Quantum Field Theory and the Standard Model

Draft version

Contents:

Appendices

- A Physics Glossary
- B Mathematics Glossary

Contents

List of Figures

Terminology and Notation

Here is a list of symbols.

$[\ ,\ ]$	commutator
$\{\ ,\ \}$	Poisson bracket
$\dagger$	Hermitian conjugation
$:=$	definition
$\equiv$	identity
$\stackrel{*}{=}$	only true in a special coordinate system
iff	If and only if
η_{ab}	Minkowski metric
$\eta(x)$	test function of a variation of action
$\mathcal{A}$	space of gauge fields or area
$A_\mu(x)$	Yang-Mills connection
D_μ	covariant derivative
$\mathcal{M}$	spacetime manifold
$\mathbf{M}$	The Master consraint
$\hat{\mathbf{M}}$	The Master consraint operator
$\omega^\alpha_{\mu\beta}$	spin connection
$\mathcal{C}$	constraint surface in phase space
S	labells spin-network
s	equivalent class of spin-networks under the action of Diff denoted $s-$ knots
$s(S)$	denotes equivalent class S to which belongs
g_{ab}	spacetime metric
K_{ab}	extrinsic curvature of Σ
G_{ab}	Einstein tensor
T_{ab}	The energy-momentum tensor
e^a_I, E^a_i	tetrad and triad
$\mathcal{L}_t$	Lie derivative with respect to t
n_a	unit normal to Σ_t
N, $(\tilde{N})$	lapse function (density)
N^a	shift vector on Σ
$\Omega_{\alpha\beta}$	sympletic form
$\mathcal{A}/\mathcal{G}$	space of gauge fields moduli gauge transformations
$[A]$	gauge equivalence classe of the connection A
$\mathcal{HA}$	the holonomy algebra
$\overline{\mathcal{HA}}$	the completion of the holonomy algebra in the norm $\Vert f \Vert := \sup_{[A]\in\mathcal{A}/\mathcal{G}} \mid f([A]) \mid$
$\overline{\mathcal{A}/\mathcal{G}}$	spectrum of $\overline{\mathcal{HA}}$

Preface

Warning: We are sure there are lots of mistakes in these notes. Use at your own risk! Corrections and other feedback would be very appreciated.

Acknowledgments

Dedicated

Paths through the report

Chapter 1

Introduction

Chapter 2

Quantum Electrodynamics

An open problem in quantum gravity is to compute particle scattering amplitudes from the background-independent theory and recover the low energy physics.

Calculations should agree with low energy conventional field theory. Here we introduce conventional scattering theory.

Feynman derived his rules in a non-rigorous fashion but it still incorporated all QED processes. These rules were shown to follow from a systematic treatment within the framework of quantum field theory. In this appendix we follow the route taken by Feynmann, we breifly demonstrate its equivalence to the more rigorous quantum field theory in the next appendix.

2.1 The Electromagnetic Field and the Photon

Light behaves as a wave as it demonstrates interference and diffraction. Maxwell's theory seemed to confirm the wave theory of light.

But then the development following the discovery of the photoelectric effect led to the realisation that sometimes light behaves like particles.

2.1.1 Review of Special Relativistic Notation

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1}$$

$$\eta^{\mu\nu} = (\eta^{-1})_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.2}$$

The world four vector

$$x^\mu = \{x^0, x^1, x^2, x^3\} = \{t, x, y, z\} \tag{2.3}$$

describes the spacetime coordinates. The covariant four vector is

$$x_\mu = \eta_{\mu\nu} x^\nu = \{t, -x, -y, -z\} = \{x_0, x_1, x_2, x_3\} \tag{2.4}$$

We have

$$\begin{aligned} x \cdot x &= x^\mu x_\mu && (2.5) \\ &= t^2 - x^2 - y^2 - z^2. && (2.6) \end{aligned}$$

The definition of four-momentum vector is analogous,

$$p^\mu = \{E, p_x, p_y, p_z\}, \tag{2.7}$$

and the scalar product $p_1 \cdot p_2$ is

$$p_1 \cdot p_2 = p_1^\mu p_{2\mu} = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 \tag{2.8}$$

and the scalar product $x \cdot p$ is

$$x \cdot p = x^\mu p_\mu = x_\mu p^\mu = Et - \mathbf{x} \cdot \mathbf{p}. \tag{2.9}$$

We use the general notions for four vectors

$$a = \{a_0, a_1, a_2, a_3\}. \tag{2.10}$$

We denote three-vectors by bold type as in

$$\mathbf{a} = \{a_1, a_2, a_3\}. \tag{2.11}$$

The components

$$a^\mu = \{a^0, a^1, a^2, a^3\}$$

2.1.2 Maxwell's Equations

Classical electromagnetism is described by Maxwell's equations. In the presence of a charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{j}(\mathbf{x}, t)$, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ satify the equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho && (2.12)\\ \nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} && (2.13)\\ \nabla \cdot \mathbf{B} &= 0 && (2.14)\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (2.15) \end{aligned}$$

From the second pair of Maxwell's equations follows the existence of scalar and vector $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ potentials defined by

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \cdot \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{2.16}$$

These equations do not determine the potential uniquely, since for an aribtrary function $\Lambda(\mathbf{x}, t)$ the transformation

$$\phi \to \phi' = \phi + \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla \Lambda \tag{2.17}$$

Expressed in terms of potentials the first Maxwell equation becomes

$$-\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \Box \phi - \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}\right) = \rho \tag{2.18}$$

For the second we need $\nabla \times (\nabla \times \mathbf{A})$

$$
\begin{aligned}
[\nabla \times (\nabla \times \mathbf{A})]_i &= \epsilon_{ijk}\partial_j(\epsilon_{kj'k'}\partial_{j'}A_{k'}) \\
&= \epsilon_{ijk}\epsilon_{ki'j'}\partial_j\partial_{i'}A_{j'} \\
&= (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'})\partial_j\partial_{i'}A_{j'} \\
&= \partial_j\partial_{i'}A_j - \nabla^2 A_i
\end{aligned} \tag{2.19}
$$

$$
\Box\mathbf{A} + \nabla\left(\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A}\right) = \mathbf{j} \tag{2.20}
$$

In four-vector notation the gauge transformations (2.17) read

$$
A^\mu \to A^\mu + \partial^\mu\Lambda(x).
$$

The first (2.18) and second (2.20) Maxwell equations can be combined into one equation

$$
\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \tag{2.21}
$$

2.1.3 Transverse Gauge field

It is always possible to find a function $\Lambda(x)$ such that the transformed potential satisfies the Lorentz gauge

$$
\partial_\mu A^\mu = 0. \tag{2.22}
$$

Only in this gauge does the wave equation have the simple form

$$
\Box A^\mu = 0. \tag{2.23}
$$

For

$$
k^\mu k_\mu = 0 \tag{2.24}
$$

its solutions are plane waves

$$
A^\mu(x,k) = \epsilon^\mu N_k(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{2.25}
$$

In the Lorentz gauge, we can make futher gauge transformations $A^\mu \to A^\mu + \partial^\mu \Lambda(x)$ provided Λ satisfies

$$\Box \Lambda(x) = 0.$$

Such regauging does obviously does not change the Lorentz condition. The radiation gauge

$$A^0 = 0, \quad \nabla \cdot \mathbf{A} = 0. \tag{2.26}$$

can be chosen. To see this consider aribtrary $\mathbf{A}'(x)$, and postulate $\Lambda(x)$ such that

$$\nabla \cdot \mathbf{A}(x) = \nabla \cdot \mathbf{A}'(x) - \nabla^2 \Lambda(x) = 0 \tag{2.27}$$

We obviously need to solve the equation

$$-\nabla^2 \Lambda(x) = \nabla \cdot \mathbf{A}'(x) \tag{2.28}$$

Notice this is just the equation

$$-\nabla^2 \Lambda(x) = f(x) \tag{2.29}$$

which we know has the solution

$$\Lambda(x) = \int d^3 r' \frac{f(r')}{4\pi |r' - r|} \tag{2.30}$$

Therefore this gauge can always be choosen. In this gauge the timelike component of ϵ^μ vanishes. ϵ satisfies

$$\epsilon \cdot \mathbf{k} = 0 \tag{2.31}$$

and normalised such that

$$\epsilon \cdot \epsilon = 1 \tag{2.32}$$

$$\begin{aligned} \epsilon^\mu k_\mu &= 0 \\ \epsilon^\mu \epsilon_\mu &= -1 \end{aligned} \tag{2.33}$$

2.1.4 Blackbody Radiation, the Photoelectric Effect and the Compton Effect

$$E = hf \tag{2.34}$$

where f is the frequency and h is Plank's constant.

Blackbody Radiation

Boltzmann statistics for a gas of free particles is

$$p(\vec{v}) = Ne^{-E/kT}$$

Classical physics can be used to derive an equation which describes the intensity of the blackbody radiation as a function of frequency for a fixed temperature - the result is known as the Rayleigh-Jeans law. Although the Rayleigh-Jeans law works for low frequencies, it diverges as f^2; this divergence for high frequencies is called the ultraviolet catastrophe.

Planck's law states that

$$I(f,T) = \frac{2hf^3}{c^2}\frac{1}{e^{\frac{hf}{kT}} - 1} \tag{2.35}$$

We analyse using Bose-Einstein statistics: a "gas" of photons. There is no constrainton the number of photons. We find that for a system of photons, that the number of photons $n(\epsilon)d\epsilon$ in a (small) energy range ϵ to $\epsilon + d\epsilon$ is given by

Consider an energy level ϵ_i with degeneracy g_i, containing n_i bosons. This can be represented by $g_i - 1$ lines, and the bosons by n_i circles. The number of distinct orderings of lines and circles is

$$\frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \tag{2.36}$$

The total number of microstates for a given distribution is

$$\prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \tag{2.37}$$

For $g_i >> 1$ this can be replaced by

$$t(\{n_i\}) = \prod_i \frac{(n_i + g_i)!}{n_i! g_i!}. \tag{2.38}$$

If we assume that both g_i and n_i are large enough for Stirling's approximation to hold for $\ln g_i!$ and $\ln n_i!$, we find that $\ln t$ is given by

$$\ln t \approx \sum_i [(n_i + g_i) \ln(n_i + g_i) - g_i \ln g_i - n_i \ln n_i] dn_i \tag{2.39}$$

We want to maximise $\ln t$ subject to the constraint

$$\sum_i \epsilon_i n_i = U. \tag{2.40}$$

If $\ln t$ were maximal the change in $\ln t$ resulting from changes dn_i in each of the n_i's would zanish:

$$d \ln t \approx \sum_i [\ln(n_i + g_i) - \ln n_i] dn_i = 0. \tag{2.41}$$

If all the dn_i's were independent from each other, each coefficient in (2.41) would have to zanish. However because of the constraint (2.40) it no longer follows that all the dn_i's are independent from each other as they have to satisfy

$$dU = \sum_i \epsilon_i dn_i = 0. \tag{2.42}$$

Adding this multiplied by the Lagrange multiplier β to (2.41) we obtain

$$\sum_i [\ln(n_i + g_i) - \ln n_i + \beta \epsilon_i] dn_i = 0 \tag{2.43}$$

which is a condition for a maximal value for $\ln t$ subject to the constraint (2.40). For appropriate value of β we can consider all dn_i independent from each other. Therefore, each coefficient must zanish separately:

$$\ln \left(\frac{n_i + g_i}{n_i} \right) + \beta \epsilon_i = 0. \tag{2.44}$$

We then find that the most probable distribution is

$$n_i = \frac{g_i}{e^{-\beta\epsilon_i} - 1}. \tag{2.45}$$

This is the Bose-Einstein distribution. β is related to the temperature, so the Bose-Einstein distribution takes the form.

$$n_i = \frac{g_i}{e^{\frac{\epsilon_i}{kT}} - 1}. \tag{2.46}$$

We consider a “gas” of photons.

$$n(\epsilon)d\epsilon = \frac{g(\epsilon)d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{2.47}$$

where $g(\epsilon)$ are the density of states. We first derive the density of states as a function of the wavevector $\vec{k}$.

In order to determine the available wavevectors, we ask what standing waves can propagate within the box subject to boundary condition that the amplitude is zero at the boundaries. Using a cube with a side of length L, we see that there must be an integer number of half wavelengths in L for each of the directions. Hence if the vector is $\vec{k}$, with Cartesian components (k_x, k_y, k_z), we must have

$$k_x = \frac{n_x\pi}{L}, \quad k_y = \frac{n_y\pi}{L}, \quad k_z = \frac{n_z\pi}{L} \tag{2.48}$$

where n_x, n_y and n_z are from the set of positive non-zero integers - these define“elementary cells”. The total number of states with wavwenumber, $|\vec{k}|$, less than some value k is found by counting the number of triples (n_x, n_y, n_z) such that

$$\frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} < k \tag{2.49}$$

We can find this by considering the octant of a sphere of radius k in $\vec{k}-$space. The volume of the sphere in $\vec{k}-$space is

$$\frac{4}{3}\pi k^3$$

and the volume of an elementary cell is

$$\left(\frac{\pi}{L}\right)^3$$

therefore the number of states satisfying (2.49) is 1/8 the volume of the sphere divided by the volume of an elementary cell,

$$N(k) = \frac{1}{8}\frac{4}{3}\pi k^3 \left(\frac{L}{\pi}\right)^3 = \frac{1}{3}4\pi\frac{V}{(2\pi)^3}k^3$$

where V is the volume of the container. We must multiply this by a factor of 2 from the fact that there are two polarisations for the photons. Using $g(k) = 2 \times dN(k)/dk$ we obtain for waves in a box

$$g(k)dk = 2 \times 4\pi\frac{V}{(2\pi)^3}k^2 dk. \tag{2.50}$$

Now we use the quantum relationship between photon energy and wave number:

$$\epsilon = hf = \frac{hc}{\lambda} = hc\frac{k}{2\pi} = \hbar ck. \tag{2.51}$$

$$g(\epsilon)d\epsilon = 8\pi\frac{V}{(2\pi)^3}\frac{\epsilon^2}{(\hbar c)^3}d\epsilon \tag{2.52}$$

Thus, the number of photons $n(\epsilon)d\epsilon$ in the energy range from ϵ to $\epsilon + d\epsilon$ is

$$n(\epsilon)d\epsilon = \frac{8\pi V}{(hc)^3}\frac{\epsilon^2 d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{2.53}$$

The energy $u(\epsilon)$ in the range ϵ and $\epsilon + d\epsilon$ is given by:

$$u(\epsilon)d\epsilon = n(\epsilon)\epsilon d\epsilon = \frac{8\pi V}{(hc)^3}\frac{\epsilon^3 d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{2.54}$$

Since the energyand frequency are related by the quantum formula,

$$\epsilon = hf,$$

we find that the density is:

$$u(f)df = \frac{8\pi V h}{c^3}\frac{f^3 df}{e^{\frac{hf}{kT}} - 1} \tag{2.55}$$

The total energy, U. This the area under the graph of the energy spectrum

$$U = \int_0^\infty u(f)df = \frac{8\pi V}{h^2c^3}\int_0^\infty \frac{(hf)^3 df}{e^{\frac{hf}{kT}} - 1} \tag{2.56}$$

Define

$$y = \frac{hf}{kT}$$

in terms of which, the expression for the total energy becomes

$$U = \int_0^\infty u(f)df = \frac{8\pi V}{(hc)^3}(kT)^4\int_0^\infty \frac{y^3 dy}{e^y - 1} \tag{2.57}$$

The integral is $\pi^4/15$, hence

$$\frac{U}{V} = \frac{8\pi^5}{15(hc)^3}(kT)^4. \tag{2.58}$$

The Photoelectric Effect

The photoelectric effect is the ejection of electrons from a metal surface exposed to electromagnetic radiation. The energy of the emitted electrons is given by the frequency of the irradiating light.

An increase in the intensity of the radiation leads to the emision of more electrons, but does not change their energy. This clearly contradict the view of Maxwell's wave theory where the energy of a wave is given by its intensity.

There is no smaller quantity of energy in radiation of a certain frequency f than the energy of a single photon. The radiation is regarded as a stream of photons, each having an energ hf.

The Compton Effect

The successes of blackbody radiation and the photoelectric effect were not sufficient to convince all scientists of the idea that radiation is quanatised. Further evidence for the photon concept came from the so-called Compton effect.

In 1923 Compton was studying the the scattering of x-rays off graphite. Classically, the charges should oscillate at the same frequency as of the incoming radiation and then give off radiation of the same frequency. However he found that radiation was being emmitted at a longer wavelength. This is called the Compton effect. Specifically, if the incoming radiation is scattered by an angle θ and if λ and λ' are wavelengths of the incident and scattered radiation, respectively, we find that

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta). \tag{2.59}$$

where m_0 is the rest mass of the electron. Thomson scattering, the classical theory of an electromagnetic wave scattered by charged particles, cannot explain the results of the experiment and demonstrates that light cannot be explained purely as a wave phenomenon.

The photon idea provides a clear explanation and provides additional direct confirmation of the quantum nature of radiation. The results can be analyzed in terms of a collision between a photon and an electron (in the experiment the energy of the photon was very much larger than the binding energy of the electron and could therefore be considered as a free electron). The incident photon collides with an at rest electron, which then recoils as a result of the impact, the scattered photon has less energy, smaller frwquency, and longer wavelength than the incident photon.

In fact we can derive (2.59) by a simple calculation. Classically we know from the equation $E^2 - c^2p^2 = m_0^2c^4$ that for a photon implies $p = E/c$. Since the energy of a photon is hf, its momentum is

$$p = \frac{hf}{c} = \frac{h}{\lambda}$$

Part of the energy of the radiation is transferred to the recoiling electron, we have

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + E_{kin} \tag{2.60}$$

where λ' is the wavelength after scattering and $E_{kin} = (\gamma - 1)m_0c^2$ is the relativistic kinematic energy of the recoiling electron. We consider the collision in the $x - y$ plane, where the incoming photon is scattered by an angle θ and the electron, initially at rest, is deflected by an angle ϕ. Conservation of momentum in the x and y directions respectively gives

$$\begin{aligned} \frac{h}{\lambda} &= \frac{h}{\lambda'}\cos\theta + p_e\cos\phi \\ 0 &= \frac{h}{\lambda'}\sin\theta - p_e\sin\phi \end{aligned} \tag{2.61}$$

where $p_e = mv = \gamma m_0 v$. By noting from (2.61) that

$$p_e^2 \sin^2\phi^2 = \frac{h^2}{\lambda'}\sin^2\theta, \qquad p_e^2\cos^2\phi^2 = h^2\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\cos\theta\right)^2$$

and adding these together we can eliminate ϕ and after some manipulation obtain

$$p_e^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta.$$

We can obtain another expression for p_e^2 by using $E^2 = p_e^2c^2 + m_0^2c^4 = (E_{kin} + m_0c^2)^2$ and (2.60)

$$\begin{aligned}
p_e^2 &= \left(\frac{E_{kin}}{c} + m_0c\right)^2 - m_0^2c^2 \\
&= \left(\frac{h}{\lambda} - \frac{h}{\lambda'} + m_0c\right)^2 - m_0^2c^2 \\
&= \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)^2 + 2\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)m_0c.
\end{aligned}$$

Equating these two expressions for p_e^2 we obtain after cancellation of terms

$$-\frac{2h^2}{\lambda\lambda'}\cos\theta = -\frac{2h^2}{\lambda\lambda'} + 2\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)m_0c^2$$

which after simplifying gives the final result

$$\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\theta).$$

the quantity $h/m_0c = 2.43 \times 10^{-12}m$ is called the Compton wavelength. The wavelength shift $\lambda' - \lambda$ is at most twice the Compton wavelength (for $\theta = 180^0$).

2.1.5 Photons

The formula

$$E = hf$$

means that the energy E carried by a photon and the frequency of the photon's electromagnetic vibration are directly proportional, the constant of proportionality being Planck's constant, h.

We find the energy of the electromagnetic field of a plane wave

$$\vec{\epsilon} N_k() \tag{2.62}$$

by using

$$E_{photon} = \frac{1}{8\pi}\int d^3x < \mathbf{E}^2 + \mathbf{B}^2 >= \frac{1}{4\pi}\int d^3x < \mathbf{B}^2 > \tag{2.63}$$

and substituiting in

$$\mathbf{B} = \nabla \times \mathbf{A} = iN_k\ \mathbf{k} \times \epsilon\ (e^{-ik\cdot x} - e^{ik\cdot x}) = 2N_k\ \mathbf{k}\times\epsilon\ \sin(k\cdot x) \tag{2.64}$$

and using

$$\begin{aligned}(\mathbf{k}\times\epsilon)\cdot(\mathbf{k}\times\epsilon) &= \epsilon\cdot\epsilon\ \mathbf{k}\cdot\mathbf{k} - (\mathbf{k}\cdot\epsilon)^2 \\ &= (\epsilon_0^2 - \epsilon\cdot\epsilon)\mathbf{k}^2 - (k_0\epsilon_0 - k\cdot\epsilon)^2 \\ &= \epsilon_0^2\mathbf{k}^2 + \mathbf{k}^2 - k_0^2\epsilon_0^2 \\ &= \mathbf{k}^2 = \omega^2.\end{aligned} \tag{2.65}$$

We find the energy to be

$$E_{photon} = \frac{4\omega^2}{4\pi}N_k^2\int d^3x < \sin^2(\omega t - \mathbf{k}\cdot\mathbf{x}) >= \frac{2\omega^2}{4\pi}N_k^2 V \tag{2.66}$$

where V the volume of the box. The condition $E_{photon} = \hbar\omega$ (where $\omega = 2\pi f$ is the angular frequency and $\hbar = h/2\pi$) leads to the normalisation constant

$$N_k = \sqrt{\frac{4\pi}{2\omega V}}. \tag{2.67}$$

We write

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\ \epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}). \tag{2.68}$$

2.2 Dirac Spinors

As a precursor to the Dirac equation, we introduce the Klein-Gordon equation which describes relativistic scalar particles.

2.2.1 Klein-Gordon Equation

From quantum mechanics we know about the correspodance between Schrodinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m_0}\nabla^2 + V(x)\right]\psi(\mathbf{x},t) \tag{2.69}$$

and the non-relativistic energy relation,

$$E = \frac{\mathbf{p}^2}{2m_0} + V(\mathbf{x}). \tag{2.70}$$

The former can be obtained from the latter via the substitutions

$$E \to \hat{E} \quad = \quad i\hbar\frac{\partial}{\partial t} \tag{2.71}$$

$$\mathbf{p} \to \hat{\mathbf{p}} \quad = \quad -i\hbar\nabla. \tag{2.72}$$

Now consider the classical relativistic equation

$$E^2 = \mathbf{p}^2 + m_0^2, \tag{2.73}$$

Make the same substitutions as before (that is 2.72 and 2.72). In terms of these operators the Einstein relation between energy, momentum, and mass can be written as

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -\hbar^2\nabla^2\phi + m_0^2\phi \tag{2.74}$$

Current density

Multiply Schrodinger's equation from the left by ψ^* and its conjugate by the left ψ then substract. One obtains

$$i\hbar\frac{\partial|\psi|^2}{\partial t} = -\frac{\hbar^2}{2m_0}\left[\psi^*(\mathbf{x},t)\nabla^2\psi(\mathbf{x},t) - \psi(\mathbf{x},t)\nabla^2\psi^*(\mathbf{x},t)\right] \tag{2.75}$$

This is the continuity equation in the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{2.76}$$

where

$$\rho = |\psi|^2 \tag{2.77}$$

is the probability density and

$$\mathbf{j} = -\frac{i\hbar}{2m_0}\left[\psi^*(\mathbf{x},t)\nabla^2\psi - \psi(\mathbf{x},t)\nabla^2\psi^*\right] \tag{2.78}$$

is the current density.

$$\int_V \frac{\partial\rho}{\partial t}d^3x = \frac{\partial}{\partial t}\int_V \rho d^3x = -\int_V \nabla\cdot\mathbf{j}d^3x = \int_S \mathbf{j}\cdot d\mathbf{S} = 0. \tag{2.79}$$

Hence,

$$\int_V \frac{\partial\rho}{\partial t}d^3x = Const. \tag{2.80}$$

By similar reasoning (see section) obtain for the Klein-Gordon equation the four-current density

$$j_\mu = \frac{i\hbar}{2m_0}(\phi^*\nabla^\mu\phi - \phi\nabla^\mu\phi^*) \tag{2.81}$$

The probability density is

$$\rho = j_0 = \frac{i\hbar}{2m_0}(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}) \tag{2.82}$$

Since the Klein-Gordon equation is second order in time, at a given time both ϕ and $\partial\phi/\partial t$ may have arbitrary values and so ρ could be negative! Also if ϕ is real then the probability density is zero. The Klein-Gordon equation also has negative energy states. This and the problem with the probability interpretation was the reason for a long time, the Klein-Gordon equation was considered physically meaningless.

2.2.2 Dirac's Equation

This difficulty led Dirac to search for a first order differential equation in t with positive definite probability density. Dirac wanted to construct a Hamiltonian that is linear in spatial derivatives aswell so that time and space are put on the same footing. He postulated an equation of the form

$$i\hbar\frac{\partial \psi}{\partial t} = [-i\hbar(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}) + \hat{\beta} m_0]\psi \tag{2.83}$$

where $\hat{\alpha}$, $\hat{\beta}$ are $N \times N$ matrices and ψ is a column vector

$$\begin{pmatrix} \psi_1(\mathbf{x}, t) \\ \psi_2(\mathbf{x}, t) \\ \vdots \\ \psi_N(\mathbf{x}, t) \end{pmatrix} \tag{2.84}$$

To find the concrete form of this equation we follow the natural requirements:

- Energy-momentum relation for relativistic free particle

$$E^2 = \mathbf{p}^2 + m_0^2, \tag{2.85}$$

- continuity equation for the density
- Lorentz covariance

Energy-momentum relation for relativistic free particle

Every component ψ_σ of the spinor must satify the Klein-gordon equation

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left(-\hbar^2 \nabla^2 + m_0^2\right) \psi_\sigma \tag{2.86}$$

$$\begin{aligned}
-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} &= i\hbar \frac{\partial}{\partial t}(i\hbar \frac{\partial}{\partial t}\psi) = i\hbar \frac{\partial}{\partial t}\hat{\mathcal{H}}\psi = \hat{\mathcal{H}}^2 \psi \\
&= \left[-i\hbar \hat{\alpha}_i \frac{\partial}{\partial x^i} + \hat{\beta} m_0\right]\left[-i\hbar \hat{\alpha}_j \frac{\partial}{\partial x^j} + \hat{\beta} m_0\right]\psi \\
&= -\hbar^2 \sum_{i,j=1}^{3} \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j} - i\hbar m_0 \sum_{i=1}^{3} (\hat{\alpha}_i \hat{\beta} + \hat{\beta}\hat{\alpha}_i) \frac{\partial \psi}{\partial x^i} + \hat{\beta}^2 m_0^2 \psi
\end{aligned} \tag{2.87}$$

Comparison with (2.86) implies the following requirements

$$\begin{aligned}
\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= 2\delta_{ij} 1, \\
\hat{\alpha}_i \hat{\beta} + \hat{\beta}\hat{\alpha}_i &= 0, \\
\hat{\alpha}_i^2 = \hat{\beta}^2 &= 1.
\end{aligned} \tag{2.88}$$

For the Hamiltonian to be Hermitian, the matrices $\hat{\alpha}_i, \hat{\beta}$ have to be Hermitian

$$\hat{\alpha}_i^\dagger = \hat{\alpha}_i, \quad \hat{\beta}^\dagger = \hat{\beta}. \tag{2.89}$$

Therefore the eigenvalues are real. Since $\hat{\alpha}_i^2 = 1$ and $\hat{\beta}^2 = 1$, it follows that the eigenvalues can only take the values ± 1. The eigenvalues are independent of the representation. Consider the diagonal representation of $\hat{\alpha}_i$, for example, we have

$$\hat{\alpha}_i = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & A_N \end{pmatrix} \tag{2.90}$$

with eigenvalues $A_1, \ldots, A_N$, and $\hat{\alpha}_i^2 = 1$ yields

$$\hat{\alpha}_i^2 = I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & 0 & \cdots & 0 \\ 0 & A_2^2 & 0 & \cdots & 0 \\ 0 & 0 & A_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & A_N^2 \end{pmatrix} \tag{2.91}$$

from which

$$A_k^2 = 1, \quad i.e. \quad A_k = \pm 1. \tag{2.92}$$

Now from the anticommutation relations we have

$$\hat{\alpha}_i = -\hat{\beta}\hat{\alpha}_i\hat{\beta}$$

and the identity

$$Tr\hat{A}\hat{B} = Tr\hat{B}\hat{A}$$

we conclude

$$Tr\hat{\alpha}_i = -Tr\hat{\beta}\hat{\alpha}_i\hat{\beta} = -Tr\hat{\alpha}_i\hat{\beta}^2 = -Tr\hat{\alpha}_i = 0. \tag{2.93}$$

We see that the matrices $\hat{\alpha}_i, \beta$ must even dimensional.

The smallest even dimension for which the (2.88) can be fulfilled is $N = 4$. In fact it is easily shown that the following is a representation

$$\hat{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\hat{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.94}$$

To see this note that

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \tag{2.95}$$

where $\hat{\sigma}_i$ are Pauli's 2×2 matrices which satify the relation

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij}\mathbf{1}, \tag{2.96}$$

as this means

$$
\begin{aligned}
\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} \hat{\sigma}_i\hat{\sigma}_j & 0 \\ 0 & \hat{\sigma}_i\hat{\sigma}_j \end{pmatrix} + \begin{pmatrix} \hat{\sigma}_j\hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_j\hat{\sigma}_i \end{pmatrix} \\
&= \begin{pmatrix} \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i \end{pmatrix} \\
&= 2\delta_{ij} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix},
\end{aligned} \tag{2.97}
$$

and also

$$
\begin{aligned}
\hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} = 0.
\end{aligned} \tag{2.98}
$$

Continuity equation for the density

We need to construct the four-current density and the equation of continuity. Let us multiply (2.83) from the left by $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$

$$
i\hbar\psi^\dagger\frac{\partial}{\partial t}\psi = -i\hbar\sum_{k=1}^{3}\psi^\dagger\hat{\alpha}_k\frac{\partial}{\partial x^k}\psi + m_0\psi^\dagger\hat{\beta}\psi \tag{2.99}
$$

Take the Hermitian conjugate of (2.83)

$$
i\hbar\frac{\partial\psi^\dagger}{\partial t} = i\hbar\sum_{k=1}^{3}\frac{\partial\psi^\dagger}{\partial x^k}\hat{\alpha}_k^\dagger + m_0\psi^\dagger\hat{\beta}^\dagger \tag{2.100}
$$

and multiply from the right by ψ, taking into account the Hermiticty of $\hat{\alpha}_i, \hat{\beta}$, we get

$$
-i\hbar\frac{\partial\psi^\dagger}{\partial t}\psi = i\hbar\sum_{k=1}^{3}\frac{\partial\psi^\dagger}{\partial x^k}\hat{\alpha}_k\psi + m_0\psi^\dagger\hat{\beta}\psi \tag{2.101}
$$

Then, subtraction of (2.101) from (2.99) yields

$$i\hbar\frac{\partial}{\partial t}(\psi^\dagger\psi) = -i\hbar\sum_{k=1}^{3}\frac{\partial}{\partial x^k}(\psi^\dagger\hat{\alpha}_k\psi) \tag{2.102}$$

which can be seen as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{2.103}$$

where

$$\rho = \psi^\dagger\psi \tag{2.104}$$

is the positive definite density and

$$j^k = \psi^\dagger\hat{\alpha}^k\psi \tag{2.105}$$

We still have to show that $\rho(x)$ is the temporal component of a four-vector so that the spatial integral $\int\rho d^3x$ becomes constant in time. Only the the probability interpretation of $\rho(x)$ ensured.

Lorentz covariance

2.2.3 Free Motion of a Dirac Particle

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\psi = \left(\hat{\alpha}\cdot\hat{\mathbf{p}} + m_0\hat{\beta}\right)\psi \tag{2.106}$$

Stationary states can be found by substituting

$$\psi(\mathbf{x},t) = \psi(\mathbf{x})\exp[-(i/\hbar)\epsilon t] \tag{2.107}$$

into the Dirac equation. We get

$$\epsilon\psi(\mathbf{x}) = \hat{\mathcal{H}}\psi(\mathbf{x}) \tag{2.108}$$

Split the four-component spinor into two two-component spinors ϕ and χ, i.e.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{2.109}$$

$$\varphi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad \chi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{2.110}$$

or

$$\begin{aligned} \epsilon\psi &= \hat{\alpha}\chi \cdot \hat{\mathbf{p}} + m_0\varphi \\ \epsilon\chi &= \hat{\alpha}\varphi \cdot \hat{\mathbf{p}} - m_0\chi \end{aligned} \tag{2.111}$$

The states

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} \exp[(i/\hbar)\mathbf{p} \cdot \mathbf{x}] \tag{2.112}$$

This results in

$$\begin{aligned} (\epsilon - m_0)I\varphi_0 - \hat{\sigma} \cdot \hat{\mathbf{p}}\chi_0 &= 0, \\ -\hat{\sigma} \cdot \hat{\mathbf{p}}\, \varphi_0 + (\epsilon + m_0)I\chi_0 &= 0. \end{aligned} \tag{2.113}$$

This linear homogeneous system of equations for φ_0 and χ_0 has notrivial solutions only in the case of a vanishing determinant of the coefficients, that is

$$\begin{vmatrix} (\epsilon - m_0)I & -\hat{\sigma} \cdot \hat{\mathbf{p}} \\ -\hat{\sigma} \cdot \hat{\mathbf{p}} & (\epsilon + m_0)I \end{vmatrix} = 0. \tag{2.114}$$

Using the relation

$$(\hat{\sigma} \cdot \mathbf{A})(\hat{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B}I + i\hat{\sigma} \cdot (\mathbf{A} \times \mathbf{B}) \tag{2.115}$$

(2.114) becomes

$$\begin{aligned} (\epsilon^2 - m_0^2)I - (\hat{\sigma} \cdot \hat{\mathbf{p}})(\hat{\sigma} \cdot \hat{\mathbf{p}}) &= (\epsilon^2 - m_0^2)I - \mathbf{p} \cdot \mathbf{p}I - i\hat{\sigma} \cdot (\mathbf{p} \times \mathbf{p}) \\ &= (\epsilon^2 - m_0^2)I - \mathbf{p} \cdot \mathbf{p}I = 0 \end{aligned} \tag{2.116}$$

or

$$\epsilon^2 = m_0^2 + \mathbf{p}^2 \tag{2.117}$$

from which it follows

$$\epsilon = \pm E_p, \quad E_p = \sqrt{\mathbf{p}^2 + m_0^2} \tag{2.118}$$

2.2.4 Positive and Negative Energy Eigenvectors

with solutions

$$\begin{aligned} \varphi(t) &= \varphi(0)e^{-im_0t} \\ \chi(t) &= \chi(0)e^{-im_0t} \end{aligned} \tag{2.119}$$

φ represents a particle, while χ represents an antiparticle.

$$\chi_0 = \frac{(\hat{\sigma} \cdot \hat{\mathbf{p}})}{m_0 + \epsilon}\varphi_0. \tag{2.120}$$

Let us denote the two-spinor φ_0 in the form

$$\varphi_0 = U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{2.121}$$

with the normalisation

$$U^\dagger U = U_1^* U_1 + U_2^* U_2 = 1, \tag{2.122}$$

where U_1, U_2 are complex numbers.

2.2.5 Helicity

There is another quantum number, the helicity, can be used to classify the free one-particle states. Its operator should commute with the operators whose eigenvalues have already been introduced to label our free solutions.

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \tag{2.123}$$

The helicity commutes with the Hamiltonian

$$\begin{aligned}
[\hat{\mathcal{H}}, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}] &= \left[\hat{\alpha} \cdot \hat{\mathbf{p}} + m_0 \hat{\beta}, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right] \\
&= \begin{pmatrix} 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \\ \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \end{pmatrix} \begin{pmatrix} \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix} - \begin{pmatrix} \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \\ \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 \\ (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 \\ (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 & 0 \end{pmatrix} \\
&= 0
\end{aligned}$$

Hence

$$\left[\hat{\mathcal{H}}, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right] = 0 \tag{2.124}$$

and obviously we have

$$\left[\hat{\mathbf{p}}, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right] = 0 \tag{2.125}$$

the helicity operator

$$\hat{\Lambda}_S = \frac{\hbar}{2}\hat{\boldsymbol{\Sigma}} \cdot \frac{\hat{\mathbf{p}}}{|\mathbf{p}|} = \hat{\mathbf{S}} \cdot \frac{\hat{\mathbf{p}}}{|\mathbf{p}|} \tag{2.126}$$

Helicity is the projection of the spin onto the direction of momentum.

If the electron wave propagates into th edirection of the $z-$axis, we have

$$\mathbf{p} = \{0, 0, p\}$$

and because of (2.126),

$$\hat{\Lambda}_S = \hat{S}_z = \frac{\hbar}{2}\hat{\Sigma} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.127}$$

with eigenvalues $\pm\hbar/2$. Clearly, the eigenvectors of $\hat{\Lambda}_S$ are

$$\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_1 \end{pmatrix} \tag{2.128}$$

with

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.129}$$

If in any particular direction a quantum state take one of two values it is likely to do with spin half particles. In fact Lorentz-covariance of Dirac's equation will imply that these two-state systmes transform under rotations as two-spinors.

2.2.6 Coupling of Dirac spinors to the Electromagnetic Field

Under a gauge transformation the wavefunction $\psi(x)$ transforms as

$$\psi(x) \to e^{i\Lambda(x)}\psi(x) \tag{2.130}$$

and

$$A^\mu(x) \to A^\mu(x) - \frac{1}{e}\partial^\mu\Lambda(x) \tag{2.131}$$

thus

$$\left(i\hbar\frac{\partial}{\partial x^\mu} - eA^\mu\right) \tag{2.132}$$

is gauge covariant. The minimal coupling prescription

$$i\hbar\frac{\partial}{\partial x^\mu} \to \left(i\hbar\frac{\partial}{\partial x^\mu} - eA^\mu\right) \tag{2.133}$$

results in the equation

$$(i\hbar\gamma^\mu\frac{\partial}{\partial x^\mu} - eA_\mu\gamma^\mu - m_0)\psi = 0. \tag{2.134}$$

2.2.7 Lorentz Transformations for Dirac Spinors

How do we construct the wave function $\psi'(x')$ in one inertial frame if we know the wave function $\psi(x)$ in another frame, where the two frames are related by the Lorentz transformation a_μ^ν? Here we construct the Lorentz transformation law between $\psi(x)$ and $\psi'(x')$.

We start from the principle of special relativity which states that the laws of physics should take the same form in all inertial systems, $\psi'(x')$ must be a solution of a Dirac equation which has the form

$$\left(i\hbar\gamma^{\mu'}\frac{\partial}{\partial x'^{\mu}} - m_0\right)\psi'(x') = 0 \tag{2.135}$$

in the primed system, where $\gamma^{\mu'}$ satisfy the same anti-commutation relations as γ^μ:

$$\gamma^{\mu'}\gamma^{\nu'} + \gamma^{\nu'}\gamma^{\mu'} = 2\eta^{\mu\nu}I \tag{2.136}$$

and

$$\gamma'^{0\dagger} = \gamma'^{0} \tag{2.137}$$

$$\gamma'^{i\dagger} = -\gamma'^{i} \quad i = 1, 2, 3. \tag{2.138}$$

It can be shown that $\gamma^{\mu'}$ that satify the above relations are identical to γ^μ up to a unitary transformation $\hat{U}$, i.e.

$$\gamma^{\mu'} = \hat{U}^\dagger\gamma^\mu\hat{U}, \quad \hat{U}^\dagger = \hat{U}^{-1}. \tag{2.139}$$

Since unitary transformations do not change the physics, we may use the same γ matrices in both Lorentz systems. From now on we just take $\gamma^\mu = \gamma^{\mu'}$.

$$\left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m_0\right)\psi(x) = 0 \quad \text{and} \quad \left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x'^{\mu}} - m_0\right)\psi'(x') = 0$$

Let $\hat{a}$ denote the matrix of the Lorentz transformation a_μ^ν. We write

$$\psi'(x') = \psi'(\hat{a}x') = \hat{S}(\hat{a})\psi(x) = \hat{S}(\hat{a}^{-1}x') \tag{2.140}$$

We must have an inverse transformation

$$\psi(x) = \hat{S}^{-1}(\hat{a})\psi'(x') = \hat{S}^{-1}(\hat{a})\psi'(\hat{a}x)$$

Start with Dirac equation

$$\left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m_0\right)\psi(x) = 0$$

expressing $\psi(x)$ by $\hat{S}^{-1}(\hat{a})\psi'(x')$ yields

$$\left(i\hbar\gamma^{\mu}\hat{S}^{-1}(\hat{a})\frac{\partial}{\partial x^{\mu}} - m_0\hat{S}^{-1}(\hat{a})\right)\psi'(x') = 0.$$

We the multiply by $\hat{S}(\hat{a})$

$$\left(i\hbar\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})\frac{\partial}{\partial x^{\mu}} - m_0\right)\psi'(x') = 0 \tag{2.141}$$

Now we transform $\partial/\partial x^{\mu}$ to x' coordinates is given by

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\nu}} = a^{\nu}{}_{\mu}\frac{\partial}{\partial x'^{\nu}} \tag{2.142}$$

So that () becomes

$$\left(i\hbar(\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})a^{\nu}{}_{\mu})\frac{\partial}{\partial x'^{\nu}} - m_0\right)\psi'(x') = 0. \tag{2.143}$$

Comparing this to Dirac's equation in the x' coordinates we see that $\hat{S}(\hat{a})$ must satisfy

$$\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})a^{\nu}{}_{\mu} = \gamma^{\nu} \tag{2.144}$$

or equivalently

$$\hat{S}(\hat{a})\gamma^{\nu}\hat{S}^{-1}(\hat{a}) = a_{\mu}{}^{\nu}\gamma^{\mu} \tag{2.145}$$

2.2.8 Infintesimal Generating Technique for Lorentz Transformations

We first give the idea of the infintesimal generating technique with a couple of simple examples.

Example 2: Lorentz transformation in x_1-direction for $2d$-spacetime

We derive the Lorentz transformation formula for boosts in the x_1-direction. Consider two inertia frames, the 'primed' frame one moving away from the 'unprimed' frame at an infintesimal velocity δv along the $x_1$direction. For an infintesimal relative velocity the spacetime transformation is Galilean:

$$x_1' = x_1 - \delta v t. \tag{2.146}$$

How is special relativity brought into the calculation? This is done by requiring that

$$x_1^2 - t^2 = x_1'^2 - t'^2. \tag{2.147}$$

From this we see that $t' \neq t$, and so t should transform some way as well. Let us write

$$t' = t + a\delta v x_1. \tag{2.148}$$

Using this in (2.147) we find $a = -1$. The two transformation equations can be combined in the matrix equation

$$\begin{aligned}\begin{pmatrix} t' \\ x' \end{pmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta v \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix} t \\ x \end{pmatrix} \\ &= (\mathbf{1} + \delta v \hat{I}_2)\begin{pmatrix} t \\ x \end{pmatrix}\end{aligned} \tag{2.149}$$

where $\hat{I}_x = -\mathbf{1}$. Now we repeat the transformation N times to generate a finite transformation with velocity pararmeter $\theta = N\delta v$. Then

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \left(\mathbf{1} + \frac{\theta}{N}\hat{I}_x\right)^N \begin{pmatrix} t \\ x \end{pmatrix} \tag{2.150}$$

In the limit $N \to \infty$,

$$\lim_{N\to\infty}\left(\mathbf{1}+\frac{\theta}{N}\hat{I}_x\right)^N = \exp(\theta\hat{I}_2). \tag{2.151}$$

Noting $\hat{I}_x^2 = \mathbf{1}$, we expand the expontential

$$\begin{aligned}
\exp(\theta\hat{I}_x) &= \mathbf{1}+\theta\hat{I}_x+\frac{\theta^2\hat{I}_x^2}{2!}+\frac{\theta^3\hat{I}_x^3}{3!}+\ldots \\
&= \mathbf{1}[1+\frac{1}{2!}\theta^2+\ldots]+\hat{I}_x[\theta-\frac{\theta^3}{3!}+\ldots] \\
&= \\
&= \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \qquad (2.152)
\end{aligned}$$

$\cosh\theta$ and $\sinh\theta$ can be identified by considering the origin of the primed coordinate system, $x' = 0$, or $x = vt$. Substituiting this into () we have

$$0 = x\cosh\theta - t\sinh\theta. \tag{2.153}$$

So

$$\tan\theta = v$$

Using $1-\tanh^2\theta = (\cosh^2\theta)^{-1}$,

$$\cosh\theta = \frac{1}{(1-v^2)^{1/2}}, \qquad \sinh\theta = \frac{v}{(1-v^2)^{1/2}}. \tag{2.154}$$

We finally obtain the known Lorentz transformations

$$t' = \frac{t-vx}{(1-v^2)^{1/2}}, \qquad x' = \frac{x-vt}{(1-v^2)^{1/2}} \tag{2.155}$$

□

This result can easily be generalised to 4-minkowski space time. We just use the generator

$$I_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.156}$$

We end up with the answer

$$t' = \frac{t - vx}{(1-v^2)^{1/2}}, \qquad x' = \frac{x - vt}{(1-v^2)^{1/2}}, \qquad y' = y, \qquad z' = z. \tag{2.157}$$

If we had wanted to do boost in the direction given by the unit vector

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma) \tag{2.158}$$

we would use the generator

$$I_n = \begin{pmatrix} 0 & -\cos\alpha & -\cos\beta & -\cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{pmatrix}. \tag{2.159}$$

Note that, to first order, $t^2 - \vec{r}^2 = (t')^2 - \vec{r}'^2$ is satified for an infintesimal relative velocity.

The reader is invited to do the full calculation and derive the Lorentz transformation formula.

Example 2: Rotations about the $z-$direction for $3d$-spacetime

It is easily seen, drawing a diagram, that under an infintesimal rotation $\delta\phi$ around the $z-$axis results in

$$x' = x + y\delta\phi, \qquad y' = y - x\delta\phi \tag{2.160}$$

The two transformation equations can be combined in the matrix equation

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (\mathbf{1} + \delta\phi i\hat{\sigma}_2) \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned} \tag{2.161}$$

where $\hat{\sigma}_2$ is a Pauli matrix. Let us now do the expontentiation using $\hat{\sigma}_2^2 = 1$:

$$\begin{aligned}
\exp(\phi i\hat{\sigma}_2) &= \mathbf{1} + \phi i\hat{\sigma}_2 - \frac{\phi^2}{2!}\hat{\sigma}_2^2 - i\frac{\phi^3}{3!}\hat{\sigma}_2^3 \\
&= \mathbf{1}(1 - \frac{\phi^2}{2!} + \ldots) + i\hat{\sigma}_2(\phi - \frac{\phi^3}{3!} + \ldots) \\
&= \mathbf{1}\cos\phi + i\hat{\sigma}_2\sin\phi \\
&= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}
\end{aligned} \tag{2.162}$$

So that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.163}$$

□

In 4d-minkowski space time we use the generator

$$I_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.164}$$

We end up with the answer

$$t' = t, \qquad x' = x\cos\phi + y\sin\phi, \qquad y' = -x\sin\phi + y\cos\phi, \qquad z' = z. \tag{2.165}$$

Proper Lorentz Transformations

$$a^{\nu}{}_{\mu} = \delta^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu} \tag{2.166}$$

We denote the inverse Lorentz Transformation as $a_{\nu}{}^{\sigma}$. Then, neglecting terms quadtratic in $\Delta\omega$,

$$\begin{aligned}
a^{\mu}{}_{\nu}a_{\mu}{}^{\sigma} &= \delta^{\sigma}_{\nu} = (\delta^{\mu}_{\nu} + \Delta\omega^{\mu}{}_{\nu})(\delta^{\sigma}_{\mu} + \Delta\omega_{\mu}{}^{\sigma}) \\
&\approx \delta^{\mu}_{\nu}\delta^{\sigma}_{\mu} + \delta^{\mu}_{\nu}\Delta\omega_{\mu}{}^{\sigma} + \delta^{\sigma}_{\mu}\Delta\omega^{\mu}{}_{\nu} \\
&= \delta^{\sigma}_{\nu} + \Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu}
\end{aligned} \tag{2.167}$$

Hence,

$$\Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu} = 0$$

or

$$\eta^{\mu\nu}\left(\Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu}\right) = 0 = \Delta\omega^{\mu\sigma} + \Delta\omega^{\sigma\mu}.$$

so we must have

$$\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu} \tag{2.168}$$

Consequently, there are six independent non-vanishing parameters $\Delta\omega^{\mu\nu}$.

2.2.9 The $\hat{S}$ Operator for Infintesimal Lorentz Transformations

We aim to determine the operator $\hat{S}$ by assertaining its infintesimal form by finding its expansion to linear order in the generators $\Delta\omega^{\mu\nu}$. We write

$$\hat{S}(\Delta\omega^{\mu\nu}) \;=\; \mathbf{1} - \frac{i}{4}\hat{\sigma}_{\mu\nu}\Delta\omega^{\mu\nu} \tag{2.169}$$

where $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$. The inverse operator being

$$\hat{S}^{-1}(\Delta\omega^{\mu\nu}) \;=\; \mathbf{1} + \frac{i}{4}\hat{\sigma}_{\mu\nu}\Delta\omega^{\mu\nu} \tag{2.170}$$

By finding $\sigma_{\alpha\beta}$ we can find $\hat{S}$. By substituting (2.169) and (2.170) into the defing equation for $\hat{S}$:

$$(\delta^{\nu}_{\mu} + \Delta\omega_{\mu}{}^{\nu})\gamma^{\mu} = \hat{S}(\Delta\omega^{\mu\nu})\gamma^{\nu}\hat{S}^{-1}(\Delta\omega^{\mu\nu})$$

we can find an equation that determines $\sigma_{\alpha\beta}$.

$$(\delta^{\nu}_{\mu} + \Delta\omega_{\mu}{}^{\nu})\gamma^{\mu} = \left(\mathbf{1} - \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)\gamma^{\nu}\left(\mathbf{1} + \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)$$

or omitting the quadratic terms in $\Delta\omega_{\mu}{}^{\nu}$,

$$\Delta\omega_{\mu}{}^{\nu}\gamma^{\mu} = -\frac{i}{4}\Delta\omega^{\alpha\beta}(\hat{\sigma}_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\hat{\sigma}_{\alpha\beta}) \tag{2.171}$$

Using the antisymmetry of $\Delta\omega_{\mu}{}^{\nu}$, the LHS becomes

$$\begin{aligned}
\Delta\omega_{\mu}{}^{\nu}\gamma^{\mu} &= \eta^{\nu}{}_{\sigma}\Delta\omega_{\mu}{}^{\sigma}\gamma^{\mu} \\
&= \Delta\omega_{\beta}{}^{\alpha}(\eta^{\nu}{}_{\alpha}\gamma^{\beta}) \\
&= -\Delta\omega^{\alpha\beta}(\eta^{\nu}{}_{\alpha}\gamma_{\beta}) \\
&= -\frac{1}{2}\Delta\omega^{\alpha\beta}(\eta^{\nu}{}_{\alpha}\gamma_{\beta} - \eta^{\nu}{}_{\beta}\gamma_{\alpha})
\end{aligned} \tag{2.172}$$

Comparing this with (2.171), we end up with the relation

$$-2i(\eta_{\nu}{}^{\alpha}\gamma_{\beta} - \eta_{\nu}{}^{\beta}\gamma_{\alpha}) = [\hat{\sigma}_{\alpha\beta}, \gamma^{\nu}] \tag{2.173}$$

It is shown in section 2.14.4 that this is solved by

$$\hat{\sigma}_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}].$$

The operator $\hat{S}(\Delta\omega^{\mu\nu})$ is now

$$\hat{S}(\Delta\omega^{\mu\nu}) = \mathbf{1} + \frac{1}{8}[\gamma_{\mu}, \gamma_{\nu}]\Delta\omega^{\mu\nu} \tag{2.174}$$

The problem of finding $\hat{S}$ for finite proper Lorentz transformations has now been essentially solved! To construct $\hat{S}$ we now sucessively apply the infintesimal operators (2.174). In order to do this we write

$$\Delta\omega^{\nu}{}_{\mu} = \Delta\omega(\hat{I}_{\mathbf{n}})^{\nu}{}_{\mu} \tag{2.175}$$

Here $\Delta\omega$ is an inintesimal parameter of the Lorentz group around an axis in the $\mathbf{n}$ direction.

2.2.10 The $\hat{S}$ Operator for Proper Lorentz Transformations

$$\begin{aligned}
\psi'(x') = \hat{S}(\hat{a})\psi(x) &= \lim_{N\to\infty}\left(\mathbf{1} - \frac{i}{4}\frac{\omega}{N}\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu}\right)^{N}\psi(x) \\
&= e^{-(i/4)\omega\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu}}\psi(x)
\end{aligned} \tag{2.176}$$

Example: Lorentz boost along x-axis

$$(I_x)^0{}_1 = (I_x)^1{}_0 = -(I_x)^{01} = +(I_x)^{10} = -1.$$

So (2.176) becomes

$$\begin{aligned} \psi'(x') &= \exp\left\{-\frac{i}{4}\omega\left[\hat{\sigma}_{01}(I_x)^{01} + \hat{\sigma}_{10}(I_x)^{10}\right]\right\}\psi(x) \\ &= \exp\left\{-\frac{i}{4}\omega\left[\hat{\sigma}_{01}(+1) + \hat{\sigma}_{10}(-1)\right]\right\}\psi(x) \\ &= \exp\left\{-\frac{i}{2}\omega\hat{\sigma}_{01}\right\}\psi(x) \end{aligned} \tag{2.177}$$

□

Example: Rotation around z-axis

Recall that

$$\Delta\omega^\nu{}_\mu = \delta\phi(\hat{I}_3)^\nu{}_\mu, \tag{2.178}$$

where $\hat{I}_3$ is given by (). Thus only the elements $(\hat{I}_3)^{12} = -(\hat{I}_3)^{21}$ are non-zero, and we get

$$\begin{aligned} \psi'(x') &= \exp\left\{-\frac{i}{4}\phi\hat{\sigma}_{\mu\nu}(\hat{I}_3)^{\mu\nu}\right\}\psi(x) \\ &= \exp\left\{-\frac{i}{4}\phi\left[\hat{\sigma}_{12}(I_x)^{12} + \hat{\sigma}_{21}(I_3)^{21}\right]\right\}\psi(x) \\ &= \exp\left\{-\frac{i}{4}\phi\left[\hat{\sigma}_{12}(-1) + \hat{\sigma}_{21}(+1)\right]\right\}\psi(x) \\ &= \exp\left\{\frac{i}{2}\phi\hat{\sigma}_{12}\right\}\psi(x) = \exp\left\{\frac{i}{2}\phi\hat{\sigma}^{12}\right\}\psi(x) \end{aligned} \tag{2.179}$$

□

Spinor for spacial rotations

2.2.11 The Four-Current Density

$$j_\mu(x) = \psi^\dagger(x)\gamma^0\gamma^\mu\psi(x). \tag{2.180}$$

This current density transforms under the Lorentz transformation as

$$\begin{aligned} j'^{\mu}(x') &= \psi^{\dagger'}(x')\gamma^0\gamma^\mu\psi'(x') \\ &= \psi^\dagger(x)\hat{S}^\dagger\gamma^0\gamma^\mu\hat{S}\psi(x) \\ &= \psi^\dagger(x)\gamma^0(\gamma^0\hat{S}^\dagger\gamma^0)\gamma^\mu\hat{S}\psi(x) \\ &= \psi^\dagger(x)\gamma^0\hat{S}^{-1}\gamma^\mu\hat{S}\psi(x) \\ &= \psi^\dagger(x)\gamma^0(a^\nu{}_\mu\gamma^\mu)\psi(x) \\ &= a^\nu{}_\mu j^\mu(x) \end{aligned} \tag{2.181}$$

and as such is identified as a four-vector.

2.2.12 Plane Waves in Arbitrary Directions

Free solutions have the form

$$\psi^r = \omega^r(0)e^{-\epsilon_r(m_0/\hbar)t} \tag{2.182}$$

We have

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega^4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.183}$$

$$\omega^r(p) = \hat{S}(-\mathbf{v})\omega^r(0) = e^{-(\omega/2)\alpha\cdot\mathbf{v}/v}\omega^r(0) \tag{2.184}$$

$$\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu} = 2\big(\hat{\sigma}_{01}(\hat{I}_{\mathbf{n}})^{01} + (\hat{I}_{\mathbf{n}})^{02} + (\hat{I}_{\mathbf{n}})^{03}\big) \tag{2.185}$$

$$\frac{\mathbf{v}}{v} = (\cos\alpha, \cos\beta, \cos\gamma) \tag{2.186}$$

Also

$$
\begin{aligned}
\hat{\sigma}_{0i} &= \frac{i}{2}(\gamma_0\gamma_i - \gamma_i\gamma_0) \\
&= i\gamma_0\gamma_i \\
&= -i\gamma^0\gamma^i = -i\gamma^0\gamma^0\alpha_i = -i\alpha_i
\end{aligned} \tag{2.187}
$$

With this the spinor transformation for Lorentz transformations to interial systems with direction of velocity $\mathbf{v}/v$ now becomes

$$
\hat{S}(-\mathbf{v}) = \hat{S}\left(-\frac{\mathbf{p}}{E}\right) = e^{-(\omega/2)\hat{\alpha}\cdot\mathbf{v}/v} \tag{2.188}
$$

When we expand $\hat{S}$ we will need the following

$$
\begin{aligned}
(\hat{\alpha}\cdot\mathbf{v})^2 &= \hat{\alpha}^i\hat{\alpha}^j v_i v_j \\
&= \gamma^0\gamma^i\gamma^0\gamma^j v_i v_j \\
&= -\gamma^i\gamma^j v_i v_j \\
&= -\frac{1}{2}(\gamma^i\gamma^j + \gamma^j\gamma^i)v_i v_j \\
&= -\frac{1}{2}2\eta^{ij}\mathbf{1}v_i v_j = +v^2\mathbf{1}
\end{aligned} \tag{2.189}
$$

We expand $\hat{S}$

$$
\begin{aligned}
\hat{S}(-\mathbf{v}) &= \mathbf{1} - \frac{\omega}{2}\frac{\hat{\alpha}\cdot\mathbf{v}}{v} + \frac{1}{2!}\frac{\omega^2}{4v^2}(\hat{\alpha}\cdot\mathbf{v})^2 - \frac{1}{3!}\frac{\omega^3}{8v^3}(\hat{\alpha}\cdot\mathbf{v})^3 + \dots \\
&= \mathbf{1}(1 + \frac{1}{2}\frac{\omega^2}{4}) - \frac{\hat{\alpha}\cdot\mathbf{v}}{v}(\frac{\omega}{2}+) \\
&= \mathbf{1}\cosh\frac{\omega}{2} - \frac{\hat{\alpha}\cdot\mathbf{v}}{v}\sinh\frac{\omega}{2}
\end{aligned} \tag{2.190}
$$

The matrix written out $\hat{\alpha}\cdot\mathbf{v}/v$

$$
\begin{aligned}
\frac{\hat{\alpha}\cdot\mathbf{v}}{v} &= \hat{\alpha}_x\frac{v_x}{v}+\hat{\alpha}_y\frac{v_y}{v}+\hat{\alpha}_z\frac{v_z}{v} \\
&= \frac{p_x}{p}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{ip_y}{p}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&\quad + \frac{p_z}{p}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{p}\begin{pmatrix} 0 & 0 & p_z & p_- \\ 0 & 0 & p_+ & -p_z \\ p_z & p_- & 0 & 0 \\ p_+ & -p_z & 0 & 0 \end{pmatrix}
\end{aligned} \tag{2.191}
$$

where $p_\pm = p_x \pm ip_y$. We obtain

$$
\hat{S}(-\mathbf{v}) = \cosh\frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \cosh\frac{\omega}{2}\frac{\tanh\frac{\omega}{2}}{p}\begin{pmatrix} 0 & 0 & p_z & p_- \\ 0 & 0 & p_+ & -p_z \\ p_z & p_- & 0 & 0 \\ p_+ & -p_z & 0 & 0 \end{pmatrix} \tag{2.192}
$$

To find expressions for $\cosh\frac{\omega}{2}$ and $\tanh\frac{\omega}{2}$ we consider only motion in the x direction. We convert the rotation angle ω with the aid of

$$
-v_x = \tanh\omega, \tag{2.193}
$$

or

$$
\omega = \tanh^{-1}(-v_x) = -\tanh^{-1}(v_x) \tag{2.194}
$$

We need the equations

$$
\begin{aligned}
\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} &= \frac{1}{2}\sinh\omega, \\
\cosh\frac{\omega}{2}\cosh\frac{\omega}{2} &= \frac{1}{2}(\cosh\omega+1), \\
\sinh\frac{\omega}{2}\sinh\frac{\omega}{2} &= \frac{1}{2}(\cosh\omega-1).
\end{aligned} \tag{2.195}
$$

Therefore

$$\tanh\frac{x}{2}=\frac{\sinh x}{\cosh x+1}=\frac{\tanh x}{1+1/\cosh x}=\frac{\tanh x}{1+\sqrt{1-\tanh^2 x}} \tag{2.196}$$

With (2.194)

$$\begin{aligned}
-\tanh\frac{\omega}{2} &= \frac{-\tanh\omega}{1+\sqrt{1-\tanh^2\omega}}\\
&= \frac{v_x}{1+\sqrt{1-v_x^2}}\\
&= \frac{m_0/\sqrt{1-v_x^2}}{m_0/\sqrt{1-v_x^2}}\frac{v_x}{1+\sqrt{1-v_x^2}}\\
&= \frac{p_x}{E+m_0}
\end{aligned} \tag{2.197}$$

taking into account that we are considering only motion in the x direction, we may write

$$-\tanh\frac{\omega}{2}=\frac{p}{E+m_0}. \tag{2.198}$$

And

$$
\begin{aligned}
\cosh\frac{\omega}{2} &= \frac{1}{\sqrt{1-\tanh\frac{\omega}{2}}} \\
&= \frac{1}{\sqrt{1-[\tanh\omega/(1+\sqrt{1-\tanh^2\omega})^2]}} \\
&= \frac{1}{\sqrt{1-[v_x/(1+\sqrt{1-v_x^2})^2]}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{(1+\sqrt{1-v_x^2})^2+v_x^2}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{(1+2\sqrt{1-v_x^2}+1-v_x^2)-v_x^2}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{2}\sqrt{1-v_x^2+\sqrt{1-v_x^2}}} \\
&= \frac{[1/\sqrt{1-v_x^2}+1]m_0}{\sqrt{1+[1/\sqrt{1-v_x^2}]}\sqrt{2}m_0} \\
&= \frac{E+m_0}{\sqrt{m_0+E}\sqrt{2m_0}} \\
&= \sqrt{\frac{E+m_0}{2m_0}} \qquad (2.199)
\end{aligned}
$$

substituting this result and(2.198) into (2.192) we obtain

$$
\begin{aligned}
\hat{S}(-\mathbf{v}) &= \sqrt{\frac{E+m_0}{2m_0}}\begin{bmatrix} 1 & 0 & \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} \\ 0 & 1 & \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} \\ \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} & 1 & 0 \\ \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} & 0 & 1 \end{bmatrix} \\
&= [\omega^1(\mathbf{p}),\omega^2(\mathbf{p}),\omega^3(\mathbf{p}),\omega^4(\mathbf{p})]. \qquad (2.200)
\end{aligned}
$$

2.2.13 Bilinear Covariants

Linear independence

$$
\begin{aligned}
\hat{\Gamma}^S &= \mathbf{I}, \quad \hat{\Gamma}^V_\mu = \gamma_\mu, \quad \hat{\Gamma}^T_{\mu\nu} = \hat{\sigma}_{\mu\nu} = -\hat{\sigma}_{\nu\mu} \\
\hat{\Gamma}^P &= i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma^5, \quad \hat{\Gamma}^a_\mu = \gamma^5\gamma_\mu \qquad (2.201)
\end{aligned}
$$

i) $(\hat{\Gamma}^n)^2 = \pm\mathbf{I}$.

Proof: Proved explicitly.

ii) To each $\hat{\Gamma}^n$ except $\hat{\Gamma}^S$ there exists at least one $\hat{\Gamma}^m$ such that

$$\hat{\Gamma}^n\hat{\Gamma}^m = -\hat{\Gamma}^m\hat{\Gamma}^n \tag{2.202}$$

Proof: Proved explicitly.

iii) $Tr(\hat{\Gamma}^n) = 0$

By (2.202)

$$-\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m = +\hat{\Gamma}^n(\hat{\Gamma}^m)^2$$

As $(\hat{\Gamma}^m)^2 = \pm\mathbf{I}$

$$\pm Tr(\hat{\Gamma}^n) = Tr(\hat{\Gamma}^n(\hat{\Gamma}^m)^2) = -Tr(\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m) = -Tr(\hat{\Gamma}^n\hat{\Gamma}^m\hat{\Gamma}^m) = 0.$$

iv) For given $\hat{\Gamma}^a$ and $\hat{\Gamma}^b$ $(a \neq b)$ there exists a $\hat{\Gamma}^n \neq \hat{\Gamma}^S$ such that

$$\hat{\Gamma}^a\hat{\Gamma}^b = f^n_{ab}\hat{\Gamma}^n. \tag{2.203}$$

Proof: Proved explicitly.

v) The $\hat{\Gamma}^n$ are linearly independent. Suppose

$$\sum_n a_n\hat{\Gamma}^n = 0. \tag{2.204}$$

Multiply from the right by $\hat{\Gamma}^m \neq \hat{\Gamma}^S$ we get

$$\begin{aligned}
0 &= \sum_n a_n Tr(\hat{\Gamma}^n\hat{\Gamma}^m) \\
&= a_m(\hat{\Gamma}^m)^2 + \sum_{n\neq m} a_n Tr(\hat{\Gamma}^n\hat{\Gamma}^m) \\
&= a_m(\hat{\Gamma}^m)^2 + \sum_{n\neq m} a_n Tr(f^\nu_{nm}\hat{\Gamma}^\nu) \\
&= \pm 4a_m.
\end{aligned} \tag{2.205}$$

Thus $a_m = 0$ for all $m \neq S$. Now in the case of $\hat{\Gamma}^m = \hat{\Gamma}^S$

$$0 = Tr\left(\sum_n a_n \hat{\Gamma}^S \hat{\Gamma}^n\right) = a_S Tr(\mathbf{I}) + \sum_{n\neq S} a_n Tr(\hat{\Gamma}^n) = 0, \tag{2.206}$$

i.e. $a_S = 0$.

□

Lorentz transformations

Under Lorentz transformations

$$\psi \to \hat{S}\psi, \qquad \overline{\psi} \to \overline{\psi}\hat{S}^{-1} \tag{2.207}$$

This is proved by

$$\begin{aligned} \overline{\psi}'(x') &= \psi'^{\dagger}(x')\gamma^0 \\ &= \psi^{\dagger}(x)\hat{S}^{\dagger}\gamma^0 \\ &= \psi^{\dagger}(x)\gamma^0(\gamma^0\hat{S}^{\dagger}\gamma^0) \\ &= \overline{\psi}(x)\hat{S}^{-1}. \end{aligned} \tag{2.208}$$

□

We now prove

$$\gamma_5\hat{S} = det|a|\ \hat{S}\gamma_5. \tag{2.209}$$

This is easily proven that for proper Lorentz transformations (here $det|a| = 1$), first

$$[\gamma_5, \hat{\sigma}_{\mu\nu}] = \frac{1}{2}(\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) - (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\gamma_5) = 0$$

where we have used $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$. So from the formula for proper Lorentz transformations

$$\hat{S}(\hat{a}) = \exp\left(-\frac{i}{4}\hat{\sigma}_{\mu\nu}(I_{\mathbf{n}})^{\mu\nu}\right),$$

we have

$$[\hat{S}(\hat{a}), \gamma_5] = 0. \tag{2.210}$$

We know prove (2.209) for spacial reflections, which are given by

$$\mathbf{x}' = -\mathbf{x}, \qquad t' = t, \tag{2.211}$$

with corresponding transformation matrix

$$a^{\nu}{}_{\mu} = \eta^{\mu\nu}. \tag{2.212}$$

The relation

$$\hat{S}^{-1}\gamma^{\mu}\hat{S} = a^{\mu}{}_{\nu}\gamma^{\nu}$$

holds for improper Lorentz transformations as well. Let us dente the parity operator by $\hat{P}$. We can then write

$$a^{\nu}{}_{\mu}\gamma^{\mu} = \hat{P}\gamma^{\nu}\hat{P}^{-1}$$

or

$$a^{\sigma}{}_{\nu}a^{\nu}{}_{\mu}\gamma^{\mu} = \hat{P}a^{\sigma}{}_{\nu}\gamma^{\nu}\hat{P}^{-1}$$

This is equivalent to

$$\delta^{\sigma}_{\mu}\gamma^{\mu} = \hat{P}(\sum_{\nu=0}^{3}\eta^{\sigma\nu}\gamma^{\nu})\hat{P}^{-1} \tag{2.213}$$

which in turn is equivalent to

$$\hat{P}^{-1}\gamma^{\sigma}\hat{P} = \eta^{\sigma\sigma}\gamma^{\sigma}. \tag{2.214}$$

This has the simple solution

$$\hat{P} = e^{i\varphi}\gamma^0, \qquad \hat{P}^{-1} = e^{-i\varphi}\gamma^0 \tag{2.215}$$

For this operator we easily have

$$\hat{P}\gamma_5 = -\gamma_5\hat{P}. \tag{2.216}$$

$\square$

i) $\overline{\psi}\psi$ is a scalar:

$$\begin{aligned}\overline{\psi}\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\hat{S}\psi \\ &= \overline{\psi}\psi\end{aligned}$$

ii) $\overline{\psi}\gamma_5\psi$ is a pseudoscalar:

$$\begin{aligned}\overline{\psi}\gamma_5\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}\psi \\ &= det|a|\ \overline{\psi}\hat{S}^{-1}\hat{S}\gamma_5\psi \\ &= det|a|\ \overline{\psi}\gamma_5\psi\end{aligned}$$

iii) $\overline{\psi}\gamma^\mu\psi$ is a vector:

$$\begin{aligned}\overline{\psi}\gamma^\mu\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma^\mu\hat{S}\psi \\ &= a^\mu{}_\nu\overline{\psi}\gamma^\nu\psi\end{aligned}$$

iv) $\overline{\psi}\gamma_5\gamma^\mu\psi$ is a pseudovector:

$$\begin{aligned}\overline{\psi}\gamma_5\gamma^\mu\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma_5\gamma^\mu\hat{S}\psi \\ &= \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}(\hat{S}^{-1}\gamma^\mu\hat{S})\psi \\ &= \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}(a^\mu{}_\nu\gamma^\nu)\psi \\ &= det|a|a^\mu{}_\nu\ \overline{\psi}\gamma^\nu\psi \\ &= det|a|a^\mu{}_\nu\ \overline{\psi}\gamma^\nu\psi\end{aligned}$$

v) $\overline{\psi}\hat{\sigma}^{\mu\nu}\psi$ is a pseudovector:

$$
\begin{aligned}
\overline{\psi}\hat{\sigma}^{\mu\nu}\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\hat{\sigma}^{\mu\nu}\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\hat{S}^{-1}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\hat{S}^{-1}(\gamma^{\mu}\hat{S}\hat{S}^{-1}\gamma^{\nu} - \gamma^{\nu}\hat{S}\hat{S}^{-1}\gamma^{\mu})\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\{(a^{\mu}{}_{\rho}\gamma^{\rho})(a^{\nu}{}_{\tau}\gamma^{\tau}) - (a^{\mu}{}_{\rho}\gamma^{\rho})(a^{\nu}{}_{\tau}\gamma^{\tau})\}\psi \\
&= a^{\mu}{}_{\rho}a^{\nu}{}_{\tau}\,\overline{\psi}\frac{i}{2}(\gamma^{\rho}\gamma^{\tau} - \gamma^{\tau}\gamma^{\rho})\psi \\
&= a^{\mu}{}_{\rho}a^{\nu}{}_{\tau}\,\overline{\psi}\hat{\sigma}^{\rho\tau}\psi.
\end{aligned}
$$

□

2.2.14 Properties of Free Solutions

$$
(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})\omega^{r}(\mathbf{p}) = 0 \tag{2.217}
$$

$$
\begin{aligned}
[(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})\omega^{r}(\mathbf{p})]^{\dagger} &= 0 = \omega^{r\dagger}(\mathbf{p})(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})^{\dagger} \\
&= \omega^{r\dagger}(\mathbf{p})(p_{\mu}\gamma^{\mu\dagger} - \epsilon_{r}m_{0}) \\
&= \omega^{r\dagger}(\mathbf{p})(p_{0}\gamma^{0} - p_{k}\gamma^{k} - \epsilon_{r}m_{0})
\end{aligned}
$$

Multiplication from the right by γ^0 yields

$$
\begin{aligned}
\omega^{r\dagger}(\mathbf{p})(p_{\mu}\gamma^{\mu\dagger} - \epsilon_{r}m_{0})\gamma^{0} &= 0 = \omega^{r\dagger}(\mathbf{p})\gamma^{0}(p_{0}\gamma^{0} + p_{k}\gamma^{k} - \epsilon_{r}m_{0}) \\
&= \overline{\omega}^{r}(\mathbf{p})(p_{\mu}\gamma^{\mu} - \epsilon_{r}m_{0})
\end{aligned}
$$

The normalisation condition

The quantity $\overline{\omega}^{r}(\mathbf{p})\omega^{r'}(\mathbf{p})$ is a Lorentz scalar and hence

$$
\overline{\omega}^{r}(\mathbf{p})\omega^{r'}(\mathbf{p}) = \overline{\omega}^{r}(0)\omega^{r'}(0) = \omega^{r\dagger}(0)\gamma^{0}\omega^{r'}(0) = \delta_{rr'}\epsilon_{r}. \tag{2.218}
$$

The completenes relation

We have

$$\omega^{r\dagger}(\epsilon_r \mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \delta_{rr'}(E/m_0)$$

This is proved in section 2.14.6.

The closure relation

In the rest frame of the electron we have

$$\sum_{r=1}^{4} \epsilon_r \omega^r(0)\overline{w}^r_\beta(0) = \delta_{\alpha\beta} \tag{2.219}$$

We know that

$$\omega^r(p) = \hat{S}\left(\frac{-\mathbf{p}}{E}\right) w^r(0)$$

and so

$$\begin{aligned}
\overline{\omega}^r(p) &= \omega^{r\dagger}(p)\gamma^0 \\
&= \left(\hat{S}\left(\frac{-\mathbf{p}}{E}\right)\omega^r(0)\right)^\dagger \gamma^0 \\
&= w^{r\dagger}(0)\gamma^0\gamma^0\hat{S}^\dagger\left(\frac{-\mathbf{p}}{E}\right)\gamma^0 \\
&= \overline{\omega}^r(0)\hat{S}^{-1}\left(\frac{-\mathbf{p}}{E}\right).
\end{aligned} \tag{2.220}$$

where we have used

$$\hat{S}^\dagger = \gamma^0\hat{S}^{-1}\gamma^0.$$

Using these we find

$$
\begin{aligned}
\sum_{r=1}^{4} \epsilon_r \omega_\alpha^r(p) \overline{\omega}_\beta^r(p) &= \sum_{r=1}^{4} \sum_{\gamma,\lambda=1}^{4} \epsilon_r \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E}\right) \omega_\gamma^r(0) \, \overline{\omega}_\lambda^r(0) \hat{S}_{\lambda\beta}^{-1} \left(\frac{-\mathbf{p}}{E}\right) \\
&= \sum_{\gamma,\lambda=1}^{4} \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E}\right) \hat{S}_{\lambda\beta} \left(\frac{-\mathbf{p}}{E}\right) \sum_{r=1}^{4} \epsilon_r \omega_\gamma^r(0) \overline{\omega}_\lambda^r(0) \\
&= \sum_{\gamma,\lambda=1}^{4} \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E}\right) \hat{S}_{\lambda\beta} \left(\frac{-\mathbf{p}}{E}\right) \delta_{\gamma\lambda} \\
&= \delta_{\alpha\beta}
\end{aligned} \tag{2.221}
$$

Therefore we have the closure relation

$$
\sum_{r=1}^{4} \epsilon_r \omega_\alpha^r(p) \overline{\omega}_\beta^r(p) = \delta_{\alpha\beta}. \tag{2.222}
$$

2.2.15 Projection Operators for Energy and Spin

Projection Operators for Energy

Recall

$$
(p_\mu \gamma^\mu - \epsilon_r m_0 c) w^r(\mathbf{p}) \quad \text{which implies} \quad \epsilon_r p_\mu \gamma^\mu w^r(\mathbf{p}) = m_0 c w^r(\mathbf{p})
$$

We immediately see that the projection operator for eigenstates with positive or negative energy is given by

$$
\hat{\Lambda}_r(p) = \frac{\epsilon_r p_\mu \gamma^\mu + m_0}{2m_0} \tag{2.223}
$$

and that it is Lorentz covariant. We check that it has all the properties of a projection operator. Obviously

$$
\hat{\Lambda}_+(p) + \hat{\Lambda}_-(p) = \frac{+p_\mu \gamma^\mu + m_0}{2m_0} + \frac{-p_\mu \gamma^\mu + m_0}{2m_0 c} = 1
$$

We now establish $(\hat{\Lambda}_+)^2 = \hat{\Lambda}_+$, $(\hat{\Lambda}_-)^2 = \hat{\Lambda}_-$, and $\hat{\Lambda}_+ \hat{\Lambda}_- = 0$. This is done with the help of

$$
\begin{aligned}
p^{\mu}\gamma_{\mu}p^{\nu}\gamma_{\nu} &= \frac{1}{2}(\gamma_{\mu}\gamma_{\nu}+\gamma_{\nu}\gamma_{\mu})p^{\mu}p^{\nu} \\
&= \eta_{\mu\nu}p^{\mu}p^{\nu} \\
&= E^2-\mathbf{p}^2 \\
&= (m_0^2+\mathbf{p}^2)-\mathbf{p}^2 = m_0^2. \qquad (2.224)
\end{aligned}
$$

Now

$$
\begin{aligned}
\hat{\Lambda}_r(p)\hat{\Lambda}_{r'}(p) &= \frac{(\epsilon_r p^{\mu}\gamma_{\mu}+m_0)(\epsilon_{r'}p^{\mu}\gamma_{\mu}+m_0)}{4m_0^2} \\
&= \frac{(\epsilon_r\epsilon_{r'}p^{\mu}p^{\nu}\gamma_{\mu}\gamma_{\nu}+m_0^2+(\epsilon_r+\epsilon_{r'})m_0p^{\mu}\gamma_{\mu}}{4m_0^2} \\
&= \frac{m_0^2(1+\epsilon_r\epsilon_{r'})+m_0p^{\mu}\gamma_{\mu}\epsilon_r(1+\epsilon_r\epsilon_{r'})}{4m_0^2} \\
&= \frac{1+\epsilon_r\epsilon_{r'}}{2}\frac{\epsilon_r p_{\mu}\gamma^{\mu}+m_0}{2m_0} = \frac{1+\epsilon_r\epsilon_{r'}}{2}\hat{\Lambda}_r(p). \qquad (2.225)
\end{aligned}
$$

Projection Operators for Spin

In the non-relativistic limit the operator for “spin up” or “spin down”

$$\hat{P}_{\pm} = \frac{1\pm\hat{\sigma}_3}{2}$$

We can generalise this to a spin-projection operator in an arbitrary direction

$$\hat{P}(\mathbf{u}) = \frac{1+\hat{\sigma}\cdot\mathbf{u}}{2} \qquad (2.226)$$

where $\mathbf{u}$ is a unit vector. We need the relativistic generalisation of this. To that end introduce the four-vector

$$u_z^{\nu} \qquad (2.227)$$

which in the rest system of the electron is

$$(u_z^{\nu})_{R.S.} = (0,0,0,1) = (0,0,0,\mathbf{u}_z) \qquad (2.228)$$

$$
\begin{aligned}
\gamma_5\gamma_3(u_z^3)_{R.S.} &= \gamma_5\gamma_3 \\
&= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma_3 \\
&= i\gamma^0\gamma^1\gamma^2 \\
&= i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}
\end{aligned}
\tag{2.229}
$$

In the rest frame we have for positive-energy $\omega^{1,2}(0)$

$$
\begin{aligned}
\hat{\Sigma}(u_z^3)\omega^{1,2}(0) &= \frac{1+\gamma_5\gamma^3(u_z^3)_{R.S.}}{2}\omega^{1,2}(0) \\
&= \frac{1}{2}\begin{pmatrix} I+\sigma_3 & 0 \\ 0 & I-\sigma_3 \end{pmatrix}\omega^{1,2}(0) \\
&= \frac{1}{2}\begin{cases} 1\cdot\omega^1(0) \\ 0\cdot\omega^2(0) \end{cases} .
\end{aligned}
\tag{2.230}
$$

In the rest frame we have for negitive-energy

$$
\begin{aligned}
\hat{\Sigma}(u_z^3)\omega^{3,4}(0) &= \frac{1+\gamma_5\gamma^3(u_z^3)_{R.S.}}{2}\omega^{3,4}(0) \\
&= \frac{1}{2}\begin{pmatrix} I+\sigma_3 & 0 \\ 0 & I-\sigma_3 \end{pmatrix}\omega^{3,4}(0) \\
&= \frac{1}{2}\begin{cases} 0\cdot\omega^3(0) \\ 1\cdot\omega^4(0) \end{cases}
\end{aligned}
\tag{2.231}
$$

The projection of negative-energy states arre opposite to those of positive-energy states. The opposite occurs because the spin of the missing particle of spin $\uparrow$ corresponds to a particle of spin $\downarrow$.

We generalise the spin projection operator for an arbitrary spin vector s^μ with $s^\mu p_\mu = 0$:

$$
\hat{\Sigma}(s) = \frac{1}{2}(1+\gamma_5 s^\mu\gamma_\mu) \tag{2.232}
$$

We show that it is a true projection operator. We have

$$
\hat{\Sigma}(s) + \hat{\Sigma}(-s) = 1 \tag{2.233}
$$

$$
\begin{aligned}
\hat{\Sigma}^2(s) &= \frac{1}{4}(1+\gamma_5 s^\mu\gamma_\mu)(1+\gamma_5 s^\nu\gamma_\nu) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu + s^\mu s^\nu\gamma_5\gamma_\mu\gamma_5\gamma_\nu) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu - s^\mu s^\nu\gamma_5^2\gamma_\mu\gamma_\nu) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu - s^\mu s^\nu\gamma_5^2\frac{1}{2}(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)) \\
&= \frac{1}{4}(1+2\gamma_5 s^\mu\gamma_\mu - s\cdot s) \\
&= \frac{1}{2}(1+\gamma_5 s^\mu\gamma_\mu) = \hat{\Sigma}(s) \qquad (2.234)
\end{aligned}
$$

Similarly $\hat{\Sigma}^2(-s) = \hat{\Sigma}(-s)$.

$$
\begin{aligned}
\hat{\Sigma}(s)\hat{\Sigma}(-s) &= \frac{1}{4}(1+\gamma_5 s^\mu\gamma_\mu)(1-\gamma_5 s^\nu\gamma_\nu) \\
&= \frac{1}{4}(1+s\cdot s) = 0. \qquad (2.235)
\end{aligned}
$$

Simjultaneous Projection Operators for Energy and Spin

$$
\begin{aligned}
p_\mu\gamma^\mu\gamma_5\gamma^\nu s_\nu &= p_\mu\gamma^\mu i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\nu s_\nu \\
&= -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^\nu s_\nu p_\mu \\
&= -\gamma_5(2\eta^{\mu\nu}-\gamma^\nu\gamma^\mu)s_\nu p_\mu \\
&= s^\mu p_\mu\gamma_5 + s_\nu p_\mu\gamma_5\gamma^\nu\gamma^\mu \\
&= \gamma_5 s_\nu\gamma^\nu p_\mu\gamma^\mu \qquad (2.236)
\end{aligned}
$$

implies

$$
\left[\hat{\Sigma}(s), \hat{\Lambda}_\pm(p)\right] = 0 \qquad (2.237)
$$

$$
\begin{aligned}
\hat{P}_1 &= \hat{\Lambda}_+(p)\hat{\Sigma}(u_z) \\
\hat{P}_2 &= \hat{\Lambda}_+(p)\hat{\Sigma}(-u_z) \\
\hat{P}_3 &= \hat{\Lambda}_-(p)\hat{\Sigma}(u_z) \\
\hat{P}_4 &= \hat{\Lambda}_-(p)\hat{\Sigma}(-u_z) \qquad (2.238)
\end{aligned}
$$

2.2.16 Summary

Maxwell's equations with source are

$$\Box A^{\mu} - \partial^{\mu}(\partial_{\nu} A^{\nu}) = j^{\mu} \tag{2.239}$$

where we are free to perform gauge transformations

$$A^{\mu} \rightarrow A^{\mu'} = A^{\mu} + \partial^{\mu}\Lambda. \tag{2.240}$$

The free Dirac equation can be written

$$i\hbar\frac{\partial\psi(x)}{\partial t} = [\alpha \cdot (-i\hbar\nabla) + \beta m_0]\psi(x) \tag{2.241}$$

where $= (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ and $\hat{\beta}$ are 4×4 Hermitian matrices satisfying

$$\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = 2\delta_{ij}, \quad \hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i = 0, \quad \beta^2 = 1, \quad i = 1, 2, 3. \tag{2.242}$$

With

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i \tag{2.243}$$

Dirac's equation becomes

$$i\hbar\gamma^{\mu}\frac{\partial\psi(x)}{\partial x^{\mu}} - m_0\psi(x) = 0 \tag{2.244}$$

with the 4×4 matrices γ^{μ}, $\mu = 0, \ldots, 3$, satifying the anticommutation relations

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} \tag{2.245}$$

and Hermiticity conditions

$$\gamma^{\mu\dagger} = \gamma^0\gamma^{\mu}\gamma^0. \tag{2.246}$$

Coupling of a Spinor to the electromagnetic field is given by

$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu(x) - m_0)\psi(x) = 0. \tag{2.247}$$

The four current density of the dirac field is given by

$$j^\mu = \overline{\psi}\gamma^\mu\psi \tag{2.248}$$

The plane wave for a photon is

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{2.249}$$

Plane waves of electrons: incoming

$$\psi(x) = \sqrt{\frac{m_0}{EV}}\, u(p,s)e^{-ip\cdot x} \tag{2.250}$$

and outgoing

$$\overline{\psi}(x) = \sqrt{\frac{m_0}{EV}}\, \overline{u}(p,s)e^{ip\cdot x}. \tag{2.251}$$

The general free solution has the form

$$\psi^r(x) = \omega^r(p)e^{-i\epsilon_r p_\mu x^\mu} \tag{2.252}$$

where

$$[\omega^1(p),\omega^2(p),\omega^3(p),\omega^4(p)] = \sqrt{\frac{E+m_0}{2m_0}}\begin{bmatrix} 1 & 0 & \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} \\ 0 & 1 & \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} \\ \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} & 1 & 0 \\ \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} & 0 & 1 \end{bmatrix} \tag{2.253}$$

The orthogonality condition for spinors

$$\omega^{r\dagger}(\epsilon_r p)\omega^{r'}(\epsilon_r{}' p) = \frac{E_p}{m_0}\delta_{rr'} \tag{2.254}$$

The completeness relation for spinors

$$\sum_{r=1}^{4} \epsilon_r \omega^r(p) \overline{\omega}^r_\alpha(p) = \delta_{\alpha\beta} \tag{2.255}$$

There are projection operators for Energy

$$\hat{\Lambda}_\pm(p) = \frac{\pm p_\mu \gamma^\mu + m_0}{2m_0} \tag{2.256}$$

and spin

$$\hat{\Sigma} = \frac{1}{2}(1 + \gamma_5 s_\mu \gamma^\mu) \tag{2.257}$$

such that

$$\hat{\Sigma} u(p, +s) = u(p, +s), \quad \hat{\Sigma} u(p, -s) = 0. \tag{2.258}$$

The basic bilinear covariants of Dirac theory are

$$\begin{array}{rl} \overline{\psi}\psi & \text{scalar} \\ \overline{\psi}\gamma^\mu\psi & \text{vector} \\ \overline{\psi}\sigma^{\mu\nu}\psi & \text{antisymmetric second-rank tensor} \\ \overline{\psi}\gamma^5\gamma^\mu\psi & \text{pseudo-vector} \\ \overline{\psi}\gamma^5\psi & \text{pseudo-scalar} \end{array} \tag{2.259}$$

2.3 Perturbation Theory

2.3.1 Non-Relativistic Green's Function

Given Schrodinger's equation

$$\left(i\hbar \frac{\partial}{\partial t} - \hat{H}_0(x) - V(x) \right) \psi(x) = 0 \tag{2.260}$$

The retarded Green's function is defined by the differential equation

$$\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0(x') - V(x')\right) G^+(x';x) = \delta^4(x'-x) \tag{2.261}$$

and the boundary condition

$$G^+(x';x) = 0 \quad \text{for } t' < t. \tag{2.262}$$

Free Green's function in momentum space

$$\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0(x')\right) G^+(x';x) = \delta^4(x'-x) \tag{2.263}$$

where

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\nabla'^2 \tag{2.264}$$

As the above differential equation can be turned into an algebraic equation in energy-momentum space, we write

$$G_0^+(x'-x) = \int \frac{d^3p dE}{(2\pi\hbar)^4} \exp\left[-\frac{i}{\hbar}E(t'-t)\right] \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right] \; G_0^+(p;E) \tag{2.265}$$

and apply

$$\begin{aligned} & \left(i\hbar\frac{\partial}{\partial t'} + \frac{\hbar^2}{2m}\nabla'^2\right) G_0^+(x'-x) \\ = & \int \frac{d^3p dE}{(2\pi\hbar)^4} \left\{\left(E - \frac{\mathbf{p}^2}{2m}\right) G_0^+(p;E)\right\} \exp\left[-\frac{i}{\hbar}E(t'-t)\right] \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right] \\ = & \; \hbar\delta^4(x'-x) \end{aligned} \tag{2.266}$$

We will recall the δ-function integral representation

$$\int \frac{d^3p dE}{(2\pi\hbar)^4} \exp\left[-\frac{i}{\hbar}E(t'-t)\right] \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right] = \delta^4(x'-x)$$

Therefore for $E \neq \mathbf{p}^2/2m$ we obtain

$$G_0^+(\mathbf{p}, E) = \frac{\hbar}{E - \frac{\mathbf{p}^2}{2m}} \tag{2.267}$$

How do we deal with the singularity when we do the inverse Fourier transformation? The clue how to proceed comes from the integral representation of the step function:

$$\Theta(\tau) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon}. \tag{2.268}$$

By adding a small imaginary part $i\epsilon$ to the energy one will obtain the retardation condition (2.262), while the resulting Green's function still satisfies the Green's function differential equation (2.263) in the limit $\epsilon \to 0$. Write

$$G_0^+(x' - x) = \hbar \int \frac{d^3p}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar}\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})\right] \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \frac{\exp\left[-iE(t' - t)/\hbar\right]}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon} \tag{2.269}$$

With the substitution $E' = E - \mathbf{p}^2/2m$ the last integral becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dE'}{2\pi\hbar} \frac{\exp[-i(E' + \mathbf{p}^2/2m)(t' - t)/\hbar]}{E' + i\epsilon} \\
&= \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \int_{-\infty}^{\infty} \frac{dE'}{2\pi\hbar} \frac{\exp[-iE'(t' - t)/\hbar]}{E' + i\epsilon} \\
&= \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \frac{-i}{\hbar} \cdot \frac{-1}{2\pi i} \int_{-\infty}^{\infty} dE' \frac{\exp[-iE'(t' - t)/\hbar]}{E' + i\epsilon} \\
&= \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \left[-\frac{i}{\hbar}\Theta\left(\frac{t' - t}{\hbar}\right)\right] \\
&= -\frac{i}{\hbar} \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t' - t)\right] \Theta(t' - t)
\end{aligned} \tag{2.270}$$

We do indeed recover the retardation condition. Now (2.269) becomes

$$G_0^+(x' - x) = -i\Theta(t' - t) \int \frac{d^3p}{(2\pi\hbar)^3} \exp\left\{\frac{i}{\hbar}\left[\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x}) - \frac{\mathbf{p}^2}{2m}(t' - t)\right]\right\} \tag{2.271}$$

This can be expressed in terms of plane waves of the free Schrodinger's equation. The δ-function normalised plane waves are

$$
\begin{aligned}
\phi_p(\mathbf{x},t) &= \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{i}{\hbar}\left((\mathbf{p}\cdot\mathbf{x} - \frac{\mathbf{p}^2}{2m}t\right)\right] \\
&= \frac{1}{\sqrt{2\pi\hbar}} \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] \qquad (2.272)
\end{aligned}
$$

where

$$
\hbar\omega = \frac{\mathbf{p}^2}{2m}, \quad \hbar\mathbf{k} = \mathbf{p}.
$$

$$
G_0^+(x'-x) = -i\Theta(t'-t)\int d^3p\, \phi_p(\mathbf{x}',t')\phi_p^*(\mathbf{x},t) \qquad (2.273)
$$

Full Green's function in terms of plane waves

We give a proof that the Green's function can be written as

$$
G^+(x';x) = -i\Theta(t'-t)\sum_n \psi_n^*(x)\psi_n(x'). \qquad (2.274)
$$

where $\psi_n(\mathbf{x},t)$ are a complete set of eigenfunctions of Schrodinger's equation. We do this using the closure relation

$$
\sum_n \psi_n^*(\mathbf{x},t)\psi_n(\mathbf{x}',t) = \delta^3(\mathbf{x}'-\mathbf{x}). \qquad (2.275)
$$

for the eigenfunctions of Schrodinger's equation

$$
\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)\psi_n(x') = 0. \qquad (2.276)
$$

Using this we have

$$
\begin{aligned}
\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)G^+(x';x) &= \hbar\delta(t'-t)\sum_n \psi_n^*(\mathbf{x},t)\psi_n(\mathbf{x}',t) \\
&\quad -i\Theta(t'-t)\sum_n\left[\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)\psi_n(x')\right]\psi_n^*(x) \\
&= \hbar\delta(t'-t)\delta^3(\mathbf{x}'-\mathbf{x}) \\
&= \hbar\delta^4(x'-x). \qquad (2.277)
\end{aligned}
$$

We have the relations describing the evolution of solutions of Schrodinger's equation:

$$\begin{aligned} i\int d^3x G^+(x';x)\psi_n(x) &= \Theta(t'-t)\sum_m \psi_m(x')\int d^3x \psi_m^*(x)\psi_n(x) \\ &= \Theta(t'-t)\psi_n(x') \end{aligned} \tag{2.278}$$

and on the other hand

$$\begin{aligned} i\int d^3x \psi_n^*(x')G^+(x';x) &= \Theta(t'-t)\sum_m \int d^3x' \psi_m(x')\psi_n^*(x')\ \psi_m^*(x) \\ &= \Theta(t'-t)\psi_n^*(x) \end{aligned} \tag{2.279}$$

The first of these relations expresses the propagation of $\psi_n(x)$ forward in time and the second corresponding backward propagation of $\psi_n^*(x')$.

Perturbation theory

$$\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0\right) G^+(x';x) = \delta^4(x'-x) + V(x')G^+(x';x) \tag{2.280}$$

The RHS can be interpreted as the source term in an inhomogeneous Schrodinger equation:

$$\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0(x')\right) G^+(x';x) = \rho(x';x) \tag{2.281}$$

Using the free Green's function G_0 the solution is

$$G^+(x';x) = \int d^4x_1 G_0^+(x';x_1)\rho(x_1;x) \tag{2.282}$$

This leads to the following integral equation for the integrating Green's function

$$\begin{aligned} G^+(x';x) &= \int d^4x_1 G_0^+(x';x_1)\big(\delta^4(x_1-x) + V(x_1)G^+(x_1;x)\big) \\ &= G_0^+(x';x) + \int d^4x_1 G_0^+(x';x_1)V(x_1)G^+(x_1;x) \end{aligned} \tag{2.283}$$

Repeatedly substituting this equation into itself we obtain perturbative series

$$\begin{aligned}
G^+(x';x) &= G_0^+(x';x) + \int d^4x_1 G_0^+(x';x_1)V(x_1)G^+(x_1;x) \\
&= G_0^+(x';x) + \int d^4x_1 G_0^+(x';x_1)V(x_1)G_0^+(x_1;x) \\
&\quad + \int d^4x_1 d^4x_2 G_0^+(x';x_1)V(x_1)G_0^+(x_1;x_2)V(x_2)G_0^+(x_2;x) \\
&\quad + \dots \quad . \qquad (2.284)
\end{aligned}$$

Boundary condition

$$\begin{aligned}
\psi(x') &= \lim_{t\to-\infty} i \int d^3x G^+(x';x)\phi(x) \\
&= \lim_{t\to-\infty} i \int d^3x \left(G_0^+(x';x) + \int d^4x_1 G_0^+(x';x_1)V(x_1)G^+(x_1;x) \right) \phi(x) \\
&= \phi(x') + \lim_{t\to-\infty} \int d^4x_1 G_0^+(x';x_1)V(x_1) i \int d^3x G^+(x_1;x)\phi(x) \\
&= \phi(x') + \lim_{t\to-\infty} \int d^4x_1 G_0^+(x';x_1)V(x_1)\psi(x_1) \qquad (2.285)
\end{aligned}$$

The second the on the RHS is the scattered wave.

We consider a scattering problem where no interaction occurs in the distant past and future:

$$V(\mathbf{x},t) \to 0 \quad \text{for} \quad t \to \mp\infty \qquad (2.286)$$

The initial wave ϕ is therefore a solution of the Schrodinger equation for free particles, which fulfills the initial conditions of the experiment. The exact wavefunction $\psi(\mathbf{x},t)$ then approaches the incoming wave $\phi(\mathbf{x},t)$ in the limit $t \to -\infty$:

$$\psi(\mathbf{x},t) \to \phi(\mathbf{x},t). \qquad (2.287)$$

The scattering matrix

Let $\phi_i(x)$ and $\phi_f(x)$ denote the intial and final free wave with quantum numbers i and f that are emitted, observed at the begining, end of the scattering process respectively. The full wavefunction $\psi_i(x)$ is given in terms the integral equation,

$$\psi_i(x) = \phi_i(x) + \int d^4x_1 G_0^+(x, x_1) V(x_1) \psi_i(x_1) \tag{2.288}$$

The wavefunction $\psi_i(x)$ satifies the boundary condition $\psi_i(\mathbf{x}, t) \to \phi_i(\mathbf{x}, t)$ for $t \to -\infty$. The scattering matrix results from the projection of $\psi_i(x)$ on the final state $\phi_f(\mathbf{x}, t)$

$$S_{fi} = \lim_{t\to+\infty} \Big\langle \phi_f(x) \Big| \psi_i(x) \Big\rangle \tag{2.289}$$

$$\begin{aligned} S_{fi} &= \lim_{t\to+\infty} \Big\langle \phi_f(x) \Big| \phi_i(x) + \int d^4x_1 G_0^+(x, x_1) V(x_1) \psi(x_1) \Big\rangle \\ &= \delta_{fi} + \lim_{t\to+\infty} \int d^3x \phi_f^*(x) \int d^4x_1 G_0^+(x, x_1) V(x_1) \psi(x_1) \\ &= \delta_{fi} + \lim_{t\to+\infty} \int d^4x_1 \left(\int d^3x \phi_f^*(x) G_0^+(x, x_1) \right) V(x_1) \psi(x_1) \end{aligned} \tag{2.290}$$

Using the equations (2.279) for free particles we obtain

$$\int d^3x \phi_f^*(x) G_0^+(x'; x_1) = -i\phi_f(x_1) \quad \text{for } t' > t_1, \tag{2.291}$$

so the x integral can be carried out resulting in

$$S_{fi} = \delta_{fi} - i \lim_{t\to+\infty} \int d^4x_1 \phi_f^*(x_1) V(x_1) \psi(x_1) \tag{2.292}$$

Now repeatedly substituting (2.288) into this we obtain

$$\begin{aligned} S_{fi} = \ & \delta_{fi} - i \int d^4x_1 \phi_f^*(x_1) V(x_1) \phi_i(x_1) \\ & -i \int d^4x_1 d^4x_2 \phi_f^*(x_1) V(x_1) G_0^+(x_1; x_2) V(x_2) \phi_i(x_2) \\ & -i \int d^4x_1 d^4x_2 \phi_f^*(x_1) V(x_1) G_0^+(x_1; x_2) V(x_2) G_0^+(x_2; x_3) V(x_3) \phi_i(x_3) \\ & + \ldots \quad . \end{aligned} \tag{2.293}$$

Each line represents a free Green's function $G_0^+(x_i; x_{i-1})$, i.e. the amplitude that a particle wave originating at the spacetime point x_{i-1} and propagates freely to the spacetime point x_i. At the point x_i the particle wave is scattered with probability amplitude $V(x_i)$ per

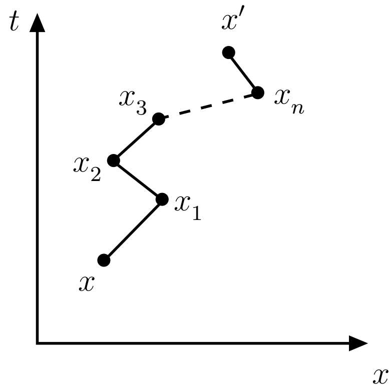


Figure 2.1: nth-order Green's function as the probability amplitude for multiple scattering.

unit spacetime volume. Such points are called interaction vertices and are denoted by filled-in circles. The resulting scattered wave then again propagates freely foward in time (recall $G_0^+(x_{i+1}; x_i) = 0$ for $t_{i+1} < t_i$) from the spacetime point x_i towards the point x_{i+1} with the amplitude $G_0^+(x_{i+1}; x_i)$ where the next interaction takes place, and so on.

2.3.2 The Electron and Positron Propagator

Differential equation for relativistic propagator

Let us introduce the relativistic propagator

$$S_F(x', x; A) \tag{2.294}$$

in analogy to the nonrelativistic propagator, by satisfying the following differential equation

$$\left[\gamma_\mu \left(i\hbar \frac{\partial}{\partial x'_\mu} - \frac{e}{c} A^\mu(x') \right) - m_0 \right] S_F(x', x; A) = \hbar \delta^4(x' - x) \mathbf{1}. \tag{2.295}$$

We from now on use natural units

$$\frac{e}{\hbar c} \to e, \quad \frac{m_0 c}{\hbar} \to m_0. \tag{2.296}$$

Thus we write

$$\left[\gamma_\mu \left(i\partial'^\mu - eA^\mu(x') \right) - m_0 \right] S_F(x', x; A) = \delta^4(x' - x). \tag{2.297}$$

where we have suppressed the unit matrix, however, it must be kept in mind that we are dealing with a matrix equation.

The free-particle propagator satisfies (2.297) with the interaction term $\gamma_\mu A^\mu(x')$ absent, i.e.

$$(i\gamma_\mu \partial'^\mu - m_0) S_F(x', x) = \delta^4(x' - x). \tag{2.298}$$

As in the non-relativistic case we calculate $S_F(x', x)$ in momentum space.

Non-interacting propagator in momentum space

$$S_F(x', x) = S_F(x' - x) = \int \frac{d^4p}{(2\pi)^4} \exp[-ip \cdot (x' - x)]\ S_F(p) \tag{2.299}$$

which implies that

$$(p^\mu \gamma_\mu + m_0) S_F(p) = \mathbf{1} \tag{2.300}$$

This can be solved for $S_F(p)$ by multiplying by $(p^\mu \gamma_\mu + m_0)$ from the left

$$(p^\mu \gamma_\mu + m_0)(p^\nu \gamma_\nu - m_0) S_F(p) = (p^\mu \gamma_\mu + m_0) \tag{2.301}$$

Since

$$p^\mu \gamma_\mu p^\nu \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) p^\mu p^\nu = \eta_{\mu\nu} p^\mu p^\nu = p_\mu p^\mu = p^2$$

we then have

$$(p^2 - m_0^2) S_F(p) = p^\mu \gamma_\mu + m_0$$

so

$$S_F(p) = \frac{p^\mu \gamma_\mu + m_0}{p^2 - m_0^2} \quad \text{for } p^2 \neq m_0^2 \tag{2.302}$$

Let us consider the inverse Fourier transformation.

$$
\begin{aligned}
S_F(x'-x) &= \int \frac{d^4}{(2\pi)^4} S_F(p) \exp[-ip\cdot(x'-x)] \\
&= \int \frac{d^4}{(2\pi)^4} S_F(p) \exp\{[-ip_0\cdot(t'-t) - \mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\} \\
&= \int \frac{d^3}{(2\pi)^3} S_F(p) \exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})] \times \int_C \frac{dp_0}{2\pi} \frac{\exp[-ip_0\cdot(t'-t)]}{p^2-m_0^2}
\end{aligned}
\tag{2.303}
$$

where C is contour of integration choosen to avoid the singularities of $S_F(p)$. As we know from the nonrelativistic case the choice of contour encodes the boundary conditions imposed on $S_F(x'-x)$.

Propagator describing positive-energy particle waves

Considering the particle's propagation forward in time implies that $t'-t$ is positive so that the p_0 integration must be performed along the contour closed in the lower half plane as this gives zanishing contribution. Then the only pole is at

$$
p_0 = +E_p = +\sqrt{\mathbf{p}^2 + m_0^2}.
$$

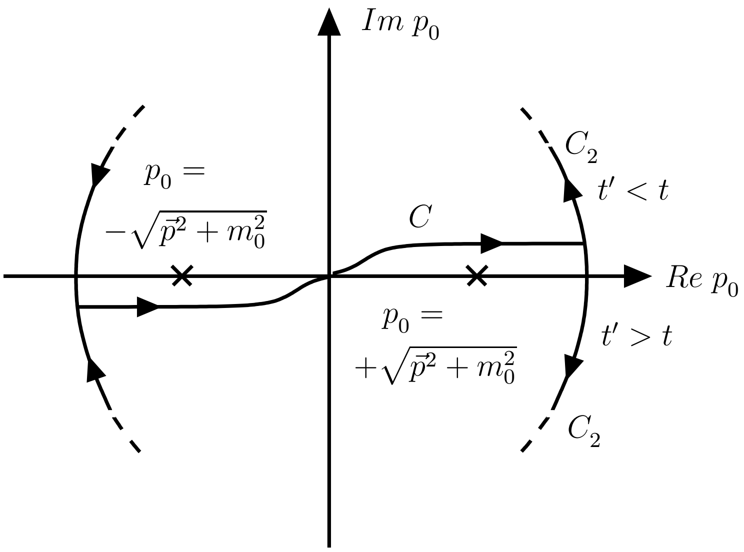


Figure 2.2:

The propagator is then

$$S_F^{(t'>t)}(x'-x) = -i\int \frac{d^3p}{(2\pi)^3}\exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\exp[-iE_p(t'-t)] \times\frac{(E_p\gamma^0-\mathbf{p}\cdot\gamma+m_0)}{2E_p} \qquad \text{for} \quad t'>t. \tag{2.304}$$

Instead of deforming the contour as in fig (2.3.2), we can move the poles an infintesimal distance η off the real axis, as shown in fig (), and perform the p_0 integration alng the whole real axis

Propagator describing negative-energy particle waves

On the other hand, considering the particle's propagation backward in time implies that $t'-t$ is negative so that the p_0 integration must be performed along the contour closed in the upper half plane as this gives zanishing contribution. Then the only pole is at

$$p_0 = -E_p = -\sqrt{\mathbf{p}^2+m_0^2}.$$

$$S_F^{(t>t')}(x'-x) = -i\int \frac{d^3p}{(2\pi)^3}\exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\exp[+iE_p(t'-t)] \times\frac{(-E_p\gamma^0-\mathbf{p}\cdot\gamma+m_0)}{2E_p} \qquad \text{for} \quad t'<t. \tag{2.305}$$

2.3.3 Propagating Positive and Negative Particles

We combine the two propagators describing positive-energy particle waves and negative-energy particle waves moving forward and backward in time, respectively.

$$S_F(x'-x) = S_F^{(t'>t)}(x'-x) + S_F^{(t>t')}(x'-x) \tag{2.306}$$

$$
\begin{aligned}
S_F(x'-x) &= -i\int \frac{d^3p}{(2\pi)^3} \\
&\quad \Bigg\{\exp[-i(+E_p)(t'-t)]\exp[+i\vec{p}\cdot(\vec{x}'-\vec{x})]\frac{(+E_p\gamma^0+p_i\gamma^i+m_0)}{2E_p}\,\Theta(t'-t) \\
&\quad +\exp[-i(-E_p)(t'-t)]\exp[-i\vec{p}\cdot(\vec{x}'-\vec{x})]\frac{(-E_p\gamma^0-p_i\gamma^i+m_0)}{2E_p}\Theta(t-t')\Bigg\} \\
&= -i\int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\Bigg\{\frac{p_\mu\gamma^\mu+m_0}{2m_0}\exp[-ip\cdot(x'-x)]\Theta(t'-t) \\
&\qquad +\frac{-p_\mu\gamma^\mu+m_0}{2m_0}\exp[ip\cdot(x'-x)]\Theta(t'-t)) \\
&= -i\int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}(\hat{\Lambda}_+(p)\exp[-ip\cdot(x'-x)]\Theta(t-t') \\
&\qquad +\hat{\Lambda}_-(p)\exp[ip\cdot(x'-x)]\Theta(t'-t))
\end{aligned}
\tag{2.307}
$$

Free propagator in terms of plane waves

This can also by written in terms of the normalised Dirac plane waves

$$
\begin{aligned}
S_F(x'-x) &= -i\Theta(t'-t)\int d^3p\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x) \\
&\quad +i\Theta(t-t')\int d^3p\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x)
\end{aligned}
\tag{2.308}
$$

Proof:

$$
\begin{aligned}
\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x) &= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\sum_{r=1}^{2}\omega^r(p)\overline{\omega}^r(p) \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\underbrace{\sum_{r=1}^{4}\epsilon_r\omega^r(p)\overline{\omega}^r(p)}_{=1}\frac{p_\mu\gamma^\mu+m_0}{2m_0} \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\,\frac{p_\mu\gamma^\mu+m_0}{2m_0} \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\,\hat{\Lambda}_+(p)
\end{aligned}
$$

A similary calulation for the secod part gives

$$\sum_{r=3}^{4} \psi_p^r(x')\overline{\psi}_p^r(x) = -\frac{1}{(2\pi)^3}\frac{m_0}{E_p}\exp[ip\cdot(x'-x)]\,\hat{\Lambda}_-(p) \tag{2.309}$$

□

Using this we easily verify:

$$\Theta(t'-t)\psi^{(+E)}(x') = i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x), \tag{2.310}$$

$$\Theta(t-t')\psi^{(-E)}(x') = -i\int d^3x S_F(x'-x)\gamma_0\psi^{(-E)}(x), \tag{2.311}$$

(see section 2.14.9). Equation (2.310) explicitly expresses the interpretation of electrons in terms of positive-energy solutions propagating forward in time and equation (2.311) the interpretation of positrons in terms of negative-energy solutions moving backward in time.

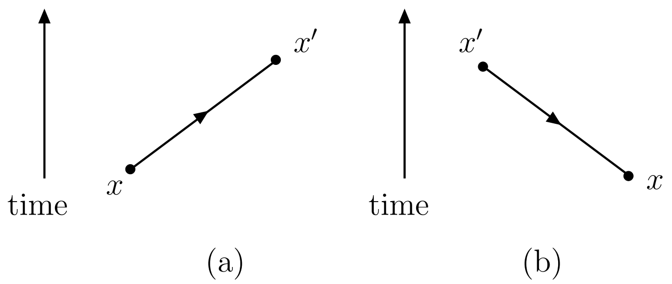


Figure 2.3: (a) $t < t'$: an electron propagated from x to x'. (b) $t > t'$: a positron propagated from x' to x.

The reader should be warnerd no to take this pictorial description of the mathematics as a literal process in space and time. For example, for x and x' with space-like separation, our naive interptetation of propagation would imply the electron/positron travels between the two points with speed greater than the speed of light.

2.3.4 Perturbation Expansion for the Stuckelberg-Feynmann Propagator

Equations () or () determine the free-particle propagator of the electron-positron theory. Here we develop a perturbative expansion for how this is modified to the exact propagator in the presence of an electromagnetic potential, the so-called Stuckelberg-Feynmann propagator, $S_F(x', x; A)$.

$$(i\gamma_\mu \partial^\mu - m_0) S_F(x', x; A) = \delta^4(x' - x) + eA_\mu(x')\gamma^\mu S_F(x', x; A) \tag{2.312}$$

This can be viewed as in inhomogeneous Dirac equation of the form

$$(i\gamma_\mu \partial^\mu - m_0)\Psi(x) = \rho(x) \tag{2.313}$$

which is solved by

$$\Psi(x) = \Psi_0(x) + \int d^4y S_F(x - y)\rho(y) \tag{2.314}$$

As is easily seen:

$$\begin{aligned}
(i\gamma_\mu \partial_x^\mu - m_0)\Psi(x) &= (i\gamma_\mu \partial_x^\mu - m_0)\Psi_0(x) + \int d^4y (i\gamma_\mu \partial_x^\mu - m_0) S_F(x - y)\rho(y) \\
&= \int d^4y \delta^4(x - y)\rho(y) \\
&= \rho(x)
\end{aligned} \tag{2.315}$$

In this way we obtain an integral equation for $S_F(x', x; A)$

$$\begin{aligned}
S_F(x', x; A) &= \int d^4y S_F(x' - y) \left[\delta^4(y - x) + eA_\mu(x')\gamma^\mu S_F(y, x; A)\right] \\
&= S_F(x' - y) + e\int d^4y S_F(x' - y) A_\mu(x')\gamma^\mu S_F(y, x; A).
\end{aligned} \tag{2.316}$$

Repeatedly substituting this equation into itself we obtain

$$\begin{aligned}
S_F(x', x; A) &= \int d^4y S_F(x' - y) + e\int d^4x_1 S_F(x' - x_1) A_\mu(x_1)\gamma^\mu S_F(x_1 - x) \\
&+ e^2 \int d^4x_1 d^4x_2 S_F(x' - x_1) A_\mu(x_1)\gamma^\mu S_F(x_1 - x_2) A_\nu(x_2)\gamma^\nu S_F(x_2 - x) \\
&+ \ldots\ .
\end{aligned} \tag{2.317}$$

Boundary condition of Feynman and Stuckelberg

$$\Psi(x) = \psi(x) + \int S_F(x-y) e\gamma_\mu A^\mu(y)\Psi(y) \tag{2.318}$$

The second term on the RHS represents the scattered wave.

Now by () $t \equiv x^0 \to +\infty$

$$S_F(x-y) \to -i \int d^3p \sum_{r=1}^{2} \psi_p^r(x)\overline{\psi}_p^r(y)$$

and $t \equiv x^0 \to -\infty$

$$S_F(x-y) \to +i \int d^3p \sum_{r=3}^{4} \psi_p^r(x)\overline{\psi}_p^r(y)$$

So that

$$\Psi(x) - \psi(x) \to \int d^3p \sum_{r=1}^{2} \psi_p^r(x) \left(-ie \int d^4y\, \overline{\psi}_p^r(x) A_\mu(y)\gamma^\mu \Psi(y) \right) \quad \text{for} \quad t \to +\infty \tag{2.319}$$

and

$$\Psi(x) - \psi(x) \to \int d^3p \sum_{r=3}^{4} \psi_p^r(x) \left(+ie \int d^4y\, \overline{\psi}_p^r(x) A_\mu(y)\gamma^\mu \Psi(y) \right) \quad \text{for} \quad t \to -\infty \tag{2.320}$$

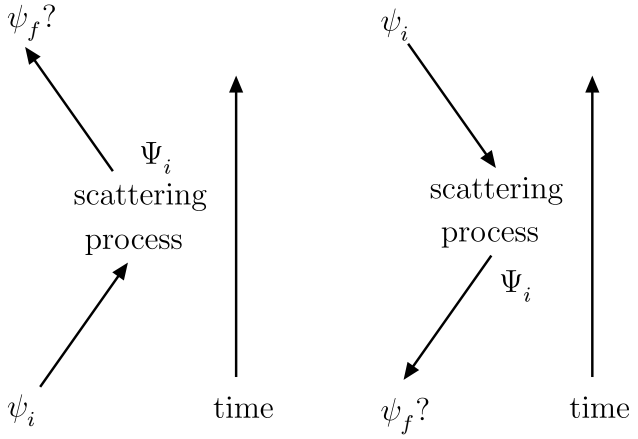
Therefore the scattered wave contains only positive frequencies

2.3.5 The $S-$Matrix Elements

The S-matrix elements are defined in the same manner as in the nonrelativistic case.

Let $\psi_f(x)$ denote the final free wave with quantum numbers f that is observed at the end of the scattering process.

$$
\begin{aligned}
S_{fi} &= \lim_{t \to \pm\infty} < \psi_f(x) | \Psi_i(x) > \\
&= \lim_{t \to \pm\infty} \left\langle \psi_f(x) \middle| \psi_i(x) + \int d^4 y S_F(x-y) e A_\mu(y) \gamma^\mu \Psi_i(x) \right\rangle
\end{aligned}
\tag{2.321}
$$



(a) (b)

Figure 2.4: $\Psi_i(x)$ stands for the incoming wave, which either reduces at $y_0 \to -\infty$ to an incident positive energy wave $\psi_i(x)$ or at $y_0 \to +\infty$ to an incident negative energy wave $\psi_i(x)$. (a) ψ_f describes an electron in the limit $t \to +\infty$. (b) ψ_f describes a positron in the limit $t \to -\infty$.

There are four basic processes to consider: (a) electron scattering; (b) positron scattering; (c) electron-positron pair creation; (d) pair annihilation.

We will need the following relations for adjoint spinors (proven in section 2.14.9).

$$
\Theta(t-t') \overline{\psi}^{(+E)}(x') = i \int d^3 x \overline{\psi}^{(+E)}(x) \gamma_0 S_F(x'-x) \tag{2.322}
$$

$$
\Theta(t'-t) \overline{\psi}^{(-E)}(x') = -i \int d^3 x \overline{\psi}^{(-E)}(x) \gamma_0 S_F(x'-x). \tag{2.323}
$$

These are the adjoint spinor versions of equations (2.310) and (2.311).

Using (2.322), for electron scattering we have

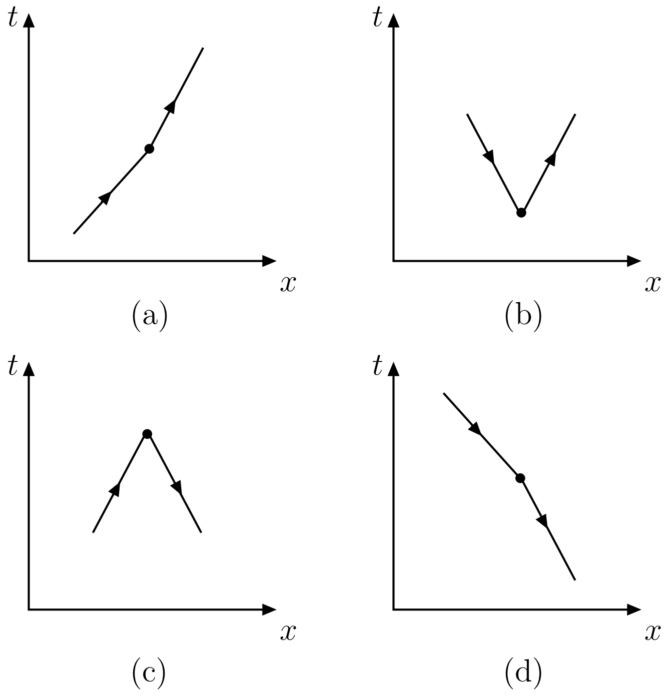


Figure 2.5: (a) electron scattering; (b) electron-positron pair creation; (c) pair annihilation (d) positron scattering.

$$
\begin{aligned}
S_{fi} &= \lim_{t\to+\infty} \left\langle \psi_f(x) \middle| \psi_i(x) + \int d^4y S_F(x-y) e A_\mu(y)\gamma^\mu \Psi_i(x) \right\rangle \\
&= \delta_{fi} + e \lim_{t\to+\infty} \int d^3x \psi_f^*(x) \int d^4y S_F(x-y) A_\mu(y)\gamma^\mu \Psi_i(x) \\
&= \delta_{fi} - ie \lim_{t\to+\infty} \int d^4y \left(i \int d^3x \overline{\psi}_f(x)\gamma^0 S_F(x-y) \right) A_\mu(y)\gamma^\mu \Psi_i(x) \\
&= \delta_{fi} - ie \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x)
\end{aligned}
$$

while, using (2.323) similarly, positron scattering is described by

$$
S_{fi} = \delta_{fi} + ie \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x)
$$

Both results can be combined by writing

$$
S_{fi} = \delta_{fi} - ie\varepsilon_f \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x) \tag{2.324}
$$

where $\varepsilon_f = +1$ for positive energy waves in the future and $\varepsilon_f = -1$ for energy waves in the past. $\Psi_i(x)$ stands for the incoming wave

Repeated substitution of (2.318)

$$
\begin{aligned}
S_{fi} &= \delta_{fi} - e\varepsilon_f \int d^4y \overline{\psi}_f(y) A_\mu \gamma^\mu(y) \Psi_i(y) \\
&= \delta_{fi} - e\varepsilon_f \left[\int d^4y_1 \overline{\psi}_f(y_1) A_\mu(y_1) \gamma^\mu \psi_i(y_1) \right. \\
&\quad \left. + \int d^4y_1 \int d^4y_2 \overline{\psi}_f(y_2) A_{\mu_2}(y_2) \gamma^{\mu_2} S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i(y_1) \right] \\
&\quad + \ldots \quad . \\
&= \delta_{fi} + \sum_{n=1}^{\infty} S_{fi}^{(n)}
\end{aligned} \tag{2.325}
$$

where

$$
\begin{aligned}
S_{fi}^{(n)} &= -ie^n \varepsilon_f \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f(y_n) A_{\mu_n}(y_n) \gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1}) \gamma^{\mu_{n-1}} \ldots \\
&\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i(y_1)
\end{aligned} \tag{2.326}
$$

"Ordinary" scattering of electrons

- $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with positive energy.

In this case

$$
\Psi_i(y) \to \psi_i^{(+E)}(y) = \sqrt{\frac{m_0}{E_-}} \frac{1}{(2\pi)^{3/2}} u(p_-, s_-) \exp(-ip_- \cdot x) \quad \text{as} \quad y_0 \to -\infty \tag{2.327}
$$

an incoming electron with positive energy E_- and momentum p_- and spin s_-

$$
\begin{aligned}
S_{fi}^{(n)} &= -ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(+E)}(y_n) A_{\mu_n}(y_n) \gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1}) \gamma^{\mu_{n-1}} \ldots \\
&\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i^{(+E)}(y_1)
\end{aligned} \tag{2.328}
$$

In addition to ordinary scattering intermediate pair creation and pair annihilation are included in the series.

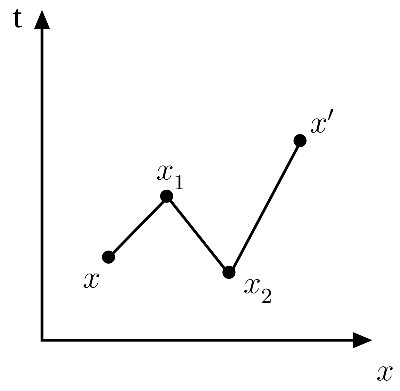


Figure 2.6: The electron at x_1 propagates backward in time from x_1 to x_2. Physically a positron-electron pair is created at x_2, the positron propagates forward in time where it anihilates with the intial electron at x_1.

Pair production processes

- $\Psi_i(y)$ in this case at $y_0 \to +\infty$ reduces to a plane wave with negative energy.

The positron state at $t \to +\infty$ is described by a plane wave of negative energy. We use the notation

By hole theory a positron is an electron with negative energy, negative momentum and negative spin.

Now we need the plane wave propagating backward in time. There will be an exponential with a positive sign in the exponent

$$e^{(i+p_+\cdot y)}$$

expressing the property that it has negative energy and momentum. It also will involve

$$v(p_+, +1/2) = \omega^4(p_+) \quad \text{and} \quad v(p_+, -1/2) = \omega^3(p_+)$$

where ω^4 is the spinor corresponding to a negative energy electron with spin up and ω^3 a negative energy electron with spin down. By using the spinors $v(p,s)$ we take care of the fact that the spin of electrons withe negative enery is $-s$. Here s is the spin of the positron.

$$\psi_i^{(electron)}(-p_f, -s_f) = Const.\ v(p_f, s_f)\exp(+ip_f \cdot x) \tag{2.329}$$

$$\Psi_i(x) \to \sqrt{\frac{m_0}{E_-}}\frac{1}{(2\pi)^{3/2}}v(p_+, s_+)e^{i+p_+\cdot y)} \quad \text{as} \quad y_0 \to \infty \tag{2.330}$$

$$\begin{aligned} S_{fi}^{(n)} &= -ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(+E)}(y_n)A_{\mu_n}(y_n)\gamma^{\mu_n}S_F(y_n - y_{n-1})A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}} \ldots \\ &\quad \times S_F(y_2 - y_1)A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i(-E)(y_1) \end{aligned} \tag{2.331}$$

Pair annihilation processes

- $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with negative energy.

$$\begin{aligned} S_{fi}^{(n)} &= +ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(-E)}(y_n)A_{\mu_n}(y_n)\gamma^{\mu_n}S_F(y_n - y_{n-1})A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}} \ldots \\ &\quad \times S_F(y_2 - y_1)A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i^{(+E)}(y_1) \end{aligned} \tag{2.332}$$

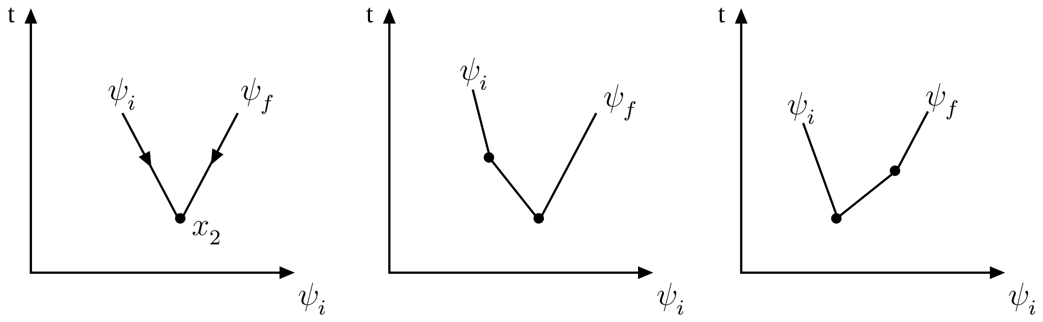


Figure 2.7:

Scattering of positrons

- $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with negative energy.

$$\begin{aligned} S_{fi}^{(n)} &= +ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(-E)}(y_n)A_{\mu_n}(y_n)\gamma^{\mu_n}S_F(y_n - y_{n-1})A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}} \ldots \\ &\quad \times S_F(y_2 - y_1)A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i^{(-E)}(y_1) \end{aligned} \tag{2.333}$$

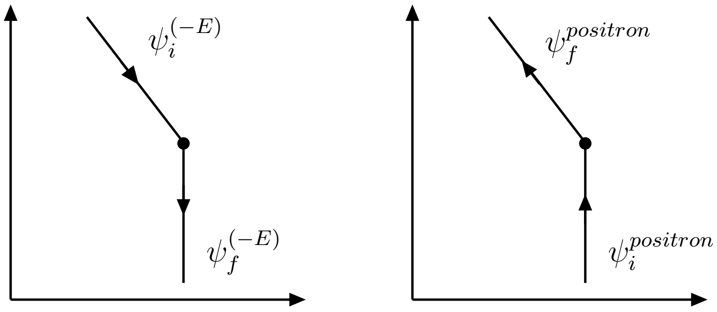


Figure 2.8: Lowest order positron scattering. (a) incoming negative energy electron $\psi_i^{(-E)}$ is scattered into an outgoing negative energy electron $\psi_f^{(-E)}$. (b) This corresponds to an incident positron $\psi_f^{positron}$ and emerging positron $\psi_f^{positron}$. This is the link between the calculational technique and the real physical picture of positron scattering.

2.4 Scattering of an Electron off a Coulomb Potential

2.4.1 The Scattering Amplitude

We calculate the Rutherford scattering of an electron at a fixed Coulomb potential to lowest order of perturbation theory. The appropriate S-Matrix element is the first order term of (2.328)

$$S_{fi} = -ie \int d^4x \, \overline{\psi}_f(x) A_\mu \gamma^\mu(x) \psi_i(x) \tag{2.334}$$

$\psi_i(x)$ is given by the incoming plane wave of an electron with momentum p_i and s_i:

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} \, u(p_i, s_i) e^{-ip_i \cdot x} \tag{2.335}$$

$\overline{\psi}_f(x)$ is given by

$$\overline{\psi}_f(x) = \sqrt{\frac{m_0}{E_f V}} \, \overline{u}(p_f, s_f) e^{ip_f \cdot x}. \tag{2.336}$$

Recall

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}, \quad \mathbf{B} = -\nabla \times \mathbf{A}$$

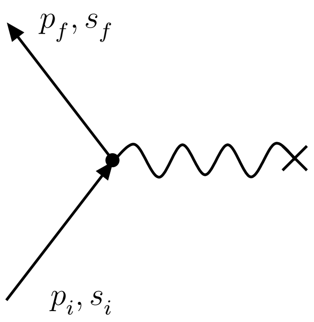


Figure 2.9:

Choosing

$$A_0(x) = A_0(\mathbf{x}) = -\frac{Ze}{|\mathbf{x}|}, \quad \mathbf{A}(x) = 0. \tag{2.337}$$

corresponds to a Coulomb force generated by a static charge $-Ze$. With these assumptions the S-Matrix element becomes

$$S_{fi} = iZe^2 \frac{1}{V} \sqrt{\frac{m_0^2}{E_f E_i}} \overline{u}(p_f, s_f) \gamma^0 u(p_i, s_i) \int d^4x \; e^{i(p_f - p_i)\cdot x} \frac{1}{|\mathbf{x}|}. \tag{2.338}$$

The integral over the time coordinate can be integrated

$$\int_{-\infty}^{\infty} dx_0 \; e^{i(E_f - E_i)t} = 2\pi\delta(E_f - E_i) \tag{2.339}$$

The remaining integral is

$$A_0(\mathbf{x}) = -Ze \int d^3x \frac{1}{|x|} e^{-i\mathbf{q}\cdot\mathbf{x}}$$

where $\mathbf{q}$ is the momentum transfer i.e. $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$. This can be evaluated using integration by parts of Poisson's formula $\Delta(1/|\mathbf{x}|) = -4\pi\delta^3(\mathbf{x})$:

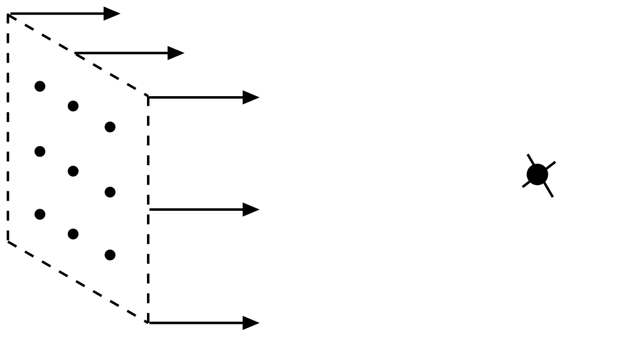
$$\int d^3x \frac{1}{|\mathbf{x}|} e^{-i\mathbf{q}\cdot\mathbf{x}} = -\frac{1}{\mathbf{q}^2}\int d^3x \frac{1}{|\mathbf{x}|}\Delta e^{-i\mathbf{q}\cdot\mathbf{x}}$$
$$= -\frac{1}{\mathbf{q}^2}\int d^3x \Delta\left(\frac{1}{|\mathbf{x}|}\right) e^{-i\mathbf{q}\cdot\mathbf{x}}$$
$$= -\frac{1}{\mathbf{q}^2}\int d^3x (-4\pi\delta^3(\mathbf{x}))e^{-i\mathbf{q}\cdot\mathbf{x}}$$
$$= \frac{4\pi}{\mathbf{q}^2} \tag{2.340}$$

Thus the $S-$matrix element becomes

$$S_{fi} = iZe^2\frac{1}{V}\sqrt{\frac{m_0^2}{E_f E_i}}\,\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\,\frac{4\pi}{q^2}\,2\pi\delta(E_f - E_i). \tag{2.341}$$

2.4.2 The Cross Section

A differential cross section σ is defined by the effective area of target particles.



target

Figure 2.10: .

$$\text{Chance of hitting a Coulomb potential} = \frac{N_T\sigma}{A} \tag{2.342}$$

Let us say that there are N_I incoming particles. The number of scattering events is then

$$\text{number of events} = N_I\frac{N_T\sigma}{A} \tag{2.343}$$

so that the cross section is then expressed as

$$\sigma = \left(\frac{\text{number of events}}{N_I N_T}\right) A. \tag{2.344}$$

We wish the express the cross section in terms of the flux of the incoming beam,

$$\text{flux} = \rho v$$

where v is the velocity of the beam moving toward the stationary target. The number of particles in the beam N_I is equal to the density of the beam ρ times the volume of the beam, vtA. The cross section can therefore be written as

$$\begin{aligned}
\sigma &= \frac{\text{number of events}/t}{(\rho v t A) N_T / t} A \\
&= \frac{\text{number of events}/t}{\rho v} \cdot \frac{1}{N_T} \\
&= \frac{\text{transition rate}}{\text{flux}} \cdot \frac{1}{N_T}
\end{aligned} \tag{2.345}$$

$$dW = \frac{|S_{fi}|^2 dN_f}{T} \frac{1}{J} \tag{2.346}$$

dN_f is now determined.

2.4.3 Transition Probability Per Particle into Final States

Standing waves in a cubical box of volume $V = L^3$ require

$$\begin{aligned}
p_x L &= n_x 2\pi, \\
p_y L &= n_y 2\pi, \\
p_z L &= n_z 2\pi,
\end{aligned} \tag{2.347}$$

with integer number n_x, n_y, n_z. For large L the discrete set of $\mathbf{p}-$values approaches a continuum. The number of states is

$$
\begin{aligned}
dN &= dn_x dn_y dn_z \\
&= \frac{1}{(2\pi)^3} L^3 dp_x dp_y dp_z \\
&= \frac{V}{(2\pi)^3} d^3p. \qquad (2.348)
\end{aligned}
$$

The transition probability per particle into these final states is

$$
\begin{aligned}
dW &= |S_{fi}|^2 \frac{V d^3p_f}{(2\pi)^3} \\
&= \frac{Z^2(4\pi\alpha)^2 m_0^2}{E_i V} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{d^3p_f}{(2\pi)^3 E_f} (2\pi\delta(E_f - E_i))^2 \qquad (2.349)
\end{aligned}
$$

2.4.4 Transition Probability Per Particle, Per Unit Time

We smear out the $\delta-$function $2\pi\delta(E_f - E_i)$:

$$
\begin{aligned}
\int_{-T/2}^{T/2} dx_0 e^{i(E_f - E_i)x_0} &= \left[\frac{1}{i(E_f - E_i)} e^{i(E_f - E_i)x_0} \right]_{-T/2}^{T/2} \\
&= \frac{2\sin(E_f - E_i)T/2}{E_f - E_i} \qquad (2.350)
\end{aligned}
$$

Thus we replace the square of the $\delta-$function is replaced by

$$
(2\pi\delta(E_f - E_i))^2 \;\Rightarrow\; 4\frac{\sin^2(E_f - E_i)T/2}{(E_f - E_i)^2} \qquad (2.351)
$$

The area of this function is

$$
\int_{-\infty}^{\infty} 4\frac{\sin^2(E_f - E_i)T/2}{(E_f - E_i)^2} dE_f = 2\pi T. \qquad (2.352)
$$

Knowing that the "area" under the square of the δ is $\lim_{T\to\infty} 2\pi T$, we make the replacement

$$(2\pi\delta(E_f - E_i))^2 \rightarrow 2\pi T\delta(E_f - E_i). \tag{2.353}$$

Denote the rate R

$$dR = \frac{dW}{T} = \frac{Z^2\alpha^2 m_0^2}{E_i V}\frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4}\frac{d^3p_f}{E_f}\delta(E_f - E_i) \tag{2.354}$$

2.4.5 Formula for Differential Cross Section

The scattering cross section can be defined as the transition probabilty per particle and per unit time divided by the incoming current of particles

$$J^a_{inc}(x) = \overline{\psi}_i(x)\gamma^a\psi_i(x) \tag{2.355}$$

Taking the spinors with spin polarisation in the z-direction we determine the current

$$\begin{aligned}
J^a_{inc} &= \overline{\psi}_i(x)\gamma^a\psi_i(x) \\
&= \frac{m_0}{E_i V}\,\overline{u}(p_i, s_i)\gamma^3 u(p_i, s_i) \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\left(1\ 0\ \frac{p_i}{E_i + m_0}\ 0\right)\,\gamma^0\gamma^3\,\begin{pmatrix} 1 \\ 0 \\ \frac{p_i}{E_i+m_0} \\ 0 \end{pmatrix} \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\left(1\ 0\ \frac{p_i}{E_i + m_0}\ 0\right)\,\begin{pmatrix} 1 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\,\begin{pmatrix} 1 \\ 0 \\ \frac{p_i}{E_i+m_0} \\ 0 \end{pmatrix} \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\left(1\ 0\ \frac{p_i}{E_i + m_0}\ 0\right)\,\begin{pmatrix} \frac{p_i}{E_i+m_0} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{p_i}{E_i}\frac{1}{V}.
\end{aligned} \tag{2.356}$$

$$|\mathbf{J}_{inc}| = \frac{|\mathbf{v}_i|}{V}. \tag{2.357}$$

The differential cross section can now be determined

$$d\sigma = \frac{dR}{|\mathbf{J}_{inc}|} = \frac{4Z^2\alpha^2 m_0^2}{E_i V \frac{|v_i|}{V}} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{d\mathbf{p}_f^3}{E_f} \delta(E_f - E_i) \tag{2.358}$$

Use

$$d^3 p_f = \mathbf{p}_f^2 d|\mathbf{p}_f| d\Omega_f \tag{2.359}$$

Then the differential cross section becomes

$$d\sigma = \frac{4Z^2\alpha^2 m_0^2}{E_i V \frac{|v_i|}{V}} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{\mathbf{p}_f^2 d|\mathbf{p}_f|}{E_f} d\Omega_f \, \delta(E_f - E_i) \tag{2.360}$$

2.4.6 Averaging Over Spin

The differential cross section above can be applied to calculate the scattering of a particle with initial polarisation (s_i) to final polarisation (s_f).

First we give a simple example. From the relation

$$\overline{w}^r (p_\mu \gamma^\mu - \epsilon_r m_0) = 0.$$

we see that

$$\overline{w}^r_\gamma(p_i)\Lambda_{\gamma\delta}(p) = 0$$

for $r = 3, 4$, where

$$\Lambda_{\gamma\delta}(p) = \frac{-p_\mu\gamma^\mu + m_0}{2m_0}$$

$$
\begin{aligned}
\sum_{s_i} u_\beta(p_i, s_i)\overline{u}_\beta(p_i, s_i) &= u_\beta(p_i, \uparrow)\overline{u}_\beta(p_i, \uparrow) + u_\beta(p_i, \downarrow)\overline{u}_\beta(p_i, \downarrow) \\
&= \sum_{r=1}^{2} w^r_\beta(p_i)\overline{w}^r_\delta(p_i) \\
&= \sum_{r=1}^{2} w^r_\beta(p_i)\sum_{\gamma=1}^{4}\overline{w}^r_\gamma(p_i)\Lambda_{\gamma\delta}(p) \\
&= \sum_{\gamma,r=1}^{4} \epsilon_r w^r_\beta(p_i)\overline{w}^r_\gamma(p_i)(\Lambda(p))_{\gamma\delta} \\
&= (\Lambda(p))_{\beta\delta} \qquad (2.361)
\end{aligned}
$$

where we used the completeness relation

$$
\sum_{r=1}^{4} \epsilon_r w^r_\beta(p_i)\overline{w}^r_\gamma(p_i) = \delta_{\beta\gamma}.
$$

Using this result and similar considerations we now calculate the spin sum.

$$
\begin{aligned}
&\sum_{\alpha,\sigma,\beta,\delta}\sum_{s_f}\overline{u}_\alpha(p_f, s_f)\gamma^0_{\alpha\beta}\left(\sum_{s_f} u_\alpha(p_i, s_i)\overline{u}_\delta(p_i, s_i)\right)\gamma^0_{\delta\sigma}u_\sigma(p_f, s_f) \\
=& \sum_{\alpha,\sigma}\sum_{s_f}\overline{u}_\alpha(p_f, s_f)\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}u_\sigma(p_f, s_f) \\
=& \sum_{\alpha,\sigma}\sum_{r=1}^{2}\overline{w}^r_\alpha(p_f)\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}w^r_\sigma(p_f) \\
=& \sum_{\alpha,\sigma}\sum_{r=1}^{4}\epsilon_r\overline{w}^r_\alpha(p_f)\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}\sum_{\tau=1}^{4}\left(\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right)_{\sigma\tau}w^r_\tau(p_f) \\
=& \sum_{\alpha,\sigma}\left(\gamma^0\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma^0\right)_{\alpha\sigma}\left(\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right)_{\sigma\alpha} \\
=& \ Tr\left[\gamma^0\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma^0\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right] \qquad (2.362)
\end{aligned}
$$

Using this the differential cross section can be written

$$
\frac{d\overline{\sigma}}{d\Omega_f} = \frac{4Z^2\alpha^2 m_0^2}{2|\mathbf{q}|^4}Tr\left[\gamma^0\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma^0\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right]. \qquad (2.363)
$$

2.4.7 Taking the Trace of the Product of Gamma Matrices in the Differential Cross Section

We first prove that the trace of an odd number of $\gamma-$matrices vanishes. To do this we use $(\gamma^5)^2 = I$ and $\gamma_\mu\gamma^5 + \gamma^5\gamma_\mu = 0$.

$$\begin{aligned}
Tr\gamma_\mu \dots \gamma_\nu &= Tr\gamma_\mu \dots \gamma_\nu\gamma^5\gamma^5 \\
&= (-1)^n Tr\gamma^5\gamma_\mu \dots \gamma_\nu\gamma^5 \\
&= (-1)^n Tr\gamma_\mu \dots \gamma_\nu\gamma^5\gamma^5 \\
&= (-1)^n Tr\gamma_\mu \dots \gamma_\nu
\end{aligned} \tag{2.364}$$

where in the third line we used the cyclic permutation of the trace. With this () reduces to

$$\frac{d\overline{\sigma}}{d\Omega_f} = \frac{4Z^2\alpha^2 m_0^2}{2|\mathbf{q}|^4}\left[Tr(\gamma^0(p_i)_\mu\gamma^\mu\gamma^0(p_f)_\nu\gamma^\nu) + m_0^2 Tr(\gamma^0)^2\right]. \tag{2.365}$$

We have $Tr(\gamma^0)^2 = TrI = 4$. To evaluate the first trace we derive a couple of results: Firstly

$$\begin{aligned}
a_\mu b_\nu Tr\gamma^\mu\gamma^\nu &= a_\mu b_\nu \frac{1}{2} Tr(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \\
&= a_\mu b_\nu \eta^{\mu\nu} TrI \\
&= 4a\cdot b.
\end{aligned} \tag{2.366}$$

where we have used μ and ν are dummy variables in the first line. Secondly, starting with

$$\begin{aligned}
a_\mu b_\nu c_\gamma d_\delta\, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta &= 2a\cdot b\, c_\gamma d_\delta\, Tr\gamma^\gamma\gamma^\delta - b_\mu a_\nu c_\gamma d_\delta\, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta \\
&= 2a\cdot b\, c_\gamma d_\delta\, Tr\gamma^\gamma\gamma^\delta - 2a\cdot c b_\mu d_\delta\, Tr\gamma^\mu\gamma^\delta \\
&\quad + b_\mu c_\nu a_\gamma d_\delta\, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta \\
&= 2a\cdot b\, c_\gamma d_\delta\, Tr\gamma^\gamma\gamma^\delta - 2a\cdot c\, b_\mu d_\delta\, Tr\gamma^\mu\gamma^\delta \\
&\quad + 2a\cdot d\, b_\mu c_\nu\, Tr\gamma^\mu\gamma^\nu - b_\mu c_\nu d_\gamma a_\delta\, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta
\end{aligned}$$

then using the cyclic property of the trace we find

$$
\begin{aligned}
a_\mu b_\nu c_\gamma d_\delta \, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta &= a\cdot b\, c_\mu d_\nu \, Tr\gamma^\mu\gamma^\nu - a\cdot c\, b_\mu d_\nu \, Tr\gamma^\mu\gamma^\nu \\
&\quad + a\cdot d\, b_\mu c_\nu \, Tr\gamma^\mu\gamma^\nu \qquad (2.367)
\end{aligned}
$$

Using the second result first with $a = c = (1, 0, 0, 0)$ we get

$$
(p_i)_\mu (p_f)_\nu \, Tr(\gamma^0\gamma^\mu\gamma^0\gamma^\nu) = a\cdot p_i \, Tr - a\cdot a \, Tr + a\cdot p_f \, Tr
$$

Now using the first result we have

$$
\begin{aligned}
(p_i)_\mu (p_f)_\nu \, Tr(\gamma^0\gamma^\mu\gamma^0\gamma^\nu) &= 4(a\cdot p_i)(a\cdot p_f) - (a\cdot a)4(p_i\cdot p_f) + 4(a\cdot p_f)(a\cdot p_i) \\
&= 4E_iE_f - 4(E_iE_f - \vec{p}_i\cdot\vec{p}_f) + 4E_iE_f \qquad (2.368)
\end{aligned}
$$

2.4.8 Mott Scattering Formula

The $\delta(E_i - E_f)$ function of the cross section ensures energy conservation $E_i = E_f$, thus $E_i^2 = E_f^2$:

$$
m_0^2 + \vec{p}_i^{\,2} = m_0^2 + \vec{p}_f^{\,2}
$$

implying

$$
|\vec{p}_i| = |\vec{p}_f| = |\vec{p}|
$$

The scalar product of initial and final momentum is the following function of the scattering angle θ

$$
\begin{aligned}
p_i\cdot p_f &= |\mathbf{p}|^2\cos\theta \\
&= |\mathbf{p}|^2\left(1 - 2\sin^2\frac{\theta}{2}\right) \\
&= \beta^2E^2\left(1 - 2\sin^2\frac{\theta}{2}\right) \qquad (2.369)
\end{aligned}
$$

From this we have for the momentum transfer

$$
\begin{aligned}
|q| &= |p_f - p_i| \\
&= \sqrt{|p_f|^2 + |p_i|^2 - p_f \cdot p_i} \\
&= \sqrt{2p^2 - |p|\cos\theta} \\
&= 2|p|\sin\frac{\theta}{2}
\end{aligned} \tag{2.370}
$$

A simple exercise, left to the reader, gives us the well known Mott scattering formula

$$
\frac{d\overline{\sigma}}{d\Omega_f} = \frac{Z^2\alpha^2(1-\beta^2\sin^2\frac{\theta}{2})}{4\beta^2|\mathbf{p}|^2\sin^4\frac{\theta}{2}} \tag{2.371}
$$

In the limit $\beta \to 0$ (small velocities) reduces to Rutherford's scattering formula

$$
\frac{d\overline{\sigma}}{d\Omega_f} = \frac{Z^2\alpha^2}{4\beta^2|\mathbf{p}|^2\sin^4\frac{\theta}{2}} \tag{2.372}
$$

2.5 Scattering of an Electron off a Free Proton

In the last example we considered scattering off a central potential. Now we make our first step toward the derivation of Feynmann's rules by considering scattering off two free particles.

2.5.1 Inhomogeneous Wave Equation and Photon Proporgator

In electromagnetism the invariance in A_μ comes about because the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is left unchanged by the gauge transformations

$$
A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)
$$

We wish to calculate the four-potential $A_\mu(x)$ produced by a source current $J^\nu(x)$ term,

$$
\Box A^\nu(x) - \partial^\mu\partial_\mu A^\nu(x) = 4\pi J^\nu(x) \tag{2.373}
$$

We are free to choose the most convenient gauge for the calculation intened to make.

We will choose the Lorentz gauge

$$\partial_\mu A^\mu(x) = 0$$

In momentum space this reads

$$k^\mu A^\mu(k) = 0.$$

$$\Box A^\nu(x) = 4\pi J^\nu(x) \tag{2.374}$$

The solution of the above equation may be systematically formulated using the appropriate Green's function which we call $D_F(x-y)$, the propogator for electromagnetism.

$$\Box D_F(x-y)(x) = 4\pi\delta^4(x-y). \tag{2.375}$$

The Fourier-transformed proporgator is defined by

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp[-iq\cdot(x-y)] D_F(q) \tag{2.376}$$

Using

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp[-iq\cdot(x-y)] \tag{2.377}$$

and making comparison we get

$$D_F(q) = -\frac{4\pi}{q^2} \tag{2.378}$$

The four-potential $A^\mu(x)$ solving (2.374) is

$$A^\mu(x) = \int d^4y D_F(x-y) J^\mu(y). \tag{2.379}$$

2.5.2 Potential of Proton Current

$$S_{fi} = -ie \int d^4x \overline{\psi}_f(x)\gamma_\mu A^\mu(x)\Psi_i(x) \tag{2.380}$$

At first order the four-potential $A^\mu(x)$ is the field produced by the proton to lowest order in α.

$$S_{fi} = -i \int d^4x d^4y \left[e\overline{\psi}_f(x)\gamma_\mu \psi_i(x) \right] D_F(x-y) J^\mu(y). \tag{2.381}$$

The term in the brackets represents the current of the electron. As the electron and proton play equivalent roles in the scattering process, the proton's current should be of the same form as the electronic current. Therefore we make the replacement

$$J^\mu(y) \to e_p \overline{\psi}^p_f(y)\gamma^\mu \psi^p_i(y) \tag{2.382}$$

where $\overline{\psi}^p_f(y)$ and $\psi^p_i(y)$ have the same for as the electron wavefunctions

$$\begin{aligned} \psi^p_i(y) &= \sqrt{\frac{M_0}{E^p_i V}} u(P_i, S_i) \exp(-iP_i \cdot y) \\ \psi^p_f(y) &= \sqrt{\frac{M_0}{E^p_f V}} u(P_f, S_f) \exp(-iP_f \cdot y) \end{aligned} \tag{2.383}$$

where P_i and P_f denote the four-momentum of the proton, S_i, S_f and E^p_i, E^p_f denote its spin and energy respectively. M_0 is the proton's rest mass. The proton's current is then

$$J^\mu_{fi}(y) = -\sqrt{\frac{M_0^2}{E^p_f E^p_i}} \frac{e}{V} \exp[i(P_f - P_i)\cdot y]\, \overline{u}(P_f, S_f)\gamma^\mu u(P_i, S_i). \tag{2.384}$$

Inserting this into the expression for the S-matrix gives

$$\begin{aligned} S_{fi} &= +i\frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E^p_f E^p_i}}\, [\overline{u}(p_f, s_f)\gamma_\mu u(p_i, s_i)] \\ &\times \int d^4x d^4y \frac{d^4q}{(2\pi)^4} \exp[-iq\cdot(x-y)] \exp[i(p_f - p_i)\cdot x] \exp[i(P_f - P_i)\cdot x] \\ &\times \left(\frac{-4\pi}{q^2 + i\epsilon}\right) [\overline{u}(P_f, S_f)\gamma_\mu u(P_i, S_i)] \end{aligned} \tag{2.385}$$

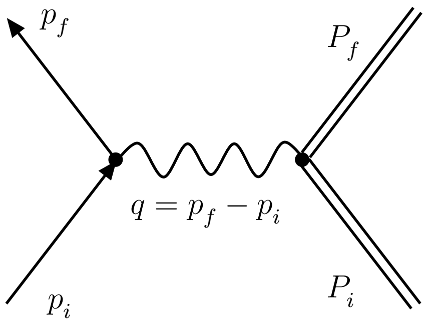


Figure 2.11: Lowest order electron-proton scattering.

2.5.3 Conservation of Four-Momentum

The $x-$ and $y-$integrations give

$$\begin{aligned}\int d^4x \exp(i(p_f - p_i - q) \cdot x) &= (2\pi)^4 \delta^4(p_f - p_i - q) \\ \int d^4y \exp(i(p_f - p_i - q) \cdot y) &= (2\pi)^4 \delta^4(P_f - P_i - q)\end{aligned} \tag{2.386}$$

The integration over q is then readily done:

$$\begin{aligned}&\int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(p_f - p_i - q)(2\pi)^4 \delta^4(P_f - P_i - q) \left[-\frac{4\pi}{q^2 + i\epsilon} \right] \\ &= (2\pi)^4 \delta^4(P_f - P_i + p_f - p_i) \left[-\frac{4\pi}{(p_f - p_i)^2 + i\epsilon} \right]\end{aligned} \tag{2.387}$$

2.5.4 Remarks on the Form of the S-matrix Element

Here we display properties of S-matrix element that are a first step toward "deriving" the Feynmann rules for QED. The total S-matrix element the reads

$$S_{fi} = i(2\pi)^4 \delta^4(P_f - P_i + p_f - p_i)\ M_{fi}\ \frac{1}{V^2} \sqrt{\frac{m_0^2}{E_f E_i}} \sqrt{\frac{M_0^2}{E_f^p E_i^p}} \tag{2.388}$$

where

$$M_{fi} = [\overline{u}(p_f, s_f)(-ie\gamma_\mu)u(p_i, s_i)] \frac{-4\pi\ \eta^{\mu\nu}}{(p_f - p_i)^2 + i\epsilon} [\overline{u}(P_f, S_f)(-ie_p\gamma_\mu)u(P_i, S_i)] \tag{2.389}$$

This describes the lowest order contribution. This is put in diagrammatic form in fig (3.7). The wavy line represents the virtual photon being exchanged between the electron and proton. The four momentum of the photon is

$$q = p_f - p_i = P_f - P_i \tag{2.390}$$

• The following factor represents the amplitude for the propagation of a photon with momentum q:

$$\frac{-4\pi\ \eta^{\mu\nu}}{q^2 + i\epsilon} \tag{2.391}$$

• There is a factor of $-ie\gamma_\mu$ for every vertex.

• These act between spinors $u(p, s)$ describing the free ingoing and outgoing Dirac particles.

• There is a four dimensional $\delta-$function, ensuring conservation of total energy and momentum in the scattering process.

2.5.5 The Scattering Cross Section

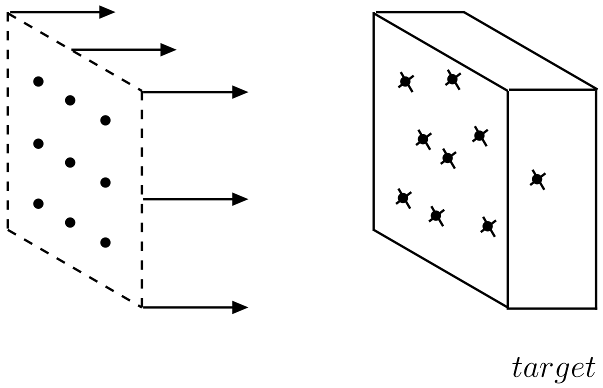


Figure 2.12: .

We divide $|S_{fi}|^2$ by the time interval and the space volume of the reaction (Dirac waves normalised so that there is one particle per unit volume)

$$W_{fi} = \frac{|S_{fi}|^2}{VT} \tag{2.392}$$

Now we come to calculating the cross section. As in section we have to consider the square of the δ^4-function

$$(2\pi)^4\delta^4\left(p_f + P_f - p_i - P_i\right) = \lim_{T\to\infty, V\to\infty} \int_{-T/2}^{T/2}\int_V d^3\mathbf{x} \exp\left[ix\cdot\left(p_f + P_f - p_i - P_i\right)\right] \tag{2.393}$$

$$\left[(2\pi)^4\delta^4\left(p_f + P_f - p_i - P_i\right)\right]^2 \to TV(2\pi)^4\delta^4\left(p_f + P_f - p_i - P_i\right) \tag{2.394}$$

To obtain the transition rate to a group of final states with momenta in the intervals $f = 1, 2$, we multiply by the number of these states which is

$$\frac{Vd^3\mathbf{P}_1}{(2\pi)^3}\frac{Vd^3\mathbf{P}_2}{(2\pi)^3} \tag{2.395}$$

number of target particles per unit volume $= 1/V$

Combining these results with (), we obtain the required formula for the differential cross section

$$\begin{aligned} d\sigma &= V^2\frac{d^3\mathbf{p}_f}{(2\pi)^3}\frac{d^3\mathbf{P}_f}{(2\pi)^3}\frac{1}{|\mathbf{J_{inc}}|}\frac{1}{1/V}W_{fi} \\ &= \end{aligned} \tag{2.396}$$

2.5.6 Lorentz Invariance

Each particle leaving the scattering process contributes a factor

$$\frac{m_0}{E_f}\frac{d^3p_f}{(2\pi)^3} \tag{2.397}$$

to the cross section. Consider

$$
\begin{aligned}
\int_{-\infty}^{\infty} d^4p \; \delta(p^2 - m_0^2)\Theta(p_0) &= \int_0^{\infty} dp_0 \; \delta(p_0^2 - \mathbf{p}^2 - m_0^2) \; d^3p \\
&= \int_0^{\infty} dp_0 \; \delta(p_0^2 - E^2) \; d^3p \\
&= \int_0^{\infty} dp_0 \; \delta[(p_0 - E)(p_0 + E)] \; d^3p \\
&= \int_0^{\infty} dp_0 \; \delta[2E(p_0 - E)] \; d^3p \\
&= \frac{d^3p}{2E}
\end{aligned}
$$

where

$$
\Theta(p_0) = \begin{cases} 1 & \text{for } p_0 > 0 \\ 0 & \text{for } p_0 < 0 \end{cases}
$$

This step function is obviously Lorentz invariant since Lorentz transformations always transform timelike four vectors into timelike four vectors. Thus we have established that $d^3p/2E$ is a Lorentz-invariant factor.

Now we consider the factor

$$
\frac{m_0}{E_i} \frac{M_0}{E_i^P} \frac{1}{|\mathbf{J}_{inc}|V}.
$$

$$
|\mathbf{J}_{inc}| = \frac{1}{V} |\mathbf{v}_i - \mathbf{V}_i| \tag{2.398}
$$

$$
\mathbf{v}_i = \frac{\mathbf{p}_i}{E_i}, \quad \mathbf{V}_i = \frac{\mathbf{P}_i}{E_i^P}
$$

This gives

$$
\begin{aligned}
\frac{m_0}{E_i} \frac{M_0}{E_i^P} \frac{1}{|\mathbf{J}_{inc}|V} &= \frac{m_0 M_0}{E_i E_i^P |\mathbf{v}_i - \mathbf{V}_i|} \\
&= \frac{m_0 M_0}{E_i E_i^P \sqrt{\mathbf{v}_i^2 + \mathbf{V}_i - 2\mathbf{v}_i \cdot \mathbf{V}_i}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + \mathbf{P}_i^2 E_i^2 - 2\mathbf{p}_i \cdot \mathbf{P}_i E_i E_i^P}}
\end{aligned} \tag{2.399}
$$

We prove that for collinear collisions that this is equivalent to the Lorentz invariant scalar

$$\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}},$$

because

$$\begin{aligned}
\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} &= \frac{m_0 M_0}{\sqrt{(E_i E_i^P - \mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{E_i^2 E_i^{P2} - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{(m_0^2 + \mathbf{p}_i^2)(M_0^2 + \mathbf{P}_i^2) - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + m_0^2 \mathbf{P}_i^2 - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2}} \qquad (2.400)
\end{aligned}$$

As the velocity vectors are collinear we have that $(\mathbf{p}_i \cdot \mathbf{P}_i)^2 = \mathbf{p}_i^2 \mathbf{P}_i^2$.

$$\begin{aligned}
\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} &= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + m_0^2 \mathbf{P}_i^2 - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + \mathbf{p}_i^2 \mathbf{P}_i^2}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + \mathbf{P}_i^2 E_i^2 - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i}} \qquad (2.401)
\end{aligned}$$

We can use this Lorentz-invariant flux factor to write the cross section in a invariant form

$$d\sigma = \frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} \ |M_{fi}|^2 \ (2\pi)^4 \delta^4(P_f - P_i + p_f - p_i) \ \frac{m_0 d^3 p_f}{(2\pi)^3 E_f} \ \frac{M_0 d^3 p_f}{(2\pi)^3 E_f^P}. \quad (2.402)$$

2.5.7 Averaging over Spin

The squared invariant matrix element averaged over initial and final spin is

$$\overline{|M_{fi}|^2} = \frac{1}{4} \sum_{S_f, S_i, s_f, s_i} \left| \overline{u}(p_f, s_f) \gamma^\mu u(p_i, s_i) \ \frac{e e_p (4\pi)}{q^2 + i\epsilon} \ \overline{u}(P_f, S_f) \gamma_\mu u(P_i, S_i) \right|^2 \qquad (2.403)$$

$$
\begin{aligned}
&\sum_{S_f,S_i,s_f,s_i} \left| [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(P_f,S_f)\gamma_\mu u(P_i,S_i)] \right|^2 \\
= &\sum_{S_f,S_i,s_f,s_i} [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(P_f,S_f)\gamma_\mu u(P_i,S_i)] \\
&[\overline{u}(p_f,s_f)\gamma^\nu u(p_i,s_i)]^*\ [\overline{u}(P_f,S_f)\gamma_\nu u(P_i,S_i)]^* \\
= &\sum_{s_f,s_i} [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(p_f,s_f)\gamma^\nu u(p_i,s_i)]^* \\
&\sum_{S_f,S_i} [\overline{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)]\ [\overline{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)]^*
\end{aligned} \tag{2.404}
$$

At this point the reader should go throught section 2.14.8. The answer according to (2.588) is

$$
Tr\left[\frac{p_{f\alpha}\gamma^\alpha + m_0}{2m_0}\ \gamma^\mu\ \frac{p_{i\beta}\gamma^\beta + m_0}{2m_0}\ \gamma^\nu\right]\ Tr\left[\frac{P_{f\gamma}\gamma^\gamma + M_0}{2M_0}\ \gamma_\mu\ \frac{P_{i\delta}\gamma^\delta + M_0}{2M_0}\ \gamma_\nu\right] \tag{2.405}
$$

$$
\overline{|M_{fi}|^2} = \frac{e^2 e_p^2 (4\pi)^2}{q^4} L^{\mu\nu} H_{\mu\nu} \tag{2.406}
$$

where we have introduced the lepton tensor $L^{\mu\nu}$ and the hadron tensor $H_{\mu\nu}$, defined as

$$
L^{\mu\nu} = Tr\left[\frac{p_{f\alpha}\gamma^\alpha + m_0}{2m_0}\ \gamma^\mu\ \frac{p_{i\beta}\gamma^\beta + m_0}{2m_0}\ \gamma^\nu\right] \tag{2.407}
$$

and

$$
H_{\mu\nu} = Tr\left[\frac{P_{f\gamma}\gamma^\gamma + M_0}{2M_0}\ \gamma_\mu\ \frac{P_{i\delta}\gamma^\delta + M_0}{2M_0}\ \gamma_\nu\right] \tag{2.408}
$$

Using methods already introduced in the previous section on Coulomb scattering, we can easily evaluate the trace in the lepton tensor $L^{\mu\nu}$ to obtain:

$$
L^{\mu\nu} = \frac{1}{2}\frac{1}{m_0^2}\left[p_f^\mu p_i^\nu + p_i^\mu p_f^\nu - \eta^{\mu\nu}(p_f \cdot p_i - m_0^2)\right] \tag{2.409}
$$

The Hadron trace has the same structure, we just replace small letters by capitals and lower the spacetime indices.

2.5.8 Differential Cross Section in Rest Frame of Proton

The calculation of $\overline{|M_{fi}|^2}$ which we leave to the reader results in

$$\begin{aligned}\overline{|M_{fi}|^2} &= \frac{e^2 e_p^2 (4\pi)^2}{4m_0^2 M_0^2 (q^2)^2}\Big[(p_i \cdot P_i)(p_f \cdot P_f) + (p_i \cdot P_f)(p_f \cdot P_i) \\ &\quad -(p_i \cdot p_f)M_0^2 - (P_i \cdot P_f)m_0^2 + 2m_0^2 M_0^2\Big]. \qquad (2.410)\end{aligned}$$

Let us work in the rest frame of the proton. We define

$$\begin{aligned} p_f &= (E', \mathbf{p}') =: p' \\ p_f &= (E, \mathbf{p}) =: p \\ P_i &= (M_0, 0) \qquad (2.411)\end{aligned}$$

We calculate the differential cross section for electron scattering into a solid angle $d\Omega'$ centered around the scattering angle θ. Thus we will integrate the differential cross section over all momentum variable except for the direction of $\mathbf{p_f}$. First we will want to write down the spin averaged differential cross section in the proton rest system. The invariant flux factor reduces to

$$\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i) - m_0^2 M_0^2}} = \frac{m_0 M_0}{\sqrt{E^2 M_0^2 - m_0^2 M_0^2}} = \frac{m_0}{|\mathbf{p}|} \qquad (2.412)$$

We will use

$$\frac{M_0 d^3 P_f}{(2\pi)^3 E_f^p} = \frac{2M_0}{(2\pi)^3}\int_{-\infty}^{\infty} d^4 P_f \; \delta(P_f^2 - M_0^2)\Theta(P_f^0).$$

The spin averaged differential cross section $d\overline{\sigma}$ is then

$$\begin{aligned} d\overline{\sigma} &= \frac{m_0}{|\mathbf{p}|}\,\overline{|M_{fi}|^2}(2\pi)^4\delta^4(P_f + p' - P_i - p) \\ &\quad \times \frac{m_0}{(2\pi)^3}|\mathbf{p}'|dE'd\Omega' \times \frac{2M_0}{(2\pi)^3}\int_{-\infty}^{\infty} d^4 P_f \; \delta(P_f^2 - M_0^2)\Theta(P_f^0) \qquad (2.413)\end{aligned}$$

Now integrating over dE' and d^3P_f we obtain

$$
\begin{aligned}
\frac{d\overline{\sigma}}{d\Omega'} &= \int dE'|\mathbf{p}'|E' \, (d\overline{\sigma}) \\
&= \frac{m_0^2 M_0}{|\mathbf{p}|2\pi^2} \int dE'|\mathbf{p}'| \, \overline{|M_{fi}|^2} \delta\left((p' - P_i - p)^2 - M_0^2\right) \Theta(P_i^0 + E - E') \\
&= \frac{m_0^2 M_0}{|\mathbf{p}|2\pi^2} \int_{m_0}^{M_0+E} dE'|\mathbf{p}'| \, \overline{|M_{fi}|^2} \delta(2m_0^2 - 2(E' - E) - 2E'E2|\mathbf{p}||\mathbf{p}'|\cos\theta)
\end{aligned}
\tag{2.414}
$$

where the upper limit in the integral comes from the step function and the lower from the fact that E' cannot be less than m_0. Furthermore, the argument of the delta function has been expressed in terms of the kinematical variables of the laboratory frame. The remaining integral over E' can be performed using

$$
\delta\left(f(x)\right) = \sum_k \frac{\delta(x - x_k)}{|\frac{df}{dx}|_{x_k}}
$$

x_i being the roots of $f(x)$ inside the range of integration. We get

$$
\frac{d\overline{\sigma}}{d\Omega'} = \frac{m_0^2 M_0}{4\pi^2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \frac{\overline{|M_{fi}|^2}}{M_0 + E - |\mathbf{p}|(E'/|\mathbf{p}'|)\cos\theta} \tag{2.415}
$$

where we have used $d|\mathbf{p}'|/dE' = E'/|\mathbf{p}'|$ and we have for E'

$$
E'(M_0 + E) - |\mathbf{p}||\mathbf{p}'|\cos\theta = EM_0 + m_0^2 \tag{2.416}
$$

For given scattering angle θ the final energy E' of the electrons can be determined as a function of E and θ. The resulting E' and the corresponding $|\mathbf{p}'| = E'^2 - m_0^2$ have to be inserted into (2.415).

2.6 Scattering of Identical Fermions

We can take over many aspects of electron-proton scattering. But now because the two particles are of the same type there is no way to tell which of the two emerging electrons was the "incident" and which was the "target" particle. This is taken into account by adding the amplitudes for both processes but incuding a change of sign since we are exchanging two identical fermions. The resulting total amplitude is then

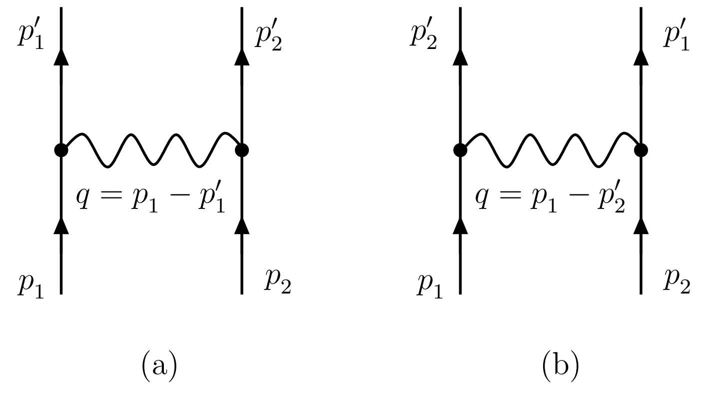


Figure 2.13:

$$\begin{aligned} S_{fi} &= -\frac{1}{2}\sqrt{\frac{m_0^2}{E_1 E_2}}\sqrt{\frac{m_0^2}{E'_1 E'_2}}(2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \\ & \times \Big\{ + [\overline{u}(p'_1, s'_1)(-ie\gamma_\mu)u(p_1, s_1)]\frac{i4\pi}{(p_1 - p'_1)^2 + i\epsilon}[\overline{u}(p'_2, s'_2)(-ie\gamma_\mu)u(p_2, s_2)] \\ & - [\overline{u}(p'_2, s'_2)(-ie\gamma_\mu)u(p_1, s_1)]\frac{i4\pi}{(p_1 - p'_2)^2 + i\epsilon}[\overline{u}(p'_1, s'_1)(-ie\gamma_\mu)u(p_2, s_2)] \Big\} \end{aligned} \tag{2.417}$$

2.6.1 Averaging over Spin

The squared invariant matrix element averaged over initial and final spin is

$$\begin{aligned} & \overline{|M_{fi}|^2} \\ = \; & e^4 (4\pi)^2 \frac{1}{4} \sum_{s'_1, s_1} \sum_{s'_2, s_2} \Bigg| \overline{u}(p'_1, s'_1)\gamma_\mu u(p_1, s_1)\frac{1}{(p_1 - p'_1)^2}\overline{u}(p'_2, s'_2)\gamma^\mu u(p_2, s_2) \\ & - \overline{u}(p'_2, s'_2)\gamma_\mu u(p_1, s_1)\frac{1}{(p_1 - p'_2)^2}\overline{u}(p'_1, s'_1)\gamma^\mu u(p_2, s_2) \Bigg|^2 \end{aligned} \tag{2.418}$$

We gain familiarity with calculating spin averaging, the reader will be left to put together the results to obtain the final answer. Consider the mod-squared terms in the square $|\cdots|^2$. Take the first such term:

$$
\begin{aligned}
&\sum_{s_1',s_1}\sum_{s_2',s_2}\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right)\\
&\qquad\times\left(\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\right)^*\,\left(\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right)^*\\
=&\left[\sum_{s_1',s_1}\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\right)^*\right]\\
&\quad\times\left[\sum_{s_2',s_2}\left(\overline{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\right)\,\left(\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right)^*\right]
\end{aligned}
\tag{2.419}
$$

It is sufficient to consider just the first sum over s_1', s_1, since the second term s_2', s_2 has the same structure. The reader should see section 2.14.8 on how to turn this into the following trace

$$
Tr\left[\gamma_\mu\frac{p_{1\alpha}\gamma^\alpha+m_0}{2m_0}\gamma_\nu\frac{p_{1\beta}'\gamma^\beta+m_0}{2m_0}\right].
\tag{2.420}
$$

Now we consider the more complicated mixed terms in the square $|\cdots|^2$. The first of these is

$$
\begin{aligned}
&\sum_{s_1',s_1}\sum_{s_2',s_2}\left[\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right)\right]\\
&\qquad\times\left[\left(\overline{u}(p_2',s_2')\gamma_\nu u(p_1,s_1)\right)\,\left(\overline{u}(p_1',s_1')\gamma^\nu u(p_2,s_2)\right)\right]^*\\
&\sum_{s_1',s_1}\sum_{s_2',s_2}\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_1,s_1)\gamma_\nu u(p_2',s_2')\right)\\
&\qquad\times\left(\overline{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\right)\,\left(\overline{u}(p_2,s_2)\gamma_\nu u(p_1',s_1')\right)\\
=&\sum_{s_1'}\sum_{s_2'}\left(\overline{u}(p_1',s_1')\gamma_\mu\frac{p_{1\alpha}\gamma^\alpha+m_0}{2m_0}\gamma_\nu u(p_2',s_2')\right)\\
&\qquad\times\left(\overline{u}(p_2',s_2')\gamma_\mu\frac{p_{2\alpha}\gamma^\alpha+m_0}{2m_0}\gamma_\nu u(p_1',s_1')\right)
\end{aligned}
\tag{2.421}
$$

where we have used the identity

$$
\sum_{s_1}u_\alpha(p_1,s_1)\overline{u}_\beta(p_1,s_1)=\left(\frac{p_{1\alpha}\gamma^\alpha+m_0}{2m_0}\right)_{\alpha\beta}
$$

We use this identity another two times to obtain the final result

$$Tr\left[\gamma_\mu \frac{p_{1\alpha}\gamma^\alpha + m_0}{2m_0}\gamma_\nu \frac{p'_{2\beta}\gamma^\beta + m_0}{2m_0}\gamma^\mu \frac{p_{2\gamma}\gamma^\gamma + m_0}{2m_0}\gamma^\nu \frac{p'_{1\delta}\gamma^\delta + m_0}{2m_0}\right]. \tag{2.422}$$

In section 2.14.1 we provide theorems on the trace of $\gamma-$matrices sufficient for the reader to evaluate the above expresions.

2.7 Electron-Positron Scattering

2.7.1 Scattering of an Positron off a Coulomb Potential

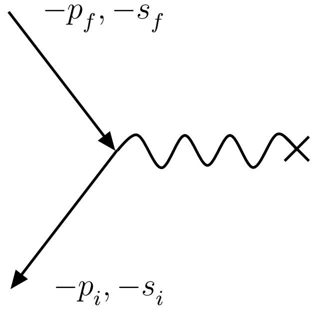


Figure 2.14: The incoming positron with momentum p_i and spin s_i is described by an outgoing electron with negative energy, with momentum $-p_i$ and spin $-s_i$. Similarly for the outgoing positron.

$$S_{fi} = -iZe^2\frac{1}{V}\sqrt{\frac{m_0^2}{E_f E_i}}\overline{v}(p_i, s_i)\gamma^0 v(p_f, s_f)\ \int d^4x\ e^{i(p_f-p_i)\cdot x}\frac{1}{|x|}. \tag{2.423}$$

2.7.2 Electron-Positron Scattering Amplitude

Make the replacements

an incoming electron spinor $u(p_i, s_i) \rightarrow$ an outgoing positron spinor $\overline{v}(p_f, s_f)$

$$
\begin{aligned}
S_{fi}(dir.) &= -\frac{1}{2}\sqrt{\frac{m_0^2}{E_1\overline{E}_2'}}\sqrt{\frac{m_0^2}{E_1'\overline{E}_2}}(2\pi)^4\delta^4(p_1+\overline{p}_2'-p_1'-\overline{p}_2) \\
&\times\left[\overline{u}(p_1',s_1')(-ie\gamma_\mu)u(p_1,s_1)\right]\frac{i4\pi}{(p_1-p_1')^2+i\epsilon}\left[\overline{v}(\overline{p}_2,\overline{s}_2)(-i(-e)\gamma_\mu)v(p_2',s_2')\right]
\end{aligned}
\tag{2.424}
$$

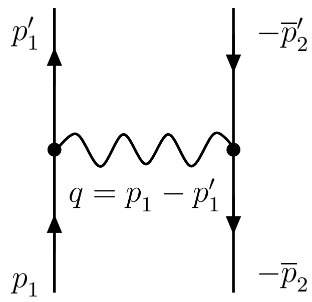


Figure 2.15:

The exchange amplitude

$$
\begin{aligned}
S_{fi}(exch.) &= \frac{1}{2}\sqrt{\frac{m_0^2}{E_1E_2}}\sqrt{\frac{m_0^2}{E_1'E_2'}}(2\pi)^4\delta^4(p_1+\overline{p}_2'-p_1'-\overline{p}_2) \\
&\times\left[\overline{v}(\overline{p}_2,\overline{s}_2)(-i(-e)\gamma_\mu)u(p_1,s_1)\right]\frac{i4\pi}{(p_1-p_1')^2+i\epsilon}\left[\overline{u}(p_1',s_1')(-ie\gamma_\mu)u(\overline{p}_2',\overline{s}_2')\right]
\end{aligned}
\tag{2.425}
$$

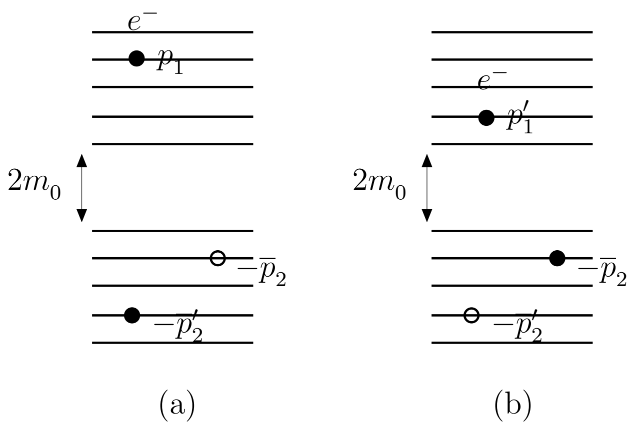


Figure 2.16: (a) The intial state. (b) The final state.

The exclusion principle requires that antisymmetric combinations of amplitudes be chosen for processes which differ only by an exchange of particles. In the final state, fig (**??**), has to be antisymmetric with respect to the exchange $p_1' \leftrightarrow -\overline{p}_2$

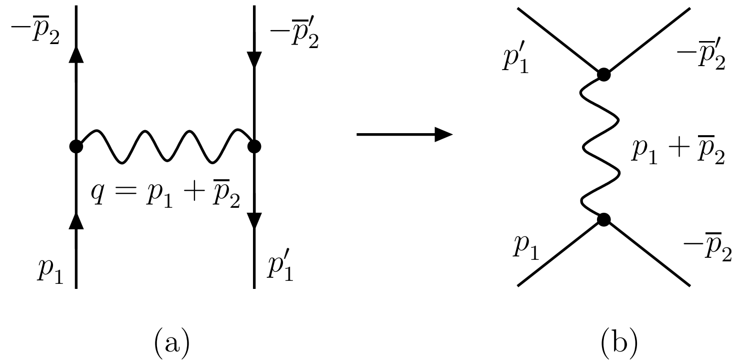


Figure 2.17: (a) . (b) The exchange graph is usually written this way.

2.7.3 Remarks on the Form of the $S-$Matrix element

2.7.4 Crossing Symmetry

The squared invariant matrix element for electron-positron scattering can be obtained from the squared invariant matrix element for electron-electron scattering by making the following substitutions of four-momenta

$$\begin{aligned} p_1 &\rightarrow p_1 \\ p_1' &\rightarrow p_1' \\ p_2 &\rightarrow -\overline{p}_2' \\ p_2' &\rightarrow -\overline{p}_2. \end{aligned} \tag{2.426}$$

2.8 Scattering of Polarised Dirac Particles

s^μ is a Lorentz vector which is properly defined in the rest system of the particle where it reduces to a spacial unit vector

$$(s^\mu)_{RS} = (0, \mathbf{s}'). \tag{2.427}$$

We wish to obtain the components of s^μ in a frame in which the particle moves with momentum $\mathbf{p}$. What is the Lorentz transformation formula os an aritrary four-vecto a^μ for when $\mathbf{v}$ is not parallel to the x-axis? Consider

$$a^{0'} = \gamma\left(a^0 - \mathbf{v}\cdot\mathbf{a}\right), \qquad \mathbf{a}' = \mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{v^2}(\gamma-1) - \gamma a^0\right)\mathbf{v} \tag{2.428}$$

where

$$\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}.$$

Specialising to $\mathbf{v} = (v,0,0)$ we find

$$a^{0'} = \gamma\left(a^0 - v a_x\right), \tag{2.429}$$

$$(a'_x, a'_y, a'_z) = (a_x, a_y, a_z) - \left(\frac{\mathbf{v}\cdot a}{v^2}(1-\gamma) - \gamma a_t\right)(v,0,0) \tag{2.430}$$

which reads separately

$$a'_t = \gamma(a_t - v a_x), \quad a'_x = \gamma(a_x - v a_t), \quad a'_y = a_y, \quad a'_z = a_z \tag{2.431}$$

We want the inverse of this transformation which is easily obtained by making the replacement $\mathbf{v} \to -\mathbf{v}$. We have that $s^{0'} = 0$. We obtain for s^μ,

$$s^\mu = \left[\gamma \mathbf{v}\cdot\mathbf{s}'\ ,\ \mathbf{s}' + \frac{\mathbf{v}\cdot\mathbf{s}'}{v^2}(\gamma-1)\mathbf{v}\right] \tag{2.432}$$

We use the following:

$$\mathbf{v} = \frac{\mathbf{p}}{E}, \tag{2.433}$$

$$E/\gamma = m_0, \tag{2.434}$$

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\mathbf{p}^2/E^2}} \\ &= \frac{E}{\sqrt{E^2-\mathbf{p}^2}} = \frac{E}{m_0},\end{aligned} \tag{2.435}$$

$$\begin{aligned}\frac{\gamma-1}{E^2-\mathbf{p}^2} &= \frac{E/m_0-1}{(E+m_0)(E-m_0)}\\ &= \frac{1}{m_0(E+m_0)},\end{aligned}\tag{2.436}$$

so that () finally becomes

$$s^\mu = \left[\frac{\mathbf{p}\cdot\mathbf{s}'}{m_0}\ ,\ \mathbf{s}' + \frac{\mathbf{p}\cdot\mathbf{s}'}{m_0(E+m_0)}\mathbf{p}\right]\tag{2.437}$$

Because of the Lorentz invariance of the four-diemnsional scalar product it follows

$$s_\mu s^\mu = (s_\mu)_{RS}(s^\mu)_{RS} = -\mathbf{s}\cdot\mathbf{s} = -1\tag{2.438}$$

and

$$p^\mu s_\mu = (p^\mu)_{RS}(s_\mu)_{RS} = (m_0, 0, 0, 0)\begin{pmatrix} 0 \\ -s_x \\ -s_y \\ -s_z \end{pmatrix} = 0.\tag{2.439}$$

So we have the normalisation and orthogonality relations

$$s^2 = -1, \quad p\cdot s = 0.\tag{2.440}$$

We will specialise to helicity states, namely states where the spin points in the direction (or opposite direction) of the momentum:

$$s'_\lambda = \lambda\frac{\mathbf{p}}{|\mathbf{p}|}, \quad \text{where} \quad \lambda = \pm 1.\tag{2.441}$$

Substituting this into (2.437) leads to the spin four-vector

$$s^\mu = \lambda\left(\frac{|\mathbf{p}|}{m_0}\ ,\ \frac{\mathbf{p}}{|\mathbf{p}|} + \frac{\mathbf{p}^2}{m_0(E+m_0)}\frac{\mathbf{p}}{|\mathbf{p}|}\right) = \lambda\left(\frac{|\mathbf{p}|}{m_0}\ ,\ \frac{E}{m_0}\frac{\mathbf{p}}{|\mathbf{p}|}\right)\tag{2.442}$$

After this preliminar work we now look the cross section for Coulomb scattering.

2.8.1 Polarised Electron Scattering of a Coulomb Potential

$$\frac{d\sigma}{d\Omega}(s_i, s_f) = \frac{4Z^4\alpha^2 m_0^2}{|q|^4}|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2 \tag{2.443}$$

We introduce auxialiary summations over the spin orientations s_i and s_f using the spin projection operator $\mathbf{\Sigma}(s)$ which suppresses the "wrong" spin state $u(p, -s)$.

$$\begin{aligned}\frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\left(\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\right)\left(u^\dagger(p_i, s_i)\gamma_0^\dagger\gamma_0^\dagger u(p_f, s_f)\right) \\ &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\sum_{s_i', s_f'}\left(\overline{u}(p_f, s_f')\gamma^0\hat{\Sigma}(s_i)u(p_i, s_i')\right)\left(\overline{u}(p_i, s_i')\gamma^0\hat{\Sigma}(s_f)u(p_f, s_f')\right)\end{aligned} \tag{2.444}$$

The same calculation as in (2.362) but with the replacement $\gamma^0_{\alpha\beta} \to \sum_{\delta=1}^4 \gamma^0_{\alpha\delta}(\hat{\Sigma})_{\delta\beta}$

$$\begin{aligned}\frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\,Tr\left[\gamma_0\hat{\Sigma}(s_i)\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\hat{\Sigma}(s_f)\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right] \\ &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\,Tr\left[\gamma_0\frac{1+\gamma_5 s_i^\sigma\gamma_\sigma}{2}\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\frac{1+\gamma_5 s_f^\delta\gamma_\delta}{2}\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right]\end{aligned} \tag{2.445}$$

2.8.2 When the Incoming Beam is Unpolarised

Before examinig the above we warm up by looking at when the scattering process in which the incoming beam is unpolarised, and ask if polarisation of the scaterred particle takes place. The above cross section is then replaced by

$$\frac{d\sigma}{d\Omega}(s_i, s_f) = \frac{1}{2}\frac{4Z^4\alpha^2 m_0^2}{|q|^4}\,Tr\left[\gamma_0\frac{1+\gamma_5 s_i^\sigma\gamma_\sigma}{2}\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right] \tag{2.446}$$

The factor 1/2 comes from averaging over the initial spins.

Expanding (2.446) we find the traces

$$Tr\gamma_0\gamma_5\gamma_\sigma\gamma_0, \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_\mu\gamma_0\gamma_\nu, \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_\mu\gamma_0, \quad \text{and} \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_0\gamma_\nu.$$

It is easily seen this reduces to evaluating

$$Tr\gamma_5\gamma_\sigma, \quad Tr\gamma_5\gamma_\sigma\gamma_\mu\gamma_\nu, \quad \text{and} \quad Tr\gamma_5\gamma_\sigma\gamma_\mu.$$

The first two obviously vanish as the trace of the product of γ_5 and an odd number of γ matrices is zero

$$Tr\gamma_5\gamma_\mu\ldots\gamma_\nu = (-1)^n Tr\gamma_\mu\ldots\gamma_\nu\gamma_5 = (-1)^n Tr\gamma_5\gamma_\mu\ldots\gamma_\nu$$

where we first used $\gamma_5\gamma_\mu = \gamma_\mu\gamma_5$ and then the cyclic property of the trace. We consider the last trace. Say first that $\mu = \nu$ then

$$Tr\gamma_5\gamma_\mu\gamma_\nu = Tr\gamma_5(\gamma_\mu)^2 = \eta^{\mu\mu}Tr\gamma_5.$$

As $\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$ and thus inparticular $\gamma^0\gamma^5 = -\gamma^5\gamma^0$ we get

$$\begin{aligned} Tr\gamma_5 &= Tr\gamma_5(\gamma^0)^2 \\ &= -Tr\gamma^0\gamma_5\gamma^0 \\ &= -Tr\gamma_5(\gamma^0)^2 \\ &= -Tr\gamma_5 = 0. \end{aligned}$$

Now if $\mu \neq \nu$ we choose λ that differs from μ and ν and use $\gamma_\lambda\gamma_5\gamma_\mu\gamma_\nu = (-1)^3\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda$

$$\begin{aligned} Tr\gamma_5\gamma_\mu\gamma_\nu &= Tr\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1}\gamma_\lambda \\ &= Tr\gamma_\lambda\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1} \\ &= (-1)^3 Tr\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\lambda^{-1} \\ &= -Tr\gamma_5\gamma_\mu\gamma_\nu = 0. \end{aligned}$$

Thus the cross section is independent of the final spin and agrees with half the unpolarised Mott scattering cross section

$$\frac{d\sigma}{d\Omega}(s_f) = \frac{1}{2}\frac{d\sigma_{Mott}}{d\Omega} \tag{2.447}$$

Thus at first order in perturbatin theory Coulomb scattering of electrons does not lead to polarisation of the incoming beam.

2.8.3 Polarised Scattering

We assume that the spin of the incoming electron is parallel to its direction of motion, i.e. it has well defined helicity $\lambda_i = +1$

$$
\begin{aligned}
s_{i_{\lambda_i}} &= \lambda_i \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0} \frac{\mathbf{p}_i}{|\mathbf{p}|} \right) \equiv \lambda_i s_i \\
s_{f_{\lambda_f}} &= \lambda_i \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0} \frac{\mathbf{p}_f}{|\mathbf{p}|} \right) \equiv \lambda_f s_f
\end{aligned} \tag{2.448}
$$

Dropping terms we know to zanish from the previous example, the polarised scattering cross section becomes

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0 \frac{1+\gamma_5 s_i^\sigma \gamma_\sigma}{2} \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma_0 \frac{1+\gamma_5 s_f^\mu \gamma_\mu}{2} \frac{(p_f)_\nu \gamma^\nu + m_0}{2m_0}\right] \\
&= \frac{4Z^4\alpha^2 m_0^2}{|q|^4} \frac{1}{4} \frac{1}{(2m_0)^2} \Big(Tr[\gamma_0((p_i)_\mu\gamma^\mu + m_0)\gamma_0((p_f)_\nu\gamma^\nu + m_0)] \\
&\quad + \lambda_i \lambda_f\, Tr[\gamma_0\gamma_5 s_i^\sigma \gamma_\sigma((p_i)_\mu\gamma^\mu + m_0)\gamma_0\gamma_5 s_f^\delta \gamma_\delta((p_f)_\nu\gamma^\nu + m_0)]\Big)
\end{aligned} \tag{2.449}
$$

Here we define the degree of polarisation P of the scattered particles by as the difference between counting rates for the positive and negative helicities, normalised by the total counting rate:

$$
P = \frac{d\sigma(\lambda_f = +1) - d\sigma(\lambda_f = -1)}{d\sigma(\lambda_f = +1) + d\sigma(\lambda_f = -1)} \tag{2.450}
$$

If the initial state is fully polarised, e.g. $\lambda_i = +1$, the final degree of polarisation becomes, using ()

$$
P = \frac{Tr[\gamma_0\gamma_5 s_i^\sigma \gamma_\sigma((p_i)_\mu\gamma^\mu + m_0)\gamma_0\gamma_5 s_f^\mu \gamma_\mu((p_f)_\nu\gamma^\nu + m_0)]}{Tr[\gamma_0((p_i)_\mu\gamma^\mu + m_0)\gamma_0((p_f)_\nu\gamma^\nu + m_0)]} \tag{2.451}
$$

The evaluation of the trace in the denominator is done along the same lines as early calculations. Expand the numerator, using that the trace of an odd number of γ matrices vanishes,

$$
\begin{aligned}
& Tr[\gamma_0\gamma_5 s_i^\sigma\gamma_\sigma((p_i)^\mu\gamma^\mu + m_0)\gamma_0\gamma_5 s_f^\mu\gamma_\mu((p_f)^\nu\gamma^\nu + m_0) \\
= \; & s_i^\sigma p_i^\mu s_f^\delta p_f^\nu \; Tr[\gamma_0\gamma_\sigma\gamma_\mu\gamma_0\gamma_\delta\gamma_\nu] + m_0^2 s_i^\sigma s_f^\delta \; Tr[\gamma_0\gamma_\sigma\gamma_0\gamma_\delta]
\end{aligned} \tag{2.452}
$$

To evaluate the first trace we generalise the result of () to arbitrary even number of γ matrices. We know

$$
Tr\gamma^{\mu_1}\dots\gamma^{\mu_n} = 2\eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\dots\gamma^{\mu_n} - Tr\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\dots\gamma^{\mu_n}
$$

Repeating this procedure we get

$$
\begin{aligned}
Tr\gamma^{\mu_1}\dots\gamma^{\mu_n} \;=\; & 2\eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\dots\gamma^{\mu_n} - \\
& +2\eta^{\mu_1\mu_n}Tr\gamma^{\mu_2}\dots\gamma^{\mu_n} \\
& -Tr\gamma^{\mu_2}\dots\gamma^{\mu_n}\gamma^{\mu_1}
\end{aligned}
$$

Using the cyclic property of traces we get

$$
Tr\gamma^{\mu_1}\dots\gamma^{\mu_n} = \eta^{\mu_1\mu_2}\; Tr\gamma^{\mu_3}\dots\gamma^{\mu_n} - \dots + \eta^{\mu_1\mu_n}\; Tr\gamma^{\mu_2}\dots\gamma^{\mu_n} \tag{2.453}
$$

To do the calculation we need the following scalar products:

It satifies the orthogonality relations. Firstly

$$
\begin{aligned}
p_i \cdot s_i \;=\; & (E, \mathbf{p}_i) \cdot \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_i}{|\mathbf{p}|} \right) \\
=\; & \frac{E}{m_0}(|\mathbf{p}| - \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{|\mathbf{p}|}) = 0.
\end{aligned} \tag{2.454}
$$

Similarly we have

$$
p_f \cdot s_f = 0. \tag{2.455}
$$

$$
\begin{aligned}
p_i \cdot s_f \;=\; & (E, \mathbf{p}_i) \cdot \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_f}{|\mathbf{p}|} \right) \\
=\; & \frac{E}{m_0}(|\mathbf{p}| - \frac{\mathbf{p}_i \cdot \mathbf{p}_f}{|\mathbf{p}|}) \\
=\; & \frac{E|\mathbf{p}|}{m_0}(1 - \cos\theta)
\end{aligned} \tag{2.456}
$$

$$p_f \cdot s_i = \frac{E|\mathbf{p}|}{m_0}(1 - \cos\theta) \tag{2.457}$$

Similarly we have

$$\begin{aligned} s_i \cdot s_f &= \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_i}{|\mathbf{p}|}\right) \cdot \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_f}{|\mathbf{p}|}\right) \\ &= \frac{1}{m_0^2}(\mathbf{p}^2 - \frac{E^2}{\mathbf{p}^2}\mathbf{p}_i \cdot \mathbf{p}_f) \\ &= \frac{1}{m_0^2}(\mathbf{p}^2 - E^2 \cos\theta) \end{aligned} \tag{2.458}$$

$$p_i \cdot p_f = E^2 - \mathbf{p}^2 \cos\theta \tag{2.459}$$

We leave the details of the calculation to the reader. The result leads to

$$P = 1 - \frac{2 \sin\frac{\theta}{2}}{\left(\frac{E}{m_0}\right)^2 \cos\frac{\theta}{2} + \sin\frac{\theta}{2}} \tag{2.460}$$

In the nonrelativistic limit $E \to m_0$ this reduces to

$$P \simeq 1 - 2\sin\frac{\theta}{2} = \cos\theta. \tag{2.461}$$

2.9 Bremsstrahlung

When electrons scatter at protons or in the field of a nucleus, they can emit real photons. Bremsstrahlung is a second order process

$$S_{fi}^{(2)} = -ie^2 \int d^4y d^4x \overline{\psi}_f(x) A_\mu(x) \gamma^\mu S_F(x-y) A_\nu(y) \gamma^\nu \psi_i(y) \tag{2.462}$$

the outgoing photon by

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\, \epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{2.463}$$

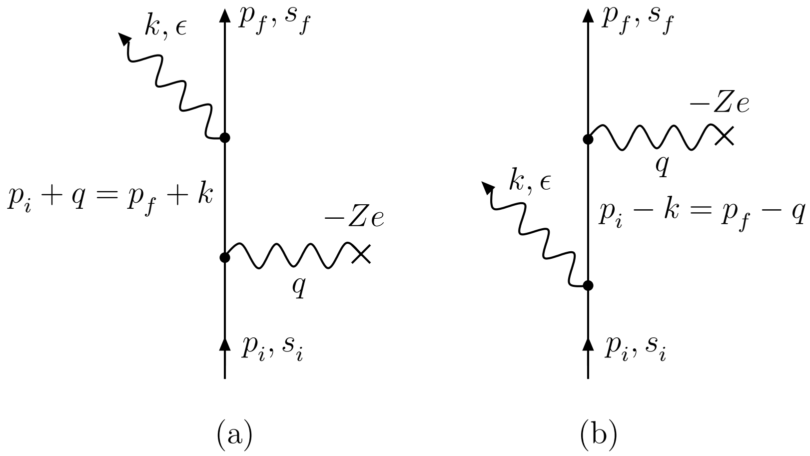


Figure 2.18: (a) . (b) .

the incoming electron

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot x} \tag{2.464}$$

the outgoing electron

$$\psi_f(x) = \sqrt{\frac{m_0}{E_f V}} u(p_f, s_f) e^{ip_f \cdot x}. \tag{2.465}$$

and the Coulomb potetial is

$$A_0^{coul}(x) = -\frac{Ze}{|\mathbf{x}|} \tag{2.466}$$

$$\begin{aligned} S_{fi} &= e^2 \int d^4y d^4x \overline{\psi}_f(x) \Big[(-iA_\mu(x,k)\gamma^\mu) iS_F(x-y)(-i\gamma^0)A_0^{coul}(y) \\ &\quad + (-i\gamma^0)A_0^{coul}(x) iS_F(x-y)(-iA_\mu(y,k)\gamma^\mu)\Big] \psi_i(x) \end{aligned} \tag{2.467}$$

Again we transform to momentum space. The Fourier transformation of the Coulomb potential

$$-\frac{Ze}{|\mathbf{x}|} = -Ze 4\pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{|\mathbf{q}|^2} e^{-iq\cdot x} \tag{2.468}$$

Substituting all of the above into (2.467)

$$\begin{aligned}
S_{fi} &= e^2 \int d^4y d^4x \left(\sqrt{\frac{m_0}{E_f V}} \overline{u}(p_f, s_f) e^{ip_f \cdot x} \right) \left[-i \left(\sqrt{\frac{4\pi}{2\omega V}}\, \epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \right) \gamma^\mu \times \right. \\
&\quad \times \left(\int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{p_\alpha \gamma^\alpha - m_0 + i\epsilon} \right) \times (-i\gamma^0) \left. \left(-Ze4\pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{|\mathbf{q}|^2} e^{-iq\cdot y} \right) \right] \\
&\quad \times \left(\sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot y} \right) + exch. \qquad (2.469)
\end{aligned}$$

rearanging

$$\begin{aligned}
S_{fi} &= -\frac{Ze^3 4\pi}{V^{3/2}} \sqrt{\frac{4\pi}{2\omega}} \sqrt{\frac{m_0^2}{E_f E_i}} \int d^4y d^4x \frac{d^3q}{(2\pi)^3} \frac{d^4p}{(2\pi)^4} \\
&\quad \times \overline{u}(p_f, s_f) e^{ip_f\cdot x} \left[-i\epsilon_\mu \gamma^\mu (e^{-ik\cdot x} + e^{ik\cdot x})\, \frac{i e^{-ip\cdot(x-y)}}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}\, (-i\gamma^0) \frac{e^{-iq\cdot y}}{|\mathbf{q}|^2} \right. \\
&\quad \left. (-i\gamma^0) \frac{e^{-iq\cdot y}}{|\mathbf{q}|^2}\, \frac{i e^{-ip\cdot(x-y)}}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}\, (-i\epsilon_\mu\gamma^\mu)(e^{-ik\cdot x} + e^{ik\cdot x}) \right] u(p_i, s_i) e^{-ip_i\cdot y}
\end{aligned}$$

(2.470)

Peforming the integrations

$$\begin{aligned}
& \int d^4x (e^{ix\cdot(p_f - k - p)} + e^{ix\cdot(p_f + k - p)}) \int d^4y e^{iy\cdot(-q+p-p_i)} \\
&= [(2\pi)^4 \delta^4(p_f - k - p) + (2\pi)^4 \delta^4(p_f + k - p)](2\pi)^4 \delta^4(q + p - p_i) \\
& \int d^4y (e^{iy\cdot(p - k - p_i)} + e^{iy\cdot(p + k - p_i)}) \int d^4x e^{ix\cdot(p_f + q - p)} \\
&= [(2\pi)^4 \delta^4(p - k - p_i) + (2\pi)^4 \delta^4(p + k - p_i)](2\pi)^4 \delta^4(-q + p - p_i)
\end{aligned}$$

The $S-$matrix becomes

$$
\begin{aligned}
S_{fi} &= -\frac{Ze^3 4\pi}{V^{3/2}}\sqrt{\frac{4\pi}{2\omega}}\sqrt{\frac{m_0^2}{E_f E_i}}\int d^4y d^4x \frac{d^3q}{(2\pi)^3}\frac{d^4p}{(2\pi)^4} \\
&\times \Big\{ [(2\pi)^4\delta^4(p_f - k - p) + (2\pi)^4\delta^4(p_f + k - p)](2\pi)^4\delta^4(q + p - p_i) \\
&\times \overline{u}(p_f, s_f)(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}(-i\gamma^0)\frac{1}{|\mathbf{q}|^2}u(p_i, s_i) \\
&+ [(2\pi)^4\delta^4(p - k - p_i) + (2\pi)^4\delta^4(p + k - p_i)](2\pi)^4\delta^4(-q + p - p_i) \\
&\times \overline{u}(p_f, s_f)(-i\gamma^0)\frac{1}{|\mathbf{q}|^2}\frac{i}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}(-i\epsilon_\mu\gamma^\mu)u(p_i, s_i)\Big\} \qquad (2.471)
\end{aligned}
$$

In the following we will need the formula

$$
\int dx\delta(x - a)\delta(x - b) = \delta(a - b).
$$

Consider the momentum integrals coming from the direct graph, we find

$$
\begin{aligned}
& \int \frac{d^3q}{(2\pi)^3}\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^4(p_f \pm k - p)(2\pi)^4\delta^4(p - q - p_i)f(p, |\mathbf{q}|) \\
&= \int \frac{d^3q}{(2\pi)^3}(2\pi)^4\delta^4(p_f \pm k - q - p_i)f(p, |\mathbf{q}|) \\
&= 2\pi\delta(E_f - E_i \pm \omega)f(p, |\mathbf{q}|) \qquad (2.472)
\end{aligned}
$$

where $q = p_f \pm k - p_i$ and $p = p_f \pm k$. There is something similary for the exchange graph.

Since we want to describe photon emmission the electron loses energy, $E_f < E_i$, which correspnds to $E_f = E_i - \omega$ - this is the arrangement measured experimentally. The $S-$matrix we require to describe emmision of a photon is then

$$
\begin{aligned}
S_{fi} &= -Ze^3 2\pi\delta(E_f + \omega - E_i)\sqrt{\frac{4\pi}{2\omega V}}\sqrt{\frac{m_0^2}{E_f E_i V^2}}\frac{4\pi}{|\mathbf{q}|^2} \\
&\times \overline{u}(p_f, s_f)\left[(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu - m_0}(-i\gamma_0)\right. \\
&\qquad \left. + (-i\gamma_0)\frac{i}{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu - m_0}(-i\epsilon_\mu\gamma^\mu)\right]u(p_i, s_i) \qquad (2.473)
\end{aligned}
$$

Using the relations $p_i^2 = p_f^2 = m_0^2$, $k^2 = 0$, we have

$$\begin{aligned}\frac{1}{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu - m_0 + i\epsilon} &= \frac{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu + m_0}{(p_f+k)^2 - m_0^2 + i\epsilon} \\ &= \frac{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu + m_0}{2p_f\cdot k + i\epsilon}. \qquad (2.474)\end{aligned}$$

2.9.1 Remarks on the Form of the $S-$Matrix element

• At the free vertex, where a free photon with polarisation vector ϵ_μ is emitted, a factor

i) $(-i\epsilon_\mu\gamma^\mu)$ occurs

ii) and the normalisation factor of the photon $\sqrt{4\pi/2\omega V}$ enters.

2.9.2 Bremsstrahlung Cross Section

We can simplify the notation by writing

$$S_{fi} = iZe^3 2\pi\delta(E_f + \omega - E_i)\sqrt{\frac{4\pi}{2\omega V}}\sqrt{\frac{m_0^2}{E_f E_i V^2}}\frac{4\pi}{|\mathbf{q}|^2}\,\epsilon^\mu M_\mu(k), \qquad (2.475)$$

where

$$M_\mu(k) = \overline{u}(p_f, s_f)\left[\gamma_\mu\frac{p_{f\alpha}\gamma^\alpha + k_\alpha\gamma^\alpha + m_0}{2p_f\cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{p_{i\alpha}\gamma^\alpha - k_\alpha\gamma^\alpha + m_0}{-2p_i\cdot k + i\epsilon}\gamma_\mu\right]u(p_i, s_i) \qquad (2.476)$$

The bremsstrahlung cross section is given by

$$\begin{aligned}d\sigma &= \frac{1}{\frac{|\mathbf{v_i}|}{V}T}\,|S_{fi}|^2\,\frac{Vd^3k}{(2\pi)^3}\frac{Vd^3p_f}{(2\pi)^3} \\ &= \frac{Z^2e^6}{|\mathbf{v_i}|}\frac{4\pi}{2\omega}\frac{m_0^2}{E_fE_i}\frac{(4\pi)^2}{|\mathbf{q}|^4}\,|\epsilon^\mu M_\mu(k)|^2\,2\pi\delta(E_f+\omega-E_i)\frac{d^3k}{(2\pi)^3}\frac{d^3p_f}{(2\pi)^3} \qquad (2.477)\end{aligned}$$

2.9.3 Sum Over Polarisations of Photon

We know that gauge invariance implies the condition of conservered current

$$\frac{\partial J_\mu(x)}{\partial x_\mu} = 0, \tag{2.478}$$

where $J_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$. In momentum space the conservation condition reads

$$k^\mu J_\mu(k) = 0 \tag{2.479}$$

Now the matrix element $M_\mu(k)$ given in () is a quantum mchanical transition current for Bremsstrahlung in lowest order perurbation theory and also satifies

$$k^\mu M_\mu(k) = 0 \tag{2.480}$$

This condition is easily verified using $k_\mu\gamma^\mu p_\nu\gamma^\nu = -p_\mu\gamma^\mu k_\nu\gamma^\nu + 2p \cdot k$ and the dirac equation:

$$\overline{u}(p_f, s_f)(p_{f\mu}\gamma^\mu - m_0) = 0, \quad (p_{i\mu}\gamma^\mu - m_0)u(p_i, s_i) = 0$$

$$\begin{aligned}
k^\mu M_\mu(k) &= \overline{u}(p_f, s_f)\left[k_\mu\gamma^\mu \frac{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu + m_0}{2p_f \cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{p_{i\mu}\gamma^\mu - k_\mu\gamma^\mu + m_0}{-2p_i \cdot k + i\epsilon}\, k_\nu\gamma^\nu\right] u(p_i, s_i) \\
&= \overline{u}(p_f, s_f)\left[\frac{-(p_{f\nu}\gamma^\nu - m_0)k_\mu\gamma^\mu + 2p_f \cdot k + k^2}{2p_f \cdot k + i\epsilon}\gamma_0\right. \\
&\quad \left. +\gamma_0\frac{-k_\nu\gamma^\nu(p_{i\mu}\gamma^\mu - m_0) + 2p_i \cdot k - k^2}{-2p_i \cdot k + i\epsilon}\right] u(p_i, s_i) \\
&= \overline{u}(p_f, s_f)\left[\frac{2p_f \cdot k}{2p_f \cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{2p_i \cdot k}{-2p_i \cdot k + i\epsilon}\right] u(p_i, s_i) \\
&= 0.
\end{aligned} \tag{2.481}$$

We now perform the summation over the photon polarisation vectors $\epsilon_\mu(\mathbf{k}, \lambda)$ with $\lambda = 1, 2$. The quantity of interest is

$$\overline{|\epsilon \cdot M_\mu(k)|^2} = \sum_{\lambda=1,2} |\epsilon_\mu(\mathbf{k}, \lambda)M^\mu(k)|^2 = \sum_{\lambda=1,2} \epsilon_\mu(\mathbf{k}, \lambda)\epsilon^*_\nu(\mathbf{k}, \lambda)M^\mu(k)M^{*\nu}(k) \tag{2.482}$$

We work in the radiation gauge and choose a particular coordinate which simplifies the calculation. Consider the coordinate system such that the momentum vector $\mathbf{k}$ points in the z-direction

$$k^\mu = \omega(1, 0, 0, 1) \tag{2.483}$$

We choose the two transverse polarisation vectors

$$\begin{aligned} \epsilon(\mathbf{k}, 1) &= (0, 1, 0, 0), \\ \epsilon(\mathbf{k}, 2) &= (0, 0, 1, 0). \end{aligned} \tag{2.484}$$

Now we use the condition of current conservation

$$k^\mu M^\mu = \omega(M^0 - M^3) = 0, \tag{2.485}$$

which implies $M^0 = M^3$. Therefore we can write

$$\overline{|\epsilon \cdot M|^2} = M^1 M^{*1} + M^2 M^{*2} + M^3 M^{*3} - M^0 M^{*0} = -M_\mu M^{*\mu} \tag{2.486}$$

Obviously this is Lorentz covariant. In general we have

$$\sum_{\lambda=1,2} \epsilon_\mu(\mathbf{k}, 1)\epsilon_\nu(\mathbf{k}, 2) = -\eta_{\mu\nu} + \text{gauge terms}. \tag{2.487}$$

The additional terms are proportional to k_μ or k_ν and thus do not contribute to any observable quantitiy since the sum is multiplied by conserved currents which satisfy $k \cdot J$.

2.9.4 The Infrared Catastrophe

A photon may be emmited which is too soft to be detected because of the energy resolution ΔE of the apparatus. Consequently, the experimental cross section is the sum the cross section for bremsstrahlung of energy less than ΔE and second order (raidiative corrected) elastic cross section, i.e.,

$$\left(\frac{d\sigma}{d\Omega'}\right)_{Exp} = \left(\frac{d\sigma}{d\Omega'}\right)_B + \left(\frac{d\sigma}{d\Omega'}\right)_{El}. \tag{2.488}$$

Here $(d\sigma/\Omega')_B$ is the soft bremsstrahlung cross section integrated over the range of photon energy $0 \ll \omega \ll \Delta E$ and $(d\sigma/\Omega')_{El}$ is the cross section for raidiative corrected elastic electron off the Coulomb potential.

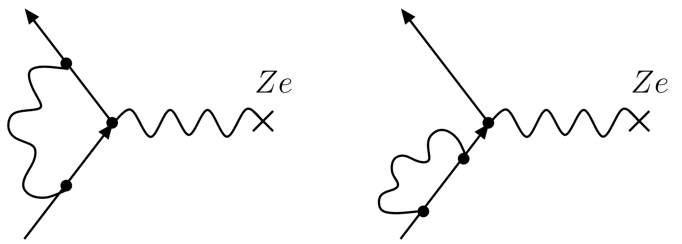


Figure 2.19: The two types of lowest order radiative corrections to elastic scattering of an electron of a Coulomb potential.

2.10 Compton Scattering

We describe the incoming photon as a plane wave:

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\,\epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{2.489}$$

and the outgoing (scattered) photon by

$$A'_\mu(x',k') = \sqrt{\frac{4\pi}{2\omega' V}}\,\epsilon_\mu(k,\lambda)(e^{-ik'\cdot x'} + e^{ik'\cdot x'}) \tag{2.490}$$

the incoming electron

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i\cdot x} \tag{2.491}$$

the outgoing electron

$$\psi_f(x) = \sqrt{\frac{m_0}{E_f V}} u(p_f, s_f) e^{ip_f\cdot x}. \tag{2.492}$$

$$\begin{aligned} S_{fi} \;\; = \;\; & e^2 \int d^4x d^4y\, \overline{\psi}_f(x) \Big[(-iA_\mu(y,k')\gamma^\mu)\; iS_F(x-y)(-iA_\nu(y,k)\gamma^\nu) \\ & + (-iA_\mu(y,k)\gamma^\mu)\; iS_F(x-y)(-iA_\nu(y,k')\gamma^\nu)\Big]\, \psi_i(y). \end{aligned} \tag{2.493}$$

From previous experience we know to write this in momentum space to be

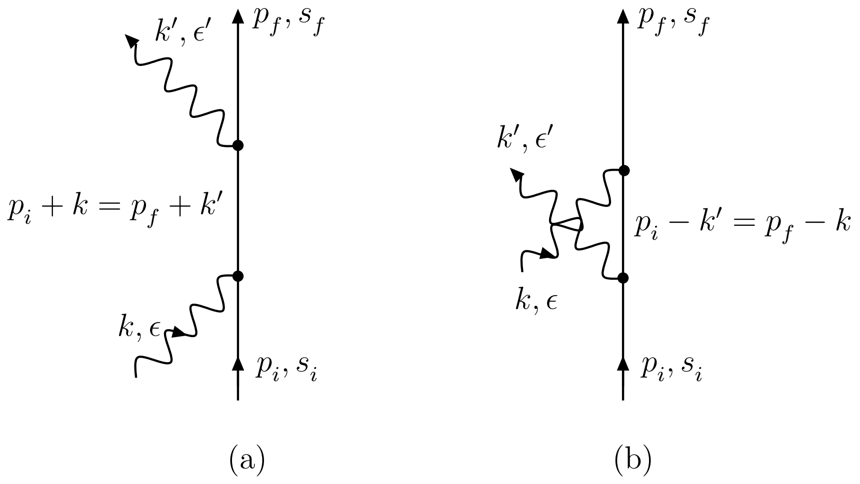


Figure 2.20: The direct and exchange diagrams describing Compton scattering.

$$
\begin{aligned}
S_{fi} &= \frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_i E_f}}\sqrt{\frac{(4\pi)^2}{2\omega 2\omega'}}\,(2\pi)^4\delta^4(p_f + k' - p_i - k) \\
&\times\ \overline{u}(p_f, s_f)\left[(-i\epsilon'_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu - m_0}(-i\epsilon_\sigma\gamma^\sigma)\right. \\
&\qquad\left. +(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu - k'_\nu\gamma^\nu - m_0}(-i\epsilon'_\sigma\gamma^\sigma)\right] u(p_i, s_i)
\end{aligned}
\tag{2.494}
$$

In going from (2.493) to (2.494) we integrated over plane waves which gave delta functions which impose energy-momentum conservation at the vertices. It results in four different processes, two of which are not allowed kinematically: the emission or absorption of two photons by a free electron. A third process is not compatible with the kinematic conditions fixed by the experiment. The process describing Compton scattering corresponds to the followings constraints on four momentum:

$$
+k + p_i = +k' + p_f \tag{2.495}
$$

The situation is similarly to what we encoutered in Bremsstrahlung in that not every term is physically relevant for the process considered. The term in the Compton scattering amplitude stems from the part $\exp(-ik \cdot x)$ of the photon field in (2.489) describing the absorption of a photon with four-momentum k^μ by the electron and from the part $\exp(-ik' \cdot x')$ of the photon field in (2.490) describing a photon emitted by the electron with four-momentum k'^μ.

2.10.1 Compton Scattering Cross Section

We split the S-matrix into two parts:

$$S_{fi} = -i\frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_i E_f}}\sqrt{\frac{(4\pi)^2}{2\omega 2\omega'}}(2\pi)^4\delta^4(p_f + k' - p_i - k)\ \epsilon^{\mu}(\mathbf{k}',\lambda')\epsilon^{\nu}(\mathbf{k},\lambda)\ M_{\mu\nu} \tag{2.496}$$

Here $M_{\mu\nu}$ is the Compton tensor

$$M_{\mu\nu} = \overline{u}(p_f,s_f)\left[\gamma_{\mu}\frac{p_{i\alpha}\gamma^{\alpha} + k_{\alpha}\gamma^{\alpha} + m_0}{2p_i\cdot k + i\epsilon}\gamma_{\nu} + \gamma_{\nu}\frac{p_{i\alpha}\gamma^{\alpha} - k'_{\alpha}\gamma^{\alpha} + m_0}{-2p_i\cdot k' + i\epsilon}\gamma_{\mu}\right]u(p_i,s_i) \tag{2.497}$$

we have

$$k'^{\mu}M_{\mu\nu} = k^{\nu}M_{\mu\nu} = 0. \tag{2.498}$$

The proof is analogous to the bremsstrahlung case.

The cross section starts as

$$d\sigma = \int\frac{|S_{fi}|^2}{T|\mathbf{v}_{rel}|/V}\frac{Vd^3p_f}{(2\pi)^3}\frac{Vd^3k'}{(2\pi)^3} \tag{2.499}$$

with

$$\frac{|S_{fi}|^2}{T} = \frac{|S_{fi}|^2}{VT/V}$$

being the transition rate per unit volume and normalised to one electron per volume. $|\mathbf{v_{rel}}|/V$ is the incoming photon flux.

$$d\sigma = \frac{e^4}{V^4}\frac{m_0^2}{E_i E_f} \tag{2.500}$$

We calculate the cross section in the laboratory frame.

$$p_i = (m_0, 0)$$

Also

$$|\mathbf{v}_{rel}| = |\mathbf{c} - \mathbf{v_e}| = |\mathbf{c}| = c$$

We use the covariant expression for the density of final states:

$$\frac{d^3p}{2E} = \int_{-\infty}^{\infty} d^4p \; \delta(p^2 - m_0^2)\Theta(p_0) \tag{2.501}$$

Averaging over initial and final electron spins and polarisations

$$\frac{1}{4}\sum_{pol}\sum_{spin} |\epsilon^\mu(\mathbf{k}', \lambda')\epsilon^\nu(\mathbf{k}, \lambda) \; M_{\mu\nu}|^2 = \frac{1}{4}\sum_{spin} M^{\mu\nu} M^*_{\mu\nu} \tag{2.502}$$

(on account of (2.498) and (2.487)).

the unpolarised differential cross section for Compton scattering is then

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m}\left(\frac{\omega'}{\omega}\right)^2 \left\{\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right\}. \tag{2.503}$$

2.11 Annihilation of Particle and Antiparticle

$$\begin{aligned} S_{fi} &= e^2 \int d^4x d^4y \; \overline{\psi}_+(x) \Big[(-iA_\mu(y, k')\gamma^\mu) \; iS_F(x-y)(-iA_\nu(y,k)\gamma^\nu) \\ &\quad + (-iA_\mu(y, k)\gamma^\mu) \; iS_F(x-y)(-iA_\nu(y,k')\gamma^\nu)\Big] \; \psi_-(y). \end{aligned} \tag{2.504}$$

To fit the experimental situation both photon outgoing plane waves should be used. We are lead to the following expression in momentum space:

$$\begin{aligned} S_{fi} &= \frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_+E_-}}\sqrt{\frac{(4\pi)^2}{\omega_1\omega_2}}(2\pi)^4\delta^4(k_1 + k_2 - p_+ - p_-) \\ &\times \; \overline{v}(p_+, s_+)\Bigg[(-i\epsilon_{2\mu}\gamma^\mu)\frac{i}{p_{-\alpha}\gamma^\alpha - k_{1\alpha}\gamma^\alpha - m_0}(-i\epsilon_{1\nu}\gamma^\nu) \\ &\quad +(-i\epsilon_{1\mu}\gamma^\mu)\frac{i}{p_{-\alpha}\gamma^\alpha - k_{2\alpha}\gamma^\alpha - m_0}(-i\epsilon_{2\nu}\gamma^\nu)\Bigg] \, u(p_-, s_-) \end{aligned} \tag{2.505}$$

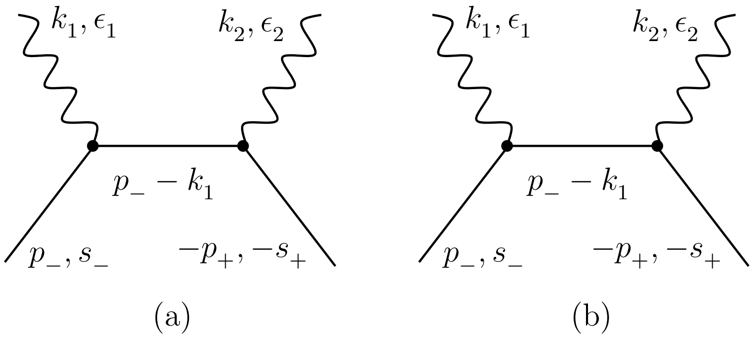


Figure 2.21: Direct and exchange graph of pair annihilation into two photons.

2.12 Second Order Electron-Proton Scattering

Before we state Feynman's rules for QED we wish to examine how to formulate the amplitude for a second order scattering problem, namely the secon order S-matrix for electron-proton scattering.

The amplitude for second order electron proton scattering

$$S_{fi}^{(2)} = -ie^2 \int d^4x d^4y \overline{\psi}_f(x) A_\mu \gamma^\mu S_F(x-y) A_\nu \gamma^\mu \psi_i(y) \tag{2.506}$$

The second order electron current is given by

$$J_{\mu\nu}^{(2)}(x,y) = ie^2 \overline{\psi}_f(x)\ \gamma_\mu S_F(x-y) \gamma_\nu\ \psi_i(y) \tag{2.507}$$

We generalise the relation

$$A^\mu(x) = \int d^4y\ D_F(x-y)\ J^\mu(y).$$

by conjecturing

$$A_\mu(x) A_\nu(y) = \int d^4X d^4Y\ D_F(x-X) D_F(y-Y) J_{\mu\nu}^{p(2)}(X,Y). \tag{2.508}$$

By symmetry the proton current $J_{\mu\nu}^{p(2)}(X,Y)$ should be

$$J_{\mu\nu}^{p(2)}(X,Y) = ie^2\ \overline{\psi}_f^p(X)\ \gamma_\mu S_F(X-Y) \gamma_\nu\ \psi_i^p(Y) \tag{2.509}$$

Substituting () and (2.508) into

$$
\begin{aligned}
S_{fi}^{(2)}(dir.) \;=\; & e^2 e_p^2 \int d^4x d^y d^4X d^4Y \\
& \times[\overline{\psi}_f(x)\gamma^\mu S_F(x-y)\gamma^\nu\, \psi_i(y)] \\
& \times D_F(x-X) D_F(y-Y) \\
& \times[\overline{\psi}^p_f(X)\gamma_\mu S_F(X-Y)\gamma_\nu\, \psi^p_i(Y)]
\end{aligned} \tag{2.510}
$$

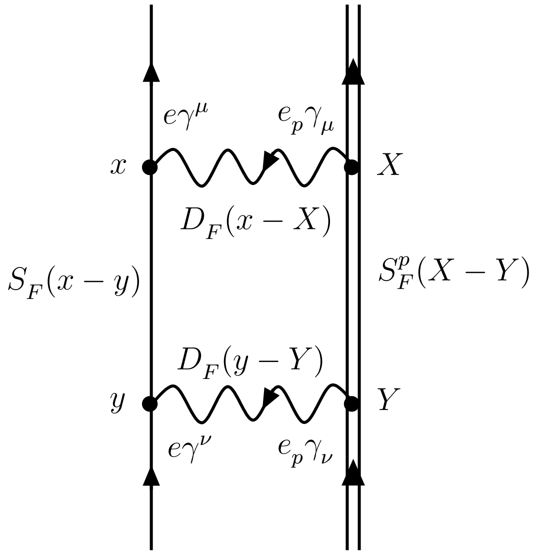


Figure 2.22:

Since the two photons emitted by the proton current are indistiguishable the electron at x does not know whether the photon absorbed there has been emitted at X or Y. According to quantum mechanics we must coherently add the contribution coming from the corresponding exchange graph in (2.12).

$$
\begin{aligned}
S_{fi}^{(2)}(exch.) \;=\; & e^2 e_p^2 \int d^4x d^y d^4X d^4Y \\
& \times[\overline{\psi}_f(x)\gamma^\mu S_F(x-y)\gamma^\nu\, \psi_i(y)] \\
& \times D_F(x-Y) D_F(y-X) \\
& \times[\overline{\psi}^p_f(X)\gamma_\nu S_F(X-Y)\gamma_\mu\, \psi^p_i(Y)]
\end{aligned} \tag{2.511}
$$

Notice how the indicies μ and ν in the 'proton current' term have been exchanged with respect to the direct term.

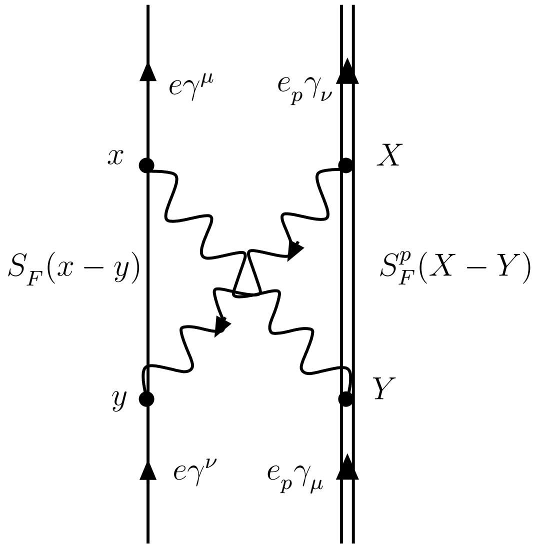


Figure 2.23:

2.12.1 Feynman Diagram in Momentum Space

As always, the external particles (incoming and outgoing electron and proton) are described by plane waves. The direct term becomes

$$\begin{aligned} S_{fi}^{(2)}(dir.) &= \frac{(4\pi)^2 e^4}{V^2} \int d^4x d^4y d^4X d^4Y \sqrt{\frac{m_0^2}{E_f E_i}} \sqrt{\frac{M_0^2}{E_f^p E_i^p}} \\ & \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4P}{(2\pi)^4} \end{aligned} \tag{2.512}$$

It is easy to perform the integration over spacetime coordinates. It results in the product δ^4-functions:

$$\begin{aligned} &(2\pi)^4\delta^4(q_1 + p - p_f)(2\pi)^4\delta^4(q_2 - p + p_i) \\ &\times(2\pi)^4\delta^4(-q_2 - P + P_i)(2\pi)^4\delta^4(-q_1 + P - P_f). \end{aligned} \tag{2.513}$$

Each δ^4-function expresses the energy-momentum conservation at one of the four vertices. Now we can integrate over

$$
\begin{aligned}
S^{(2)}_{fi}(dir.) &= \frac{(4\pi)^2 e^4}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E^p_f E^p_i}}\,(2\pi)^4\delta^4(P_f + p_f - P_i - p_i) \\
&\times \int \frac{d^4 q_1}{(2\pi)^4}\frac{1}{q_1^2 + i\epsilon}\frac{1}{(q-q_1)^2 + i\epsilon} \\
&\times \left[\overline{u}(p_f, s_f)\gamma^\mu \frac{1}{p_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - m_0 + i\epsilon}\gamma^\nu u(p_i, s_i)\right] \\
&\times \left[\overline{u}(P_f, S_f)\gamma_\mu \frac{1}{P_{f\alpha}\gamma^\alpha + q_{1\alpha}\gamma^\alpha - M_0 + i\epsilon}\gamma_\nu u(P_i, S_i)\right] \qquad (2.514)
\end{aligned}
$$

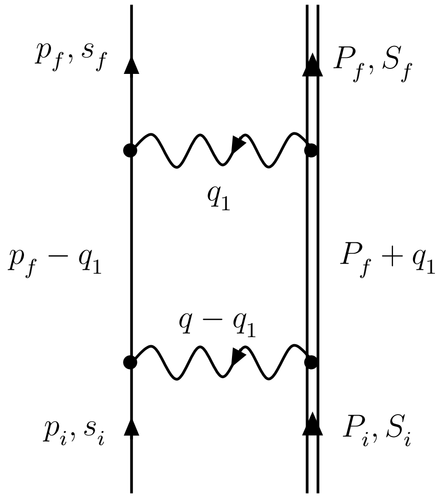


Figure 2.24:

$$
\begin{aligned}
S^{(2)}_{fi}(exch.) &= \frac{(4\pi)^2 e^4}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E^p_f E^p_i}}\,(2\pi)^4\delta^4(P_f + p_f - P_i - p_i) \\
&\times \int \frac{d^4 q_1}{(2\pi)^4}\frac{1}{q_1^2 + i\epsilon}\frac{1}{(q-q_1)^2 + i\epsilon} \\
&\times \left[\overline{u}(p_f, s_f)\gamma^\mu \frac{1}{p_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - m_0 + i\epsilon}\gamma^\nu u(p_i, s_i)\right] \\
&\times \left[\overline{u}(P_f, S_f)\gamma_\nu \frac{1}{P_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - M_0 + i\epsilon}\gamma_\mu u(P_i, S_i)\right] \qquad (2.515)
\end{aligned}
$$

2.12.2 Remarks on form of scattering Matrix

• each vertex contributes a factor of the form $-ie\gamma_\mu ...$

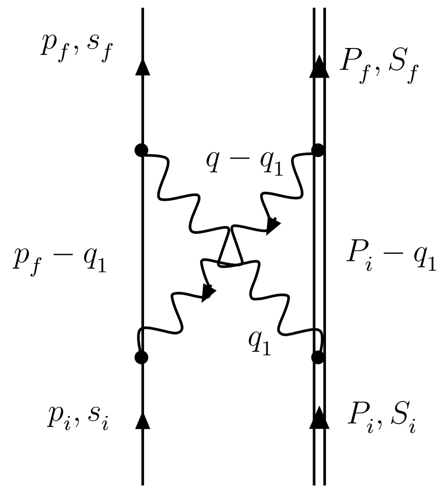


Figure 2.25:

• each external particle yields a facttor $\sqrt{m_0/E}$.

2.13 Feynmann Rules of QED

There origins should be clear to the reader.

2.13.1 Scattering Amplitudes

We consider a scattering process in which two particles, they may be electrons, positrons or photons, with four-momenta

$$p_i = (E_i, \mathbf{p}_i), \quad i = 1, 2,$$

collide and produce N final particles with momenta

$$p_f = (E_f, \mathbf{p}_f), \quad f = 1, \ldots, N.$$

Individual energy-momentum conservation at each verrtex leads to conservation of total energy-momentum, represented by the delta function

$$\delta^4 \left(p_1 + p_2 - \sum_{i=1}^{n} p_i' \right).$$

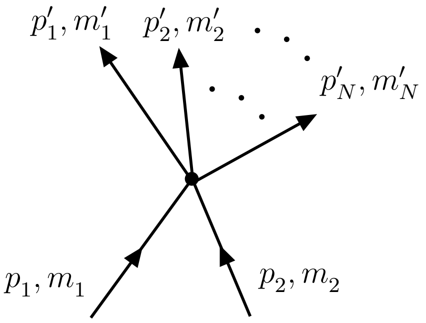


Figure 2.26: We are considering reactions in which there are two particles in the initial state and n particles in the final state.

The scattering matrix element S_{fi} is given by

$$S_{fi} = i(2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{i=1}^{n} p_i' \right) \; M_{fi} \; \prod_{i=1}^{2} \sqrt{\frac{N_i}{2E_i V}} \prod_{i=1}^{n} \sqrt{\frac{N_i'}{2E_i' V}} \tag{2.516}$$

The normalisation factors N_i:

$$N_i = \begin{cases} 4\pi & photons \\ 2m_0 & spin - \frac{1}{2} particles \end{cases} \tag{2.517}$$

After drawing any Feynman diagram in momentummm space we see clearly how to translate the various lines in the graph directly into mathematical expressions.

The Feynman rules concern the calculation of the reduced scattering matrix element M_{fi}. A Feynman graph describing a scattering process consists of three parts:

(1) the external lines representing the wave functions of incomingand outgoing particles,

(2) the internal lines described by propagators, and

(3) the vertices representing the interactions between the particles.

With each external line one associates the following factors:

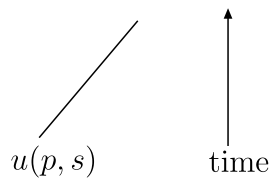


Figure 2.27: An electron entering an interaction.

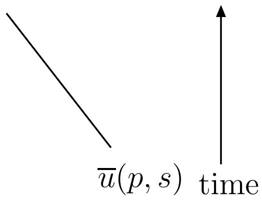


Figure 2.28: An electron leaving an interaction.

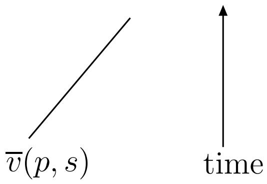


Figure 2.29: .

electron:

$$iS_F = \frac{i(p_\mu \gamma^\mu + m_0)}{p^2 - m_0^2 + i\epsilon} \tag{2.518}$$

photon:

$$D_F^{\mu\nu}(k) = \frac{-i4\pi\ \eta^{\mu\nu}}{k^2 + i\epsilon} \tag{2.519}$$

Each vertex is associated with a factor

$$-ie\gamma_\mu. \tag{2.520}$$

a) a factor of -1 for each incoming positron (outgoing electtron with negative energy)

b) a factror of -1 in the case that two graphs which differ only by the exchange of two fermion lines.

c) a factor of -1 for each closed fermion loop.

For each internal loop, integrate over

$$\int \frac{d^4q}{(2\pi)^4}$$

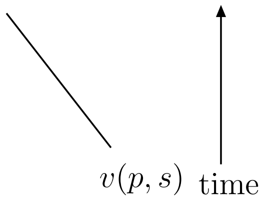


Figure 2.30: .

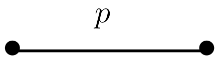


Figure 2.31: Electron propagator.

2.13.2 Differential Cross Section

To obtain the transition rate to a group of final states with momenta in the intervals $f = 1, \ldots, N$, we multiply by the number of these states which is

$$\prod_f \frac{V d^3\mathbf{p}'_f}{(2\pi)^3} \tag{2.521}$$

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} N_1 N_2 (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{i=1}^{N} p'_i) S |M_{fi}|^2 \prod_f \frac{d^3\mathbf{p}'_f}{2E'_f(2\pi)^3} \tag{2.522}$$

The degeneracy factor S exists when the final state contains identical particles. Its taken into account by

$$S = \prod_k \frac{1}{g_k!}, \tag{2.523}$$

where g_k particles of the kind k in the final state.

External static electromagnetic fields

2.14 Details

2.14.1 Traces of Products of $\gamma-$Matrices

1. The trace of an odd number of $\gamma-$matrices vanishes.

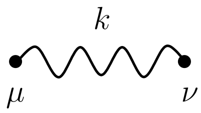


Figure 2.32: Photon propagator.

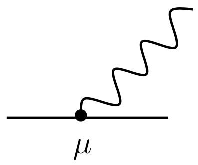


Figure 2.33: Vertex

2. $a_\mu b_\nu\ Tr\gamma^\mu\gamma^\nu = 4a\cdot b$.

3.

$$\begin{aligned} Tr\gamma^{\mu_1}\ldots\gamma^{\mu_n} &= \eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\ldots\gamma^{\mu_n} \\ &\quad -\eta^{\mu_1\mu_3}Tr\gamma^{\mu_2}\gamma^{\mu_4}\ldots\gamma^{\mu_n} \\ &\quad +\eta^{\mu_1\mu_n}Tr\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}} \end{aligned} \tag{2.524}$$

4. $Tr\gamma^5 = 0$.

5. $Tr\gamma^5\gamma^\mu\gamma^\nu = 0$.

6. $a_\mu b_\nu c_\sigma d_\gamma Tr\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\gamma = -4i\epsilon^{\mu\nu\sigma\gamma}a_\mu b_\nu c_\sigma d_\gamma$.

7. $Tr\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_{2n}} = Tr\gamma_{\mu_{2n}}\ldots\gamma_{\mu_1}$.

8.

i) $\gamma_\mu\gamma^\mu = 4\mathbf{1}$

ii) $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$

iii) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu = 4\eta^{\nu\sigma}\mathbf{1}$

iv) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\mu = -2\gamma^\gamma\gamma^\sigma\gamma^\nu$

v) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\delta\gamma^\mu = 2\gamma^\delta\gamma^\nu\gamma^\sigma\gamma^\gamma + 2\gamma^\gamma\gamma^\sigma\gamma^\nu\gamma^\delta$

Proof.

1. See ()-()

2. See ()-()

3. See ()-()

4. See ()-()

5. See ()-()

6. See ()-()

7. We make use of the matrix $\hat{C}$, involved in charge conjugation. $\hat{C}$ has the property $\hat{C}\gamma_\mu\hat{C}^{-1} = -\gamma_\mu^T$. It follows that

$$\begin{aligned}
Tr\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_{2n}} &= Tr(\hat{C}\gamma_{\mu_1}\hat{C}^{-1})(\hat{C}\gamma_{\mu_2}\hat{C}^{-1})\ldots(\hat{C}\gamma_{\mu_{2n}}\hat{C}^{-1}) \\
&= (-1)^{2n}\, Tr\gamma_{\mu_1}^T\gamma_{\mu_2}^T\ldots\gamma_{\mu_{2n}}^T \\
&= Tr[\gamma_{\mu_{2n}}\ldots\gamma_{\mu_1}]^T \\
&= Tr\gamma_{\mu_{2n}}\ldots\gamma_{\mu_1}.
\end{aligned}$$

8.

i) $\gamma_\mu\gamma^\mu = \frac{1}{2}(\gamma_\mu\gamma^\mu + \gamma^\mu\gamma_\mu) = \frac{1}{2}2g^\mu{}_\mu 1 = 41$

ii)

$$\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\mu &= \gamma_\mu(2\eta^{\mu\nu} - \gamma^\mu\gamma^\nu) \\
&= 2\gamma^\nu - 4\gamma^\nu \\
&= -2\gamma^\nu.
\end{aligned}$$

iii)

$$\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu &= \gamma_\mu\gamma^\nu(2\eta^{\mu\sigma} - \gamma^\mu\gamma^\sigma) \\
&= 2\gamma^\sigma\gamma^\nu - \gamma_\mu\gamma^\nu\gamma^\mu\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu + 2\gamma^\nu\gamma^\sigma \\
&= 2(2\eta^{\nu\sigma} - \gamma^\nu\gamma^\sigma) + 2\gamma^\nu\gamma^\sigma \\
&= 4\eta^{\nu\sigma}.
\end{aligned}$$

$\square$

iv)

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\sigma (2\eta^{\mu\gamma} - \gamma^\mu \gamma^\gamma) \\
&= 2\gamma^\gamma \gamma^\nu \gamma^\sigma - (\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\mu)\gamma^\gamma \\
&= 2\gamma^\gamma \gamma^\nu \gamma^\sigma - 4\eta^{\nu\sigma}\gamma^\gamma \\
&= 2\gamma^\gamma (2\eta^{\nu\sigma} - \gamma^\sigma \gamma^\nu) - 4\eta^{\nu\sigma}\gamma^\gamma \\
&= -2\gamma^\gamma \gamma^\sigma \gamma^\nu .
\end{aligned}
$$

□

v)

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\delta \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma (2\eta^{\mu\delta} - \gamma^\mu \gamma^\delta) \\
&= 2\gamma^\delta \gamma^\nu \gamma^\sigma \gamma^\gamma - (\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\mu)\gamma^\delta \\
&= 2\gamma^\delta \gamma^\nu \gamma^\sigma \gamma^\gamma + 2\gamma^\gamma \gamma^\sigma \gamma^\nu \gamma^\delta .
\end{aligned}
$$

□

2.14.2 Complete Set of 4×4 Matrices

Any 4×4 matrix can be written as

$$
\sum_{A=1}^{16} a_A \hat{\Gamma}_A \tag{2.525}
$$

where

$$
\begin{aligned}
\hat{\Gamma}_A = \; & \mathbf{I}, \\
& \gamma_0,\; i\gamma_1,\; i\gamma_2,\; i\gamma_3, \\
& i\gamma_2\gamma_3,\; i\gamma_3\gamma_1,\; i\gamma_1\gamma_2,\; \gamma_1\gamma_0,\; \gamma_2\gamma_0,\; \gamma_3\gamma_0, \\
& \gamma_1\gamma_2\gamma_3,\; i\gamma_1\gamma_2\gamma_0,\; i\gamma_3\gamma_1\gamma_0,\; i\gamma_2\gamma_3\gamma_0 \\
& i\gamma_1\gamma_2\gamma_3\gamma_0
\end{aligned} \tag{2.526}
$$

Proof:

Note

$$\hat{\Gamma}_A^2 = \mathbf{I} \quad (A = 1, \dots, 16) \tag{2.527}$$

For all $\hat{\Gamma}_A$ but $\mathbf{I}$ there exists a $\hat{\Gamma}_B$ with

$$\hat{\Gamma}_B \hat{\Gamma}_A \hat{\Gamma}_B = -\hat{\Gamma}_A \tag{2.528}$$

The trace of all $\hat{\Gamma}_A$ $(A = 2, \dots, 16)$ are zero,

$$Tr(\hat{\Gamma}_A) = -Tr(\hat{\Gamma}_B \hat{\Gamma}_A \hat{\Gamma}_B) = -Tr(\hat{\Gamma}_B^2 \hat{\Gamma}_A) = -Tr(\hat{\Gamma}_A).$$

Linear independence

Say

$$\sum_{A=1}^{16} a_A \hat{\Gamma}_A = 0.$$

Multiply this sum from the right by $\hat{\Gamma}_B$,

$$a_B \mathbf{I} + \sum_{A \neq B} a_A \hat{\Gamma}_A \hat{\Gamma}_B = 0. \tag{2.529}$$

Then take the trace

$$4a_B + \sum_{A \neq B} a_A Tr(\hat{\Gamma}_A \hat{\Gamma}_B) = 0. \tag{2.530}$$

Now from (2.526) we see that $\hat{\Gamma}_A \hat{\Gamma}_B = Const.\hat{\Gamma}_C$. In the case where $A \neq B$, $\hat{\Gamma}_C \neq \mathbf{I}$. This implies in (2.530) that $a_B = 0$.

Expansion of 4×4

Each 4×4 matrix can be expanded as

$$\hat{X} = \sum_{A=1}^{16} x_A \hat{\Gamma}_A. \tag{2.531}$$

This is evident since 4×4 matrices represents a 16-dimension space and the $\hat{\Gamma}_A$ are linearly independent. The coefficents are then given by

$$x_B = \frac{1}{4}Tr(\hat{\Gamma}_A\hat{X}).$$

□

2.14.3 Unitary Equivalence of Representations of the Dirac Algebra

All representations of the Dirac algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{I}$ which satify $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$ are unitary equivalent.

Proof: The proof is split into five parts.

i) First we prove that each 4×4 matrix which commutes with all $\hat{\Gamma}_A$ is a multiple of $\mathbf{I}$. Consider such a the matrix which write

$$\hat{X} = x_B\hat{\Gamma}_B + \sum_{A\neq B} x_A\hat{\Gamma}_A \tag{2.532}$$

where we have picked out a particular matrix which is not $\mathbf{I}$. We choose $\hat{\Gamma}_C$ such that

$$\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C = -\hat{\Gamma}_B. \tag{2.533}$$

Since $\hat{X}$ comutes wth all $\hat{\Gamma}_A$,

$$\hat{X} = \hat{\Gamma}_C\hat{X}\hat{\Gamma}_C$$

we have

$$\begin{aligned} x_B\hat{\Gamma}_B + \sum_{A\neq B} x_A\hat{\Gamma}_A &= x_B\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C + \sum_{A\neq B} x_A\hat{\Gamma}_C\hat{\Gamma}_A\hat{\Gamma}_C \\ &= x_B\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C + \sum_{A\neq B} x_A(\pm\hat{\Gamma}_A\hat{\Gamma}_C)\hat{\Gamma}_C \\ &= -x_B\hat{\Gamma}_B + \sum_{A\neq B}(\pm)x_A\hat{\Gamma}_A \end{aligned} \tag{2.534}$$

where we have used $\hat{\Gamma}_C\hat{\Gamma}_A = (\pm)\hat{\Gamma}_A\hat{\Gamma}_C$ (this established by inspection (2.526)). Next multiply by $\hat{\Gamma}_B$ and take the trace

$$4x_B + \sum_{A\neq B} x_A Tr(\hat{\Gamma}_A\hat{\Gamma}_B) = -4x_B + \sum_{A\neq B}(\pm)x_A Tr(\hat{\Gamma}_A\hat{\Gamma}_B) \tag{2.535}$$

implying

$$x_B = -x_B = 0.$$

So we conclude that

$$\hat{X} = x_1\mathbf{I}. \tag{2.536}$$

This result is actually just a special case of Schur's lemma which states that every matrix which commutes with every element of and irreducible representation must be a muliply of the identity matrix; we know 4×4 matrix reprersentations of the Dirac algebra is irreducible as there are no lower dimensional representations.

ii) Let γ_μ and γ'_μ be two representations of the Dirac algebra and $\hat{\Gamma}_A, \hat{\Gamma}'_A$ are respectively their basis. We wish to prove that

$$\hat{\Gamma}'_A\hat{S} = \hat{S}\hat{\Gamma}_A \tag{2.537}$$

where

$$\hat{S} = \sum_{B=1}^{16}\hat{\Gamma}'_B\hat{F}\hat{\Gamma}_B \tag{2.538}$$

and $\hat{F}$ is an aritrary 4×4 matrix. To this end, consider the matrix

$$\hat{\Gamma}'_A\hat{S}\hat{\Gamma}_A = \sum_{B=1}^{16}\hat{\Gamma}'_A\hat{\Gamma}'_B\hat{F}\hat{\Gamma}_B\hat{\Gamma}_A. \tag{2.539}$$

By inspection, from (2.526) we have $\hat{\Gamma}_B\hat{\Gamma}_A = \alpha_C\hat{\Gamma}_C$ where $\alpha_C \in \{\pm 1, \pm i\}$.

$$\hat{\Gamma}'_B\hat{\Gamma}'_A = \alpha_C\hat{\Gamma}'_C. \tag{2.540}$$

Multiply from the right by $\hat{\Gamma}'_A$

$$\hat{\Gamma}'_B = \alpha_C \hat{\Gamma}'_C \hat{\Gamma}'_A$$

then the right by $\hat{\Gamma}'_C$

$$\hat{\Gamma}'_C \hat{\Gamma}'_B = \alpha_C \hat{\Gamma}'_A$$

and from the left by $\hat{\Gamma}'_B$ gives

$$\hat{\Gamma}'_A \hat{\Gamma}'_B = \frac{1}{\alpha_C} \hat{\Gamma}'_C. \tag{2.541}$$

Substituting this into (2.539) then gives

$$\hat{\Gamma}'_A \hat{S} \hat{\Gamma}_A = \sum_{C=1}^{16} \left(\frac{1}{\alpha_C} \hat{\Gamma}'_C \right) \hat{F} \left(\alpha_C \hat{\Gamma}'_C \right) = \hat{S} \tag{2.542}$$

proving (2.537).

Suppose we could choose $\hat{F}$ (recall $\hat{F}$ is completely arbitrary) so that $\hat{S}$ is non-singular, we would then have in particular

$$\hat{\Gamma}'_2 = \hat{S}\hat{\Gamma}_2\hat{S}^{-1}, \quad \hat{\Gamma}'_3 = \hat{S}\hat{\Gamma}_3\hat{S}^{-1}, \quad \hat{\Gamma}'_4 = \hat{S}\hat{\Gamma}_4\hat{S}^{-1}, \quad \hat{\Gamma}'_5 = \hat{S}\hat{\Gamma}_5\hat{S}^{-1}$$

equivalently

$$\gamma'_\mu = \hat{S} \gamma_\mu \hat{S}^{-1}.$$

The next two steps are to prove we can choose $\hat{F}$ so the above conditions for $\hat{S}$ are fulfilled. In the final step we show that the additional conditions $\gamma_0 = \gamma_0^\dagger$, $\gamma_i = -\gamma_i^\dagger$, $\gamma'_0 = \gamma_0'^\dagger$, $\gamma'_i = -\gamma_i'^\dagger$ imply that $\hat{S}$ can be choosen to be unitary, proving the entire result.

iii) The matrix $\hat{F}$ can be choosen so that $\hat{S}$ does not vanish. We prove this by contraction. Say $\hat{S} = 0$ held for all choices of $\hat{F}$, then

$$0 = (\hat{S})_{\mu\rho} = \sum_{B=1}^{16} \sum_{\alpha,\beta=1}^{4} (\hat{\Gamma}'_B)_{\mu\alpha} (\hat{F})_{\alpha\beta} (\hat{\Gamma}_B)_{\beta\rho}. \tag{2.543}$$

for all μ and ρ. Now let us choose $\hat{F}$ such that a single element has the value 1 with all other elements being zero. Say it is the element $(\hat{F})_{\nu\sigma}$ that is equal to 1, then (2.543) reads

$$\sum_{B=1}^{16} (\hat{\Gamma}'_B)_{\mu\nu} (\hat{\Gamma}_B)_{\sigma\rho} = 0 \tag{2.544}$$

This equation can be written for all possible choices of ν and σ, so we can infer

$$\sum_{B=1}^{16} (\hat{\Gamma}'_B)_{\mu\nu} \, \hat{\Gamma}_B = 0 \tag{2.545}$$

holds for all μ and ν. Since $\hat{\Gamma}_B^2 = \mathbf{I}$ this would imply

$$\sum_{B=1}^{16} (\hat{\Gamma}'_B)_{\mu\nu} = 0 \quad \text{for all} \quad \mu, \nu. \tag{2.546}$$

As $(\hat{\Gamma}'_B)_{\mu\nu}$ cannot be equal to zero simultaneously, we have a contradiction to the linear independence of the $\hat{\Gamma}_B$.

iv) Now we prove that $\hat{S}$ is not singular with appropriate choice of $\hat{F}$. To this end construct

$$\hat{T} = \sum_{B=1}^{16} \hat{\Gamma}_B \hat{G} \hat{\Gamma}'_B \tag{2.547}$$

where $\hat{G}$ is arbitrary. Obviously we have

$$\hat{\Gamma}_A \hat{T} = \hat{T} \hat{\Gamma}'_A \tag{2.548}$$

(same argument as in iii) which together with (2.537) implies

$$(\hat{\Gamma}_A \hat{T})\hat{S} = (\hat{T}\hat{\Gamma}'_A)\hat{S} = \hat{T}(\hat{\Gamma}'_A \hat{S}) = \hat{T}(\hat{S}\hat{\Gamma}_A)$$

i.e.

$$\hat{\Gamma}_A (\hat{T}\hat{S}) = (\hat{T}\hat{S})\hat{\Gamma}_A, \tag{2.549}$$

accordingly $\hat{T}\hat{S}$ must be a multiple of the identity

$$\hat{T}\hat{S} = k\,\mathbf{I}. \tag{2.550}$$

Obviously we can choose $\hat{G}$ so that $\hat{T} \neq 0$ (same argument as iii). With the same kind of choice of $\hat{F}$ as in part iv) we will now show that we must have $k \neq 0$, and hence that $\hat{S}$ is not singular! We prove it by contradiction

$$\sum_{B=1}^{16} \hat{T}\hat{\Gamma}'_B \hat{F}\hat{\Gamma}_B = 0 \tag{2.551}$$

with the choice of $(\hat{F})_{\nu\rho} = 1$ with all other terms zero,

$$\sum_{B=1}^{16} (\hat{T}\hat{\Gamma}'_B)_{\mu\nu} (\hat{\Gamma}_B)_{\rho\sigma} = 0 \tag{2.552}$$

or

$$\sum_{B=1}^{16} (\hat{T}\hat{\Gamma}'_B)_{\mu\nu} \hat{\Gamma}_B = 0. \tag{2.553}$$

From $\hat{\Gamma}'^2_A = \mathbf{I}$ and the fact that $(\hat{T}\hat{\Gamma}'_B)_{\mu\nu}$ cannot all be simulateously zero as $\hat{\Gamma}'_1 = \mathbf{I}$ and $\hat{T} \neq 0$. This is in contradiction to the linear independence of the $\hat{\Gamma}_B$.

v) We now show that in the case of

$$\gamma_0 = \gamma_0^\dagger, \quad \gamma_i = -\gamma_i^\dagger, \quad \gamma'_0 = \gamma_0'^\dagger, \quad \gamma'_i = -\gamma_i'^\dagger,$$

equivalently

$$\gamma_\mu^\dagger = \eta_{\mu\mu}\gamma_\mu, \quad \gamma_\mu'^\dagger = \eta_{\mu\mu}\gamma'_\mu, \tag{2.554}$$

then $\hat{S}$ can be choosen as a unitary operator. To see this put $\hat{V} \equiv (det\,\hat{S})^{-1}\hat{S}$ then

$$\gamma'_\mu = \hat{V}\gamma_\mu\hat{V}^{-1}, \quad det\,\hat{V} = 1. \tag{2.555}$$

Let us see if there exist another choice for $\hat{V}$. We must have $det\,\hat{V}_1 = det\,\hat{V}_2 = 1$ and

$$\gamma'_\mu = \hat{V}_1\gamma_\mu\hat{V}_1^{-1} = \hat{V}_2\gamma_\mu\hat{V}_2^{-1}. \tag{2.556}$$

Eq (2.556) implies

$$\hat{V}_1 \hat{\Gamma}_A \hat{V}_1^{-1} = \hat{V}_2 \hat{\Gamma}_A \hat{V}_2^{-1}, \tag{2.557}$$

for example

$$\begin{aligned} \hat{V}_1 (i\gamma_1\gamma_2) \hat{V}_1^{-1} &= i(\hat{V}_1 \gamma_1 \hat{V}_1^{-1})(\hat{V}_1 \gamma_2 \hat{V}_1^{-1}) \\ &= i(\hat{V}_2 \gamma_1 \hat{V}_2^{-1})(\hat{V}_2 \gamma_2 \hat{V}_2^{-1}) \\ &= \hat{V}_2 (i\gamma_1\gamma_2) \hat{V}_2^{-1}. \end{aligned} \tag{2.558}$$

Eq (2.557) rearranged becomes

$$(\hat{V}_2^{-1} \hat{V}_1) \hat{\Gamma}_A = \hat{\Gamma}_A (\hat{V}_2^{-1} \hat{V}_1). \tag{2.559}$$

By result i) (Schur's lemma)

$$\hat{V}_2^{-1} \hat{V}_1 = k' \mathbf{I}$$

hence

$$\hat{V}_1 = k' \hat{V}_2. \tag{2.560}$$

As $det\ \hat{V}_2 = det\ \hat{V}_1 = k'^4 det\ \hat{V}_2$, we must have $k' \in \{\pm 1, \pm i\}$. Now take the Hermitian conjugate of (2.555),

$$\gamma_\mu{}' \dagger = (\hat{V}^{-1})^\dagger \gamma_\mu^\dagger \hat{V}^\dagger \tag{2.561}$$

then by means of (2.554),

$$\gamma'_\mu = (\hat{V}^\dagger)^{-1} \gamma_\mu \hat{V}^\dagger \tag{2.562}$$

We see that $(\hat{V}^\dagger)^{-1}$ fulfills (2.555) as does $\hat{V}$. From (2.560) ($k' \in \{\pm 1, \pm i\}$) it follows

$$\begin{aligned} (\hat{V}^\dagger)^{-1} &= k' \hat{V}, \quad \hat{V}^\dagger = k'^{-1} \hat{V}^{-1} \\ \hat{V}^\dagger \hat{V} &= k'^{-1} \mathbf{I}. \end{aligned} \tag{2.563}$$

Since

$$(\hat{V}^\dagger\hat{V})_{ii} = \sum_j (\hat{V}^\dagger)_{ij}(\hat{V})_{ji} = \sum_j |V_{ji}|^2 = k'^{-1} \tag{2.564}$$

Hence k'^{-1} must be real and positive, i.e. $k'^{-1} = 1$. Hence,

$$\hat{V}^\dagger\hat{V} = \mathbf{I}. \tag{2.565}$$

□

2.14.4 Coefficients of Infintesimal Lorentz Transformation

We prove that the

$$\hat{\sigma}_{\alpha\beta} = \frac{i}{2}(\gamma_\alpha\gamma_\beta - \gamma_\alpha\gamma_\beta)$$

fulfill ()

Proof: Insert the above expression in the RHS of ()

$$\begin{aligned}
[\gamma^\nu, \hat{\sigma}_{\alpha\beta}] &= \frac{i}{2}\left[\gamma^\nu, [\gamma_\alpha, \gamma_\beta]\right] \\
&= \frac{i}{2}\left(\left[\gamma^\nu, \gamma_\alpha\gamma_\beta\right] - \left[\gamma^\nu, \gamma_\beta\gamma_\alpha\right]\right) \\
&= \frac{i}{2}\left(2\left[\gamma^\nu, \gamma_\alpha\gamma_\beta\right] - 2\left[\gamma^\nu, \eta_{\alpha\beta}\right]\right) \\
&= i[\gamma^\nu, \gamma_\alpha\gamma_\beta]
\end{aligned} \tag{2.566}$$

where we used $\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\eta_{\alpha\beta}$. Furthermore we have

$$\begin{aligned}
i[\gamma^\nu, \gamma_\alpha\gamma_\beta] &= i(\gamma^\nu\gamma_\alpha\gamma_\beta - \gamma_\alpha\gamma_\beta\gamma^\nu) \\
&= i(\gamma^\nu\gamma_\alpha\gamma_\beta - 2\eta^\nu{}_\beta\gamma_\alpha + \gamma_\alpha\gamma^\nu\gamma_\beta) \\
&= i(\gamma^\nu\gamma_\alpha\gamma_\beta - 2\eta^\nu{}_\beta\gamma_\alpha + 2\eta^\nu{}_\alpha\gamma_\beta - \gamma^\nu\gamma_\alpha\gamma_\beta) \\
&= 2i(\eta^\nu{}_\alpha\gamma_\beta - \eta^\nu{}_\beta\gamma_\alpha).
\end{aligned} \tag{2.567}$$

□

2.14.5 Proof of Relation $\hat{S}^{-1} = \gamma_0 \hat{S}^\dagger \gamma_0$

We show that for

$$\hat{S} = \exp\left(-\frac{i}{4}\omega\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu}\right)$$

the inverse operator is given by

$$\hat{S}^{-1} = \gamma_0 \hat{S}^\dagger \gamma_0 \tag{2.568}$$

Proof:

(i) Rotations:

For spacial rotations we can write:

$$\hat{S} = \exp\left(-\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}\right). \tag{2.569}$$

The $\hat{\sigma}_{ij}$ are Hermitian because

$$\begin{aligned}
\hat{\sigma}_{ij}^\dagger &= -\frac{i}{2}\left\{(\gamma_i\gamma_j)^\dagger - (\gamma_j\gamma_i)^\dagger\right\} \\
&= -\frac{i}{2}\left\{\gamma_j\gamma_i - \gamma_i\gamma_j\right\} \\
&= \hat{\sigma}_{ij}.
\end{aligned} \tag{2.570}$$

This implies

$$\hat{S}^\dagger = \exp\left(\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}^\dagger\right) = \exp\left(\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}\right). \tag{2.571}$$

Obviously, γ_0 commutes with $\hat{\sigma}_{ij}$ and thus with $\hat{S}^\dagger$. Hence we have

$$\gamma_0 \hat{S}^\dagger \gamma_0 = \hat{S}^\dagger = \hat{S}^{-1}. \tag{2.572}$$

(ii) Lorentz boosts:

Given that a general Loreentz transformation can be decomposed into first a rotation, a Lorentz boost along the x-direction and then undoing the rotation, and given that

$\gamma^i\gamma^0 + \gamma^0\gamma^i = 0$, it suffices to prove the result for a simple boost along the $x-$direction. For this transformation we have

$$\hat{S} = \exp\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)$$

$\hat{\sigma}_{01}$ is antihermitian because

$$\begin{aligned}
\hat{\sigma}_{01}^{\dagger} &= -\frac{i}{2}\left\{(\gamma_0\gamma_1)^{\dagger} - (\gamma_1\gamma_0)^{\dagger}\right\} \\
&= \frac{i}{2}\left\{\gamma_1\gamma_0 - \gamma_0\gamma_1\right\} \\
&= -\hat{\sigma}_{01}.
\end{aligned} \tag{2.573}$$

Therefore

$$\hat{S}^{\dagger} = \exp\left(\frac{i}{2}\omega\hat{\sigma}_{01}^{\dagger}\right) = \exp\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right) = \hat{S}. \tag{2.574}$$

From

$$\begin{aligned}
\gamma_0\hat{\sigma}_{01} &= \frac{i}{2}\left\{\gamma_0\gamma_0\gamma_1 - \gamma_0\gamma_1\gamma_0\right\} \\
&= \frac{i}{2}\left\{\gamma_1\gamma_0\gamma_0 - \gamma_0\gamma_1\gamma_0\right\} \\
&= \hat{\sigma}_{10}\gamma_0 = -\hat{\sigma}_{01}\gamma_0.
\end{aligned} \tag{2.575}$$

we get

$$\begin{aligned}
\gamma_0\hat{S}^{\dagger}\gamma_0 &= \gamma_0\left[\sum_{n=0}^{\infty}\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)^n\right]\gamma_0 \\
&= \sum_{n=0}^{\infty}\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)^n\gamma_0 \\
&= \sum_{n=0}^{\infty}\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)\gamma_0\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)\gamma_0\ldots\gamma_0\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)\gamma_0 \\
&= \sum_{n=0}^{\infty}\left(+\frac{i}{2}\omega\hat{\sigma}_{01}\right)^n \\
&= \exp\left(\frac{i}{2}\omega\hat{\sigma}_{01}\right) = \hat{S}^{-1}.
\end{aligned} \tag{2.576}$$

$\square$

2.14.6 Proof of the Completenes Relation for spinors

Proof of completenes relation: $\omega^{r\dagger}(\epsilon_r \mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \delta_{rr'}(E/m_0)$

Proof:

We calculate some examples

$r = 1, r' = 1$:

$$\begin{aligned}
& \frac{E+m_0}{2m_0}\left(1, 0, \frac{p_z}{E+m_0}, \frac{p_-}{E+m_0}\right)\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m_0} \\ \frac{p_-}{E+m_0} \end{pmatrix} \\
= \ & \frac{E+m_0}{2m_0}\left\{1 + \frac{\mathbf{p}^2}{(E+m_0)^2}\right\} \\
= \ & \frac{E+m_0}{2m_0}\left\{\frac{(E+m_0)^2 + \mathbf{p}^2}{(E+m_0)^2}\right\} \\
= \ & \left\{\frac{2E + 2m_0 E}{2m_0(E+m_0)}\right\} \\
= \ & \frac{E}{m_0}\delta_{11}
\end{aligned} \tag{2.577}$$

$r = 2, r' = 4$:

$$\begin{aligned}
& \frac{E+m_0}{2m_0}\left(0, 1, \frac{p_+}{E+m_0}, -\frac{p_z}{E+m_0}\right)\begin{pmatrix} -\frac{p_-}{E+m_0} \\ \frac{p_z}{E+m_0} \\ 0 \\ 1 \end{pmatrix} \\
= \ & \frac{E+m_0}{2m_0}\left\{\frac{p_z}{E+m_0} - \frac{p_z}{E+m_0}\right\} = 0.
\end{aligned} \tag{2.578}$$

$r = 4, r' = 4$:

$$\begin{aligned}
& \frac{E+m_0}{2m_0}\left(1, 0, \frac{p_+}{E+m_0}, -\frac{p_z}{E+m_0}\right)\begin{pmatrix} -\frac{p_-}{E+m_0} \\ \frac{p_z}{E+m_0} \\ 0 \\ 1 \end{pmatrix} \\
= & \frac{E}{m_0}\delta_{44} \qquad (2.579)
\end{aligned}$$

The other combinations can be calculated similarly.

□

2.14.7 Integral representation for the step function

We show that

$$\Theta(\tau) = -\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon} \qquad (2.580)$$

Proof: We can evaluate the integral by means of complex integration in the complex $\omega-$plane. This can be done if we can show if the contribution from the upper (lower), infinitely distant half circle vanishes. Let I_R be the integral along the upper (lower) semicircle, then we can write

$$\begin{aligned}
I_R &= \lim_{R\to\infty}\int_0^{\pm\pi}[zf(z)]\frac{dz}{z} \\
&\leq \lim_{R\to\infty} Max[zf(z)]\int_0^{\pm\pi}\frac{dz}{z} \\
&= \pm i\pi \lim_{R\to\infty} Max[zf(z)] \qquad (2.581)
\end{aligned}$$

Hence, if we can show that $\lim_{R\to\infty} Max\,|[zf(z)]| \to 0$ then the integral over the semicircle can be ignored and the integral along the real line can be converted into a closed contour integral.

For $\tau < 0$ we show that the contribution from the upper, infinitely distant half circle vanishes

$$
\begin{aligned}
f(R,\theta) &= \frac{e^{-i\omega\tau}}{\omega} \\
&= \frac{e^{-iR\tau(\cos\theta + i\sin\theta)}}{re^{i\theta}} \\
&= e^{-iR\tau\cos\theta}\frac{e^{+R\tau\sin\theta}}{Re^{i\theta}} \qquad (2.582)
\end{aligned}
$$

and

$$
\lim_{R\to\infty} |Rf(R,\theta)| = e^{-R|\tau|\sin\theta} = 0
$$

For $\tau < 0$ we close the contour in the upper half plane. There is only a first order pole at $-i\epsilon$. Therefore this integral will be zero.

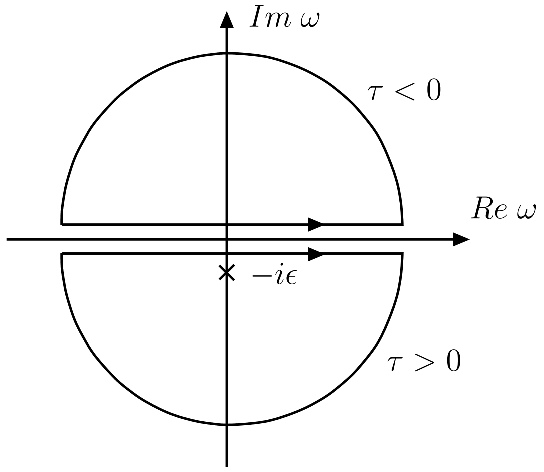


Figure 2.34:

In the case $\tau > 0$ for similar reasons one can close the contour by means of an infinitely large half circle below the real axis. Cauchy's integral theorem says that the integrand at the pole

$$
\begin{aligned}
\Theta(\tau > 0) &= -\frac{1}{2\pi i}(-1)2\pi i \lim_{\epsilon\to 0} Res\left[\frac{e^{-i\omega\tau}}{\omega + i\epsilon}\right] \\
&= e^{-\epsilon\tau}\Big|_{\epsilon=0} = 1. \qquad (2.583)
\end{aligned}
$$

where we have a minus sign coming from the clockwise direction of the integration.

□

2.14.8 Averaging over Spin

We have

$$\left(\overline{u}(f)\;\hat{\Gamma}_1 u(i)\right)\;\left(\overline{u}(f)\;\hat{\Gamma}_2 u(i)\right)^* = (\overline{u}(f)\;\hat{\Gamma}_1 u(i))\;(\overline{u}(i)\;\overline{\hat{\Gamma}}_2 u(f)) \tag{2.584}$$

where

$$\overline{\hat{\Gamma}} = \gamma^0 \hat{\Gamma}^\dagger \gamma^0. \tag{2.585}$$

Proof:

First note what the complex conjugate $(\overline{u}(f)\hat{\Gamma}u(i))^*$ of the number $\overline{u}(f)\hat{\Gamma}u(i)$ is equal to

$$\begin{aligned}(\overline{u}(f)\hat{\Gamma}u(i))^\dagger &= \left(\overline{u}^\dagger(f)\gamma^0\hat{\Gamma}u(i)\right)^\dagger \\ &= u(i)^\dagger\hat{\Gamma}^\dagger\gamma^{0\dagger}\;u^{\dagger\dagger}(f) \\ &= \overline{u}(i)(\gamma^0\hat{\Gamma}^\dagger\gamma^0)u(f) \\ &= \overline{u}(i)\overline{\hat{\Gamma}}u(f)\end{aligned} \tag{2.586}$$

where we have used $\gamma^{0\dagger} = \gamma^0$ and $(\gamma^0)^2 = 1$.

$$\left(\overline{u}(f)\;\hat{\Gamma}_1 u(i)\right)\;\left(\overline{u}(f)\;\hat{\Gamma}_2 u(i)\right)^* = (\overline{u}(f)\;\hat{\Gamma}_1 u(i))\;(\overline{u}(i)\;\overline{\hat{\Gamma}}_2 u(f)) \tag{2.587}$$

□

The barred matrices $\overline{\hat{\Gamma}}$ can be directly calculated for a number of operators:

(i) $\overline{\gamma}^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$

(ii) $\overline{i\gamma^5} = i\gamma^5$

(iii) $\overline{\gamma^\mu\gamma^5} = \gamma^\mu\gamma^5$

(iv) $\overline{\gamma^\mu\gamma^\nu\ldots\gamma^\lambda} = \gamma^\lambda\ldots\gamma^\nu\gamma^\mu$

Proof:

(i) First

$$\gamma^0\gamma^{0\dagger}\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$$

and secondly

$$\gamma^0\gamma^{i\dagger}\gamma^0 = -\gamma^0\gamma^i\gamma^0 = \gamma^i\gamma^0\gamma^0 = \gamma^i.$$

(ii) As $i\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3$ and

$$\overline{i\gamma^5} = -\gamma^0\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger}\gamma^0 = +\gamma^0\gamma^3\gamma^2\gamma^1 = -\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^5.$$

(iii) Similar to (ii).

(iv) Proved using (i).

□

Spin summation of the general squared matrix element

$$\sum_{s_f\ s_i} \left(\overline{u}(p_f, s_f)\ \hat{\Gamma}_1\ u(p_i, s_i)\right)\ \left(\overline{u}(p_f, s_f)\ \hat{\Gamma}_2\ u(p_i, s_i)\right)^* = Tr\left[\hat{\Gamma}_1\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\ \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right] \tag{2.588}$$

A special case is

$$\sum_{s_f\ s_i} |\overline{u}(p_f, s_f)\ \hat{\Gamma}\ u(p_i, s_i)|^2 = Tr\left[\hat{\Gamma}\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}\ \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right] \tag{2.589}$$

Proof: We use Einstein's summation convention.

$$\begin{aligned}
& \sum_{s_f\ s_i} \left(\overline{u}_\alpha(p_f, s_f)\ (\hat{\Gamma}_1)_{\alpha\beta}\ u_\beta(p_i, s_i)\right)\ \left(\overline{u}_\gamma(p_i, s_i)\ (\hat{\overline{\Gamma}}_2)_{\gamma\delta}\ u_\delta(p_f, s_f)\right) \\
= & \sum_{s_f} \overline{u}_\alpha(p_f, s_f)\ (\hat{\Gamma}_1)_{\alpha\beta} \left(\sum_{s_i} u_\beta(p_i, s_i)\overline{u}_\gamma(p_i, s_i)\right) \hat{\overline{\Gamma}}_{\gamma\tau}\ u_\tau(p_f, s_f) \\
= & \sum_{s_f} \overline{u}_\alpha(p_f, s_f)\ \left(\hat{\Gamma}_1\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\right)_{\alpha\beta} u_\beta(p_f, s_f) \\
= & \sum_{r=1}^{4} \epsilon_r \overline{\omega}^r_\alpha(p_f)\ \left(\hat{\Gamma}_1 \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\right)_{\alpha\beta} \left(\frac{p_{f\mu}\gamma^\mu + m_0}{2m_0}\right)_{\beta\gamma} \omega^r_\gamma(p_f) \\
= & \ Tr\left[\hat{\Gamma}_1\ \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\ \hat{\overline{\Gamma}}_2\ \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right]
\end{aligned} \tag{2.590}$$

□

2.14.9 Proof of Equations (2.310) and (2.311)

$$\Theta(t-t')\psi^{(-E)}(x') = -i\int d^3x S_F(x'-x)\gamma_0\psi^{(-E)}(x)$$

We prove (2.310):

$$\Theta(t'-t)\psi^{(+E)}(x') = i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x)$$

Proof:

Any wave packet of positive energy may be expressed in terms of normalised plane waves

$$\psi^{(+E)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b(p,r)\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x) \tag{2.591}$$

where $E_p = \sqrt{\mathbf{p}^2 + m_0^2}$ and $\epsilon_1 = \epsilon_2 = +1$.. We will need to make use of the orthogonality condition

$$\omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E_p}{m_0}\delta_{rr'}. \tag{2.592}$$

We start with the plane-wave representation of the Feynman propagator

$$S_F(x'-x) = -i\Theta(t'-t)\int d^3p\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x) + i\Theta(t-t')\int d^3p\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x) \tag{2.593}$$

where

$$\psi_p^r = \sqrt{\frac{m_0}{E_p}}\frac{1}{(2\pi)^{3/2}}\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x). \tag{2.594}$$

Inserting the above into the RHS of ()

$$
\begin{aligned}
& i \int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x) \\
= \; & \Theta(t'-t)\int d^3x \int d^3p \sum_{r=1}^{2} \psi_p^r(x')\overline{\psi}_p^r(x)\gamma_0\psi^{(+E)}(x) \\
& -\Theta(t'-t)\int d^3x \int d^3p \sum_{r=3}^{4} \psi_p^r(x')\overline{\psi}_p^r(x)\gamma_0\psi^{(+E)}(x) \\
= \; & \Theta(t'-t)\int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\sum_{r=1}^{2}\omega^r(p)\overline{\omega}^r(p)\gamma_0\exp[-i\epsilon_r p\cdot(x'-x)] \\
& \times \int \frac{d^3p'}{(2\pi)^3}\frac{m_0}{E_{p'}}\sum_{r=1}^{2} b(p',r')\omega^{r'}(p')\exp(-i\epsilon_{r'}p'\cdot x) \\
& -\Theta(t'-t)\int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\sum_{r=3}^{4}\omega^r(p)\overline{\omega}^r(p)\gamma_0\exp[-i\epsilon_r p\cdot(x'-x)] \\
& \times \int \frac{d^3p'}{(2\pi)^3}\frac{m_0}{E_{p'}}\sum_{r=1}^{2} b(p',r')\omega^{r'}(p')\exp(-i\epsilon_r p'\cdot x) \\
= \; & \Theta(t'-t)\int \frac{d^3p d^3p'}{(2\pi)^3}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_p}}\sum_{r=1,2;r'=1,2}\omega^r(p)\omega^{r\dagger}(p)\omega^{r'}(p')b(p',r')\exp(-i\epsilon_{r'}p'\cdot x) \\
& \quad \times \int \frac{d^3x}{(2\pi)^3}\exp[i(\epsilon_r p-\epsilon_{r'}p')\cdot x] \\
- \; & \Theta(t'-t)\int \frac{d^3p d^3p'}{(2\pi)^3}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_{p'}}}\sum_{r=3,4;r'=1,2}\omega^r(p)\omega^{r\dagger}(p)\omega^{r'}(p')b(p',r')\exp(-i\epsilon_r p\cdot x') \\
& \quad \times \int \frac{d^3x}{(2\pi)^3}\exp[i(\epsilon_r p-\epsilon_{r'}p')\cdot x]
\end{aligned}
\tag{2.595}
$$

Performing the x integration in the Θ term yields

$$
\exp[i(E_p - E_{p'})t]\delta^3(\mathbf{p}-\mathbf{p}') \to \delta^3(\mathbf{p}-\mathbf{p}') \tag{2.596}
$$

Performing the x integration in the Θ term yields

$$
\exp[-i(E_p + E_{p'})t]\delta^3(\mathbf{p}+\mathbf{p}') \to \exp(-2iE_p t)\delta^3(\mathbf{p}+\mathbf{p}'). \tag{2.597}
$$

Integrating over $\mathbf{p}$ and relabelling $\mathbf{p'}$ as $\mathbf{p}$ we find

$$
\begin{aligned}
& i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x) \\
= \;& \Theta(t'-t)\int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2}\sum_{r=1,2;r'=1,2}\omega^r(\mathbf{p})\omega^{r\dagger}(\mathbf{p})\omega^{r'}(\mathbf{p})b(p,r')\exp(i\epsilon_r p\cdot x') \\
& -\Theta(t-t')\int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2}\sum_{r=3,4;r'=1,2}\omega^r(-\mathbf{p})\omega^{r\dagger}(-\mathbf{p})\omega^{r'}(+\mathbf{p})b(p,r') \\
& \times\exp(i\epsilon_r p\cdot x')\exp(-2iE_p t) \qquad (2.598)
\end{aligned}
$$

Now we make use of the orthogonality relation. For $r, r' = 1, 2$

$$
\omega^{r\dagger}(\mathbf{p})\omega^{r'}(\mathbf{p}) = \omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E_p}{m_0}\delta_{rr'} \qquad (2.599)
$$

and for $r = 3, 4$ and $r' = 1, 2$,

$$
\omega^{r\dagger}(-\mathbf{p})\omega^{r'}(\mathbf{p}) = \omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = 0 \qquad (2.600)
$$

The second term vanishes. The remaining term gives

$$
\begin{aligned}
i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x) &= \Theta(t'-t)\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b(p,r)\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x') \\
&= \Theta(t'-t)\psi^{(+E)}(x') \qquad (2.601)
\end{aligned}
$$

□

Similar relations can be deduced the propagation of adjoint spinors $\overline{\psi}^{(+E)}(x), \overline{\psi}^{(-E)}(x)$:

$$
\Theta(t-t')\overline{\psi}^{(+E)}(x') = i\int d^3x\overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \qquad (2.602)
$$

and

$$
\Theta(t'-t)\overline{\psi}^{(-E)}(x') = -i\int d^3x\overline{\psi}^{(-E)}(x)\gamma_0 S_F(x'-x) \qquad (2.603)
$$

Proof:

Any adjoint wave packet of positive energy may be expressed in terms of normalised plane waves

$$\overline{\psi}^{(+E)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=1}^{2} b^*(p,r)\overline{\omega}^r(\mathbf{p}) \exp(+ip \cdot x) \tag{2.604}$$

Consider the integral

$$\begin{aligned} & i \int d^3x \overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \\ = \quad & i \int d^3x \int \frac{d^3p'}{(2\pi)^{3/2}} \frac{m_0}{E_p} \sqrt{\frac{m_0}{E_{p'}}} \int \frac{d^3p}{(2\pi)^3} \sum_{r'=1}^{2} \overline{\omega}^{r'}(p') \exp(ip' \cdot x)\gamma_0 \\ & \times \left\{ -i\Theta(t-t') \sum_{r=1}^{2} \omega^r(p)\overline{\omega}^r(p) \ \exp[-ip \cdot (x-x')] \right. \\ & \left. + \, i\Theta(t-t') \sum_{r=3}^{4} \omega^r(p)\overline{\omega}^r(p) \ \exp[+ip \cdot (x-x')] \right\} \end{aligned} \tag{2.605}$$

Again we do the x integration and use the orthogonality relations for spinors. We obtain

$$\int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=1}^{2} b^*(p,r)\overline{\omega}(\mathbf{p}) \exp(ip \cdot x)\Theta(t-t'). \tag{2.606}$$

This is just the expansion of the adjoint spinor $\overline{\psi}^{(+E)}$ multiplied by the step function $\Theta(t-t')$.

□

2.15 Scalar Quantum Electrodynamics

The Klein-Gordon equation results from substitution of quantum mechanical operators for energy and momentum into the Einstein relation $E^2 - \mathbf{p}^2 = m_0^2$ of special relativity. As we shall seee this leads to inconsistences such as negative probabilities and negative energy states. These inconsistencies come from interpreting the equation as a single particle wave equation.

Instead we will interpret ϕ as a field, much as the electron field, adopting a description in which particles are created and destroyed at vertices and negative energy solutions propagating backward in time as antiparticles propagating forward in time. We reintepret probability density as charge density, with the existence of particles and antiparticles.

2.15.1 Klein-Gordon Equation

Four current

We construct the four current j_μ for the Klein-Gordon equation. The current will satify a conservation equation. Take the complex conjugate of

$$(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi = 0,$$

i.e.

$$(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi^* = 0,$$

Multiplying the first equation from the left by ϕ^* and the second from the left by ψ and take the difference

$$\phi^*(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi - \phi(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\phi^* = 0,$$

or

$$-\phi^*(\nabla_\mu \nabla^\mu)\phi - \phi(\nabla_\mu \nabla^\mu)\phi^* = 0,$$

which implies

$$\nabla_\mu(\phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^*) = 0 \tag{2.607}$$

We define the four-current density

$$j_\mu = \frac{i\hbar}{2m_0}(\phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^*) \tag{2.608}$$

which by (2.607) satifies

$$\nabla_\mu j^\mu = 0 \tag{2.609}$$

The constant $i\hbar/2m_0$ was choosen so that j_0 has the dimentions of a probability density.

Charge density

$$\rho^Q = \frac{i\hbar e}{2m_0c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \tag{2.610}$$

Minimal coupling

minimal coupling

$$[(\hat{p}^\mu - eA^\mu)(\hat{p}_\mu - eA_\mu) - m_0^2]\phi(x) \tag{2.611}$$

rearanging gives

$$[\partial^\mu\partial_\mu - m_0^2]\phi(x) = -\hat{V}\phi(x) \tag{2.612}$$

where

$$\hat{V}\phi(x) = ie(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi - e^2A^\mu A_\mu\phi. \tag{2.613}$$

2.15.2 Current density in presence of electromagnetic potential

$$\begin{aligned} 0 &= \phi^*[\partial^\mu\partial_\mu\phi + ie(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi] - \\ &\qquad \phi[\partial^\mu\partial_\mu\phi - ie(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi^*] \\ &= \phi^*\partial^\mu\partial_\mu\phi - \phi\partial^\mu\partial_\mu\phi + 2ie\phi^*\phi\partial^\mu A^\mu + 2ieA_\mu\phi^*\partial^\mu\phi + 2ieA_\mu\phi\partial^\mu\phi^* \\ &= \partial^\mu[\phi^*\partial^\mu\phi - \phi\partial_\mu\phi^* + 2ie\phi^*\phi A^\mu] \end{aligned} \tag{2.614}$$

$$j_\mu = ie\phi^* \overleftrightarrow{\partial} \phi - 2e^2A_\mu\phi^*\phi \tag{2.615}$$

2.15.3 Feynman Propagator for Scalar Particles

$$(\Box + m_0)\Delta_F(x-y) = -\delta^4(x-y). \tag{2.616}$$

$$\Delta_F(p) = \frac{1}{p^2 - m_0^2 + i\epsilon}. \tag{2.617}$$

2.15.4 Perturbative Series

$$
\begin{aligned}
S_{fi} &= \lim_{t\to+\infty} \left(\varphi^{(+)}_{p_f}(x)\middle|\phi_{p_i}(x)\right) \\
&= \lim_{t\to+\infty} \int d^3x\, \varphi^{(+)*}_{p_f}(\mathbf{x},t) i \overleftrightarrow{\partial}\, \phi_{p_i}(\mathbf{x},t) \qquad (2.618)
\end{aligned}
$$

2.15.5 Scattering off a Coulomb Potential

Scattering off a Coulomb Potential

$$
S_{fi} = -ie \int d^4x \varphi^*_f(x)(\partial^\mu A_\mu(x) + A_\mu(x)\partial_\mu)\phi_i(x) \qquad (2.619)
$$

To first order $\phi_i(x)$ is given by the incoming plane wave $\varphi_i(x)$ of a scalar particle with momentum p_i:

$$
\varphi_i(x) = \sqrt{\frac{1}{2E_pV}} e^{-ip_i\cdot x} \qquad (2.620)
$$

$\varphi^*_f(x)$ is given by

$$
\varphi_f(x) = \sqrt{\frac{1}{2E_pV}} e^{ip_f\cdot x} \qquad (2.621)
$$

$$
\begin{aligned}
S_{fi} &= e\frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}} \int d^4x e^{+ip_f\cdot x}(\partial^\mu A_\mu(x) + A_\mu(x)\partial_\mu)e^{-ip_i\cdot x} \\
&= e\frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}} \int d^4x[-(ip^\mu_f) + (-ip^\mu_i)]A_\mu(x)e^{i(p_f-p_i)\cdot x} \\
&= [(-ie)(p^\mu_f + p^\mu_i)]\frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}} A_\mu(p_f - p_i) \\
&= iZe^2\frac{1}{V}\frac{1}{\sqrt{2E_f 2E_i}}\frac{4\pi}{q^2}\, 2\pi\delta(E_f - E_i). \qquad (2.622)
\end{aligned}
$$

Formula for Differential Cross Section

2.15.6 Scattering of Identical Bosons

$$A^{\mu}(x) = \int d^4y D_F(x-y) j^{\mu}_{fi}(y). \tag{2.623}$$

Chapter 3

Quantum Field Theory: Functional Integral and Canonical Approach

3.1 Lagrangian Field Theory

$$
\begin{aligned}
0 &= \delta S \\
&= \delta \int d^4 \mathcal{L} \\
&= \int d^4 \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right\}
\end{aligned}
\tag{3.1}
$$

use

$$
\delta(\partial_\mu \varphi) = \partial_\mu(\delta \varphi)
$$

and apply integration by parts, the boundary terms vanish because the end points are fixed:

$$
\begin{aligned}
\int d^4 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) &= \int d^4 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta \varphi) \\
&= -\int d^4 \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta \varphi
\end{aligned}
\tag{3.2}
$$

Altogether, the variation of the action is

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right\} \delta\varphi \tag{3.3}$$

Either the integrand takes positive and negative values or the integrand is zero over the domain of integration. As $\delta\varphi$ is arbitrary we know that the integrand must be zero. The terminside the braces vanishes. This gives the Euler-Lagrange equations for the field φ

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \tag{3.4}$$

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L} \tag{3.5}$$

$$H = \int \mathcal{H} d^3x \tag{3.6}$$

following

$$p_r(t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_r}$$

conjugate momentum defined in the usual way

$$\pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \tag{3.7}$$

3.2 Bosonic Integration

3.2.1 N Real variables

$$Z(j) = \int \prod_{i=1}^{N} dx_i \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_{i=1}^{N} j_i x_i \right) \tag{3.8}$$

where the matrix A_{ij} is symmetric and strictly positive.A more compact notation is to represent the column vectors $(x_i ... x_N)$ and $(j_1 ... j_N)$ as x and j, then the row vectors would be x^T and j^T where T=transpose. We then have:

$$\sum_{i,j=1}^{N} x_i A_{ij} x_j = x^T A x, \qquad \sum_{i=1}^{N} j_i x_i = j^T x \tag{3.9}$$

Change variables to x' given by

$$x = x' + A^{-1} j, \tag{3.10}$$

(the matrix A^{-1} exists because A is assumed positive) then

$$-\frac{1}{2} x^T A X + J^T x = -\frac{1}{2} x'^T A x' + \frac{1}{2} j^T A^{-1} j, \tag{3.11}$$

The integral then becomes

$$Z(j) = \exp(\frac{1}{2} j^T A^{-1} j) Z(0) \tag{3.12}$$

In many cases equation () is all one needs (for example in calculating correlation functions, from which Z(0) cancels). Z(0) reads

$$Z(0) = \int \prod_{i=1}^{N} dx'_i \exp(-\frac{1}{2} x^T A x), \tag{3.13}$$

Let R be an orthogonal transformation ($RR^T = I$) diagonalising A,

$$A = \mathbf{R}^T \mathbf{D} \mathbf{R}, \qquad \mathbf{D} = \begin{pmatrix} d_1 & & & \\ & d_2 & & 0 \\ 0 & & \ddots & \\ & & & d_N \end{pmatrix}, \qquad d_i > 0 \qquad \forall i. \tag{3.14}$$

Make the following change of variables with unit Jacobian:

$$x' = \mathbf{R} x \qquad (det \mathbf{R} = 1), \tag{3.15}$$

$$\int \prod_{i=1}^{N} dx_i \exp(\frac{1}{2} x^T \mathbf{A} x) = \int \prod_{i=1}^{N} dx'_i \exp(\frac{1}{2} x'^T \mathbf{D} x') \tag{3.16}$$

The last integral is the product of N independent gaussian integrals, and is given by

$$(2\pi)^{N/2}\prod_{i=1}^{N}(d_i)^{-1/2} = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{3.17}$$

$$Z(0) = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{3.18}$$

$$\int \prod_{i=1}^{N} dx_i \exp(-x^T \mathbf{A} x + j^T x +) = \frac{\pi^{N/2}}{det\mathbf{A}^{1/2}} \exp((1/2) j^T A^{-1} j), \tag{3.19}$$

3.2.2 Complex variables

First consider the case of a single complex variable z=x+iy with integral

$$I = \int d^2 z e^{-z^* A z + z^* j + z j^*} = \int dx dy e^{-A(x^2+y^2)+2j_1 x + 2j_2 y} \tag{3.20}$$

where $j = j_1 + ij_2$ and $A = a_1 + ia_2$, with $a_1 > 0$. Then I follows immediately:

$$I = \frac{\pi}{A} a^{j^* A^{-1} j} \tag{3.21}$$

Next, consider the case of N complex variables z_i,

$$I = \int \prod_{i=1}^{N} d^2 z_i e^{-z^\dagger \mathbf{A} z + z^\dagger j + j^\dagger z}, \tag{3.22}$$

where transposes have been replaced by Hermitian conjugates. Assume that $\mathbf{A}$ can be diagonalized by a unitary transformation U,

$$A = U^\dagger D U, \tag{3.23}$$

where D is a diagonal matrix with elements d_i whose real parts are positive. Write

$$U = R + iS, \tag{3.24}$$

where R and S are real matrices; the relation $U^\dagger U = I$ entails

$$RR^T + SS^T = I, \qquad RS^T - SR^T = 0. \tag{3.25}$$

The transformation $z' = Uz$ amounts to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} R & -S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.26}$$

and the matrix which transforms (x,y) into (x', y') is orthogonal so that the jacobian of the transformation is 1, which leads to the end result

$$\int \prod_{i=1}^{N} e^{-z^\dagger \mathbf{A} z + z^\dagger j + j^\dagger z} = \frac{\pi^N}{det\mathbf{A}} e^{j^\dagger \mathbf{A^{-1}} j} \tag{3.27}$$

3.3 Feynmann Rules for Scalar Quantum field theory

3.4 Perturbation Theory

3.4.1 Diagrammatic Perturbation Theory

In this section we investigate general rules for the perturbative calculation of correlation functions, rules designed to yield the result in the form of an expansion in powers of g,

$$G = G_0 + gG_1 + g^2 G_2 + g^3 G_3 + ... + g^n G_n + ... \tag{3.28}$$

Where G_0 is the correlation function of the Gaussian model, (non-interacting model). These rules are easily represented in diagrammatic form. These diagrams are the so-called *Feynman Diagrams*. As a simple example we examine the Ginzburg-Landau theory (see eq.(**??**)). It is impossible to find an exact closed formula for Z(0), but if g is small on can expand $\exp(-g \int d^d x \phi^4(x)/4!)$.

The calculation of $\mathbf{G^{(2)}}$ to order g

First we calculate the 2-point greens function to order g. One must evaluate the integral

$$I(x,y) = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H} = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H_0} \left[1 - \frac{g}{4!} \int d^d z \phi^4(z) + \cdots \right]. \tag{3.29}$$

The first term in the square brackets merely yields

$$\mathcal{N} \langle \phi(x)\phi(y) \rangle_0 = \mathcal{N} G_0(x-y) \quad \text{where } \mathcal{N} = Z_0(j=0). \tag{3.30}$$

To evaluate the integral in the second term,

$$\int \mathcal{D}\phi\phi(x)\phi(y)e^{-H_0}\int d^dz\phi^4(z), \tag{3.31}$$

we use Wick's theorem (**??**). There are two types of result from the contractions:

$$(a)\ \ \langle\phi(x)\phi(y)\rangle_0\left\langle\phi^4(z)\right\rangle \quad \text{and} \quad (b)\ \ \langle\phi(x)\phi(z)\rangle_0\left\langle\phi^2(z)\right\rangle_0\langle\phi(y)\phi(z)\rangle_0 \tag{3.32}$$

in the wick expansion there were $4\times = 12$ terms of type (a) and 3 terms of type (b). It is convenient to represent these contractions as diagrams, by drawing two "external" points x and y ("external" means that they refer to the arguments of the correlation function), and "internal" point or "vertex" z, which stems from the expansion of exp(-V), and over which we integrate. Every contraction is represented by a line joining arguments of ϕ. The two types of terms possible in (3.32) are drawn

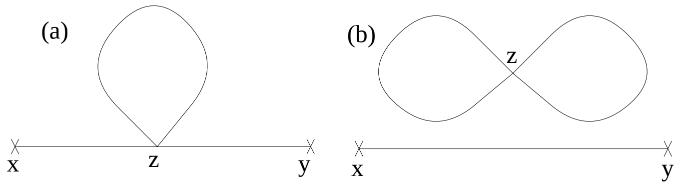


Figure 3.1: The two diagrams of order g

These diagrams are called *Feynman diagrams (or graphs)*; one such diagram corresponds to every distinct group of terms of the perturbation expansion. The integral I reads

$$I(x,y)=\mathcal{N}\left[G_0(x-y)-\frac{1}{2}g\int d^dzG_0(x-z)G_0(0)G_0(z-y)-\frac{1}{8}gG_0(x-y)(G_0(0))^2\int d^dz\right] \tag{3.33}$$

In order to obtain the correlation function, we must divide by Z(0):

$$Z(0)=\int\mathcal{D}\phi e^{-H_0}\left(1-\frac{g}{4!}\int d^dz\phi^4(z)+\cdots\right)=\mathcal{N}\left[1-\frac{g}{8}(G_0(0))^2\int d^dz+\cdots\right]. \tag{3.34}$$

The second term in the square brackets is represented by the diagram.

Dividing (3.33) by (3.34) we obtain the correlation function to order g

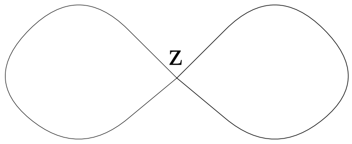


Figure 3.2: The vacuum-fluctuation diagram

$$G^{(2)}(x-y) = \frac{I(x,y)}{Z(0)} = G_0(x-y) - \frac{1}{2}g \int d^d z G_0(x-z) G_0(0) G_0(z-y) + \mathcal{O}(g^2). \quad (3.35)$$

The graph(b) from fig.(3.1) does not feature in the perturbation expansion of G. Diagrams of this type are called "vacuum-fluctuation" (sub)diagrams, meaning a subgraph that is completely disconnected from the "external" points x and y. The sum of all vacuum-fluctuation diagrams is equal to $Z(0) = \mathcal{D}\phi e^{-H}$. **Division by $Z(0)$ cancels all graphs containing "vacuum-fluctuations" parts disconnected from the rest of the diagram**. A proof is given in citeBellac (p 160).

On taking the Fourier transform, eq.(3.35) becomes

$$G^{(2)}(k) = G_0(k) - \frac{1}{2}g G_0(k) \left[\int \frac{d^d q}{(2\pi)^d} G_0(q) \right] G_0(k). \quad (3.36)$$

The factor in front of the second term on the r.h.s. is called the *symmetry factor* of the diagram. To become familiar with the "Feynman rules", i.e. the rules for associating diagrams with the perturbation expansion, we move to the calculation of $G^{(2)}$ to order g^2.

The calculation of $\mathbf{G^{(2)}}$ to order $\mathbf{g^2}$

We use Wick's theorem to compute the expression

$$\left\langle \phi(x)\phi(y) \int d^d z d^d u \phi^4(z) \phi^4(u) \right\rangle_0 . \quad (3.37)$$

Eliminating the terms that contain vacuum-fluctuation parts, one finds three types of graphs shown in fig.(3.3), with their symmetry factors given in brackets:

The vertices z and u may be permuted, which yields a multiplicative factor 2!; however this is exactly cancelled by the factor 1/2! from the expansion of the exponential. This is the same kind of cancellation happens in the nth order.

We shall settle for examining the contribution $\bar{G}(x-y)$ to the correlation function from graph (a) in fig.(3.3). Thus

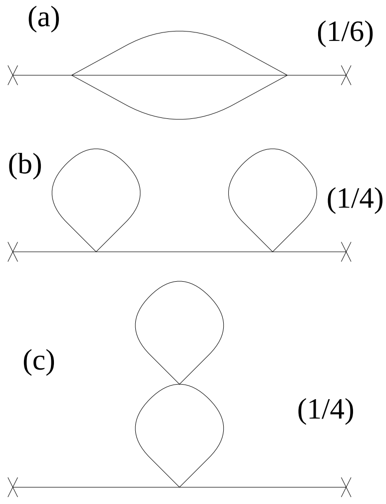


Figure 3.3: The vacuum-fluctuation diagram

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^d z d^d u G_0(x-z)[G_0(z-u)]^3 G_0(u-y). \tag{3.38}$$

Let us write $\bar{G}(x-y)$ as a Fourier transform, by replacing every factor G_0 by its Fourier representation

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^d z d^d u \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\prod_{l=1}^{3}\left\{\frac{d^d q_l}{(2\pi)^d}e^{i\sum_{l=1}^{3} q_l\cdot(z-u)}\right\}$$
$$\times e^{ik\cdot(x-z)}e^{ik'\cdot(u-y)}G_0(k)G_0(k')\prod_{l=1}^{3}G_0(q_l). \tag{3.39}$$

The integration over z and u yield a product of two delta functions

$$2\pi)^d\delta^d(k-q_1-q_2-q_3)\times(2\pi)^d\delta^d(k'-q_1-q_2-q_3) \tag{3.40}$$

which represent "momentum conservation" at the two vertices. Hence

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int \frac{d^d k}{(2\pi)^d}e^{ik\cdot(x-y)}[G_0(k)]^2 \times \int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}G_0(q_1)G_0(q_2)G_0(k-q_1-q_2). \tag{3.41}$$

The last expression shows that $\bar{G}(x-y)$ is the Fourier transform of the function $\bar{G}(k)$,

$$\bar{G}(k) = \frac{1}{6} g^2 G_0(k) \left[\int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} G_0(q_1) G_0(q_2) G_0(k - q_1 - q_2) \right] G_0(-k), \qquad (3.42)$$

(where we have used $G_0(k) = G_0(-k)$). (3.42) can be represented diagrammatically in fig.(3.4). The graph shown there has two external propagators $G_0(k)$ and $G_0(-k)$, and three internal propagators; because of the delta-functions $\delta^d(\ldots)$ ("momentum conservation"), only two of the three internal lines are independent. By following the internal propagators one can describe three different closed loops, but because of "momentum conservation" only two of these are independent; i.e. there are only two integration variables in (3.42).

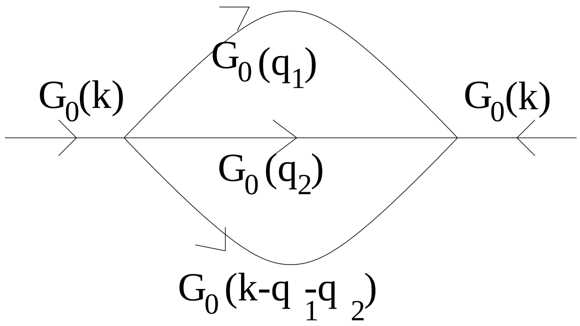


Figure 3.4: Diagrammatic representation of (3.42)

Our experience with the previous examples suggest the following "Feynman rules" in k-space ("momentum space"):

1. We draw the Feynman diagram with a momentum assigned to each line. We must have overall momentum conservation and conservation at each vertex.

2. To every vertex we assign a factor -g

3. To every line we assign a factor $G_0(k)$

4. To every independent loop there corresponds an integration $\int d^d q/(2\pi)^d$.

5. Finally, every graph is multiplied by a symmetry factor.

3.4.2 The Generating Functional of Connected Diagrams

We start with an example, by investigating the correlation function $G^{(4)}$. It subdivides into one connected and three disconnected diagrams,

$$G^{(4)}(1,2,3,4) = G_c^{(4)}(1,2,3,4) + \{G_c^{(2)}(1,2)G_c^{(2)}(3,4) + permutations\}, \tag{3.43}$$

where G_c denotes a connected correlation function. (note $G_c^{(2)} = G^{(2)}$. In terms of graphs this is represented as in fig.(refbubble0)

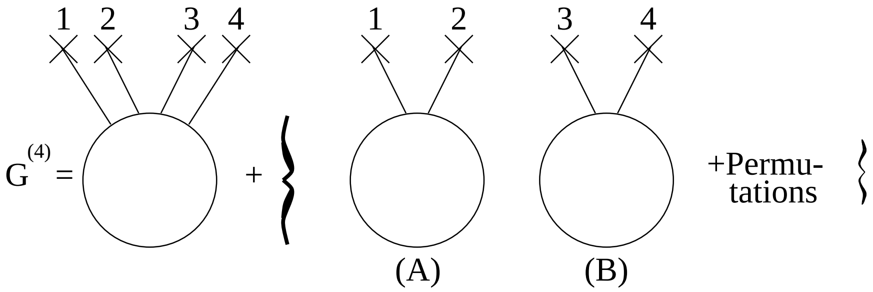


Figure 3.5:

The number of disconnected terms is $3 = 4!/[(2!)^2 \times (2!)]$. 4! is the number of permutations of the external points (1,2,3,4); but the result is unaffected by permuting (1,2), or (3,4), or the two bubbles (A) and (B), hence a factor $(2!)^2 \times 2!$.

We have been only considering theories where the n-point correlation functions with n odd vanish: $G^{(2k+1)} = 0$. For more generality, we shall assume that the interaction contains terms inφ^{2p+1}. Consider a disconnected diagram of $G^{(N)}$ corresponding to the subdivision into connected diagrams (fig 3.6):

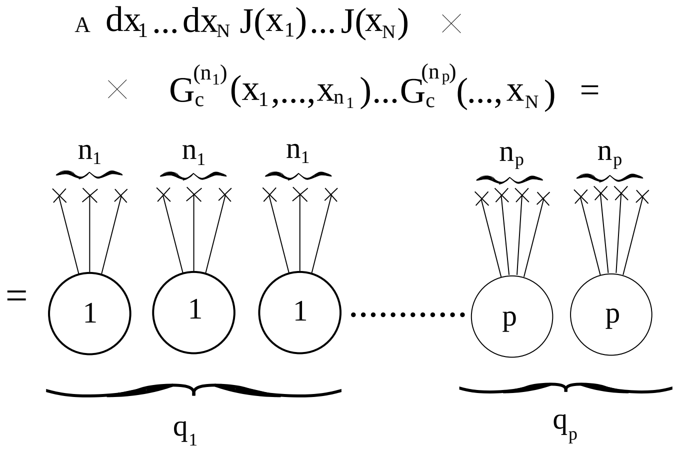


Figure 3.6:

There are q_l bubbles connected to n_l external points,...., q_p bubbles connected to n_p external points, with

$$q_1 n_1 + \cdots + q_p n_p = N. \tag{3.44}$$

The number of independent terms is

$$\frac{N!}{[(n_1!)^{q_1} q_1!] \ldots [(n_p!)^{q_p} q_p!]} \tag{3.45}$$

It is found that the Functional that generates just connected diagrams is the logarithm of the normalised Generating functional. Hence, consider the exponential of the generating functional of connected diagrams:

$$\exp \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \ldots dx_N j(x_1) \ldots j(x_N) G_c^N(x_1 \ldots x_N) \tag{3.46}$$

This should give the expansion for the generating function of all possible diagrams. When the exponential is expanded it is obvious that the amplitude for every possible disconnected diagram will be produced. To complete the proof that this is the correct Generating Functional we need to check each diagram comes with the correct prefactor.(i.e. equation (3.45). So expanding equation (3.46)

$$\sum_{q=0}^{\infty} \frac{1}{q!} \left(\sum_{n=1}^{\infty} \int dx_1 \ldots . dx_n j(x_1) \ldots . j(x_n) G_c^{(n)}(x_1 \ldots . x_n) \right)^q \tag{3.47}$$

We convert this sum into a summation over N, the number of legs of the disconnected diagrams (figure).

$$\sum_{N=0}^{\infty} \sum_{q_1 n_1 + \ldots + q_p n_p = N} \prod_{i=1}^{p} \frac{1}{q_i!} \left[\frac{\int dx_1 \ldots . dx_{n_i} j(x_1) \ldots . j(x_{n_i}) G_c^{n_i}(x_1 \ldots . x_{n_i})}{n_i!} \right]^{q_i} \tag{3.48}$$

Now we use (3.45) and the symmetry of G_c with respect to its arguments to rewrite the above equation as

$$\sum_{N=0}^{\infty} \frac{1}{N!} \int dx_1 \ldots . dx_N j(x_1) \ldots . j(x_N) \sum_{q_1 n_1 + \ldots + q_p n_p = N} G_c^{n_i}(x_1 \ldots . x_{n_i}) \ldots . G_c^{n_p}(\ldots . x_N) \tag{3.49}$$

Which is the correct form for the generating functional. Thus we have found that the generating functional of connected diagrams W(j) is indeed ln[Z(j)/Z(0)],

$$W(j) = ln \frac{Z(j)}{Z(0)} = \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \ldots dx_N j(x_1) \ldots j(x_N) G_c^N(x_1 \ldots x_N) \tag{3.50}$$

3.4.3 Generating functional of proper vertices

We have seen that to find $\log Z$ we only need consider the connected diagrams. The number of diagrams that need to be calculated can be reduced further. This is possible because some connected diagrams are made up of two or more connected diagrams joined together in a simple way. The following examples from ϕ^4 theory illustrates this redundancy connected diagrams can still have:

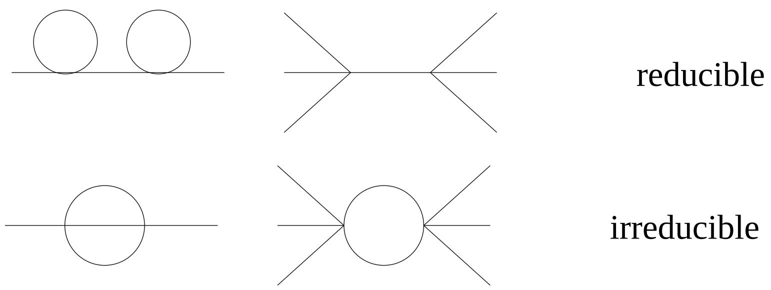


Figure 3.7: Examples of reducible and irreducible diagrams from ϕ^4 theory.

The diagrams in the first row can be split into two disjoint parts by cutting a single line, thus they are called *reducible*. The examples on the second row are called *one-particle irreducible*, 1PI, since they cannot be dissected in this way. The irreducible diagrams will play an important role later because they are closely related to the parameters of the theory and also they play an important role for the systematic construction of diagrams in higher loop orders. We define an *irreducible vertex function* which is an n-point function $\Gamma^{(n)}(x_1, \ldots, x_n)$. It is possible to construct a generating functional from which the irreducible vertex functions can be obtained. This functional is defined by a Legendre transformation:

$$\Gamma[\langle\phi\rangle] = W[J] - \int d^d x J(x) \langle\phi(x)\rangle \tag{3.51}$$

where

$$\langle\phi(x)\rangle = \frac{\delta W}{\delta J(x)} \tag{3.52}$$

This functional Γ has no explicit dependence on $j(x)$. It is a functional of the expectation value of the operator $\phi(x)$ in the presence of the source J denoted $\langle\phi\rangle$. It is easy to find from Eq.(3.51) that:

$$J(x) = -\frac{\delta\Gamma}{\delta\langle\phi(x)\rangle} \tag{3.53}$$

If we functionally differentiate equation 3.52 by $J(y)$ and use the chain rule we get the identity

$$\int d^d z \frac{\delta^2 W}{\delta j(x)\delta j(z)} \frac{\delta^2 \Gamma}{\delta \langle\phi(z)\rangle \, \delta \langle\phi(y)\rangle} = \int d^d z G^{(2)}(x-z)\Gamma^{(2)}(z-y) = -\delta^{(d)}(x-y) \quad (3.54)$$

Which shows that $-\Gamma^{(2)}(x-y)$ is the inverse of the connected Green's function $G^{(2)}(x-y)$

The self-energy $\Sigma(k)$ is defined as the sum of all two point 1PI diagrams shorn of their external lines. The correlation function $G^{(2)}(k)$ can be written in terms of $\Sigma(k)$ as

$$G^{(2)}(k) = G_0(k) + G_0(k)\Sigma(k)G_0(k) + \cdots = \left[G_0^{-1}(k) - \Sigma(k)\right]^{-1} \quad (3.55)$$

We have the relationships in ϕ^4 theory:

$$G^{(2)}(k) = \frac{1}{k^2 + m^2 - \Sigma(k)}, \qquad \Gamma^{(2)}(k) = -(k^2 + m^2 - \Sigma(k)) \quad (3.56)$$

We have a new effective mass $-\Gamma^{(2)}(0) = m^2 - \Sigma(0)$.

One can begin to see how the functions $\Gamma^{(n)}$ are related to the running parameters of the model.

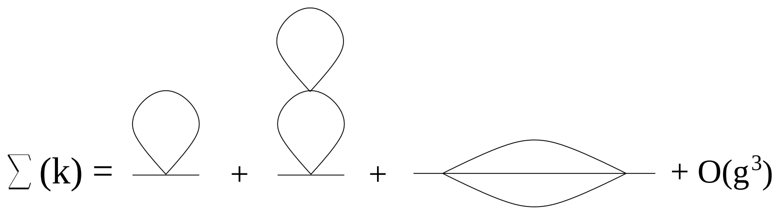


Figure 3.8: The self-energy in ϕ^4 theory to order g^2

To ease the notation in the proof of what follows, we use

$$\frac{\delta}{\delta J(x)} \to \frac{\delta}{\delta j_i}, \qquad \int d^d x \to \sum_i \quad (3.57)$$

We have just met the identity

$$\sum_l \frac{\delta^2 W}{\delta j_i \delta j_l} \frac{\delta^2 \Gamma}{\delta \bar\phi_l \delta \bar\phi_k} = \sum_l G^{(2)}_{il} \Gamma^{(2)}_{lk} = -\delta_{ik} \quad (3.58)$$

which shows that $-\Gamma^{(2)}_{lk}$ is the inverse of the(connected) Green's function $G^{(2)}_{kl}$.

We now proceed to Green's functions of higher order, by differentiating the identity Eq.(3.58) with respect to j_m:

$$\sum_l \frac{\delta^3 W}{\delta j_i \delta j_l \delta j_m} \frac{\delta^2 \Gamma}{\delta \langle \phi \rangle_l \delta \langle \phi \rangle_k} + \sum_l \frac{\delta^2 W}{\delta j_i \delta j_l} \frac{\delta^3 \Gamma}{\delta j_m \delta \langle \phi \rangle_l \delta \langle \phi \rangle_k} = 0 \tag{3.59}$$

Since Γ is a function of the $\langle \phi \rangle_i$, we must transform the second derivative in Eq.(3.59); we do this for the general case ($\Gamma^{(N)}_{i_1 \dots i_N} = \delta^{(N)}\Gamma / \delta\langle\phi\rangle_{i_1} \dots \delta\langle\phi\rangle_{i_N}$):

$$\frac{\delta}{\delta j_m} \Gamma^{(N)}_{i_1 \dots i_N} = \sum_n \frac{\delta\langle\phi\rangle_n}{\delta j_m} \frac{\delta \Gamma^{(N)}_{i_1 \dots i_N}}{\delta\langle\phi\rangle_n} = \sum_n G^{(2)}_{mn} \Gamma^{(N+1)}_{n i_1 \dots i_N}. \tag{3.60}$$

Equations (3.59) and (3.60) can be put into a graphical form if we represent the $\Gamma^{(N)}$ by shaded bubbles Fig.(3.9) (Here we have used Eq.(3.60) in order to transform Eq.(3.59).)

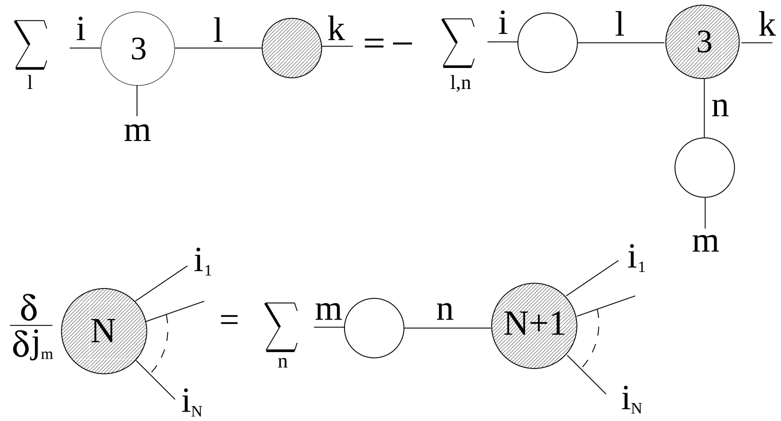


Figure 3.9: Graphical representation of Eq.(3.59) and Eq.(3.60).

Multiplying the two terms in Eq.(3.59) from the right by G_{kp} and summing over k we find the relation between $G^{(3)}_c$ and $\Gamma^{(3)}$,

$$G^{(3)}_{ijk} = \sum_{l,m,n} G_{il} G_{jm} G_{kn} \Gamma^{(3)}_{lmn} \tag{3.61}$$

see Fig(3.10). Clearly, $\Gamma^{(3)}$ represents the connected, truncated 3-point function since $G^{(3)}$ is connected, and each $\Gamma^{(2)}$ above chops off one of the external legs. $\Gamma^{(3)}$ is automatically 1PI, because it is a 3-point function.

We continue the process by differentiating Eq.(3.61) once more with respect to j_l. Using Eq.(3.59) and Eq.(3.60) we obtain the relation between $G^{(4)}$ and $\Gamma^{(4)}$, Fig.(3.11)

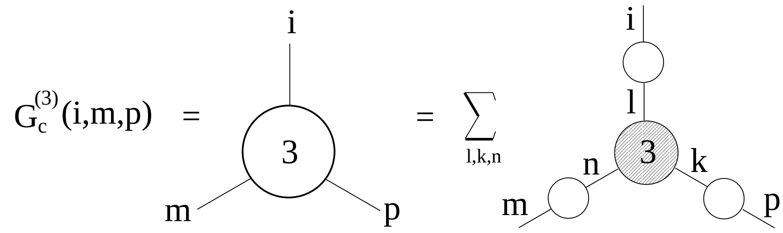


Figure 3.10: 3-point Green's function, $G^{(3)}$, written in terms of the proper 3-point vertex and 2-point Green's functions $G^{(2)}$

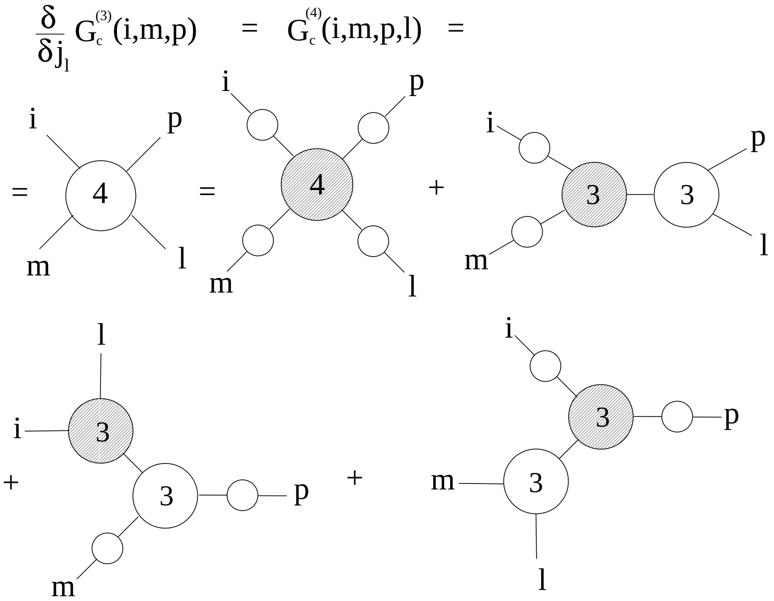


Figure 3.11: Relation between $G^{(4)}$ and $\Gamma^{(4)}$, $\Gamma^{(3)}$ and $G^{(2)}$.

As we know that $\Gamma^{(3)}$ is a sum of all 3-point 1PI diagrams, the last three terms of Fig.(3.11) supply all the diagrams of the 4-point Green's functions that can rendered disconnected by cutting one internal line. So the first term in Fig.(3.11) must be the summation over those diagrams of $G^{(4)}$ that don't become disconnected by cutting an internal line. Hence, we can identify the fourth derivative of the Effective action, $\Gamma^{(4)}$, as the summation over all 4-point 1PI diagrams.

It transpires that the higher order derivatives of Γ are the summation over 1PI diagrams. These can be joined together with 2-point Green's functions to construct the four and higher point connected Green's functions. The special cases N=3 and N=4 which we have just studied enable us to see how one can set out to prove the above statement.

One can prove by induction, assuming that an equation like that of Fig(3.11) can be written to order N, and that the N-th order proper vertex can be identified with the Nth derivative of the generating functional. Differentiating this equation with respect to j_i,

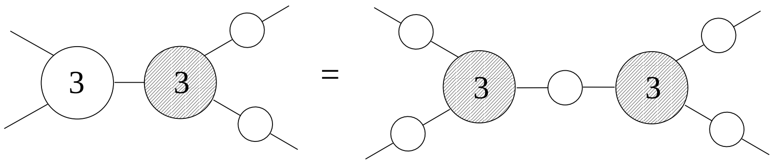


Figure 3.12: We can replace each $G^{(3)}$ for $\Gamma^{(3)}$ in Fig.(3.11) using Fig.(3.10).

one finds: see Fig.(3.13)

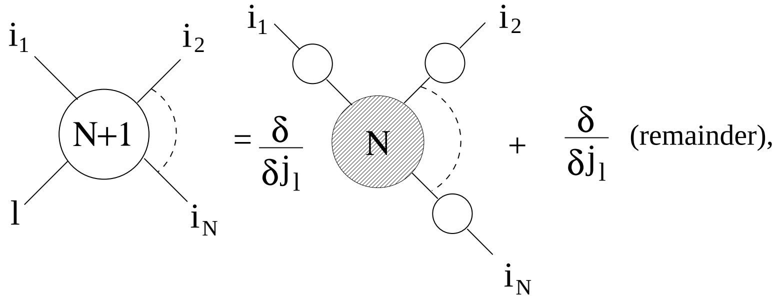


Figure 3.13: Differentiating $G^{(N)}$ with respect to source j_i.

where the “remainder” does not contain $\Gamma^{(N)}$, but only $\Gamma^{(N)}$ $\Gamma^{(N-2)}$, etc. (see Fig.(3.11)). Hence we obtain

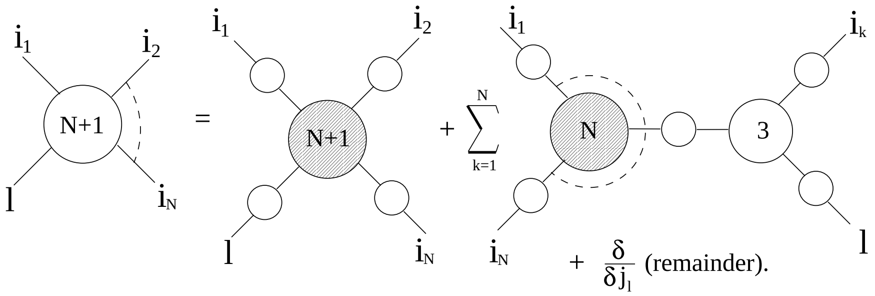


Figure 3.14: All possible Green's functions with N external legs.

On removing the full external propagators from both sides, we can identify $\Gamma^{(N+1)}$ as the N+1-point 1PI diagram.

There is an algebraic proof that by cutting one line in all possible ways in a diagram contributing to $\Gamma[\langle\phi\rangle]$ does not produce disconnected diagrams that is given in [**?**] (p. 135). This proof is easily generalsied to models that are more complicated than ϕ^4. Their proof is presented in Appendix C.

3.5 Grassmann Integration

Grassmann Algebra

We consider a set of anticommutating Grassmann variables $\{\zeta_i\}_{i=1,...,n}$, with complex linear coefficients, where n is the dimension of the algebra. The decisive relation defining the structure of the algebra is the anticommutation relation

$$\zeta_i\zeta_j + \zeta_j\zeta_i = 0 \tag{3.62}$$

for all i and j. As a particular consequence of this condition the square and all higher powers of a variable vanish,

$$\zeta_i^2 = 0 \tag{3.63}$$

The Grassmann algebra generate a Grassmann algebra of functions which have the form

$$f(\zeta) = f^{(0)} + \sum_i f_i^{(1)} + \sum_{i_1<i_2} f^{(2)}_{i_1i_2}\zeta_{i_1}\zeta_{i_2} + ... + f^{(n)}\zeta_{i_1}\zeta_{i_2}...\zeta_{i_n} \tag{3.64}$$

where the coefficients $f^{(k)}$ are ordinary complex numbers.

On this algebra we will need to define the derivative. We first consider a simple Grassmann algebra of order $n = 2$ with the variables ζ_1 and ζ_2.

$$f(\zeta_1, \zeta_2) = f^{(0)} + f_1^{(1)}\zeta_1 + f_2^{(1)}\zeta_2 + f^{(2)}\zeta_1\zeta_2$$

$$\frac{\partial f}{\partial \zeta_1} = f_1^{(1)} + f^{(2)}\zeta_2, \qquad \frac{\partial f}{\partial \zeta_2} = f_2^{(1)} - f^{(2)}\zeta_1. \tag{3.65}$$

Note the minus sign in the last equation of (3.65),

$$\frac{\partial}{\partial \zeta_j}\zeta_1\zeta_2 = \delta_{j1}\zeta_2 - \delta_{j2}\zeta_1.$$

The general rule for differentiation of a product is given by

$$\frac{\partial}{\partial \zeta_j}\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m} = \delta_{ji_1}\zeta_{i_2}...\zeta_{i_m} - \delta_{ji_2}\zeta_{i_1}\zeta_{i_3}...\zeta_{i_m} + ... + (-1)^{m-1}\delta_{ji_2}\zeta_{i_1}\zeta_{i_2}...\zeta_{i_{m-1}} \tag{3.66}$$

The respective factor ζ_{i_k} is anticommuted to the left until the derivative operator can be directly applied. We may prove the following properties of the derivatives

$$\frac{\partial}{\partial\zeta_i}\frac{\partial}{\partial\zeta_j}+\frac{\partial}{\partial\zeta_j}\frac{\partial}{\partial\zeta_i}=0 \tag{3.67}$$

$$\frac{\partial}{\partial\zeta_i}\zeta_j+\zeta_j\frac{\partial}{\partial\zeta_i}=0 \tag{3.68}$$

Grassmann integration

An attempt to to introduce an indefinite integral as the inverse of differentiation is bound to fail. This illustrated by the fact that according to (3.67) the second derivative of any Grassmann function vanishes, so that the inverse operation does not exist, for if there was an inverse to $\frac{\partial^2 F}{\partial\zeta^2}$ it should give

$$\int d\zeta\frac{\partial^2 F}{\partial\zeta^2}=\frac{\partial F}{\partial\zeta}$$

However as

$$0=\int d\zeta 0=\int d\zeta\frac{\partial^2 F}{\partial\zeta^2}$$

this would imply we always have

$$\frac{\partial F}{\partial\zeta}=0$$

which is not true in general.

We must be content with some formal definition. One way to arrive at it is to require that integration be translationally invariant. For an arbitrary function $g(\zeta)=g_1+g_2\zeta$ we have

$$\begin{aligned}\int d\zeta g(\zeta+\eta) &= \int d\zeta\,[g_1+g_2(\zeta+\eta)]=\int d\zeta\,[g_1+g_2\zeta]+\int d\zeta g_2\eta\\ &= \int d\zeta g(\zeta)+\left[\int d\zeta 1\right]g_2\eta=\int d\zeta g(\zeta)\end{aligned} \tag{3.69}$$

The translational invariance requires the integral of 1 is zero. The following postulates uniquely fix the value of any integral.

$$\int d\zeta 1 = 0, \tag{3.70}$$

$$\int d\zeta\zeta = 1. \tag{3.71}$$

Eq. (3.70) comes from the condition of translational invariance. The sole non-vanishing integral $\int d\zeta\zeta$ arbitrarily is assigned the value 1. This is a convenient normalisation condition.

We see that integration is equivalent to differentiation. Generalising integration rules to higher dimensions straightforward

$$\int d\zeta_i 1 = 0, \tag{3.72}$$

$$\int d\zeta_i\zeta_j = \delta_{ij}. \tag{3.73}$$

Note that the differentials $d\zeta_i$ must anticommute with all other Grassmann variables as integration is equivalent to differentiation. In order to obtain analog results of conventional integration we introduce complex Grassmann variables. Let us start with two disjoint sets of Grassmann variables $\zeta_1^*, ..., \zeta_n^*$ and $\zeta_1, ..., \zeta_n$, which are all mutually anticommutating

$$\{\zeta_i, \zeta_j\} = \{\zeta_i^*, \zeta_j^*\} = \{\zeta_i, \zeta_j^*\} = 0 \tag{3.74}$$

The two sets are related, using complex cunjugation, according to

$$\begin{aligned}
(\zeta_i)^* &= \zeta_i^*, \\
(\zeta_i^*)^* &= -\zeta_i, \\
(\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m})^* &= \zeta_{i_m}^*...\zeta_{i_2}^*\zeta_{i_1}^* \\
(\lambda\zeta_i)^* &= \lambda^*\zeta_i^*
\end{aligned} \tag{3.75}$$

where λ is a complex number.

In order to develop functional integral formalism for Grassmann fields we need to solve *Gaussian integrals.*

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp\left\{ - \sum_{k,l=1}^{N} \zeta_k^* M_{kl} \zeta_l \right\} \tag{3.76}$$

To simplify the notation, let us write this as

$$I = \int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} \tag{3.77}$$

The calculation in principle is very simply because grassmann functions can at worst be linear in each variable, causing the series expansion of the exponential function to terminate. On the other hand , according to the rules for Grassmann integration, the integrand must contain as many different Grassmann variables as there are integrals or else the overall integration vanishes.

Let us consider the case where we have two pairs of variables. The exponential then reads

$$e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2 - \frac{1}{3!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^3 + \ldots \tag{3.78}$$

Obviously this series terminates beyond second order, so we have

$$e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2. \tag{3.79}$$

Let us consider the integration of the first two terms, obviously we have

$$\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; 1 = 0$$
$$\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b) = 0$$

as the number of variables integrated over is greater than the number of variables appearing in the integrand. For the case of two pairs of variables one effectively has

$$\begin{aligned} e^{-\zeta^* M \zeta} &\rightarrow \frac{1}{2!} (\zeta^* M \zeta)^2 \\ &= \frac{1}{2!} (\zeta_1^* M_{11} \zeta_1 + \zeta_1^* M_{12} \zeta_2 + \zeta_2^* M_{21} \zeta_1 + \zeta_2^* M_{22} \zeta_2)^2 \\ &= (M_{11} M_{22} - M_{12} M_{21}) \zeta_1^* \zeta_1 \zeta_2^* \zeta_2 \end{aligned} \tag{3.80}$$

where the last line follows from the anticommutating character of the Grassmann numbers. The integration of $\zeta_1^*\zeta_1\zeta_2^*\zeta_2$, gives unity, and so for this case

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = detM \tag{3.81}$$

One should suspect that this result holds in general. For the case of N pairs of variables, only the term of order $(\zeta^* M\zeta)^N$ survives in the expansion of the exponential and contributes to the integral:

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = \frac{(-1)^N}{N!} \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k)(\zeta^* M\zeta)^N. \tag{3.82}$$

In view of the anticommutativity of the Grassmann variables, these terms contain the appropriately signed products of matrix elements which define the determinant. But rather than go through this combinatorial exercise we will follow the method given in (Brown QFT).(page83) which is presented in Appendix(B). The integral is some function $I(M)$ of the matrix M, let us derive a differential equation for this function. Since $\zeta^* M\zeta$ contains the product of two anti commutating variables and thus a commutating variable itself,

$$\begin{aligned} \delta_M (\zeta^* M\zeta)^N &= (\zeta^*(M+\delta M)\zeta)^N - (\zeta^* M\zeta)^N \\ &= n(\zeta^* \delta M\zeta)(\zeta^* M\zeta)^{n-1} \end{aligned} \tag{3.83}$$

$$\begin{aligned} \delta I &= \int [d\zeta^* d\zeta] \, (e^{-\zeta^*(M+\delta M)\zeta} - e^{-\zeta^* M\zeta}) \\ &= \int [d\zeta^* d\zeta] \sum_{n=0} \frac{(-1)^n}{n!} [(\zeta^*(M+\delta M)\zeta)^n - (\zeta^* M\zeta)^n] \\ &= -\int [d\zeta^* d\zeta] \, (\zeta^* \delta M\zeta) \sum_{n=0} \frac{(-1)^{n-1}}{(n-1)!} (\zeta^* M\zeta)^{n-1} \\ &= -\int [d\zeta^* d\zeta] \, \zeta^* \delta M\zeta \; e^{-\zeta^* M\zeta} \\ &= -\int [d\zeta^* d\zeta] \sum_{m,l=1}^{n} \zeta_m^* (\delta M)_{ml} \zeta_l \; e^{-\zeta^* M\zeta} \end{aligned} \tag{3.84}$$

Since $\zeta^* M\zeta$ commute, the derivative of $e^{-\zeta^* M\zeta}$ is given by

$$
\begin{aligned}
\frac{\partial}{\partial \zeta_k^*} e^{-\zeta^* M \zeta} &= \frac{\partial}{\partial \zeta_k^*} \sum_{n=0} \frac{(-1)^n}{n!} (\zeta^* M \zeta)^n \\
&= \sum_{n=1} \frac{(-1)^n}{n!} \Big([\frac{\partial}{\partial \zeta_k^*} (\zeta^* M \zeta)] (\zeta^* M \zeta)^{n-1} + \\
&\quad + (\zeta^* M \zeta) [\frac{\partial}{\partial \zeta_k^*} (\zeta^* M \zeta)] (\zeta^* M \zeta)^{n-2} + \cdots + (\zeta^* M \zeta)^{n-1} [\frac{\partial}{\partial \zeta_k^*} (\zeta^* M \zeta)] \Big) \\
&= -\sum_{l=1} M_{kl} \zeta_l \sum_{n=1} \frac{(-1)^{n-1}}{(n-1)!} (\zeta^* M \zeta)^{n-1} \\
&= -\sum_{l=1} M_{kl} \zeta_l e^{-\zeta^* M \zeta}. \qquad (3.85)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\delta I &= -\int [d\zeta^* d\zeta] \sum_{m,l=1} \zeta_m^* (\delta M)_{ml} \zeta_l \; e^{-\zeta^* M \zeta} \\
&= -\int [d\zeta^* d\zeta] \sum_{m,k,l=1} \zeta_m^* (\delta M M^{-1})_{mk} M_{kl} \zeta_l e^{-\zeta^* M \zeta} \\
&= \int [d\zeta^* d\zeta] \sum_{m,k=1} \zeta_m^* (\delta M M^{-1})_{mk} \frac{\partial}{\partial \zeta_k^*} e^{-\zeta^* M \zeta}. \qquad (3.86)
\end{aligned}
$$

This integral may be evaluated by "integration by parts" by using the following relation, where F is an arbitrary function of Grassmann variables

$$
\frac{\partial}{\partial \zeta_k^*} (\zeta_m^* F) = \delta_{km} F - \zeta_m^* \frac{\partial}{\partial \zeta_k^*} F. \qquad (3.87)
$$

To prove the above equation we need only consider three cases

(i) $k = m$ and F is a function of ζ_k^* as well. The r.h.s. vanishes as it should.

(ii) $k = m$ and F is not a function of ζ_k^* as well. Then we should just be left with F on thee r.h.s.

(iii) $k \neq m$. For this case the equation gives the correct answer again. The minus sign is there as the derivative has to pass through the first Grassmann variable.

Therefore,

$$\delta I = \int [d\zeta^* d\zeta] \sum_{k,m} (\delta M M^{-1})_{mk} \left\{ \delta_{km} e^{-\zeta^* M \zeta} - \frac{\partial}{\partial \zeta_k^*} (\zeta_m^* e^{-\zeta^* M \zeta}) \right\} \tag{3.88}$$

Since the Grassmann integral of a derivative vanishes,

$$\delta I = (Tr \delta M M^{-1}) I, \tag{3.89}$$

which gives

$$\delta \ln I = \frac{\delta I}{I} = Tr(\delta M M^{-1}), \tag{3.90}$$

It turns out that

$$Tr(\delta M M^{-1}) = \delta(\ln \ detM) \tag{3.91}$$

which we digress to prove.

$$detM = \sum_n M_{kn} C_{nk} \tag{3.92}$$

Taking the derivative of this, noting that C_{mk} is independent of the element M_{km},

$$\frac{\partial}{\partial M_{km}} detM = C_{mk} \tag{3.93}$$

Now the well known formul for the inverse matrix M^{-1} is

$$(M^{-1})_{kl} = \frac{C_{kl}}{detM} \tag{3.94}$$

So that we have

$$\frac{\partial}{\partial M_{km}} detM = (M^{-1})_{km} detM \tag{3.95}$$

or

$$\frac{\partial}{\partial M_{km}} \ln detM = (M^{-1})_{km}. \tag{3.96}$$

Accodingly,

$$\begin{aligned}
\delta \ln detM &= \sum_{k,m} \delta M_{km} \frac{\partial \ln detM}{\partial M_{km}} \\
&= \sum_{k,m} \delta M_{km} (M^{-1})_{mk} \\
&= Tr(\delta M M^{-1}) \\
&= Tr(M^{-1} \delta M).
\end{aligned} \tag{3.97}$$

Comparing this with (3.91), we see the equation (3.90) for $I(M)$ becomes

$$\delta \ln I = \delta(\ln\ detM), \tag{3.98}$$

with the solution

$$I(M) = Const.\ detM. \tag{3.99}$$

Treating the problem as a differential equation for $I(M)$, we set $M = 1$ in order to determine the proportionalitiy constant,

$$\begin{aligned}
I(1) &= Const. = \left[\int d\zeta^* d\zeta e^{-\zeta^* \zeta} \right]^N \\
&= \prod_{k=1}^N \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^N \zeta_i^* \zeta_i)^N \\
&= \frac{1}{N!} \prod_{k=1}^N \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^N \zeta_i^* \zeta_i)^N \\
&= (-1)^N \int d\zeta_N^* d\zeta_N \ldots d\zeta_1^* d\zeta_1 (\zeta_N^* \zeta_N \ldots \zeta_1^* \zeta_1) \\
&= 1^N = 1.
\end{aligned} \tag{3.100}$$

Hence the constant is unity and we do obtain the expected result:

$$I = \int \prod_{k=1}^N (d\zeta_k^* d\zeta_k) e^{-\zeta^* M \zeta} = detM \tag{3.101}$$

This should be compared to the ordinary integration where the corresponding integral gives $\det M^{-1}$.

Grassmann generating Functional

It is not surprising that the Gaussian integral formula (3.101) can be generalised to the case of general bilinear forms in the exponent:

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp - \left(\zeta^* M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho\right) = \det M \exp(\rho^\dagger M^{-1} \rho). \qquad (3.102)$$

Here ρ is an n-component vector of Grassmann variables. Equation (3.102) is obtained by translating the integration variable, $\zeta' = \zeta + M^{-1}\rho$,

$$\begin{aligned}
\det M &= \int \prod_{k=1}^{N} (d\zeta'^*_k d\zeta'_k) \exp -\zeta'^* M \zeta' \\
&= \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp -(\zeta^\dagger + \rho^\dagger M^{-1}) M (\zeta + M^{-1}\rho) \\
&= \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp - \left(\zeta^\dagger M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho + \rho^\dagger M^{-1} \rho\right) \\
&= \left[\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp - \left(\zeta^\dagger M \zeta + \rho^\dagger \zeta + \zeta^\dagger \rho\right)\right] \exp -(\rho^\dagger M^{-1} \rho) \qquad (3.103)
\end{aligned}$$

The construction of functional integration in section (4.1.2) did not make use of any special properties of the integration over field variables which might restrict the validity to ordinary c-numbers.

$$\boxed{\begin{aligned} &\int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\left[-\int d^d x'\, d^d x \bar{\chi}(x') A(x',x) \chi(x) + \int d^d x (\bar{\rho}(x)\chi(x) + \bar{\chi}(x)\rho(x))\right] \\ &\qquad = detA \exp\left[\int d^d x'\, d^d x \bar{\rho}(x') A^{-1}(x',x) \rho(x)\right]. \end{aligned}} \qquad (3.104)$$

in which the measure is $\propto \prod_{\mathbf{r}} d\bar{\varphi}(r) d\varphi(r)$ and $Z(\rho = 0) = detA$. Note that to normalise the functional we divide by $detA$ as apposed to $det(A^{-1})$ in the bosonic case (**??**).

It is rather straightforward to extend the results of esction 4.1 to the fermionic case: The Grassmann functional derivative is defined

$$\frac{\delta G[\chi(y)]}{\delta \chi(x)} = \lim_{\Delta V_i \to 0} \frac{\partial G}{\partial \chi_i} \quad \text{where } \mathbf{x} \text{ is located in cell } \Delta V_i \qquad (3.105)$$

The (2n)-point correlators

$$G^{(2n)}(y_1,\ldots,y_n;x_1,\ldots,x_n) = \langle\chi(y_n),\ldots,\chi(y_1);\bar{\chi}(x_1),\ldots,\bar{\chi}(x_n)\rangle \tag{3.106}$$

can now be obtained by forming derivatives of the generating functional [1]

$$G^{(2n)}(y_1,\ldots,y_n;x_1,\ldots,x_n) = \left.\frac{\delta^{2n}Z[\rho,\bar{\rho}]}{\delta\rho(x_n)\cdots\delta\rho(x_1)\delta\bar{\rho}(y_1)\cdots\delta\bar{\rho}(y_n)}\right|_{\rho=\bar{\rho}=0}. \tag{3.107}$$

3.6 QED from a Functional Integral

3.6.1 Photon Propagator in Different Gauges

The form of the propagator we used in our QED calculations was (in momentum space)

$$D_{F\mu\nu} = -\frac{4\pi}{k^2+i\varepsilon}\eta_{\mu\nu} \tag{3.108}$$

which is only one of many choices of defining the photon propagator.

When we construct S-matrix elemenet, e.g.

$$j^{\mu}_{43}(p_4,p_3)D_{F\mu\nu}(k)j^{\nu}_{21}(p_2,p_1) \tag{3.109}$$

where $p_2 = p_1 - k$ and $p_4 = p_3 + k$. Now the transition currents obey the equation of continuity. Their four divergencies vanish, i.e. in momentum space

$$\begin{aligned} k_\nu j^{\nu}_{21}(p_1-k,p_1) &= 0 \\ k_\nu j^{\nu}_{43}(p_3+k,p_3) &= 0. \end{aligned} \tag{3.110}$$

Therefore one can add to $D_{F\mu\nu}$ the expression $k_\mu f_\nu(k) + k_\nu g_\mu(k)$ with arbitrary functions $f_\nu(k)$ and $g_\mu(k)$ without changing the result of the calculation.

Keeping the symmetry between the transition currents, we generalise () to

$$D_{F\mu\nu} = -\frac{4\pi}{k^2+i\varepsilon}\eta_{\mu\nu} + k_\mu f_\nu(k) + k_\nu f_\mu(k). \tag{3.111}$$

[1] The order of the derivatives was chosen in such that we get agreement with the bosionic case. This is not a trivial matter as the Grassmann derivatives $\delta/\delta\rho(x)$ and $\delta/\delta\rho(x)$ anticommute with the field variables $\chi(x)$ and $\bar{\chi}(x)$. One can show, however, that there is an even number of commutations when we carry out the differentiations of (3.107) and write the result in the form (3.106).

The origin of this amniguity of the photon propagator is the gauge degrees of freedon of the electromagnetic field:

$$A_\mu(x) \to A_\mu(x) + \frac{\partial}{\partial x^\mu}\chi(x). \tag{3.112}$$

The Coulomb gauge

The propagator takes the form

$$\begin{aligned} D_{Cij}(k) &= \frac{4\pi}{k^2 + i\varepsilon}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \\ D_{C0j}(k) &= D_{Ci0}(k) = 0 \\ D_{C00}(k) &= \frac{4\pi}{\mathbf{k}^2} \end{aligned} \tag{3.113}$$

The component $D_{C00}(k)$ is just the fourier transform of the electrostatic potential $1/r$.

3.6.2 The Generating Functional Integral of QED

We include coupling of the electromagnetic field to the Dirac field. Since our previous work is valid in any sort of current coupling to the photon field

$$Z[J, \eta, \overline{\eta}] = \int [\mathcal{D}A][\mathcal{D}\psi][\mathcal{D}\overline{\psi}] \exp\left\{ i \int \mathcal{L}_{QED} \right\}, \tag{3.114}$$

where

$$\begin{aligned} \mathcal{L}_{QED} &= -\frac{1}{4e^2}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \overline{\psi}\left[\gamma_\mu\left(\frac{1}{i}\partial^\mu - A^\mu\right) + m_0\right]\psi + \\ &\qquad J^\mu A_\mu + \overline{\psi}\eta_0 + \overline{\eta_0}\psi. \end{aligned} \tag{3.115}$$

3.7 Canonical Quantisation of Scalar Field

We promote fields to operators, and impose equal time commutation relations on the fields and their conjugate momenta.

Position and momentum p are not operrators, instead they are just numbers.

The fields $\varphi(x,t)$ and their conjugate momentum fields $\pi(x,t)$ are operators

We have states as we do in ordinary quantum mechanics, but these are states of the field

$$\varphi = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\varphi(\vec{k})e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + \varphi^*(\vec{k})e^{i(\omega_k t - \vec{k}\cdot\vec{x})}\right] \tag{3.116}$$

We promote φ to an operator by promoting the coefficients $\varphi(\vec{k})$ and $\varphi^*(\vec{k})$:

$$\begin{aligned} \varphi(\vec{k}) &= \hat{a}(\vec{k}) \\ \varphi^*(\vec{k}) &= \hat{a}^\dagger(\vec{k}) \end{aligned} \tag{3.117}$$

Conjugate momentum

$$\begin{aligned} \partial_t\hat{\varphi} &= \partial_t \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger(\vec{k})e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})(-i\omega_k)e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger(\vec{k})(+i\omega_k)e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \\ &= -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}} \left[\hat{a}(\vec{k})e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} - \hat{a}^\dagger(\vec{k})e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \end{aligned} \tag{3.118}$$

3.7.1 Commutatin Relations

To quantise the scalar field we postulate the standard commutatin relations

$$\begin{aligned} [\hat{\varphi}(x), \hat{\pi}(y)] &= i\delta^3(\mathbf{x}-\mathbf{y}) \\ [\hat{\varphi}(x), \hat{\varphi}(y)] &= 0 \\ [\hat{\pi}(x), \hat{\pi}(y)] &= 0 \end{aligned} \tag{3.119}$$

3.7.2 Bose Statistics

$$\begin{aligned} |\vec{k}_1, \vec{k}_2> &= \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)|0> \\ &= \hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0> \\ &= |\vec{k}_2, \vec{k}_1> \end{aligned} \tag{3.120}$$

which implies

$$|\vec{k}_1, \vec{k}_2 >= |\vec{k}_2, \vec{k}_1 > \tag{3.121}$$

3.8 Perturbation Theory in Canonical Approach

Any physical process is a transition from an initial state $|i >= |A(t_0) >$ to a final state $|f >= |A(t) >$

3.8.1 Pictures

The Schrodinger picture

In the Schrodinger picture the time dependence is carried by the states according to Schrodinger's equation

$$i\hbar\frac{d}{dt}|A(t) >_S = \hat{H}|A(t) > \tag{3.122}$$

This can be formally solved in terms of the state of the system at an arbitrary initial time t_0

$$|A(t) >_S = \hat{U}|A(t_0) >_S \tag{3.123}$$

where $\hat{U}$ is the unitary operator:

$$\hat{U} := \hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar} \tag{3.124}$$

The Heisenberg picture

Using $\hat{U}$ we perform the following transformations, defining the state $|A(t) >_H$ and operator $O^H(t)$:

$$|A(t) >_H = \hat{U}^\dagger|A(t) >_S = |A(t_0) >_S \tag{3.125}$$

and

$$O^H(t) = \hat{U}^\dagger O^S \hat{U} \tag{3.126}$$

The H indicates that this is the Heisenberg picture. At $t = t_0$ states and operators in the two pictures are the same. We see that in the Heisenberg picture states are constant in time.

Differentiation of (3.126) gives the Heisenberg equation of motion

$$\begin{aligned}
i\hbar\frac{d}{dt}O^H(t) &= (i\hbar\frac{d}{dt}\hat{U}^\dagger)O^S\hat{U} + \hat{U}^\dagger O^S(i\hbar\frac{d}{dt}\hat{U}) \\
&= \hat{U}^\dagger\hat{O}^S\hat{U}\hat{H} - \hat{H}\hat{U}^\dagger\hat{O}^S\hat{U} \\
&= [\hat{O}^H, \hat{H}]
\end{aligned} \tag{3.127}$$

The Interaction picture

The Interaction picture arises if the Hamiltonian is split into two parts

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \tag{3.128}$$

The interaction picture is related to the Schrodinger picture by the unitary transformation

$$\hat{U}_0 = \hat{U}_0(t, t_0) = e^{-i\hat{H}_0(t-t_0)/\hbar} \tag{3.129}$$

i.e.

$$|A(t)>_I = \hat{U}_0^\dagger|A(t)>_S \tag{3.130}$$

and

$$O^I(t) = \hat{U}_0^\dagger O^S \hat{U}_0 \tag{3.131}$$

Differentiation of (3.131) gives the equation of motion in the interaction picture

3.8.2 Perturbation Theory

3.9 QED from Interaction Picture

$$\mathcal{H}_{int} = -ie\overline{\psi}\gamma^{\mu}\psi A_{\mu}. \tag{3.132}$$

There are various factrs of $\mathcal{H}_{int}$ in the expansion of thr interacting Lagrangian. We use Wick's theorem to pair off the various fermion photon lines to for propagators and vertices.

Chapter 4

Electro-Weak Theory

4.1 Fermi Interactions

Recall because of crossing symmetry, processes which differ in the grouping of incoming and outgoing particles are related to each other. In particular the matrix element of three body decay can be derived from that of the two-body scattering process. For example fig (4.1)

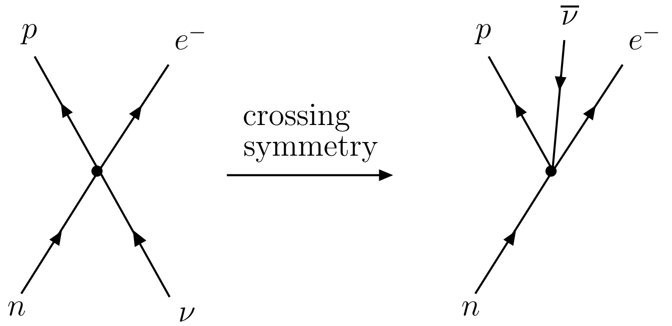


Figure 4.1: .

From parity violatoin in nuclear β decay interaction was postulated to be of the form

$$H_{int} = \frac{G}{\sqrt{2}} \int d^3x [\overline{u}_p(x)\gamma_\mu(C_V + C_A)u_n(x)] \times [\overline{u}_e(x)\gamma^\mu(1-\gamma_5)u_\nu(x)] \tag{4.1}$$

where the leptonic contribution

$$\overline{u}_e(x)\gamma^\mu(1-\gamma_5)u_{\nu_e}(x) \tag{4.2}$$

resembles the electromagnetic transition current.

By analogy with the electromagnetic current, we therefore introduce the weak leptonic current:

$$\begin{aligned} J_\mu^{(L)}(x) &= J_\mu^{(e)}(x) + J_\mu^{(\mu)}(x) + J_\mu^{(\tau)}(x) \\ &= \overline{u}_e(x)\gamma_\mu(1-\gamma_5)u_{\nu_e}(x) + \overline{u}_\mu(x)\gamma_\mu(1-\gamma_5)u_{\nu_\mu}(x) \\ &\quad + \overline{u}_\tau(x)\gamma_\mu(1-\gamma_5)u_{\nu_\tau}(x) \end{aligned} \tag{4.3}$$

Motivated by (4.1)

$$H_{int} = \frac{G}{\sqrt{2}} \int d^3x J_\mu^{(L)\dagger}(x) J^\mu_{(L)}(x). \tag{4.4}$$

We consider purely leptonic processes.

4.2 Intermediate Vector Gauge Boson Theory

4.2.1 Free Massive Vector Boson

We construct wave functions which describe particles with spin 1 out of solutions of the Dirac equation and the wave equation it generates.

In the rest system we find the following linearly independent combinations

$$\begin{aligned} \omega_{\alpha\beta}^{(+)}(0, i=0) &= \delta_{\alpha 1}\delta_{\beta 1} \\ \omega_{\alpha\beta}^{(+)}(0, i=1) &= \delta_{\alpha 2}\delta_{\beta 1} + \delta_{\alpha 1}\delta_{\beta 2} \\ \omega_{\alpha\beta}^{(+)}(0, i=2) &= \delta_{\alpha 2}\delta_{\beta 2} \end{aligned} \tag{4.5}$$

Each of these spinors represents an eigenvector of the operator of total spin, which in the rest system is defined by

$$\frac{1}{2}\hbar\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'} = \frac{1}{2}\hbar\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\beta\beta'} + \frac{1}{2}\hbar\hat{\Sigma}^3_{\beta\beta'}\delta_{\alpha\alpha'} \tag{4.6}$$

where

$$\hat{\Sigma}^3_{\alpha\alpha'} = \begin{pmatrix} \hat{\sigma}_3 & 0 \\ 0 & \hat{\sigma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.7}$$

We verify that the multispinors $\omega^{(+)}(0,4)$ fulfill the eigenvector equation

$$\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i) = \hbar(s-i)\omega^{(+)}(0,i)$$

Obviously we have

$$\begin{aligned} \hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'1} &= +\delta_{\alpha 1} \\ \hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'2} &= -\delta_{\alpha 2} \end{aligned} \tag{4.8}$$

$$\begin{aligned} \left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=0)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\ \omega^{(+)}_{\alpha'\beta'}(0,i=0) \\ &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'1}\delta_{\beta 1} + \delta_{\alpha 1}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'1} \\ &= \frac{\hbar}{2}2(\delta_{\alpha 1}\delta_{\beta 1}) \\ &= \hbar(3-2)\omega^{(+)}_{\alpha\beta}(0,i=0) \end{aligned} \tag{4.9}$$

$$\begin{aligned} \left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=1)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\ \omega^{(+)}_{\alpha'\beta'}(0,i=1) \\ &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'2}\delta_{\beta 1} + \delta_{\alpha 2}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'1} \\ &+ \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'1}\delta_{\beta 2} + \delta_{\alpha 1}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'2} \\ &= -\frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 1} + \frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 1} \\ &+ \frac{\hbar}{2}\delta_{\alpha 1}\delta_{\beta 2} - \frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 2} = 0 \end{aligned} \tag{4.10}$$

$$
\begin{aligned}
\left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=2)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\ \omega^{(+)}_{\alpha'\beta'}(0,i=2) \\
&= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'2}\delta_{\beta 2} + \delta_{\alpha 2}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta' 2} \\
&= -\frac{\hbar}{2}2(\delta_{\alpha 2}\delta_{\beta 2}) \\
&= -\hbar(3-2)\omega^{(+)}_{\alpha\beta}(0,i=2) \qquad (4.11)
\end{aligned}
$$

Now we can transform these multispinors into an arbitrary frame of reference via the inverse Lorentz transform.

$$
\hat{S}_{\alpha\alpha'\beta\beta'}\left(\frac{\mathbf{p}}{E}\right) = \hat{S}_{\alpha\alpha'}\left(\frac{\mathbf{p}}{E}\right)\hat{S}_{\beta\beta'}\left(\frac{\mathbf{p}}{E}\right) \qquad (4.12)
$$

Applying this we get

$$
\omega^{(+)}(x;p,i) = \hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^{(+)}(0,i), \qquad \omega^{(-)}(x;p,i) = \hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^{(-)}(0,i) \qquad (4.13)
$$

Now every wave function can be written as a superposition of plane waves:

$$
\begin{aligned}
\Psi^{(+)}_{\alpha\beta}(x;p,i) &= \omega^{(+)}_{\alpha\beta}(x;p,i)e^{-ip\cdot x/\hbar} \\
\Psi^{(-)}_{\alpha\beta}(x;p,i) &= \omega^{(-)}_{\alpha\beta}(x;p,i)e^{+ip\cdot x/\hbar} \qquad (4.14)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\Psi_{\alpha\beta}(x) &= \sum_i \int c^{(+)}(p,i)\Psi^{(+)}_{\alpha\beta}(x;p,i)d^3p \\
&+ \sum_i \int c^{(-)}(p,i)\Psi^{(-)}_{\alpha\beta}(x;p,i)d^3p. \qquad (4.15)
\end{aligned}
$$

It is easily seen that the plane wave satify Diracs equation. We consider one particular example

$$
(i\hbar\gamma\cdot\partial - m_0c)(\omega^{(+)}(x;p,i=1)e^{-ip\cdot x/\hbar}).
$$

first recall that

$$(i\hbar\gamma\cdot\partial - m_0c)\left[\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^1(0)e^{-ip\cdot x/\hbar}\right] = 0$$

where

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now

$$\begin{aligned}
& (i\hbar\gamma\cdot\partial - m_0c)_{\alpha\alpha'}(\omega^{(+)}_{\alpha'\beta}(x;p,i=1)e^{-ip\cdot x/\hbar}) \\
= \; & (i\hbar\gamma\cdot\partial - m_0c)_{\alpha\alpha'}\left(\hat{S}_{\alpha'\gamma}\left(\frac{\mathbf{p}}{E}\right)\hat{S}_{\beta\delta}\left(\frac{\mathbf{p}}{E}\right)\omega^{(+)}_{\gamma\delta}(0,i=1)e^{-ip\cdot x/\hbar}\right) \\
= \; & (i\hbar\gamma\cdot\partial - m_0c)_{\alpha\alpha'}\left(\hat{S}_{\alpha'1}\left(\frac{\mathbf{p}}{E}\right)e^{-ip\cdot x/\hbar}\right)\hat{S}_{\beta1}\left(\frac{\mathbf{p}}{E}\right) \\
= \; & \underbrace{\left[(i\hbar\gamma\cdot\partial - m_0c)\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^1(0)e^{-ip\cdot x/\hbar}\right]_\alpha}_{=0}\times\hat{S}_{\beta1}\left(\frac{\mathbf{p}}{E}\right)
\end{aligned} \tag{4.16}$$

Let us apply the Dirac equation to this first index α

$$\begin{aligned}
(i\hbar\gamma\cdot\partial - m_0c)_{\alpha\alpha'}\Psi_{\alpha'\beta}(x) \; = \; & \sum_i\int c^{(+)}(p,i)(i\hbar\gamma\cdot\partial - m_0c)_{\alpha\alpha'}\Psi^{(+)}_{\alpha'\beta}(x;p,i)d^3p \\
+ \; & \sum_i\int c^{(-)}(p,i)(i\hbar\gamma\cdot\partial - m_0c)_{\alpha\alpha'}\Psi^{(-)}_{\alpha'\beta}(x;p,i)d^3p
\end{aligned} \tag{4.17}$$

The multispinors therefore fulfill the following Dirac equations separately:

$$\begin{aligned}
(i\hbar\gamma\cdot\partial - m_0c)_{\alpha\alpha'}\Psi_{\alpha'\beta}(x) &= 0 \\
(i\hbar\gamma\cdot\partial - m_0c)_{\beta\beta'}\Psi_{\alpha\beta'}(x) &= 0
\end{aligned} \tag{4.18}$$

These are the so-called Bargmann-Wigner equations. Each component is also a solution of the Klein-Gordon equation

$$\left(\Box + \frac{m_0^2c^2}{\hbar^2}\right)\Psi_{\alpha\beta}(x) = 0. \tag{4.19}$$

The two Dirac equations for the symmetric matrix $\Psi_{\alpha\beta}(x)$ may be written as

Since the 4×4 spinor is symmetric, it may be expanded in terms of a complete set of symmetric elements of the Clifford algebra representation

$$\gamma^\mu \hat{C}, \quad \hat{\sigma}^{\mu\nu}\hat{C} \tag{4.20}$$

We define

$$\Psi(x) = m_0\gamma_\mu \hat{C} W^\mu(x) + \frac{1}{2}\hat{\sigma}^{\mu\nu}\hat{C} G_{\mu\nu}(x) \tag{4.21}$$

where thye coefficients $W^\mu(x)$ and $G^{\mu\nu}(x)$ are generally complex and transform under Loretz transformations like a vector and an anti-symmetric tensor, respectively. The Bargmann-Winger equations now become

$$\begin{aligned}
(i\hbar\cdot\partial\gamma - m_0c)\left(\frac{1}{\hbar}m_0c\gamma_\mu W^\mu(x) + \frac{1}{2}\hat{\sigma}_{\mu\nu}G^{\mu\nu}(x)\right)\hat{C} &= 0 \\
\left(\frac{1}{\hbar}m_0c\gamma_\mu W^\mu(x) + \frac{1}{2}\hat{\sigma}_{\mu\nu}G^{\mu\nu}(x)\right)\hat{C}(i\hbar\gamma^T\cdot \overleftarrow{\partial} - m_0c) &= 0
\end{aligned} \tag{4.22}$$

Using that $\hat{C}\gamma_\mu^T = -\gamma_\mu \hat{C}$

$$\begin{aligned}
&\Big(im_0c\partial_\alpha W_\mu(x)\gamma^\alpha\gamma^\mu - m_0^2c^2\gamma_\mu W^\mu(x) + \frac{1}{2}i\hbar\gamma_\alpha\hat{\sigma}^{\mu\nu}\partial^\alpha G_{\mu\nu}(x) \\
&\quad -\frac{1}{2}m_0c\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x)\Big)\hat{C} = 0,
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
&\Big(im_0c\partial_\alpha W_\mu(x)\gamma^\mu\gamma^\alpha - m_0^2c^2\gamma_\mu W^\mu(x) + \frac{1}{2}i\hbar\gamma_\alpha\partial^\alpha\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x) \\
&\quad +\frac{1}{2}m_0c\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x)\Big)\hat{C} = 0,
\end{aligned} \tag{4.24}$$

Subtracting (4.24) from (4.23) gives

$$
\begin{aligned}
&i m_0 \partial_\alpha W_\mu(x)\{\gamma^\alpha\gamma^\mu - \gamma^\mu\gamma^\alpha\}\hat{C} - \frac{2m_0^2c^2}{\hbar}\gamma_\mu \hat{C} W_\mu(x) \\
&+\frac{1}{2} i\hbar\{\gamma_\alpha\hat{\sigma}^{\mu\nu} - \hat{\sigma}^{\mu\nu}\gamma_\alpha\}\partial_\alpha G_{\mu\nu}(x)\hat{C} - m_0 c\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x)\hat{C} = 0.
\end{aligned} \tag{4.25}
$$

Using

$$
\begin{aligned}
i m_0 \partial_\alpha W_\mu(x)\{\gamma^\alpha\gamma^\mu - \gamma^\mu\gamma^\alpha\}\hat{C} &= 2m_0\partial_\alpha W_\mu(x)\hat{\sigma}^{\alpha\mu}\hat{C} \\
&= m_0(\partial^\alpha W^\mu(x) - \partial^\mu W^\alpha(x))\hat{\sigma}_{\alpha\mu}\hat{C}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2} i\hbar\{\gamma_\alpha\hat{\sigma}^{\mu\nu} - \hat{\sigma}^{\mu\nu}\gamma^\alpha\}\partial^\alpha G_{\mu\nu}(x)\hat{C} &= \frac{1}{2} i\hbar 2i(\eta^{\alpha\mu}\gamma^\nu - \eta^{\alpha\nu}\gamma^\mu)\hat{C}\partial_\alpha G_{\mu\nu}(x) \\
&= -2\hbar\gamma_\mu\hat{C}\partial_\alpha G^{\alpha\mu}(x)
\end{aligned}
$$

It follows that

$$
m_0 c(\partial^\alpha W^\mu(x) - \partial^\mu W^\alpha(x) - G^{\alpha\mu})\hat{\sigma}_{\alpha\mu}\hat{C} - 2\gamma_\mu\hat{C}\left(\hbar\partial_\alpha G^{\alpha\mu} + \frac{m_0^2c^2}{\hbar}W^\mu\right) = 0 \tag{4.26}
$$

The coefficients of the linearly independent matrices $\hat{C}$ must vanish separately. Hence, for $m_0 \neq 0$, this implies that

$$
G^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu, \tag{4.27}
$$

$$
\partial_\mu G^{\mu\nu} = -\frac{m_0^2c^2}{\hbar^2}W^\nu. \tag{4.28}
$$

Expressed in terms of the vector field W^μ

$$
\Box W^\mu(x) - \partial^\mu(\partial_\nu W^\nu(x)) + \frac{m_0^2c^2}{\hbar^2}\, W^\mu(x) = 0 \tag{4.29}
$$

On taking the divergence of this equation, one automatically obtains the Lorentz condition

$$\frac{m_0^2 c^2}{\hbar^2}\,\partial_\mu W^\mu(x) = 0 \tag{4.30}$$

This is in contrast to the photon case, where the Lorentz condition must be imposed as a subsidiary condition. The Proca equation then reduces to

$$\Box W^\mu(x) + \frac{m_0^2 c^2}{\hbar^2}\,W^\mu(x) = 0 \tag{4.31}$$

The Propagator

$$\eta^{\nu\lambda}\Box W_\lambda(x) - \partial^\nu(\partial^\lambda W_\lambda) + \frac{m_0^2 c^2}{\hbar^2}\eta^{\nu\lambda}W_\lambda(x) = J^\nu \tag{4.32}$$

In momentum space the left hand side reads

$$[\eta^{\nu\lambda}(-q^2 + M^2) + q^\nu q^\lambda]W_\lambda \tag{4.33}$$

The inverse operator $D_{\lambda\mu}(q)$ will have the structure

$$A(q^2)\eta_{\lambda\mu} + B(q^2)q_\lambda q_\mu \tag{4.34}$$

because there are only two second-rank tensors that can be formed.

$$\begin{aligned} & [\eta^{\nu\lambda}(-q^2 + M^2) + q^\nu q^\lambda]\;[A(q^2)\eta_{\lambda\mu} + B(q^2)q_\lambda q_\mu] \\ = \;& A(q^2)\;[(-q^2 + M^2)\delta^\nu_\mu + q^\nu q_\mu] + B(q^2)\;[q^\nu q_\mu(-q^2 + M^2) + q^2 q^\nu q_\mu] \\ = \;& \delta^\nu_\mu . \end{aligned}$$

Thus we get

$$A(q^2)\;(-q^2 + M^2) = 1$$

for $\mu = \nu$, and

$$A(q^2)\;[q^\nu q^\lambda] + B(q^2)\;[q^\nu q_\mu(-q^2 + M^2) + q^2 q^\nu q_\mu] = 0. \tag{4.35}$$

for $\mu \neq \nu$. Hence

$$A(q^2) = -\frac{1}{-q^2 + M^2} \tag{4.36}$$

and

$$B(q^2) = -\frac{1/M^2}{-q^2 + M^2} \tag{4.37}$$

The propagator

$$\begin{aligned} D^{\lambda\mu}(q) &= A(q^2)\eta^{\lambda\mu} + B(q^2)q^\lambda q^\mu \\ &= -\frac{\eta^{\lambda\mu}}{q^2 - M^2} + \frac{q^\lambda q^\mu/M^2}{q^2 - M^2} \\ &= \frac{-\eta^{\lambda\mu} + q^\lambda q^\mu/M^2}{q^2 - M^2} \end{aligned} \tag{4.38}$$

4.2.2 Interactions via Massive Bosons

vector meson exchange

$$\mathcal{L} = g_W^2(\overline{\Psi}_p\gamma^\mu\Psi_n)\frac{\eta_{\mu\nu} - q_\mu q_\nu/M_W^2}{q^2 - M_W^2 + i\epsilon}(\overline{\psi}_e\gamma^\nu\psi_\nu) \tag{4.39}$$

4.3 Lagrangian for Yang-Mills Theory

Of particular interest here are so called Yang-Mills theories which are a special example of gauge theory with a non-abelian group symmetry group. A nice introduction to the Yang-Mills field, with calculational details can be found in the book by Greiner (W. Greiner and B. Muler, "Gauge Theory of Weak Interactions", Springer-Verlag). For non-Abelian gauge theory a gauge transformation is given by

$$A_\mu^U \cdot \hat{T} = \hat{U}(A_\mu \cdot \hat{T})\hat{U}^{-1} + \frac{i}{g}\hat{U}\left(\partial_\mu \hat{U}^{-1}\right) \tag{4.40}$$

where

$$\hat{U} = \exp\left(i\theta^a(x)T^a\right). \tag{4.41}$$

with the generators of the Lie algebra corresponding to

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c \tag{4.42}$$

where the T^a are matrices, and as such they in general do not commute with each other. The curvature or field-strength tensor $\hat{F}_{\mu\nu} \equiv \sum_a F^a_{\mu\nu} T^a$. The index a is sometimes referred to a "colour" index. The Field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{4.43}$$

The Lagrangian is given by

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{Tr}(\hat{F}^2) = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} \tag{4.44}$$

The gauge field is called non-Abelian as we have

$$[A^a_\mu T^a, A^b_\nu T^b] = A^a_\mu A^b_\nu [T^a, T^b] = i A^a_\mu A^b_\nu f^{abc} T^c. \tag{4.45}$$

In fact, this is the origin of the quadratic term in the field strength tensor. This term results in the gauge bosons of the theory to interact with themselves, complicating the theory.

It can be shown that under a gauge transformation that the Field strength tensor transforms as:

$$\hat{T} \cdot \mathbf{F}'_{\mu\nu} = \hat{U}(\hat{T} \cdot \mathbf{F}_{\mu\nu})\hat{U}^{-1}, \tag{4.46}$$

This means that the Field strength tensor is not gauge invariant, however the quantity $Tr(\hat{F}^{\mu\nu} \cdot \hat{F}_{\mu\nu})$ is gauge invariant due to the cyclic property of the trace operation.

A method of quantizing the Yang-Mills theory is by functional methods, i.e. Path integral formulation. One introduces a generating functional for "n"-point functions as

$$Z[J] = \int \mathcal{D}[A] \exp\left[-\frac{i}{2}\int d^4x\, \mathrm{Tr}(\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}) + i\int d^4x\, J^a_\mu(x) A^{a\mu}(x)\right], \tag{4.47}$$

Quantum electrodynamics is the most famous example an Abelian Group—Abelian gauge theory. It relies on the symmetry group $U(1)$ and has one massless gauge field, the $U(1)$ gauge symmetry, dictating the form of the interactions involving the electromagnetic field, with the photon being the gauge boson. In this case $\hat{U} = \exp\{i\theta(x)\}$. The gauge field transforms as

$$A_\mu^U(x) = A_\mu(x) + \frac{1}{g}\partial_\mu\theta(x). \tag{4.48}$$

In QED there is no "colour" index, and no matrices, and the gauge group is $U(1)$ which corresponds to complex "numbers" of modulus (or Absolute value) 1, and hence the Abelian nature of QED:

$$[A_\mu, A_\nu] = A_\mu(x)A_\nu(x) - A_\nu(x)A_\mu(x) = 0. \tag{4.49}$$

Here the field strength tensor is simply given by,

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{4.50}$$

Note the absence of a term quadratic in the gauge fields. Also, note that here the field strength tensor is trivially gauge invariant.

Under the composition of two non-abelian gauge transformations we have

$$A_\mu^{U'U} \cdot \hat{T} = \hat{U}''(A_\mu \cdot \hat{T})\hat{U''}^{-1} + \frac{i}{g}\hat{U}''\left(\partial_\mu \hat{U''}^{-1}\right) \tag{4.51}$$

where

$$U'' = UU' \tag{4.52}$$

The infinitesimal form of the gauge transformation is

$$A_\mu'^a = A_\mu^a + \frac{1}{g}\partial_\mu\theta^a + f^{abc}A_\mu^b\theta^c. \tag{4.53}$$

4.4 Fadeev-Popov Gauge Fixing

For many years the naive quantization of a Yang-Mills theory (Yang-Mills theories now a cornerstone of particle physics) was flawed, and it remained unclear how to resolve the problem. It was Ludvig Faddeev and Victor Popov who developed the correct procedure for properly defining the functional integral quantization, now known as Faddeev-Popov gauge fixing.

The Standard Model of particle physics is a theory concerning the electromagnetic, weak, and strong nuclear interactions, as well as classifying all the matter particles known, i.e. scalar fields (in particular the charged Higgs field of the Standard Model) and spinor fields describing fermionic matter, the most familiar examples of fermionic matter being electrons and up quark and down quarks (the up and down quarks are from which which protons and neutrons are comprised).

The calculation of probability amplitudes in theoretical particle physics as calculated as a perturbative series requires the use of rather large and complicated integrals over a large number of variables. These integrals do, however, have a regular structure, and may be represented graphically as Feynman diagrams. The Feynman diagrams are drawn according to the Feynman rules, which depend upon the interaction Lagrangian.

It is possible to read off the Feynman rules of quantum field theory from a functional integral - in particularly from the Lagrangian. Here we are concerned with gauge fields, which mediate the forces between material particles, the simplest example of which would be the electromagnetic gauge potential. It turns out that, because of our freedom to make gauge transformations, one encounters difficulties not present with the description of matter fields (this problem turns out to be trivial in the very first case of a quantum field theory, i.e. quantum electrodynamics, involving the electromagnetic gauge potential, which is known as an Abelian gauge field. Here we will show how to properly apply the functional integral method to gauge fields, a method essential to non-abelian gauge theory also known as Yang-Mills theory which is used in the description of the unification of the electromagnetic and weak forces (electroweak theory) as well as the strong force (quantum chromodynamics) which together form the interactions of the Standard Model of particle physics. The method is known as Faddeev-Popov gauge fixing.

Historically, for many years, however before the Faddeev-Popov method, the quantization of Yang-Mills theory was not clear for some time. In 1965 Feynman (Feynman, R. P. 1963, *Acta Physica Polonica* **24**, *697.*) showed that the naive quantization of the theory was not unitary (implying non conservation of probability). In order to to cancel the nonunitary terms from the theory, Feynman was lead to postulate the existence of a term that did not emerge from the naive quantization. The procedure for properly defining the functional integral was developed by Faddeev and Popov (Faddeev, L. D., and Popov, V. N., 1967, *Phys. Lett.* **25B** 29; Se also : Mandelstam, S., 1962, *Ann. Phys.* **19**, 1.), from which Feynman's postulated term naturally emerges, now know as the Faddeev-Popov ghost.

4.4.1 Gauge Fixing: Analogy in a simple context

The Faddeev-Popov procedure is just a change of coordinates, but since it is for a functional integral, it may at first sight seem unfamiliar. Therefore we begin with a simple example. Consider the integral

$$Z = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-(x-y)^2} \tag{4.54}$$

The integral is invariant of the transformation

$$x \mapsto x + a \tag{4.55}$$
$$y \mapsto y + a \tag{4.56}$$

This represents the gauge symmetry of this toy model. The action $-(x-y)^2$ is gauge invariant, with a an element of the gauge group.

To impose a gauge constraint,

$$g(x, y) = 0 \tag{4.57}$$

we cant just insert the delta function $\delta(g = 0)$. For example, if we have a function $g(x)$ that has a zero at $x = c$, we have that:

$$\delta(g(x)) = \frac{\delta(x - C)}{|g'(x)|}. \tag{4.58}$$

Instead, let us consider the inclusion of the integral

$$1 = \int da \; \delta(g(a)) \, \frac{dg}{da} \tag{4.59}$$

into the expression.

This expression generalizes to

$$1 = \left(\prod_i^n \int da_i \right) \delta^n(g(a)) \; det \left(\frac{\partial g_i}{\partial a_j} \right) \tag{4.60}$$

where $\frac{\partial g_i}{\partial a_j}$ is an $n-$dimensional matrix M

$$
\begin{aligned}
\Delta_g^{-1}(x,y) &= \int da\, \delta\left(g(x(a),y(a))\right) \\
&= \int dg \;\det\left|\frac{da}{dg}\right| \delta(g) \\
&= \det\left|\frac{da}{dg}\right|_{g=0}
\end{aligned}
\tag{4.61}
$$

First we prove that $\Delta_g(x,y)$ is gauge invariant. We do this by proving $\Delta_g(x(a'),y(a')) = \Delta_g(x,y)$ for constant a',

$$
\begin{aligned}
\Delta_g^{-1}(x(a'),y(a')) &= \int da\, \delta\left(g(x+a'+a, y+a'+a)\right) \\
&= \int d(a+a')\, \delta\left(g(x+a+a', y+a+a')\right) \\
&= \int da''\, \delta\left(g(x(a''),y(a''))\right) \\
&= \Delta_g^{-1}(x,y)
\end{aligned}
\tag{4.62}
$$

where we have used $da = da''$

We insert

$$
1 = \Delta_g(x,y) \int da\; \delta(g(x(a),y(a)) = 0) \tag{4.63}
$$

into Z, +

$$
Z = \int dxdy\; \left\{\Delta_g(x,y) \int da\; \delta(g(a)=0)\right\} e^{-(x-y)^2} \tag{4.64}
$$

One then performs a gauge transformation taking $x(a)$ and $y(a)$ to $x = x(0)$ and $y = y(0)$ respectively, and using $dx(a) = dx, dy(a) = dy$, the action is gauge invariant, and so is $\Delta_g(x,y)$ we get

$$
Z = \int dxdy\; \left\{\Delta_g(x,y) \int da\; \delta(g(x(a),y(a))=0)\right\} e^{-(x-y)^2} \tag{4.65}
$$

$$
= \left(\int da\right) \int dxdy\; \Delta_g(x,y)\; \delta(g(x,y)=0) e^{-(x-y)^2} \tag{4.66}
$$

where we have been able to take out the factor $\int da$ since the integrand is independent of a.

$$\Delta_g = \left|\frac{\partial g}{\partial a}\right|_{g=0} = M \tag{4.67}$$

where M is a constant.

We now have to deal with the $\delta(g=0)$. Note that the result is independent of the choice of slice, so we could just as well have chosen $g=c$, where c is a real constant.

Here we would insert

$$1 = \left|\frac{\partial g}{\partial a}\right|_{g=c} \int da\ \delta(g(x(a), y(a)) - c = 0) \tag{4.68}$$

and obtain:

$$Z = \int da \int dxdy\ \left\{\delta(g(x,y) - c)\left|\frac{dg}{da}\right|_{g=c}\right\} e^{-(x-y)^2} \tag{4.69}$$

We take advantage of the fact that the integrand is independent of the c and integrate over this parameter with convenient weight. We take this weight to be the normalized Gaussian

$$1 = \frac{1}{\sqrt{\pi}} \int dc\ \exp(-c^2). \tag{4.70}$$

We obtain for Z,

$$\begin{aligned}
Z &= \int dxdy \int dc\ \left\{\delta(g-c=0)\left|\frac{dg}{da}\right|_{g=c}\right\} \frac{e^{-c^2}}{\sqrt{\pi}} e^{-(x-y)^2} && (4.71)\\
&= \int dxdy\ \left\{\int dc\ \delta(g-c=0)\frac{e^{-c^2}}{\sqrt{\pi}}\right\}\left|\frac{dg}{da}\right|_{a=0} e^{-(x-y)^2} && (4.72)\\
&= \int dxdy\ \frac{1}{\sqrt{\pi}}\left|\frac{dg}{da}\right|_{a=0} \exp\left(-(x-y)^2 + g^2\right) && (4.73)
\end{aligned}$$

where g^2 is the gauge fixing term.

4.4.2 Fadeev-Popov Gauge Fixing

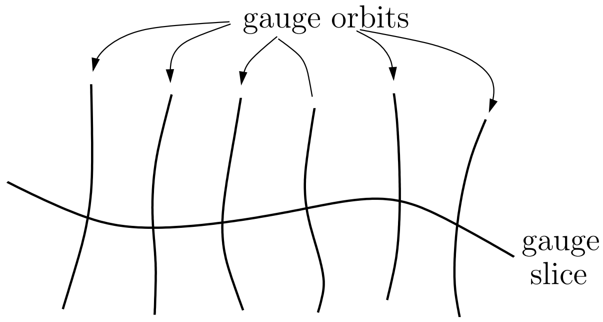


Figure 4.2: Schematic representation of the vector potential gauge field configuration space.

The line running upward represents the gauge orbit of the vector potential A_μ^Λ as the gauge transformation function Λ varies. By gauge invariance, all point along these lines are physically equivalent. Functionally integrating over all configurations overcounts the integrand repeatedly an infinite number of times giving us the non-sensical result

$$\int \mathcal{D}A_\mu \exp(iS[A]) = \infty. \tag{4.74}$$

This surface, the gauge slice, is the surface in function space that is described a gauge-fixing constraint. One might be tempted to solve the problem by simply inserting a gauge fixing factor

$$\delta\left(F(A)\right).$$

into the functional integral, forcing it to respect the gauge choice. However, this is inconsistent. We know that the delta function changes when we make changes in it. For example, if we have a function $f(x)$ that has a zero at $x = a$, we recall that:

$$\delta(f(x)) = \frac{\delta(x-a)}{|f'(x)|}. \tag{4.75}$$

Consider the integral

$$1 = \int da\ \delta(g(a))\ \frac{dg}{da}. \tag{4.76}$$

$Z[J]$, in this form, is not particularly easy to calculate into the functional integral

$$1 = \int \mathcal{D}U(x)\ \delta(G(A^U))\ det\left(\frac{\delta G(A^U)}{\delta U}\right). \tag{4.77}$$

This is the continuum generalisation of

$$1 = \left(\prod_i \int da_i\right) \delta^n(g(a))\ det\left(\frac{\partial g_i}{\partial a_j}\right) \tag{4.78}$$

Define

$$\Delta_{FP}^{-1}(A_\mu) := \int \mathcal{D}U\ \delta\left[G(A_\mu^U)\right] = \left(det\left|\frac{\delta G(A^U)}{\delta U}\right|_{G=0}\right)^{-1} \tag{4.79}$$

This is the Fadeev-Popov determinant. First we prove that it is gauge independent. This is done by proving that $\Delta_{FP}^{-1}(A_\mu) = \Delta_{FP}^{-1}(A_\mu^{U'})$ where $A_\mu^{U'}$ is obtained from A_μ via a gauge transformation corresponding to U':

$$\begin{aligned} \Delta_{FP}^{-1}(A_\mu^{U'}) &= \int \mathcal{D}U\ \delta\left[G(A_\mu^{U'U})\right] \\ &= \int \mathcal{D}[U'U]\ \delta\left[G(A_\mu^{U'U})\right] \\ &= \int \mathcal{D}[U'']\ \delta\left[G(A_\mu^{U''})\right] \\ &= \Delta_{FP}^{-1}(A_\mu), \end{aligned} \tag{4.80}$$

where we have used that (for compact groups) the volume element in group space defines and invariant measure (See, for example: Hamermesh, M. (1962), "Group Theory and its Application to Physical Problems", Addison-Wesley or Talman, J.D. (1968), "Special functions: A group Theoretical Approach', Benjamin):

$$\mathcal{D}U = \mathcal{D}U''. \tag{4.81}$$

We have

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}U(x)\ \delta(G(A^U)) \tag{4.82}$$

Following Faddeev and Popov we insert this identity into the functional integral $Z[J]$,

$$Z[J] = \int \mathcal{D}A \, \left\{ \Delta_{FP}(A_\mu) \int \mathcal{D}U(x) \, \delta(G(A^U)) \right\} \exp(iS[A]) \tag{4.83}$$

One then performs a gauge transformation taking A_μ^Λ to A_μ and using $\mathcal{D}A^\Lambda = \mathcal{D}A$, the action S is gauge invariant, and so is $\Delta_{FP}(A)$ we get

$$\begin{aligned} Z &= \int \mathcal{D}A \, \left\{ \Delta_{FP}(A_\mu) \int \mathcal{D}U(x) \, \delta(G(A)) \right\} \exp(iS[A]) \\ &= \left(\int \mathcal{D}U(x) \right) \int \mathcal{D}A \, \Delta_{FP}(A_\mu) \delta(G(A)) \exp(iS[A]) \end{aligned} \tag{4.84}$$

where we have been able to take out the factor $\mathcal{D}U(x)$ since the integrand is independent of U. Therefore, summation over gauge equivalent configurations has been factored out so that the divergent integral $\int \mathcal{D}U(x)$ gives a simply multiplicative factor. The correct expression to use is therefore:

$$Z = \int \mathcal{D}A \, \Delta_{FP}(A_\mu) \delta(G(A)) \exp(iS[A]) \tag{4.85}$$

The essential point is that the factor Δ_{FP} gives the correct measure in the functional integral, the factor needed for so many years to cure previous attempts to quantize gauge theories.

and so in (**??**) summation over gauge equivalent configurations has been factored put so that the divergent integral $\int \mathcal{D}U(x)$ gives a simply multiplicative factor.

The essential point is that the factor Δ_{FP} gives the correct measure in the functional integral, the factor needed for so many years to cure previous attempts to quantise gauge theories.

We have deal with the delta function. Note that $Z[J]$ as originally defined, is completely independent of the arbitrary function $\chi(x)$. It is convenient computationally to add together the contribution from all slices labeled by $\chi(x)$, each slice weighted with a gaussian function centred on $\chi = 0$.

4.4.3 Faddeev-Popov ghost fields

We now need an expression for

$$\Delta_{FP} = \det \left| \frac{\delta G(A^\Lambda)}{\delta \Lambda} \right|_{G=0} = \det M \tag{4.86}$$

Following Faddeev-Popov we convert the determinant of the matrix M into a Gaussian integral over two fields η and $\eta^\dagger$;

$$\Delta_{FP} = \det M = \int \mathcal{D}\overline{\eta}\mathcal{D}\eta \exp\left(\int d^4x d^4y \overline{\eta}^a(x) M_{ab}(x,y)\eta^b(y)\right) \tag{4.87}$$

The determinant appears in the numerator of the functional integral, rather than the denominator, which means we must integrate over Grassmann variables, rather than bosonic variables. The fields η and $\overline{\eta}$ are called"ghost fields", that is, scalar fields obeying Fermi-Dirac statistics. This is the origin of the Faddeev-Popov ghosts.

The Faddeev-Popov ghosts violate the *spin-statistics theorem*, and they are regarded as "non-physical" and only occur as *virtual particles* in Feynman diagrams.

We have deal with the delta function. We can modify the the gauge fixing term $\delta(G(A))$ by making the exchange:

$$G^a \mapsto G^a - \chi^a(x) \tag{4.88}$$

where $\chi^a(x)$ is an arbitrary function. Note that $Z[J]$ as originally defined, is completely independent of the arbitrary function $\chi(x)$. It is convenient computationally to multiply Z by an overall factor

$$\exp\left(-\frac{i}{2\xi}\int \chi^2 dx\right) \tag{4.89}$$

and then integrating over χ,

$$\begin{aligned} Z &= \int \mathcal{D}\chi \int \mathcal{D}A\ \Delta_{FP}(A_\mu)\delta(G(A)-\chi)\exp\left(-\frac{i}{2\xi}\int \chi^2 dx\right)\exp(iS[A]) \\ &= \int \mathcal{D}A\ \Delta_{FP}(A_\mu)\left\{\int \mathcal{D}\chi\ \delta(G(A)-\chi)\exp\left(-\frac{i}{2\xi}\int \chi^2 dx\right)\right\}\exp(iS[A]) \\ &= \int \mathcal{D}A \int \mathcal{D}\overline{\eta}\mathcal{D}\eta\ \exp\left(iS_{eff}[A]\right) \end{aligned} \tag{4.90}$$

where

$$S_{eff} = \int \left(-\frac{1}{2}Tr(F^{\mu\nu}F_{\mu\nu}) - \frac{1}{2\xi}G^2\right)dx + \int \overline{\eta}^a(x) M_{ab}(x,y)\eta^b(y) dx dy \tag{4.91}$$

As we obtain $M_{ab}(x,y)$ by taking a functional derivative, it will get put in the form: $\delta(x-y)\tilde{M}_{ab}$ in the integral. Therefore, we can write

$$Z = \int \mathcal{D}A \int \mathcal{D}\overline{\eta}\mathcal{D}\eta \, \exp\left(i \int \mathcal{L}_{eff}[A]dx\right) \tag{4.92}$$

where this effective Lagrangian, $\mathcal{L}_{eff}$ is given by

$$\begin{aligned} \mathcal{L}_{eff} &= \mathcal{L}[A] - \tfrac{1}{2\xi} G^2[A] + \mathcal{L}_{FPG}[\partial\overline{\eta}, \partial\eta, \overline{\eta}, \eta; A] & (4.93) \\ &= \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FPG} & (4.94) \end{aligned}$$

where $\mathcal{L}_{GF}$ is the gauge fixing term and $\mathcal{L}_{FPG}$ is the Faddeev-Popov ghost term. The naive quantization missed off the Faddeev-Popov ghost term.

The gauge fixing and this ghost Lagrangian modify the original theory in a compensating manner which allows one to define Feynman rules of the theory and carry out any perturbative calculation.

4.4.4 The R_ξ gauge

The exact form or formulation of ghosts is dependent on the particular gauge chosen, although the same physical results are obtained with all the gauges. The Gauge fixing R_ξ gauge is usually the simplest gauge for this purpose. It is a generalization of the Lorentz gauge, and obtained by putting

$$G^a = \partial^\mu A^a_\mu. \tag{4.95}$$

In evaluating the determinant, we start from the configuration satisfying the gauge constraint, and then perform an infinitesimal gauge transformation, given by

$$(A^\theta)^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \theta^a + f^{abc} A^b_\mu \theta^c. \tag{4.96}$$

then

$$\begin{aligned} G(A^\theta_\mu) - \chi^a(x) &= \partial^\mu A^a_\mu + \frac{1}{g}\left(\partial^\mu \partial_\mu \theta^a + g f^{abc} \partial^\mu (A^b_\mu \theta^c)\right) - \chi^a(x) \\ &= \frac{1}{g}\left(\partial^\mu \partial_\mu \theta^a + g f^{abc} \partial^\mu (A^b_\mu \theta^c)\right) & (4.97) \end{aligned}$$

and

$$\left|\frac{\delta G(A_\mu^\theta)}{\delta\theta^a}\right|_{\theta=0} = \frac{1}{g}\left(\delta^{ac}\partial^\mu\partial_\mu + gf^{abc}\partial^\mu A_\mu^b\right)\delta^4(x-y) \tag{4.98}$$

We convert this into a Grassmann integration

$$det\tilde{M} = \int \mathcal{D}\overline{\eta}\mathcal{D}\eta\ \exp\left(i\int -\overline{\eta}^a(\delta^{ac}\partial^\mu\partial_\mu + gf^{abc}\partial^\mu A_\mu^b)\eta^c\ dx\right) \tag{4.99}$$

(The factor of $1/g$ is absorbed into the normalization of the fields η and $\overline{\eta}$.) The effective Lagrangian, $\mathcal{L}_{eff}$, is then mde up of the parts:

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \tag{4.100}$$

for the field,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial\cdot A)^2 \tag{4.101}$$

for the gauge fixing and

$$\mathcal{L}_{FPG} = -(\bar{\eta}^a\partial_\mu\partial^\mu\eta^a + g\bar{\eta}^a f^{abc}\partial_\mu A^{b\mu}\eta^c) \tag{4.102}$$

for the ghost.

We obtain the quantization of Yang-Mills theory given in section.

4.4.5 Decoupling of ghost fields in QED

In particle physics, quantum electrodynamics (QED) is the relativistic quantum field theory of electrodynamics. In essence, it describes how the electromagnetic field and charged matter (electrons and protons as well as their anti-particles) interact and is the first theory where full agreement between quantum mechanics and special relativity is achieved. It is the simplest example of a gauge theory.

To obtain some familiarity with the Faddeev-Popov term, let us commute the Faddeev-Popov determinant for this simplest of gauge theory, Maxwell's theory. As it is an Abelian gauge theory we are lucky and the Faddeev-Popov ghost fields completely decouple from the theory and are not required; $\theta(x)$ has no "colour" index and the structure constants are zero. The infinitesimal form of the transformation is

$$(A^\theta)_\mu = A_\mu + \frac{1}{g}\partial_\mu \theta. \tag{4.103}$$

Then choosing

$$G = \partial^\mu A_\mu \tag{4.104}$$

we have

$$\begin{aligned} G(A^\theta) - \chi &= \partial^\mu A_\mu + \frac{1}{g}\partial^\mu \partial_\mu \theta - \chi \\ &= \frac{1}{g}\partial^\mu \partial_\mu \theta \end{aligned} \tag{4.105}$$

So in QED the determinant

$$det \left| \frac{\delta G(A^\theta)}{\delta \theta} \right|_{\theta=0} = \frac{1}{g}\partial^\mu \partial_\mu \delta^4(x-y) \tag{4.106}$$

is independent of A_μ and therefore just contributes to the overall normalization constant:

$$det\tilde{M} = \int \mathcal{D}\overline{\eta}\mathcal{D}\eta \; \exp\left(i \int -\overline{\eta}(\partial^\mu \partial_\mu)\eta \; dx \right). \tag{4.107}$$

4.5 Example: QED

4.5.1 Photon propagator

We have

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \\
&= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \\
&= -\frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \\
&= \frac{1}{2}[A^\nu \partial^\rho \partial_\rho A_\nu - \partial^\mu(A^\nu \partial_\mu A_\nu)] - \frac{1}{2}[A^\mu \partial_\mu \partial_\nu A^\nu - \partial_\mu(A^\nu \partial_\nu A^\mu)] \\
&\quad + \frac{1}{2\xi}[A^\mu \partial_\mu \partial_\nu A^\nu - \partial_\mu(A^\mu \partial_\nu A^\nu)] \\
&= \frac{1}{2}A^\mu \left[\Box\eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right] A^\nu - \frac{1}{2}\partial^\mu \left[A^\nu \partial_\mu A_\nu - A^\nu \partial_\nu A_\mu + \frac{1}{\xi}A_\mu \partial_\nu A^\nu\right]
\end{aligned}
\tag{4.108}
$$

We may write $\mathcal{L}$ as

$$
\mathcal{L} = \frac{1}{2}A^\mu \left[\Box\eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right] A^\nu = \frac{1}{2}A^\mu \tilde{P}_{\mu\nu} A^\nu \tag{4.109}
$$

In momentum space the operator $\tilde{P}_{\mu\nu}$ becomes

$$
(-q^2\eta^{\nu\lambda} + (1 - \xi^{-1})q^\nu q^\lambda) \tag{4.110}
$$

The inverse operator $D_{\lambda\mu}(q)$ will have the structure

$$
(A(q^2)\eta_{\lambda\mu} + B(q^2)q_\mu q_\lambda) \tag{4.111}
$$

$$
\begin{aligned}
& [(-q^2\eta^{\nu\lambda} + (1 - \xi^{-1})q^\nu q^\lambda)]\ [(A(q^2)\eta_{\lambda\mu} + B(q^2)q_\mu q_\lambda)] \\
=\ & A(q^2)\ [-q^2\delta^\nu_\mu + (1 - \xi^{-1})q^\nu q_\mu] + B(q^2)\ [-\xi^{-1}q^2 q^\nu q_\mu] \\
=\ & \delta^\nu_\mu .
\end{aligned}
\tag{4.112}
$$

Thus we get

$$
A(q^2) \cdot -q^2 = 1
$$

for $\mu = \nu$, and

$$A(q^2)\ [(1-\xi^{-1})q^\nu q_\mu] - \xi^{-1}B(q^2)\ [q^2 q^\nu q_\mu] = 0 \tag{4.113}$$

for $\mu \neq \nu$. Hence

$$A(q^2) = -\frac{1}{q^2} \tag{4.114}$$

and

$$B(q^2) = \frac{1-\xi}{q^4} \tag{4.115}$$

So that

$$(A(q^2)\eta_{\mu\nu} + B(q^2)q_\mu q_\nu) = \left(-\frac{\eta_{\mu\nu}}{q^2} + \frac{(1-\xi)q_\mu q_\nu}{q^4}\right). \tag{4.116}$$

The propagator is then

$$D_{\mu\nu}(q) = -\frac{1}{q^2+i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2+i\epsilon}\right). \tag{4.117}$$

4.6 Non-Abelian Case

In QED, the determinant was independent of A, so could be factorised out as another unimportant overall constant

In the non-Abelian case $Z[J]$ then becomes

$$\begin{aligned} Z[J] &= N(\xi)\int \mathcal{D}\chi \exp\left[-i\int d^4x \frac{Tr\chi^2(x)}{2\xi}\right]\int \mathcal{D}U(x) \\ &\quad \times \int \mathcal{D}A\, \delta(F(A)-\xi(x))\ det\left(\frac{\delta F^a(A)}{\delta\Lambda^b}\right)\exp(iS[A]) \\ &= N(\xi)\int \mathcal{D}U(x)\int \mathcal{D}A\ det\left(\frac{\delta F^a(A)}{\delta\Lambda^b}\right)\exp\left(iS[A] - i\int d^4x \frac{Tr(F[A])^2}{2\xi}\right) \end{aligned} \tag{4.118}$$

where $N(\xi)$ is an unimportant noralisation constant.

$$[T^a, T^b] = i\epsilon^{abc}T^c$$

$$A_\mu^\Lambda = UA_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger, \tag{4.119}$$

where

$$U(x) = \exp(ig\Lambda^a(x)T^a) \tag{4.120}$$

We have

$$\begin{aligned}
A_\mu^{c\Lambda}T^c(x) &= (\mathbf{1} + ig\Lambda^a(x)T^a)A_\mu^b(x)T^b(\mathbf{1} - ig\Lambda^a(x)T^a) + \frac{i}{g}(ig\partial_\mu\Lambda^c(x)T^c)(\mathbf{1} - ig\Lambda^b(x)T^b) \\
&= A_\mu^c(x)T^c - \partial_\mu\Lambda^c(x)T^c + ig\Lambda^a(x)A_\mu^b(x)[T^a, T^b] + \mathcal{O}(\Lambda^2) \\
&= A_\mu^c(x)T^c - \partial_\mu\Lambda^c(x)T^c - g\Lambda^a(x)A_\mu^b(x)\epsilon^{abc}T^c + \mathcal{O}(\Lambda^2)
\end{aligned} \tag{4.121}$$

or

$$A_\mu^{a\Lambda}(x) = A_\mu^a(x) - \partial_\mu\Lambda^a(x) \tag{4.122}$$

$$\begin{aligned}
M_{ab} = \frac{\delta F_a(x)}{\delta\Lambda^b(y)} &= \int d^4z \frac{\delta F_a(x)}{\delta A_c^\mu(z)}\frac{\delta A_c^\mu(z)}{\delta\Lambda^b(y)} \\
&= \int d^4z[-\partial_\mu\delta_{ac}\delta^4(x-z)][] \\
&= \\
&= \int d^4z\partial_\mu\delta_{ac}\delta^4(x-z)(D^\mu)_{cb}\delta^4(z-y) \\
&= (\partial_\mu D^\mu)_{ab}\delta^4(x-y)
\end{aligned} \tag{4.123}$$

where

$$(D^\mu)_{ab} = \partial_\mu\delta_{ab} + ig\epsilon_{abc}A^{\mu c}$$

The determinant in terms of a fermionic Gaussian integral over complex anticommutationg functions of the operator put in between the fields,

$$det\ M = \int \mathcal{D}\eta^* \mathcal{D}\eta \exp\left(\int d^4x d^4x' \eta^*(x) M(x-x') \eta(x')\right) \tag{4.124}$$

The Jacobian, Δ^F, may be represented as

$$\begin{aligned} & det\ |M_{ab}|_{F=0} \\ = & \int \mathcal{D}\eta^{*a} \mathcal{D}\eta^a \exp\left(\int d^4x d^4x' \eta^{*a}(x) M_{ab}(x-x') \eta^b(x')\right) \end{aligned} \tag{4.125}$$

So far we have worked only with the integral $\int \mathcal{D}A \exp(iS[A])$. Say we wanted to calculate the quantity

$$\frac{\int \mathcal{D}A \mathcal{O}(A) \exp(iS[A])}{\int \mathcal{D}A \exp(iS[A])} \tag{4.126}$$

where the operator is gauge invariant. The same procedure goes through with the numerator as the replacement of A by A^Λ works. We find for the correlation function

$$\frac{\int \mathcal{D}A \int \mathcal{D}\eta^* \mathcal{D}\eta\ \mathcal{O}(A) \exp(iS[A] - \frac{1}{2\xi}(F[A])^2 + iS_g[\eta,\eta^*;A])}{\int \mathcal{D}A \int \mathcal{D}\eta^* \mathcal{D}\eta \exp(iS[A] - \frac{1}{2\xi}(F[A])^2 + iS_g[\eta,\eta^*;A])} \tag{4.127}$$

where the awkard constant factors have canceled.

4.6.1 Final Lagrangian for Yang-Mills

The quadratic in the fields part of the actio, the part that gives the propagators, is given by

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \eta^\dagger_a \partial^2 \eta_a. \tag{4.128}$$

The interaction part of the Lagrangian is given by,

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{1}{2} g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \epsilon^{abc} A^{b\mu} A^{c\nu} \\ &\quad + \frac{1}{4} g^2 \epsilon^{abc} \epsilon^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \\ &\quad - ig\eta^{a\dagger} \epsilon^{abc} \partial^\mu A^c_\mu \eta^b. \end{aligned} \tag{4.129}$$

Boson propagator

The boson lagrangian is given by

$$\mathcal{L}_B = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \tag{4.130}$$

In analogy to the calculation (4.108), we may write $\mathcal{L}_B$ as

$$\mathcal{L}_B = \frac{1}{2} A^{a\mu}\left[\Box\eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right] A^{a\nu} = \frac{1}{2} A^{a\mu}\tilde{P}^{ab}_{\mu\nu}A^{b\nu}. \tag{4.131}$$

In momentum space the operator $\tilde{P}^{ac}_{\mu\nu}$ becomes

$$(-q^2\eta^{\nu\lambda} + (1-\xi^{-1})q^\nu q^\lambda)\delta^{ac}. \tag{4.132}$$

The inverse operator $D^{cb}_{\lambda\mu}(q)$ will have the structure

$$(A(q^2)\eta_{\lambda\mu} + B(q^2)q_\mu q_\lambda)\delta^{cb} \tag{4.133}$$

Performing the analogous calculation as in section 4.5.1, the propagator is

$$D^{ab}_{\mu\nu}(q) = -\frac{1}{q^2 + i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2 + i\epsilon}\right)\delta^{ab}. \tag{4.134}$$

Ghost Propagator

The free ghost lagrangian is given by

$$\mathcal{L}_{FP} = -\eta^\dagger_a \Box \eta_a \tag{4.135}$$

The ghost field propagates like a masslees spin-zero field:

$$\Delta^{ab}(p) = \frac{\delta^{ab}}{p^2 + i\epsilon}. \tag{4.136}$$

As we saw the functional integral method allows us to read off the Feynmann rules for vertices directly from the interacting field theory.

4.6.2 Interaction Vertices of Gauge Fields

The fermion and Yang-Mills vertex

In QED the interaction term of the Lagrangian is given by

$$\mathcal{L}_{int}^{EM} = -e\overline{\Psi}\gamma^\mu \Psi A_\mu \tag{4.137}$$

The vertex function is just

$$\Gamma^\mu_{EM} = -e\gamma^\mu. \tag{4.138}$$

In the same way we can read off from the interaction Lagrangian the vertex function for the coupling of fermions and Yang-Mills field

In general the Feynman rules are obtained by varying the corresponding action integral in momentum space.

Triple vertex

$$S_{int}^{trip} = \int d^4x \mathcal{L}_{int}^{trip} \tag{4.139}$$

$$\begin{aligned}\mathcal{L}_{int}^{quad} &= -\frac{1}{2}g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\epsilon_{abc}A^\mu_b A^\nu_c \\ &= -g\partial_\mu A^a_\nu \epsilon_{abc} A^\mu_b A^\nu_c\end{aligned} \tag{4.140}$$

$$A^a_\mu(x) = \int \frac{d^4p}{(2\pi)^4} A^a_\mu(p) e^{-ip\cdot x} \tag{4.141}$$

in momentum space

$$
\begin{aligned}
S_{int}^{trip} &= \int d^4x \mathcal{L}_{int}^{trip} \\
&= -g\epsilon_{abc} \int d^4x \partial_\mu A_\nu^a A_b^\mu A_c^\nu \\
&= -g\epsilon_{abc} \int d^4x \frac{d^4p d^4k d^4q}{(2\pi)^{12}} (-ip_\mu) A_\nu^a(p) A_b^\mu(q) A_c^\nu(k) e^{-i(p+q+k)\cdot x} \\
&= ig\epsilon_{abc} \int \frac{d^4p\ d^4k\ d^4q}{(2\pi)^8} p_\mu A_\nu^a(p) A_b^\mu(q) A_c^\nu(k) \delta(p+q+k) \\
&= ig\epsilon_{abc} \int \frac{d^4p\ d^4k\ d^4q}{(2\pi)^8}\ \delta(p+q+k) \\
&\quad \frac{1}{3!}(p_\mu A_\nu^a(p) A_b^\mu(q) A_c^\nu(k) + \\
&\quad\quad p_\mu A_\nu^a(p) A_b^\mu(k) A_c^\nu(q) + \\
&\quad\quad q_\mu A_\nu^a(q) A_b^\mu(k) A_c^\nu(p) + \\
&\quad\quad q_\mu A_\nu^a(q) A_b^\mu(p) A_c^\nu(k) + \\
&\quad\quad k_\mu A_\nu^a(k) A_b^\mu(p) A_c^\nu(q) + \\
&\quad\quad k_\mu A_\nu^a(k) A_b^\mu(q) A_c^\nu(p))
\end{aligned} \tag{4.142}
$$

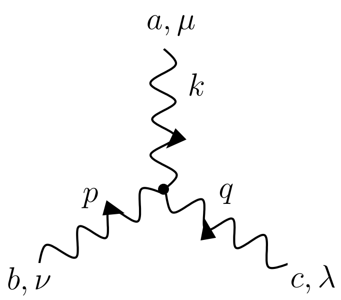


Figure 4.3: .

Variation yields

$$
\frac{\delta A_\mu^a}{\delta A_\nu^b} = \eta_\beta^\alpha\ \delta_{ab} \tag{4.143}
$$

in momentum space

Quadruple vertex

$$
S_{int}^{quad} = \int d^4x \mathcal{L}_{int}^{quad} \tag{4.144}
$$

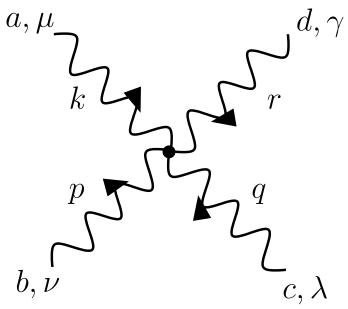


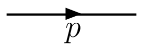
Figure 4.4: .

4.6.3 Ghosts and Coupling to the Gauge Field

$$\mathcal{L}_g = \eta_a^\dagger \partial^2 \eta_a + g\eta^{a\dagger}\epsilon^{abc}\partial^\mu A_\mu^c \eta^b \tag{4.145}$$

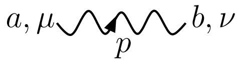
4.6.4 Feynmann Rules for Yang-Mills Theory

Fermion:



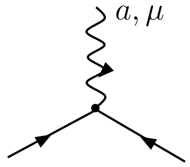
$$S_F(p) = \frac{\gamma^\mu p_\mu + M}{p^2 - M^2 + i\epsilon}$$

Boson:



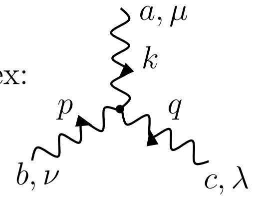
$$(D_F)^{ab}_{\mu\nu} = \frac{-iP_{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$$

Fermion vertex:



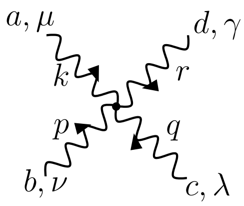
$$ig\gamma^\mu T^a$$

triple vertex:



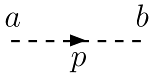
$$ig\epsilon_{abc}[\eta_{\mu\nu}(k-p)_\lambda + \eta_{\nu\lambda}(p-q)_\mu + \eta_{\lambda\mu}(q-k)_\nu]$$

quadruple vertex:



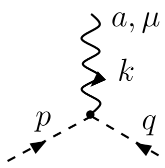
$$-ig^2[\epsilon_{abe}\epsilon_{cde}(\eta_{\mu\lambda}\eta_{\nu\gamma} - \eta_{\mu\gamma}\eta_{\nu\lambda}) + \epsilon_{ace}\epsilon_{bde}(\eta_{\mu\nu}\eta_{\lambda\gamma} - \eta_{\mu\gamma}\eta_{\mu\lambda}) + \epsilon_{ade}\epsilon_{cbe}(\eta_{\mu\lambda}\eta_{\mu\gamma} - \eta_{\mu\nu}\eta_{\lambda\gamma})]$$

Ghost:



$$\Delta_{ab}(p) = \frac{\delta_{ab}}{p^2 + i\epsilon}$$

Ghost-vertex:



$$g\epsilon_{abc}q^\mu$$

Figure 4.5: .

4.6.5 The Axial Gauge and the Temporal Gauge

The axial gauge

The axial gauge is defined by the condition

$$n^\mu A^a_\mu = 0, \quad n^\mu n_\mu = 1 \tag{4.146}$$

where n^μ is a spacelike vector. The gauge constraint is then

$$G = n^\mu A^a_\mu \tag{4.147}$$

and

$$
\begin{aligned}
G^a(A_\mu^\theta) - \chi^a &= n^\mu A_\mu^a + \frac{1}{g} n^\mu \partial_\mu \theta^a + f^{abc} n^\mu A_\mu^b \theta^c - \chi^a \\
&= \frac{1}{g} n^\mu \partial_\mu \theta^a + f^{abc} \chi^b \theta^c
\end{aligned} \tag{4.148}
$$

In this gauge the ghost ffield decouples from the gauge field.

We have

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} - \frac{1}{2\xi}(n^\mu A_\mu^a)^2 \\
&= \frac{1}{2}[A^{a\nu}\partial^\rho\partial_\rho A_\nu^a - \partial^\mu(A^{a\nu}\partial_\mu A_\nu^a)] - \frac{1}{2}[A^{a\mu}\partial_\mu\partial_\nu A^{a\nu} - \partial_\mu(A^{a\nu}\partial_\nu A^{a\mu})] - \frac{1}{2\xi}(n^\mu A_\mu^a)^2 \\
&= \frac{1}{2} A^{a\mu}\left[\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu - \frac{1}{\xi} n_\mu n_\nu\right] A^{a\nu} - \frac{1}{2}\partial^\mu\left[A^{a\nu}\partial_\mu A_\nu^a - A^{a\nu}\partial_\nu A_\mu^a\right]
\end{aligned} \tag{4.149}
$$

We may write $\mathcal{L}$ as

$$
\mathcal{L} = \frac{1}{2} A^{a\mu}\left[\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu - \frac{1}{\xi} n_\mu n_\nu\right] A^{a\nu} = \frac{1}{2} A^{a\mu} \tilde{P}^{ab}_{\mu\nu} A^{b\nu}. \tag{4.150}
$$

The operator $\tilde{P}^{ab}_{\mu\nu}$ in momentum space,

$$(-q^2\eta_{\mu\nu} + q_\mu q_\nu - \frac{1}{\xi}n_\mu n_\nu)\delta^{ab}. \tag{4.151}$$

It can be demonstrated that the inverse is

$$-\frac{1}{q^2}\left(\eta^{\mu\nu} + \frac{(n^2+\xi q^2)q^\mu q^\nu}{(q\cdot n)^2} - \frac{q^\mu n^\nu + n^\mu q^\nu}{(q\cdot n)}\right)\delta^{ab}. \tag{4.152}$$

i.e.

$$\begin{aligned}
& [(-q^2\eta_{\mu\lambda} + q_\mu q_\lambda - \frac{1}{\xi}n_\mu n_\lambda)\delta^{ac}] \cdot -\left[\frac{1}{q^2}\left(\eta^{\lambda\nu} + \frac{(n^2+\xi q^2)q^\lambda q^\nu}{(q\cdot n)^2} - \frac{q^\lambda n^\nu + n^\lambda q^\nu}{(q\cdot n)}\right)\delta^{cb}\right] \\
= & \left(\delta^\nu_\mu + \cancel{\frac{(n^2+\xi q^2)q_\mu q^\nu}{(q\cdot n)^2}} - \frac{q_\mu n^\nu + n_\mu q^\nu}{(q\cdot n)} - \frac{q_\mu q^\nu}{q^2} - \cancel{\frac{(n^2+\xi q^2)q_\mu q^\nu}{(q\cdot n)^2}} + \frac{q_\mu n^\nu + (q\cdot n)q_\mu q^\nu/q^2}{(q\cdot n)}\right. \\
& \left. + \cancel{\frac{n_\mu n^\nu}{\xi q^2}} + \frac{(\cancel{n^2}+\xi q^2)n_\mu q^\nu}{\xi q^2(q\cdot n)} - \frac{(q\cdot n)\cancel{n_\mu n^\nu} + \cancel{n^2}n_\mu q^\nu}{\xi q^2(q\cdot n)}\right)\delta^{ab} \\
= & \left(\delta^\nu_\mu - \frac{q_\mu n^\nu + n_\mu q^\nu}{(q\cdot n)} - \frac{q_\mu q^\nu}{q^2} + \frac{q_\mu n^\nu + (q\cdot n)q_\mu q^\nu/q^2}{(q\cdot n)} + \frac{n_\mu q^\nu}{(q\cdot n)}\right)\delta^{ab} \\
= & \delta^\nu_\mu \delta^{ab}.
\end{aligned} \tag{4.153}$$

The Temporal Gauge

$$n^\mu A^a_\mu = 0, \quad n^\mu n_\mu = -1 \tag{4.154}$$

where n^μ is a timelike vector.

In particualar, with $n_\mu = (1,0,0,0)$,

$$A^a_0 = 0. \tag{4.155}$$

4.7 Electro-Weak Theory

Veltman: I do not care what or how, but what we must have is at least one renormalisable theory with massive charged bosons, and whether that looks like Nature is of no concern, those are details that will be fixed later by some model freak...

't Hooft: I can do that.

Veltman: What do you say?

't Hooft: I can do that.

4.7.1 Introduction

The Higgs field is introduced into the model causing spontaneous symmetry breaking. This leads to the electron gaining mass.

4.7.2 Massless Dirac Lagrangian

The Dirac Lagrangian with zero mas is given by

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi \tag{4.156}$$

$$\begin{aligned}
\mathcal{L} &= i\overline{\psi}\gamma^\mu \partial_\mu \psi \\
&= i(\overline{\psi}_L + \overline{\psi}_R)\gamma^\mu \partial_\mu (\psi_L + \psi_R) \\
&= i\overline{\psi}_L\gamma^\mu \partial_\mu \psi_L + i\overline{\psi}_R\gamma^\mu \partial_\mu \psi_R + i(\overline{\psi}_L\gamma^\mu \partial_\mu \psi_R + \overline{\psi}_R\gamma^\mu \partial_\mu \psi_L)
\end{aligned} \tag{4.157}$$

The last term vanishes as

$$\begin{aligned}
\overline{\psi}_L\gamma^\mu \partial_\mu \psi_R &= \left(\frac{1-\gamma_5}{2}\right)\overline{\psi}\gamma^\mu \partial_\mu \left(\frac{1+\gamma_5}{2}\right)\psi \\
&= \frac{1}{4}(1-\gamma_5+\gamma_5-\gamma_5^2)\,\overline{\psi}\gamma^\mu \partial_\mu \psi \\
&= 0
\end{aligned} \tag{4.158}$$

so the mixed terms vanish and we are left

$$\mathcal{L} = i\overline{\psi}_L\gamma^\mu \partial_\mu \psi_L + i\overline{\psi}_R\gamma^\mu \partial_\mu \psi_R. \tag{4.159}$$

We see that the Lagrangian splits up into left- and right-handed parts.

4.7.3 Leptonic Fields in Electro-Weak Theory

$$\Psi_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \tag{4.160}$$

where ν_e is the electron neutrino and e_L is the left-handed electron field.

$$e_L = \left(\frac{1-\gamma_5}{2}\right) e, \qquad e_R = \left(\frac{1+\gamma_5}{2}\right) e \tag{4.161}$$

If we take the nuetrino be massless, the there is only the left-handed component of the neutrino field. Since the field is entirely left-handed, it satifies the equation

$$\left(\frac{1-\gamma_5}{2}\right) \nu_e = \nu_e. \tag{4.162}$$

With no right-handed component of the neutrino field, we define

$$\Psi_R = \begin{pmatrix} 0 \\ e_R \end{pmatrix} \tag{4.163}$$

4.7.4 Charges of the Electroweak Interaction

4.7.5 Higgs Field

In the standard model of particle physics, which obviusly contains electroweak theory, the massess of all the particles are zero. an extra field, the so-called Higgs field, is inserted by hand to give the particles mass.

4.7.6 Feynmann Rules for Electroweak Theory

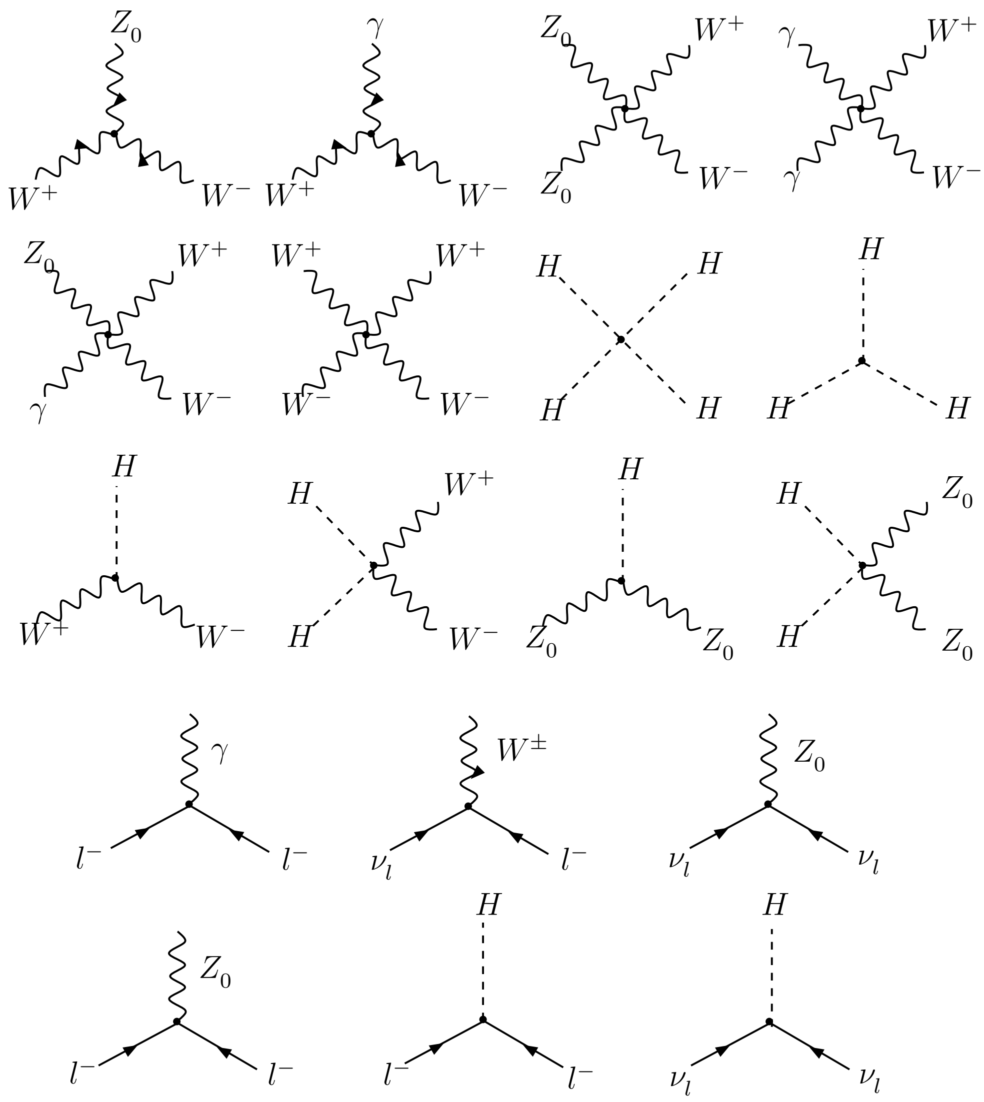


Figure 4.6: The 18 vertices of standard electroweak theory.

Chapter 5

The Standard Model of Particle Physics

5.0.7 The Weak Force

The gauge group of the weak force is $SU(2)$. The gauge bosons which mediate the weak force are denoted W^+, W^-, and Z. As the range of the weak force is short, these gauge bosons are massive.

5.0.8 The Strong Force

The gauge group of the strong force is $SU(3)$. The gauge bosons which mediate this force are called gluons.

5.0.9 Leptons

Leptons interact via the eletromagnetic and weak interaction, but do not participate in the sttrng interaction.

5.0.10 Higgs Field

In the standard model of particle physics the massess of all the particles are zero. An extra field, the so-called Higgs field, is inserted by hand to give the particles mass.

Bibliography

[1] Kenyon I R, *General Relativity* (Oxford University Press, New York).

[2] Ray D'Invero, *Introducing Einstein's Relativity* (Oxford University Press, New York).

[3] Robert M. Wald, *General Relativity*, (The University of Chicago Press).

[4] E. Poisson, *A Relativist's Toolkit - The Mathematics of Black-Hole Mechanics*,

[5] Misner C W, Thorne K S, Wheeler J A, *Gravitation*, (W. H. Freeman and Com)

[6] F. De Felice, C. J. S. Clarke, *Relativity on Curved Manifolds*, (Cambridge University Press 1990).

[7] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press; (February 1975)

[8] Penrose R., *Techniques of Differential Topology in Relativity*, A.M.S. Colloquium Publications, SIAM, Philadelphia.

[9] Geroch R., *Domain of Dependence*, Jour. Math. Phys. 11, p. 437.

[10] Alan D. Rendall, *The Cauchy problem for the Einstein equations*, available online.

[11] Robert M. Wald, *Quantum Field Theory in Curved Sapcetime and Black Hole Thermodynamics*

[12] L Smolin, *The Life of the Cosmos*, (Phoenix)

[13] L Smolin, *Three Roads to Quantum Gravity*, (Weidenfeld & Nicolson)

[14] M.B. Green, J. Schwarz, E. Witten, *Superstring Theory*, vol 1, 2, Cambridge University Press, Cambridge, 1987.
J. Polchinski, *String Theory*, vol 1 : *An Introduction to the Bosonic String*, vol. 2 : *Superstring theory and Beyond*, Cambridge University Press, Cambridge, 1998.

[15] B. Zwiebach, *A first course in string theory*, Cambridge University Press, Cambridge, 2004.; Graduate Course in String Theory by Angel M. Uranga, http://gesalerico.ft.uam.es/personales/angeluranga/firstpage.html.

[16] F. David Peat, *Superstrings and the Search of the Theory of Everything*

[17] Roger Penrose *The Road to Reality, A Complete Guide to the Laws of Physics* (Published by Jonathan Cape 2004).

[18] *Not Even Wrong...*

[19] L. Smolin *THE TROUBLE WITH PHYSICS - The Rise of String Theory, the Fall of Science, and What Comes Next*

[20] C. Rovelli, *Quantum Gravity*, (Cambridge University Press, Cambridge, 2004) to appear; a draft version of the book can be downloaded at http://www.cpt.univ_mrs.fr/ rovelli

[21] Klaus Kiefer, *Quantum Gravity* (International Series of Monographs on Physics), (Oxford University Press, ???, 2004).

[22] C. Rovelli Strings, loops and others: a critical survey of the present approaches to quantum gravity [gr-qc/9803024]:

[23] C. Rovelli. *Loop Quantum Gravity.* Living Reviews in General Relativity, electronic journal, http://www.livingreviews.org/Articles/Volume1/1998-1rovelli, [9709008].

[24] A. Ashtekar. *Lecture Notes on non Perturbative Canonical Gravity*, World Scientific, 1991.

[25] R. Gambini, J. Pullin, *Loops, Knots, Gauge Theories and Quantum Gravity*, Cambridge University Press, Cambridge, 1996.

[26] Jorge Pullin, Knot theory and quantum gravity in loop space: a primer, *AIP Conf.Proc.* 317 (1994) 141 [hep-th/9301028].

[27] L. Smolin. *An Invitation to Loop Quantum Gravity* [hep-th/0408048]

[28] A. Perez, *Introduction to Loop Quantum Gravity and Spin Foams*, [gr-qc/0409061].

[29] L. Smolin, *How far are we from a quantum theory of gravity*, [hep-th/0303185].

[30] Carlo Rovelli, *A dialog on quantum gravity*, Int.J.Mod.Phys. D12 (2003) 1509-1528, [hep-th/0310077].

[31] T. Thiemann, Modern Canonical Quantum General Relativity General Relativity

[32] T. Thiemann, Introduction to Modern Canonical Quantum General Relativity, [gr-qc/0110034].

[33] T. Thiemann, Lectures on Loop Quantum Gravity *Lect.Notes Phys. 631 (2003)* 41-135 [gr-qc/0210094].

[34] A. Ashtekar, J. Lewandowski, *Background Independent Quantum Gravity: A Status Report*, Class.Quant.Grav. 21 (2004) R53, [gr-qc/0404018].

[35] Jorge Pullin, Canonical quantization of general relativity: the last 18 years in a nutshell, *AIP Conf.Proc. 668 (2003) 141-153*, [gr-qc/0209008].

[36] C. Rovelli *background-independent* (McGraw-Hill)

[37] C. Rovelli *background-independent*

[38] C. Rovelli, *Loop Quantum Gravity and the Meaning of Diffeomorphism Invariance*, [gr-qc/ 9910079].

[39] C. Rovelli, Half way through the woods, Contemporary Research on Space and Time. The Cosmos of Science: Essays of Exploration, ed. J. Earman & J.D. Norton, (Pittsburgh: U Pitt Press: 1997), pp. 180-223.

[40] C. Rovelli, *Time in quantum gravity: An hypothesis.* Phys. Rev. D, 43:442 456, 1991.

[41] K. Kuuchař "Time and interpretations of quantum gravity" in "Proceedings of the 4th Canadian conference on general relativity and relativistic astrophysics", G. Kunstater, D. Vincent, J. Williams (editors), World scientific, Singapore (1992), online at http://www.phys.lsu.edu/faculty/pullin/kvk.pdf

[42] CJ Isham, "Canonical quantum gravity and the problem of time", Presented at 19th International Colloquium on Group Theoretical Methods in Physics, Salamanca, Spain (1992), 157-288; [arXiv:gr-qc/9210011].

[43] Johannes Simon, *Change Without Time*, PhD dissertation.

[44] L. Smolin *The case for background independence*, [hep-th/0507235].

[45] *Conceptual Problems of Quantum Gravity*, ed. by A. Ashtekar, J. Stachel, (Birkhar, Boston, 1991).

[46] C Rovelli, Relation Quantum Mechanics, [quant-ph/9609002], N Grot, C Rovelli and R S Tate, Time-of-arrival in quantum mechanics, [quant-ph/9603021]

[47] D. N. Page and W. K. Wootters, Phys. Rev. D **27** 2885 (1983).

[48] Kai S Lam *Topics in Conteporary Mathematical Physics* (World scientific publishing Co.Pte.Ltd)

[49] C.Rovelli, *GPS observables in general relativity*, Phys.Rev. D65 (2002) 044017, [gr-qc/0110003].

[50] , *Introducing Relativity in GNSS*, submitted for publication, [gr-qc/507121].

[51] C.Rovelli, *Partial observables*, Phys.Rev. D65 (2002) 124013, [gr-qc/0110035].

[52] Kauffmann L H, Lins S L, *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds*, (Princeton University Press)

[53] Haag R. Hugenholtz N M and Winnik M 1967 *Commun. Math. Phys.* **5** 215.

[54] C. Rovelli and L. Smolin, Loop space representation of quantum general relativity, *Nucl. Phys.* B331 (1990) 80-152

[55] Carlo Rovelli, Lee Smolin, Spin Networks and Quantum Gravity, Phys.Rev. D52 (1995) 5743-5759, [gr-qc/9505006]

[56] Seth Major, *A Spin Network Primer*, [gr-qc/9905020].

[57] Carlo Rovelli, Lee Smolin, *Discreteness of area and volume in quantum gravity*, Nucl.Phys. B442 (1995) 593-622; Erratum-ibid. B456 (1995) 753, [gr-qc/9411005].

[58] C. Rovelli, S. Speziale, *Reconcile Planck-scale discreteness and the Lorentz-Fitzgerald contraction*, Phys.Rev. D67 (2003) 064019, [gr-qc/0205108].

[59] S. Frittelli, L. Lehner, C. Rovelli, *The complete spectrum of the area from recoupling theory in loop quantum gravity*, Class.Quant.Grav. **13** (1996) 2921-2932, [gr-qc/9608043].

[60] T. Thiemann, *A length operator for canonical quantum gravity*, J.Math.Phys. **38** (1997) 4730-4740, [gr-qc/9606092].

[61] C. Rovelli, The projector on physical States in loop quantum gravity, [gr-qc/9806121]

[62] Keith Hannabuss, *An introduction to Quantum Theory* (Oxford University Press, New York)

[63] P. A. M. Dirac, *Lecture notes on Quantum Mechanics* (Belfer Graduate Scholl of Science, New York. 1964)

[64] V. Husain, Nucl. Phys. **B313** (1989) 711-724; B. Bruegmann and J. Pullin, Nucl. Phys. **B363** (1991) 221.

[65] B. Brgmann, R. Gambini, and J. Pullin , *Knot invariants as nondegenerate quantum geometries*, Phys. Rev. Lett. 68, 431-434 (1992).

[66] C. Rovelli and L. Smolin, *Knot Theory and Quantum Gravity*, Phys. Rev. Lett **61**, 1155 (1988); *Loop representation for quantum General Relativity*, Nucl. Phys. **B133** (1990) 80.

[67] T. Jacobson and L. Smolin, Nucl. Phys. B 299 (1988) 295.

Classical Relativity

[68] Robert Bartnik, Jim Isenberg, *The Constraint Equations*, to appear in the proceedings of the 2002 Cargese meeting "50 Years of the Cauchy Problem, in honour of Y.Choquet-Bruhat", editors P.T.Chruściel and H. Friedrich, [gr-qc/0405092].

[69] A. Einstein, *Relativity: The Special and General Theory; A Popular Exposition*, translated by R.W. Lawson (Crown Publishers, New York, 1961).

[70] J. David Brown, Donald Marolf, *On Relativistic Material Reference Systems*, Phys.Rev. D53 (1996) 1835-1844, [gr-qc/9509026].

[71] J.D. Brown, K.V. Kuchar, *Dust as a Standard of Space and Time in Canonical Quantum Gravity*, Phys.Rev. D51 (1995) 5600-5629, [gr-qc/9409001].

Quantum mechanics???

[72] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yesiva University Press, 1964).

[73] quant-ph/0110165 29 Oct 2001 *Rigged Hilbert Space Approach to the Schrödinger Equation*

[74] J. C. Baez, An Introduction to Spin Foam Models of Quantum Gravity and BF Theory, Lect.Notes Phys. 543 (2000) 25-94, [gr-qc/9905087]

[75] Michael Murray, Differential Geometry - Lecture notes for an honors course at the University of Adelaide by Michael Murray in HTML with GIFs.

[76] K. Reidermeister, Knotentheorie (Chelsa Publishing Co., New York, 1948), original printing (Springer, Berlin, 19 32). See also Kauffman, *Knots and Physics, pp.16*

[77] T. Thiemann *Phoenix Project: Master Constraint Programme for Loop Quantum Gravity*, Class.Quant.Grav. 23 (2006) 2211-2248, [gr-qc/0305080]

[78] A. Ashetekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann *A Maifestly Gauage-Invariant Approach to Quantum Theories of Gauge Fields* [hep-th/9408108]

[79] http://arcturus.mit.edu/8.962

[80] Jorge V. Jose, Eugene J. Saletan (Cambridge Univesity Press 1998) *Classical Dynamics A Contemporary Approach*

[81] Descartes, R. 1983 [1644]. *Principles of Philosophy.* Translated by V. R. Miller and R.P.Miller. Dordrecht: D. Reidel.

[82] M. Montesinos gr-qc/0002023 16 Jan 2001 *Relational evolution of the degrees of freedom of generally covariant quantum theories*

[83] K. Hannabuss, *An Introduction to Quantum Theory*, (Oxford University Press Inc., New York).

[84] Abhay Ashtekar, Troy A. Schilling, Geometrical Formulation of Quantum Mechanics, [gr-qc/9706069]

[85] Quantum Spin Dynamics (QSD), *Class. Quantum Grav.* 15 (1998) 839-873, [gr-qc/9606089]

[86] QSD V: Quantum Gravity as the Natural Regulator of Matter Quantum Field Theories, *Class. Quantum Grav.* 15 (1998) 1281-1314, [gr-qc/9705019].

[87] M. Han, Y. Ma, *Dynamics of Scalar Field in Polymer-like Representation*, accepted for pubilcation in Class. Quant. Grav, [gr-qc/0602101].

The Kinematic Theory

[88] Okolow, A., and Lewandowski, J. (2003): Diffeomorphism covariant representations of teh holonomy-flux $^*-$algebra. *Class. Quant. Grav.* **20**, 3543-3568 [gr-qc/0302059].

[89] J. Lewandowski, A. Okolow, H. Sahlmann, T. Thiemann, *Uniqueness of diffeomorphism invariant states on holonomy-flux algebras*, Comm. Math. Phys., Vol. 267 No. 3 (2006), 703-733, [gr-qc/0504147].

[90] C. Fleischhack, *Representations of the Weyl algebra in quatum geometry*, [math-ph/0407006].

[91] L. Smolin, *Time, measurement and information loss in quantum cosmology*, [gr-qc/9301016].

[92] L. Smolin, *Finite, diffeomorphism invariant observables in quantum gravity*, Phys.Rev. D49 (1994) 4028-4040, [gr-qc/9302011].

[93] Ashtekar, A. (1986). "Self-duality and spinorial techniques." In *Quantum Concepts in Space and Time*, eds. C.J. Isham and R. Penrose, Oxford: Oxford University Press, 302-317.

[94] C. Rovelli, L. Smolin, *Discreteness of area and volume in quantum gravity*, Nucl.Phys. B442 (1995) 593-622; Erratum-ibid. B456 (1995) 753, [gr-qc/9411005].

[95] C. Rovelli, L. Smolin, *Spin Networks and Quantum Gravity*, Phys.Rev. D52 (1995) 5743-5759, [gr-qc/9505006].

[96] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, T. Thiemann, *Quantization of diffeomorphism invariant theories of connections with local degrees of freedom*,J.Math.Phys. 36 (1995) 6456-6493, [gr-qc/9504018].

[97] A. Ashtekar, J. Lewandowski, *Quantum Theory of Gravity I: Area Operators*, Class.Quant.Grav. 14 (1997) A55-A82, [gr-qc/9602046].

[98] A. Ashtekar, J. Lewandowski, *Quantum Theory of Geometry II: Volume operators*, [gr-qc/9711031].

[99] K. Giesel, T. Thiemann, em Consistency Check on Volume and Triad Operator Quantisation in Loop Quantum Gravity I, Class.Quant.Grav. 23 (2006) 5667-5692, [gr-qc/0507036].

[100] K. Giesel, T. Thiemann, em Consistency Check on Volume and Triad Operator Quantisation in Loop Quantum Gravity II, Class.Quant.Grav. 23 (2006) 5693-5772, [gr-qc/0507037].

[101] W. Fairbairn, C. Rovelli, *Separable Hilbert space in loop quantum gravity*, J.Math.Phys.45:2802-2814,2004, [gr-qc/0403047].

[102] T.Koslowski, *Physical Diffeomorphisms in Loop Quantum Gravity*, [gr-qc/0610017].

[103] T.Koslowski, *Reduction of a Quantum Theory*, [gr-qc/0612138].

Measure Theory, Functional Integration

[104] A. Ashtekar, D. Marolf, J. Mourao, *Integration on the space of Connections Modulo Gauge Transformations*, Proc. of the Cornelius Lanczos Centenary Conference ed. by J.D. Brown et. al. (SIAM, Philadelphia, 1994), [gr-qc/9403042].

[105] D. Marolf, J. M. Mourao, *On the support of the Ashtekar-Lewandowski measure*, Commun.Math.Phys. 170 (1995) 583-606, [hep-th/9403112].

The Hamiltonian Constraint

[106] T. Thiemann, *Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity*, Phys.Lett. B380 (1996) 257-264, [gr-qc/9606088].

[107] T. Thiemann, *Quantum Spin Dynamics (QSD)*, Class.Quant.Grav. 15 (1998) 839-873, [gr-qc/9606089].

[108] T. Thiemann, *Quantum Spin Dynamics (QSD) II*, Class.Quant.Grav. 15 (1998) 875-905, [gr-qc/9606090].

QSD III : T. Thiemann, *Quantum Constraint Algebra and Physical Scalar Product in Quantum General Relativity*, Class.Quant.Grav. 15 (1998) 1207-1247, [gr-qc/9705017].

[109] T. Thiemann, *QSD IV : 2+1 Euclidean Quantum Gravity as a model to test 3+1 Lorentzian Quantum Gravity*, Class.Quant.Grav. 15 (1998) 1249-1280, [gr-qc/9705018].

[110] T. Thiemann, *QSD V : Quantum Gravity as the Natural Regulator of Matter Quantum Field Theories*, Class.Quant.Grav. 15 (1998) 1281-1314, [gr-qc/9705019].

[111] T. Thiemann, *QSD VI : Quantum Poincaré Algebra and a Quantum Positivity of Energy Theorem for Canonical Quantum Gravity*, Class.Quant.Grav. 15 (1998) 1463-1485, [gr-qc/9705020].

[112] T. Thiemann, *Quantum Spin Dynamics (QSD) : VII. Symplectic Structures and Continuum Lattice Formulations of Gauge Field Theories*, Class.Quant.Grav. 18 (2001) 3293-3338, [hep-th/0005232].

[113] M. Bojowald, A. Perez, *Spin Foam Quantization and Anomalies*, [gr-qc/0303026].

[114] R. Haag, *Local Quantum Physics*, Springer Verlag, 2nd ed., Berlin-Heidelberg-New York (1996).

[115] V. Hussain, O. Wimkler, *On Singularity Resolution in Quantum Gravity*, Phys.Rev. D69 (2004) 084016, [gr-qc/0312094].

[116] Roumen Borissov, Roberto De Pietri, Carlo Rovelli, *Matrix Elements of Thiemann's Hamiltonian Constraint in Loop Quantum Gravity*, Class.Quant.Grav. 14 (1997) 2793-2823, [gr-qc/9703090].

[117] T. Thiemann, O. Winkler, Gauge Field Theory Coherent States (GCS) : IV. Infinite Tensor Product and Thermodynamical Limit, *Class.Quant.Grav. 18* (2001) 4997-5054, [hep-th/0005235]

[118] T. Thiemann, Towards the QFT on Curved Spacetime Limit of QGR. I: A General Scheme, [gr-qc/0207030]

[119] T. Thiemann, Towards the QFT on Curved Spacetime Limit of QGR. II: A Concrete Implementation, [gr-qc/0207031].

Rovelii smolin Hamiltonian Consraint

[120] A. Perez, *On the regularization ambiguities in loop quantum gravity*, [gr-qc/0509118].

Introduction to Hopf algebreas and renormalization:

[121] H. Figueroa, J. M. Gracia-Bondia, *Combinatorial Hopf algebras in quantum field theory I* [hep-th/0408145].

[122] Fontini Markopoulou, *An algebraic approach to coarse graining*, [hep-th/0006199].

[123] Fontini Markopoulou, *Course graining in spin foam models*, Class.Quant.Grav. 20 (2003) 777-800, [gr-qc/0203036].

[124] R. Oeckl, *Renormalization of discrete models without background*, Nucl. Phys. **B657**, 107-138 (2003), [gr-qc/0212047].

[125] Rodolfo Gambini, Jorge Pullin, Discrete quantum gravity: a mechanism for selecting the value of fundamental constants, *Int.J.Mod.Phys. D12 (2003) 1775-1782* [gr-qc/0306095]

Group averaging:

[126] Marolf D. *Group averaging and refined algebraic quantization: Where are we now?*, [gr-qc/0011112].

[127] D. Giulini and D. Marolf *On the generality of refined algebraic quantization*, Class. Quantum Grav. 16 2479, [gr-qc/9812024].

Black Holes, Isolated Horizons and Dynamical Horizons

[128] Black Holes Lecture notes by Dr. P.K. Townsend, University of Cambridge, U.K.

[129] Bekenstein J D 1973 Black holes entropy *Phys. Rev.* **D7** 2333
Bekenstein J D 1974 Generalized second law of thermodynamics in black hole physics *Phys. Rev.* **D9** 3292
Bardeen J W, Carter B and Hawking S W 1973 The four laws of black hole mechanics *Commun. Math. Phys* **31** 161.

[130] J. Traschen, *An Introduction to Black Hole Evaporation*, Published in Mathematical Methods of Physics, proceedings of the 1999 Londrina Winter School, editors A. Bytsenko and F. Williams, World Scientific (2000), [gr-qc/0010055].

[131] Olaf Dreyer, Badri Krishnan, Eric Schnetter, Deirdre Shoemaker *Introduction to Isolated Horizons in Numerical Relativity*, Phys.Rev. D67 (2003) 024018, [gr-qc/0206008].

[132] A. Ashtekar, J Baez, A Corichi and K. Krasnov, "Quantum Geometry and Blackhole Entropy", *Phys. Rev. Lett.* **80** (1998) 904, [gr-qc/9710007]

[133] Abhay Ashtekar, Badri Krishnan, *Dynamical Horizons and their Properties*, Phys.Rev. D68 (2003) 104030, [gr-qc/0308033].

[134] A. Ashtekar, B. Krishnan, *Isolated and Dynamical Horizons and Their Applications*, Living reviews at http://relativity.livingreviews.org/Articles/lrr-2004-10/title.html.

[135] E. Gourgoulhon, J.L. Jaramillo *A 3+1 perspective on null hypersurfaces and isolated horizons*, [gr-qc/0503113].

[136] Abhay Ashtekar, Jonathan Engle, Tomasz Pawlowski, Chris Van Den Broeck, *Multipole Moments of Isolated Horizons*, Class.Quant.Grav. 21 (2004) 2549-2570, [gr-qc/0401114].

[137] A. Ashtekar, and B. Krishnan *Isolated Horizons and Dynamical Horizons and Their Applications*, [gr-qc/0407042]

[138] Marcin Domagala, Jerzy Lewandowski, *Black hole entropy from Quantum Geometry*, [gr-qc/0407051].

[139] A. Meissner, *Blachole Entropy in Loop Quantum Gravity,* Class. Quant. Grav. **21** (2004) 5245, [gr-qc/0407052].

[140] V. Hussain, O. Wimkler, *On Singularity Resolution in Quantum Gravity*, Phys.Rev. D69 (2004) 084016, [gr-qc/0312094].

[141] Abhay Ashtekar, Jonathan Engle, Chris Van Den Broeck, *Quantum horizons and black hole entropy: Inclusion of distortion and rotation*, [gr-qc/0412003].

[142] Badri Krishnan, PhD theses: *Isolated Horizons in Numerical Relativity*, available from http://cgpg.gravity.psu.edu/archives/thesis/index.shtml.

[143] Jonathan Engle, PhD theses: *Black Hole Entropy, Constraints, and Symmetry in quantum gravity*, available from http://cgpg.gravity.psu.edu/archives/thesis/index.shtml.

[144] Viqar Husain, Oliver Winkler, *Quantum resolution of black hole singularities*, [gr-qc/0410125].

[145] Viqar Husain, Oliver Winkler, *Quantum black holes*, [gr-qc/0412039].

[146] Ted Jacobson, Donald Marolf, Carlo Rovelli, *Black hole entropy: inside or out?*, [hep-th/0501103].

[147] B. Dittrich, R. Loll, *Counting a black hole in Lorentzian product triangulations*, [gr-qc/0506035].

[148] A. Corichi, . Diaz-Polo, E. Fernandez-Borja, *lack hole entropy quantization*, [gr-qc/0609122].

[149] Chris Van Den Broeck, thesis: *Black Holes and Neutron stars: Fundamental and Phenomenological Issues*, available to download at http://cgpg.gravity.psu.edu/archives/thesis/index.shtml.

[150] Carlo Rovelli, Thomas Thiemann, The Immirzi parameter in quantum general relativity, *Phys.Rev. D57 (1998) 1009-1014*, [gr-qc/9705059].

[151] M. Reisenberger, C. Rovelli, *Sum over Surfaces form of Loop Quantum Gravity*, Phys. Rev., D56, 3490-3508, (1997). For a related online version see: M. Reisenberger, et al., *Sum over Surfaces form of Loop Quantum Gravity*, [gr-qc/9612035].

[152] A. Perez, *Spin foam models for quantum gravity.* Class. Quant. Grav. 20 (2003), R43. [gr-qc/0301113]

[153] S. Alexandrov, *"$SO(4, C)$-covariant Ashtekar Barbero gravity and the Immirzi parameter"*, Class. Quantum Grav. 17, 4255 (2000) [gr-qc/0005085]; S. Alexandrov and D. Vassilevich, *Area spectrum in Lorentz covariant loop gravity*, Phys. Rev. D 64 , 044023 (2001) [gr-qc/0103105]; S. Alexandrov, On choice of connection in loop quantum gravity, Phys. Rev. D 65 , 024011 (2002) [gr-qc/0107071]; S. Alexandrov, *Hilbert space structure of covariant loop quantum gravity*, Phys. Rev. D 66 , 024028 (2002) [gr-qc/0201087]; S. Alexandrov and E.R. Livine, *$SU(2)$ loop quantum gravity seen from covariant theory)*, Phys. Rev. D 67 , 044009 (2003) [gr-qc/0209105].

[154] Sergei Alexandrov, Zoltan Kadar, *Timelike surfaces in Lorentz covariant loop gravity and spin foam models*, [gr-qc/0501093].

[155] Olaf Dreyer, New Hints from General Relativity, [gr-qc/0401035]

[156] A. Ashtekar, and Bojojwald. M. In preparation.

[157] Ola Bratteli and Derek W. Robinson, Operator Algebras and Quantum Statistical Mechanics I

[158] *A black hole mass threshold from non-singular quantum gravitational collapse*, [gr-qc/0503041].

[159] Abhay Ashtekar, Martin Bojowald, *Black hole evaporation: A paradigm*, [gr-qc/0504029].

[160] Sean A. Hayward, *The disinformation problem for black holes (pop version)*, [gr-qc/0504038].

[161] Sean A. Hayward, *The disinformation problem for black holes (conference version)*, [gr-qc/0504037].

[162] A. Ashtekar, M. Bojowald, *Quantum geometry and the Schwarzschild singularity*, [gr-qc/0509075].

Loop Quantum Cosmology

[163] David Lerner, *Techniques of Topology and Differential Geometry in General Relativity.*

[164] Roger Penrose, *Techniques of Topology and Differential Geometry in General Relativity.*

[165] S. W. Hawking, R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. Lond. A. **314**, 529-548 (1970).

[166] S. W. Hawking, Hawking on the Big Bang and Black Holes, Advanced Series in Astrophysics and Cosmology - Vol. 8, (Published by World Scientific Publishing Co. Pte. Ltd.).

[167] R.Brandenberger, *Topis in Cosmology*, [hep-th/0701157].

[168] Simone C (World Scientific Publishing Co. Pte. Ltd) *Deparametrizable and Path Integral Quanization of Cosmological Models*

[169] M. Bojowald, Loop Quantum Cosmology I: Kinematics, *Class.Quant.Grav. 17 (2000) 1489-1508* [gr-qc/9910103]

[170] M. Bojowald, Loop Quantum Cosmology II: Volume Operators, *Class.Quant.Grav. 17 (2000) 1509-1526* [gr-qc/9910104]

[171] M. Bojowald, Loop Quantum Cosmology III: Wheeler-DeWitt Operators, *Class.Quant.Grav. 18 (2001) 1055-1070*, [gr-qc/0008052]

[172] M. Bojowald, Loop Quantum Cosmology IV: Discrete Time Evolution, *Class.Quant.Grav. 18 (2001) 1071-1088*, [gr-qc/0008053]

[173] M. Bojowald and Kastrup, Symmetry Reduction for Quantized Diffeomorphism Invariant Theories of Connections, *Class. Quantum Grav. 17* (2000) 3009-3043, [hep-th/9907042]

[174] M. Bojowald *Absence of singularity in Loop Quantum Cosmology, Phys.Rev Lett.86(2001)* 5227-5230, [gr-qc/0102069]

[175] M. Bojowald *Dynamical Intial Conditions in Quantum Cosmology, Phys. Rev. Lett. 87 (2001)* 121301, [gr-qc/0104072]

[176] M. Bojowald, *Initial Conditions for the Universe*, *Gen. Rel. Grav. 35* (2003) 1877-1883, [gr-qc/0305069]

[177] M. Bojowald, *The semiclassical Limit of Loop Quantum Cosmology*, *Class. Quantum Grav. 18* (2001) L109-L116, [gr-qc/0105113]

[178] M. Bojowald, *Elements of Loop Quantum Cosmology*, Chapter contributed to "100 Years of Relativity - Space-time Structure: Einstein and Beyond", Ed. A. Ashtekar (World Scientific), [gr-qc/0505057].

[179] M. Bojowald, *Quantum Cosmology*, published in Encyclopedia of Mathematical Physics, eds. J.-P. Franccoise, G. L. Naber and Tsou S. T., Oxford: Elsevier, 2006 (ISBN 978-0-1251-2666-3), volume 4, page 153, [gr-qc/0603110].

[180] Merced Montesinos, Carlo Rovelli, Thomas Thiemann, SL(2,R) model with two Hamiltonian constraints [gr-qc/9901073]

[181] Thiomas F. Jordon, *Linear Operators for Quantum Mechanics*, (Published by F. Jordon, 2249 Dunedin Ave. Duluth, Minnesota U.S.A.)

[182] S. Tsujikawa, P. Singh, R. Maartens, *Loop quantum gravity effects on inflation and the CMB*, [astro-ph/0311015].

[183] A. Ashtekar, T. Pawlowski, P. Singh, *Quantum Nature of the Big Bang: An Analytical and Numerical Investigation I*, [gr-qc/0604013].

[184] D. A. Easson and R. H. Brandenberger, JHEP **0106**, 024 (2001).

[185] L. Crane, *Clock and Category; IS QUANTUM GRAVITY ALGEBRAIC*, J.Math.Phys. 36 (1995) 6180-6193, [gr-qc/9504038].

[186] L. Smolin, *The Bekenstein Bound, Topological Quantum Field Theory and Pluralistic Quantum Field Theory*, [gr-qc/9508064].

[187] M. Bojowald, *Large scale effective theory for cosmological bounces*, [gr-qc/0608100].

[188] M. Bojowald, *Loop quantum cosmology and inhomogeneities*, Gen.Rel.Grav. 38 (2006) 1771-1795, [gr-qc/0609034].

[189] M. Bojowald, H.H. Hernandez, M. Kagan, P. Singh, A. Skirzewski, *Hamiltonian cosmological perturbation theory with loop quantum gravity corrections*, Phys.Rev. D74 (2006) 123512, [gr-qc/0609057].

[190] A. Ashtekar, M. Campiglia, A. Henderson, *Casting Loop Quantum Cosmology in the Spin Foam Paradigm*, Gen.Rel.Grav. **27**, 135020 (2010), [gr-qc/10015147].

Spin Foams

[191] C. Gomez, M. Ruiz-Altaba, G. Sierra, *Quantum Groups in Two-dimensional Physics*, Cambridge University Press 1996.

[192] William Gordon Ritter, *Introduction to Quantum Group Theory*, [math.QA/0201080].

[193] Jaganathan, *Some introductory notes on quantum groups, quantum algebras, and their applications*, [math-ph/0105002].

[194] D. Oriti, *Quantum Gravity as a quantum field theory of simplicial geometry*, [qr-qc/0512103].

[195] J. Daniel Christensen, *Finiteness of Lorentzian 10j symbols and partition functions*, [gr-qc/0512004].

[196] E Brezin, C Itzykson, G Parisi, J Zuber, Commun Math 59 (1978) 35. F David, Nucl Phys B257 (1985) 45.

[197] J.W. Barrett and L. Crane, *Relativistic spin networks and quantum gravity*, J. Math. Phys. **39**, 3296 (1998), [qr-qc/9709028].

[198] J.W. Barrett and L. Crane, *A Lorentzian signature model for quantum general relativity*, Class. Quant. Grav. **17**, 3101 (2000), [qr-qc/9904025].

[199] S. Alexandrov, E.R. Livine, *$SU(2)$ Loop Quantum Gravity seen from a Covariant Theory*, Phys. Rev. D67 (2003) 044009, [qr-qc/0209105].

[200] E.R. Livine, S. Speziale, *A new spin foam vertex for quantum gravity*, [qr-qc/0705.0674].

[201] J. Engle, R. Pereira, C. Rovelli, *Flipped spinfoam vertex for loop gravity*, [qr-qc/0705.0674].

[202] L. Freidel, K. Krasnov, *A new spin foam vertex for quantum gravity*, [qr-qc/0708.1236].

[203] E.R. Livine, S. Speziale, *Consistently Solving the Simplicity Constraints for Spinfoam Quantum Gravity*, [qr-qc/0708.1915].

[204] S. Alexandrov, *Spin foam model from canonical quantization*, [qr-qc/0705.3892].

CLQG

[205] E.R. Livine, em Towards a covariant loop quantum gravity, [gr-qc/0608135].

Master Constraint Programme

[206] J. Brunnemann, T. Thiemann, *Simplification of the Spectral Analysis of the Volume Operator in Loop Quantum Gravity, Class. Quant. Grav* **23** (2006), [gr-qc/0405060].

[207] J. Brunnemann, K. Giesel, T. Thiemann, *Semiclassical Properties of the* ***Master Constraint Operator*** *for Loop Quantum Gravity*, in preparation.

[208] B. Dittich, T. Thiemann, *Path Integral Formulation of the* ***Master Action*** *for Loop Quantum Gravity*, in preparation.

[209] B. Dittich, T. Thiemann, *Testing the Master Constraint Programme for Loop Quantum Gravity I. General Framework*, Class.Quant.Grav. 23 (2006) 1025-1066, [gr-qc/0411138].

[210] B. Dittich, T. Thiemann, *Testing the Master Constraint Programme for Loop Quantum Gravity II. Finite Dimensional Systems*, Class.Quant.Grav. 23 (2006) 1067-1088, [gr-qc/0411139].

[211] B. Dittich, T. Thiemann, *Testing the Master Constraint Programme for Loop Quantum Gravity III. SL(2,R) Models*, Class.Quant.Grav. 23 (2006) 1089-1120, [gr-qc/0411140].

[212] B. Dittich, T. Thiemann, *Testing the Master Constraint Programme for Loop Quantum Gravity IV. Free Field Theories*, Class.Quant.Grav. 23 (2006) 1121-1142, [gr-qc/0411141].

[213] B. Dittich, T. Thiemann, *Testing the Master Constraint Programme for Loop Quantum Gravity V. Interacting Field Theories*, Class.Quant.Grav. 23 (2006) 1143-1162, [gr-qc/0411142].

[214] T. Thiemann, *Quantum Spin Dynamics VIII. The Master Constraint*, Class.Quant.Grav. 23 (2006) 2249-2266, [gr-qc/0510011].

[215] M. Han, W. Huang, Y. Ma, *Fundamental Structure of Loop Quantum Gravity*, [gr-qc/0509064].

[216] B. Bahr, T. Thiemann, *Approximating the physical inner product of Loop Quantum Cosmology*, [gr-qc/0607075].

bf Other Thiemann Stuff

[217] J. Brunnemann, T. Thiemann, *On (Cosmological) Singularity Avoidance in Loop Quantum Gravity*, Class.Quant.Grav. 23 (2006) 1395-1428, [gr-qc/0505032].

[218] J. Brunnemann, T. Thiemann, *Unboundedness of Triad – Like Operators in Loop Quantum Gravity*, Class.Quant.Grav. 23 (2006) 1429-1484, [gr-qc/0505033].

[219] T. Thiemann, *The LQG – String: Loop Quantum Gravity Quantization of String Theory I. Flat Target Space*, [gr-qc/0401172].

[220] T. Thiemann, *The LQG – String: Loop Quantum Gravity Quantization of String Theory II. Curved Target Space*, in prearation.

[221] K. Giesel, T. Thiemann, *Algebraic Quantum Gravity (AQG) I. Conceptual Setup*, [gr-qc/0607099].

[222] K. Giesel, T. Thiemann, *Title: Algebraic Quantum Gravity (AQG) II. Semiclassical Analysis*, [gr-qc/0607100].

[223] K. Giesel, T. Thiemann, *Algebraic Quantum Gravity (AQG) III. Semiclassical Perturbation Theory*, [gr-qc/0607101].

[224] K. Giesel, T. Thiemann, *Algebraic Quantum Gravity (AQG) IV. Reduced Phase Space Quantization of Loop Quantum Gravity*, [gr-qc/0711.0119].

[225] T. Thiemann, Loop Quantum Gravity: An Inside View, [hep-th/0608210]

Semiclassics

[226] A. Ashtekar, C. Rovelli, L. Smolin, *Gravitons and Loops*, Phys.Rev. D44 (1991) 1740-1755, [hep-th/9202054].

[227] A. Ashtekar, C. Rovelli, L. Smolin, *Weaving a classical geometry with quantum threads*, Phys.Rev.Lett. 69 (1992) 237-240, [hep-th/9203079].

[228] L. Smolin, *Quantum gravity with a positive cosmological constant,* [hep-th/0209079].

[229] L. Freidel, L. Smolin,*The linearization of the Kodama state*, [hep-th/0310224].

[230] Lee Smolin, Chopin Soo, *The Chern-Simons Invariant as the Natural Time Variable for Classical and Quantum Cosmology*, Nucl.Phys. B449 (1995) 289-316, [gr-qc/9405015].

[231] C. Soo, *Wave function of the Universe and Chern-Simons Perturbation Theory*, Class.Quant.Grav. 19 (2002) 1051-1064, [gr-qc/0109046].

[232] Chopin Soo, Lay Nam Chang, *Superspace Dynamics and Perturbations Around "Emptiness"*, Int.J.Mod.Phys. D3 (1994) 529-544, [gr-qc/9307018].

[233] A. Mikovic, *Quantum Gravity Vacuum and Invariants of Embedded Spin Networks*, [gr-qc/0301047].

[234] A. Mikovic, *Flat Spacetime Vacuum in Loop Quantum Gravity*, Class.Quant.Grav. 21 (2004) 3909-3922, [gr-qc/0404021].

[235] Abhay Ashtekar, Carlo Rovelli, Lee Smolin, *Gravitons and Loops*, Phys.Rev. D44 (1991) 1740-1755, [hep-th/9202054].

[236] M. Varadarajan, *Fock representations from U(1) holonomy algebras*, Phys.Rev. D61 (2000) 104001, [gr-qc/0001050].

[237] M. Varadarajan, *Photons from quantized electric flux representations*, Phys.Rev. D64 (2001) 104003, [gr-qc/0104051].

[238] M. Varadarajan, *Gravitons from a loop representation of linearised gravity*, Phys.Rev. D66 (2002) 024017, [gr-qc/0204067].

[239] Abhay Ashtekar, Jerzy Lewandowski, *Relation between polymer and Fock excitations*, Class.Quant.Grav. 18 (2001) L117-L128, [gr-qc/0107043].

[240] Abhay Ashtekar, Stephen Fairhurst, Joshua L. Willis, *Quantum gravity, shadow states, and quantum mechanics*, Class.Quant.Grav. 20 (2003) 1031-1062, [gr-qc/0207106].

[241] Abhay Ashtekar, Jerzy Lewandowski, Hanno Sahlmann, *Polymer and Fock representations for a Scalar field*, Class.Quant.Grav. 20 (2003) L11-1, [gr-qc/0211012].

[242] Josh Willis: *On the Low-Energy Ramifications and a Mathematical Extension of Loop Quantum Gravity* PhD Thesis available at http://cgpg.gravity.psu.edu/archives/thesis/index.shtml.

[243] B. Hall, *The Segal-Bargmann "coherent state" transform for compact Lie groups*, Journ. Funct. Analysis **122** (1994) 103-151.

[244] T. Thiemann, *Gauge Field Theory Coherent States (GCS) : I. General Properties*, Class.Quant.Grav. 18 (2001) 2025-2064, [hep-th/0005233].

[245] T. Thiemann, O. Winkler, *Gauge Field Theory Coherent States (GCS) : II. Peakedness Properties*, Class.Quant.Grav. 18 (2001) 2561-2636, [hep-th/0005237].

[246] T. Thiemann, O. Winkler, *Gauge Field Theory Coherent States (GCS) : III. Ehrenfest Theorems*, Class.Quant.Grav. 18 (2001) 4629-4682, [hep-th/0005234].

[247] T. Thiemann, O. Winkler, *Gauge Field Theory Coherent States (GCS) : IV. Infinite Tensor Product and Thermodynamical Limit*, Nucl.Phys. B606 (2001) 401-440, [gr-qc/0102038].

[248] H. Sahlmann, T. Thiemann, O. Winkler, *Coherent States for Canonical Quantum General Relativity and the Infinite Tensor Product Extension*, Nucl.Phys. B606 (2001) 401-440, [gr-qc/0102038].

[249] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mouro, T. Thiemann, *Coherent State Transforms for Spaces of Connections*, Journ. Math. Analysis, **135** (1996) 519-551, [gr-qc/9412014].

[250] T. Thiemann, *Complexifier Coherent States for Quantum General Relativity*, [gr-qc/0206037].

[251] A. Ashtekar, D. Marolf, J. Mouro, T. Thiemann, *Osterwalder-Schrader Reconstruction and Diffeomorphism Invariance*, [quant-ph/9904094].

[252] Hanno Sahlmann, Thomas Thiemann, *Towards the QFT on Curved Spacetime Limit of QGR. I: A General Scheme*, [gr-qc/0207030].

[253] Hanno Sahlmann, Thomas Thiemann, *Towards the QFT on Curved Spacetime Limit of QGR. II: A Concrete Implementation*, [gr-qc/0207031].

[254] B. Bahr, T. Thiemann, *Gauge-invariant coherent states for Loop Quantum Gravity I: Abelian gauge groups*, [gr-qc/07094619].

[255] B. Bahr, T. Thiemann, *Gauge-invariant coherent states for Loop Quantum Gravity II: Non-abelian gauge groups*, [gr-qc/07094636].

[256] Florian Girelli, Etera R. Livine, *Reconstructing Quantum Geometry from Quantum Information: Spin Networks as Harmonic Oscillators*, [gr-qc/0501075].

[257] Etera R. Livine, *Some Remarks on the Semi-Classical Limit of Quantum Gravity*, [gr-qc/0501076].

[258] Mohammad H. Ansari, Lee Smolin, *Self-organized criticality in quantum gravity*, [hep-th/0412307].

[259] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59** (1987) 381.

[260] P. Bak, How Nature Works, Springer-Verlag, New York 1996.

[261] E. Hawkins, F. Markopoulou, H. *Sahlmann, Evolution in Quantum Causal Histories*, Class.Quant.Grav. 20 (2003) 3839, [hep-th/0302111].

[262] Rodolfo Gambini, Jorge Pullin, *Making classical and quantum canonical general relativity computable through a power series expansion in the inverse cosmological constant*, Phys.Rev.Lett. 85 (2000) 5272-5275, [gr-qc/0008031].

[263] Rodolfo Gambini, Jorge Pullin, The large cosmological constant approximation to classical and quantum gravity: model examples, Class.Quant.Grav. 17 (2000) 4515-4540, [gr-qc/0008032].

[264] F. Conrady, *Free vacuum for loop quantum gravity*, [gr-qc/0409036].

[265] J. Ambjorn, J. Jurkiewicz, R. Loll, *3d Lorentzian, Dynamically Triangulated Quantum Gravity*, Nucl.Phys.Proc.Suppl. 106 (2002) 980-982 [hep-lat/0201013].

[266] J. Ambjorn, J. Jurkiewicz, R. Loll, *Emergence of a 4D World from Causal Quantum Gravity*, [hep-th/0404156]

[267] Olaf Dreyer, *Background Independent Quantum Field Theory and the Cosmological Constant Problem* [hep-th/0409048].

should this go here?

[268] L. Freidel, A. Starodubtsev, *Quantum gravity in terms of topological observables*, [hep-th/0401076].

[269] A. Ashtekar, L. Bombelli, A. Corichi, *Semiclassical States for Constrained Systems*, [gr-qc/0504052].

[270] G. W. Gibbons, S. W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev.* **D15** (1977) 2752-2756.

Noiseless subsystems

[271] O. Dreyer, F. Markopoulou, L. Smolin, *Symmetry and entropy of black hole horizons*, Nucl.Phys. B744 (2006) 1-13, [hep-th/0409056].

[272] D. W. Kribs, F. Markopoulou, *Geometry from quantum particles*, [gr-qc/0510052].

[273] T Konopka, F. Markopoulou, *Constrained Mechanics and Noiseless Subsystems*, [gr-qc/0601028].

[274] S. O. Bilson-Thompson, F. Markopoulou, L. Smolin, *Quantum gravity and the standard model*, [hep-th/0603022].

[275] F. Markopoulou, *Towards Gravity from the Quantum*, Expanded version of the contribution to "Towards Quantum Gravity", edited by D.Oriti, to be published by C.U.P, [hep-th/0604120].

[276] Sundance O. Bilson-Thompson, Fotini Markopoulou, Lee Smolin, *Quantum gravity and the standard model*, [hep-th/0603022].

Preon models

[277] Sundance O.Bilson-Thompson, *A topological model of composite preons*, submitted to Phys. Lett. B, [hep-ph/0503213].

Timeless approach

[278] C. Rovelli, Phy. Rev. D. **42**, 2638 (1990).

[279] C.Rovelli *A simple background-independent Hamiltonian quantum model*, 13 Jun 2003 [gr-qc/0306059]

[280] C. Rovelli, 21 Feb 2002 *A note on the foundation of relativistic mechanics. I: Relativistic observables and relativistic states*, [gr-qc/0111037]

[281] Leonardo Modesto, Carlo Rovelli, *Particle scattering in loop quantum gravity*, [gr-qc/0502036].

[282] F. Mattei., C. Rovelli, S. Speziale, M.Testa , *From 3-geometry transition amplitudes to graviton states*, [gr-qc/0508007].

[283] C. Rovelli, *Graviton propagator from background-independent quantum gravity*, [gr-qc/0508124].

[284] E. Bianchi, L. Modesto, C. Rovelli, S. Speziale *Graviton propagator in loop quantum gravity*, [gr-qc/0604044].

[285] B. Dittrich, L. Freidel, S. Speziale, *Linearized dynamics from the 4-vertex Regge action*, [gr-qc/0707.4513].

[286] E. Alesci, C. Rovelli, *The complete LQG propagator I. Difficulties with the Barrett-Crane vertex*, [gr-qc/0708.0883].

[287] L. Freilel, K. Krasnov, *A new Spin Foam Model for 4d Gravity*, [gr-qc/0708.1595].

[288] C Rovelli, "What is observable in classical and quantum gravity?", Class. Quant. Grav. 8 (1991), 297-316.

[289] C. Rovelli, "Statistical mechanics of gravity and thermodynamical origin of time", Class. Quant. Grav. 10 (1993), 1549-1566.

[290] *The Statistical State of the Universe*, C. Rovelli, Class. and Quant. Grav. 10, 1567 (1993).

[291] A Connes and C Rovelli: "Von Neumann algebra and automorphisms and time versus thermodynamics relation in general covariant quantum theories", Class and Quantum Grav 11(1994) 2899.

[292] Rovelli C. *Diamonds's Temperature: Unruh effect for bounded trajectories and thermal time hypothesis*, [gr-qc0212074].

[293] C. Rovelli *Covariant hamiltonian formulism for field theory: Hamilton-Jacobi equation on the space* $\mathcal{M}$ [gr-qc/0306095].

[294] B. Sayandeb, *Quantum theory using classical perturbation: First steps towards a covariant approximation scheme in quantum gravity*, [gr-qc/0410015].

[295] Donald Marolf, *Quantum Observables and Recollapsing Dynamics*, Class.Quant.Grav. 12 (1995) 1199-1220, [gr-qc/9404053].

[296] Donald Marolf, *Observables and a Hilbert Space for Bianchi IX*, Class.Quant.Grav. 12 (1995) 1441-1454, [gr-qc/9409049].

[297] Donald Marolf, *Almost Ideal Clocks in Quantum Cosmology: A Brief Derivation of Time*, Class.Quant.Grav. **12** (1995) 2469-2486, [gr-qc/9412016].

[298] Carl E Dolby, *The Conditional Probability Interpretation of the Hamiltonian Constraint*, [gr-qc/0406034].

[299] F. Hellmann, M. Mondragon, A. Perez, C. Rovelli, *Multiple-event probability in general-relativistic quantum mechanics*, [gr-qc/0610140].

[300] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press 1992).

[301] B. Dittrich, *Partial and Complete Observables for Hamiltonian Constrained Systems*, [gr-qc/0411013].

[302] T. Thiemann, *Reduced Phase Space Quantization and Dirac Observables*, Class.Quant.Grav. 23 (2006) 1163-1180, [gr-qc/0411031].

[303] B. Dittrich, *Partial and Complete Observables for Canonical General Relativity*, Class.Quant.Grav. 23 (2006) 6155-6184, [gr-qc/0507106].

[304] B. Dittrich, J. Tambornino, *A perturbative approach to Dirac observables and their space-time algebra*, Class.Quant.Grav. 24 (2007) 757-784, [gr-qc/0610060].

[305] B. Dittrich, J. Tambornino, *Gauge invariant perturbations around symmetry reduced sectors of general relativity: applications to cosmology*, [gr-qc/0702093].

[306] R. Arnowitt, S. Deser, C.W. Misner, *The Dynamics of General Relativity* in Gravitation: An Introduction to Current Research, ed. by L. Witten (Wiley, New York, 1962), [gr-qc/0405109].

[307] S.B. Giddings, D. Marolf, J.B. Hartle *Observables in effective gravity*, [hep-th/0512200].

[308] T. Thiemann, *Solving the Problem of Time in General Relativity and Cosmology with Phantoms and k-Essence*, [astro-ph/0607380].

Background Independent Scattering Amplitudes

[309] Carlo Rovelli, *A note on the foundation of relativistic mechanics. II: Covariant hamiltonian general relativity*, [gr-qc/0202079].

[310] Carlo Rovelli, *A note on the foundation of relativistic mechanics. I: Relativistic observables and relativistic states*, [gr-qc/0111037].

[311] Carlo Rovelli (CPT), Daniele Colosi (CPT), Luisa Doplicher (CPT), Winston Fairbairn (CPT), Leonardo Modesto (CPT), Karim Noui, *Background independence in a nutshell*, [gr-qc/0408079].

[312] Daniele Colosi, Carlo Rovelli, *Global particles, local particles*, [gr-qc/0409054].

[313] EP Wigner, "On unitary representations of the inhomogeneous Lorentz group", Ann of Math **40** (1939), 149-204.

[314] Leonardo Modesto, Carlo Rovelli, *Particle scattering in loop quantum gravity*, [gr-qc/0502036].

[315] *Relativistic quantum measurement*, Phys.Rev. D66 (2002) 023510, [gr-qc/0203056].

[316] Michael Reisenberger, Carlo Rovelli, *Spacetime states and covariant quantum theory*, Phys.Rev. D65 (2002) 125016, [gr-qc/0111016].

[317] Etera R. Livine, Simone Speziale, *Group Integral Techniques for the Spinfoam Graviton Propagator*, [gr-qc/0608131].

[318] John W. Barrett, *Geometrical measurements in three-dimensional quantum gravity*,Int.J.Mod.Phys. A18S2 (2003) 97-113, [gr-qc/0203018].

[319] John W. Barrett, *Feynman loops and three-dimensional quantum gravity*, [gr-qc/0412107].

[320] John W. Barrett, *Feynman diagrams coupled to three-dimensional quantum gravity*, [gr-qc/0502048].

[321] S. Speziale, *Towards the graviton from spinfoams: the 3d toy model*, [gr-qc/0512102].

Consistent discretization

[322] Rodolfo Gambini, Jorge Pullin, *Consistent discretizations for classical and quantum general relativity*, In "Gravity, astrophysics and strings @ the Black Sea", P. Fiziev, M. Todorov editors, St Kliment Ohridski University Press (2003), [gr-qc/0108062].

[323] Rodolfo Gambini, Jorge Pullin, *Canonical quantization of general relativity in discrete space-times*, Phys.Rev.Lett. **90** (2003) 021301, [gr-qc/0206055].

[324] G. Calabrese, J. Pullin, O. Reula, O. Sarbach, M. Tiglio, *Well posed constraint-preserving boundary conditions for the linearized Einstein equations*, Commun.Math.Phys. 240 (2003) 377-395, [gr-qc/0209017].

[325] Rodolfo Gambini, Jorge Pullin, *Discrete quantum gravity: applications to cosmology*, Class.Quant.Grav. **20** (2003) 3341, [gr-qc/0212033].

[326] Rodolfo Gambini, Rafael Porto, Jorge Pullin, *Consistent discrete gravity solution of the problem of time: a model*, [gr-qc/0302064].

[327] G. Calabrese, L. Lehner, D. Neilsen, J. Pullin, O. Reula, O. Sarbach, M. Tiglio, *Novel finite-differencing techniques for numerical relativity: application to black hole excision*, Class.Quant.Grav. **20** (2003) L245-L252, [gr-qc/0302072].

[328] Rodolfo Gambini, Rafael Porto, Jorge Pullin, *Loss of coherence from discrete quantum gravity*, Class.Quant.Grav. 21 (2004) L51-L57, [gr-qc/0305098].

[329] Rodolfo Gambini, Jorge Pullin, *Discrete quantum gravity: a mechanism for selecting the value of fundamental constants*, Int.J.Mod.Phys. D12 (2003) 1775-1782, [gr-qc/0306095].

[330] Rodolfo Gambini, Jorge Pullin, *Canonical quantum gravity and consistent discretizations*, to appear in Pramana, [gr-qc/0402062].

[331] Rodolfo Gambini, Rafael Porto, Jorge Pullin, *A relational solution to the problem of time in quantum mechanics and quantum gravity induces a fundamental mechanism for quantum decoherence*, New J.Phys. 6 (2004) 45, [gr-qc/0402118].

[332] Cayetano Di Bartolo, Rodolfo Gambini, Jorge Pullin, *Consistent and mimetic discretizations in general relativity*, [gr-qc/0404052].

[333] Cayetano Di Bartolo, Rodolfo Gambini, Rafael Porto, Jorge Pullin, *Dirac-like approach for consistent discretizations of classical constrained theories*, [gr-qc/0405131].

[334] Rodolfo Gambini, Rafael Porto, Jorge Pullin, *No black hole information puzzle in a relational universe*, [hep-th/0405183].

[335] Rodolfo Gambini, Rafael Porto, Jorge Pullin, *Realistic clocks, universal decoherence and the black hole information paradox*, [hep-th/0406260].

[336] Rodolfo Gambini, Rafael Porto, Jorge Pullin, *Fundamental decoherence from relational time in discrete quantum gravity: Galilean covariance*, [gr-qc/0408050].

[337] Rodolfo Gambini, Jorge Pullin, *Consistent discretization and loop quantum geometry*, [gr-qc/0409057].

[338] Rodolfo Gambini, Jorge Pullin, *Consistent discretizations and quantum gravity*, to appear in the proceedings of The II International Conference on Fundamental Interactions, O. Piguet (editor), Editora UNESP, Sao Paulo, [gr-qc/0408025].

[339] R. Gambini, S. Jay Olson, J. Pullin, *Unified model of loop quantum gravity and matter*, [gr-qc/0409045].

[340] R. Gambini, J. Pullin *Discrete space-time*, [gr-qc/0505023]. Submitted to the volume "100 Years of Relativity - Space-time Structure: Einstein and Beyond", A. Ashtekar, ed., to be published by World Scientific. "

[341] M. Campiglia, C. Di Bartolo, R. Gambini, J. Pullin *Uniform discretizations: a quantization procedure for totally constrained systems including gravity*, [gr-qc/0606121].

Quantum Gravity Phenomenology

[342] R. Gambini, J. Pullin, *Nonstandard optics from quantum spacetime*, Phys.Rev. D59 (1999) 124021, [gr-qc/9809038].

[343] , Quantum gravity corrections to neutrino propagation, Phys.Rev.Lett. 84 (2000) 2318-2321, [gr-qc/9909079].

[344] J. Alfaro, H.A. Morales-Tcotl, L.F. Urrutia *Loop quantum gravity and light propagation*, Phys.Rev. D65 (2002) 103509, [hep-th/0108061].

[345] J. Alfaro, H.A. Morales-Tecotl, L.F. Urrutia, *Quantum gravity and spin 1/2 particles effective dynamics*, Phys.Rev. D66 (2002) 124006, [hep-th/0208192].

[346] Christoph Simon, Dieter Jaksch, *Could Energy Decoherence due to Quantum Gravity be observed?*, [quant-ph/0406007].

[347] M. Reuter, H. Weyer *Quantum Gravity at Astrophysical Distances?*, [hep-th/0410119]. ;M. Reuter, H. Weyer *Running Newton Constant, Improved Gravitational Actions, and Galaxy Rotation Curves*, 72pp, to appear in Phys. Rev. D, [hep-th/0410117].

[348] S. Hofmann, O. Winkler, *The Spectrum of Fluctuations in Singularity-free Inflationary Quantum Cosmology*, [astro-ph/0411124].

[349] Joy Christian, *Testing Quantum Gravity via Cosmogenic Neutrino Oscillations*, [gr-qc/0409077].

[350] G. Amelino-Camelia, C. Lmmerzahl, A. Macias, H. Mller, *The Search for Quantum Gravity Signals*, submitted to AIP Conference Proceedings of the 2nd Mexican Meeting on Mathematical and Experimental Physics , [gr-qc/0501053].

[351] A. Perez, C. Rovelli, *Physical effects of the Immirzi parameter*, [gr-qc/0505081].

Do I want to keep this one?

[352] Theodore G. Pavlopoulos, *Are we observing Lorentz violation in gamma ray bursts?*, [astro-ph/0508294].

[353] W.A. Christiansen, Y. Jack Ng, H. van Dam, *Probing spacetime foam with extragalactic sources*, [gr-qc/0508121].

[354] P. J. Salzman, S. Carlip, *A possible experimental test of quantized gravity*, [gr-qc/0606120].

[355] M. Bojowald, H. Hernandez, M. Kagan, P. Singh, A. Skirzewski, *Formation and Evolution of Structure in Loop Cosmology*, [astro-ph/0611685].

[356] M. Bojowald, *Quantum gravity and cosmological observations*, plenary talk at the VIth Latin American Symposium on High Energy Physics (Puerto Vallarta, Mexico, Nov. 2006), [gr-qc/0701142].

Algebraic Structurers, Category Theory and Topos

[357] Hector Figueroa, Jose M. Gracia-Bondia, *Combinatorial Hopf algebras in quantum field theory I*, [hep-th/0408145].

[358] Mittelstaedt P 1978 Quantum Logic (Dordrecht: Reidel); Beltrametti E G and Cassinelli G 1981 The Logic of Quantum Mechanics (Reading, MA: Addison-Wesley)

[359] C. McLarty, *Elementary Categories, Elementary Toposes* (1992) (Oxford University Press).

[360] J. Butterfield , *Topos Theory as a Framework for Partial Truth*, found at http://philsci-archive.pitt.edu/archive/00000192/01/PARTIALCRACOW.PS, (16 January 2000).

[361] J. Butterfield, *Topos Theory as a Framework for Partial Truth*, To appear in Proceedings of the 11th International Congress of Logic, Methodology and Phi- losophy of Science, edited by Peter Gardenfors, Katarzyna Kijania-Placek and Jan Wolenski, Kluwer Academic.

[362] C J Isham, *Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity*, Found.Phys. 30 (2000) 1707-1735, [gr-qc/9910005].

[363] C J Isham, *Some reflections on the Status of Conventional Quantum Theory when Applied to Quantum Gravity*, [quant-ph/0206090].

[364] C J Isham, *Lectures on Quantum Theory Mathematical and Structural Foundations*, (Published by Imperial College Press 1995).

[365] C J Isham, *Quantum logic and decohering histories*, Writeup of lecture given at conference "Theories of fundamental interactions", Maynooth Eire 24–26 May 1995, [quant-ph/9506028].

[366] F. Dowker and A. Kent, *On the consistent histories approach to quantum theory* to appear Jour. Stat, Phys. (1995), [gr-qc/9412067].

[367] F. Markopoulou, *The internal description of a causal set: What the universe looks like from the inside*, Commun.Math.Phys. 211 (2000) 559-583, [gr-qc/9811053].

[368] F. Markopoulou, *Quantum causal histories*, Class.Quant.Grav. 17 (2000) 2059-2072, [hep-th/9904009].

[369] F. Markopoulou, *An insider's guide to quantum causal histories*, Nucl.Phys.Proc.Suppl. 88 (2000) 308-313, [hep-th/9912137].

[370] F. Markopoulou, *Planck-scale models of the Universe*, Invited contribution to "Science & Ultimate Reality: From Quantum to Cosmos (Explorations Celebrating the Vision of John Archibald Wheeler)", [gr-qc/0210086].

[371] E. Hawkins, F. Markopoulou, H. Sahlmann, *Evolution in Quantum Causal Histories*, Class.Quant.Grav. 20 (2003) 3839, [hep-th/0302111].

[372] Carlo Rovelli, *Relational Quantum Mechanics*, Int. J. of Theor.Phys. 35 (1996) 1637, [quant-ph/9609002].

[373] Federico Laudisa, *The EPR Argument in a Relational Interpretation of Quantum Mechanics*, Foundations of Physics Letters, vol. 14 (2) 2001, pp. 119-132, [quant-ph/0011016].

[374] M. Smerlak, C. Rovelli, *Relational EPR*, [quant-ph/0604064].

[375] Federico Laudisa, C. Rovelli, *Relational Quantum Mechanics*, http://plato.stanford.edu/entries/qm-relational/

[376] A. Grinbaum, *Elements of information-theoretic derivation of the formalism of quantum theory*, International Journal of Quantum Information 1(3) 2003, pp. 289-300, [quant-ph/0306079].

[377] A. Grinbaum, *The Significance of Information in Quantum Theory*, 191 pages, 13 figures. Ph.D. dissertation (Ecole Polytechnique, Paris), [quant-ph/0410071].

[378] A. Grinbaum, *On the notion of reconstruction of quantum theory*, [quant-ph/0509104].

[379] A. Grinbaum, *Information-theoretic principle entails orthomodularity of a lattice*, Found. Phys. Lett. 18 (6) 2005, pp. 563-572, [quant-ph/0509106].

[380] P.M. Cohn. *Universal algebra*. Harper and Row, New York, 1965.

[381] A. Gleason, Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6: 885-894, 1967.

[382] Thomas Streicher Publications, Lecture Notes etc. -

[383] something.

[384] John C. Baez, *Quantum Quandaries: a Category-Theoretic Perspective*, [quant-ph/0404040].

Appendices

[385] Jan Derezi'nski, *Notes on Quantization and Canonical Commutation Relations*, http://www.fuw.edu.pl/ derezins/ccr.ps.

[386] W.G. Unruh, *Phys. Rev. D* **14**, 870 (1976).

[387] P.C.W. Davies, *J. Phys. A.* **8(4)**, 609 (1975).

[388] E. Witten, "$(2+1)$-dimensional gravity as an exactly solvable system", Nucl. Phys. **B311**: 46, (1988).

[389] E. Witten, "Quantum field theory and the Jones polynomial", *Commun. Math. Phys.*, 121, 351-399, (1989). On-line open access at:

http://projecteuclid.org/Dienst/UI/1.0/Summarize/euclid.cmp/1104178138

[390] Bastian Wemmenhove, *Quantisation of 2+1 dimensional Gravity as a Chern-Simons theory*, 2002.

[391] Joseph D. Romano, *Geometrodynamics vs. Connection dynamics*, Department of physics, University of Maryland.

[392] Enore Guadagnini, "The Link Invaraints of the Chern-Simons Field Theory", Berlin ; New York: de Gruyter, 1993.

[393] R. Haag and D. Kastler. J. Math. Phys. **5** (1964) 848.

[394] Martin Rainer *Algebraic Quantum Theory on Manifolds: A Haag-Kastler Setting for Quantum Geometry*, [gr-qc/9911076]

[395] Martin Rainer *Is Loop Quantum Gravity a QFT ?* [gr-qc/9912011]

[396] R. Brunetti, M. Porrmann, G. Ruzzi, *General Covariance in Algebraic Quantum Field Theory*, [math-ph/0512059].

[397] Hendryk Pfeiffer *Quantum general relativity and the classification of smooth manifolds* [gr-qc/0404088].

[398] Arthur Jaffe, *Constructive Quantum Field Theory*, http://www.arthurjaffe.com/Assets/pdf/CQFT.pdf.

[399] Tomohiro Harada, *Gravitational collapse and naked singularities*, [gr-qc/0407109].

[400] T. Padmanabhan, *From Gravitons to Gravity: Myths and Reality*, [gr-qc/0409089].

[401] J. Baez and J. Dolan, *Higher-Dimensional Algebra and Topological Quntum Field Theory*, [q-alg/9503002].

[402] M. Dutdevich, *Quantum Geometry and new concept of space.*

[403] John R. Klauder, *Overview of Affine Quantum Gravity*, invited contribution to IJGMMP, [gr-qc/0507113].

Maths

[404] J.S. Milne , *Group Theory.*

[405] . M. Capiński and E. Kopp, *Measure, Integral and Probability*, (Springer-Verlag London 2004).

[406] M. Vonk, *A mini-course on topological strings*, [hep-th/0504147].

[407] J.L. Bell, *General topology* Download from ...

[408] Aisling McCluskey and Brian McMaster, *Topology Course Lecture Notes*, (August 1997), can be found in....

[409] "Topological spaces", King's College, London. Can be downloaded at http://www.mth.kcl.ac.uk/ iwilde/notes/fa2/fa2.pdf.

[410] Y. Yamasaki, *Measures on Infinite Dimensional Spaces*, World Scientific, Singapore, 1985.

[411] **Stuff**

Penn State Gravitation Seminars (www.phys.psu.edu/events/index.html)

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[412] State Sum Models for Quantum Gravity IGPG Seminar by Prof. John Barrett from University of Nottingham Thursday at 1:00 PM in (8/27/1998)

[413] Internal Logic of Casual Sets: What the Universe Looks Like from the Inside Relativity Seminar by Dr. Fotini Markopoulou from Penn State Thursday at 1:00 PM in (10/29/1998)

couldn't listen to this one - software problem

[414] Group Averaging: A Uniqueness Theorem IGPG Seminar by Dr. Donald Marolf from Syracuse University Monday at 1:00 PM in (11/2/1998)

may have listened to this one - can't remember

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[426] Quantum Cosmology Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (11/8/2002)

haven't listened to this one yet

[427] Towards QFT on Quantum Geometry I: Ideas, Formalism, Conceptual Issues, Open Questions Gravity Seminar by Dr. Hanno Sahlmann from Penn State University Friday at 11:00 AM in 318 Osmond Laboratory (12/6/2002)

haven't listened to this one yet

[428] Towards QFT on Quantum Geometry II: Some Concrete Calculations Gravity Seminar by Dr. Hanno Sahlmann from Penn State University Friday at 11:00 AM in 318 Osmond Laboratory (12/13/2002)

[429] Black Hole Entropy: Inclusion of Distortion and Angular Momentum Gravity Seminar by Dr. Abhay Ashtekar from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/17/2003)

[430] Modeling the Low energy Limit of Loop Quantum Gravity: Shadow States and Quantum Mechanics Gravity Seminar by Josh Willis from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/24/2003)

[431] Quantum Cosmology: An Overview IGPG Seminar by Dr. Martin Bojowald from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (1/27/2003)

[432] Quantum Cosmology: Formalism Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (1/31/2003)

haven't listened to this one yet

[433] Quantum Gravity: Overview and Status IGPG Seminar by Dr. Abhay Ashtekar from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (2/3/2003)

haven't listened to this one yet

[434] Quantum Cosmology: Applications Gravity Seminar by Dr. Martin Bojowald from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (2/7/2003)

haven't listened to this one yet

[435] Representations of the Sahlmann Star Algebra Gravity Seminar by Dr. Jerzy Lewandowski from University of Warsaw (Poland) Friday at 11:00 AM in 318 Osmond Laboratory (2/28/2003)

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[436] How Black Holes Grow Gravity Seminar by Dr. Badri Krishnan from Albert Einstein Institute Friday at 11:00 AM in 318 Osmond Laboratory (4/11/2003)

haven't listened to this one yet

[437] Diffeomorphism Invariance in the Smooth Category Gravity Seminar by Christian Fleischhack from Leipzig University and Penn State Monday at 3:00 PM in 318 Osmond Lab (4/28/2003)

haven't listened to this one yet

[438] An Update on Dynamical Horizons: Applications and Open Issues Gravity Seminar by Abhay Ashtekar from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (9/19/2003)

haven't listened to this one yet

[439] 2+1 Gravity and the Physical Scalar Product from Spin Foams Gravity Seminar by Alejandro Perez from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (10/24/2003)

haven't listened to this one yet

[440] Quadrupole Moments in Isolated Horizons Gravity Seminar by Jonathan Engle from Penn State Friday at 11:00 AM in 318 Osmond Laboratory (10/31/2003)

haven't listened to this one yet

[441] Dynamics in Loop Quantum gravity as a Sum Over Histories of Quantum Geometry IGPG Seminar by Alejandro Perez from Penn State Monday at 3:00 PM in 318 Osmond Laboratory (1/26/2004)

haven't listened to this one yet

[442] Discrete Quantum Gravity: Recent Developments and Applications Gravity Seminar by Rodolfo Gambini from University of Uruguay Friday at 11:00 AM in 318 Osmond Laboratory (2/6/2004)

[443] Master Constraint Programme for Loop Quantum Gravity Gravity Seminar by Thomas Thiemann from Perimeter Institute Monday at 12:00 PM in 318 Osmond Laboratory (2/9/2004) Audio, Presentation

[444] Testing the Master Constraint Programme for Loop Quantum Gravity Gravity Seminar by Bianca Dittrich from Perimeter Institute Friday at 11:00 AM in 318 Osmond Laboratory (2/13/2004)

[445] Goals and Achievements of Twistor Theory IGPG Seminar by Sir Roger Penrose from Oxford University and Penn State Monday at 3:00 PM in 318 Osmond (4/12/2004)

[446] Black Hole Entropy Gravity Seminar by Jerzy Lewandowski from University of Warsaw (Poland) Friday at 11:00 AM in 318 Osmond (9/3/2004)

[447] Black Hole Evaporation and Information Loss: Recent Advances IGPG Seminar by Abhay Ashtekar from Penn State Monday at 3:00 PM in 318 Osmond (9/20/2004)

[448] Free Vacuum for Loop Quantum Gravity Gravity Seminar by Florian Conrady from University of Marseille Friday at 9:00 AM in 318 Osmond (10/22/2004)

[449] Jorge Pullin, *Consistent discretization for Classical and Quantum General Relativity*, (17/01/05).

Index