac{1}{2} J_i^j \epsilon_j^i + \mathcal{O}, \tag{C.193}$$

where the J_i^j are $N \times N$ matrices

$$[J_i^j, J_k^l] = \delta_k^j J_i^l - \delta_l^i J_k^j + \delta_k^i J_j^l - \delta_l^j J_i^k \,, \quad i, j, kl = 1, \ldots, n. \tag{C.194}$$

$$SO(4) = SO(3) \oplus SO(3). \tag{C.195}$$

$$L^{\pm} := \frac{1}{2}(L \pm M') \tag{C.196}$$

The structure constants with respect to this new basis are given by

$$[(L^{\pm})^i, (L^{\pm})^j] = i\hbar\epsilon^{ij}{}_k (L^{\pm})^k \,, \quad [(L^+)^i, (L^-)^j] = 0 \tag{C.197}$$

C.3.9 Conformal Group

In Minkowskian spacetime a *conformal transformation* are coordinate transformations $x \to x'(x)$ which are such that the induced change in the metric is a positive rescaling by a positive function:

$$ds'^2 = \Omega(x)^2 \, ds^2 \tag{C.198}$$

where $\Omega(x)$ is a real-valued function. Geometrically conformal transformations leaves thae *angles* but changes distances; it involves dilations (rescalings). Tranformations with this property are used in the design of geographic maps. Penrose diagrams bring infinity onto a page.

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin\theta d\phi^2) \tag{C.199}$$

We introduce double null coordinates

$$u = t + r \tag{C.200}$$

$$v = t - r \tag{C.201}$$

$dudv = (dt + dr)(dt - dr) = dt^2 - dr^2$ and $(u - v)^2 = (t + r - t + r)^2 = 4r^2$ so the line element becomes

$$ds^2 = dudv - \frac{1}{4}(u - v)^2(d\theta^2 + \sin\theta d\phi^2). \tag{C.202}$$

$$p = \tan^{-1} u, \quad q = \tan^{-1} v. \tag{C.203}$$

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{4}\sec^2 p \sec^2 q[4dpdq - \sin^2(p - q)(d\theta^2 + \sin\theta d\phi^2)] \tag{C.204}$$

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}dx^\alpha dx^\beta = dpdq - \sin^2(p - q)(d\theta^2 + \sin\theta d\phi^2) \tag{C.205}$$

$$\Omega = \frac{1}{4}\sec^2 p \sec^2 q \tag{C.206}$$

Now finally we introduce the coordinates

$$
\begin{aligned}
t' &= p+q && \text{(C.207)}\\
r' &= p-q && \text{(C.208)}
\end{aligned}
$$

with the coordinate range

$$
\begin{aligned}
-\pi < t'+r' < \pi, && \text{(C.209)}\\
-\pi < t'-r' < \pi, && \text{(C.210)}\\
r' > 0 && \text{(C.211)}
\end{aligned}
$$

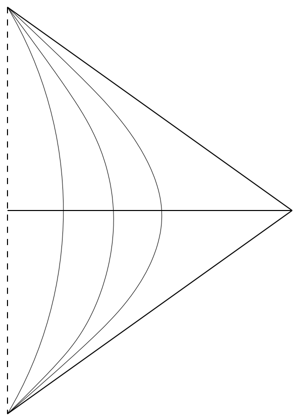


Figure C.6: Penrose diagram for Minkowskian spacetime.

C.3.10 Group Integration: The Haar Measure

For a compacct group, we can replace the sums over group elements that occur in the representation theory of finite groups, by convergent integrals over the group elements using the invariant Haar measure, which is usually denoted by $d[g]$. The invariance property is expressed by $d[g_1 g] = d[g]$. This allows us to make a change of variables transfromation, $g \to g_1 g$, identaical to that which played such an important role in deriving the finite group theorems. Consequently, all the results from finite groups, such as the existence of an invariant inner product and the orthoganality theorems, acn be taken over by the simple replacement of a sum by an integral.

As a vector space, $R(G)$ has a distinguished basis given by the characters $\chi_i(g)$ of the irreducible representations. Recall that the orthog-onality relations for characters are essentially the same as in the finite group case, with the sum over group elements replace by an integral

$$\int \overline{\chi_i(g)}\chi_j(g)dg = \delta_{ij} \tag{C.212}$$

where i and j are labels for irreducible representations and dg is the standard Haar measure, normalized so that the volume of g is 1.

For the case of finite group one important property of these groups is the rearrangement theoremis given for fixed S,

$$\sum_R f(R) = \sum_R f(SR) \tag{C.213}$$

$$\int f(R)dR = \int f(SR)dR \tag{C.214}$$

To clarify the relation to the finite group rearangement theorem we write the integral on the left-hand side of (C.213) as an integral over the parameters. the density of group elements is arranged so that the density of points at SR is the same as that at R.

$$\int f(R)\, dR = \int f(R)\mu(R)\, da, \tag{C.215}$$

where $\mu(R)$ is the density of group elements in the parameter space in the neighborhood of R.

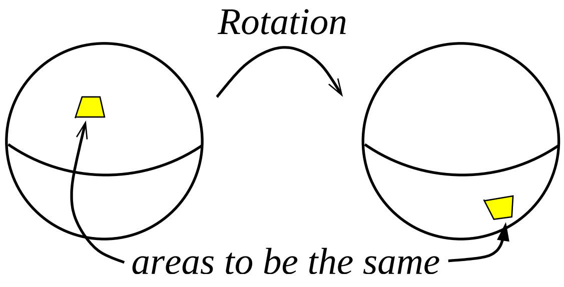


Figure C.7: Haarmeas1.

uniform measure over a 3-sphere S^3

Of course the measure is

$$dU = d\theta \cos\theta d\phi \tag{C.216}$$

as under a rotation

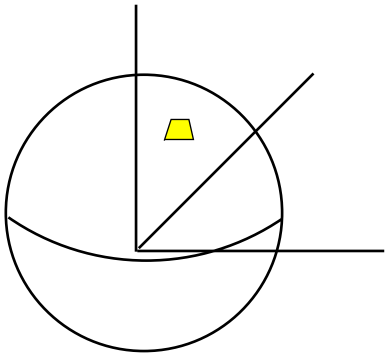


Figure C.8: HaarmeasSO3.

$$d\theta \cos\theta d\phi \mapsto d\theta' \cos\theta' d\phi' \tag{C.217}$$

Approach that is easily extended to other Lie groups.

$$dU = \pi^{-2} da_1 da_2 da_3 \delta\left(\sum_{i=1}^{2} a_i^2 - 1\right) \tag{C.218}$$

$SU(2)$ a 3-sphere in 4-dimensional Euclidean space.

$$dU = \pi^{-2} da_0 da_1 da_2 da_3 \delta\left(\sum_{i=0}^{3} a_i^2 - 1\right) \tag{C.219}$$

$$\mu(C) = \mu(0) / \left(\frac{\partial c}{\partial a}\right)_{a=0} \tag{C.220}$$

$$\begin{aligned}
&\int dU U_{ij} = 0 \\
&\int dU U_{i_1 j_1} U_{i_2 j_2} = \frac{1}{2} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \\
&\int dU U_{ij} (U^{-1})_{kl} = \frac{1}{2} \delta_{jk} \delta_{il}
\end{aligned} \tag{C.221}$$

$$\int dU |\mathrm{Tr} U|^2 = 1 \tag{C.222}$$

Details

$$\int dUU_{ij}(U^{-1})_{kl} = A\delta_{jk}\delta_{il} \tag{C.223}$$

The constant A is easily found by setting $j = k$ summing over the index

$$\int dU\delta_{il} = A\delta_{jj}\delta_{il} = 2A\delta_{il}, \tag{C.224}$$

noting $\int dU = 1$ implies $A = 1/2$.

$$\int dU|\mathrm{Tr}U|^2 = \int dUU_{ij}(U^\dagger)_{kl} = \int dUU_{ij}(U^{-1})_{kl} = \frac{1}{2}\delta_{ji}\delta_{ij} = \frac{1}{2}\delta_{jj} = 1 \tag{C.225}$$

For any compact group G the Haar measure is the unique measure dU on G which obeys invarianc (C.214) and normalization: $\int_G dU = 1$.

$$dc = J\left[\frac{\partial c}{\partial a}\right] da \tag{C.226}$$

where $J[\partial c/\partial a]$ is the *Jacobian* defined by

$$J\left[\frac{\partial c}{\partial a}\right] = \frac{\partial(c_1, c_2, \ldots, c_h)}{\partial(a_1, a_2, \ldots, a_h)} \equiv det\left|\frac{\partial c_i}{\partial a_i}\right| \tag{C.227}$$

Example 1: $SO(2)$ $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$. Thus, from $R(\theta') = R(\theta)R(\epsilon) = R(\theta + \epsilon)$, where $R(\epsilon)$ is the infintesimal transformation, we have

$$\theta' = \theta + \epsilon \tag{C.228}$$

Accordingly $d\theta'/d\epsilon = 1$, so we have that $\mu(\theta) = \mu_0$. Through normalization

$$\int_{-\pi}^{\pi} \mu(\theta)d\theta = 2\pi\mu_0 = 1 \tag{C.229}$$

we obtain $\mu(\theta) = 1/(2\pi)$.

Unitary Group $U(2)$

The unitary group $U(N)$ has N^2 generators

$$[\mathcal{C}_{im}, \mathcal{C}_{jn}] = \delta_{jm}\mathcal{C}_{in} - \delta_{in}\mathcal{C}_{jm}. \tag{C.230}$$

a representation of $U(2)$ is

$$(\mathcal{C}_{im}) = (\delta_{im}) \tag{C.231}$$

where (δ_{im} denotes the matrix which has a "1" at the intersection of the ith row and the mth column and zeros everywhere else i.e.

$$\begin{aligned} \mathcal{C}_{11} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{C}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{C}_{21} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{C}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{C.232}$$

An arbitrary group element of $U(2)$ is given by

$$\exp\left(-i\sum_{k,l=1}^{2}\theta_{kl}\mathcal{C}_{kl}\right) \tag{C.233}$$

where the "angles" θ parametrize the group tranformation.

The transition to $SU(2)$ can be made by constructing traceless matrices from the $\mathcal{C}_{kl}$

All in all

$$\begin{aligned} \tilde{\mathcal{C}}_{11} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \tilde{\mathcal{C}}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \tilde{\mathcal{C}}_{21} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\mathcal{C}}_{22} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{aligned} \tag{C.234}$$

Finally for $\mu(c)$ we obtain

$$\mu(c)1/J[\partial c/\partial a]_{a=0} \tag{C.235}$$

U(2), SU(2) The measure is given by

$$\begin{aligned}
\mu(U(2)) &= |\epsilon_1 - \epsilon_2|^2 = |\exp(-i\theta_{11}) - \exp(-i\theta_{22})|^2 \\
&= |\exp\left(-\frac{1}{2}i(\theta_{11}+\theta_{22})\right)|^2 \times |\exp\left(-\frac{1}{2}i(\theta_{11}-\theta_{22})\right) \\
&\quad |\exp\left(\frac{1}{2}i(\theta_{11}-\theta_{22})\right)|^2 \\
&= 4\sin^2\left(\frac{1}{2}(\theta_{11}-\theta_{22})\right).
\end{aligned} \tag{C.236}$$

restriction to $SU(2)$ yields $\theta_{11} + \theta_{22} = 0$. With $\theta_{11} - \theta_{22} = \phi$ then

$$\mu(SU(2)) = 4\sin^2\frac{1}{2}\phi. \tag{C.237}$$

Normalization $C\int_0^{4\pi}\mu(SU(2))\,d\phi = 1$ determines constant C;

$$C\int_0^{4\pi} d\phi\sin^2\frac{1}{2}\phi = \frac{C}{2}\int_0^{4\pi} d\phi\,[1-\cos\phi] = \frac{C}{2}\int_0^{4\pi} d\phi = 2\pi \tag{C.238}$$

C.3.11 Peter-Weyl theorem

The irreducible representation functions $D^l(R)_m^{m'}$ satisfy orthogonality and completeness relations. In fact, they form a complete basis in the space of square integrable functions defined on the group parameter space. This is the Peter-Weyl theorem. That the irreducible representation functions form a complete basis for square-integrable functions $f(R) \in L^2$ can be expressed as

$$f(R) = \sum_{\alpha ab} f_\alpha^{ab} D_{ab}^{(\alpha)}(R) \tag{C.239}$$

Moreover the Peter-Weyl theorem states that the χ_r form a basis of the space of square-integrable class functions on the group G. That means that every square-integrable function $f(U)$, obeying

$$f(U) = f(VUV^{-1}), \tag{C.240}$$

can be expanded into characters:

$$f(U) = \sum_{r\in G} f_r\chi_r(U) \tag{C.241}$$

$$f_r = \int dU \overline{\chi_r}(U) f(U). \tag{C.242}$$

(compare with discrete case: $\lambda(c_l) = \sum_\alpha a_\alpha \chi^{(\alpha)}(c_l)$ where $a_\alpha = \sum_k N_k \chi^{(\alpha)}(c_k)^* \lambda(c_k)$) the completeness relation can be written as

$$\sum_{r \in G} \chi_r(U) \overline{\chi_r}(V) = \delta(UV^{-1}). \tag{C.243}$$

$$\int dU \overline{\chi_r}(U) \chi_s(U) = \delta_{rs}. \tag{C.244}$$

The invariant $\delta-$function on G is defined by means of

$$\int dU f(U) \delta(UV^{-1}) = f(V), \tag{C.245}$$

and obeys

$$\delta(U) = \delta(U^{-1}). \tag{C.246}$$

The charactres are

$$\chi_j(U) = \frac{\sin\left(j + \frac{1}{2}\right)\phi}{\sin\frac{1}{2}\phi}. \tag{C.247}$$

$$f_j = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \sin\frac{1}{2}\phi \, \sin\left(j + \frac{1}{2}\right) \phi \, f(\phi). \tag{C.248}$$

As an example consider the group $U(1)$

$$D_{(\mu)}(\theta) = e^{i\mu\theta}, \tag{C.249}$$

where $\mu = 0, \pm 1, \pm 2, \ldots$.

$$D_{(\mu)}(\theta + 2\pi) = D_{(\mu)}(\theta). \tag{C.250}$$

The Haar measure is

$$\Omega = \frac{1}{2\pi}\, d\theta. \tag{C.251}$$

Theorem applied to $U(1)$ gives

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i\mu\theta} e^{i\nu\theta} = \delta_{\mu\nu} \tag{C.252}$$

where $0, \pm 1, \pm 2, \dots$.

The *Peter-Weyl Theorem* applied to $U(1)$ gives the Fourier series theory:

$$f(\theta) = \sum_{-\infty}^{\infty} f_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \tag{C.253}$$

where $f(\theta) \in L^2(U(1))$.

C.3.12 Analogies

$$\int dx\delta(x)f(x) = f(0) \quad \Longleftrightarrow \quad \int_G dU\delta(U)f(U) = f(I) \tag{C.254}$$

$$\int dy\delta(y-a)f(y) = f(a) \quad \Longleftrightarrow \quad \int_G dU\delta(U\gamma^{-1})f(U) = f(\gamma) \tag{C.255}$$

$$f(x) = \sum_n a_n e_n(x) \quad \Longleftrightarrow \quad f(g) = \sum_j a_j \chi_j(g) \quad (f(g) = f(\gamma g \gamma^{-1}))$$

$$a_n = \int \overline{e_n(x)} f(x) dx \quad \Longleftrightarrow \quad a_j = \int_G \overline{\chi_j(g)} f(g) dg \tag{C.256}$$

$$\delta(x-y) = \sum_n \overline{e_n(x)} e_n(y) \quad \Longleftrightarrow \quad \delta(gx^{-1}) = \sum_j \overline{\chi_j(g)} \chi_j(x) \tag{C.257}$$

$$\delta(x+a-a) \quad \Longleftrightarrow \quad \delta(xgx^{-1}) = \delta(g) \tag{C.258}$$

$$\int dx \overline{e_m(x)} e_n(x) = \delta_{mn} \quad \Longleftrightarrow \quad \int D^j_{mn}(g) \overline{D^{j'}_{m'n'}(g)} dg = \delta_{jj'}\delta_{mm'}\delta_{nn'} \tag{C.259}$$

$$\int dy \overline{e_m(y)} e_n(x-y) = \delta_{mn} e_n(x) \quad \Longleftrightarrow \quad \int_G \overline{\chi_{j'}(g)} \chi_j(gx) dg = V_G \delta_{jj'} \frac{\chi_j(x)}{d_j} \tag{C.260}$$

Exercise

(a):

Recall how we prove $\int dx\delta(x)f(x) = f(0)$ implies $\int dy\delta(y-a)h(y) = h(a)$ where $h(y) = f(y-a)$

make the substitution $x = y - a$

$$h(a) = f(0) = \int dy\delta(y-a)f(y-a) = \int dy\delta(y-a)h(y) \tag{C.261}$$

prove

$$\int_G dU\delta(U)f(U) = f(I) \quad \Rightarrow \int_G dU\delta(U\gamma^{-1})f(U) = f(\gamma) \tag{C.262}$$

define $h(U\gamma^{-1}) = g(U)$

$$h(\gamma) = f(1) \tag{C.263}$$

(b):

$$f(g) = \sum_{\Lambda} \phi_{\Lambda}^{\alpha\beta} D_{\alpha\beta}^{(\Lambda)}(g) \tag{C.264}$$

$$f(g_1, g_2) = \sum_{\Lambda_1\Lambda_2} \phi_{\Lambda_1\Lambda_2}^{\alpha_1\alpha_2\beta_1\beta_2} D_{\alpha_1\beta_1}^{(\Lambda_1)}(g_1) D_{\alpha_2\beta_2}^{(\Lambda_2)}(g_2) \tag{C.265}$$

$$f(g_1, g_2 = C) = \sum_{\Lambda_1} a_{\Lambda_1}^{\alpha_1\beta_1}(g_2 = C) D_{\alpha_1\beta_1}(g_1) \tag{C.266}$$

$$a_{\Lambda_1}^{\alpha_1\beta_1}(g_2) = \sum_{\Lambda_2}^{\alpha_2\beta_2} D_{\alpha_2\beta_2}(g_2) \tag{C.267}$$

$$a_{\Lambda_1}^{\alpha_1\beta_1} =: \phi_{\Lambda_1\Lambda_2}^{\alpha_1\alpha_2\beta_1\beta_2} \tag{C.268}$$

Using $D_{ml}^j(g)D_{ln}^j(x) = D_{mn}^j(gx)$

$$\int \overline{D_{m'n'}^{j'}(g)} D_{mn}^j(gx)dg = \delta_{jj'}\delta_{mm'}D_{nn'}^j(x) \tag{C.269}$$

contracting m, m' and nn' we egt (C.260).

Riesz-Fishcher theorem

Parseval's equation :

$$\sum_{-\infty}^{\infty} |c_n|^2 = \int |f(x)|^2 dx. \tag{C.270}$$

$$f_n(x) = \sum_{k=-n}^{n} c_k e_k(x) \tag{C.271}$$

converge to the vector f in the sense of L_2:

$$||f - f_n|| \to 0. \tag{C.272}$$

limit in the mean of the f_n's.

$$||f_m - f_n||^2 = \sum_{|k|=n+1}^{m} |c_k|^2. \tag{C.273}$$

if c_n are given complex numbers for which $\sum_{-\infty}^{\infty} |c_n|^2$ converges, then there exists a function f in L_2. If we grant the completeness of L_2 as a metric space this is easy to prove.

This tells us that the f's form a Cauchy sequence in L_2; and since L_2 is complete, there exists a function f in L_2 such that $f_n \to f$.

It is apparent that

C.3.13 Clebsch-Gordan

$$D_{(\mu)} \times D_{(\nu)} = \sum_{\otimes\sigma} n_{\mu\nu}^{(\Lambda)} D_{(\Lambda)} \tag{C.274}$$

where $n_{\mu\nu}^{(\Lambda)}$ is the number of times that the Λ-th irreducible representation occurs in the product representation (**??**).

$$\chi_{(\mu\times\nu)}(g) = \chi_{(\mu)}(g)\chi_{(\nu)}(g). \tag{C.275}$$

Thus by

$$n_{\mu\nu}^{(\Lambda)} = \int_G dg \overline{\chi_{(\Lambda)}(g)} \chi_{(\mu)}(g)\chi_{(\nu)}(g) \tag{C.276}$$

C.3.14 Semi-direct Products

A full Lorentz transformation can be decomposed into an ordinary spacial rotaion, followed by a boost, followed by a further ordinary rotation.

Definition (First definition)

Suppose N is a normal subgroup of G and H is another subgroup of G such that $N \cap H = E$ (the identity of G) and every element of G can be written in a unique way as

$$g = ba, \quad b \in H, \ a \in N,$$

then G is said to be a **semi-direct product** of H and N, written $G = H \otimes_S N$.

Definition (Second definition)

We form a new group whose elements are the elements of $H \times N$ and multiplication given by

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1 h_2 \ , \ n_1 \rho_{h_1}(n_2)) \quad \text{with} \quad \rho_{h_1}(n_2) = h_1 n_2 h_1^{-1}. \tag{C.277}$$

Note $h_1 n_2 h_1^{-1} \in N$ as N is a normal subgroup of G.

Check it forms a group

The identity element of this group is (E, E):

$$\begin{aligned} (h, n) \cdot (E, E) &= (h, h\ hEh^{-1}) = (h, n), \\ (E, E) \cdot (h, n) &= (h, E\ EnE^{-1}) = (h, n). \end{aligned}$$

The inverse of (h, n) is (h^{-1}, n') where $n' = h^{-1} n^{-1} h$:

$$\begin{aligned} (h, n) \cdot (h^{-1}, n') &= (E, n\ hn'h^{-1}) = (E, E), \\ (h^{-1}, n') \cdot (h, n) &= (E, n'\ h^{-1} nh) = (E, E). \end{aligned}$$

Associativity $[(h_1, n_1) \cdot (h_2, n_2)] \cdot (h_3, n_3) = (h_1, n_1) \cdot [(h_2, n_2) \cdot (h_3, n_3)]$:

$$\begin{aligned} [(h_1, n_1) \cdot (h_2, n_2)] \cdot (h_3, n_3) &= (h_1 h_2, n_1\ h_1 n_2 h_1^{-1}) \cdot (h_3, n_3) \\ &= (h_1 h_2 h_3, n_1\ h_1 n_2 h_1^{-1}\ h_1 h_2 n_3 (h_1 h_2)^{-1}) \end{aligned}$$

$$
\begin{aligned}
(h_1, n_1) \cdot [(h_2, n_2) \cdot (h_3, n_3)] &= (h_1, n_1) \cdot (h_2 h_3, n_2\ h_2 n_3 h_2^{-1}) \\
&= (h_1 h_2 h_3, n_1\ h_1 (n_2\ h_2 n_3 h_2^{-1}) h_1^{-1})
\end{aligned}
$$

$\square$

Equivalence of the two defintions

for example $H \cap N$ might be empty.

for each $h \in H$ the inner automorphism $x \to hxh^{-1}$ takes N to N and defines an automorphism

$$\rho_h(n) = hnh^{-1}$$

Moreover,

$$
\begin{aligned}
\rho_{h_1}(\rho_{h_2}(n)) &= \rho_{h_1}(h_2 n h_2^{-1}) = h_1 h_2 n h_2^{-1} h_1^{-1} \\
&= \rho_{h_1 \cdot h_2}(n)
\end{aligned}
$$

Thus $h \to \rho_h$ is a homomorphism of H into the group of automorphisms of N, we write $\rho \in Hom(H, Aut(N))$.

Sub-groups

The group G is the semigroup product of N by H with homomorphism ρ.

Recall $\rho(h)(n) = hnh^{-1}$. Note that if $\rho \in Hom(H, Aut(N))$

Obviously there is the subgroup of G composed of elements of the form:

$$(h_i, E) \quad \text{for all } h_i \in H.$$

$$H \otimes_S I \simeq H$$

there is the subgroup of G composed of elements of the form:

$$(E, n_i) \quad \text{for all } n_i \in N.$$

as

$$(E, n_1) \cdot (E, n_2) = (E, E, n_1 \, Eh_2E) = (E, n_1 n_2)$$

$$I \otimes_S N \simeq N$$

Conversely, suppose that we are given two groups N and H and a homomorphism $h \to \rho_h$ of H into the group of all automorphisms of N. We may then define a group $H \otimes_S N$ with respect to ρ as follows.

$\square$

Example The group RT is a semi-direct product of the rotation group $R(2)$ and the group of all trnaslations $T(2)$

Proof:

Example The group ?? is a semi-direct product of the rotation group $R(3)$ and the group of all trnaslations $T(3)$

Proof:

The group of all translations and rotations has six generators $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{J}_1, \hat{J}_2, \hat{J}_3$. More concisely, it is called the transaltion-rotaion group and has the translations $(\hat{p}_\nu)$ as an abelian subgroup. It is obvious that

$$\hat{R}\hat{T}\hat{R}^{-1} = \hat{T}',$$

where $\hat{R}$ is a rotation and $\hat{T}$ a translation, is again a pure translation $\hat{T}'$ (see fig (M.-19)). Consequently the translation group is an invariant abelian subgroup of the translation-rotation group.

The group consists of pairs of the pairs (r, t) with $t \in T$, $r \in O(3, \mathbb{R})$ and multiplication rule

$$(r_1, t_1) \cdot (r_2, t_2) = (r_1 r_2, t_1 r_1 t_2 r_1^{-1}). \tag{C.278}$$

Example The Poincare group P includes the abelian subgroup $T(4)$ of all translations in Minkowski spacetime in addition to the Lorentz transformations. The Poincare group P is a semi-direct product of the Lorentz group $L(4)$ and the group of all translations $T(4)$ on Minkowski spaectime, that is,

$$P = L(4) \otimes_S T(4). \tag{C.279}$$

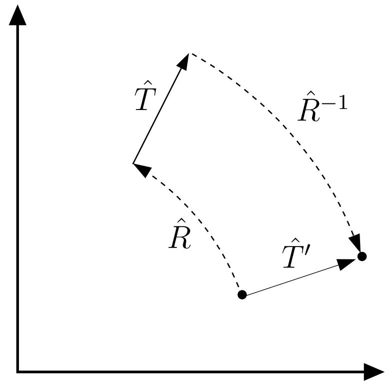


Figure C.9: TranRotGrF.

Proof:

$T(4)$ is a normal abelian subgroup of P as for $t \in T(4)$ and any $p \in P$, $ptp^{-1} \in T(4)$.

An element $p \in P$ is then denoted $p = (\Lambda^\mu_\nu, a^\mu)$.

Examples from Quantum Gravity

(i) Three diemnsional gravity:

where $ISU(2)$ is the (universal cover of the) group of Euclidean transformations. It is the semigroup:

$$ISU(2) = SU(2) \otimes_S \mathbb{R}^3 \tag{C.280}$$

Its elements are written as $(u, \vec{a})$ where u is an element of $SU(2)$ and $\vec{a}$ is a vector in $\mathbb{R}^3$. We have

$$(u_1, \vec{a}_1) \cdot (u_2, \vec{a}_2) = (u_1 u_2, u_1 \vec{a}_2 + \vec{a}_1), \tag{C.281}$$

the notation $u\vec{a}$ means $U(u)\vec{a}$ where U is the vectorial representation of $SU(2)$.

(ii) black hole gauge group is the semi-direct product of ....

(iii) Gauge group of LQG

The group of these symmetries is the semi-direct product the group of smooth local gauge transformations with the group of smooth diffeomorphisms on Σ.

C.4 Infinite-Dimensional Group Representations

C.4.1 Group Actions

If G is a permutation group, then every element of the group permutates elements of the set $\{1, 2, \cdots n\}$. We say that the group G acts on the set $\{1, 2, \cdots n\}$.

Group action on a set Let G be a group and let A be a some set. Suppose that we have a map $T : G \times A \to A$ such that for every fixed $g \in G$ the map $a \to T(g, a)$ is a permutation of the set A.

We require the compatibility conditions:

(i) $T(g_1, T(g_2, a)) = T(g_1 g_2, a)$ for all $g_1, g_2 \in G$ and $a \in A$.

(ii) $T(e, a) = a$ for all $a \in A$

where $e = e_G$ the identity of G.

define what is called T is a **group action** and that G acts on A by the action of T.

We can shorten the notion by writing $g \cdot a$ instead of $T(g, a)$.

These conditions are designed so that the map $G \to S_A$ is a group homomorphism.

C.4.2 Countable and Locally Compact Topological Groups

Let S be a set on which some group acts as a group of transformations. Let us write sx for the transfrom by x of $s \in S$. Then

$$(sx)y = s(xy) \qquad \text{for all } s \in S, x, y \in G.$$

and

$$se = s$$

where e is the identity element of G.

We assume that G is a topological group which is separable and locally compact and S comes with a 'volume element'. We assume that S is a Borel space with a measure μ; that is, that we are given a family $\mathcal{B}$ of subsets of S, closed with respect to completions and countable uniions, and measure μ assigning a nonnegative real number or ∞ to each subset $E \in \mathcal{B}$ so that

$$\mu(E_1 + E_2 + \cdots) = \mu(E_1) + \mu(E_2) + \cdots$$

whenever $E_i \cap E_j = 0$ for all $i \neq j$. The members of $\mathcal{B}$ are called *Borel sets*. We assume S is a union of countably many sets of finite measure and that

If S_1 and S_2 are Borel spaces, then a function from S_1 to S_2 is a Borel function if $g^{-1}(E)$ is a Borel set in S_1 whenever E is a Borel set in S_2. Any topological space becomes a Borel space if we define the Borel sets to be the sets one can obtain from closed sets by the taking of complements, countable unions, and countaale intersections. We extend theorems already proven to much more general context as the notion of Borel space and Borel function are quite general. A function can be widly discontinuous and still be a Borel function.

We shall say that the measure μ is *invariant* if

$$\mu(Ex) = \mu(E) \qquad \text{for all Borel sets } E \text{ and all } x \in G. \tag{C.282}$$

C.4.3 Haar Measure

Given an $n-$dimensional manifold and a nowhere-vanishing orieneted $n-$form η, we can make a measure on $\mathcal{M}$ by defining the integral of f against μ to be the integral of the $n-$form $f\eta$.

It is not hard to show that on an $n-$dimensional Lie group G, there exists a nowhere-vanishing $n-$form that is invariant under left translations and that this form is unique up to a constant. Integrating fucntions against this form gives a letf-invariant measure (i.e., a left Haar measure).

C.4.4 Summary of Group theory

- Unitrary representations.
- Irreduciable representations.
- Schur's lemmas.
- Orthoganality theorems.

C.5 Manifolds and Elementary Topology

Roughly, a manifolds are sets on which, at least around each point, everything looks Euclidean.

Organized set of points with a structure - a division into convenient family of subset families.

It is this need for care, to ensure we can rely on calculations we do, that motivates much of this course, illustrates why we empathize accurate argument as well as getting the "correct" answers, and explains why in the rest of this section we need to revise elementary notions.

C.5.1 Sets and Mappings Between Sets

We need to be able to talk easily about certain subsets of R. We say that I is an **open interval** if

$$I = (a, b) = x \in R : a < x < b. \tag{C.283}$$

Thus an open interval excludes its end points, but contains all the points in between. x is always separated from F by

In contrast a **closed interval** contains both its end points, and is of the form

$$I = [a, b] = \{x \in R : a \leq x \leq b\}. \tag{C.284}$$

It is also sometimes useful to have **half-open intervals** like $(a, b]$ and $[a, b)$. It is trivial that $[a, b] = (a, b) \cup_{\{a\}} \cup_{\{b\}}$.

The two end points a and b are points in R. It is sometimes convenient to allow also the possibility $a = -\infty$ and $b = +\infty$; it should be clear from the context whether this is being allowed. If these extensions are being excluded, the interval is sometimes called a *finite interval*, just for emphasis.

Of course we can easily get to more general subsets of R. So $(1, 2) \cup [2, 3] = (1, 3]$ shows that the union of two intervals may be an interval, while the example $(1, 2) \cup (3, 4)$ shows that the union of two intervals *need not* be an interval.

An open subset V of X is a subset where if $x \in V$, then there is a $\delta > 0$ such that an open ball $B_\delta(x)$ is entirely contained in V.

Notation	Meaning
$a \in A$	a belongs to A
$A \subset B$	A is included in B
$A = B$	A is identical to B
$A \cup B$	The union of A and B ($\cup$ is for **u**inion)
$A \cap B$	The intersection A and B ($\cap$ is for **in**tersection)
$A - B$	The set of elements of A not included in B

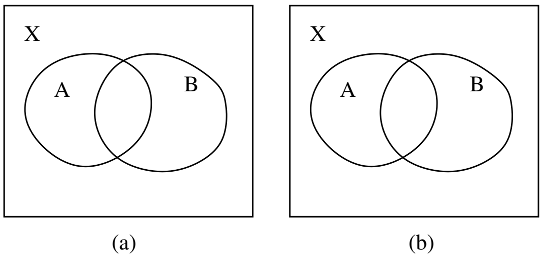


Figure C.10: .

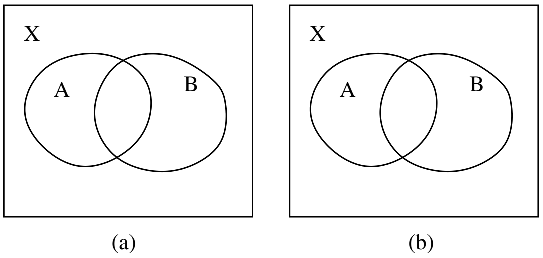


Figure C.11: .

C.5.2 Continuity

A function f is continuous at a point p if whenever we can force the distance between $f(x)$ and $f(p)$ to be as small as desired by taking the distance between x and p to be small enough.

The definition of continuity of a function f on the real line: $f : R \mapsto R$ is continuous at x_0 if for any positive number ϵ, there exists a positive number δ such that if $|y - x_0| < \delta$, then $|f(y) - f(x_0)| < \epsilon$. The setting for this definition is the real line, a Euclidean space.

We can generalize this definition a little by considering mapping between spaces with a metric. Let X, Y be two spaces with metrics d_1 and d_2, respectively. Then, $f : X \rightarrow Y$ is continuous at $x_0 \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that if $d_1(x, x_0) < \delta$, then $d_2(f(x), f(x_0)) < \epsilon$.

However, the notion of continuity does not depend on a metric.

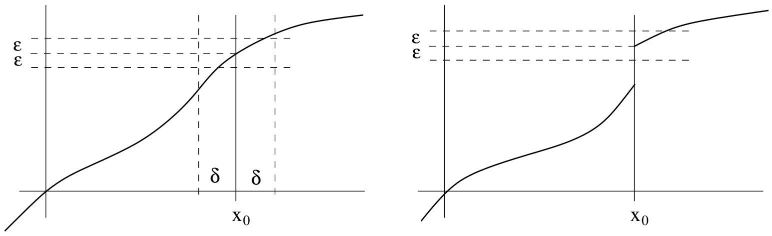


Figure C.12: (a) (b) f is discontinuous at p_1.

We need to be able to talk about a function near a point a. If we want to look at the points a distance less than d for a. we are looking at an interval $(a-d, a+d)$. We call such an interval a neighbourhood of a.

Definition A subset $\mathcal{U}$ is **open** if given $a \in \mathcal{U}$, there is some $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq \mathcal{U}$.

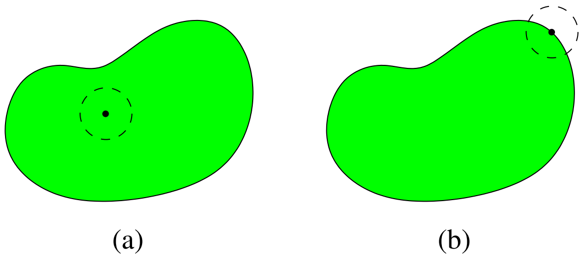


Figure C.13: Open sets interior points (b) .

In fact this is the same as saying that given $a \in \mathcal{U}$, there is some open interval containing a which lies in $\mathcal{U}$ - a set is open if it contains a neighbourhood of each of its points. This definition has the effect that if a function is defined on an open set that its behaviour near point a of interest from both sides.

We can transfer such things as limits and calculus from Euclidean space. A ***topology*** is a structure added to an arbitrary point set which enables one to define a convergent sequence and to define a continuous function in a general setting.

A ***topology***, $\mathcal{T}$, on a set X is defined to be a specified family of open subsets on X satisfying the following 3 properties:
(i) The empty set, $\emptyset$, and the space X belong to $\mathcal{T}$.
(ii) The union of any number (possibly infinite) of open subsets belonging to $\mathcal{T}$ is also in $\mathcal{T}$.
(iii) The intersection of any finite number (nor infinite) open subsets in $\mathcal{T}$ also belongs to $\mathcal{T}$.

The set X together with a topology $\mathcal{T}$ is called a topological space.

Let $f : x \to Y$ be a function between two topological spaces $(X, \mathcal{T}_X)$, and $(Y, \mathcal{T}_Y)$. If $x \in X$, then $f(x) \in Y$ is the image of x under f. Let U be an open set in X (i.e. $U \in \mathcal{T}_X$). The image of U under f is the subset $V = f(U) \subset Y$, the range of f with domain U. If V is a subset of Y, then the inverse image of V by f. The mapping f is defined to be *continuous* when the inverse image of any open set is open. That is, f is continuous if $U = f^{-1}(V) \in \mathcal{T}_X$ when $V \in \mathcal{T}_Y$.

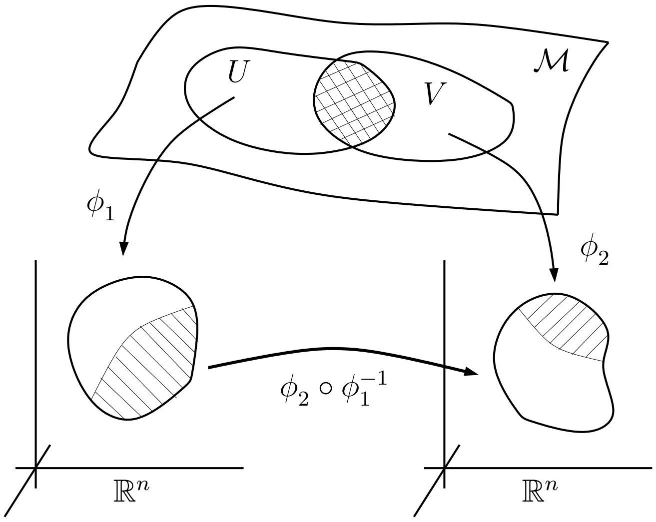


Figure C.14: (U, ϕ_1) and (V, ϕ_2) are two coordinate patches on X. Transition functions, $\phi_2 \circ \phi_2^{-1}$, are ordinary functions that go from points of one R^n space onto another, i.e. $\phi_2 \circ \phi_2^{-1} : R^n \mapsto R^n$. The domain and range of the transition function are the shaded regions in R^n.

The transition functions transform the coordinates of one overlapping patch into another.

The usual Jacobian of ψ_{ij} is

$$
\begin{aligned}
(\Lambda^a{}_b) &= \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} \\
&= \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^n} \\ \cdots & \cdots & \\ \frac{\partial x'^n}{\partial x^1} & \cdots & \frac{\partial x'^n}{\partial x^n} \end{pmatrix}
\end{aligned}
\tag{C.285}
$$

C.6 Elementary Tensor Analysis

One often needs to do a change of coordinates, either because we prefer to use a different choice of coordinates valid in some patch, or because we need to transform to new patch which covers a different portion of the manifold.

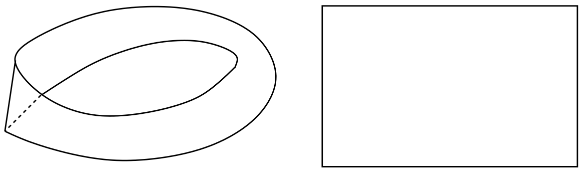


Figure C.15: Möbius.

Einstein summation convention

$$\sum_a (..)^a \, (..)_a \to (..)^a \, (..)_a \tag{C.286}$$

Contravariant Vectors

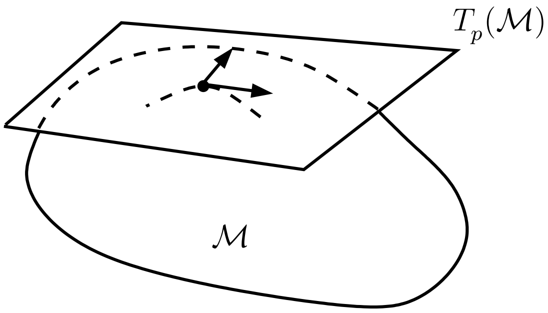


Figure C.16: tangtoM.

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{C.287}$$

the infinitesimal displacement in two different coordinate systems x^a and x'^a is related by

$$dx'^a = \sum_b \frac{\partial x^{a\prime}}{\partial x^b} dx^b \tag{C.288}$$

where $x^{a\prime}/x^b$ is evaluated at the point $p = x^a = x'^a(x^a)$. **Einstein summation convention**

$$dx^{a\prime} = \frac{\partial x^{a\prime}}{\partial x^b} dx^b \tag{C.289}$$

We set

$$\Lambda^a_b = \frac{\partial x^{a\prime}}{\partial x^b} \tag{C.290}$$

The infinitesimal displacement is the prototype of a geometric object which is called a contravariant vector. A set of quantities X_a are said to be the components of a contravariant vector if they transform, under a change of coordinates, as

$$X'^a = \frac{\partial x^{a\prime}}{\partial x^b} X^b \tag{C.291}$$

Covariant Vectors

$$df = \frac{\partial f}{\partial x^a} dx^a = 0 \tag{C.292}$$

$$\frac{\partial f'}{\partial x^{a\prime}} = \frac{\partial f}{\partial x^b} \frac{\partial x^b}{\partial x^{a\prime}} \tag{C.293}$$

$f = f(x^a) = f(x^{a\prime}) = f'$.

A set of quantities X_a are said to be the components of a covariant vector if they transform as

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b \tag{C.294}$$

Covariant vectors can be interpreted as linear functionals mapping vectors to $\mathbb{R}$.

$$\mathbf{V} = V^a \mathbf{e}_a = V'^a \mathbf{e}'_a \tag{C.295}$$

It is evident that if the vector $\mathbf{V}$ is to be invariant the basis vectors $\{\mathbf{e}_a\}$ must transform under coordinate transformations as the components of a covector. Similarly, basis covectors $\{\mathbf{e}^a\}$ must change under a coordinate transform as the components of a contravariant vector.

Tensors

A set of quantities T_{ab} are said to be the components of a covariant tensor of second?? if they transform as

$$T'_{ab} = \frac{\partial x^c}{\partial x'^b} \frac{\partial x^d}{\partial x'^a} T_{cd} == \tag{C.296}$$

Contractions of Tensors

$$\frac{\partial x^{a\prime}}{\partial x^c}\frac{\partial x^c}{\partial x^{b\prime}} = \delta_a^b \tag{C.297}$$

C.6.1 Affine Connection

$$\begin{aligned}
dX^a &= X^a(Q) - X^a(P) \\
&= X^a(x^b + dx^b) - X^a(x^b) \\
&= \frac{\partial X^a}{\partial x^b} dx^b
\end{aligned}$$

is not tensorial. Under a coordinate transformation a partial derivative of a vector X^a transforms as

$$\frac{\partial X'^a}{\partial x'^b} = \frac{\partial x^c}{\partial x'^b}\frac{\partial}{\partial x^c}\left(\frac{\partial x'^a}{\partial x^d}X^d\right) = \underbrace{\frac{\partial x^c}{\partial x'^b}\frac{\partial x'^a}{\partial x^d}\frac{\partial X^d}{\partial x^c}}_{tensorial} + \underbrace{\frac{\partial^2 x'^a}{\partial x^c \partial x^d}\frac{\partial x^c}{\partial x'^b}X^d}_{inhomogeneous} \tag{C.298}$$

This is not a tensorial transformation because of the inhomogeneous term.

have

$$dx^b \nabla_b X^a = \frac{\partial X}{\partial x^b} dx^b + \Gamma^a_{cb} X^c dx^b \tag{C.299}$$

For $\nabla_b X^a$ to transform as a tensor of type $(1,1)$ then the connection must transform as

$$\Gamma'^{\alpha}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial x^{\sigma}}{\partial x'^{\beta'}}\frac{\partial x^{\tau}}{\partial x'^{\gamma}}\Gamma^{\delta}_{\sigma\tau} + \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial^2 x^{\delta}}{\partial x'^{\beta}\partial x'^{\gamma}} \tag{C.300}$$

Any quantity that transforms as (C.300) is called an affine connection. If the second term on the right-hand side were absent, then this would be the transformation law of a tensor of type (1,2).

$$T^{\alpha}_{\beta\gamma} = \Gamma'^{\alpha}_{\beta\gamma} - \Gamma'^{\alpha}_{\gamma\beta} \tag{C.301}$$

is a tensor called the *torsion tensor*. If the torsion tensor vanishes then the connection is said to be *symmetric* or (*torsion-free*), i.e.

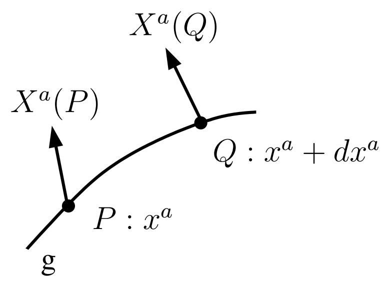


Figure C.17: connection.

$$\Gamma'^{\alpha}_{\beta\gamma} = \Gamma'^{\alpha}_{\gamma\beta} \tag{C.302}$$

Covariant differentiation can be extended to other types of tensors by demanding that the covariant derivative obeys the product rule of differential calculus. The covariant derivative of a scalar field is the same as its partial derivative,

$$\nabla\phi = \partial_a \phi. \tag{C.303}$$

Demanding that the covariant derivative obeys the Leibniz rule, then we find

$$\nabla_b X_a = \partial_b X_a - \Gamma^c_{ab} X_c \tag{C.304}$$

$$\nabla_c T^{a\ldots}_{b\ldots} = \partial_c T^{a\ldots}_{b\ldots} + \Gamma^a_{dc} T^{d\ldots}_{b\ldots} + \cdots - \Gamma^d_{bc} T^{a\ldots}_{d\ldots} - \ldots. \tag{C.305}$$

The covariant derivative contracted with X.

$$\nabla_X T^{a\cdots}_{b\cdots} := X^c \nabla_c T^{a\cdots}_{b\cdots} \tag{C.306}$$

$$\frac{D}{Du} T^{a\cdots}_{b\cdots} = \nabla_X T^{a\cdots}_{b\cdots} \tag{C.307}$$

C.6.2 Affine Geodesic

A geodesic is the closest thing there is to a straight line curved space time. In we are given a metric we can define a geodesic as the shortest distance between two points. However, there is a more general definition of a geodesic is its velocity vector is parallel transported along the curve it traces out in spacetime (it follows its own nose, so to speak). In other words, the parallely propagated vector at any point of the curve is parallel, that is, proportional to the vector at this point:

$$\frac{dx^b}{du}\nabla_b\left(\frac{dx^a}{du}\right) = \lambda(u)\frac{dx^a}{du} \tag{C.308}$$

The L.H.S. becomes

$$\begin{aligned}\frac{dx^b}{du}\nabla_b\left(\frac{dx^a}{du}\right) &= \frac{dx^b}{du}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{du}\right) + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^a}{du} \\ &= \frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du}.\end{aligned}$$

We have the geodesic equation

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \lambda(u)\frac{dx^a}{du}. \tag{C.309}$$

Note that this definition did not involve a metric, it only refers to a connection. Other geometric notions can also be defined using a connection only. Such description are desirable when we come to formulate the quantum theory where there is background metric. In fact LQG is formulated with a connection of a frame field (which we will be coming to presently). In the next section we shall verify that the differential equation derived from the principle of shortest distance is as in (C.309) but with the connection Γ^a_{bc} as a particular function of the metric. This connection is called the metric connection.

C.6.3 The Metric Connection

The vanishing of the covariant derivative of the metric is equivalent to requiring that the length of a vector is unchanged under parallel propagation. To see this first consider the covariant derivative of the length squared l^2 of a arbitrary vector l^μ,

$$\begin{aligned}0 = \partial_\sigma(l^2) = \nabla_\sigma(l^2) &= \nabla_\sigma(g_{\mu\nu}l^\mu l^\nu) \\ &= l^\mu l^\nu \nabla_\sigma g_{\mu\nu} + g_{\mu\nu}l^\nu\nabla_\sigma l^\mu + g_{\mu\nu}l^\mu\nabla_\sigma l^\nu \\ &= l^\mu l^\nu\nabla_\sigma g_{\mu\nu} + 2l_\mu\nabla_\sigma l^\mu \end{aligned} \tag{C.310}$$

By assumption the vector vector l^μ is parallel propagated i.e. $\nabla_\sigma l^\mu = 0$

$$l^\mu l^\nu \nabla_\sigma g_{\mu\nu} = 0 \tag{C.311}$$

since l^μ was arbitrary this implies,

$$\nabla_\sigma g_{\mu\nu} = 0 \tag{C.312}$$

(It should be noted that the vanishing of the covariant derivative was motivated by a physical requirement and not a mathematical one). This condition is sufficient to determine the connection as a function of the metric. We show this by using

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} + \Gamma^\rho_{\sigma\mu} g_{\nu\rho} + \Gamma^\rho_{\sigma\nu} g_{\mu\rho} = 0 \tag{C.313}$$

$\Gamma_{\mu\nu,\sigma} = g_{\sigma\lambda}\Gamma^\lambda_{\mu\nu}$. Written out explicitly, with cyclic rotating of indices,

$$\begin{aligned}
\partial_a g_{bc} + \Gamma_{ab,c} + \Gamma_{ac,b} = 0 \quad &\leftrightarrow \quad (\nabla_\sigma g_{ab} = 0) \\
\partial_\nu g_{\lambda\mu} + \Gamma_{\nu\lambda,\mu} + \Gamma_{\lambda\nu,\mu} = 0 \quad &\leftrightarrow \quad (\nabla_\nu g_{\lambda\mu} = 0) \\
\partial_\lambda g_{\mu\nu} + \Gamma_{\lambda\mu,\nu} + \Gamma_{\nu\mu,\lambda} = 0 \quad &\leftrightarrow \quad (\nabla_\lambda g_{\mu\nu} = 0)
\end{aligned} \tag{C.314}$$

These three equations are identical. But by adding the first two equations and subtracting the last (and remembering that the Christoffel symbol is symmetric in the lower case indices), we find:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\{\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\} \tag{C.315}$$

Covariant Derivative

$$\nabla_a V_b = \partial_a V_b + \Gamma^c_{ab} V_c \tag{C.316}$$

Metric Geodesic

$$\left(\frac{ds}{du}\right)^2 = g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}. \tag{C.317}$$

$$s = \int_{p_1}^{p_2} ds = \int_{p_1}^{p_2} \frac{ds}{du} du = \int_{p_1}^{p_2} \left(g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}\right)^{1/2} du. \tag{C.318}$$

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \left(\frac{d^2s}{du^2}\Big/\frac{ds}{du}\right)\frac{dx^a}{du}, \tag{C.319}$$

$$u = \alpha s + \beta, \tag{C.320}$$

where α and β are constants, then the righthand side vanishes.

we assume $ds \neq 0$

equations for a metric geodesic

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{C.321}$$

and

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = -1, \tag{C.322}$$

where Γ^a_{bc} is given by (C.315).

Proof.

$$\frac{d}{du}\frac{\partial \mathcal{L}}{\partial \dot{x}^a} - \frac{\partial \mathcal{L}}{\partial x^a} = 0 \tag{C.323}$$

We instead minimize this instead. So we use $\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b$ in

$$2\mathcal{L}\left[\frac{d}{du}\frac{\partial \mathcal{L}}{\partial \dot{x}^a} - \frac{\partial \mathcal{L}}{\partial x^a}\right] = 0 \tag{C.324}$$

which can be rewritten as

$$\frac{d}{du}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) - \frac{\partial \mathcal{L}^2}{\partial x^a} = 2\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\frac{d\mathcal{L}}{du} \tag{C.325}$$

Substituting for $\mathcal{L}^2$, the left hand side gives

$$
\begin{aligned}
\frac{d}{du}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) - \frac{\partial \mathcal{L}^2}{\partial x^a} &= \frac{d}{du}\left[\frac{\partial}{\partial \dot{x}^a}(g_{bc}\dot{x}^b\dot{x}^c)\right] - \frac{\partial}{\partial x^a}(g_{bc}\dot{x}^b\dot{x}^c) \\
&= \frac{d}{du}(2g_{ab}\dot{x}^b) - (\partial_a g_{bc})\dot{x}^b\dot{x}^c \\
&= 2g_{ab}\ddot{x}^b + 2\partial_c g_{ab} - \partial_a g_{bc}\dot{x}^b\dot{x}^c \\
&= 2g_{ab}\ddot{x}^b + 2\dot{x}^b\dot{x}^c[\frac{1}{2}(\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{bc})] \qquad \text{(C.326)}
\end{aligned}
$$

$$
\begin{aligned}
2\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\frac{d\mathcal{L}}{du} &= 2\frac{\partial}{\partial \dot{x}^a}(g_{bc}\dot{x}^b\dot{x}^c)\frac{d}{du}\frac{ds}{du} \\
&= 2(g_{bc}\dot{x}^b\dot{x}^c)^{-1/2}g_{ad}\dot{x}^d\frac{d^2s}{du^2} \\
&= 2\left(\frac{d^2s}{du^2}\Big/\frac{ds}{du}\right)g_{ab}\dot{x}^b. \qquad \text{(C.327)}
\end{aligned}
$$

multiplying through by g^{ad}

Lighlike inertial motion cannot be characterized with reference to proper time parameterization since the proper time along a null curve vanishes. However, this does not prevent us from characterizing such motion in the same manner as in the timelike case. To this end we go back to the more general definition of a geodesic as a curve $x^a(\lambda)$ with the property that the coordinate acceleration $d^2x^a/d\lambda^2$ at every point p is parallel to the tangent vector.

It is independent of the parameterization of the curve

C.6.4 Curvature

Let ξ^a be a contravariant vector field with

$$\dot{\xi}^c(x(\tau)) = 0. \qquad \text{(C.328)}$$

We take the curve to be small so that we can write

$$\xi^c(x) = \xi^c + \xi^c_{,b}x^b + \mathcal{O}(x^2). \qquad \text{(C.329)}$$

In an affine space without metric the term 'small' and 'large' appear to be meaningless. However, since differentiability is required, the small size limit *is* well defined. Thus, it is more precise to state that the curve is *infinitesimally* small.

If there is a strong gravitational field the contravariant vector may not return to its original value going around the loop once and have deviation $\delta\xi^a$. We find:

$$\begin{aligned}
\oint d\tau\dot{\xi} &= 0 \\
\delta\xi^c &= \oint d\tau\frac{d}{d\tau}\xi^c(x(\tau)) = -\oint \Gamma^c_{ab}\frac{dx^b}{d\tau}\xi^a(x(\tau))d\tau \\
&= -\oint d\tau\big(\Gamma^c_{ab} + \Gamma^c_{ab,d}x^d\big)\frac{dx^b}{d\tau}\big(\xi^a + \xi^a_{,d}x^d\big). \qquad \text{(C.330)}
\end{aligned}$$

where we chose the function $x(\tau)$ to be v. small, so that terms $\mathcal{O}(x^2)$ can be neglected. We have for a closed curve,

$$\oint d\tau\frac{dx^b}{d\tau} = 0 \quad \text{and} \quad \nabla_a\xi^c \approx 0 \to \xi^c_{,a} \approx -\Gamma^c_{ab}\xi^b, \qquad \text{(C.331)}$$

so that (M.-19) becomes

$$\delta\xi^c = \frac{1}{2}\left(\oint x^d\frac{dx^b}{d\tau}d\tau\right)R^c_{abd}\xi^a + \mathcal{O}(x^2). \qquad \text{(C.332)}$$

covariant derivative of $T_{\nu\gamma}$ is given by

$$\nabla_\mu T_{\nu\gamma} = \partial_\mu T_{\nu\gamma} + \Gamma^\delta_{\mu\nu}T_{\delta\Gamma} + \Gamma^\delta_{\mu\gamma}T_{\nu\delta} \qquad \text{(C.333)}$$

$$\begin{aligned}
\nabla_c(\nabla_d V^a) &= \partial_c(\nabla_d V^a) + \Gamma^e_{cd}(\nabla_e V^a) - \Gamma^a_{ce}(\nabla_b V^e) \qquad \text{(C.334)} \\
&= \partial_c(\partial_d V^a + \Gamma^a_{df}V^f) + \Gamma^e_{cd}(\partial_e V^a + \Gamma^c_{df}V^f) - \Gamma^a_{ce}(\partial_b V^d + \Gamma^d_{bf}V^f)
\end{aligned}$$

$$\begin{aligned}
\nabla_c\nabla_d V^a - \nabla_d\nabla_c V^a &= \partial_c\partial_d V^a - \partial_d\partial_c V^a \ \text{(we take} = 0) \\
&+ \ \partial_c\Gamma^a_{df}V^f - \partial_d\Gamma^a_{cf}V^f \\
&+ \ \Gamma^e_{cd}\partial_e V^a - \Gamma^e_{dc}\partial_e V^a \quad (= 0 \ \text{as} \ \Gamma^e_{cd} = \Gamma^e_{dc})
\end{aligned} \qquad \text{(C.335)}$$

$$\nabla_c\nabla_d V_a - \nabla_d\nabla_c V^a = \qquad \text{(C.336)}$$

where $R^{\alpha}_{\beta\gamma\delta}$

$$R^{\alpha}_{\beta\gamma\delta} \tag{C.337}$$

$$R^{\rho}_{\mu\nu\sigma} = \partial_{[\alpha}\Gamma_{\ \alpha} \tag{C.338}$$

C.6.5 Gaussian Normal Coordinates

$$x'^{a} = x^{a} + Q^{a}_{bc}\frac{x^{b}x^{c}}{2} \text{ where } Q^{a}_{bc} = Q^{a}_{cb} \tag{C.339}$$

$$\frac{\partial x'^{a}}{\partial x^{d}} = \delta^{a}_{d} + Q^{a}_{bd}x^{b} \tag{C.340}$$

$$\frac{\partial^{2} x'^{a}}{\partial x^{d}\partial x^{e}} = Q^{a}_{de} \tag{C.341}$$

$$x^{a} \stackrel{*}{=} 0 \tag{C.342}$$

$$\left[\frac{\partial x'^{a}}{\partial x^{b}}\right]_{P} = \delta^{a}_{b}, \quad \left[\frac{\partial x^{b}}{\partial x'^{a}}\right]_{P} = \delta^{a}_{b} \tag{C.343}$$

$$\Gamma'^{a}_{bc} = \frac{\partial x'^{a}}{\partial x^{d}}\frac{\partial x^{e}}{\partial x'^{b}}\frac{\partial x^{f}}{\partial x'^{c}}\Gamma^{d}_{ef} - \frac{\partial x^{d}}{\partial x'^{b}}\frac{\partial x^{e}}{\partial x'^{c}}\frac{\partial^{2} x'^{a}}{\partial x^{d}\partial x^{e}} \tag{C.344}$$

$$\left[\Gamma'^{a}_{\ bc}\right]_{P} = [\Gamma^{a}_{bc}]_{P} - Q^{a}_{bc} \tag{C.345}$$

choose $Q^{a}_{bc} = [\Gamma^{a}_{bc}]_{P}$

Mathematically this says that there exists a coordinate system the space time manifold is locally Minkowskian - one can always find a local coordinate system in which the metric tensor takes the pseudo-Euclidean form and the connection Γ^{a}_{bc} vanish at a point. In geometric terms, such a freely-falling frame of reference represents a local coordinate system.

Defined only in the immediate vicinity of a physically defined location (for example the place where two particles intersect), whose coordinate axes are very close to straight line and mutually perpendicular.

The physical statement is the gravitational field can be made to vanish at the place where two particles intersect, by going into free-fall.

C.6.6 Bianchi Identities

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} \tag{C.346}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\beta\alpha\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{C.347}$$

Cyclic Bianchi identity

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} \equiv 0 \tag{C.348}$$

$$\nabla_{\alpha} R_{\delta\sigma\beta\gamma} + \nabla_{\gamma} R_{\delta\sigma\alpha\beta} + \nabla_{\beta} R_{\delta\sigma\gamma\alpha} \equiv 0. \tag{C.349}$$

Define the tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R, \tag{C.350}$$

the so-called Einsten tensor (its name comming from that it appears on the left-hand side of Einstein's field equations of general relativity). We have the contracted Bianchi identity

$$\nabla_{\alpha} G^{\alpha\beta} = 0. \tag{C.351}$$

which follows from the Bianchi identity,

$$\nabla_{\sigma} R^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu} R^{\alpha}{}_{\beta\nu\sigma} + \nabla_{\nu} R^{\alpha}{}_{\beta\sigma\mu} = 0$$

Contracting α and μ.

$$\nabla_{\sigma} R^{\mu}{}_{\beta\mu\nu} + \nabla_{\mu} R^{\mu}{}_{\beta\nu\sigma} + \nabla_{\nu} R^{\mu}{}_{\beta\sigma\mu} = 0$$

then contracting with $g^{\sigma\beta}$ gives

$$\nabla_{\sigma} R^{\mu\sigma}{}_{\mu\nu} + \nabla_{\mu} R^{\mu\sigma}{}_{\nu\sigma} - g^{\sigma\beta} \nabla_{\nu} R^{\mu}{}_{\beta\mu\sigma} = 0$$

or

$$\nabla_{\mu}(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}) = 0.$$

C.6.7 Conformal Tensor, Ricci tensor and Ricci Scalar

C.6.8 The Weyl Tensor

The conformal tensor describes the components of the Riemann tensor, that are not contained in the Ricci tensor. The Ricci tensor being the contraction of the Riemann tensor, the rest of the information of the curvature is contained in the trace free part of the Riemann tensor, called the Weyl tensor,

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{bd} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R. \quad \text{(C.352)}$$

Constructed to have the same symmetries of the curvature tensor:

$$\begin{aligned} &C_{abcd} = -C_{abdc} = -C_{bacd} = C_{cdab}, \\ &C_{abcd} + C_{adbc} + C_{acdb} \equiv 0. \end{aligned} \quad \text{(C.353)}$$

$$C^{a}{}_{bad} \equiv 0 \quad \text{(C.354)}$$

C.6.9 Index Free Formulism

necessary to exploit coordinate systems. In this approach, tensor quantities are defined in terms of their components and the transformation rules for the latter under coordinate changes.

Physical quantities are (coordinate free) geometric objects scalar fields, vectors (not there components) and so on. The laws of physics are expressible as geometric relationships between these geometric objects.

$$\begin{aligned} X[f] &= X^a \frac{\partial}{\partial x^a} f \\ &= \\ &= X'^a \frac{\partial}{\partial x'^a} f \\ &= X'[f] \end{aligned} \quad \text{(C.355)}$$

the torsion $T(X, Y) = T^a_{\ bc} X^b Y^c$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{C.356}$$

$R(X,Y)Z = R^a{}_{bcd} X^c Y^d Z^b$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \tag{C.357}$$

An element $\xi \in T_p\mathbb{R}^n$ is identified with a mapping which takes every smooth function f, defined on any neighbourhood of p, to its directional derivative at p along ξ, denoted $\xi[f]$; that is

$$\xi[f] = \frac{d}{dt} f(p + t\xi)\Big|_{t=0}. \tag{C.358}$$

Notation for vector and covector coordinates basis

Suppose we have a vector ξ and a coordinate system x^a with basis vectors $\{\mathbf{e_a}\}_x$. Recall that for the vector ξ to remain unchanged under a coordinate transformation the basis vectors $\{\mathbf{e_a}\}_x$ should transform as the components of a covector, this suggests the following alternative notation

$$\{\mathbf{e_a}\}_x \quad \equiv \quad \{\frac{\partial}{\partial \mathbf{x^a}}\} \tag{C.359}$$

so that we would write

$$\xi = \xi^a \frac{\partial}{\partial \mathbf{x^a}},$$

and under a transformation to the new coordinates x' with new basis vectors we have

$$\xi = \xi^a \frac{\partial}{\partial \mathbf{x^a}} = \xi^b \Big(\frac{\partial x'^a}{\partial x^b} \frac{\partial}{\partial \mathbf{x'^a}} \Big) = \xi'^a \frac{\partial}{\partial \mathbf{x'^a}}$$

and hence

$$\xi'^a = \xi^b \frac{\partial x'^a}{\partial x^b}$$

which corresponds to the definition of the components of a contravariant vector. For covector ω and a coordinate system with basis covectors $\{\mathbf{e^a}\}_x$. For the covector ω to remain unchanged under a coordinate transformation the co-basis vectors $\{\mathbf{e^a}\}_x$ should transform as the components of a vector, this suggests the following alternative notation

$$\{\mathbf{e^a}\}_x \quad \equiv \quad \{\mathbf{dx^a}\} \tag{C.360}$$

so that we would write

$$\omega = \omega_a \mathbf{dx^a}.$$

$$\big(\mathbf{e^a}, \mathbf{e_b}\big) = \delta^a_b \quad \equiv \quad \big(\mathbf{dx^a}, \frac{\partial}{\partial \mathbf{x^b}}\big) = \delta^a_b \tag{C.361}$$

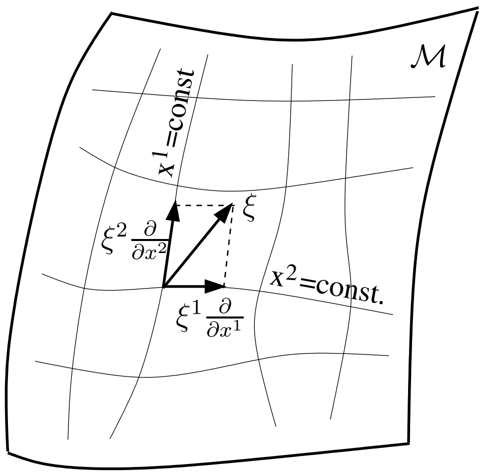


Figure C.18: coordbasevec. The vector ξ may be thought of as being composed of $\xi = \xi^1\partial/\partial\mathbf{x^1} + \xi^2\partial/\partial\mathbf{x^2}$. ξ^1 and ξ^2 are the components of ξ in the $(x^1, x^2)-$coordinate system.

The contraction of the vector and covector would go as

$$\begin{aligned}\big(\omega, \xi\big) &= \omega_a \xi^b \big(\mathbf{dx^a}, \frac{\partial}{\partial \mathbf{x^b}}\big) \\ &= \omega_a \xi^b \delta^a_b \\ &= \omega_a \xi^a \end{aligned} \tag{C.362}$$

A tensor with raised indices has contravariant components and is therefore expanded in terms of basis vectors, i.e.

$$\mathbf{T} = T^{ab} e_a \otimes e_b$$

A tensor with lowered indices has covariant components and is therefore expanded in terms of basis one-forms, i.e.

$$\mathbf{T} = T^{ab}\mathbf{e_a} \otimes \mathbf{e_b}$$

We have already seen an example of a tensor with lowered indicies, the metric tensor

$$\mathbf{g} = g_{ab}\omega^{\mathbf{a}} \otimes \omega^{\mathbf{b}}.$$

We have notation

$$\mathbf{T}(\omega_a, \omega_b) = T^{ab}$$

In the $x-$coordinates the metric tensor $\mathbf{g}$ would be written as

$$\mathbf{g} = g_{ab}\mathbf{dx^a} \otimes \mathbf{dx^b}.$$

In this notation an arbitrary $(p, q)-$tensor T is expanded in the above notation as

$$\mathbf{T} = T^{a_1 \ldots a_p}_{b_1 \ldots b_q} \frac{\partial}{\partial \mathbf{x^{a_1}}} \otimes \cdots \otimes \frac{\partial}{\partial \mathbf{x^{a_p}}} \otimes \mathbf{dx^{b_1}} \otimes \cdots \otimes \mathbf{dx^{b_q}} \tag{C.363}$$

C.7 Differential Geometry

C^r-function f if all its representatives $\psi \circ f \circ \phi^{-1} : R^m \rightarrow R^n$: (**??**)

In many undergraduate texts, one fixes a covering and coordinate patches and writes any tensor in terms of its value in some coordinate system. This approach is convenient an teaching of elementary GR, but it can obscure the coordinate independent meaning of important concepts. A more preferable formulation of the principle is based on modern differentiable geometry: such a formulation is coordinate free. Physical quantities are (coordinate free) geometric objects scalar fields, vectors (not there components) and so on. The laws of physics are expressible as geometric relationships between these geometric objects.

More advanced texts tend to use a coordinate free formulation, which is what we will present in the next few sections. We will see how the coordinate free approach replaces the 'tensor component' description.

Derive coordinate-free form of equations. These equations will involve objects such as vector fields, one-forms and scalar functions and geometric operations such as . The geometric objects belong to the manifold itself, be it a space-time or phase space or others, rather than any coordinate system on it. We will often use formulation in derivations employing local coordinate systems, but the definition used will hold on every chart, and hence they hold globally, making them chart independent. So we will have shown that they can be written in a consistent coordinate-independent manner.

C.7.1 Tangent Vectors

There are different ways of defining tangent vectors. Smooth manifolds embedded in Euclidean space

We don't want a definition dependent on embedding out space into a larger space. We can define tangent vectors in a way that is defined intrinsically to the manifold. a tangent vector being tangent to a curve in the manifold.

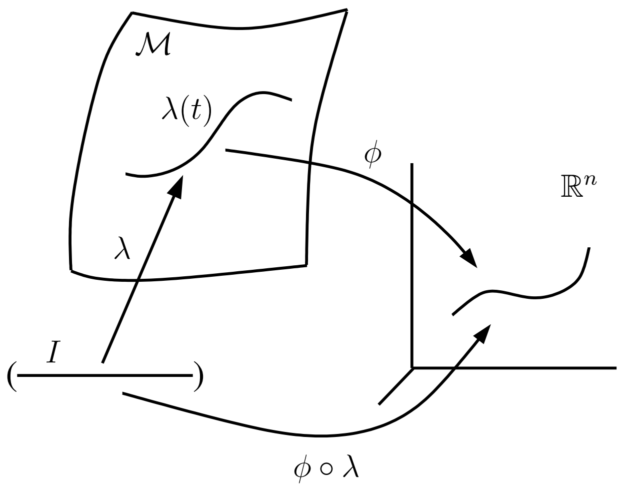


Figure C.19: tangvector. maps the tangent spaces of $\mathcal{M}$ *linearly* into the .

We will generalize the notion of a tangent vector to manifolds in a coordinate free way.

consider a curve in $\mathbb{R}^n$ $\lambda : (0, 1) \to \mathbb{R}^n$. The tangent vector to a point $p = \lambda(t_1)$ is

$$\left(\frac{d\lambda^1}{dt}\bigg|_{t=t_1}, \ldots, \frac{d\lambda^n}{dt}\bigg|_{t=t_1} \right) \tag{C.364}$$

where $\lambda(t) = (\lambda^1(t), \ldots, \lambda^n(t)) \in \mathbb{R}^n$.

Basis Vectors

The rate of change of $f(\lambda(t))$ at $t = 0$ along the curve is

$$\frac{d}{dt}f(\lambda(t))\Big|_{t=0} \tag{C.365}$$

In terms of a coordinate system, this becomes

$$\left[\frac{\partial}{\partial x^a}f(x^a)\right]\frac{dx^a}{dt}(\lambda(t))\Big|_{t=0} \tag{C.366}$$

In other words, $f(\lambda(t))$ at $t=0$ is given by applying the differential operator X to f, where

$$\frac{d\lambda(t)}{dt} = X^a\frac{\partial f}{\partial x^a} =: X[f]. \tag{C.367}$$

Thus are the components of X_p in the basis

$$\frac{\partial}{\partial x^a}\Big|_p. \tag{C.368}$$

The coordinate transformation

$$y^b = y^b(x^1,\ldots,x^n), \quad b = 1,\ldots,n.$$

If we have a coordinate system in a neighbourhood U of P, then the coordinate basis $\{\frac{\partial}{\partial x^a}\}$.

$$\hat{V} = V^a\frac{\partial}{\partial x^a} = V'^b\hat{e}_b. \tag{C.369}$$

The numbers $\{V^a\}$ are the components of $\hat{V}$ on $\{\frac{\partial}{\partial x^a}\}$. The numbers $\{V^a\}$ are the components of $\hat{V}$ on $\{\frac{\partial}{\partial y^a}\}$.

$$\frac{\partial}{\partial x^1} = \frac{\partial y^1}{\partial x^1}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^1}\frac{\partial}{\partial y^2} + \cdots + \frac{\partial y^n}{\partial x^1}\frac{\partial}{\partial y^n}$$

and similarly for other x^as.

$$X = X^a\frac{\partial}{\partial x^a} = X'^a\frac{\partial}{\partial y^a} \tag{C.370}$$

This shows that X^a and X'^a are related by

$$X'^a = X^b \frac{\partial y^a}{\partial x^b}. \tag{C.371}$$

the components of the vector transform in such a way that the vector itself is left invariant.

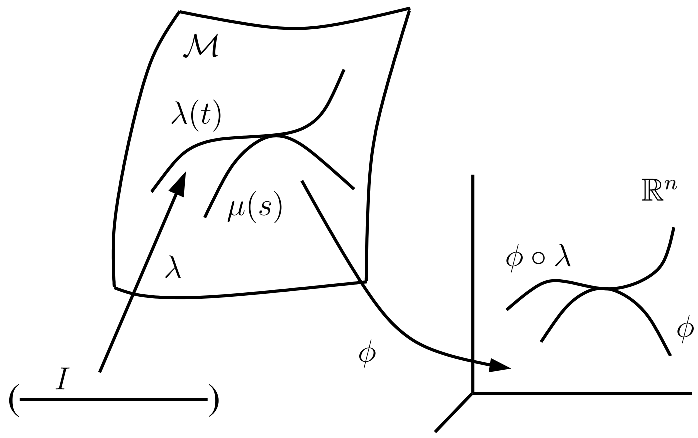


Figure C.20: Two curves $\lambda(t)$ and $\mu(t)$ are tangent at p if and only if their images are tangent at $\phi(p)$ in $\mathbb{R}^n$.

The basis need not be $\{e_a\}$, we can form linear combinations $\hat{e}_i := E_i^a e_a$, where $E = (E_i^a)$ are matrices that . These are referred to as **non-coordinate basis** or a **frame**. Note that they transform as scalars under coordinate transformations. These are very important as frames fields will be basic variables in a formulation of GR that resembles Gauge field theories that are adopted for the quantum theory

C.7.2 Covectors

it is natural to regard $\{dx^a\}$ as a basis of $T_p^*\mathcal{M}$.

$$< dx^b, \partial/\partial_a >= \delta_a^b. \tag{C.372}$$

An arbitrary covector can be written

$$\omega = \omega_a dx^a \tag{C.373}$$

where ω_a are the components of ω.

Mixed tensors

The set of type (p, q). A tensor is written in terms of the coordinate basis as

$$T = T^{a_1 \dots a_p}_{b_1 \dots b_q} \frac{\partial}{\partial x^{a_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{a_p}} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_q}. \tag{C.374}$$

The various connections are defined by index structure, i.e. how they transform, and by restrictions placed on them.

C.7.3 Induced Metric and Other Objects on Sub-manifolds

If we denote the normal to the surface as n^a ($n^a n_a = 1$) then the induced metric can be written as

$$h_{ab} = g_{ab} - n_a n_b \tag{C.375}$$

So that h_{ab} projects out the components of a vector normal to the hypersurface. Say $\xi^b = Cn^b$

$$h^a{}_b \xi^b = \delta^a{}_b \xi^b - n^a n_b \xi^b = 0. \tag{C.376}$$

Let $\mathcal{N}$ be a $n-$dimensional manifold of an $m-$dimensional manifold . The hypersurface on which we have coordinates y^α. If $\mathcal{N}$ is a hypersurface in $\mathcal{M}$, for the point labelled by y^α corresponds to the point of labelled by $\mathcal{M}$. Thus the hypersurface is described by the equations

$$x^a(y^\alpha) \tag{C.377}$$

tangent vector e^α_a, and induced metric $h_{\alpha\beta}$

$$e^\alpha_a = \frac{\partial y^\alpha}{\partial x^a} \tag{C.378}$$

ds^2 for an infinitesimal curve lying in the hypersurface:

$$ds^2 = h_{\alpha\beta} dy^\alpha dy^\beta = h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^a} \frac{\partial y^\beta}{\partial x^b} dx^a dx^b \tag{C.379}$$

$$g_{ab} = h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^a} \frac{\partial y^\beta}{\partial x^b} \tag{C.380}$$

or

$$g_{\mathcal{M}ab}(x) = g_{\mathcal{N}\alpha\beta}(f(x))\,\frac{\partial f^\alpha}{\partial x^a}\frac{\partial f^\beta}{\partial x^b} \tag{C.381}$$

where f^α denote the coordinates of $f(x)$.

Definition $g_{\mathcal{N}\alpha\beta}(f(x))$ is said to be the pull-back of $g_{\mathcal{M}ab}(x)$, denoted f_*g

$$\begin{aligned} g_{ab}dx^a \otimes dx^b &= \delta_{\alpha\beta}\,\frac{\partial f^\alpha}{\partial x^a}\frac{\partial f^\beta}{\partial x^b}\,dx^a \otimes dx^b \\ &= d\theta \otimes \theta + \sin^2\theta\, d\phi \otimes \phi. \end{aligned} \tag{C.382}$$

where

$$h_{\alpha\beta} = g_{ab}e^a_\alpha e^b_\beta \tag{C.383}$$

is the induced metric or the first fundamental form of the hypersurface. It is a scalar with respect to coordinate transformations $x^a \to x'^a$ on $\mathcal{M}$. behaves like a tensor under coordinate transformations of the manifold $\mathcal{N}$.

C.8 Active Diffeomorphisms and the Lie Derivative

Up until now we have only considered coordinate transformation, that is, passive diffeomorphisms. We now move onto active diffeomorphisms. As their formulas look very alike the two are easily mixed up. But as we have seen in chapter 1 they are quite different, active diffeomorphisms relate *distinct* spacetime geometries, whereas a coordinate transformation merely represents the same spacetime geometry in a different coordinate system.

In chapter 1 we defined an active diffeomorphism as simultaneously dragging the metric and matter fields over the spacetime manifold while keeping the coordinate lines 'attached' (fig C.8). This is called a **pushforward**.

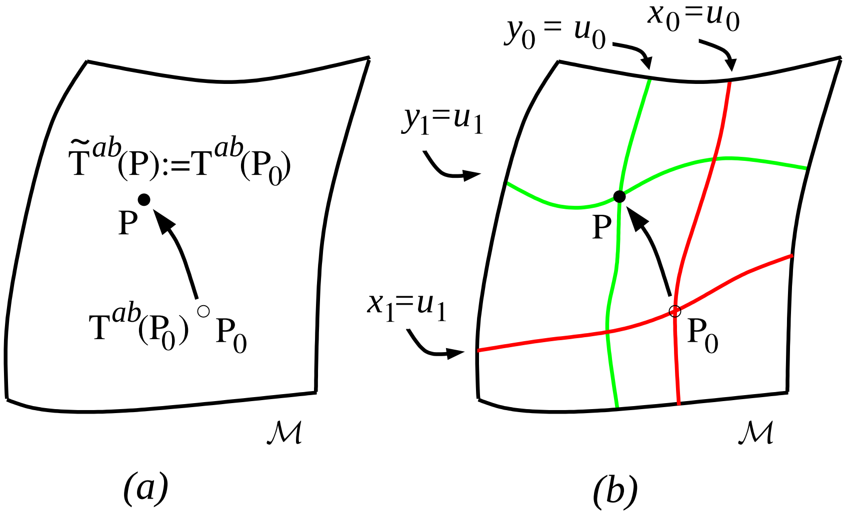


Figure C.21: activeDiffGeom. A pushforward of the tensor $T_{ab}(x)$, i.e. $T_{ab}(x) \to \tilde{T}_{ab}(y)$.

Let us slightly modify the definition of an active diffeomorphism by requiring that after we have dragged the fields across the manifold we perform a coordinate transformation back to the original coordinates. An active diffeomorphism defined this way then relates different space-time geometries and matter field configurations in the same coordinate system.

They relate $g_{ab}(x)$ to $\tilde{g}_{ab}(h(x))$ by the Jacobian matrix of the coordinate transformation $x \mapsto h(x)$,

$$\tilde{g}_{ab}(h(x)) = \Lambda^a_c \Lambda^b_d g_{ab}(x) \tag{C.384}$$

Two metrics related by an active diffeomorphism, viewed in the same coordinate system, compared at the same point also have 'transformation matrices', however, these have a different geometric interpretation!

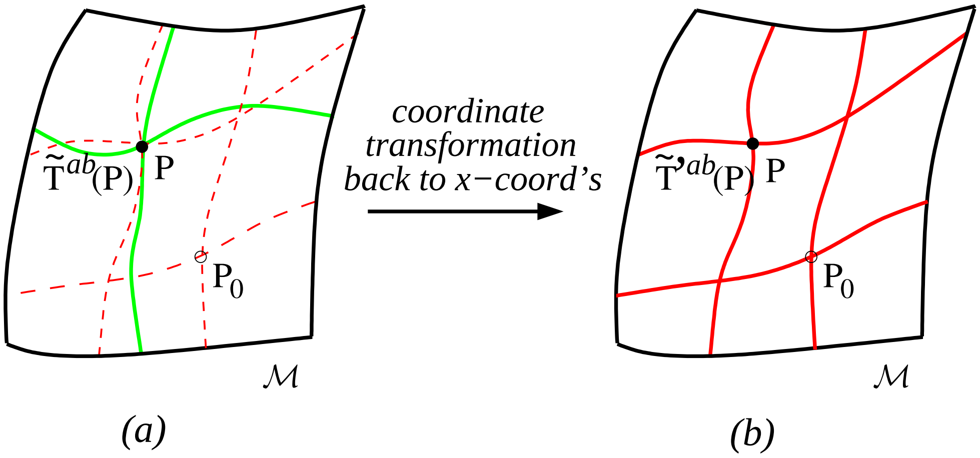


Figure C.22: activeDiffGeom1. The red dashed lines in (a) are the x−coordinate lines of the point P. We perform a coordinate transformation back to the original coordinate system. The pushed-forward tensor $\tilde{T}_{ab}(y)$ transforms to $\tilde{T}'_{ab}(x)$, i.e. $\tilde{T}_{ab}(y) \to \tilde{T}'_{ab}(x)$.

$$\tilde{g}_{ab}(x) = \frac{\partial h^c(x)}{\partial x^a}\frac{\partial h^d(x)}{\partial x^b} g_{cd}(h(x)) \tag{C.385}$$

The fact that the coordinate values do not change, while the tensor fields do, distinguishes the active diffeomorphism from a simple coordinate transformation.

Passive diffeomorphism invariance refers to invariance under change of coordinates, i.e. the same object represented in different coordinate systems. Choose a (local) coordinate system for S in which the metirc $g_{ab}(x)$. (If the map h sends each point to the same point of the manifold $\mathcal{M}$, then in the second system S' the metric given by $\tilde{g}_{ab}(h(x))$, $f(x)$ being the coordinates on $\mathcal{M}$ of the second system.)

Any theory can be made invariant under passive diffeomorphisms because a dynamical system doesn't care which coordinate system you use to describe it. However, general relativity is the only theory invariant under active diffeomorphisms and this invariance is a property of the dynamical theory itself.

Maths Tools for Manifold Without a Metric

In the previous section we reviewed metrics on manifolds, these are important in classical general relativity and are what a physicists is most likely to be familiar with. As we have learned, in reality it is only **geometry up to active diffeomorphisms** that has physical meaning. As we have empathized, in formulating the quantum theory we prefer not to employ metrics with its direct relation to the notion of distance.

There is a rich geometric structure of the manifold without a metric defined on it. Important tools of the Lie derivative and differential forms which have nothing to do with metrics. These will be important in the quantum theory where we will avoid introducing a background metric whenever possible.

C.8.1 Mapping a Manifold to Itself Along Integral Curves

We start by considering a congruence of curves defined such that only one curve goes through each point in the manifold. Then, given any one curve of the congruence,

$$x^\mu = x^\mu(u), \tag{C.386}$$

we can use it to define the tangent vector field dx^μ/du along the curve. If we do this for every curve in the congruence, then we end up with a vector field X^μ (given by dx^μ/du at every point) defined over the whole manifold, then this can be used to define a congruence of curves in the manifold called the orbits or trajectories of X^μ.

a smooth, non-intersecting family of curves on a manifold then the tangent vectors at each point can be taken together to form a vector field on the manifold.

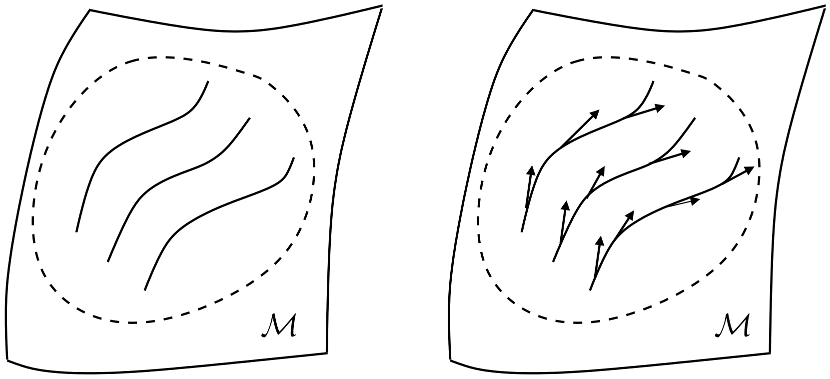


Figure C.23: The tangent vector field resulting from a congruence of curves.

These curves are obtained by solving differential equations

$$\frac{dx^\mu}{du} = X^\mu(x(u)) \tag{C.387}$$

Let x^i be a local coordinate system and let x^i_p be the coordinates of p. The equation of the integral curve is

$$\frac{d}{dt}x^i(t) = X^i(x^m(t)),$$

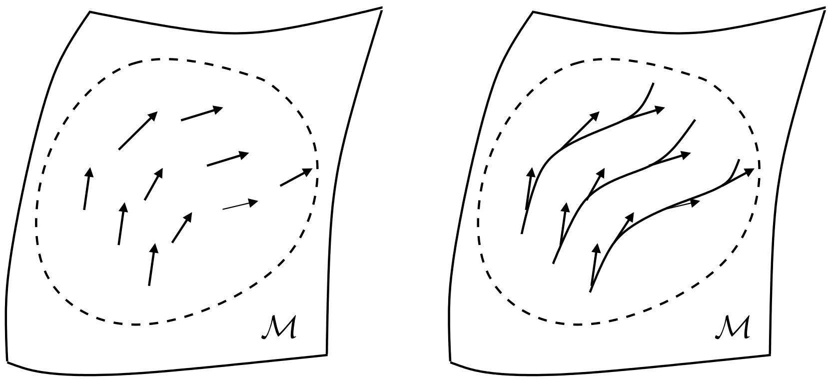


Figure C.24: The local congruence of curves resulting from vector field.

with initial conditions $x^i(0) = x^i_p$. Provided X is smooth the theory of ordinary differential equations guarantees the existence and uniqueness, (at least locally, i.e., for small t), of a solution. Uniqueness implies that no two curves in the congruence intersect (at least locally).

Definition A congruence of curves is a family of curves such that precisely one curve of the family passes through each point. It is a geodesic congruence if the curves are geodesics.

Active Diffeormorphisms

C.8.2 The Lie Derivative

This is called an *active transformation.* The *passive transformation* is a coordinate transformation.

A contravariant vector flow determines a local congruence of curves,

$$x^a = x^a(u),$$

where the tangent vector field to the congruence is

$$\frac{dx^a}{du} = X^a.$$

at least locally, a vector field generates a unique integral flow about any given point p. We use this flow to take a tensor to a nearby point and hence form a derivative. This derivative is called the Lie derivative.

$$\sigma^b(\epsilon, p) = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.388}$$

The Lie derivative of a scalar field $f \in \mathcal{C}^\infty(\mathcal{M})$. Let X be a vector field on $\mathcal{M}$ we define the Lie derivative of f along X to be

$$\mathcal{L}_X f(p) = \lim_{\epsilon \to 0} \frac{f(\sigma(\epsilon, p)) - f(p)}{\epsilon} \tag{C.389}$$

which is the usual directional derivative along X.

point transform

$$x'^b = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.390}$$

We generate a new vector (with vector components in the x' coordinates). By definition its components are related to $T^a(x)$ by a pushforward

$$\tilde{T}'^a(x') := T^a\left(x^c + \epsilon X^c(x)\right) = T^a(x) + \epsilon X^c(x)\partial_c T^{ab}(x) + \mathcal{O}(\epsilon^2). \tag{C.391}$$

We now wish to transform this tensor to the $x-$coordinates so we can compare it with the original tensor $T^{ab}(x)$. Using (C.390) we have

$$\frac{\partial x^a}{\partial x'^c} = \delta^a_c - \epsilon \partial_c X^a + \mathcal{O}(\epsilon^2) \tag{C.392}$$

The parameter distance derivative of an object along the vector field is the *Lie derivative.*

$$\begin{aligned}
\tilde{T}^a(x) &= \frac{\partial x^a}{\partial x'^c}\tilde{T}'^c(x') \\
&= (\delta^a_c - \epsilon\partial_c X^a)(T^c(x) + \epsilon X^e \partial_e T^c) + \mathcal{O}(\epsilon^2) \\
&= T^a(x) + [X^e \partial_e T^a - \partial_c X^a T^c(x)]\epsilon + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{C.393}$$

$$\mathcal{L}_X T^a = \lim_{\epsilon \to 0} \frac{\tilde{T}^a(x) - T^a(x)}{\epsilon} \tag{C.394}$$

$$\mathcal{L}_X T_a(x) = X^c \partial_c T_a + T_b \partial_c X^a \tag{C.395}$$

What is the Lie derivative for a tensor $T^{ab}(x)$? We generate a new tensor (with tensor components in the x' coordinates). By definition its components are related to $T^{ab}(x)$ by a pushforward

$$\tilde{T}'^{ab}(x') := T^{ab}\left(x^c + \epsilon X^c(x)\right) = T^{ab}(x) + \epsilon X^c(x)\partial_c T^{ab}(x) + \mathcal{O}(\epsilon^2). \tag{C.396}$$

We now wish to transform this tensor to the $x-$coordinates so we can compare it with the original tensor $T^{ab}(x)$. Using (C.392) again. The parameter distance derivative of an object along the vector field is the *Lie derivative.*

$$\begin{aligned}
\tilde{T}^{ab}(x) &= \frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}\tilde{T}'^{cd}(x') \\
&= (\delta^a_c - \epsilon\partial_c X^a)(\delta^b_d - \epsilon\partial_d X^b)(T^{cd}(x) + \epsilon X^e\partial_e T^{cd}) + \mathcal{O}(\epsilon^2) \\
&= T^{ab}(x) + [X^e\partial_e T^{ab} - \partial_c X^a T^{cb}(x) - \partial_d X^b T^{ad}(x)]\epsilon + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{C.397}$$

$$\mathcal{L}_X T^{ab} = \lim_{\epsilon\to 0}\frac{\tilde{T}^{ab}(x) - T^{ab}(x)}{\epsilon} \tag{C.398}$$

$$\mathcal{L}_X T^{ab} = X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a. \tag{C.399}$$

$$\mathcal{L}_X T_a(x) = X^c\partial_c T_a + T_b\partial_c X^a, \qquad \mathcal{L}_X T_{ab}(x) = X^c\partial_c T_{ab} + T_{cb}\partial_a X^c + T_{ac}\partial_b X^c$$

The first term of the Lie derivative, $X^c\partial_c$, corresponds to the pushforward, shifting the tensor to another point in the manifold. The remaining terms arise from the coordinate transformation back to the original coordinates. Is it coordinate invariant? Does it have the same form in all coordinate systems? In fact (C.395) is equivalent to:

$$X^c\nabla_c T^a - T^c\nabla_c X^a \tag{C.400}$$

since

$$\begin{aligned}
\mathcal{L}_X T^a(x) &= X^c\partial_c T^a - T^c\partial_c X^a \\
&= X^c(\partial_c T^a + \Gamma^a_{dc}T^d) - T^c(\partial_c X^a + \Gamma^a_{dc}X^d) \\
&= X^c\nabla_c T^a - T^c\nabla_c X^a
\end{aligned} \tag{C.401}$$

where we have used that the connection is symmetric in its lower indices. Similarly, (C.399) is equivalent to:

$$
\begin{aligned}
& X^c\nabla_c T^{ab} - T^{ac}\nabla_c X^b - T^{cb}\nabla_c X^a \\
= \;& X^c(\partial_c T^{ab} + \Gamma^a_{dc}T^{db} + \Gamma^b_{dc}T^{ad}) - T^{ac}(\partial_c X^b + \Gamma^b_{dc}X^d) - T^{cb}(\partial_c X^a + \Gamma^a_{dc}X^d) \\
= \;& X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a + T^{db}X^c(\Gamma^a_{dc} - \Gamma^a_{cd}) + T^{ad}X^c(\Gamma^b_{dc} - \Gamma^b_{cd}) \\
= \;& X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a
\end{aligned} \tag{C.402}
$$

In general, the partial derivatives appearing in Lie derivatives can be replaced by covariant derivatives. Hence, the combination of pushback and coordinate transformation make the Lie derivative a tensor in the tangent space at x^a.

$$
\tilde{T}'(q) = T(h_\epsilon(p)) \qquad \tilde{T}(p) = h_{\epsilon *}[T(h_\epsilon(p))] \tag{C.403}
$$

We can write down the coordinate free equation

$$
(\mathcal{L}_X T)(p) = \frac{h_{\epsilon *}[T(h_\epsilon(p))] - T(p)}{\epsilon} \tag{C.404}
$$

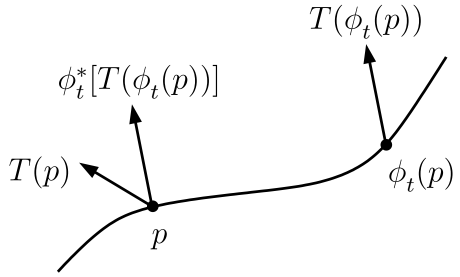


Figure C.25: .

the original tensor components at a different point. distinguishes the Lie derivative from the directional derivative.

the curve passing through P is given by x^1 varying, with x^2, x^3, x^4 all constant along the curve, and such that

$$
X^\alpha \stackrel{*}{=} \delta^\alpha_1 = (1, 0, 0, 0) \tag{C.405}
$$

along this curve. The notation used in means that the equation holds only in a particular coordinate system. Then it follows that

$$
X = X^\alpha \partial_\alpha = \partial_1, \tag{C.406}
$$

and equation reduces to

$$L_X T_{\alpha\beta} \stackrel{*}{=} \partial_1 T_{\alpha\beta} \tag{C.407}$$

Thus, in this special coordinate system, Lie differentiation reduces to ordinary differentiation.

If we have a map ϕ from a manifold $\mathcal{M}$ to another manifold $\mathcal{N}$, and we choose a point $x \in \mathcal{M}$, we can *push forward* a vector from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$, by a head-to-head and tail-to-tail map. If the vector has components X^μ and the map takes the point with coordinates x^μ to one with coordinates $\xi(x)$, the vector $\phi_* X$ has components

$$(\phi_* X)^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} X^\nu. \tag{C.408}$$

This looks like the transformation formula for contravariant vector components under a coordinate transformation, but we are doing an active transformation, changing a vector into a different one.

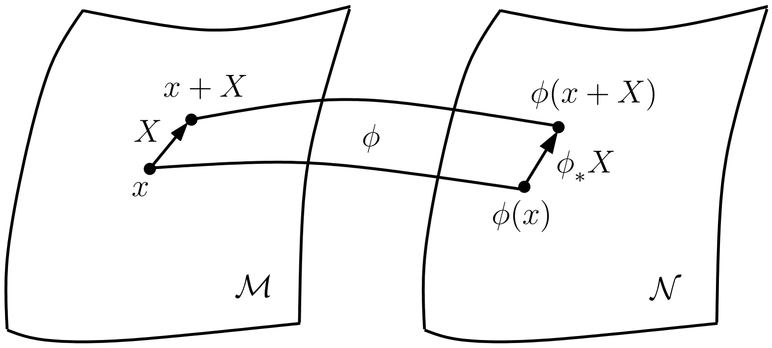


Figure C.26: pullbackDef0. Pushing forward a vector X from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$.

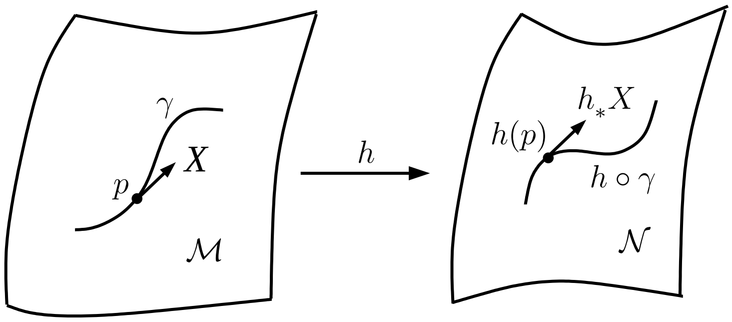


Figure C.27: The **push-forward map** h_* that maps the tangent spaces of $\mathcal{M}$ *linearly* into the tangent spaces of $\mathcal{N}$.

pushforward ϕ_* (**??**)

Recall that a one-form maps a vector to a number. Given a one-form ω on $\mathcal{N}$, we define $\phi^*\omega$ as a one-form on $\mathcal{M}$ by specifying what we get when we plug the vector X at $x \in \mathcal{M}$ into it. This we do by pushing the X forward to $T\mathcal{N}_{\phi(x)}$, plugging it into ω, and declaring the result to be the evaluation of $\phi^*\omega$ on the X. Symbolically

$$[\phi^*\omega](X) = \omega(\phi_* X). \tag{C.409}$$

or in components

$$[\phi^*\omega]_a X^a = \omega_a [\phi_* X]^a. \tag{C.410}$$

We work a coordinate system $(x^1, \ldots, x^m)$, such that x^1 is the parameter along the integral curves and the other coordinates are choosen any way. In this coordinate system and the components of the tensor pulled back from $\phi_t(p)$ to p are simply

$$\phi_{t\star}[T^{a_1\ldots a_k}_{b_1\ldots b_l}(\phi_t(p))] \stackrel{*}{=} T^{a_1\ldots a_k}_{b_1\ldots b_l}(x^1 + t, x^2, \ldots, x^n). \tag{C.411}$$

In this coordinate system the Lie derivative becomes

$$\mathcal{L}_V T^{a_1\ldots a_k}_{b_1\ldots b_l} \stackrel{*}{=} \frac{\partial}{\partial x^1} T^{a_1\ldots a_k}_{b_1\ldots b_l}, \tag{C.412}$$

Coordinate-Free Description

We will prove

$$\mathcal{L}_X Y = [X, Y]. \tag{C.413}$$

$$\sigma^b(\epsilon, p) = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.414}$$

for any f

$$
\begin{aligned}
Y_{\sigma(\epsilon)}f &= \sum_a Y^b(\sigma(\epsilon))\frac{\partial}{\partial x^a}\Big|_{\sigma(\epsilon)} f \\
&= \sum_a (Y^b + \epsilon X^c \partial_c Y^a)\Big|_{\sigma(\epsilon)} f + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b + \epsilon X^c \partial_c Y^a)(\partial f + \epsilon X^c \partial_a \partial_c f) + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b \partial_a + \epsilon X^c \partial_c Y^a \partial_a + \epsilon Y^a X^c \partial_c \partial_a) f + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{C.415}
$$

Therefore

$$
\begin{aligned}
\sigma(-\epsilon)_* Y_{\sigma(\epsilon)} f &= Y_{\sigma(\epsilon)}(f \circ \sigma(-\epsilon)_*) \\
&= \sum_a Y^a \; (h \circ \sigma(\epsilon))\frac{\partial}{\partial x^a}\Big|_p (h \circ \sigma(-\epsilon)) + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b \partial_a + \epsilon X^c \partial_c Y^a \partial_a + \epsilon Y^a X^c \partial_c \partial_a)(f - \epsilon \partial_b f X^b) + \mathcal{O}(\epsilon^2) \\
&= \sum_a Y^b \partial_a f + \epsilon(\partial_c Y^a X^c - Y^c \partial_c X^a)\partial_a f + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{C.416}
$$

From which

$$
\begin{aligned}
\frac{\sigma(-\epsilon)_* Y_{\sigma(\epsilon)}(f) - Y(f)}{\epsilon} &= (\partial_c Y^a X^c - Y^c \partial_c X^a)\partial_a f + \mathcal{O}(\epsilon^2) \\
&= [X, Y]^a \partial_a f + \mathcal{O}(\epsilon^2) \\
&= [X, Y]^a (f) + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{C.417}
$$

The Lie derivative of a covariant tensors

$$
L_X Y^\alpha = X^\beta \partial_\beta Y^\alpha - Y^\beta \partial_\beta X^\alpha \tag{C.418}
$$

The Lie derivative of a covariant vector field Y_α is given by

$$
L_X Y_\alpha = X^\beta \partial_\beta Y_\alpha + Y^\beta \partial_\alpha X^\beta \tag{C.419}
$$

The Lie Derivative

there is a coordinate system in which

$$\mathcal{L}_{\vec{N}}\vec{M} = N^a\partial_a M_b - M^a\partial_a N_b \tag{C.420}$$

It satisfies the Leibniz rule

$$L_X(Y^a Z_{bc}) = Y^a(L_X Z_{bc}) + (L_X Y^a)Z_{bc}. \tag{C.421}$$

It is type-preserving; that is, the Lie derivative of s tensor of type(p,q) is again a tensor of type (p,q).

The Lie derivative of a scalar field ϕ is simply an ordinary derivative in the direction of X

$$L_X\phi = X\phi = X^a\partial_a\phi \tag{C.422}$$

Now, given the Lie derivative of a vector and a scalar, we can apply the Leibniz rule to deduce the Lie derivative of a covariant vector field Y_a: consider the Lie derivative of the scalar formed by the contraction of an arbitrary vector Z^a with an arbitrary covector Y_a.

$$L_X(Y_c Z^c) = X^b\partial_b(Y_c Z^c) = Z^c X^b\partial_b Y_c + Y_c X^b\partial_b Z^c \tag{C.423}$$

whereas the Leibniz rule gives

$$L_X(Y_c Z^c) = Y_c L_X Z^c + (L_X Y_c)Z^c. \tag{C.424}$$

$$Z^c L_X Y_c = Z^c X^b\partial_b Y_c + Z^c Y_b\partial_a X^c. \tag{C.425}$$

but as Z^c is arbitrary this means

$$L_X Y_a == X^b\partial_b Y_a + Y_b\partial_a X^b. \tag{C.426}$$

$$\bar{T} := \phi_{\Delta\lambda}T(p_0) \quad \overline{x}^a(P) = x^a(P_0) \tag{C.427}$$

$$\mathcal{L}_\xi T := \lim_{\Delta\lambda}\frac{\phi_{\Delta\lambda}T(x) - T(x)}{\Delta\lambda} \tag{C.428}$$

$$\mathcal{L}_X T^{a\ldots}_{b\ldots} = X^c\partial_c T^{a\ldots}_{b\ldots} - T^{c\ldots}_{b\ldots}\partial_c X^a - \cdots + T^{a\ldots}_{c\ldots}\partial_b X^c + \cdots \tag{C.429}$$

C.8.3 Pull-back and Lie Derivative of a co-vector

The pullback of the function f by ϕ, denoted $\phi^* f$, is defined by

$$\phi^* f = (f \circ \phi). \tag{C.430}$$

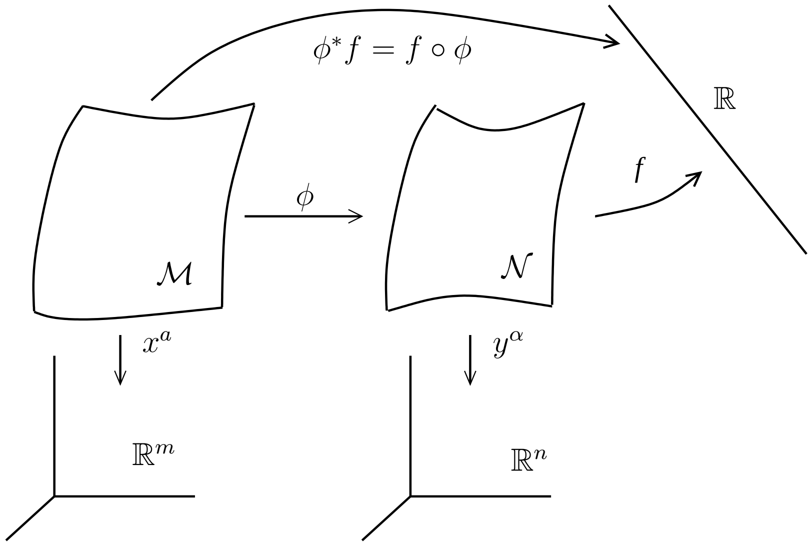


Figure C.28: The **pullback map** ϕ^* of a function f from $\mathcal{N}$ to $\mathcal{M}$ by a map $\phi : \mathcal{M} \to \mathcal{N}$ is the composition of ϕ with f.

Suppose we have a smooth map $h : \mathcal{M} \to \mathcal{N}$ as in section C.7.1. We saw that we could push-forward a vector $X_p \in T_p\mathcal{M}$ to a vector $h_* X_{h(p)} \in T_{h(p)}\mathcal{N}$ by

$$h_* X_{h(p)}(g) = X_p(g \circ h) \tag{C.431}$$

There should be a dual map which maps co-vectors in $\mathcal{N}$ to co-vectors in $\mathcal{M}$.

$$h_p^* \omega(X_p) =< h^* \omega, X_p > = < \omega, h_* X_{h(p)} > = \omega(h_* X_{h(p)}) \tag{C.432}$$

Let $h : \mathcal{M} \to \mathcal{N}$, $(y^1, \dots y^m)$ be local coordinates on $V \subset \mathcal{N}$ and $(x^1, \dots x^n)$ be local coordinates on $U \cap h^{-1}(V) \subset \mathcal{M}$. If

$$\omega = \sum_{b=1}^{n} \omega_b dy^b \Big|_p \tag{C.433}$$

We have

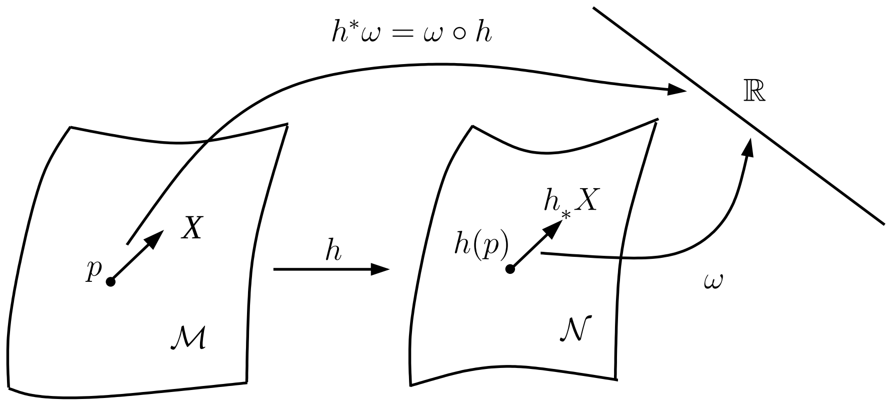


Figure C.29: pushLie. The maps the co-tangent spaces of $\mathcal{M}$ *linearly* into the co-tangent spaces of $\mathcal{N}$.

$$< \omega, h_* X_{f(p)} > \;=\; \sum_{b=1}^{n} \omega_b (f^* \omega X_{f(p)})^b \tag{C.434}$$

C.8.4 More on Lie Derivative

Definition A one-parameter group of diffeomorphisms. A one-parameter family of maps $\{\phi_t\}_{t\in\mathbb{R}}$ is said to be a one parameter group of diffeommorphism if:

(i) Each $\phi_t : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism;

(ii) $\phi_0 = \mathrm{id}$;

(iii) $\phi_{s+t} = \phi_s \circ \phi_t$ for all $s, t \in \mathbb{R}$.

That is we have a group action of $\mathbb{R}$ on $\mathcal{M}$.

□

Lemma C.8.1 *Let φ_t be the one parameter group of diffeomorphisms generated by the complete vector field X on the manifold $\mathcal{M}$, and $\psi : \mathcal{M} \to \mathcal{M}$ is a diffemorphism on $\mathcal{M}$. Then $\psi \circ \varphi_t \circ \psi^{-1}$ is the one-parameter group of diffeomorphisms generated by $\psi_* X$.*

Proof: We show that the tangent to the curve $\phi_p(t) := (\psi \circ \varphi_t \circ \psi^{-1})(p)$

$$
\begin{aligned}
(\psi_* X)_p f \quad &:= \quad X_q(f \circ \psi) = \frac{d}{dt} f \circ \psi(\varphi_t(q))\Big|_{t=0} \\
&= \quad \frac{d}{dt} f\big(\psi \circ \varphi_t \circ \psi^{-1}(\psi(q))\big)\Big|_{t=0} \\
&= \quad \frac{d}{dt} f\big(\psi \circ \varphi_t \circ \psi^{-1}(p)\big)\Big|_{t=0} \\
&= \quad \frac{d}{dt} f\big(\phi_p(t)\big)\Big|_{t=0}
\end{aligned}
\tag{C.435}
$$

□

Corollary C.8.2 *A complete tangent vector field is invariant under a diffeomorphism $\psi : \mathcal{M} \to \mathcal{M}$ if and only if the one-parameter group of diffeomorphisms φ_t generated by X commutes with ψ.*

Proof:

□

C.8.5 Isometries and Killing Vector Fields

when the metric is the same. If you move along the direction of a Killing vector field, the metric diesn't change.

A space time possess a symmetry if there exists a coordinate system such that the components of the components $g_{ab}(x)$ of the metric are independent of at least one or more of the coordinates. Then the metric has a symmetry under translations by this coordinate holding the remaining coordinates fixed.

see M. Gockeler, T. Schucker, *Differential geometry, gauge theories, and gravity*

in general these symmetries go when we go to curved space-time with fixed metric and cannot exist in general relativity where the metric becomes a dynamical variable.

Then a transformation leaving $g_{ab}(x)$ invariant is called an **isometry**.

The Lie derivative is natural to express the invariance of a tensor under a change of position. The vector ξ^a that generates the symmetry is called a Killing vector. In the original coordinates the components of the Killing vector are simply $\xi^a = \delta^a_b$. A coordinate covariant characterization of a

Killing vector is

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g'_{ab}(x') \tag{C.436}$$

will be an isometry if

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{ab}(x') \tag{C.437}$$

There is a coordinate system in which $g_{ab}(x)$ is the same function that $g'_{ab}(x')$, where $x'^a = x^a + \epsilon K^a$

For example

$dl^2 = dr^2 + r^2\, d\theta^2$

$g_{ab}(r, \theta) = D(1, r^2)$ and $g_{ab}(1, \theta') = D(1, r'^2)$ where $\theta' = \theta + \epsilon K_\alpha$

and give curves along which the geometrical environment is unchanged.

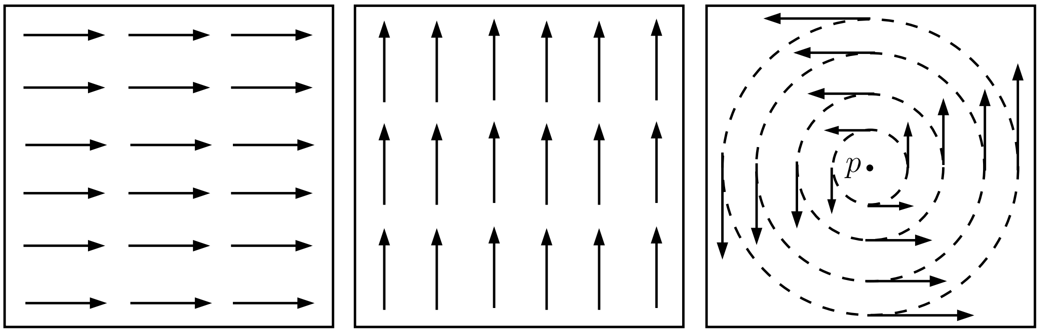


Figure C.30: The Killing vector field resulting from a congruence of curves.

Fig.(C.8.5). The first two are the "infinitesimal generators" of horizontal and vertical translations. The third is the generator of counter-clockwise rotations centered at p.

The three dimensional rotation group $O(3)$ is the isometry group for of the ordinary round sphere S^2.

$$\frac{\partial x^a}{\partial x^c}\frac{\partial x^b}{\partial x^d} \tag{C.438}$$

$$\frac{\partial x'^a}{\partial x^b} = d^\alpha_b + \epsilon \partial_b K^a \tag{C.439}$$

$$g'_{ab}(x') = g_{ab}(x) \;\text{ when } x' = xK_a. \tag{C.440}$$

$$g'_{ab}(x^a + \epsilon K^a) = g_{ab}(x) \tag{C.441}$$

$$g'_{\alpha\beta}(x') = (\delta^\gamma_\alpha + \partial_\alpha K^\gamma)(\delta^\delta_\beta + \partial_\beta K^\delta) g_{\gamma\delta}(x^\sigma + \epsilon K^\sigma) \tag{C.442}$$

$$\begin{aligned}
&= (\delta^\gamma_\alpha + \partial_\alpha K^\gamma)(\delta^\delta_\beta + \partial_\beta K^\delta)[g_{\gamma\delta}(x^\sigma) + \epsilon K^\sigma \partial_\sigma g_{\gamma\delta} + \ldots] \\
&= g_{\alpha\beta}(x) + \epsilon[g_{\alpha\delta}\partial_\beta K^\delta + g_{\beta\delta} + K^\sigma \partial_\sigma g_{\alpha\beta}] + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{C.443}$$

$$\mathcal{L}_X = X^\sigma \partial_\sigma g_{\alpha\beta} + g_{\alpha\delta}\partial_\beta X^\delta + g_{\beta\delta}\partial_\alpha X^\delta. \tag{C.444}$$

$$\mathcal{L}_K g_{\alpha\beta}(x) = 0 \tag{C.445}$$

$$\mathcal{L}_K g_{\alpha\beta}(x) = \nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0, \qquad \nabla_{(\beta} K_{\alpha)} = 0 \tag{C.446}$$

An isometry is generated by a Killing vector field K^α satisfying $\mathcal{L}_K g_{\alpha\beta}(x) = 0$

C.8.6 Conserved Quantities

$\nabla_{(a} k_{b)} = 0$

Consider a freely falling particle whose worldline has tangent vector X. Define the quantity $E = X^a k_a$, where k is a Killing vector. Then

$$\begin{aligned}
X^a \nabla_a (X^c k_c) &= X^a X^b \nabla_a k_b + k_b \underbrace{X^a \nabla_a X^b}_{=0} \\
&= X^a X^b \nabla_a k_b = 0 \\
&= X^a X^b \nabla_{(a} k_{b)} = 0
\end{aligned} \tag{C.447}$$

Thus E is conserved along the worldline of X. Given the energy-momentum tensor of a continuous distribution of matter, satisfying $\nabla_b T^{ac} = 0$. Define $J^a := T^{ac} k_c$. Then

$$
\begin{aligned}
\nabla_c(T^{bc}k_b) &= k_b \underbrace{\nabla_c T^{bc}}_{=0} + T^{bc}\nabla_c k_b \\
&= T^{bc}\nabla_c k_b \\
&= T^{bc}\nabla_{(c}k_{b)} = 0
\end{aligned}
\tag{C.448}
$$

Thus $\nabla_a J^a = 0$, i.e., the current is conserved.

Notion of energy and angular momentum have played a key role in analyzing behaviour of physical theories. For theories of fields on a fixed, background spacetime, a locally conserved stress-energy tensor, T_{ab}, normally can be defined. If the background spaceime has a Killing field, k^a, then $J^a = T^a_{\ b}k^b$ is a locally conserved current. If Σ is a Cauchy surface, then $q = \int_\Sigma J^a d\Sigma_a$ defines a conserved quantity associated with k^a; if Σ is a timelike or null surface, then $\int_\Sigma J^a d\Sigma_a$ has the interpretation of the flux of this quantity through Σ.

However, in diffeomorphism covariant theories such as general relativity, there is no notion of the local stress-energy tensor of the gravitational field, so conserved quantities () cannot and their fluxes cannot be defined by the above procedures, even when Killing fields are present.

C.8.7 Adapted Coordinates

These symmetries are removed by active diffeomorphisms, symmetries help us find simple solutions to Einstein's equation but strictly it is only those properties shared by all the spacetimes in the symmetry class that have physical meaning.

However, if we are neglecting the dynamics of gravity and only consered with dynamical theories over curved spacetime, then the particulary simple spacetime in the equivalence class can be used to get your physical properties.

Spherically symmetric spacetimes

$$
ds^2 = g_{00}dt^2 + 2g_{oi}dtdx^i + dr^2 + r^2\sin^2\theta d\theta^2 + r^2 d\phi^2. \tag{C.449}
$$

Axisually symmetric spacetimes

$$
ds^2 = dr^2 + r^2\sin^2\theta d\theta^2 + r^2 d\phi^2. \tag{C.450}
$$

A gravitational field is said to be sationary when a reference frame exists in which all the components g are independent of the time coordinate g_{ab}. This coordinate, by the way, is usually referred to as coordinate time. Sationary spacetimes.

$$ds^2 = g_{00}dt^2 + 2g_{oi}dtdx^i + g_{ij}dx^i dx^j. \tag{C.451}$$

Static spacetimes we when changing $dt \to -dt$ ds should remain unchanged.

$$ds^2 = g_{00}dt^2 + g_{ij}dx^i dx^j. \tag{C.452}$$

C.8.8 Properties of Killing Fields

A very important and immediate result is the following. If $k^b k_b = 0$, then from $\nabla_a k_b = \nabla_b k_a$ we have $k^b \nabla_b k_a = -k^b \nabla_a k_b = \nabla_a (k^b k_b) = 0$, i.e., the curve to which k is tangent is a null geodesic.

$$\nabla_a \nabla_b v_c = R_{abc}{}^d v_d \tag{C.453}$$

Proof:

$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = -R_{abc}{}^d v_d, \tag{C.454}$$

which on using Killing's equation, gives

$$\nabla_a \nabla_b v_c + \nabla_b \nabla_c v_a = -R_{abc}{}^d v_d \tag{C.455}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{C.456}$$

we have that

$$\begin{aligned} 2\nabla_b \nabla_c v_a &= -(R_{abc}{}^d + R_{bca}{}^d - R_{cab}{}^d) v_d \\ &= 2R_{cab}{}^d v_d. \end{aligned} \tag{C.457}$$

C.8.9 Diffeomorphism Gauge Group - Symmetry of GR Under Active Diffeomorphisms

It is often stated that coordinate transformations are the gauge symetries of GR. Then move onto the diffeomorphism group, however, the diff group is formed by active diffeomorphisms not coordinate transformations! The gauge symmetry referred to is GR's invariance under active diffeomorphisms!

Let us consider an infinitessimal point transformation

$$x'^a = x^a + \xi^a(x) \tag{C.458}$$

We have already proven that the metric $g_{ab}(x)$ gets mapped to the metric $g_{ab}(x) - 2D_{(a}\xi_{b)}$ under this point transformation.

Lie algebra of vector fields ξ

A vector space V with elements $x, y, z, \dots$ and bilinear bracket $[\cdots, \cdot]$, that takes two elements of V and returns another element of V, is a Lie algerbra if the bracket is anti-symmetric and the Jacobi identity holds for all elements in V.

vector field $\xi^a(x)$ generates an infinitesimal active diffeomorphism. Has a Lie algebra

$$[\xi_{(1)}, \xi_{(2)}]f = \tag{C.459}$$

C.9 Frame Fields

That is absolutely crucial to the loop quantum gravity programme is that GR can be put into a form that strongly resembles gauge theories in particle physics.

In appendix we introduced a natural basis for the tangent space T_P at a point P that were induced by the coordinates. We consider a set of basis vectors *not* derived from any coordinate system. Say we are given a time-like vector field v^α which defines a congruence of curves. For each of these curves, take any point P. We introduce a orthonormal frame of three unit space-like vectors.

$$e_I^\alpha = (e_1^\alpha, e_2^\alpha, e_3^\alpha) \tag{C.460}$$

which are orthogonal to v^α and where I is a label running from 0 to 3.

We define

$$e_0^\alpha := v^\alpha \tag{C.461}$$

orthonomality relations

$$e_1^a e_{1a} = e_2^a e_{2a} = e_3^a e_{3a} = -e_0^a e_{0a} = 1$$
$$e_0^a e_{1a} = e_0^a e_{2a} = e_0^a e_{3a} = e_1^a e_{2a} = e_1^a e_{3a} = e_2^a e_{3a} = 0 \tag{C.462}$$

The four vectors are said to form a *frame* or *tetrad* at P, and the orthonormality relations can be succinctly summarized as

$$e_i^a e_{ja} = \eta_{ij} \tag{C.463}$$

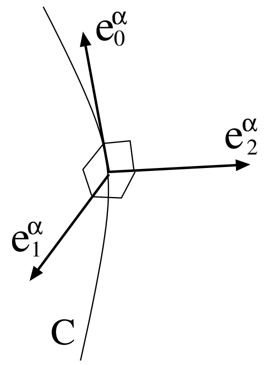


Figure C.31: Framefield or tetrad with one spatial dimension suppressed.

C.10 The Spin Connection

The triad is not a vector basis induced by a coordinate system. It turns out that the use of such frame fields brings out a different point of view on the connection and curvature, one in which GR has a strong resemblance to particle physics field theories.

Instead of a basis determined by coordinates, $\partial/\partial x^a$, we may choose any other n linearly independent vectors $\mathbf{e}_\alpha(\alpha = 1, \ldots, n)$, with components $\mathbf{e}_\alpha^\mathbf{a}(\alpha = 1, \ldots, n)$ with respect to the coordinate basis.

We have the freedom to choose a different basis. The metric $g_{ab}(x)$ is left invariant under local $SO(3,1)$ transformations such that

$$e_I^a(x^a) \to e'^a_I(x^a) = O_i^J(x^a) e_J^a(x^a), \tag{C.464}$$

where $O^j_{\ i}(x^a)$ is a matrix in $SO(3,1)$ which depends on position in space. When "Physical quantities" are left invariant, such transformations are known as *gauge transformations*, and theories invariant under them are called *gauge theories.*

Now we have introduced these frame fields we now need to know how to compare vectors in frames at different points. Put another way; a difficulty arise when one considers partial derivatives, $\partial_a V^i$. Because the matrix $O_i^j(x^a)$ depends on spacetime, it will contribute an inhomogeneous term to the transformation of the partial derivative,

$$\begin{aligned}
\partial_a V'^I(x) &= \partial_a \left(O^I{}_J(x^a) V^J(x^a) \right) \\
&= O^I j(x^a) \partial_a V^J(x^a) + V^J(x^a) \partial_a O^I{}_J(x^a)
\end{aligned} \tag{C.465}$$

The same sort of problem is encountered when considering the transformation of the partial differentiation of vector fields $\partial_a V^b(x)$. The solution there is to add the connection $\Gamma^a_{bc}(x)$ to correct for the inhomogeneous term in the transformation law, giving us the covariant derivative, $\nabla_a V_b = \partial_a V_b + \Gamma^c_{ab} V_c$. The same remedy is applied to $\partial_a V^i(x^a)$ and we introduce a connection

$$\omega_a{}^I{}_J(x) \tag{C.466}$$

with two tetrad indicies and one spacetime index.

$$\mathcal{D}_a V^I(x) = \partial_a V^I(x) + \omega_a{}^I{}_J V^J(x) \tag{C.467}$$

We require $\mathcal{D}_a V^i(x)$ to transform as a vector in internal space,

$$\mathcal{D}'_a V'^I(x) = O^I{}_J \mathcal{D}_a V^I(x). \tag{C.468}$$

Therefore the connection transforms as

$$\omega'_a{}^I{}_J = O^I{}_K \omega_a{}^K{}_J - \partial_a O^I{}_J \tag{C.469}$$

$$\mathcal{D}_a V^I = \partial_a V^I + \omega^I_{aJ} V^J \tag{C.470}$$

$$\mathcal{D}_a V^I_b = \partial_a V^I_b + \Gamma^c_{ab} V^I_c + \omega^I_{aJ} V^J_b \tag{C.471}$$

This covariant derivative is said to be compatible to the tetrad metric η_{IJ} if,

$$\mathcal{D}_a \eta_{IJ} = 0 \tag{C.472}$$

This implies,

$$\partial_a \eta^{IJ} + \omega^I_{aK} \eta^{JK} + \omega^I_{aK} \eta^{KJ} = 0, \tag{C.473}$$

This implying that the connection is antisymmetric in its tetrad indices,

$$\omega_a^{IJ} = -\omega_a^{JI}. \tag{C.474}$$

C.10.1 The Spin Connection in terms of the Tetrads

The connection $\Gamma^\alpha_{\mu\nu}$ is uniquely determined by the requirement

$$\mathcal{D}_\mu e^I_\nu(x) = 0 \tag{C.475}$$

that is,

$$\partial_\mu e^I_\nu + \omega^I_{\alpha J} e^j_b + \Gamma^\gamma_{\alpha\beta} e^I_\gamma = 0 \tag{C.476}$$

It is said to be compatible to the co-triad.

The connection field $\Gamma^\alpha_{\mu\nu}$ can be calculated in much the same way as the $\Gamma^\gamma_{\alpha\beta}$ was calculated.

Consider the anti-symmetrized covariant derivative of the tetrad

$$\mathcal{D}_{[a} e^I_{b]} = \partial_{[a} e^I_{b]} + \omega_{[a}{}^{IL} e_{b]L} = 0 \tag{C.477}$$

We can solve for ω in the same kind of way we derive the Christoffel connection. First we contract the above expression with $e^a_J e^b_K$ to obtain

$$e^a_J e^b_K \left(\partial_{[a} e^I_{b]} + \omega_{[a}{}^{IL} e_{b]L} \right) = 0. \tag{C.478}$$

Let us define

$$\Omega_{IJK} = e^a_I e^b_J \partial_{[a} e_{b]K}.$$

This is obviously anti-symmetric in the first ttwo indices. Performing rotations of indices in (C.477) we get three equations,

$$\begin{aligned}\Omega_{JKI} + e^a_J e^b_K \omega_{[a|I|}{}^L e_{b]L} &= 0 \\ \Omega_{IJK} + e^a_I e^b_J \omega_{[a|K|}{}^L e_{b]L} &= 0 \\ \Omega_{KIJ} + e^a_K e^b_I \omega_{[a|J|}{}^L e_{b]L} &= 0\end{aligned} \tag{C.479}$$

Adding the first two and subtracting the last,

$$\begin{aligned}&\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} \\ &+ e^a_J e^b_K \frac{1}{2}(\omega_{aI}{}^L e_{bL} - \omega_{bI}{}^L e_{aL}) \\ &+ e^a_I e^b_J \frac{1}{2}(\omega_{aK}{}^L e_{bL} - \omega_{bK}{}^L e_{aL}) \\ &- e^a_K e^b_I \frac{1}{2}(\omega_{aJ}{}^L e_{bL} - \omega_{bJ}{}^L e_{aL}) = 0\end{aligned} \tag{C.480}$$

This simplifies to

$$\begin{aligned}&\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} \\ &+ \frac{1}{2}(e^a_J \omega_{aIK} - e^b_K \omega_{bIJ}) \\ &+ \frac{1}{2}(e^a_I \omega_{aKJ} - e^b_J \omega_{bKI}) \\ &- \frac{1}{2}(e^a_K \omega_{aJI} - e^b I \omega_{bJK}) = 0\end{aligned} \tag{C.481}$$

Using $\omega_{aIJ} = -\omega_{aJI}$ the above reduces to

$$\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} + e^a_J \omega_{aIK} = 0. \tag{C.482}$$

Let us swap the dummy variables J and K and replace a with b, then contract with e^K_a gives

$$\omega_{aIJ} = e^K_a(-\Omega_{KJI} - \Omega_{IKJ} + \Omega_{JIK}). \tag{C.483}$$

or by using the anti-symmetry of ω_{aIJ},

$$\omega_{aIJ} = e^K_a(\Omega_{KIJ} + \Omega_{JKI} - \Omega_{IJK}). \tag{C.484}$$

We now wish to use this to express ${\omega_a}^{IJ}$ in terms of the tetrads. Note

$$\omega_a{}^{IJ} = e_a^K[\Omega_K{}^{IJ} + \Omega^J{}_K{}^I - \Omega^{IJ}{}_K]$$

and

$$\begin{aligned}
\Omega_K{}^{IJ} &= e_K^b e^{cI}\partial_{[b}e_{c]}^J \\
\Omega^J{}_K{}^I &= e^{bJ}e_K^c\partial_{[b}e_{c]}^I \\
\Omega^{IJ}{}_K &= e^{bI}e^{cJ}\partial_{[b}e_{c]K}.
\end{aligned} \tag{C.485}$$

Using these we get:

$$\begin{aligned}
\omega_a{}^{IJ} &= e_a^K\left(e_K^b e^{cI}\partial_{[b}e_{c]}^J + e^{bJ}e_K^c\partial_{[b}e_{c]}^I - e^{bI}e^{cJ}\partial_{[b}e_{c]K}\right) \\
&= e^{cI}\partial_{[a}e_{c]}^J + e^{bJ}\partial_{[b}e_{a]}^I - e^{bI}e^{cJ}e_a^K\partial_{[b}e_{c]K} \\
&= e^{bI}\partial_{[a}e_{b]}^J - e^{bJ}\partial_{[a}e_{b]}^I - e^{b[I}e^{|c|J]}e_a^K\partial_{[b}e_{c]K} \\
&= e^{b[I}\left(2\partial_{[a}e_{b]}^{J]} + e^{|c|J]}e_a^K\partial_c e_{bK}\right)
\end{aligned} \tag{C.486}$$

So finally we arrrive at

$$\omega_a{}^{IJ} = e^{c[I}\left(2\partial_{[a}e_{c]}^{J]} + e^{|b|J]}e_a^K\partial_b e_{cK}\right) \tag{C.487}$$

C.10.2 Curvature Associated with the Spin Connection.

Let us work out the commutator in the case of a vector λ_I.

We will need the covariant derivative of $\mathcal{D}_\beta\lambda_I$. This is a tensor with one internal space index I and one space-time index β. For covariant derivative of such an object is the standard formula is,

$$\mathcal{D}_\alpha T_{\beta I} = \partial_\alpha T_{\beta I} + \Gamma^\rho_{\alpha\beta}T_{\rho I} + \omega^J_{\beta I}T_{\beta J}. \tag{C.488}$$

Applying this to $\mathcal{D}_\beta\lambda_I$ gives

$$\begin{aligned}
\mathcal{D}_\alpha(\mathcal{D}_\beta\lambda_I) &= \partial_\alpha(\mathcal{D}_\beta\lambda_I) + \omega^J_{\beta I}(\mathcal{D}_\beta\lambda_J) + \Gamma^\rho_{\alpha\beta}(\mathcal{D}_\rho\lambda_I) \\
&= \partial_\alpha[\partial_\beta\lambda_I + \omega^J_{\beta I}] + \omega^K_{\alpha I}[\partial_\beta\lambda_I + \omega^J_{\beta I}] + \Gamma^\rho_{\alpha\beta}\mathcal{D}_\rho\lambda_I
\end{aligned} \tag{C.489}$$

$$
\begin{aligned}
\mathcal{D}_\alpha \mathcal{D}_\beta \lambda_I - \mathcal{D}_\beta \mathcal{D}_\alpha \lambda_I &= \Gamma^\rho_{\alpha\beta} \mathcal{D}_\rho \lambda_I - \Gamma^\rho_{\beta\alpha} \mathcal{D}_\rho \lambda_I \\
&= \partial_\alpha \partial_\beta \lambda_I - \partial_\beta \partial_\alpha \lambda_I \\
&+ \partial_\alpha (\omega^J_{\beta I} \lambda_I) - \partial_\beta (\omega^J_{\alpha I} \lambda_I) \\
&+ \omega^K_{\alpha I} \partial_\beta \lambda_K - \omega^K_{\beta I} \partial_\alpha \lambda_K \\
&+ \omega^K_{\alpha I} \omega^J_{\beta K} \lambda_J - \omega^K_{\beta I} \omega^J_{\alpha K} \lambda_J
\end{aligned} \tag{C.490}
$$

The first line is zero as we assume partial derivatives commute. The terms in the third line are cancelled by terms in the second line. So we obtain the result

$$
\mathcal{D}_\alpha \mathcal{D}_\beta \lambda_I - \mathcal{D}_\beta \mathcal{D}_\alpha \lambda_I = R_{\alpha\beta\, I}{}^J \lambda_J \tag{C.491}
$$

where $R_{\alpha\beta\, I}{}^J$ is defined by

$$
R_{\alpha\beta\, I}{}^J = \partial_\alpha \omega_{\beta I}{}^J - \partial_\beta \omega_{\alpha I}{}^J + \omega_{\alpha I}{}^K \omega_{\beta K}{}^J - \omega_{\beta I}{}^K \omega_{\alpha K}{}^J \tag{C.492}
$$

which can be written in the more compact form

$$
R_{\alpha\beta}{}^{IJ} = \partial_{[\alpha} \omega_{\beta]}{}^{IJ} + \omega_{[\alpha}{}^{IK} \omega_{\beta]K}{}^J \tag{C.493}
$$

By considering the covariant derivative of $K_\alpha = e^I_\alpha \lambda_I$ we can find the relation between the spin curvature with the

$$
\begin{aligned}
R^\alpha_{\beta\gamma\rho}(e^J_\alpha \lambda_J) = R^\alpha_{\beta\gamma\rho} K_\alpha &= \mathcal{D}_\beta \mathcal{D}_\rho K_\gamma - \mathcal{D}_\rho \mathcal{D}_\beta K_\gamma \\
&= \mathcal{D}_\beta \mathcal{D}_\rho (e^I_\gamma \lambda_I) - \mathcal{D}_\rho \mathcal{D}_\beta (e^I_\gamma \lambda_I) \\
&= e^I_\gamma [\mathcal{D}_\beta \mathcal{D}_\rho \lambda_I - \mathcal{D}_\rho \mathcal{D}_\beta \lambda_I] \\
&= e^I_\gamma R_{\beta\rho I}{}^J \lambda_J
\end{aligned} \tag{C.494}
$$

this becomes

$$
R^\alpha_{\beta\gamma\rho} e^J_\alpha = e^I_\gamma R_{\beta\rho I}{}^J \tag{C.495}
$$

which implies

$$
R^\alpha_{\beta\gamma\rho} = R_{\beta\rho I}{}^J e^I_\gamma e^\alpha_J \tag{C.496}
$$

$$
R = R_{\mu\nu}{}^{IJ} e^\mu_I e^\nu_J \tag{C.497}
$$

C.10.3 Palantini action for GR

Using $R = R_{\mu\nu}{}^{IJ} E^\mu_I E^\nu_J$ and $\sqrt{-g} = E$ we can write the Einstein Hilbert action in terms of the connection and tetrad:

$$S_{EH}[e,\omega] = \frac{1}{4\kappa}\int d^4x \epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}E^I_\nu E^J_\nu F_{\alpha\beta}{}^{KL} \tag{C.498}$$

where $F^{KL}_{\gamma\delta}$ is the curvature of the spin-connection.

Variation with respect to $\omega_\nu{}^{IJ}$ gives

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}D_\nu(E^I_\alpha E^J_\beta) = 0 \tag{C.499}$$

and variation with respect to E^ν_I gives

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}E^J_\nu F_{\alpha\beta}{}^{KL} = 0 \tag{C.500}$$

(C.497)

$$\epsilon_{\mu\nu\alpha\beta} = \frac{1}{4!}\epsilon_{PQRS}E^P_{[\nu}E^Q_\nu E^R_\alpha e^S_{\beta]} \tag{C.501}$$

Hamiltonian constraints in the Palatini formalism

We can do the 3+1 split for the Palantini action (C.498) and obtain a Hamiltonian constraint and other constraints. While seems simpler than the of the metric variables

There are second class constraints which when solved give back the same set of constraints as obtained from the ADM framework. And so their not much improvement over the ADM formalism when it comes to quantizing it.

C.10.4 Ashtekar's New Variables

In the Palantini formalism the phase space variables are (E^a_i, Γ^i_a) (from these we get the intrinsic metric of the spacelike manifold Σ and its extrinsic curvature respectively).

We consider a canonical transformation from the phase space variables in the Palantini formalism (E^a_i, Γ^i_a) Ashtekar's variables

$$\Gamma^i_a \to \Gamma^i_a + \beta K^i_a \qquad E^a_i \to \frac{1}{\beta} E^a_i \tag{C.502}$$

GR action written as

$$S_{EH}[E,\omega] = \frac{1}{2\kappa}\int d^4x \tilde{E}^\mu_I \tilde{E}^\nu_J F[\omega]_{\mu\nu}{}^{IJ} \tag{C.503}$$

C.10.5 Cartan Structure Equations

Let $\{\hat{e}_i\}$ be the non-coordinate basis and $\{\hat{\theta}^i\}$ the dual basis. The vector fields satisfy

$$[\hat{e}_i, \hat{e}_j] = c_{ij}{}^k \hat{e}_k. \tag{C.504}$$

The connection coefficients with respect to the basis $\{\hat{e}_i\}$ by

$$\nabla_i \hat{e}_j \equiv \omega^k{}_{ij} \hat{e}_k \tag{C.505}$$

$$\omega^i{}_j := \omega^i{}_{kj} \hat{\theta}^k. \tag{C.506}$$

The 2−forms of torsion are given by the first **Cartan's equations of structure**, which read

$$T^k = d\hat{\theta} + \omega^k{}_i \wedge \hat{\theta}^i, \tag{C.507}$$

and the 2-forms of curvature by the second Cartan's equations of structure

$$\Omega^k{}_i = d\omega^k{}_i + \omega^k{}_j \wedge \omega^j{}_i. \tag{C.508}$$

where $T^i \equiv \frac{1}{2} T^i{}_{jk} \hat{\theta}^j \wedge \hat{\theta}^k$ is the torsion two-form and $R^i{}_j \equiv \frac{1}{2} R^i{}_{jk\ell} \hat{\theta}^k \wedge \hat{\theta}^\ell$ the curvature two-form

Proof. Let the LHS of (C.507) act on the basis vectors $\hat{e}_k$ and $\hat{e}_\ell$,

$$\begin{aligned}
& d\hat{\theta}^i(\hat{e}_k, \hat{e}_\ell) + [< \omega^i{}_j, \hat{e}_k >< \hat{\theta}^j, \hat{e}_\ell > - < \hat{\theta}^j, \hat{e}_k >< \omega^i{}_j, \hat{e}_\ell >] \\
= \; & \{\hat{e}_k[< \hat{\theta}^i, \hat{e}_\ell >] - \hat{e}_k[< \hat{\theta}^i, \hat{e}_k >] - < \hat{\theta}^i, [\hat{e}_k, \hat{e}_\ell] >\} + \{< \omega^i{}_\ell, \hat{e}_k > - < \omega^i{}_k, \hat{e}_\ell >\} \\
= \; & -c_{k\ell}{}^i + \omega^i_{k\ell} - \omega^i_{\ell k} = T^i_{k\ell}
\end{aligned} \tag{C.509}$$

where we have made use of (**??**). The RHS acting on $\hat{e}_k$ and $\hat{e}_\ell$ yields

$$\frac{1}{2}T^i{}_{jm}[<\hat{\theta}^j,\hat{e}_k><\hat{\theta}^m,\hat{e}_\ell> - <\hat{\theta}^m,\hat{e}_k><\hat{\theta}^j,\hat{e}_\ell>] = T^i{}_{k\ell}$$

which completes the proof. Equation may be proven similarly.

$\square$

Taking the exterior derivatives of (C.507) and (C.10.5), we have the **Bianchi identities**

$$\begin{aligned} dT^i + \omega^i_{\ j}\wedge T^j &= R^i_{\ j}\wedge\hat{\theta}^j && \text{(C.510)}\\ dR^i_{\ j} + \omega^i{}_k\wedge R^k_{\ j} - R^i_{\ k}\wedge\omega^k{}_j &= 0. && \text{(C.511)}\end{aligned}$$

These are the non-coordinate basis versions of $R^a{}_{[bcd]} = 0$ and $\nabla_{[e|}R^a{}_{b|cd]} = 0$.

C.10.6 A Differential Geometry Translator

$$\begin{aligned} dA &= (\frac{\partial A_x}{\partial x}dx + \frac{\partial A_y}{\partial x}dy + \frac{\partial A_z}{\partial x}dz)\wedge dx\\ &= \frac{\partial A_y}{\partial x}\,dy\wedge dx + \frac{\partial A_z}{\partial x}\,dz\wedge dx && \text{(C.512)}\end{aligned}$$

The gravitational field e

$$e^I(x) = e^I_a dx^a \tag{C.513}$$

The spin connection

$$\omega^I{}_J(x) = \omega^I{}_{aJ}(x)dx^a \tag{C.514}$$

$$\mathcal{D}v^I = dv^I + \omega^I{}_J v^J \tag{C.515}$$

$$R^I{}_J = R^I{}_{J\,ab}dx^a\wedge dx^b \tag{C.516}$$

$T(X,Y) = T^a_{bc}X^bY^c$

$$T^a = \frac{1}{2}T^a{}_{bc}\omega^b \wedge \omega^c \tag{C.517}$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{C.518}$$

$R(X,Y)Z = R^a{}_{bcd}X^cY^dZ^b$

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z \tag{C.519}$$

Hodge-Star operation:

$$\mathcal{S}[e^I,\omega^{IJ}] = \frac{1}{4\kappa}\int_{\mathcal{M}} \epsilon_{IJKL}e^I_a e^J_b \left(R^{KL}_{\sigma\rho} - \frac{\Lambda}{6}e^K_\sigma e^L_\rho\right) dx^\mu dx^\nu dx^\sigma dx^\rho \tag{C.520}$$

$$\mathcal{S}[e^I,\omega^{IJ}] = \frac{1}{4\kappa}\int_{\mathcal{M}} \epsilon_{IJKL}(e^I \wedge e^J \wedge R[\omega]^{KL} + \frac{\Lambda}{6}e^I \wedge e^J \wedge e^K \wedge e^L) \tag{C.521}$$

C.11 More on Lie groups

For $g \in G$ we define the adjoint isomorphism $ad_g : G \to G$ by

$$ad_g(h) := ghg^{-1} = L_g \circ R_{g^{-1}}h, \tag{C.522}$$

for all $h \in G$.

It is a group homomorphism as

$$\begin{aligned}
ad_{[g_1,g_2]}(g_3) &= [[g_1,g_2],g_3] \\
&= [g_1,[g_2,g_3]] + [g_2,[g_3,g_1]] \\
&= ad(g_1)(ad(g_2)(g_3)) - ad(g_2)(ad(g_1)(g_3)) \\
&= [ad(g_1), ad(g_2)](g_3)
\end{aligned} \tag{C.523}$$

To see this is an isomorphism write $ad_g(h) = g'$ for any $g' \in G$, this has the solution $h = g^{-1}g'g$. This solution is unique as $ad_g(h) = ad_g(h')$ implies $h = h'$. This is called the adjoint representation of the Lie group G

The map ad_g fixes the neutral element e therefore the adjoint isomorphism ad_g on the Lie group G induces an isomorphism on $T_e(G)$

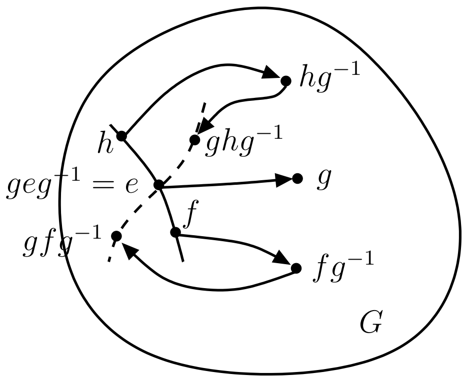


Figure C.32: A curve through e under the map $h \mapsto ghg^{-1}$, first a right action $R_{g^{-1}}$ as $h \mapsto hg^{-1}$ followed by the left action L_g as $hg^{-1} \mapsto ghg^{-1}$. The identity e is mapped to itself but points h and f near it are generally changed, so that a tangent vector at e, in $T_e(G)$, is mapped to another one in $T_e(G)$.

$$\mathcal{A}d_g := (ad_g)_* : T_e(G) \to T_e(G)$$

or

$$\mathcal{A}d_g : \mathcal{G} \to \mathcal{G}$$

on the Lie algebra $\mathcal{G}$.

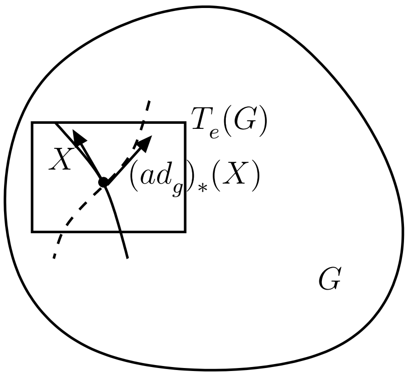


Figure C.33: A vector $X \in T_e(G)$ is mapped to another one in $\mathcal{A}d_g(X) \in T_e(G)$. Written formally as $(ad_g)_* : T_e(G) \to T_e(G)$.

This is a group homomorphism between groups

$$\begin{aligned} \mathcal{A}d(g_1 g_2) &= (ad_{g_1 g_2})_* = (ad_{g_1} \circ ad_{g_2})_* \\ &= (ad_{g_1})_* \circ (ad_{g_2})_* = \mathcal{A}d(g_1) \circ \mathcal{A}d(g_2) \end{aligned} \tag{C.524}$$

Note by injectivety that $\mathcal{A}d(g)$ maps any vector $X \in T_e(G)$ to a non-zero vector for all $g \in G$. Also we have $\mathcal{A}d(g)(X+Y) = \mathcal{A}d(g)(X) + \mathcal{A}d(g)(Y)$. We thus have the map

$$\mathcal{A}d : G \to GL(n, \mathbb{R}), \tag{C.525}$$

where $n = dim\mathcal{G}$.

This homomorphisms in turn induces a homomorphism between Lie algebras

$$Ad := \mathcal{A}d_* : \mathcal{G} \to \mathcal{GL}(n, \mathbb{R}), \quad n = dim\mathcal{G} \tag{C.526}$$

called the adjoint representation of the Lie algebra $\mathcal{G}$.

If G is a matrix group, the adjoint representation becomes a simple matrix operation.

Definition The kernel of a group homomorphism $\varphi : G \to H$ is defined by

$$\ker \varphi := \varphi^{-1}(\{e_H\}) = \{x \in G : \varphi(x) = e_H\}.$$

Proposition C.11.1 *A group homomorphism $\varphi : G \to H$ is injective if and only if its kernel is trivial, i.e., ker $\varphi = \{e_G\}$.*

Proof:

Assume φ is injective. Since we must have $e_G \in \ker \varphi$, $\ker \varphi = \{e_G\}$. Assume $\ker \varphi = \{e_G\}$. Say φ is not injective, i.e., there is $x \neq y$ such that $\varphi(x) = \varphi(y)$. Then $\varphi(xy^{-1}) = e_H$, implying $xy^{-1} = e_G$ or $x = y$.

□

If G is a group, then by an automorphism of G we mean an isomorphism of G onto itself. The collection of such automorphisms, denoted Aut(G), is a subgroup of Sym (G).

The kernel of ad_x is the subgroup of G consisting of the elements $x \in G$ with the property that $xyx^{-1} = y$ for all $y \in G$, or, equivalently that $xy = yx$ for all $y \in G$. Thus the kernel of I equals the center $Z(G)$ of G.

C.11.1 Discrete Groups

A discrete group is a group with the discrete topology.

For example any finite group is a discrete group.

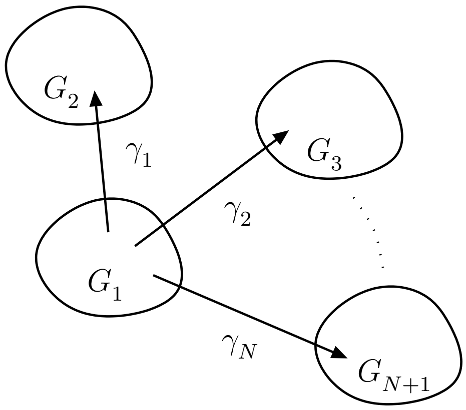


Figure C.34: If a Lie group is a direct product of the proper subgroup and some discrete subgroup then each connected component G_i is obtained from the proper subgroup G_1 by applying some discrete transformation γ_i of a discrete subgroup Γ.

C.11.2 Universal Covering Group

Two curves $g(\tau)$ and $g'(\tau)$ connecting the elements g_0 and g_1 are said to be homotomic if there exists a continuous deformation of one curve into the another, which leaves the end points g_0 and g_1 unaltered, i.e., there exists a continous function $h(\tau, s)$ of two parameters τ and s such that

$$h(0, s) = g_0, \qquad h(1, s) = g_1$$

$$h(\tau, 0) = g(\tau), \qquad h(\tau, 1) = g'(\tau).$$

A Lie group is said to be simply connected if every loop is homotopic to the null loop, i.e., every loop is contractable to one point.

A topological set not able to be partitioned into non-empty open subsets each of which has no points in common with the closure of the other. A topological space X is **connected** if $\emptyset$ and X are the only subsets of X that are both open and closed.

C.11.3 Decomposition of a Lie Group into Abelian and Non-Abelian Parts

The result of this subsection is employed in the proof of the uniqueness of Ashtekar-Lewandenski representation in LQG and the irreducibility of this representation.

Every connected compact Lie group is a quotient by a finite central subgroup of the product of a connected compact semisimple Lie group with a torus.

for a compact connected Lie group the exponential map is onto.

We prove it for the case of $SU(2)$.

Let G be any compact connected Lie group and T a maximal torus in G. We claim first that the following two statements are equivalent.

(a) the exponential map $\exp : \mathcal{L}(G) \to G$ is surjective.

(b) every element of G lies in a conjugate of T, i.e., the map $\psi : G \times T \to G$ given by $\psi(g,t) = gtg^{-1}$ is surjective.

The prove the result it is enough to establish (b) for $SU(2)$.

Proof: If (a) holds, for any $g \in G$ we have $g = \exp(\xi)$ for some $\xi \in \mathcal{L}(G)$. Therefore, g lies in the one-parameter subgroup $\{\exp(t\xi) : t \in \mathbb{R}\}$. The closure of this one-parameter subgroup is a torus in G. Therefore it is contained in a maximal torus T' in G.

...

□

Lemma C.11.2 *Let G be a connected compact Lie group. Let $\mathfrak{h}$ be an ideal in $\mathcal{L}(G)$. Then $\mathcal{L}(G)$ is the product of $\mathfrak{h}$ with the complementary ideal $\mathfrak{h}^\perp$.*

by () there exists an invariant inner product on $\mathcal{L}(G)$, i.e., $\mathcal{A}d$ is a Lie group homomorphism of G into $O(\mathcal{L}(G))$. This implies that Ad is a Lie algebra morphism of $\mathcal{L}(G)$ into the Lie algebra of $O(\mathcal{L}(G))$, i.e., all linear transformations $Ad(\xi)$, $\xi \in \mathcal{L}(G)$, are skewadjoint.

Let $\mathfrak{h}$ be an ideal in $\mathcal{L}(G)$. Then it is invariant under all $Ad(\xi)$, $\xi \in \mathcal{L}(G)$. This implies...

On the other hand, for $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}$, we have $[\xi, \eta] \in \mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$, that is, $\mathcal{L}(G)$ is the product of $\mathfrak{h}$ and $\mathfrak{h}^\perp$ as a Lie algebra.

□

Lemma C.11.3 *Every compact connected Lie group, G is isomorphic to a quotient $\tilde{G}/M$, where M is a central discrete subgroup of $\tilde{G}$, and $\tilde{G}$ is a simple product*

$$\tilde{G} = T \times P,$$

(that is, any $h \in \tilde{G}$ can be written as tp where $t \in T$ and $p \in P$), of an abelian group T and a semisimple group P.

Proof:

Recall that each Lie group possesses a Lie algebra $\mathfrak{g}$ isomorphic to the tangent vector space at the identity element of the Lie group. An ideal in a Lie algebra is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$. An ideal is said to be an invariant subalgebra.

An ideal is the Lie algebra equivalent of a closed, normal subgroup of a connected Lie group.

A connected Lie group can be defined to be simple if its Lie algebra is simple, or equivalently, if it contains no non-trivial, closed, connected normal subgroups. Under this definition, a simple connected Lie group can possess non-trivial, closed, normal subgroups, but if they exist they must be discrete.

A semisimple Lie algebra can be defined as a Lie algebra which has no non-trivial abelian ideals, but here we wish to characterise it as a Lie algebra which is the direct sum of simple Lie algebras. Semisimple Lie groups arae the direct products of simple Lie groups.

Clearly, a simple Lie algebra is semisimple.

Lemma C.11.4 *The ideal $[\mathcal{L}(G), \mathcal{L}(G)]$ in $\mathcal{L}(G)$ is a semisimple Lie algebra.*

Proof:

Let G be a compact connected Lie group.

Let $\mathfrak{h}$ be an ideal in $\mathcal{L}(G)$. Then it is invariant under all $\mathrm{ad}(\xi)$, $\xi \in \mathcal{L}(G)$. This implies that the orthogonal complement $\mathfrak{h}^{\perp}$ of $\mathfrak{h}$ is invariant under all $\mathrm{ad}(\xi)$, $\xi \in \mathfrak{g}$, i.e., $\mathfrak{h}^{\perp}$ is an ideal in $\mathfrak{g}$. It follows that $\mathcal{L}(G) = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ as a linear space. On the other hand, for $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp}$, we have $[\xi, \eta] \in \mathfrak{h} \cap \mathfrak{h}^{\perp} = \{0\}$, i.e., $\mathcal{L}(G)$ is the product of $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$ as a Lie algebra.

Let $\mathfrak{a}$ be an abelean ideal in $[\mathcal{L}(G), \mathcal{L}(G)]$. Then $\mathfrak{a}^{\perp}$ is an ideal in $\mathcal{L}(G)$ and $\mathcal{L}(G)$ is the product of $\mathfrak{a}$ and $\mathfrak{a}^{\perp}$. This implies that $\mathfrak{a}$ is in the center $\mathcal{L}(Z)$ of $\mathcal{L}(G)$. It follows that $\mathfrak{a} = \{0\}$. Therefore $[\mathcal{L}(G), \mathcal{L}(G)]$ in $\mathcal{L}(G)$ is semisimple.

□

Proposition C.11.5 *Let G be connected compact Lie group. Let $C = K \cap Z_0$ and $D = \{(c, c^{-1} \in K \times Z_0 : c \in C\}$. Then $\varphi : K \times Z_0$ given by $\varphi(k, z) = kz$ induces an isomorphism of the Lie group $(K \cap Z_0)/D$ with G.*

Consider the connected compact lie group $K \times Z_0$ and the differentiable map $\varphi : K \times Z_0 \to G$ given by $\varphi(k, z) = kz$ for $k \in K$ and $z \in Z_0$. φ is a Lie group homomorphism and $L(\varphi)$ is an isomorphism of lie algebras. Therefore, φ is a covering projection.

The kernel of φ is a finite central subgroup of $K \times Z_0$.

$$\ker\varphi = \{(k, z) \in K \times Z_0 : kz = 1\} = \{(c, c^{-1}) \in K \times Z_0 : c \in K \cap Z_0\}.$$

□

Therefore, any connected compact Lie group is a quotient by a finite central subgroup of the product of a connected compact semisimple Lie group with a torus.

C.12 Group Actions on Sets

Action of a group

$$\sigma(g, x) = y \tag{C.527}$$

$$\sigma(t, x) \tag{C.528}$$

if the flow $\sigma(t, x)$ is periodic with period T.

We can construct a new action whose group is $U(1)$

$$\overline{\sigma}(\exp(2\pi i t/T)y, x) = \sigma(t, x) \tag{C.529}$$

and one whose group is $SO(2)$

$$\tilde{\sigma}(\begin{pmatrix} \cos(2\pi t/T) & \sin(2\pi t/T) \\ -\sin(2\pi t/T) & \cos(2\pi t/T) \end{pmatrix}, x) = \sigma(t, x). \tag{C.530}$$

The action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$

$$\sigma(\mathbf{M}, x) = \mathbf{M} \cdot x \tag{C.531}$$

where $\cdot$ is the usual matrix multiplication on a vector. The action of subgroups of $GL(n, \mathbb{R})$ is defined similarly. $O(n)$ acts on $S^{(n-1)}(r)$, an $(n-1)-$sphere of radius r,

$$\sigma : O(n) \times S^{(n-1)}(r) \to S^{(n-1)}(r). \tag{C.532}$$

Formal definition of the action of a group on a manifold:

Definition Let G be a Lie group and $\mathcal{M}$ be a manifold. The action of G on $\mathcal{M}$ is a differentional map $\sigma : G \times \mathcal{M} \to \mathcal{M}$ which satisfies the conditions

(i) $\sigma(e, p) = p \qquad$ for any $p \in \mathcal{M}$

(ii) $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$.

We as well define the **orbit** of a point x of $\mathcal{M}$ as

$$orb(x) = \{gx \mid g \in G\} \tag{C.533}$$

i) The action of the group is *transitive* if any orbit is the whole of X.

ii) The action is *effective*, or *faithful*, if the trivial action on X, i.e.,if $\sigma(g, p) = p$ for all $p \in X$, implies $g = e$.

If the action is not effective, the set of g corresponding to the trivial action is an invariant subgroup H of G, and we can take G/H as having a faithful action.

iii) The action is *free* if the existence of an p such that $gp = p$ implies that $g = p$.

The **stabalizer** of x as

$$Stab(x) = \{g \in G \mid gx = x\} \tag{C.534}$$

The orbits are equivalence classes - we are often interested in the quotient space.

Isotropy group

The identity element e is obviously in $H(p)$. Now, let $g_1, g_2 \in H(p)$

$$\sigma(g_1 g_2, p) = \sigma(g_1, \sigma(g_2, p)) = \sigma(g_1, p) = p$$

$g^{-1} \in H(p)$ because

$$p = \sigma(e, p) = \sigma(g^{-1} g, p) = \sigma(g^{-1}, \sigma(g, p)) = \sigma(g^{-1}, p)$$

One can consider the quotient $G/H(p)$

Example $SO(3)/SO(2)$

we have

$$SO(3)/SO(2) \cong S^2$$

Definition A group action is *effective* if the identity element is the only element that, that is, if $\sigma(g, x) = x$ for all $x \in \mathcal{M}$, then $g = e$.

C.12.1 Transitive Actions

Properties:

i) Orbits are disjoint,

ii) $\mathcal{M}$ is the union of of the orbits.

Example. Let $G = O(n)$ and $\mathcal{M} = S^n$.

Definition A group action is *transitive* if, for any $x_1, x_2 \in \mathcal{M}$, there exists a $g \in G$ such that $\sigma(g, x_1) = x_2$.

There is only one orbit!

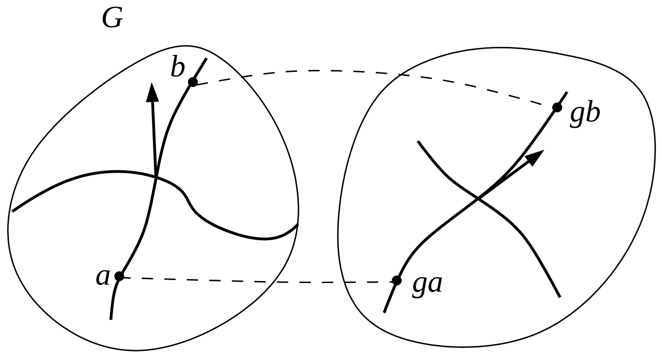


Figure C.35: leftTran. The left translation along g maps a neighbourhood of e onto one of g. There is a natural map of a vector at e to one at g.

C.12.2 Faithful Actions

The action of G on X is said to be faithful if $gx = hx$ for all $x \in X$ implies that $g = h$.

C.12.3 Free Actions

The action of G on X is said to be free if for all $g \in G$, $g \neq e$, and for all $x \in X$, $gx \neq x$.

Definition A group action is *free* if every element apart from the identity of G has no fixed points in M, that is, if there exists an element $x \in M$ such that $\sigma(g, x) = x$, then $g = e$.

C.12.4 Introduction to Gauge Invariance of the Yang-Mills Equations

$$\hat{U} = \exp\left(\frac{i}{2}\mathbf{a} \cdot \mathbf{T}\right) \tag{C.535}$$

Minimal coupling term is

$$\begin{aligned} \mathcal{L}_{int} &= g\sum_{i=1}^{3}\overline{\Psi}\gamma^{\mu}\frac{T^i}{2}A^i_{\mu}\Psi \\ &= g\overline{\Psi}\gamma^{\mu}\mathbf{A}_{\mu} \cdot \frac{\mathbf{T}}{2}\Psi \end{aligned} \tag{C.536}$$

$$\Psi \rightarrow \Psi' = \hat{U}\Psi = \exp\left(i\mathbf{a}(x) \cdot \hat{\mathbf{T}}\right)\Psi(x) \tag{C.537}$$

$$\begin{aligned} \partial_{\mu}\Psi \rightarrow \hat{U}\partial_{\mu}\Psi &= \hat{U}\partial_{\mu}(\hat{U}^{-1}\hat{U}\Psi) \\ &= \partial_{\mu}(\hat{U}\Psi) + \hat{U}(\partial_{\mu}\hat{U}^{-1})\hat{U}\Psi \\ &= \partial_{\mu}\Psi' + \hat{U}(\partial_{\mu}\hat{U}^{-1})\Psi' \\ &= \left[\partial_{\mu} + \hat{U}(\partial_{\mu}\hat{U}^{-1})\right]\Psi' \end{aligned} \tag{C.538}$$

By adding the coupling (C.536) to the free Dirac equation the additional term $\hat{U}(\partial_{\mu}\hat{U}^{-1})$, occurring for $\partial_{\mu}\mathbf{a}$, can be obsorbed by gauging the fields A^i_{μ} simultaneously. The Dirac Lagrangean is the cast into a gauge invariant form

$$\mathcal{L}(\Psi, \mathbf{A}_{\mu}) = i\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + g\overline{\Psi}\gamma^{\mu}\mathbf{A}_{\mu} \cdot \hat{\mathbf{T}}\Psi \tag{C.539}$$

and the gauged density

$$\mathcal{L}' = i\overline{\Psi}'\gamma^\mu\partial_\mu\Psi' + g\overline{\Psi}'\gamma^\mu\mathbf{A}'_\mu\cdot\hat{\mathbf{T}}\Psi' \tag{C.540}$$

$$\begin{aligned}
\mathcal{L} &= i\overline{\Psi}\gamma^\mu\partial_\mu\Psi + g\overline{\Psi}\gamma^\mu\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\Psi \\
&= i\overline{\Psi}\hat{U}^{-1}\hat{U}\gamma^\mu\partial_\mu(\hat{U}^{-1}\hat{U}\Psi) + g\overline{\Psi}\hat{U}^{-1}\hat{U}\gamma^\mu\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1}\hat{U}\Psi \\
&= i\overline{\Psi}'\hat{U}\gamma^\mu\partial_\mu(\hat{U}^{-1}\Psi') + g\overline{\Psi}'\hat{U}\gamma^\mu\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1}\Psi' \\
&= i\overline{\Psi}'\gamma^\mu\hat{U}\partial_\mu(\hat{U}^{-1}\Psi') + g\overline{\Psi}'\gamma^\mu\hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1}\Psi' \\
&= i\overline{\Psi}'\gamma^\mu\partial_\mu\Psi' + i\overline{\Psi}'\gamma^\mu[\hat{U}(\partial_\mu\hat{U}^{-1})]\Psi' + g\overline{\Psi}'\gamma^\mu(\hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1})\Psi' \\
&= i\overline{\Psi}'\gamma^\mu\partial_\mu\Psi' + g\overline{\Psi}'\gamma^\mu\left[\hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1} + \frac{i}{g}\hat{U}(\partial_\mu\hat{U}^{-1})\right]\Psi'
\end{aligned} \tag{C.541}$$

for this to be identical to the original action we must have:

$$\mathbf{A}'_\mu\cdot\hat{\mathbf{T}} = \hat{U}\mathbf{A}_\mu\cdot\hat{\mathbf{T}}\hat{U}^{-1} + \frac{i}{g}\hat{U}(\partial_\mu\hat{U}^{-1}). \tag{C.542}$$

We are forced to incorporate the term $\hat{U}(\partial_\mu\hat{U}^{-1})$, which is generated by gauging the kinematic energy of the field Ψ into the gauge transformation of the $\mathbf{A}_\mu$ fields. Let us look at the significance of this term in electrodynamics. In that case $\hat{U}$ is just $\hat{U} = \exp(ia(x))$. Then the gauge transformation reads

$$A'_\mu(x) = A_\mu(x) + \frac{1}{g}\partial_\mu a(x).$$

We can write the Lagrangean in the concise form

$$\begin{aligned}
\mathcal{L} &= i\overline{\Psi}(\partial_\mu - ig\mathbf{A}_\mu\cdot\hat{\mathbf{T}})\Psi \\
&\equiv i\overline{\Psi}\nabla_\mu\Psi
\end{aligned} \tag{C.543}$$

Here we have introduced tha covariant derivative

$$\nabla_\mu = \partial_\mu - ig\mathbf{A}_\mu\cdot\hat{\mathbf{T}} \tag{C.544}$$

The gauge transformation properties of $\mathbf{A}_\mu$ can be summarised as follows

$$\nabla_\mu \to \nabla'_\mu = \hat{U}\nabla_\mu\hat{U}^{-1} \tag{C.545}$$

The kinetic energy term of the $\mathbf{A}_\mu$ fields is missing. By analogy with electrodynamics we write

$$\mathbf{F}_{\mu\nu} = \nabla_\mu \mathbf{A}_\nu - \nabla_\nu \mathbf{A}_\mu \tag{C.546}$$

Consider

$$\begin{aligned}
\hat{F}_{\mu\nu} &= \mathbf{F}_{\mu\nu} \cdot \hat{\mathbf{T}} = \sum_{i=1}^{3} F^i_{\mu\nu} \hat{T}^i \\
&= \nabla_\mu(\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) - \nabla_\nu(\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) \\
&= \partial_\mu(\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) + ig(\mathbf{A}_\mu \cdot \hat{\mathbf{T}})(\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) - \partial_\nu(\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) - ig(\mathbf{A}_\nu \cdot \hat{\mathbf{T}})(\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) \\
&= (\partial_\mu \mathbf{A}_\nu) \cdot \hat{\mathbf{T}} - (\partial_\nu \mathbf{A}_\nu) \cdot \hat{\mathbf{T}} - ig\big[\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \mathbf{A}_\nu \cdot \hat{\mathbf{T}}\big]
\end{aligned} \tag{C.547}$$

From (C.545) we also have

$$[\nabla'_\mu, \nabla'_\nu] = \hat{U}[\nabla_\mu, \nabla_\nu]\hat{U}^{-1}$$

An explicit calculation of the commutator yields

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] &= [\partial_\mu - ig\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \partial_\nu - ig\mathbf{A}_\nu \cdot \hat{\mathbf{T}}] \\
&= [\partial_\mu, \partial_\nu] + \partial_\mu(-ig\mathbf{A}_\mu \cdot \hat{\mathbf{T}}) - (-ig\mathbf{A}_\mu \cdot \hat{\mathbf{T}})\partial_\mu \\
&\quad + \partial_\nu(-ig\mathbf{A}_\nu \cdot \hat{\mathbf{T}}) - (-ig\mathbf{A}_\nu \cdot \hat{\mathbf{T}})\partial_\nu \\
&\quad + (-ig)^2[\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \mathbf{A}_\nu \cdot \hat{\mathbf{T}}] \\
&= -ig\{(\partial_\mu A_\nu \cdot \hat{T}) - (\partial_\nu A_\mu \cdot \hat{T}) - ig[\mathbf{A}_\mu \cdot \hat{\mathbf{T}}, \mathbf{A}_\nu \cdot \hat{\mathbf{T}}]\}
\end{aligned} \tag{C.548}$$

Notice that in the last line the derivatives act only on the gauges fields.

$$\mathbf{F}_{\mu\nu} = \frac{i}{g}[\nabla_\mu, \nabla_\nu] \tag{C.549}$$

Then $\mathbf{F}_{\mu\nu}$ transforms as

$$\mathbf{F}'_{\mu\nu} = \hat{U}\mathbf{F}_{\mu\nu}\hat{U}^{-1} \tag{C.550}$$

We obtain a gauge invariant Lagrangian by performing the trace over the internal indices

$$
\begin{aligned}
\mathcal{L}'_A &= -\frac{1}{2}Tr\{(\mathbf{F}'_{\mu\nu} \cdot \hat{\mathbf{T}})(\mathbf{F}'^{\mu\nu} \cdot \hat{\mathbf{T}})\} \\
&= -\frac{1}{2}Tr\{\hat{U}(\mathbf{F}'_{\mu\nu} \cdot \hat{\mathbf{T}})\hat{U}^{-1}\hat{U}(\mathbf{F}'^{\mu\nu} \cdot \hat{\mathbf{T}})\hat{U}^{-1}\} \\
&= -\frac{1}{2}Tr\{\hat{U}^{-1}\hat{U}(\mathbf{F}'_{\mu\nu} \cdot \hat{\mathbf{T}})(\mathbf{F}'^{\mu\nu} \cdot \hat{\mathbf{T}})\} \\
&= \mathcal{L}_A
\end{aligned}
\tag{C.551}
$$

C.13 Principle Bundles and Connections

A principle bundle is a fibre bundle $\pi : P \to E$ with fibre F equal to the structure group G and having the property that for all U_a and U_b with $U_a \cap U_b \neq \emptyset$,

$$\varphi_{ba} : U_a \cap U_b \to Left(F) \subset Diff(F),$$

where $Left(F) = \{L_g | L_g(h) = gh, \text{ for all } h \in G, g \in G\}$. In other words, changing coordinates corresponds to multiplying the fibre on the left by some element of G.

Lemma C.13.1 *For every $G-$principle bundle, G acts naturally on P on the right.*

Given $u \in P$, we want to define ug, for each $g \in G$. Let U be a neighbourhood about $\pi(u)$ that has a trivialization. Using these coordinates, represent u as $(\pi(u), h)$ where $h \in G$. Then define ug to be a point of P that has the coordinates $(\pi(u), hg)$. It is not hard to check that this definition is independent of coordinates, and then it is clear that it is a right action.

To place gauge theory in a more general perspective, it is helpful to consider fibre bundle formulism. The idea in gauge theory is to consider group bundles, where each fibre is a copy of the internal symmetry group, and where the base space corresponds to spacetime.

think about gauge theory geometrically - to understand the gauge field as a connection on the principal bundle.

We have already encountered one fibre bundle, from general relativity: the tangent space at a point in spacetime is a fibre, with the fibre bundle

the gauge fields play the same role in gauge theory as the Christoffel symbols play in the tangent in general relativity.

A principal fibre bundle allows one to simultaneously view the physical space, $\mathcal{M}$, referred to as the base space, and the bundle space E, where the bundle space generally reflects the symmetry group of the theory by associating with each point $x \in \mathcal{M}$ a fibre in E diffeomorphic to some Lie Group $\mathcal{G}$, refined to as the gauge group or structure group.

Bundles

A principal bundle $P(\mathcal{M}, G)$, where G is a Lie group and $\mathcal{M}$ is a compact manifold.

covering $\mathcal{M}$ with topological trivial open subsets U_a and giving a set of transition functions

$$t_{ab} : U_a \cap U_b \to G. \tag{C.552}$$

The transition functions are to satisfy three conditions:

$$\begin{aligned} t_{aa}(x) &= 1, \\ t_{ab}(x)t_{ba}(x) &= \mathbf{1}, \\ t_{ab}(x)t_{bc}(x)t_{ca}(x) &= \mathbf{1}, \end{aligned} \tag{C.553}$$

for all points $x \in \mathcal{M}$ where the functions are all defined.

A gauge transformation is defined to be a collection of maps

$$\lambda_a : U_a \to G. \tag{C.554}$$

acting on transition functions in the following way,

$$t_{ab}(x) \longmapsto \lambda_a^{-1}(x)t_{ab}(x)\lambda_b(x). \tag{C.555}$$

Gauge-transformed transition functions define the same bundle. A bundle is characterized by a gauge equivalence class of transition functions satisfying (C.553).

A connection ω is defined globally is represented by a connection one-form ω_a on each U_a, with values in the Lie algebra of G, where, on each overlap $U_a \cap U_b$,

$$\omega_a = \mathrm{Ad}_{t_{ab}^{-1}}\omega_b + t_{ab}^{-1}dt_{ab} \tag{C.556}$$

where Ad is the adjoint representation.

$$[T_a, T_b] = C^{abc}T_c, \tag{C.557}$$

C^{abc} forms a representation of (D.256), the matrix $\mathcal{T}^a$ with components given by

$$(\mathcal{T}^a)_{bc} = iC^{abc} \tag{C.558}$$

This representation is called the *regular* representation or the *adjoint* representation.

C.14 Fibre Bundles

We have seen that a manifold has an associated with it a tangent space at *each* point p in the manifold. One can combine these spaces with the underlying manifold $\mathcal{M}$ to make one big manifold. Mathematically this combining (union) is expressed as,

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}. \tag{C.559}$$

This is known as the **tangent bundle space**. In fact we can combine spaces other than the tangent space with the manifold $\mathcal{M}$ to make a bigger topological space. These other spaces need not be vector spaces as the the tangent space is. The other space is, generally, referred to as the **fibre**. The whole space is known as a **fibre bundle**.

The simplest example of a fibre bundle, which is not a vector bundle, is the cylinder formed by choosing $\mathcal{M}$ as the real line R^1 and A as the circle S^1; one constructs the cylinder as a product between these two spaces, (actually, one should really say: *one constructs the cylinder as a product over the base manifold* R^1). This **product bundle**, also referred to as a **trivial bundle**, is denoted $(\mathcal{M} \times A, \mathcal{M}, \pi)$.The first entry refers to the fibre bundle, and the second refers to the base manifold. The third is a $\pi : \mathcal{M} \times A \to \mathcal{M}$. The projection map $\pi : E \to \mathcal{M}$ associates each fibre with a point in the base manifold. The inverse of the projection map π^{-1} associates a copy of the internal symmetry group (a fibre) with each point in the base manifold.

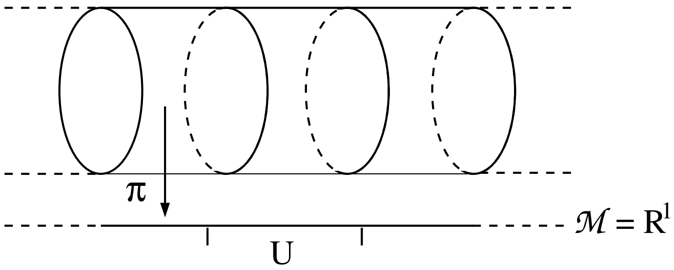


Figure C.36: Fibre bundle, $TS^1 = S^1 \times R$. The base manifold $\mathcal{M}$ (the real line R^1). The circle is the fibre. The fibre bundle consists of a manifold and a projection map π. $\pi^{-1}(U)$ is the local product space.

A fibre bundle is *locally a product bundle.* A fibre bundle which is not a product bundle known as a ***non-trivial bundle***. We give an example which shows that if a manifold is locally the direct product of two other manifolds, it is nevertheless not, in general, a product manifold. Möbius strip

The tangent bundle $T\mathcal{M}$ is locally of the form $U \times \mathcal{M}$. A ***section*** of this tangent bundle is simply a vector field $\tilde{\mathbf{w}}$ on $\mathcal{M}$. In the coordinate patch $(U, u_1, \ldots, u^n)$, $\tilde{\mathbf{w}}$ is given by its component functions $w^1_U(u), \ldots, w^n_U(u)$ with respect to the coordinate basis $\partial/\partial u$, but

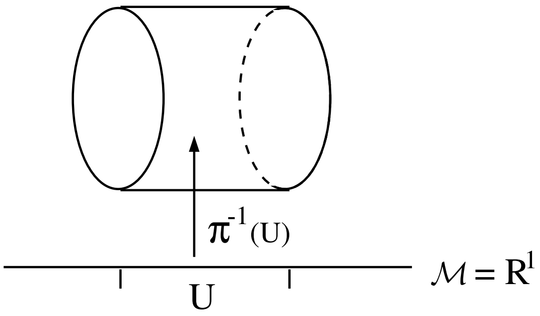


Figure C.37: The inverse map $\pi^{-1}(U)$ is the local product space.

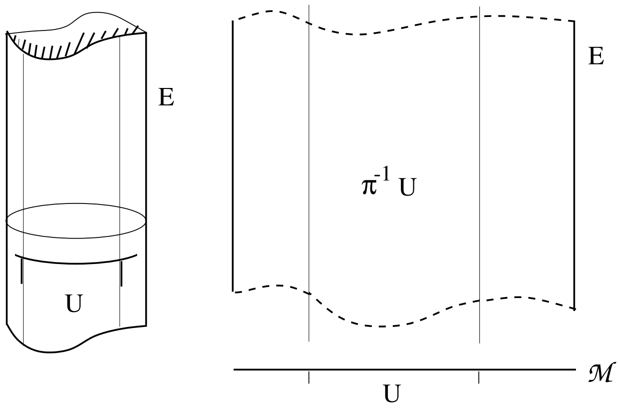


Figure C.38: tangent bundle, $TS^1 = S^1 \times R$. The base manifold $\mathcal{M}$ (the circle S^1) consists of a manifold and a projection map π. $\pi^{-1}(U)$ is the local product space.

of course these functions are defined on U only, not on all of $\mathcal{M}$. In another patch V, the same field is described by another set of component functions $w_V^1(u), \ldots, w_V^n(u)$. At a point p in the overlap $U \cap V$ these two sets of vector component functions transform as

$$w_V^a(p) = [c_{UV}(p)]_b^a w_U^b(p) \tag{C.560}$$

where $c_{UV} = \partial v / \partial u$ the Jacobian matrix.

cotangent fibre bundle

$$T^*\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p^*\mathcal{M}. \tag{C.561}$$

Phase space is the cotangent fibre bundle.

$$w_V^a(p) = [c_{UV}(p)^T]_b^a w_U^b(p) \tag{C.562}$$

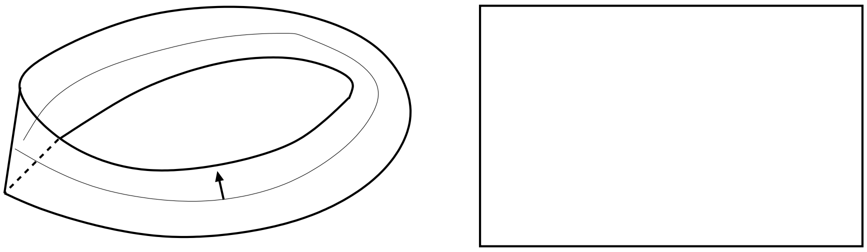


Figure C.39: Möbius.

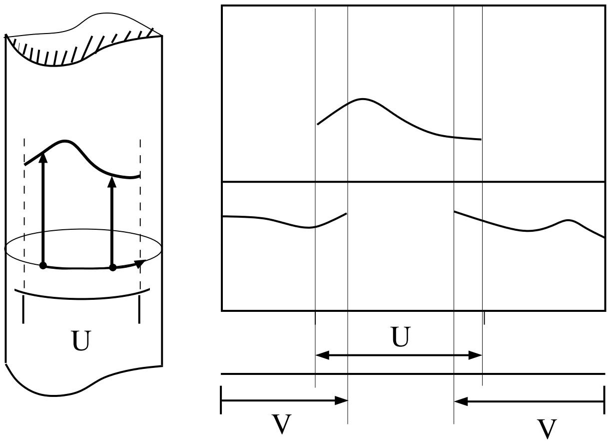


Figure C.40: tangent bundle, $TS^1 = S^1 \times R$. The curved line is a section.

Some examples:

$$\begin{array}{rcl} \pi & \approx & R^n \\ S^{n-1} & \subset & \pi^{-1}(p) \end{array} \tag{C.563}$$

$T_0\mathcal{M}$ is atypical, since it is a *subbundle* of the vector bundle $T\mathcal{M}$.

Let $M(n \times n)$ be the set of all $n \times n$ real matrices. We associate to the matrix $\underline{\underline{x}}$ the point in the n^2-dimensional Euclidean space whose coordinates are $x_{11}, x_{12}, \ldots x_{nn}$. The topological of the parameter space is R^n. The ***general linear group*** $Gl(n, R)$ is the group of all real $n \times n$ matrices $x = (x_{ij})$ with determinate $\det x \neq 0$.

$$X^i \mathbf{e_i} = x_{11}\mathbf{e_{11}} + x_{12}\mathbf{e_{12}} + x_{21}\mathbf{e_{21}} + x_{22}\mathbf{e_{22}} + \cdots + x_{nn}\mathbf{e_{nn}}. \tag{C.564}$$

It is clear from $(xy)_{ij} = \sum_k x_{ik}y_{kj}$ that the product matrix has coordinates that are smooth functions of the coordinates of $\underline{\underline{x}}$ and $\underline{\underline{y}}$

a vector tangent to G is itself a matrix

Definition The ***left (right) translation*** is $G \mapsto G$, $l_a(g) = ag$, and the ***right translation***

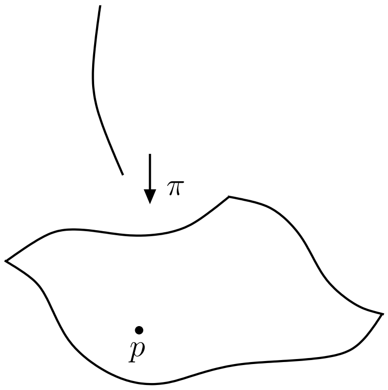


Figure C.41: Travelling up fibre.

$$\varphi_U : U \times GL(R, n) \to \pi^{-1}(U) \tag{C.565}$$

a local trivialization, such for any $x \in U$ and $(X_i^k) \in GL(R, n)$,

$$\varphi_U(x, (X_i^k)) = (x; e_1, \ldots, e_n) \tag{C.566}$$

$$y^a = y^a(x^1, \ldots, x^n), \;\; i = 1, \ldots, n \tag{C.567}$$

$$\varphi_V : V \times GL(R, n) \to \pi^{-1}(V) \tag{C.568}$$

$$\mathbf{e_i} = X_i^k \frac{\partial}{\partial x^k} = Y_i^k \frac{\partial}{\partial y^k} \tag{C.569}$$

or

$$Y_i^k = X_i^k \left(\frac{\partial y^k}{\partial x^j} \right)_x \tag{C.570}$$

by () the right translation of X is given by the Jacobian matrix $\left(\frac{\partial y^k}{\partial x^j} \right)_x$

one can identify a family of transition functions

$$c_{UV} : U \cap V \to GL(R, n) \tag{C.571}$$

given by

$$c_{UV}(x) = \varphi_V^{-1} \circ \varphi_U \tag{C.572}$$

In fact $\varphi_V^{-1} \circ \varphi_U$ is precisely the right translation of X

Details

$$L_{g_1} \circ L_{g_2} = L_{g_1 g_2} \tag{C.573}$$

or

$$\begin{aligned} L_{g_1} \circ L_{g_2} &= L_{g_1}(x; (g_1 e)_1, \ldots, (g_1 e)_n) \\ &= (x; (g_1(g_2 e))_1, \ldots, (g_1(g_2 e))_n) \\ &= (x; (g_1 g_2 e)_1, \ldots, (g_1 g_2 e)_n) \end{aligned} \tag{C.574}$$

The left translation along g maps a neighborhood of e onto one of g. There is a natural map of a vector at e to one on g. a vector tangent to G is itself a matrix

The one-parameter subgroup generated by any matrix A is the integral curve through e of the left-invariant vector field whose tangent at e is A.

As matrices

$$g(t+s) = g(t)g(s), \;\Leftrightarrow\; g_{ij}(t+s) = \sum_k g_{ik}(t) g_{kj}(s) \tag{C.575}$$

$$g_A(t + \Delta t) = g_A(t) g_A(\Delta t) \tag{C.576}$$

$$\frac{dg_A}{dt} = g_A(t) A \tag{C.577}$$

Differentiate both sides with respect to s and put $s = 0$

$$g'(t) = g(t) g'(0). \tag{C.578}$$

Since $g'(0)$ is a constant matrix, the solution to this is

$$g(t) = g(0)\exp(tg'(0)) \tag{C.579}$$

$$1 + tg + \frac{1}{2!}t^2 g^2 + \dots \tag{C.580}$$

is the most general form for a 1-parameter subgroup of a matrix group G.

$$\Phi(x, y) = \exp_p(x\mathbf{e} + y\mathbf{f}) \tag{C.581}$$

$$\frac{\partial}{\partial x}\Phi(x, y)\bigg|_{(x,y)} = \frac{\partial}{\partial x}\exp_p(x\mathbf{e})\bigg|_{(x,y)} = \mathbf{e}. \tag{C.582}$$

Theorem Let X, Y be a pair of left (right) invariant vector fields. Then $[X, Y]$ is left (right) invariant.

Proof:

$$\begin{aligned}
[X', Y']_i &= X'_j \partial'^j Y'_i - Y'_j \partial'^i X'_i \\
&= X_j \partial^j Y'_i - Y_j \partial^j X'_i \\
&= X_j \partial^j \left(\frac{\partial y_i}{\partial x_k} Y_k\right) - Y_j \partial^j \left(\frac{\partial y_i}{\partial x_k} Y_k\right) \\
&= \frac{\partial y_i}{\partial x_k}(X_j \partial^j Y_k - Y_j \partial^j X_k) + \frac{\partial^2 y_i}{\partial x_j \partial x_k}(X_j Y_k - Y_j X_k).
\end{aligned} \tag{C.583}$$

C.14.1 The Structure Group of a Bundle

In a vector bundle each $c_{UV} \in Gl(n)$. We have seen that for a Riemannian manifold $\mathcal{M}$ of dimension n, we may choose $c_{UV}(x) \in O(n)$ by using orthonormal frames. In a general bundle it may be possible to choose the $c_{UV}(x)$ such that they all lie in a specific Lie group G

$$c_{UV} : U \cap V \to G \tag{C.584}$$

We then say that G is the ***structure group*** of the bundle.

$$c_{UV} = \begin{pmatrix} \cos\alpha(x) & -\sin\alpha(x) \\ \sin\alpha(x) & \cos\alpha(x) \end{pmatrix} \in SO(2). \tag{C.585}$$

The orthonormal frames have allowed us to ***reduce*** the structure group from $Gl(2,R)$ to $SO(2)$.

C.14.2 Frame Bundle

Frame bundles are fundamental because from this we can construct the tangent bundle, and by a similar construction the cotangent bundle and all the tensor bundles.

If $\{v_1,\ldots,v_n\}$ is a local field of linear frames on a neighbourhood U in $\mathcal{M}$, and $\{e_1,\ldots,e_n\}$ is a frame at $x \in U$, then $\lambda_a^b V_{bx}$, where the number λ_a^b are the entries of a non-singular matrix λ. Thus relative to the local field, each linear frame determines an element of $GL(n,\mathbb{R})$; and each element of $GL(n,\mathbb{R})$ determines a linear frame. We therefore take $GL(n,\mathbb{R})$ as the typical fibre bundle.

The frame bundle is clearly not a vector bundle. Its typical fibre has instead the structure of a group.

C.14.3 The Idea of a Principal Bundle

If we have a connection in a principle bundle $P \to \mathcal{M}$ then a choice of a horizontal subspace at u is equivalent to a choice of projection

$$proj_u : T_uP \to V_u.$$

Then the vertical space (i.e. a fiber of the vertical bundle) may be viewed as the tangent space to the fibre, G. In turn, this an be viewed as g, the Lie algebra of G, so we really have

$$proJ_u : T_uP \to g.$$

Therefore a connection is equivalent to having a g valued 1-form on P which is invariant under the right action of G.

Let G be a Lie group and $\mathcal{M}$ a smooth manifold. A ***principle G bundle*** is over $\mathcal{M}$ is a manifold which locally looks like $\mathcal{M} \times G$.

We have already found the transition functions for this fibre bundle. We consider the intersection of two coordinate patches, and a point p

The transition functions are given by the Jacobian matrix. They are elements of the group $GL(n, \mathbf{R})$, the general linear transformations on an n dimensional real vector space.

The bundles $\mathcal{T}(\mathcal{M})$ and $\mathcal{T}^*(\mathcal{M})$ are vector bundles. a tangent bundle and a co-tangent bundle. Phase space is an example of a co-tangent bundle.

A principal bundle is a bundle whose fibres are the transition functions themselves - points in the fibre are elements of the structure group. We have just considered a principal fibre bundle, that of a frame space.

Definition The right action of G on $\pi^{-1}(U)$ is defined by $\phi_i^{-1}(ua) = (p, g_i a)$, that is, (fig C.14.3)

$$ua := \phi_i(p, g_i a) \tag{C.586}$$

for any $a \in G$ and $u \in \pi^{-1}(p)$.

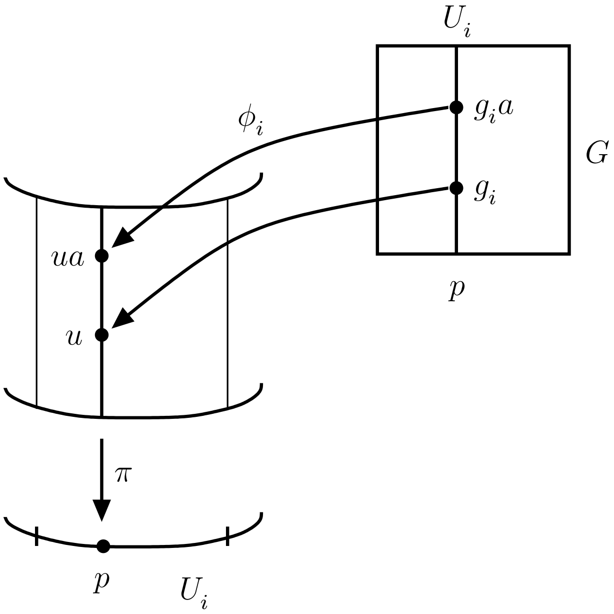


Figure C.42: Defintion of the right action of G on the principal fibre bundle P.

$\square$

Definition Principal fibre bundle.

i) A differentiable manifold E called the total space;

ii) A differentiable manifold $\mathcal{M}$ called the base space;

iii) A surjection $\pi : E \to \mathcal{M}$ called the projection. The inverse image $\pi^{-1}(x) \equiv G_x \simeq G$ called the fibre at x;

iv) A Lie group G called the structure group, which acts on the fibre G on the left;

v) An open covering $\{U_I\}$ of $\mathcal{M}$ with a diffeomorphism $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ such that $\pi\phi_i(x, g) = x$. The map is called the local gauge or local trivialisation since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times G$;

vi) If we write $\phi_i(x, g) = \phi_{i,x}(g)$, the map $\phi_{i,x} : G \to G_x$ is a diffeommorphism. On $U_i \cap U_j \neq \emptyset$, we require that $t_{ij}(x) \equiv \phi_{i,x}^{-1}\phi_{j,x} : G \to G$ be an element of the structure group G. The ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ such that $\phi_j(x, g) = \phi_i(x, t_{ij}(x)g)$. $\{t_{ij}\}$ are the transition functions.

□

C.14.4 Principal Bundle

We are now ready to state the precise definition of a principal bundle.

We will say that a fibre bundle

$$\{P, \mathcal{M}, \pi, F, G\} \tag{C.587}$$

is a **principal bundle** *if the fibre F is the same as the group G* and if the transition functions $c_{UV}(x)$ act on $F = G$ by *left translations.*

and

C.14.5 Action of the structure Group on a Principal Bundle

In practical situations transition functions are the gauge transformations required for pasting local charts together.

C.14.6 Connections on Principal Bundles

When we make comparison of objects in two different spaces is made by a prescribed mapping, and the mappings that connect the various spaces are called connections.

The first definition of a connection has a clear geometric meaning,

1. a $G-$invariant horizontal distribution $H_uP \subset T_uP$.

An equivalent, less geometric definition of a connection consistent with the first definition is:

2. a one-form $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying $\omega(s(X)) = X$.

We come to the final definition of the connection on a principal bundle, expressed in the form of $\mathcal{G}-$valued one-forms on the base manifold $\mathcal{M}$, instead of one-forms on the total space P. This definition is most suitable in physics applications.

3. a family of one-forms $A_i \in \Omega^1(P; \mathfrak{g})$ satisfying equation (for simplicity here in the case of matrix groups G):

$$A_j = t_{ij} A_i t_{ij}^{-1} + t_{ij}^{-1} dt_{ij}.$$

Definition (1): A connection on a principal bundle $\pi : P \to \mathcal{M}$ with group G is a smooth assignment to each $u \in P$ of an $n-$dimensional subspace $\mathcal{H}(p) \subset T_u P$ of the tangent space to P ($n = dim\ \mathcal{M}$) such that

$$T_u P = \mathcal{H}(p) \oplus \mathcal{V}(p), \tag{C.588}$$

where $\mathcal{V}(p)$ is the subspace

$$\mathcal{V}(p) := \{X \in T_u P : \pi_*(X) = 0\}. \tag{C.589}$$

The subspace field $\mathcal{H}$ is also required to be invariant under the left action L_g ($g \in G$):

$$(L_g)_* \mathcal{H}(p) = \mathcal{H}(gp), \tag{C.590}$$

for all $g \in G$. $\mathcal{V}(p)$ and $\mathcal{H}(p)$ are called the vertical and horizontal subspace of $T_u P$, respectively.

□

Definition (2): Suppose $\mathcal{G}$ is the Lie algebra of a Lie group G, which is the structure group of a principal fibre bundle $\pi : P \to \mathcal{M}$. Let $A \in \mathcal{G}$, and $A^\sharp$, called a fundamental field, be the vector field on P defined by

$$A^\sharp(p) := \frac{d}{dt} \exp(At)p \Big|_{t=0}. \tag{C.591}$$

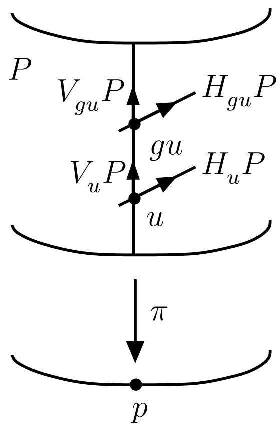


Figure C.43: The horizontal subspace $H_{gu}P$, defining by the connection in definition (1), is obtained from H_uP by the left action.

A connection on the principal bundle $\pi : P \to \mathcal{M}$ is a $\mathcal{G}-$valued one for satisfying the following two properties:

$$(i) \qquad \omega(A^\sharp(p)) = A \tag{C.592}$$

$$(ii) \qquad (L_g)^*(\omega) = \mathcal{A}d(g)(\omega). \tag{C.593}$$

□

Definition (3): A connection on a principal bundle is an assignment to each local trivialisation $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ (a choice of gauge in physics terms) a Lie algebra one-form ω_i on U_i which satisfies the following rule between different local trivialisations,

$$\omega_j(X_p) = (R_{t_{ij}^{-1}(p)})_*((t_{ij})_*(X_p)) + \mathcal{A}d(t_{ij}(p))(\omega_i(X_p)), \tag{C.594}$$

for all $X_p \in T_p\mathcal{M}$ and $p \in U \cap V$.

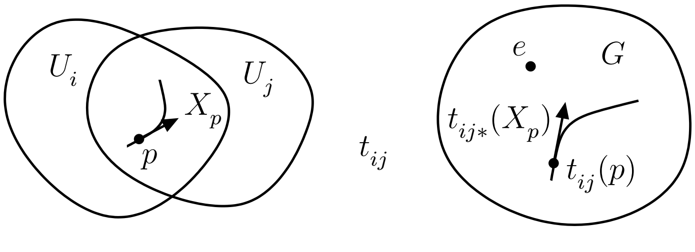


Figure C.44:

$\square$

Since a non-trivial principal bundle does not admit a global section, the pull-back $A_i = s_i^* \omega$ exists locally but not necessarily globally.

Lemma C.14.1 *The two definitions (1) and (2) of a connection on a principal fibre bundle are equivalent.*

Proof:

(2) $\Rightarrow$ **(1)**:

Suppose we are given a $\mathcal{G}-$valued connection one-form ω, as in definition (2). Consider the field of subspaces defined by

$$\mathcal{H}(u) = \{X \in T_u P : \omega_u(\mathbb{X}) = 0\}.$$

By (), $\omega(\mathbb{Y})A \in \mathcal{G}$ for any $\mathbb{Y} \in \mathcal{V}(p)$. Hence $\mathcal{V}(p) \cap \mathcal{H}(p) \neq \emptyset$ and $T_p P = \mathcal{V}(p) \oplus \mathcal{H}(p)$. We must prove that

$$L_{g*}\mathcal{H}(u) = \mathcal{H}(gu).$$

Fix a point $u \in P$ and define $\mathcal{H}(u)$ as above. Take $\mathbb{X} \in \mathcal{H}(u)$ and construct $L_{g*}\mathbb{X} \in T_{gu}$. We find

$$\omega_{gu}(L_{g*}\mathbb{X}) = L_g^* \omega_u(\mathbb{X}) = g\omega_u(\mathbb{X})g^{-1} = 0$$

since $\omega(\mathbb{X}) = 0$. Therefore, $L_{g*}\mathbb{X} \in \mathcal{H}(gu)$. Note that L_{g*} is an invertible linear map. Hence any vector $\mathbb{Y} \in \mathcal{H}(gu)$ is expressed as $\mathbb{Y} = L_{g*}\mathbb{X}$ for some $\mathbb{X} \in \mathcal{H}(u)$. Thus $L_{g*}\mathcal{H}(u) = \mathcal{H}(gu)$.

(1) $\Rightarrow$ **(2)**:

We define a $\mathcal{G}-$valued one-form ω by

$$\omega(A^\sharp + X) = A \tag{C.595}$$

Note $0 \in \mathcal{H}(u)$ and setting $\mathbb{X} = 0$, we have $\omega(A^\sharp) = A$, and as the connection is linear this implies $\omega(\mathbb{X}) = 0$ for all $\mathbb{X} \in \mathcal{H}(u)$.

To complete the proof we must establish property (ii) (C.593). Any $\mathbb{X} \in T_u P$ can be written $\mathbb{X} = a\mathbb{H} + b\mathbb{V}$, where $H \in \mathcal{H}(u)$ and $V \in \mathcal{V}(u)$. This allows us to need only verify

(C.593) separetly for arbitrary horizontal and vertical vectors. Suppose $\mathbb{H} \in \mathcal{H}(u)$. Then $\omega(\mathbb{H}) = 0$. By (C.590), $\omega((L_g)_*\mathbb{H}) = 0$ and (C.593) is verified for $\mathbb{H} \in \mathcal{H}(u)$. Consider $\mathbb{V} = A^\sharp(u) \in \mathcal{V}(u)$, for some $A \in \mathcal{G}$. We compute

$$\begin{aligned}
\omega_{gp}((L_g)_* A^\sharp(u)) &= \omega_{gp}\left((L_g)_* \frac{d}{dt}(\exp(At)u)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(g\exp(At)u)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(g\exp(At)g^{-1}\cdot gu)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(ad_g(\exp(At))\ gu)\Big|_{t=0}\right) \\
&= \omega_{gu}\left(\frac{d}{dt}(\exp(\mathcal{A}d(g)At)\ gu)\Big|_{t=0}\right) \\
&= \omega_{gu}(\ (\mathcal{A}d(g)A)^\sharp\ gu)
\end{aligned} \tag{C.596}$$

As $\omega(A^\sharp) = A$

$$\omega_{gu}(\ (\mathcal{A}d(g)A)^\sharp\ gu) = \mathcal{A}d(g)A$$

and then

$$\mathcal{A}d(g)A = (\mathcal{A}d(g))(\omega(A^\sharp(u)))$$

therefore we have

$$\omega_{gp}((L_g)_* A^\sharp(u)) = (\mathcal{A}d(g))(\omega(A^\sharp(u))),$$

that is, we have verified (C.593) for vertical vectors $A^\sharp(u)$ as well.

□

In the next section provide the proof of the equivalence between of definitions (2) and (3).

C.14.7 Gauge Fields

We now make contact with the more familiar notion of gauge fields as used in physics, which live on $\mathcal{M}$ instead of P.

Theorem C.14.2 *Given a $\mathfrak{g}-$valued one-form A_i on U_i and a local section $s_i : U_i \to \pi^{-1}(U_i)$, there exists a connection one-form ω such that $A_i = s_i^*\omega$ on U_i*

Proof:

(2) $\Rightarrow$ (3):

.....

Let ω be a $\mathcal{G}-$valued connection one-form as given in definition (2), and $\phi_i : \pi^{-1}(U) \to U \times G$ be a local trivialisation. Associated with ϕ_i there is a local section given by $s_i(p) = \phi_i^{-1}(p, e)$. We define a $\mathcal{G}-$valued one-form A_i on U by

$$A_i = s_i^*\omega,$$

then we show that it is the connection one-form in the sense of definition (3). To do this we have to show that the gauge transformation

$$\omega_i \to \omega_j = s_j^*\omega$$

is given by (C.594).

....

Recall that we have local sections $s_i : U_i \to \pi^{-1}(U_i)$ associated canonically to the trivialisation of the bundle. g_i is the canonical local trivialisation defined by $\phi_i(u) = (p, g_i)$ for $u = s_i(p)g_i$. Let us define a $\mathfrak{g}-$valued one-form ω_i on P by

$$\omega_i = g_i^{-1}\pi^* A_i g_i + g_i^{-1} d_P g_i \tag{C.597}$$

where d_P is the exterior derivative on P.

We first show that $s_i^*\omega_i = A_i$. For $X \in T_pM$. Note

$$s_i^*\omega(X) = \omega_i(s_{i*}X)$$

As $g_i = e$ at s_i

$$\begin{aligned} s_i^*\omega(X) &= \pi^* A_i(s_{i*}X) + d_P g_i(s_{i*}X) \\ &= A_i(\pi_{i*}s_{i*}X) + d_P g_i(s_{i*}X) \end{aligned} \tag{C.598}$$

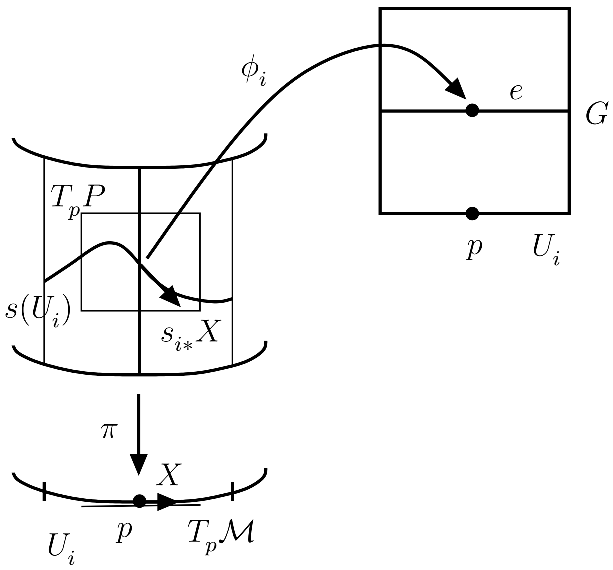


Figure C.45:

We have

$$d_P g_i(s_{i*}X) = \frac{d}{dt} g_i(s_i(t))\Big|_{t=0} = 0$$

as $g \equiv e$ along $s_i(t)$. Thus we have obtained $s_i^* \omega_i(X) = A_i(X)$.

Next we show that ω_i satisfies the axioms of a connection one-for given in definition...

We consider the subspace of T_uP tangent to the fibres. Let $X = A \in V_uP$, $A \in \mathfrak{g}$. It follows from $\pi_* X = 0$. Now we have from $g(u \exp(tA)) = g(u)\exp(tA)$

$$\begin{aligned}
\omega_i(A^\sharp) &= g_i^{-1}(d_P g_i)(A^\sharp) \\
&= g_i(u)^{-1} \frac{dg(u\exp(tA))}{dt}\Big|_{t=0} \\
&= g_i(u)^{-1} g_i(u) \frac{dg\exp(tA)}{dt}\Big|_{t=0} = A.
\end{aligned}$$

Take $X \in T_P P$ and $h \in G$. We have

$$\begin{aligned}
R_h^* \omega_i(X) &= \omega_i(R_{h*}X) \\
&= g_i^{-1}(uh) A_i(\pi_{i*} R_{i*} X) g_i(uh) + g_i^{-1}(uh)(d_P g_i(uh))(R_{h*}X)
\end{aligned}$$

Since $g_i(uh) = g_i(u)h$ and $\pi_{i*} R_{i*} X = \pi_{i*} X$ the first term above can be written

$$h^{-1}g_i^{-1}(u)A_i(\pi_{i*}X)g_i(u)h$$

For the second term we work out

$$\begin{aligned} g_i^{-1}(uh)(d_P g_i(uh))(R_{h*}X) &= g_i^{-1}(uh)\frac{d}{dt}g_i(\gamma(t)h)\Big|_{t=0} \\ &= h^{-1}g_i^{-1}(u)\frac{d}{dt}g_i(\gamma(t))h \\ &= h^{-1}g_i^{-1}(u)(d_P g_i(u))(X)h. \end{aligned}$$

Here $\gamma(t)$ is a curve through $u = \gamma(0)$, whose tangent vector at u is X. Therefore

$$R_h^*\omega_i(X) = h^{-1}\omega_i(X)h.$$

Hence the $\mathfrak{g}-$valued one-form ω_i defined by () satisfies $A_i = s_i^*\omega$ and the axioms of a connection one-form.

Global one-form on P?

Suppose we are given local one-forms A_i, A_j, etc., on $\mathcal{M}$ that satify the gauge transformation given in definition (3). First we will construct local one-forms ω_U, $\omega_V, \ldots,$ on P from the local one-forms A_U,$A_V, \ldots,$ on $\mathcal{M}$. We then need to show that $\omega_U = \omega_V$ on $\pi^{-1}(U \cap V)$, in order for the local one-forms ω_U, ω_V, etc., to collectively define a global connection one form ω as in definition (2).

.....

(3) $\Rightarrow$ (2):

We first define the map $\tilde{\omega}_i : T_uP \to \mathcal{G}$ by

$$\tilde{\omega}_i((s_i)_*X_p + A^\sharp) = A_i(X_p) + A. \tag{C.599}$$

A local $\mathcal{G}-$valued one-form on all of $\pi^{-1}(U_i)$ can be constructed from this map via

$$\omega_i(\mathbb{X}_{gu}) = (\mathcal{A}d(g))\ \tilde{\omega}_i\ ((L_{g^{-1}})_*\mathbb{X}_{gu}) \tag{C.600}$$

for $\mathbb{X}_{gu} \in T_{gu}P$. This one-form reduces to the one form $\tilde{\omega}_i$ on T_uP for $g = e$. The one-form ω_i satifies condition (ii) of definition 2 on the restricted bundle $\pi^{-1}(U)$

$$\begin{aligned}
\omega_{igu}((L_g)_*\mathbb{X}_u) &= \mathcal{A}d(g))\,\tilde{\omega}_{iu}\,((L_{g^{-1}})_*((L_g)_*\mathbb{X}_u)) \\
&= \mathcal{A}d(g))\,\tilde{\omega}_{iu}\,(\mathbb{X}_u) \\
&= \mathcal{A}d(g))\,\omega_{iu}\,(\mathbb{X}_u),
\end{aligned}$$

and thus is a connection in the sense of definition (2) on this restricted bundle.

Suppose s_j is another local trivialisation, where $U_i \cap U_j \neq \emptyset$. We can similarly define a local connect ω_j on $s_{U_j}(U_i \cap U_j)$, as then they would agree on all of $\pi^{-1}(U \cap V)$ by virtue of (C.600). That is, we wish to show that

$$\tilde{\omega}_i((s_j)_* X_p + A^\sharp) = \tilde{\omega}_j((s_j)_* X_p + A^\sharp)$$

Now since $\omega_i(A^\sharp) = A = \omega_j(A^\sharp)$, it is sufficient to check that $\tilde{\omega}_i((s_j)_* X_p) = \tilde{\omega}_j((s_j)_* X_p)$, for $x \in U \cap V$ and $X_p \in T_p\mathcal{M}$.

suppose $\gamma(t)$ is a curve in $\mathcal{M}$ with $p = \gamma(0)$ and $X_p = \gamma'(0)$. We compute $(s_j)_*(X_p)$ by

$$\begin{aligned}
(s_j)_*(X_p) &= \frac{d}{dt} s_j(\gamma(t))\Big|_{t=0} \\
&= \frac{d}{dt}\big(t_{ij}(\gamma(t)) s_j(\gamma(t))\big)\Big|_{t=0} \\
&= \frac{d}{dt}\big(t_{ij}(p) s_j(\gamma(t))\big)\Big|_{t=0} + \frac{d}{dt}\big(t_{ij}(\gamma(t)) s_j(p)\big)\Big|_{t=0} \\
&= (L_{t_{ij}(p)})_*(s_i)_*(X_p) + \frac{d}{dt}\big(t_{ij}(\gamma(t)) t_{ij}^{-1}(p) s_j(p)\big)\Big|_{t=0} \\
&= (L_{t_{ij}(p)})_*(s_i)_*(X_p) + \big((R_{t_{ij}^{-1}(p)})_*(t_{ij})_*(X_p)\big)^\sharp_{s_j(p)}.
\end{aligned}$$

Using this and by using the gauge transformation formula we then have

$$\begin{aligned}
\tilde{\omega}_i((s_j)_* X_p) &= \tilde{\omega}_i\Big(\big((R_{t_{ij}^{-1}(p)})_*(t_{ij})_*(X_p)\big)^\sharp_{s_j(p)} + (L_{t_{ij}(p)})_*(s_i)_*(X_p)\Big) \\
&= (R_{t_{ij}^{-1}(p)})_*(t_{ij})_*(X_p) + \mathcal{A}d(t_{ij})\,(A_i(X_p)) \\
&= \tilde{\omega}_i((s_j)_* X_p) = A_j(X_p)
\end{aligned} \tag{C.601}$$

By () $\tilde{\omega}_j((s_j)_* X_p) = A_j(X_p)$ the proof is complete.

□

If the principal bundle is non-trivial, it is not generally possible to describe the connection ω on P in terms of a single Yang-Mills $A_\mu(x)$ field on $\mathcal{M}$. Instead, one must cover $\mathcal{M}$ with local trivalising charts, and then local Yang-Mills fields associated with any pair of overlapping charts U_i, U_j will be related on $U_i \cap U_j$ by

$$A^{(j)}_\mu(x) = \Omega(x)^{-1} A^{(i)}_\mu(x) \Omega(x) + \Omega(x)^{-1} \partial_\mu \Omega(x)$$

with the corresponding local gauge function $\Omega_{ij}(x)$ satifying the $s_i(x) = s_j(x)\Omega_{ij}(x)$. Note that these functions $\Omega_{ij} : U_i \cap U_j \to G$ are precisely the bundle transition functions.

Let $P(\mathcal{M}, G)$ be a principal bundle over $\mathcal{M}$ and U a chart of $\mathcal{M}$. Take two local sections s_1 and s_2 over U such that $s_2(p) = s_1(p)g(p)$. The corresponding local forms A_1 and A_2 are related as

$$A_2 = g^{-1} A_1 g + g^{-1} dg. \tag{C.602}$$

In components, this reads

$$A_{2\mu}(p) = g^{-1}(p) A_{1\mu}(p) g(p) + g^{-1}(p) \partial_\mu g(p). \tag{C.603}$$

which is simply the gauge transformation ().

Example Electrodynamics:

Let P be a U(1) bundle over $\mathcal{M}$. Take overlapping charts U_i and U_j. Let A_i (A_j) be a local connection form on U_i (U_j). The transition function $t_{ij} : U_i \cap U_j \to U(1)$ is given by

$$t_{ij}(p) = \exp(i\theta(p)) \qquad \theta(p) \in \mathbb{R}. \tag{C.604}$$

A_i and A_j are related by

$$\begin{aligned} A_j(p) &= t_{ij}(p)^{-1} A_i(p) t_{ij}(p) + t_{ij}(p)^{-1} dt_{ij}(p) \\ &= A_i(p) + i d\theta(p) \end{aligned} \tag{C.605}$$

In components, we have

$$A_{j\mu} = A_i \mu + i \partial_\mu \theta. \tag{C.606}$$

□

Example General relativity:

The important example is a connection in the principal $GL(n,\mathbb{R})-$bundle $\mathbf{B}(\mathcal{M})$ of frames on a $n-$dimensional manifold $\mathcal{M}$. Any local coordinate chart (U,φ) on $\mathcal{M}$ provides a local section $s : U \to \mathbf{B}(\mathcal{M})$ by associating with $x \in U \subset \mathcal{M}$, the local frame $(\partial_1, \partial_2, \ldots, \partial_n)_x$. If ω is a connection one-form on $\mathbf{B}(\mathcal{M})$, let $\Gamma := s^*\omega$ denote the associated $\mathcal{L}(GL(n,\mathbb{R}))-$valued one-form on U, and consider the relation between Γ and the local one-form Γ' associated with another coordinate chart (U',φ') such that $U \cap U' \neq \emptyset$.

The local section of $\mathbf{B}(\mathcal{M})$ associated with s' is $(\partial/\partial x'^1, \ldots, \partial/\partial x'^m)$, and the transition function $J : U \cap U' \to GL(n,\mathbb{R})$ is just the Jacobian of the coordinatre transformations:

$$(\partial_{\mu'})_x = (\partial_\nu)_x J^\nu{}_\mu(x)$$

where

$$J^\nu{}_\mu(x) := \partial x^\nu / \partial x'^\mu.$$

Then,

$$\begin{aligned}
\Gamma'_\mu(x) &= (s^*\omega)_x(\partial_{\mu'}) \\
&= J^\alpha_{\ \mu}(x)(s'^*\omega)_x(\partial_\alpha) \\
&= J^\alpha_{\ \mu}(x)\big(J^{-1}(x)\Gamma_\alpha(x)J(x) + J^{-1}(x)\partial_\alpha J(x)\big) \qquad \text{(C.607)}
\end{aligned}$$

where we have used the result () for a connection in a bundle whose srtucture group is a matrix group. The second term reads

$$\begin{aligned}
J^\alpha_{\ \mu}(x)(J^{-1}(x)\partial_\alpha J(x))^\epsilon{}_\delta &= \frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial x'^\epsilon}{\partial x^\lambda}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\lambda}{\partial x'^\delta}\right)\right) \\
&= \frac{\partial x'^\epsilon}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\delta}.
\end{aligned}$$

If $G_\lambda{}^\rho$ is some basis for the Lie algebra $M(n,\mathbb{R})$ (the set of all $n \times n$ real matrices) of $GL(n,\mathbb{R})$ then we can write the matrix-valued one-form Γ_μ as

$$(\Gamma_\mu)^\epsilon{}_\delta = \Gamma^\lambda_{\mu\rho}(G_\lambda{}^\rho)^\epsilon{}_\delta.$$

The first term in () is then

$$
\begin{aligned}
J^{\alpha}{}_{\mu}(x)(J^{-1}(x)\Gamma_{\alpha}(x)J(x))^{\epsilon}{}_{\delta} &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\left(\frac{\partial x'^{\epsilon}}{\partial x^{\beta}}(\Gamma_{\alpha})^{\beta}{}_{\lambda}(x)\frac{\partial x^{\lambda}}{\partial x'^{\delta}}\right) \\
&= \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\left(\frac{\partial x'^{\epsilon}}{\partial x^{\beta}}(G_{\gamma}{}^{\rho})^{\beta}{}_{\lambda}\frac{\partial x^{\lambda}}{\partial x'^{\delta}}\right)\Gamma^{\gamma}_{\alpha\rho}(x).
\end{aligned}
$$

In particular, if we pick the natural basis set $(G_{\gamma}{}^{\rho})^{\beta}{}_{\lambda} = \delta^{\beta}_{\gamma}\delta^{\rho}_{\lambda}$ then () becomes the well known transformation law for the components $\Gamma^{\epsilon}_{\mu\delta}$ of an affine connection on $\mathcal{M}$:

$$
\Gamma'^{\epsilon}_{\mu\delta}(x) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\rho}}{\partial x'^{\delta}}\frac{\partial x'^{\epsilon}}{\partial x^{\gamma}}\Gamma^{\gamma}_{\alpha\rho}(x) + \frac{\partial x'^{\epsilon}}{\partial x^{\lambda}}\frac{\partial^2 x^{\lambda}}{\partial x'^{\mu}\partial x'^{\delta}}.
$$

□

C.14.8 Parallel Transport in a Principal Bundle

Parallel transport was defined as transport without change.

What is parallel transport of an element of a principal bundle along a curve in $\mathcal{M}$?

We use the notion of a horizontal lift.

Horizontal vector fields are fields whose flow lines move from one fibre into another.

Definition Let P be a principal fibre bundle and let $\gamma[0,1] \to \mathcal{M}$ be a curve in $\mathcal{M}$. A curve $\tilde{\gamma}[0,1] \to P$ is said to a horizontal lift of γ if $\pi\tilde{\gamma} = \gamma$ and the tangent vector to $\tilde{\gamma}(t)$ always belongs to $\mathcal{H}_{\tilde{\gamma}(t)}P$

□

C.14.9 Curvature on a Principal Bundle

Suppose a connection is given on a principal bundle $\pi : P \to \mathcal{M}$, with group G. Then $X \in T_u P$ can be written uniquely as the sum of a vertical and a horizontal vector

$$
D\varphi \equiv (d\varphi)^H
$$

where

$$(d\varphi)^H(\mathbb{X}_1, \ldots, \mathbb{X}_{i+1}) \equiv d\varphi(\mathbb{H}_1, \ldots, \mathbb{H}_{i+1})$$

and $\mathbb{H}_1, \ldots, \mathbb{H}_{i+1}$ are the horizontal vectors of $\mathbb{X}_1, \ldots, \mathbb{X}_{i+1} \in T_u P$.

Lemma C.14.3 *If $A, B \in \mathcal{G}$, the map $\sharp$ preserves the Lie algebra structure:*

$$[A^\sharp, B^\sharp] = [A, B]^\sharp \tag{C.608}$$

Proof:

$$[A^\sharp, B^\sharp]_p = \lim_{t\to 0} \frac{1}{t}\{B^\sharp - (L_{g(t)})_* B^\sharp\}$$

$$\begin{aligned}
[A^\sharp, B^\sharp]_p &= \lim_{t=0}\{(\sharp)_* B - (\sharp)_* \mathcal{A}d(g(t))B\} \\
&= (\sharp)_* \left(\lim_{t=0} \frac{1}{t}\{B - \mathcal{A}d(g(t))B\}\right)
\end{aligned} \tag{C.609}$$

$$\mathcal{A}d(g(t))B = (L_{g(t)})_*(R_{g^{-1}(t)})_* B.$$

by $(R_{g^{-1}(t)})_* B = B$,

$$\mathcal{A}d(g(t))B = (L_{g(t)})_* B$$

$$[A^\sharp, B^\sharp]_p = (\sharp_p)_* \left(\lim_{t\to 0} \frac{1}{t}\{B - (L_{g(t)})_* B\}\right) = (\sharp_p)_*([A, B]) = [A, B]^\sharp(p),$$

□

C.14.10 Extension and Reduction of Principal Bundles

Given any bundle E we can construct a principal bundle $P(E)$, by replacing the fibres in E with the transition functions, while keeping the transition functions the same.

Let $H \subset G$ be a closed subgroup. P is reduced to a principal H bundle Q if:

(i) $Q \subset P$ is a submanifold,

(ii) $qh \in Q$ for all $q \in Q$, $h \in H$, such that

(1) $\pi(Q) = M$,

(2) H acts transitively in each fibre $Q_x = \pi^{-1}(x) \cap Q$, (remember $\pi^{-1}(x) \subset P$); by acts transitively we mean given any two points $p, q \in Q_x$ there exists at least one $h \in H$ such that $hp = q$.

$$\{v \in H(P) | < \omega | v >= 0\} \tag{C.610}$$

$$< dx^\nu | \frac{\partial}{\partial x^\mu} >= \delta_{\mu\nu}, \quad < dg_{kj} | \frac{\partial}{\partial g_{lm}} >= \delta_{kl}\delta_{jm}, \quad < dx^\nu | \frac{\partial}{\partial g_{lm}} >=< dg_{kj} | \frac{\partial}{\partial x^\mu} >= 0. \tag{C.611}$$

$$\begin{aligned} < \omega_{ij} | X_\mu > &= < i(g^{-1})_{ik} dg_{kj} + (g^{-1} A^a_\nu T^a g)_{ij} dx^\nu | \frac{\partial}{\partial x^\mu} + iB_{\mu lm} \frac{\partial}{\partial g_{lm}} > \\ &= (ig^{-1})_{ik} < dg_{kj} | \frac{\partial}{\partial g_{lm}} > iB_{\mu lm} + (g^{-1} A^a_\nu T^a g)_\nu < dx^\nu | \frac{\partial}{\partial x^\mu} > \\ &= (g^{-1} A^a_\mu T^a g - g^{-1} B_\mu)_{ij} \end{aligned} \tag{C.612}$$

Hence

$$B_{\mu ij} = \left(A^b_\mu T^b g \right)_{ij}. \tag{C.613}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} + i \left(A^b_\mu T^b g \right)_{ij} \frac{\partial}{\partial g_{ij}} \tag{C.614}$$

$$D_\mu (gWg^{-1}) = \frac{\partial}{\partial x^\mu} \tilde{W} + iA^b_\mu T^b gWg^{-1} - igWg^{-1} A^b_\mu T^b = [\partial_\mu + A^a_\mu T^a, \tilde{W}], \tag{C.615}$$

$$ig^{-1}\mathbf{d}g + g^{-1}\mathbf{A}g = ig^{-1}h^{-1}\mathbf{d}(hg) + g^{-1}h^{-1}\mathbf{A}'hg. \tag{C.616}$$

This gives

$$\mathbf{A}' = -i\mathbf{d}hh^{-1} + h\mathbf{A}h^{-1} \tag{C.617}$$

$$[D_\mu, D_\nu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu) g\frac{\partial}{\partial g} - A^a_\mu A^b_\nu \left(T^a g\frac{\partial}{\partial g} T^b g\frac{\partial}{\partial g} - T^b g\frac{\partial}{\partial g} T^a g\frac{\partial}{\partial g} \right). \tag{C.618}$$

$$[D_\mu, D_\nu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]) g\frac{\partial}{\partial g} = iF_{\mu\nu} g\frac{\partial}{\partial g} \tag{C.619}$$

Thus we see that the curvature of the principal bundle is associated with the field strength tensor.

C.14.11 The Complex Line Bundle

$$z = a + ib \quad \leftrightarrow \quad a\mathbf{e_1} + b\mathbf{e_2}, \text{ or in component form } \begin{pmatrix} a \\ b \end{pmatrix} \tag{C.620}$$

We multiply a complex number $z = a + ib$ by i, $-b + ia$

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad J^2 = -I \tag{C.621}$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{C.622}$$

$$iz = -b + ia \quad \leftrightarrow \quad \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{C.623}$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y) \tag{C.624}$$

Of course, this can be expressed entirely in real terms

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \circ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \tag{C.625}$$

C.15 Summary of Differential Geometry

- Coordinate independent formalism.
- Anti-symmetric objects are important - unifying notations.

C.16 Some Algebraic Geomerty

C.16.1 Homology

boundaries $\subset$ cycles .

We are interested in things that do not have boundaries, but are not themselves boundaries of anything. In other words, we are interested in chains that are elements of $Z_n(K)$ but not in $B_n(K)$.

We define the $n-th$ Homology group, denoted $H_n(K)$, as

The following theorem simplifies calculations of Homology groups.

$$H_n(K) \equiv Z_n(K)/B_n(K). \tag{C.626}$$

Lemma C.16.1 *Let $f : G_1 \to G_2$ be a homomorphism.*

(i) $ker(f) = \{x : x \in G_1, f(x) = 0\}$ is a subgroup of G_1

(ii) $im(f) = \{x : x \in f(G_1) \subset G_2\}$ is a subgroup of G_2.

Proof: (i) Let $x, y \in ker(f)$. Then $x+y \in ker(f)$ since $f(x+y) = f(x)+f(y) = 0+0 = 0$. Now $0 \in ker(f)$ as $f(0) = f(0 + 0) = f(0) + f(0)$. We also have $-x \in ker(f)$ since $f(0) = f(x - x) = f(x) + f(-x) = 0$.

(ii) Let $y_1 = f(x_1), y_2 = f(x_2) \in im(f)$ where $x_1, x_2 \in G_1$. Since f is a homomorphism we have $y_1 + y_2 = f(x_1) + f(x_2) = f(x_1 + x_2) \in im(f)$.

$\square$

Note that $G_1/ker(f)$ is a quotient group. In fact we have the following.

Theorem C.16.2 Fundamental theorem of homomorphisms. *Let $f : G_1 \to G_2$ be a homomorphism. Then*

$$G_1/ker(f) \simeq im(f) \tag{C.627}$$

where $im(f)$ is the image of f in G_2.

Proof: Define a map $\varphi : G_1 \to im(f)$ by

$$\varphi([x]) = f(x)$$

This map is well defined since for $x' \in [x]$, there exists $h \in ker(f)$ such that $x' = x + h$ and

Now we show that φ is an isomorphism. Firstly φ is a homomorphism,

$$\begin{aligned}
\varphi([x]+[y]) &= \varphi([x+y]) \\
&= f(x+y) \\
&= f(x)+f(y) = \varphi([x]) + \varphi([y]).
\end{aligned} \tag{C.628}$$

Next we show that it is one-to-one. If $\varphi([x]) = \varphi([y])$, then $f(x) = f(y)$ or $f(x) - f(y) = f(x-y) = 0$. This shows that $x - y \in ker(f)$ and $[x] = [y]$. Finally, we show φ is onto.

□

If we can express $B_n(K)$ as the kernel of some map f, then

$$H_n(K) = Z_n(K)/B_n(K) = Z_n(K)/\ker(f) \simeq im(f) \tag{C.629}$$

which is generally much easier to calculate.

Examples

Let $K = \{p_0, p_1, (p_0p_1)\}$. We have

$$\begin{aligned}
C_0(K) &= \{ip_0 + jp_1 : i, j \in \mathbb{Z}\} \\
C_1(K) &= \{k(p_0p_1) : k \in \mathbb{Z}\}.
\end{aligned}$$

Since (p_0p_1) is not the boundary of any simplex in K, $B_1(K) = 0$, and

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(K).$$

If $z = m(p_0p_1) \in Z_1(K)$, it satifies

$$\partial_1 z = m\partial_1(p_0p_1) = m\{p_1 - p_0\} = mp_1 - mp_0 = 0.$$

Thus we must have $m = 0$ and so $Z_1(K) = 0$, hence

$$H_1(K) = \{0\} \tag{C.630}$$

Theorem C.16.3 *Let K be a connected simplicial complex. Then*

$$H_0(K) = \mathbb{Z}. \tag{C.631}$$

Proof: Since K is connected, for any pair of 0-simplexies p_i and p_j, there exists a sequence of 1-simplexies $(p_ip_k), (p_kp_l), \ldots, (p_mp_j)$ such that $\partial_1((p_ip_k)+(p_kp_l)+\cdots+(p_mp_j)) = p_i-p_j$. It follows

$$p_i = p_j + \partial_1(p_{ij}).$$

where p_{ij} is a 1-simplex.

Suppose

$$z = \sum_i^{I_0} n_i p_i$$

where I_0 is the number of 0-simplexies in K. Then

$$\begin{aligned} z &= \sum_i^{I_0} n_i p_i = \sum_i^{I_0} n_i p_1 + \sum_i^{I_0} n_i \partial_1(p_{1j}) \\ &= \sum_i^{I_0} n_i p_1 + \partial_1[\sum_i^{I_0} n_i p_{1j}] \end{aligned} \tag{C.632}$$

It is clear that $z \in B_0(K)$ if $\sum n_i = 0$.

Let $\sigma_j = (p_{j,1}p_{j,2})$ $(1 \le j \le I_1)$ be 1-simplexies in K, I_1 being the number of

$$\begin{aligned} B_0(K) &= im\partial_1 \\ &= \{\partial_1[n_1\sigma_1 + \cdots + n_{I_1}\sigma_{I_1}] : n_1, \ldots, n_{I_1} \in \mathbb{Z}\} \\ &= \{n_1(p_{1,2} - p_{1,1}) + \cdots + n_{I_1}(p_{I_1,2} - p_{I_1,1}) : n_1, \ldots, n_{I_1} \in \mathbb{Z}\} \end{aligned} \tag{C.633}$$

$$z = \sum_i n_i p_i$$

We have proved for a connected complex K that $z = \sum n_i p_i \in B_0(K)$ if and only if $\sum n_i = 0$.

Define a surjective homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(n_1 p_1 + \cdots + n_{I_0} p_{I_0}) = \sum_{i=1}^{I_0} n_i.$$

We then have $ker(f) = f^{-1}(0) = B_0(K)$. From theorem , it follows that $H_0(K) = Z_0(K)/B_0(K) = Z_0(K)/ker(f) = \mathbb{Z}$.

□

If also

$$H_1(K) = \{0\} \tag{C.634}$$

then it is a simply connected complex K.

Meaning of the $nth-$homology group

$$H_0(K) = \mathbb{Z} \tag{C.635}$$

for any connected complex K.

C.16.2 De'Rham Cohomology

Two gauge potentials are gauge equivalent if they differ by a gauge potential

$$B_\mu(x) = A_\mu(x) + \partial_\mu \phi(x) \tag{C.636}$$

we write

$$A_\mu(x) \sim B_\mu(x) = A_\mu(x) + \partial_\mu \phi(x) \tag{C.637}$$

We denote a gauge equivalence class $[A_\mu(x)]$ where $A_\mu(x)$ is a representative of the equivalence class. Say the gauge potentials $A_\mu(x)$ and $A'_\mu(x)$ are not related to each by a gauge transformation then their sum

$$A''_\mu \equiv A_\mu + A'_\mu \tag{C.638}$$

A''_μ cannot be related to either A_μ or A'_μ by a gauge transformation and hence belongs to another distinct equavalence class, $[A''_\mu]$. $[A''_\mu] = [A_\mu] + [A'_\mu]$. With this definition class addition, it is not hard to see that these classes form an addative abelian group with the identity $[\partial_\mu \phi(x)]$. We need to establish this we need to show that that the addittion operation is independent representative choosen.

Also can have $aA_\mu + bA'_\mu$ where a and b are real numbers.

$$\epsilon^{\mu\nu}\partial_\nu\partial_\mu\phi(x) \equiv 0 \tag{C.639}$$

In differential geometry notation Eq (C.636) is written

$$A \sim B = A + d\phi \tag{C.640}$$

$$H^p(\mathcal{M}) = \frac{Z^p(\mathcal{M})}{B^p(\mathcal{M})} \tag{C.641}$$

Two closed forms (elements of $Z^p(\mathcal{M})$) define the same cohomology class (elements of $H^p(\mathcal{M})$) if they differ by an exact form (an element of $B^p(\mathcal{M})$).

Consider all $r-$forms λ that satisfy $d\lambda = 0$. Such forms are called closed. Certainly forms that satisfy $\lambda = d\Phi$ are themselves closed bacause of the niloptence of d. Such forms are called exact. not all closed forms are exact. We will denote $Z^r(\mathcal{M})$ refers to all closed forms and $B^r(\mathcal{M})$ refers to all exact $r-$forms.

we consider all closed forms modded out by exact forms. In other words two forms are said to be equivalent if

$$\lambda_1 = \lambda_2 + d\Phi \quad \rightarrow \quad [\lambda_1] = [\lambda_2] \tag{C.642}$$

for any Φ. two closed forms are called cohomology. Elements of the cohomology are the closed forms modded out by all exact forms. This we write as

$$H^r(\mathcal{M}, \mathbb{R}) = Z^r(\mathcal{M})/B^r(\mathcal{M}), \tag{C.643}$$

Definition (i) An n-form is called a cocycle if it is closed, that is, $d\omega = 0$.

(ii) An n-form is called a coboundary if it is exact, that is, if there is an $(n-1)$-form σ such that $\omega = d\sigma$.

(iii) The vector spaces of closed (exact) n-forms are denoted $Z^n(\mathcal{M})$ and $B^n(\mathcal{M})$ respectively.

(iv) The de Rham cohomology group (under addition) is given by

$$H^n(\mathcal{M}) := Z^n(\mathcal{M})/B^n(\mathcal{M}).$$

□

Examples

(i) We first take the example $\mathcal{M} = \mathbb{R}$. First let us find $H^0(\mathbb{R})$. The set $B^0(\mathbb{R})$ has no meaning since there are no (-1)-form. We define $\Omega^{-1}(\mathcal{M})$ to be empty, hence $H^0(\mathbb{R}) = Z^0(\mathbb{R})$. The set $Z^0(\mathbb{R})$ is the set of all 0-forms f that are closed, i.e. $df = 0$. The only way to have $df = \frac{df}{dx}dx = 0$ is if f is constant and so the set of 0-forms are isomorphic to $\mathbb{R}$. So

$$H^0(\mathbb{R}) = Z^0(\mathbb{R})/B^0(\mathbb{R}) \simeq \mathbb{R}/\{0\} = \mathbb{R}.$$

In 1-dimension a 1-form is $f(x)dx$, if we take the differential operator we get

$$d(f(x)dx) = \frac{\partial f}{\partial x} dx \wedge dx = 0.$$

So in 1-dimension, any 1-form on $\mathbb{R}$ is closed. Furthermore, we can take any 1-form $f dx$ and integrate it:

$$F = \int f dx$$

so that

$$f = dF.$$

Therefore all 1-forms are exact. So, all one forms are closed and all one forms are exact therefore the quotient is the identity:

$$H^1(\mathbb{R}) = Z^1(\mathbb{R})/B^1(\mathbb{R}) = \{0\}.$$

Obviously there can be no 2-forms, 3-forms, ... etc in one dimension so that $H^n(\mathbb{R}) = 0$ for $n \geq 2$.

Summarising:

$$\begin{aligned} H^0(S^1) &= \mathbb{R} \\ H^1(S^1) &= \{0\} \\ H^n(S^1) &= 0, \quad n \geq 2. \end{aligned} \tag{C.644}$$

(ii) $\mathcal{M} = S^1$. Let $S^1 = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. Again the only way to have $df = \frac{df}{d\theta} d\theta = 0$ is if f is constant and so the set of 0-forms are isomorphic to $\mathbb{R}$. Therefore

$$H^0(S^1) = \mathbb{R}.$$

This is always the case when $\mathcal{M}$ is connected. Next we compute $H^1(S^1)$. Let $\omega = f(\theta)d\theta$ be a 1-form. Is it possible to, as in the previous example, to write any closed form as dF? If $\omega = dF$, then F must be given by

$$F(\theta) = \int_0^\theta f(\theta')d\theta'.$$

Any such F must be uniquely defined on S^1. For this to be the case F must satisfy the periodicity: $F(\theta) = F(\theta + 2\pi)$. $F(2\pi) = F(0) = \int_0^0 f(\theta')d\theta' = 0$. Namely F must satisfy

$$F(2\pi) = \int_0^{2\pi} f(\theta')d\theta' = 0.$$

Conversely, let ω is a closed form whos integration is zero. Define

$$g(\theta) = c + \int_0^{2\pi} \omega$$

We have

$$
\begin{aligned}
g(\theta + 2\pi) &= c + \int_0^{\theta+2\pi} \omega \\
&= c + \int_0^{\theta} \omega + \int_{\theta}^{\theta+2\pi} \omega \\
&= g(\theta) + \int_0^{2\pi} \omega = g(\theta).
\end{aligned} \tag{C.645}
$$

Therefore g is well defined with $\omega = dg$. We find that a one form is exact if and only if its integration is zero. As with the computation of homology groups, we can employ the fundamental theorem of homomorphisms to simplify the calculation. Let us define the map $\lambda : \Omega^1(S^1) \to \mathbb{R}$

$$
\lambda : \omega = f d\theta \mapsto \int_0^{2\pi} f(\theta')d\theta'
$$

We then have $\ker(\lambda) = \lambda^{-1}(0) = B^1(S^1)$. By theorem (C.16.2),

$$
H^1(S^1) = \Omega^1(S^1)/ker(\lambda) \simeq im(\lambda) = \mathbb{R}.
$$

Just as in the case of $\mathcal{M} = \mathbb{R}$ there can be no 2-forms, 3-forms, ... etc in one dimension so that $H^n(S^1) = 0$ for $n \geq 2$. Summarising:

$$
\begin{aligned}
H^0(S^1) &= \mathbb{R} \\
H^1(S^1) &= \mathbb{R} \\
H^n(S^1) &= 0, \quad n \geq 2.
\end{aligned} \tag{C.646}
$$

(iii) $\mathcal{M} = S^2$.

$$
\begin{aligned}
H^0(S^2) &= \mathbb{R} \\
H^1(S^2) &= \{0\} \\
H^2(S^2) &= \mathbb{R} \\
H^n(S^2) &= \{0\}, \quad n \geq 3
\end{aligned} \tag{C.647}
$$

(iv) $\mathcal{M} = S^n$.

$$\begin{aligned}
H^0(S^1) &= \mathbb{R} \\
H^n(S^1) &= \mathbb{R} \\
H^j(S^1) &= \{0\}, \quad j \neq 0, n
\end{aligned} \tag{C.648}$$

□

Meaning of $nth-$cohomology group

The Duality of Homology and Cohomology

As the name suggests, the cohomology group is a dual space of the homology group.

To do with how Stokes' theorem can be used to study those properties of a manifold which determine the relation between closed and exact forms.

Stokes' theorem establishes a precise duality between $H^n(\mathcal{M})$ and $H_n(\mathcal{M})$ via

$$(\omega, \mathcal{M}) := \int_{\mathcal{M}} \omega \tag{C.649}$$

$$\begin{aligned}
\int_{\mathcal{M}} \omega &= (d\omega, \mathcal{M}) \\
\int_{\partial\mathcal{M}} \omega &= (\omega, \partial\mathcal{M})
\end{aligned} \tag{C.650}$$

$$(d\omega, \mathcal{M}) = (\omega, \partial\mathcal{M}) \tag{C.651}$$

C.16.3 Cohomology and Homology

One can determine if two principals are equivalent or not with the use of something called *characteristic classes.*

second chern class

Roughly speaking, the cohomology $H^p(M, \mathbb{R})$ counts the number of noncontractable p-dimensional surfaces in M.

[384])

$$b^p = \dim H^p(M) = \dim H_p(M). \tag{C.652}$$

$$\Omega(c_p, \omega_q) = \int_{c_p} \omega_q. \tag{C.653}$$

Ω period matrix.

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i. \tag{C.654}$$

Definition The set U is smoothly contractible to p_o if there exists a smooth map $F : U \times I \to U$ such that

$$F(x, 0) = x, \quad F(x, 1) = p_0 \text{ for } x \in U.$$

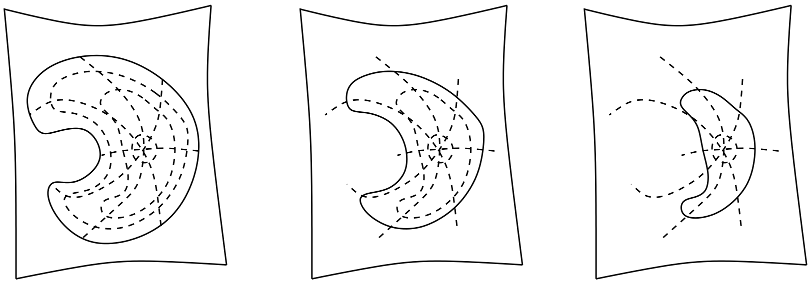


Figure C.46: .

$\square$

Theorem C.16.4 (Poincare's lemma). *If a coordinate neighbourhood U of a manifold M is contractible to a point $p_0 \in M$, any closed r-form on U is also exact.*

Proof: We may choose coordinates such that $x(p_0) = 0$ and use $F(x, t) := tx$. We claim that

$$\sigma(p) := \int_0^1 dt \frac{t^{n-1}}{(n-1)!} \, x^\nu \omega_{\nu\mu_2\ldots\mu_n}(tx(p)) \, dx^{\mu_2}(p) \wedge \cdots \wedge dx^{\mu_n}(p) \tag{C.655}$$

satifies $d\sigma = \omega$. We compute

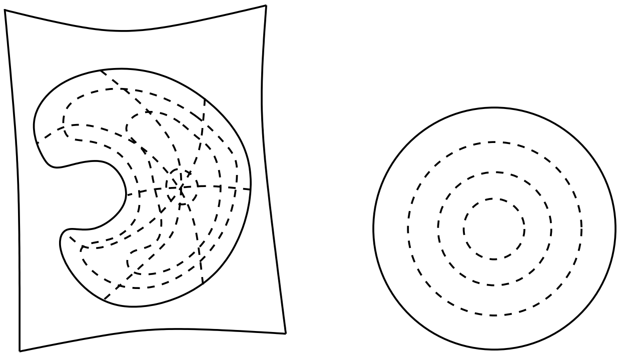


Figure C.47: .

$$
\begin{aligned}
d\sigma(p) &= \int_0^1 dt \frac{t^{n-1}}{(n-1)!} \left[\omega_{\mu_1\ldots\mu_n}(tx(p)) + t^n x^\nu(p)(\partial_{\mu_1}\omega_{\nu\mu_2\ldots\mu_n})(tx(p))\right] \\
&\quad \times dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p). && \text{(C.656)}
\end{aligned}
$$

As the r-form ω is closed we have $\partial_{[\nu}\omega_{\mu_1\ldots\mu_n]} = 0$, from which we see that

$$
\begin{aligned}
0 &= (\partial_{[\nu}\omega_{\mu_1\ldots\mu_n]})dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \frac{1}{n!}[\partial_\nu\omega_{\mu_1\ldots\mu_n} - \partial_{\mu_1}\omega_{\nu\mu_2\ldots\mu_n} + \partial_{\mu_2}\omega_{\nu\mu_1\ldots\mu_n} - \partial_{\mu_3}\omega_{\nu\mu_1\mu_2\ldots\mu_n} + \ldots] \\
&\quad \times dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \frac{1}{n!}[\partial_\nu\omega_{\mu_1\ldots\mu_n} - n\partial_{\mu_1}\omega_{\nu\ldots\mu_n}]dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p). && \text{(C.657)}
\end{aligned}
$$

Using this, (C.656) simplifies to

$$
\begin{aligned}
d\sigma(p) &= \int_0^1 dt \frac{1}{n!}\left[nt^{n-1}\omega_{\mu_1\ldots\mu_n}(tx(p)) + t^n x^\nu(p)(\partial_\nu\omega_{\mu_1\ldots\mu_n})(tx(p))\right] \\
&\quad \times dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \left[\int_0^1 dt\frac{1}{n!}\frac{d}{dt}[t^n\omega_{\mu_1\ldots\mu_n}(tx(p))]\right] dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \left[\frac{1}{n!}[t^n\omega_{\mu_1\ldots\mu_n}(tx(p))]\right]_0^1 dx^{\mu_1}(p)\wedge\cdots\wedge dx^{\mu_n}(p) \\
&= \omega(p). && \text{(C.658)}
\end{aligned}
$$

□

The theroem shows that every closed form is locally exact. The de Rham cohomology obstructs the global exactness of closed forms and giving us global information about M. For instance, if M is not simply connected (not all points are contractable to a point - homology is non-trivial) then the cohomology will be non-trivial.

C.16.4 Homotopy

two spaces *homeomorphic* need invariant so that spaces that are homeomorphic have the same result

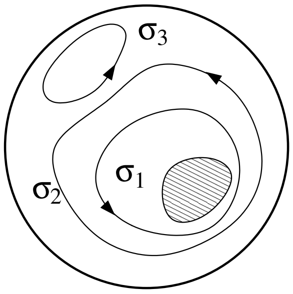


Figure C.48: σ_1 and σ_2 are homotopic to each other but not to σ_3.

$$(a \cdot b)(t) := \begin{cases} a(t), & 0 \leq t \leq \|a\| \\ b(t - \|a\|) & \|a\| \leq t \leq \|a\| + \|b\| \end{cases} \tag{C.659}$$

The product of two paths is not defined if the terminal point of the first is the same as the intial point of the other.

For any path α, we denote by α^{-1} the *inverse path* formed by transversing α in the opposite direction.

Classes of Paths and Loops

Homotopy1.fig

Figure C.49: Homotopy1A.

Proposition C.16.5 *The relation $\sim$ is*

*(i) $a \sim a$ (*reflexive*)*

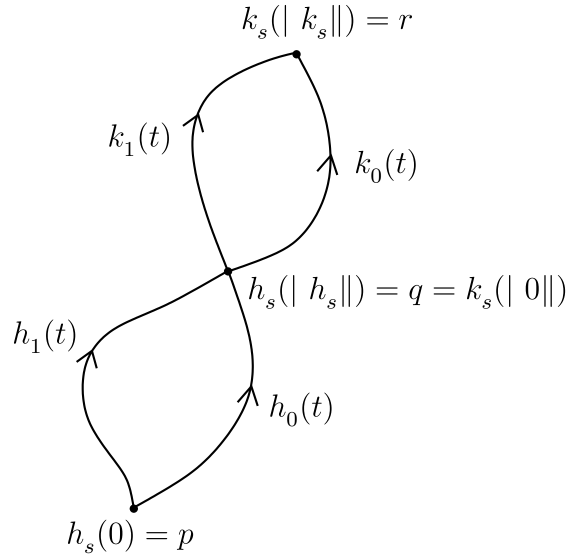


Figure C.50: Homotopy2. The product of two equivalence classes of paths $[a]$ and $[b]$.

(ii) $a \sim b$ *implies* $b \sim a$ *(*symmetric*)*

(iii) $a \sim b$ *and* $b \sim c$ *imply* $a \sim c$ *(*transitive*).*

$$j_s(t) = \begin{cases} h_{2s}(t), & 0 \leq s \leq \frac{1}{2}, \\ k_{2s-1}(t), & \frac{1}{2} \leq s \leq 1, \end{cases} \tag{C.660}$$

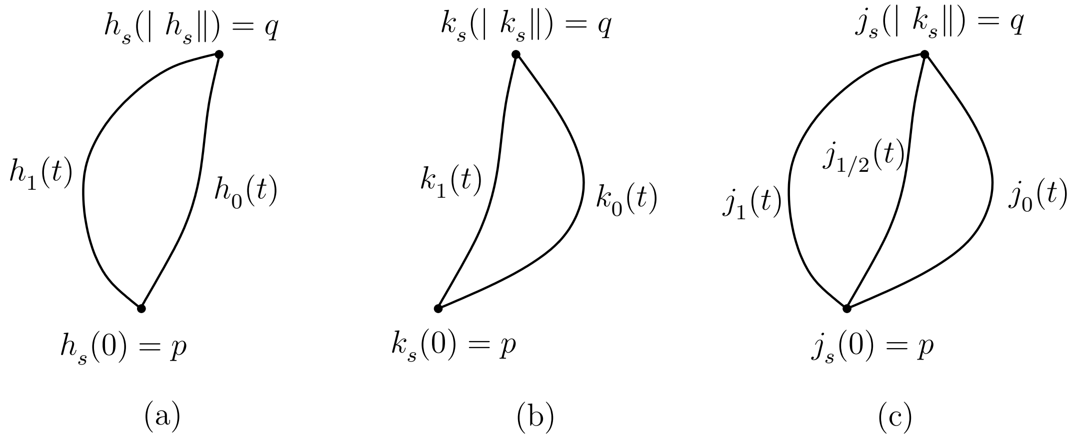


Figure C.51: transitivity. (a) $0 \leq s \leq 1$. (b) $0 \leq s \leq 1/2$. (c) $1/2 \leq s \leq 1$.

paths a and b are equivalent if and only if one can be continuously deformed into the other in X. $a \sim b$. We denote the equivalence class containing the path a by $[a]$.

$$[a] = [b] \;\; if\ and\ only\ if \;\; a \sim b \tag{C.661}$$

The collection of all equivalence classes of paths in a topological space X is called the *fundamental groupoid*, denoted $\Gamma(X)$.

The fundamental Homotopy group

Proposition C.16.6 *For any paths a, a', b, b' in X, if $a \cdot b$ is defined and $a \sim a'$ and $b \sim b'$, then $a' \cdot b'$ is defined and $a \cdot b \sim a' \cdot b'$.*

$$[a] \cdot [b] = [a \cdot b]. \tag{C.662}$$

Proposition C.16.7 *For any path a in X, there exist identity paths e_1 and e_2 such that $a \cdot a^{-1} \sim e_1$ and $a^{-1} \cdot a \sim e_2$.*

$$\begin{aligned}
h_0 &= e_1 \\
h_1(t) &= \begin{cases} a(t), & 0 \le t \le \|a\| \\ a(2\|a\| - t), & \|a\| \le t \le 2\|a\| \end{cases} \\
&= \begin{cases} a(t), & 0 \le t \le \|a\| \\ a^{-1}(t - \|a\|), & \|a\| \le t \le 2\|a\| \end{cases} \\
&= (a \cdot a^{-1})(t),
\end{aligned} \tag{C.663}$$

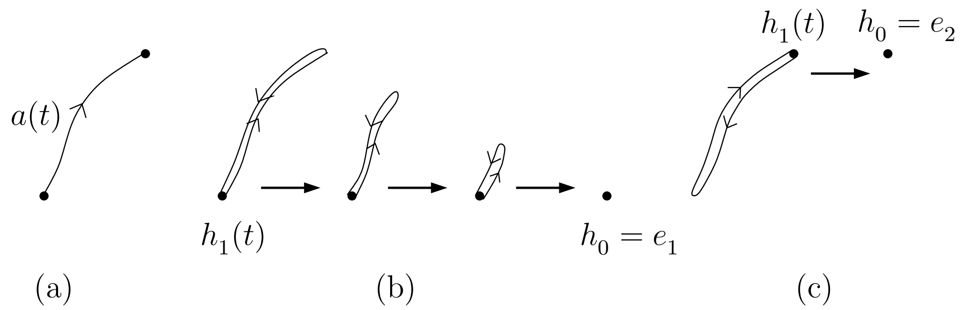


Figure C.52: (a) $a(t)$. (b) For decreasing s, $a \cdot a^{-1} \sim e_1$. (c) $a^{-1} \cdot a \sim e_2$.

Proposition C.16.8 *The set $\pi(X, p)$, together with the multiplication defined, is a group.*

This is the *fundamental group of X relative to the basepoint p.*

Sets and a binary operation. The minimal requirement

Type	identity	inverse	associative	closed	all products defined
group	yes	yes	yes	yes	yes
groupoid	no	no	no	yes	no
semi-group	no	no	yes	yes	
monoid	yes	no	yes	yes	

Obviously a group qualifies as a semi-group, a monoid and groupoid. A monoid is a semi-group with identity.

C.16.5 Homeomorphisms

So that all surfaces that can be obtained from rectangle and gluing together are homeomorphic.

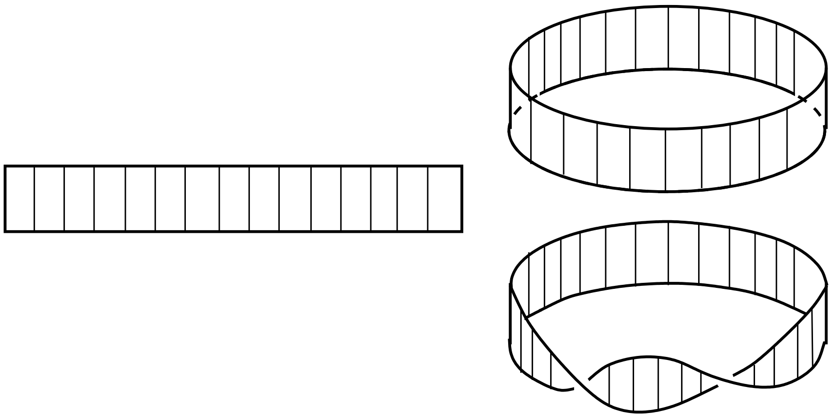


Figure C.53: HomeoFigDef.

Two genometric objects are homeomorphic if there exists a continuous one-to-one mapping $f : X \to Y$ such that the inverse mapping $f^{-1}Y :\to X$ is also continuous.

Twist homeomorphisms

canonical homology basis $\{a_1, a_2, b_1, b_2\}$

C.17 Summary

- Tensor Calculus
- Group theory

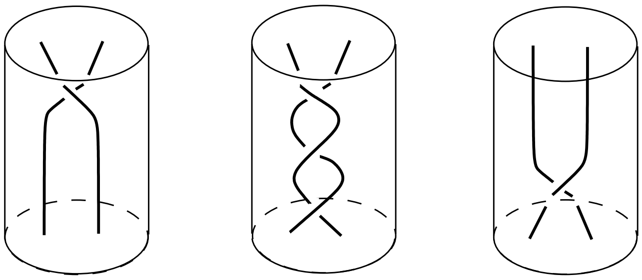


Figure C.54: homeoClylind.

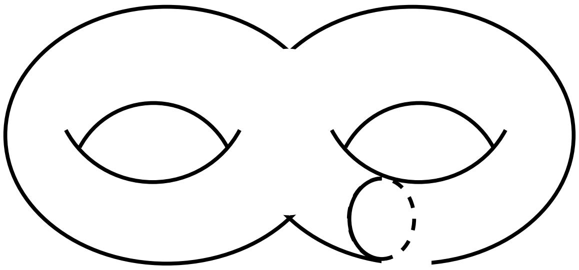


Figure C.55: homDehnfig1.

C.18 Biblioliographical notes

In this chapter I have relied on the following references:

Dana P. Williams notes on the Spectral Theorem for bounded normal operators

Introduction to topology and modern analysis G.F Simons.

Manifolds at Dr Neil Lambert Department of Mathematics King's College

C.19 Worked Exercises and Details

Commutativity of exterior derivative and $\mathcal{L}_\omega$.

Use the fact that $X^a_{;\,b}$ transforms as a tensor to deduce the transformation law of the connection given by (C.300).

Proof. $\Gamma^a_{cb} X^c = X^a_{;\,b} - X^a_{,b}$ we get

$$\Gamma'^a_{cb} X'^c = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\,\Gamma^d_{fe} X^f - \frac{\partial x'^b}{\partial x^e}\frac{\partial^2 x^a}{\partial x'^f \partial x'^b} X^f. \tag{C.664}$$

Expressing X'^a in terms of X^f and since X^f is an arbitrary vector field, we have

$$\Gamma'^a_{cb} \frac{\partial x'^c}{\partial x^f} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \Gamma^d_{fe} - \frac{\partial x'^b}{\partial x^e} \frac{\partial^2 x^a}{\partial x'^f \partial x'^b}. \tag{C.665}$$

Multiplying through by $\partial x^c / \partial_g$ and rearranging the indices, we arrive at

$$\Gamma'^a_{cb} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^e}{\partial x'^c} \Gamma^d_{fe} - \frac{\partial x'^b}{\partial x^e} \frac{\partial x^e}{\partial x'^c} \frac{\partial^2 x^a}{\partial x'^f \partial x'^b}. \tag{C.666}$$

Commutativity of exterior derivative and $\mathcal{L}_\omega$.

C.19.1 Dynamical and Non-Dynamical Symmetries

Dynamical symmetries constrain the solutions of the equations of motion. *Non-Dynamical symmetries* are redundantcies of the mathematical formulation of the theory - these are often referred to as *Gauge symmetries*.

Example:

(a) Dynamical symmetry: Time translation invariance.

The Lagrangian does not explicitly depend on time

$$\mathcal{L} = \mathcal{L}(q_n, \dot{q}_n). \tag{C.667}$$

From this it follows that

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q_n} \dot{q}_n + \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \ddot{q}_n \tag{C.668}$$

Now, at this point we make use of *the Euler-Lagrange equations of motion*

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n} \right) - \frac{\partial \mathcal{L}}{\partial q_n} = 0 \tag{C.669}$$

one finds

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \dot{q}_n \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right) + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\ddot{q}_n \\
&= \frac{d}{dt}\left(\dot{q}_n \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right) \qquad \text{(C.670)}
\end{aligned}
$$

or

$$
\frac{d}{dt}\left(\dot{q}_n \frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \mathcal{L}\right) \qquad \text{(C.671)}
$$

That is, the Hamiltonian is conserved in time

$$
\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L}. \qquad \text{(C.672)}
$$

This reflects the conservation of energy E for isolated systems. Note that, as we needed to use the equations of motion this condition only holds for trajectories that extremalize the action.

(b) Non-dynamical symmetry: reparameterization invariance.

$$
\begin{aligned}
S[q(t+\epsilon)] - S[q(t)] &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t}\epsilon + \frac{\partial \mathcal{L}}{\partial q^i}\dot{q}^i\epsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\frac{d}{dt}(\dot{q}^i\epsilon)\right] dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t}\epsilon + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)\frac{d\epsilon}{dt}\right] dt \\
&= [\mathcal{L}\epsilon]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L}\right)\frac{d\epsilon}{dt}\, dt \qquad \text{(C.673)}
\end{aligned}
$$

Parameterization-invariance means that the integral must vanish for arbitrary $d\epsilon/dt$, so that we have

$$
\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L} = 0. \qquad \text{(C.674)}
$$

Even if the action is not extremal for some trajectory, it is still invariant under reparameterization of that trajectory.

Example: Diff (S^1) - the Virasoro algebra

As will be explained in chapter M.-19, in closed string theory there is an invariance under active diffeomorphisms acting on a circle, the corresponding group is denoted Diff (S^1). If $\theta \in (0, 2\pi]$ is a coordinate on S^1,

$$\tau(\theta) \to \tau'(\theta) = \tau(\theta) + f(\tau(\theta)) \tag{C.675}$$

where f is periodic, i.e., $f(\tau + 2\pi) = f(\tau)$. There is a complete Fourier series expansion

$$f(\tau(\theta)) = \sum_{n=0}^{\infty} A_n \cos n\tau(\theta) + B_n \sin n\tau(\theta)$$

For an infinitesmal form transformation

$$\tau(\theta) \to \tau'(\theta) = \tau(\theta) + V(\theta)\frac{\partial}{\partial\theta}(\tau(\theta)) \tag{C.676}$$

where $V(\theta)$ is the vector generating the infinitesimal diffeomorphism. As $V(\theta) = V(\theta+2\pi)$ we can expand $V(\theta)$,

$$\begin{aligned} V(\theta)\frac{\partial}{\partial\theta} &= (\sum_{n=0}^{\infty} b_n \cos n\theta + c_n \sin n\theta)\frac{\partial}{\partial\theta} \\ &= \sum_{n=0}^{\infty} [\frac{(b_n - ic_n)}{2} e^{in\theta} + \frac{(b_n + ic_n)}{2} e^{-in\theta}]\frac{\partial}{\partial\theta} \\ &= \sum_{n=-\infty}^{\infty} a_n i e^{in\theta}\frac{\partial}{\partial\theta} \end{aligned} \tag{C.677}$$

where $a_n = (c_n - ib_n)/2$ and $a_{-n} = a_n^*$ for $n > 0$.

$$\tau'(\theta) = \tau(\theta) + \sum_{n\in\mathbb{Z}} a_n i e^{in\theta}\theta^\alpha \partial_\alpha \tau \tag{C.678}$$

Therefore one has a basis indexed by $n \in \mathbb{Z}$

$$D_n = ie^{in\theta}\frac{\partial}{\partial\theta} \tag{C.679}$$

The Lie algebra of Diff (S^1) is

$$
\begin{aligned}
[D_n, D_m]f &= [ie^{in\theta}\theta^\alpha\partial_\alpha ie^{im\theta}\theta^\beta\partial_\beta - ie^{im\theta}\theta^\alpha\partial_\alpha ie^{in\theta}\theta^\beta\partial_\beta]f \\
&= e^{i(n+m)\theta}\big[(in - im)\theta^\alpha\partial_\alpha f + i(\theta^\alpha\partial_\alpha\theta^\beta\partial_\beta - \theta^\beta\partial_\beta\theta^\alpha\partial_\alpha)f\big] \\
&= (n-m)D_{m+n}f
\end{aligned}
\tag{C.680}
$$

or

$$
[D_n, D_m] = (n-m)D_{m+n}. \tag{C.681}
$$