Muirhead's Theorem

We assume

$$F(a_1, a_2, \ldots, a_n) = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \quad x_\nu > 0,\ \alpha_\nu \geq 0. \tag{1}$$

We write

$$[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_n] = \frac{1}{n!} \Sigma! a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}. \tag{2}$$

It is clear that $[\alpha]$ is invariant under any permutation of the $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and therefore two sets of a are the same if they only differ in arrangement.

In general $[\alpha']$ is not comparable to $[\alpha]$, in the sense that there is an inequality between their associated expressions valid for all n-tuples of non-negative real numbers $a_1, a_2, \ldots, a_n$.

We say that (α') is majorised by (α), and write

$$(\alpha') \prec (\alpha),$$

when the (α) and (α') can be arranged so as to satisfy the following three conditions:

$$\alpha'_1 + \alpha'_2 + \cdots + \alpha'_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n; \tag{3}$$

$$\alpha'_1 \geq \alpha'_2 \geq \cdots \geq \alpha'_n \quad , \quad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n; \tag{4}$$

$$\alpha'_1 + \alpha'_2 + \cdots + \alpha'_\nu \leq \alpha_1 + \alpha_2 + \cdots + \alpha_\nu \quad (1 \leq \nu < n). \tag{5}$$

The second condition is in itself no restriction, since we may rearrange (α') and (α) in any order, but it is essential to the statement of the third.

Theorem: A necessary and sufficient condition that $[\alpha']$ should be comparable with $[\alpha]$, for all positive values of the a, is that one of the (α') and (α) should be majorised by the other. If $(\alpha') \prec (\alpha)$ then

$$[\alpha'] \leq [\alpha_1] \tag{6}$$

There is equality only when (α') and (α) are identical or when all the a are equal.

Proof:

The condition of necessary. Suppose that (4) is satisifed, and that (6) holds for all positive a. Taking all the a equal to x, we obtain

$$x^{\sum_{i=1}^n \alpha_i'} = [\alpha'] \le [\alpha] = x^{\sum_{i=1}^n \alpha_i}.$$

If $x > 1$ then we must have $\sum_{i=1}^n \alpha_i \ge \sum_{i=1}^n \alpha_i'$. If $x < 1$ then we must have $\sum_{i=1}^n \alpha_i \le \sum_{i=1}^n \alpha_i'$. Thus the above condition can only be true for all x if $\sum_{i=1}^n \alpha_i' = \sum_{i=1}^n \alpha_i$.

Next take

$$a_1 = a_2 = \cdots = a_\nu = x, \quad a_{\nu+1} = \cdots = a_n = 1,$$

and x large. Since (α') and (α) are in decreasing order, the indices of the highest powers of x in $[\alpha']$ and $[\alpha]$ are

$$\alpha_1' + \alpha_2' + \cdots + \alpha_\nu', \quad \alpha_1 + \alpha_2 + \cdots + \alpha_\nu$$

respectively. Thus, it is clear that the first sum can not be greater than the second sum and this proves (5).

The condition is sufficient. The proof is rather more difficult to establish, and we will need a new defintionn and two lemmas.

We define a special type of linear transformation T of the α's, as follows.

Suppose that $\alpha_k > \alpha_l$, then let us write

$$\alpha_k = \rho + \tau, \quad \alpha_l = \rho - \tau \quad (0 < \tau \le \rho). \tag{7}$$

If now

$$0 \le \sigma < \tau \le \rho \tag{8}$$

then a transformation T is defined by

$$\begin{aligned} T(\alpha_k) &= \alpha_k' = \rho + \sigma = \frac{\tau+\sigma}{2\tau}\alpha_k + \frac{\tau-\sigma}{2\tau}\alpha_l \\ T(\alpha_l) &= \alpha_l' = \rho - \sigma = \frac{\tau-\sigma}{2\tau}\alpha_k + \frac{\tau+\sigma}{2\tau}\alpha_l \\ T(\alpha_\nu) &= \alpha_\nu' = \alpha_\nu \quad (\nu \ne k,\ \nu \ne l). \end{aligned} \tag{9}$$

If (α') arises from (α) by a transformation T, we write $\alpha' = T\alpha$. The definition does not necessarily imply that either the (α) or the (α') are in decreasing order.

The suffciency of our comparability condition will be established if we can prove the following two lemmas.

Lemma 1. If $\alpha' = T\alpha$ then $[\alpha'] \leq [\alpha]$, with equality only when all a are equal.

Lemma 2. If $(\alpha') \prec (\alpha)$, but (α') is not identical with (α), then (α') can be derived from (α) by the successive applications of a finite number of transformations T.

Proof of Lemma 1: We may rearrange (α) and (α') so that $k = 1$, $l = 2$. Then

$$\begin{aligned} & n!2[\alpha] - n!2[\alpha'] \\ = \; & n!2[\rho+\tau, \rho-\tau, \alpha_3, \ldots] - n!2[\rho+\sigma, \rho-\sigma, \alpha_3, \ldots] \\ = \; & \Sigma! a_3^{\alpha_3} \cdots a_n^{\alpha_n} (a_1^{\rho+\tau} a_2^{\rho-\tau} + a_1^{\rho-\tau} a_2^{\rho+\tau} - a_1^{\rho+\sigma} a_2^{\rho-\sigma} - a_1^{\rho-\sigma} a_2^{\rho+\sigma}) \\ = \; & \Sigma! (a_1 a_2)^{\rho-\tau} a_3^{\alpha_3} \cdots a_n^{\alpha_n} (a_1^{\tau+\sigma} - a_2^{\tau+\sigma})(a_1^{\tau-\sigma} - a_2^{\tau-\sigma}) \geq 0, \end{aligned} \tag{10}$$

with equality only when all the a's are equal.

Proof of Lemma 2: We suppose that condition (4) is satisfied, and call the number of differences $\alpha_\nu - \alpha'_\nu$ which are not zero the discrepancy of (α) and (α'); if the discrepancy is zero the sets are identical. We will prove the lemma by induction, assuming it to be true when the discrepancy is less than r and proving that it is also true when the discrepancy is r.

Suppose then that $(\alpha') \prec (\alpha)$ and that the discrepancy is $r > 0$. Since $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \alpha'_i$ and $\sum(\alpha_\nu - \alpha'_\nu) = 0$, and not all of these differences are zero, there must be positive and negative differences, and the first which is not zero must be positive because of the second condition of $(\alpha') \prec (\alpha)$. We can therefore find k and l such that

$$\alpha'_k < \alpha_k, \quad \alpha'_{k+1} = \alpha_{k+1}, \ldots, \alpha'_{l-1} = \alpha_{l-1}, \quad \alpha'_l = \alpha_l. \tag{11}$$

We take

$$\alpha_k = \rho + \tau, \quad \alpha_l = \rho - \tau, \tag{12}$$

as in (7), and define σ by

$$\sigma = \mathrm{Max}(|\alpha'_k - \rho|, |\alpha'_l - \rho|). \tag{13}$$

Then $0 < \tau \leq \rho$, since $\alpha_k > \alpha_l$ ($\rho + \tau > \rho - \tau$ implies $\tau > 0$ and $\tau \leq \rho$ becasue $\alpha_l \geq 0$). Also, one (possibly both) of

$$\alpha_l' - \rho = -\sigma, \quad \alpha_k' - \rho = \sigma,$$

is true, since $\alpha_k' \geq \alpha_l$ (); and $\sigma < \tau$, since $\alpha_k' < \alpha_k$ and $\alpha_l' > \alpha_l$ ($\rho + \sigma < \rho + \tau$ implies $\sigma < \tau$ and $\rho - \sigma > \rho - \tau$ implies $\sigma < \tau$). Hence

$$0 \leq \sigma < \tau \leq \rho, \tag{14}$$

as in (8).

We now write

$$\alpha_k'' = \rho + \sigma, \quad \alpha_l'' = \rho - \sigma, \quad \alpha_\nu'' = \alpha_\nu \quad (\nu \neq k,\ \nu \neq l). \tag{15}$$

If $\alpha_k' - \rho = \sigma$, $\alpha_k'' = \alpha_k'$; if $\alpha_l' - \rho = -\sigma$, $\alpha_l'' = \alpha_l'$. Since the pairs α_k, α_k' and α_l, α_l' each contribute a unit to the discrepancy r between (α') and (α), the discrepancy between (α') and (α'') is smaller, being $r - 1$ or $r - 2$.

Next, comparing (15) with (9), and observing that (8), we see that (α'') arises from (α) by a transformation T.

Finally, (α') is majorised by (α''). To prove this we must verify that the conditions corresponding to (3), (4) and (5), with regard to α'' for α, are satisfied. First we have

$$\alpha_k'' + \alpha_l'' = 2\rho = \alpha_k + \alpha_l, \quad \sum_{i=1}^{n} \alpha_i' = \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha_i''. \tag{16}$$

For the second condition. First $\alpha_k' - \rho \leq |\alpha_k' - \rho|$ and $\alpha_l' - \rho \geq -|\alpha_l' - \rho|$ implies

$$\begin{aligned} \alpha_k' &\leq \rho + |\alpha_k' - \rho| \leq \rho + \sigma = \alpha_k'', \\ \alpha_l' &\geq \rho - |\alpha_l' - \rho| \geq \rho - \sigma = \alpha_l'' \end{aligned} \tag{17}$$

where we have used (13). We have

$$
\begin{aligned}
\alpha''_{k-1} &= \alpha_{k-1} \quad \text{by (15)} \\
&\geq \alpha_k \quad \text{by (4)} \\
&= \rho + \tau \quad \text{by (12)} \\
&> \rho + \sigma \quad \text{by (14)} \\
&= \alpha''_k \quad \text{by (15)} \\
&\geq \alpha'_k \quad \text{by (17)} \\
&\geq \alpha'_{k+1} \quad \text{by (4)} \\
&= \alpha_{k+1} \quad \text{by (11)} \\
&= \alpha''_{k+1} \quad \text{by (15)},
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
\alpha''_{l-1} &= \alpha_{l-1} \quad \text{by (15)} \\
&= \alpha'_{l-1} \quad \text{by (11)} \\
&\geq \alpha'_l \quad \text{by (4)} \\
&\geq \alpha''_l \quad \text{by (17)} \\
&= \rho - \sigma \quad \text{by (15)} \\
&> \rho - \tau \quad \text{by (14)} \\
&= \alpha_l \quad \text{by (12)} \\
&\geq \alpha_{l+1} \quad \text{by (4)} \\
&= \alpha''_{l+1} \quad \text{by (15)},
\end{aligned} \tag{19}
$$

and the inequalities affecting the α'' are those required. Finally, we have to prove that

$$\alpha'_1 + \alpha'_2 + \cdots + \alpha'_\nu \leq \alpha''_1 + \alpha''_2 + \cdots + \alpha''_\nu \quad (1 \leq \nu < n). \tag{20}$$

Now, this is true if $\nu < k$ because of the definition of (α''), (15), and also the third condition of $(\alpha') \prec (\alpha)$,

$$\alpha'_1 + \alpha'_2 + \cdots + \alpha'_\nu \leq \alpha_1 + \alpha_2 + \cdots + \alpha_\nu = \alpha''_1 + \alpha''_2 + \cdots + \alpha''_\nu \quad (1 \leq \nu < k).$$

Condition (20) is also true for $\nu \geq l$. This follows from $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \alpha''_i$, because of the definition of (α''), (15), and also the third condition of $(\alpha') \prec (\alpha)$,

$$\begin{aligned}
\alpha'_1 + \alpha'_2 + \cdots + \alpha'_{n-1} &\leq \sum_{i=1}^{n} \alpha_i - \alpha_n = \sum_{i=1}^{n} \alpha''_i - \alpha''_n \\
\alpha'_1 + \alpha'_2 + \cdots + \alpha'_{n-2} &\leq \sum_{i=1}^{n} \alpha_i - \alpha_{n-1} - \alpha_n = \sum_{i=1}^{n} \alpha''_i - \alpha''_{n-1} - \alpha''_n \\
&\vdots \\
\alpha'_1 + \alpha'_2 + \cdots + \alpha'_l &\leq \sum_{i=1}^{n} \alpha_i - \sum_{i=l+1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha''_i - \sum_{i=l+1}^{n} \alpha''_i. \qquad (21)
\end{aligned}$$

Condition (20) is true for $\nu = k$, because it is true for $\nu = k - 1$ and $\alpha'_k \leq \alpha''_k$, and it is true for $k < \nu < l$ because it is valid for $\nu = k$ and the intervening α' and α'' are identical from (11) and (15).

We have proved that (α') is majorised by (α''), a set arising from (α) by a transformation T and having a discrepancy from (α') less than r. This proves Lemma 2.

Therefore the theorem is proved because if $(\alpha') \prec (\alpha)$ then

$$(\alpha') = (T_m \ldots T_2 T_1 \alpha) \qquad (22)$$

and

$$[\alpha'] = [T_m \ldots T_2 T_1 \alpha] \leq \cdots \leq [T_2 T_1 \alpha] \leq [T_1 \alpha] \leq [\alpha] \qquad (23)$$

so finally

$$[\alpha'] \leq [\alpha]. \qquad (24)$$