# Irreducible Representations of $SU(2)$ and Racah's Coefficients

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# Chapter 1

# Wigner Matrices

## 1.1 Introduction

We write a vector in $\mathbb{C}^2$ as a pair of complex numbers $(z_1, z_2)$. We define the element of a Hilbert space $\mathcal{H}_\ell$ as an polynomial in $z_1$ and $z_2$ that is a linear combination of polynomials

$$P \in \mathbb{C}[z_1, z_2], \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad z = (z_1, z_2), \tag{1.1}$$

and

$$zg = (az_1 + cz_2, bz_1 + dz_2). \tag{1.2}$$

$$P_k(z_1, z_2) = z_1^k z_2^{n-k}, \quad 0 \geq k \geq n, \tag{1.3}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z_1^k z_2^{n-k} \quad = \quad (az_1 + cz_2)^r (bz_1 + dz_2)^{n-r} \tag{1.4}$$

Could take the basis of mononials of degree $n$ in the order

$$z_1^n, z_1^{n-1} z_2, \ldots, z_1 z_2^{n-1}, z_2^n. \tag{1.5}$$

OR

$$P_\ell(z_1, z_2) = z_1^{\ell+j} z_2^{\ell-j}, \quad -\ell \geq j \geq \ell, \quad \ell = 0, \frac{1}{2}, 1, \ldots \tag{1.6}$$

where $\ell$ is an integer or half integer

$$\frac{z_1^{2\ell}}{\sqrt{(2\ell)!}}, \frac{z_1^{2\ell-1} z_2}{\sqrt{(2\ell-1)!(1)!}}, \frac{z_1^{2\ell-2} z_2^2}{\sqrt{(2\ell-2)!(2)!}}, \ldots, \frac{z_1 z_2^{2\ell-1}}{\sqrt{(1)!(2\ell-1)!}}, \frac{z_2^{2\ell}}{\sqrt{(2\ell)!}}. \tag{1.7}$$

These are the basis vector of a vector space we denote $V_\ell$. We easily see that it is closed under the linear transformation

$$z_1^{\ell+j} z_2^{\ell-j} \to (az_1 + bz_2)^{\ell+j} (-cz_1 + dz_2)^{\ell-j}$$

as

$$\begin{aligned}
& (az_1 + bz_2)^{\ell+j}(-cz_1 + dz_2)^{\ell-j} \\
= & \sum_{r=0}^{\ell+j} \sum_{q=0}^{\ell-j} (az_1)^{\ell+j-r} (bz_2)^r (-cz_1)^q (dz_2)^{\ell-j-q} \\
= & \sum_{r=0}^{\ell+j} \sum_{q=0}^{\ell-j} \begin{pmatrix} \ell+j \\ r \end{pmatrix} \begin{pmatrix} \ell-j \\ q \end{pmatrix} \left(a^{\ell+j-r} b^r (-c)^q d^{\ell-j-q}\right) \cdot z_1^{\ell+(j-r+q)} z_2^{\ell-(j-r+q)}
\end{aligned} \tag{1.8}$$

Does $j - r + q$ vary between $-\ell$ and $\ell$? The minimum value $j - r + q$ can take is when $r = \ell + j$ and $q = 0$, and hence is $-\ell$. The maximum value $j - r + q$ can take is when $r = 0$ and $q = \ell - j$, and hence is $\ell$.

Since the monomial set is closed under the linear transformation $g$, it will provide a $(2\ell + 1) \times (2\ell + 1)$ matrix representation.

For any $g \in SU(2)$, let $U_\ell(g)$ be the linear transformation of $\mathcal{H}_\ell$ given by

$$(U_\ell(g)f)(v) = f(g^{-1}v) \tag{1.9}$$

for all $f \in \mathcal{H}_\ell$ and $v \in \mathbb{C}^2$. This is a representation: $U_\ell(1)$ is the identity, and for any $g, h \in SU(2)$ we have

$$\begin{aligned}
(U_\ell(g)U_\ell(h)f)(v) &= (U_\ell(h)f)(g^{-1}v) \\
&= f(h^{-1}g^{-1}v) \\
&= f((gh)^{-1}v) \\
&= (U_\ell(gh)f)(v)
\end{aligned} \tag{1.10}$$

for all $f \in \mathcal{H}_\ell$, $v \in \mathbb{C}^2$.

## 1.2 Matrix elements

We derive the symmetric version, based on the basis vectors (1.7),

$$\pi_\ell(g)_{mn} = \alpha^{m+n}(-\overline{\beta})^{\ell-m}\beta^{\ell-n}\sum_{s=0}^{\ell-n}\frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}{(m+n+s)!(\ell-m-s)!(\ell-n-s)!s!}\left(-\frac{|\alpha|}{|\beta|}\right)^s \tag{1.11}$$

Setting

$$\phi_k^\ell(z_1, z_2) = \frac{z_1^{\ell+k}z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}} \tag{1.12}$$

The matrix elements are defined by

$$U_\ell(g)\phi_n^\ell = \sum_{k=\ell}^{-\ell}\phi_k^\ell(z_1, z_2)\ \pi_\ell(g)_{kn} \tag{1.13}$$

From (1.10)

$$\begin{aligned}
U_\ell(g)U_\ell(h)\phi_n^\ell &= U_\ell(g)\sum_{j=-\ell}^{\ell}\phi_j^\ell\ \pi_\ell(h)_{jn} \\
&= \sum_{j=-\ell}^{\ell}\sum_{k=-\ell}^{\ell}\phi_k^\ell\ \pi_\ell(g)_{kj}\pi_\ell(h)_{jn} \\
&= U_\ell(gh)\phi_n^\ell \\
&= \sum_{k=-\ell}^{\ell}\phi_k^\ell\ \pi_\ell(h)_{kn}(gh)
\end{aligned} \tag{1.14}$$

that is

$$\sum_{k=-\ell}^{\ell}\frac{z_1^{\ell+k}z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}}\left(\sum_{j=-\ell}^{\ell}\pi_\ell(g)_{kj}\pi_\ell(h)_{jn} - \pi_\ell(h)_{kn}\right) \tag{1.15}$$

Setting $z_2 = 1$ and multiplying both sides by $z_1^{-\ell-m-1}$, integrating $z_1$ around the unit circle about the origin of the $z_1$-plane we obtain

$$\sum_{j=-\ell}^{\ell} \pi_\ell(g)_{mj}\pi_\ell(h)_{jn} = \pi_\ell(h)_{mn} \tag{1.16}$$

i.e. $\pi_\ell(g)_{kj}$ is a matrix representation of $SU(2)$.

We defined the representation $\pi_\ell(g)_{mn}$ by

$$\begin{aligned} U_\ell(g)\phi_n^\ell(z_1,z_2) &= \frac{1}{\sqrt{(\ell+n)!(\ell-n)!}}U_\ell(g)z_1^{\ell+n}z_2^{\ell-n} \\ &= \frac{(\alpha z_1-\overline{\beta}z_2)^{\ell+n}(\beta z_1+\overline{\alpha}z_2)^{\ell-n}}{\sqrt{(\ell+n)!(\ell-n)!}} \\ &= \sum_{m=\ell}^{-\ell}\phi_m^\ell(z_1,z_2)\ \pi_\ell(g)_{mn} \\ &= \sum_{m=\ell}^{-\ell}\frac{z_1^{\ell+m}z_2^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}\ \pi_\ell(g)_{mn} \end{aligned} \tag{1.17}$$

To arrive at (1.11), first we expand into the binomial expansion

$$\begin{aligned} &(\alpha z_1-\overline{\beta}z_2)^{\ell+n}(\beta z_1+\overline{\alpha}z_2)^{\ell-n} \\ =& \left(\sum_{t=0}^{\ell+n}(-1)^{\ell+n-t}\begin{pmatrix}\ell+n\\ t\end{pmatrix}\alpha^t\overline{\beta}^{\ell+n-t}z_1^t z_2^{\ell+n-t}\right)\left(\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\beta^{\ell-n-k}\overline{\alpha}^k z_1^{\ell-n-k}z_2^k\right) \\ =& \sum_{t=0}^{\ell+n}\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ t\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\alpha^t(-\overline{\beta})^{\ell+n-t}\beta^{\ell-n-k}\overline{\alpha}^k\cdot z_1^{\ell+t-n-k}z_1^{\ell-t+n+k} \end{aligned} \tag{1.18}$$

We will show that this is equal to

$$\sum_{m=-\ell}^{\ell}\alpha^{m+n}(-\beta^*)^{\ell-m}\beta^{\ell-n}\left(\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ m+n+k\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\left(-\left|\frac{\alpha}{\beta}\right|\right)^k\right)z_1^{\ell+m}z_1^{\ell-m} \tag{1.19}$$

equivalently

$$\sum_{m=-\ell}^{\ell}\sum_{k=0}^{\ell-n}\left(\begin{array}{c}\ell+n\\ m+n+k\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{m,n,k} \tag{1.20}$$

where $Q_{m,n,k}=\alpha^{m+n+k}(-\beta^*)^{\ell-m-k}(\alpha^*)^k\beta^{\ell-n-k}\cdot z_1^{\ell+m}z_1^{\ell-m}$. We do this by working backwards. First write

$$\begin{aligned}
B &:= \sum_{m=-\ell}^{\ell}\sum_{k=0}^{\ell-n}\left(\begin{array}{c}\ell+n\\ m+n+k\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{m,n,k}\\
&= \sum_{k=0}^{\ell-n}\sum_{m=-\ell}^{\ell}\left(\begin{array}{c}\ell+n\\ m+n+k\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{m,n,k}\\
&= \sum_{k=0}^{\ell-n}\sum_{t=n+k-\ell}^{\ell+n+k}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{t-n-k,n,k}
\end{aligned} \tag{1.21}$$

but note that

$$\left(\begin{array}{c}\ell+n\\ \ell+n+k\end{array}\right)=0 \quad \text{if} \quad k>0,$$

because $1/(-N)!=0$ for positive integer $N$. As such the upper limit on the $t$ summation will always be $\ell+n$. We now explicitly write out each term in the $k$ summation. In doing so we will use

$$\left(\begin{array}{c}\ell+n\\ n+k-\ell\end{array}\right)=0 \quad \text{if} \quad k<\ell-n$$

which again follows from $1/(-N)!=0$ for positive integer $N$. Hence,

$$\begin{aligned}
B &= \sum_{t=n-\ell}^{\ell+n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ 0\end{array}\right)Q_{t-n,n,0}+\sum_{t=n+1-\ell}^{\ell+n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ 1\end{array}\right)Q_{t-n-1,n,1}+\dots\\
&\quad \dots+\sum_{t=0}^{\ell+n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ \ell-n\end{array}\right)Q_{t-\ell,n,\ell-n}\\
&= \sum_{t=0}^{\ell+n}\sum_{k=0}^{\ell-n}\left(\begin{array}{c}\ell+n\\ t\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)Q_{t-n-k,n,k}
\end{aligned} \tag{1.22}$$

Now $Q_{t-n-k,n,k} = \alpha^t \beta^{\ell-n-k} \overline{\alpha}^k \overline{\beta}^{\ell+n-t} \cdot z_1^{\ell+t-n-k} z_1^{\ell-t+n+k}$ and we have proven (1.18).

We divide the last line above by $[(\ell+n)!(\ell-n)!]^{1/2}$ and get $(\alpha z_1 - \overline{\beta} z_2)^{\ell+n} (\beta z_1 + \overline{\alpha} z_2)^{\ell-n} / [(\ell+n)!(\ell-n)!]^{1/2}$. From (1.17)

$$\begin{aligned} & \sum_{m=-\ell}^{\ell} \alpha^{m+n} (-\beta^*)^{\ell-m} \beta^{\ell-n} \left( \sum_{k=0}^{\ell-n} \begin{pmatrix} \ell+n \\ m+n+k \end{pmatrix} \begin{pmatrix} \ell-n \\ k \end{pmatrix} \left( - \left| \frac{\alpha}{\beta} \right| \right)^k \right) \frac{z_1^{\ell+m} z_1^{\ell-m}}{\sqrt{(\ell+n)!(\ell-n)!}} \\ = & \sum_{m=\ell}^{-\ell} \frac{z_1^{\ell+m} z_2^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}} \pi_\ell(g)_{mn} \end{aligned} \tag{1.23}$$

We can then read off the matrix elements (1.11) .

□

## 1.3 Examples

$\ell = 0, 1/2, 1, 3/2, \ldots$

For $2\ell =$ to an even integer, the representatioon is the $2\ell+1$ dimensional tensorial representation of $SO(3)$. For $2\ell =$ to an odd integer, $\pi_\ell$ is a spinor representation. For matter, 1/2 describe elementary particles of half-integer spin.

**(1)** $\ell = 1/2$

$$\begin{aligned} \pi_{1/2}(g)_{\frac{1}{2}\frac{1}{2}} &= \alpha \sum_{k=0}^{0} = \alpha \\ \pi_{1/2}(g)_{-\frac{1}{2}-\frac{1}{2}} &= \alpha^{-1}(-\overline{\beta})\beta \sum_{k=0}^{1} \frac{1}{(k-1)!(1-k)!(1-k)!k!} \left( - \left| \frac{\alpha}{\beta} \right| \right)^k = \overline{\alpha} \\ \pi_{1/2}(g)_{-\frac{1}{2}\frac{1}{2}} &= = \beta \end{aligned} \tag{1.24}$$

$$\pi_{1/2} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{1.25}$$

**(2)** $\ell = 1$:

$$\pi_1 \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\overline{\beta} & |\alpha|^2 - |\beta|^2 & \sqrt{2}\overline{\alpha}\beta \\ \overline{\beta}^2 & -\sqrt{2}\overline{\alpha}\overline{\beta} & \overline{\alpha}^2 \end{pmatrix} \tag{1.26}$$

**Unitarity of the representation**

the transpose $g \longrightarrow g^T$

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \longrightarrow \pi_\ell(g^T)_{mn} = \pi_\ell(g)_{nm} = (\pi_\ell(g)^T)_{mn} \tag{1.27}$$

the complex conjugate $g \longrightarrow \overline{g}$

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \rightarrow \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \longrightarrow \pi_\ell(\overline{g})_{mn} = (\overline{\pi_\ell(g)})_{mn} \tag{1.28}$$

Combining both operations $g \longrightarrow g^\dagger$, this induces $\pi_\ell(g^\dagger) = \pi_\ell(g)^\dagger$ and so we can write

$$\pi_\ell(g)^\dagger = \pi_\ell(g^\dagger) = \pi_\ell(g^{-1}) = \pi_\ell(g)^{-1}. \tag{1.29}$$

Therefore the representation is unitary.

$\square$

According to Euler's theorem, every rotation $R$ in $\mathbb{R}^3$ can be written as $R = R_3(\phi)R_2(\theta)R_3(\psi)$, see fig (1.1).

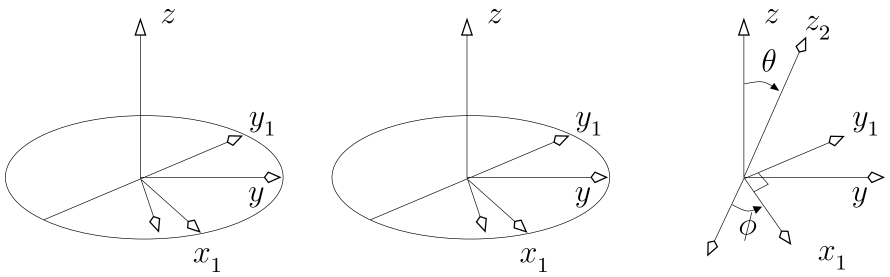


Figure 1.1: EulerTherm.

$$\begin{aligned} \pi_{\frac{1}{2}}(\phi, \theta, \psi) &= \exp\left(-i\frac{\phi}{2}\sigma_3\right)\exp\left(-i\frac{\theta}{2}\sigma_2\right)\exp\left(-i\frac{\psi}{2}\sigma_3\right) \\ &= \begin{pmatrix} \exp(-i\frac{\phi}{2}) & 0 \\ 0 & \exp(i\frac{\phi}{2}) \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} \exp(-i\frac{\psi}{2}) & 0 \\ 0 & \exp(i\frac{\psi}{2}) \end{pmatrix} \end{aligned} \tag{1.30}$$

Together

$$\begin{pmatrix} \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & -\exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \\ \exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \end{pmatrix} \tag{1.31}$$

**Irreducibility**

According to Schur's lemma, the representation is irreducible if the matrix which commutes with all the elements of the representation is a constant matrix. This we will use to prove the representations (labelled by $\ell$) are irreducible.

Consider special case in (1.11) $\alpha = e^{-im\theta/2}$ $\beta = 0$

$$\pi_\ell(0, 0, R_3(\phi))_{mn} = \delta_{mn} e^{-im\phi} \tag{1.32}$$

where $m, n = \ell, \ell - 1, \cdots - \ell + 1, -\ell,$

$$(e^{-in\theta} - e^{-im\theta})M_{nm} = 0. \tag{1.33}$$

implying that $M$ is diagonal

Thus any matrix $M$ that commutes with the representation is a constant matrix.

# Chapter 2

# Angular Momentum Operators

## 2.1 The Angular Momentum Operator

In quantum mechanics, (any) Hermitian vector operator $\vec{J}$ whose components satisfy the commutation relation

$$[J_i, J_j] = i\epsilon_{ij}{}^k J_k \tag{2.1}$$

is called the angular momentum operator. All components of $\vec{J}$ commute with the squared angular momentum

$$J^2 = J_x^2 + J_y^2 + J_z^2. \tag{2.2}$$

Therefore one can simultaneously diagonalise $J^2$ and one of the components, conventionally chosen as $J_z$. Let us write the eigenvalue equations

$$J^2|\lambda, \mu\rangle = \lambda|\lambda, \mu\rangle, \tag{2.3}$$
$$J_z|\lambda, \mu\rangle = \mu|\lambda, \mu\rangle. \tag{2.4}$$

To derive the solution of the eigenvalue problem, it is appropriate to use the following raising and lowering operators

$$J_\pm = J_x \pm iJ_y. \tag{2.5}$$

There commutators with $J^2$ and $J_z$ are given by

$$[J^2, J_\pm] = 0, \qquad [J_z, J_\pm] = \pm J_\pm. \tag{2.6}$$

These realtions imply that the raising and lowering operatorsraise and lower the eigenvalue of $J_z$ by one, while leaving the eigenvalue of $J^2$ unchanged:

$$\begin{aligned} J^2(J_\pm|\lambda,\mu\rangle) &= \lambda J_\pm|\lambda,\mu\rangle, \\ J_z(J_\pm|\lambda,\mu\rangle &= (\mu \pm 1)J_\pm|\lambda,\mu\rangle. \end{aligned} \tag{2.7}$$

This means that the states $J_\pm|\lambda,\mu\rangle$ must be proportional to $|\lambda,\mu\pm\rangle$:

$$J_\pm|\lambda,\mu\rangle = C_\pm(\lambda,\mu)|\lambda,\mu\pm\rangle. \tag{2.8}$$

Thus if any one of the eigenstates $|\lambda,\mu\rangle$ is known, then all the eigenstates for that of $\lambda$ can be derived by repeatedly applying $J_+$ and $J_-$.

On the other hand, the inequality

$$\langle\lambda,\mu|J^2 - J_z^2|\lambda,\mu\rangle = \langle\lambda,\mu|J_x^2 + J_y^2|\lambda,\mu\rangle \geq 0 \tag{2.9}$$

implies that the eigenvalue $\mu$ is restrcited by

$$\mu^2 \leq \lambda. \tag{2.10}$$

This condition is satisfied only if there is a maximal eigenvalue $\mu_{max}$, such that the action of the raising operator on the state $|\lambda,\mu_{max}\rangle$ doesnot give a new eigenvalue, but instead

$$J_+|\lambda,\mu_{max}\rangle = 0. \tag{2.11}$$

Similarly, there must exist a minimal eigenvalue $\mu_{min}$, such that

$$J_-|\lambda,\mu_{min}\rangle = 0. \tag{2.12}$$

Using the identities

$$\begin{aligned} J_-J_+ &= J^2 - J_z^2 - J_z, \\ J_+J_- &= J^2 - J_z^2 + J_z. \end{aligned} \tag{2.13}$$

in (2.11) and (2.12), one finds the relation

$$\lambda = \mu_{max}(\mu_{max} + 1) = -\mu_{min}(-\mu_{min} + 1), \tag{2.14}$$

solving the quadratic equations gives

$$\begin{aligned} \mu_{min} &= \frac{1}{2} \pm \sqrt{\lambda + \frac{1}{4}} \\ \mu_{max} &= -\frac{1}{2} \pm \sqrt{\lambda + \frac{1}{4}}. \end{aligned} \tag{2.15}$$

Obviously we must take $\mu_{min} = \frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}$ and in turn we must take $\mu_{max} = -\frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}$. Showing that $\mu_{max}$ and $\mu_{min}$ are related to each other by

$$\mu_{max} = -\mu_{min}. \tag{2.16}$$

Consider now acting repeatedly with $J\_$ on the state $|\lambda, \mu_{max}\rangle$. After a certain numbe of actions, say $N$, one must arrive at a state $|\lambda, \mu_{min}\rangle$. Since the eigenvalue $\mu$ is lowered by one at each step, the difference $\mu_{max} - \mu_{min}$ must be equal to $N$, giving

$$\mu_{max} = \frac{N}{2}. \tag{2.17}$$

The eigenvalue $\lambda$ is then

$$\lambda = \mu_{max}(\mu_{max} + 1) = \frac{N}{2}\left(\frac{N}{2} + 1\right). \tag{2.18}$$

The possible eigenvalues of the operators $J^2$ and $J_z$ have therefore been determined. The eigenvalue equations (2.3) and (2.4) can be written as

$$\begin{aligned} J^2|jm\rangle &= j(j+1)|jm\rangle \\ J_z|jm\rangle &= m|jm\rangle, \end{aligned} \tag{2.19}$$

where $j$ may be any integer or half integer, and $m$ ranges from $-j$ to $j$ in steps of 1.

To complete the solution of the eigenvalue problem, it remains to find the coefficients $C_{\pm}(\lambda, \mu) \equiv C_{\pm}(j, m)$ in (2.8), since this will show how the angular momentumoperator acts on the eigenstates $|jm\rangle$. By multiplying the equation with its adjoint, we find

$$
\begin{aligned}
|C_{\pm}(j,m)|^2 &= \langle J_{\mp} J_{\pm} | jm \rangle \\
&= \langle jm | J^2 - J_z^2 \mp J_z | jm \rangle \\
&= j(j+1) - m(m \pm 1), \qquad (2.20)
\end{aligned}
$$

which determines $C_{\pm}(j,m)$ up to a phase. The usual convention is to choose the phase such that $C_{\pm}(j,m)$ is real and positive. This implies

$$
J_{\pm}|jm\rangle = \sqrt{j(j+1) - m(m \pm 1)}|jm \pm 1\rangle. \qquad (2.21)
$$

## 2.2 Angular momentum operators as operators on scalar function of two complex variables

Here we derive the functions used by Wigner to generate the rotation matrices.

The angular momentum operators properties are faithfully reproduced when you take

$$
J_x = -\frac{1}{2}\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right), \quad J_y = \frac{i}{2}\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right), \quad J_z = -\frac{1}{2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right). \qquad (2.22)
$$

These act upon scalar functions $f(\eta,\xi)$ of the two complex variables $\xi$, $\eta$. Let us verify that they indeed satisfy the angular momentum commutaion relations,

$$
\begin{aligned}
& [J_x, J_y] f(\eta,\xi) = \\
=\; & \left[-\frac{1}{2}\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right), \frac{i}{2}\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right)\right] f(\eta,\xi) \\
=\; & -\frac{i}{4}\left\{\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right)\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right) - \left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right)\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right)\right\} f(\eta,\xi) \\
=\; & -\frac{i}{4}\left\{\left(\eta^2\frac{\partial^2}{\partial\xi^2} - \eta\frac{\partial}{\partial\eta} - \eta\xi\frac{\partial^2}{\partial\xi\partial\eta} + \xi\frac{\partial}{\partial\xi} + \xi\eta\frac{\partial^2}{\partial\eta\partial\xi} - \xi^2\frac{\partial^2}{\partial\eta^2}\right)\right. \\
& \left. - \left(\eta^2\frac{\partial^2}{\partial\xi^2} + \eta\frac{\partial}{\partial\eta} + \eta\xi\frac{\partial^2}{\partial\xi\partial\eta} - \xi\frac{\partial}{\partial\xi} - \xi\eta\frac{\partial^2}{\partial\eta\partial\xi} - \xi^2\frac{\partial^2}{\partial\eta^2}\right)\right\} f(\eta,\xi) \\
=\; & i \times -\frac{1}{2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) f(\eta,\xi) \\
=\; & iJ_z f(\eta,\xi). \qquad (2.23)
\end{aligned}
$$

next

$$
\begin{aligned}
& [J_z, J_x] f(\eta,\xi) = \\
= \;& \left[ -\frac{1}{2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right), -\frac{1}{2}\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right)\right] f(\eta,\xi) \\
= \;& \frac{1}{4}\left\{ \left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right) - \left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right)\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\right\} f(\eta,\xi) \\
= \;& \frac{1}{4}\left\{ \left(\xi\eta\frac{\partial^2}{\partial\xi^2} + \xi\frac{\partial}{\partial\eta} + \xi^2\frac{\partial^2}{\partial\xi\partial\eta} - \eta\frac{\partial}{\partial\xi} - \eta^2\frac{\partial^2}{\partial\eta\partial\xi} - \eta\xi\frac{\partial^2}{\partial\eta^2}\right)\right. \\
& \quad \left. - \left(\eta\frac{\partial}{\partial\xi} + \eta\xi\frac{\partial^2}{\partial\xi^2} - \eta^2\frac{\partial^2}{\partial\xi\partial\eta} + \xi^2\frac{\partial^2}{\partial\eta\partial\xi} - \xi\frac{\partial}{\partial\eta} - \xi\eta\frac{\partial^2}{\partial\eta^2}\right)\right\} f(\eta,\xi) \\
= \;& i \times \frac{i}{2}\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right) f(\eta,\xi) \\
= \;& iJ_y f(\eta,\xi).
\end{aligned} \tag{2.24}
$$

lastly

$$
\begin{aligned}
& [J_y, J_z] f(\eta,\xi) = \\
= \;& \left[ \frac{i}{2}\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right), -\frac{1}{2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\right] f(\eta,\xi) \\
= \;& -\frac{i}{4}\left\{ \left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right)\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) - \left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right)\right\} \\
= \;& -\frac{i}{4}\left\{ \left(\eta\frac{\partial}{\partial\xi} + \eta\xi\frac{\partial^2}{\partial\xi^2} - \eta^2\frac{\partial^2}{\partial\xi\partial\eta} - \xi^2\frac{\partial^2}{\partial\eta\partial\xi} + \xi\frac{\partial}{\partial\eta} + \xi\eta\frac{\partial^2}{\partial\eta^2}\right)\right. \\
& \quad \left. - \left(\xi\eta\frac{\partial^2}{\partial\xi^2} - \xi\frac{\partial}{\partial\eta} - \xi^2\frac{\partial^2}{\partial\xi\partial\eta} - \eta\frac{\partial}{\partial\xi} - \eta^2\frac{\partial^2}{\partial\eta\partial\xi} + \eta\xi\frac{\partial^2}{\partial\eta^2}\right)\right\} f(\eta,\xi) \\
= \;& i \times -\frac{1}{2}\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right) f(\eta,\xi) \\
= \;& iJ_x f(\eta,\xi).
\end{aligned} \tag{2.25}
$$

The expressions for $J_\pm = J_x \pm iJ_y$ and $J^2 = J_x^2 + J_y^2 + J_z^2$ follows from (2.22):

$$J_+ = -\eta\frac{\partial}{\partial\xi}, \qquad J_- = -\xi\frac{\partial}{\partial\eta} \tag{2.26}$$

$$\begin{aligned}
J^2 &= J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) \\
&= \frac{1}{4}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) + \frac{1}{2}\left(\eta\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\eta} + \xi\frac{\partial}{\partial\eta}\eta\frac{\partial}{\partial\xi}\right) \\
&= \frac{1}{4}\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right)\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right) - \frac{1}{2}\left(\xi\frac{\partial}{\partial\xi}\eta\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\eta}\xi\frac{\partial}{\partial\xi}\right) \\
&+ \frac{1}{2}\left(\eta\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\eta} + \xi\frac{\partial}{\partial\eta}\eta\frac{\partial}{\partial\xi}\right) \\
&= \frac{1}{4}\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right)\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta} + 2\right).
\end{aligned} \tag{2.27}$$

There are simultaneous eigenstates $f_m^{\ j}$ of $J^2$ and $J_z$ which satisfy

$$J^2 f_m^{\ j} = j(j+1) f_m^{\ j}, \tag{2.28}$$

$$J_z f_m^{\ j} = m f_m^{\ j} \tag{2.29}$$

where $j$ is an integer or half-integer and $m = -j, -j+1, \ldots j-1, j$. Also, adopting the Condon-Shortley convention for relative phases of the $f_m^{\ j}$ with fixed $j$,

$$\begin{aligned}
J_\pm f_m^{\ j} &= \left[(j \mp m)(j \pm m + 1)\right]^{1/2} f_{m\pm1}^{\ j}; \\
J_\pm f_{\pm j}^{\ j} &= 0.
\end{aligned} \tag{2.30}$$

To determine the $f_m^{\ j}$ explicitly, we first find $f_j^{\ j}$. Since it is an eigenfunction of $J^2$ it must by (2.27) be eigenfunction of

$$\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}.$$

By Euler's theorem (see appendix A)

$$\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right) f_j^{\ j} = D f_j^{\ j}$$

So by (2.27) we have

$$
\begin{aligned}
J^2 {f_j}^j &= \frac{1}{4}\left(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}\right)\left(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}+2\right){f_j}^j \\
&= \frac{1}{4}D(D+2)
\end{aligned} \tag{2.31}
$$

Comparing it to $J^2 {f_j}^j = j(j+1)$, we obtain $D = 2j$. From $J_+ {f_j}^j = 0$,

$$
J_+ {f_j}^j(\eta,\xi) = -\eta\frac{\partial}{\partial\xi}{f_j}^j(\eta,\xi) = 0
$$

it follows that ${f_j}^j$ is independent of $\xi$. As ${f_j}^j$ is a homogeneous function of one variable (from appendix A) we conclude that

$$
{f_j}^j = \frac{\eta^{2j}}{(2j)!}; \tag{2.32}
$$

the phase and amplitude are chosen arbitrarily. Using (2.30) it can be shown by induction that

$$
{f_m}^j = (-1)^{j-m}\frac{\eta^{j+m}\xi^{j-m}}{[(j+m)!(j-m)!]^{\frac{1}{2}}}. \tag{2.33}
$$

Let us do that. The base case is $m = j$ for which we see that (2.33) coincides with (2.32). We assume (2.33) and prove the result for $m-1$,

$$
\begin{aligned}
J_- {f_m}^j &= J_-\left\{(-1)^{j-m}\frac{\eta^{j+m}\xi^{j-m}}{[(j+m)!(j-m)!]^{\frac{1}{2}}}\right\} \\
&= -\xi\frac{\partial}{\partial\eta}\left\{(-1)^{j-m}\frac{\eta^{j+m}\xi^{j-m}}{[(j+m)!(j-m)!]^{\frac{1}{2}}}\right\} \\
&= (-1)^{j-(m-1)}\frac{(j+m)\eta^{j+m}\xi^{j-m}}{[(j+m)!(j-m)!]^{\frac{1}{2}}} \\
&= \sqrt{(j+m)(j-(m-1))}(-1)^{j-(m-1)}\frac{\eta^{j+(m-1)}\xi^{j-(m-1)}}{[(j+(m-1))!(j-(m-1))!]^{\frac{1}{2}}} \\
&= \sqrt{(j+m)(j-(m-1))}{f_{m-1}}^j
\end{aligned}
$$

from which we read off

$$f_{m-1}{}^{j} = (-1)^{j-(m-1)} \frac{\eta^{j+(m-1)} \xi^{j-(m-1)}}{[(j+(m-1))!(j-(m-1))!]^{\frac{1}{2}}}$$

which is the required result. We easily see that $J_{\_} f_{-j}{}^{j} = 0$.

The functions (2.33) were used by Wigner to generate rotation matrices.

# Chapter 3

# A Simple Derivation of the Clebsch-Gordan coefficients

## 3.1 Introduction

We know consider systems with two different angular momentum. Consider the tensor product space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. An obvious basis on this space is provided by the tensor product states $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$.

Angular momentum operators are defined to act upon these states in the following manner

$$J^{(1)}(|j_1 m_1\rangle \otimes |j_2 m_2\rangle) := (J \otimes \mathbb{1})(|j_1 m_1\rangle \otimes |j_2 m_2\rangle) = J|j_1 m_1\rangle \otimes |j_2 m_2\rangle \qquad (3.1)$$

and

$$J^{(2)}(|j_1 m_1\rangle \otimes |j_2 m_2\rangle) := (\mathbb{1} \otimes J)(|j_1 m_1\rangle \otimes |j_2 m_2\rangle) = |j_1 m_1\rangle \otimes J|j_2 m_2\rangle \qquad (3.2)$$

where $\mathbb{1}$ is the identity operator.

Angular momentum in different spaces commute:

$$[J^{(1)} \otimes \mathbb{1}, \mathbb{1} \otimes J^{(2)}] = 0 \qquad (3.3)$$

The total angular momentum operators are defined by

$$J := J^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J^{(2)}. \qquad (3.4)$$

The total angular momentum operators satisfy the very same commutation operators,

$$[J_i, J_j] = i\epsilon_{ijk} J_k^{(a)}. \tag{3.5}$$

For example,

$$\begin{aligned}
[J_x, J_y] &= [J_x^{(1)} + J_x^{(2)}, J_y^{(1)} + J_y^{(2)}] \\
&= [J_x^{(1)}, J_y^{(1)}] + [J_x^{(1)}, J_y^{(2)}] + [J_y^{(1)}, J_x^{(2)}] + [J_y^{(1)}, J_y^{(2)}] \\
&= [J_x^{(1)}, J_y^{(1)}] + [J_y^{(1)}, J_y^{(2)}] \\
&= iJ_z^{(1)} + J_z^{(2)} \\
&= iJ_z.
\end{aligned} \tag{3.6}$$

## 3.2 Clebsch-Gordan coefficients

The tensor product states $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$, which are eigenstates of the mutually commuting operators

$$\left(J^{(1)}\right)^2, \quad \left(J^{(2)}\right)^2, \quad J_z^{(1)}, \quad J_z^{(2)}. \tag{3.7}$$

Another complete set of commuting operators on $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ is formed by the operators

$$\left(J^{(1)}\right)^2, \quad \left(J^{(2)}\right)^2, \quad \left(J^{(1)} + J^{(2)}\right)^2, \quad J_z^{(1)} + J_z^{(2)}. \tag{3.8}$$

Let us check that $\left[\left(J^{(1)}\right)^2, \left(J^{(1)} + J^{(2)}\right)^2\right] = 0$:

$$\begin{aligned}
&\left[\left(J^{(1)}\right)^2, \left(J^{(1)} + J^{(2)}\right)^2\right] = \\
= &\left[\left(J^{(1)}\right)^2 \otimes \mathbb{1}^{(2)},\ \left(J_x^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J_x^{(2)}\right)^2\right. \\
&\left. + \left(J_y^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J_y^{(2)}\right)^2 + \left(J_z^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J_z^{(2)}\right)^2\right] \\
= &\left[\left(J^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}, \left(J^{(1)}\right)^2 \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes \left(J^{(2)}\right)^2\right] \\
&+ 2\left[\left(J^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}, J_x^{(1)} \otimes J_x^{(2)} + J_y^{(1)} \otimes J_y^{(2)} + J_z^{(1)} \otimes J_z^{(2)}\right] \\
= &\ 2\left[\left(J^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}, J_x^{(1)} \otimes J_x^{(2)} + J_y^{(1)} \otimes J_y^{(2)} + J_z^{(1)} \otimes J_z^{(2)}\right]
\end{aligned} \tag{3.9}$$

where we have used that operators in different spaces commute and where $\left(J^{(1)}\right)^2 = \left(J_x^{(1)}\right)^2 \otimes \mathbb{1}^{(2)} + \left(J_y^{(1)}\right)^2 \otimes \mathbb{1}^{(2)} + \left(J_z^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}$.

$$\begin{aligned}
&\left[\left(J^{(1)}\right)^2, \left(J^{(1)}+J^{(2)}\right)^2\right] = \\
= \;& 2\left[\left(J_x^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}, J_y^{(1)} \otimes J_y^{(2)} + J_z^{(1)} \otimes J_z^{(2)}\right] \\
& + 2\left[\left(J_y^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}, J_x^{(1)} \otimes J_x^{(2)} + J_z^{(1)} \otimes J_z^{(2)}\right] \\
& +2\left[\left(J_z^{(1)}\right)^2 \otimes \mathbb{1}^{(2)}, J_x^{(1)} \otimes J_x^{(2)} + J_y^{(1)} \otimes J_y^{(2)}\right]
\end{aligned} \tag{3.10}$$

We have

$$\begin{aligned}
[\left(J_x^{(1)}\right)^2, J_y^{(1)}] &= J_x^{(1)}[J_x^{(1)}, J_y^{(1)}] + [J_x^{(1)}, J_y^{(1)}]J_x^{(1)}) \\
&= i(J_x^{(1)}J_z^{(1)} + J_z^{(1)}J_x^{(1)})
\end{aligned} \tag{3.11}$$

etc. So

$$\begin{aligned}
&\left[\left(J^{(1)}\right)^2, \left(J^{(1)}+J^{(2)}\right)^2\right] = \\
= \;& 2\Big(i(J_x^{(1)}J_z^{(1)} + J_z^{(1)}J_x^{(1)}) - i(J_x^{(1)}J_y^{(1)} + J_y^{(1)}J_x^{(1)}) \\
& -i(J_y^{(1)}J_z^{(1)} + J_z^{(1)}J_y^{(1)}) + i(J_y^{(1)}J_x^{(1)} + J_x^{(1)}J_y^{(1)}) \\
& +i(J_z^{(1)}J_y^{(1)} + J_y^{(1)}J_z^{(1)}) - i(J_z^{(1)}J_x^{(1)} + J_x^{(1)}J_z^{(1)})\Big) \\
= \;& 0.
\end{aligned} \tag{3.12}$$

Thus

$$\left[\left(J^{(1)}\right)^2, \left(J^{(1)}+J^{(2)}\right)^2\right] = 0 \tag{3.13}$$

Let us check that $\left[\left(J^{(1)}+J^{(2)}\right)^2, J_z^{(1)}+J_z^{(2)}\right] = 0$:

$$
\begin{aligned}
& \left[\left(J^{(1)}+J^{(2)}\right)^2, J_z^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J_z^{(2)}\right] \\
= & \left[J_x^{(1)} \otimes J_x^{(2)} + J_y^{(1)} \otimes J_y^{(2)} + J_z^{(1)} \otimes J_z^{(2)}, J_z^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes J_z^{(2)}\right] \\
= & [J_x^{(1)}, J_z^{(1)}] \otimes J_x^{(2)} + [J_y^{(1)}, J_z^{(1)}] \otimes J_y^{(2)} + J_x^{(1)} \otimes [J_x^{(2)}, J_z^{(2)}] + J_y^{(1)} \otimes [J_y^{(2)}, J_z^{(2)}] \\
= & -iJ_y^{(1)} \otimes J_x^{(2)} + iJ_x^{(1)} \otimes J_y^{(2)} - iJ_x^{(1)} \otimes J_y^{(2)} + iJ_y^{(1)} \otimes J_x^{(2)} \\
= & \ 0.
\end{aligned}
\tag{3.14}
$$

Let us denote their eigenstates by $|j_1 j_2; jm\rangle$. Since both sets of staes span the space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$, they must be related to each other by unitary transformation of the form

$$
|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} |j_1 j_2; jm\rangle \tag{3.15}
$$

and

$$
|j_1 j_2; jm\rangle = \sum_{m_1 m_2} C^{(j_1 j_2 j) m_1 m_2}{}_{m} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \tag{3.16}
$$

The coefficients in these expansions are known as the Clebsch-Gordon coefficients.

We can derive orthogonality relations. First

$$
\begin{aligned}
|j_1 m_1\rangle \otimes |j_2 m_2\rangle &= \sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} |j_1 j_2; jm\rangle \\
&= \sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} \sum_{m_1' m_2'} C^{(j_1 j_2 j) m_1' m_2'}{}_{m} |j_1 m_1'\rangle \otimes |j_2 m_2'\rangle \\
&= \sum_{m_1' m_2'} \left( \sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} C^{(j_1 j_2 j) m_1' m_2'}{}_{m} \right) |j_1 m_1'\rangle \otimes |j_2 m_2'\rangle \\
&= \sum_{m_1' m_2'} \delta_{m1}^{m_1'} \delta_{m2}^{m_2'} \ |j_1 m_1'\rangle \otimes |j_2 m_2'\rangle,
\end{aligned}
\tag{3.17}
$$

implying

$$
\sum_{jm} C^{(j_1 j_2 j)}{}_{m_1 m_2}{}^{m} C^{(j_1 j_2 j) m_1' m_2'}{}_{m} = \delta_{m1}^{m_1'} \ \delta_{m2}^{m_2'}. \tag{3.18}
$$

Second

$$
\begin{aligned}
|j_1j_2;jm\rangle &= \sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_m|j_1m_1\rangle\otimes|j_2m_2\rangle \\
&= \sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_m \sum_{j'm'} C^{(j_1j_2j)}{}_{m_1m_2}{}^{m'}|j_1j_2;j'm'\rangle \\
&= \sum_{j'm'}\left(\sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_m C^{(j_1j_2j)}{}_{m_1m_2}{}^{m'}\right)|j_1j_2;j'm'\rangle \\
&= \sum_{j'm'} \delta_{jj'}\delta^m_{m'}\ |j_1j_2;j'm'\rangle,
\end{aligned}
\tag{3.19}
$$

implying

$$
\sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_m C^{(j_1j_2j)}{}_{m_1m_2}{}^{m'} = \delta_{jj'}\delta^m_{m'}. \tag{3.20}
$$

We can work out $\mathbf{J}_z|j_1j_2;jm\rangle$ two different ways:

$$
\begin{aligned}
\mathbf{J}_z|j_1j_2;jm\rangle &= \left(\mathbf{J}_z^{(1)}+\mathbf{J}_z^{(2)}\right)\sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_m|j_1m_1\rangle\otimes|j_2m_2\rangle \\
&= \sum_{m_1m_2}(m_1+m_2)C^{(j_1j_2j)m_1m_2}{}_m|j_1m_1\rangle\otimes|j_2m_2\rangle
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
\mathbf{J}_z|j_1j_2;jm\rangle &= m|j_1j_2;jm\rangle \\
&= \sum_{m_1m_2} mC^{(j_1j_2j)m_1m_2}{}_m|j_1m_1\rangle\otimes|j_2m_2\rangle.
\end{aligned}
\tag{3.22}
$$

By identifying the above relations we obtain, because of linear independence, the condition

$$
(m-m_1-m_2)C^{(j_1j_2j)m_1m_2}{}_m = 0. \tag{3.23}
$$

Thus it follows that the Clebsch-gordon coefficients vanish if $m \neq m_1+m_2$. Thus the double sum in (3.16) reduces to a single sum

$$
|j_1j_2;jm\rangle = \sum_{m_1} C^{(j_1j_2j)m_1m_2}{}_m|j_1m_1\rangle\otimes|j_2m-m_1\rangle \tag{3.24}
$$

The conservation of angualr momentum (more precisely the projection onto the quantisation direction) is expressed by the relation $m = m_1 + m_2$.

Next we want to work out the possible values of $j$ defined by

$$\mathbf{J}^2 |j_1 j_2; jm\rangle = j(j+1)|j_1 j_2; jm\rangle.$$

Since $\mathbf{J}$ is an anugular momentum operator we have

$$-j \leq m \leq j.$$

From the relation $m = m_1 + m_2$ it follows that the greatest value of $m$, which we denote $m_{max}$ is

$$m_{max} = j_1 + j_2. \tag{3.25}$$

This value appears only once in (3.24), namely if $m_1 = j_1$ and $m_2 = j_2$. This shows that the largest eigenvalue$j$, which we denote by $j_{max}$, must be

$$j_{max} = j_1 + j_2. \tag{3.26}$$

The second largest value of $m$ is $m_{max} - 1$. It occurs twice, for

$$m_1 = j_1, \quad m_2 = j_2 - 1 \qquad \text{and} \qquad m_1 = j_1 - 1, \quad m_2 = j_2 \tag{3.27}$$

corresponding to the states

$$|j_1 j_1\rangle \otimes |j_2 j_2 - 1\rangle \qquad \text{and} \qquad |j_1 j_1 - 1\rangle \otimes |j_2 j_2\rangle \tag{3.28}$$

respectively. One of the two linearly independent combinations of the two states must belong to $|j_1 j_2; j = j_1 + j_2 m\rangle$, since $m$ assumes all values $j_1 + j_2 \geq m \geq -j_1 - j_2$ in integer steps. The other linear combination necessarily belongs to the state

$$|j_1 j_2; jm\rangle \quad \text{with} \quad j = j_1 + j_2 - 1 \tag{3.29}$$

because there are no states with $j > j_{max} = j_1 + j_2$.

Continuing the argument one finds that for $j$, all values

$$j = \begin{cases} j_1 + j_2 \\ j_1 + j_2 - 1 \\ \vdots \\ |j_1 - j_2| \end{cases} \tag{3.30}$$

appears precisely once. This is the so called trianglar conditions.

**Total number of total angular momumentum eigenstates**

An easy way of checking that $|j_1 - j_2|$ is the lower limit is by counting the number of states $|j_1 j_2; jm\rangle$. This is done by calculating the sum $\sum_{J=|j_1-j_2|}^{j_1+j_2}(2J+1)$.

First consider the case where $J = j_1 + j_2$ is an integer, (say $j_1 > j_2$), then

$$\begin{aligned}
\sum_{J=|j_1-j_2|}^{j_1+j_2}(2J+1) &= \sum_{J=|j_1-j_2|}^{j_1+j_2} 2J + [(j_1+j_2) - |j_1-j_2| + 1] \\
&= \sum_{J=0}^{j_1+j_2} 2J - \sum_{J=0}^{|j_1-j_2|-1} 2J + [(j_1+j_2) - |j_1-j_2| + 1] \\
&= (j_1+j_2)(j_1+j_2+1) - (j_1-j_2-1)(j_1-j_2) + 2j_2 + 1 \\
&= (j_1+j_2)^2 - (j_1-j_2)^2 + (j_1+j_2) + (j_1-j_2) + 2j_2 + 1 \\
&= 4j_1j_2 + 2j_1 + 2j_2 + 1 \\
&= (2j_1+1)(2j_2+1).
\end{aligned} \tag{3.31}$$

Now consider the case where $J = j_1 + j_2$ is a half-integer. We will need to know how to perform the summation $\sum_{n=1/2}^{N/2} 2n$ where increments are in 1. This is easily,

$$\begin{aligned}
\sum_{n=1/2}^{N/2} 2n &= 1 + 3 + \cdots + (N-2) + N \\
&= (1 + 2 + 3 + \cdots + (N-1) + N) - 2(1 + 2 + \cdots (N-1)/2) \\
&= \frac{1}{2}N(N+1) - \left(\frac{N-1}{2}\right)\left(\frac{N+1}{2}\right) \\
&= \frac{1}{4}2(N^2+N) - \frac{1}{4}(N^2-1) \\
&= \frac{1}{4}(N^2+2N+1) \\
&= \frac{1}{4}(N+1)^2.
\end{aligned} \tag{3.32}$$

Assuming $j_1 > j_2$, then

$$
\begin{aligned}
\sum_{J=j_1-j_2}^{j_1+j_2}(2J+1) &= \sum_{J=|j_1-j_2|}^{j_1+j_2} 2J + [(j_1+j_2) - |j_1-j_2| + 1] \\
&= \sum_{J=1/2}^{2(j_1+j_2)/2} 2J - \sum_{J=1/2}^{2(j_1-j_2-1)/2} 2J + [(j_1+j_2)-(j_1-j_2)+1] \\
&= \frac{1}{4}[2(j_1+j_2)+1]^2 - \frac{1}{4}[2(j_1-j_2-1)+1]^2 + 4j_2 + 1 \\
&= (j_1+j_2)^2 + (j_1+j_2) + \frac{1}{4} - (j_1-j_2)^2 + (j_1-j_2) - \frac{1}{4} + 4j_2 + 1 \\
&= 4j_1j_2 + 2j_1 + 2j_2 + 1 \\
&= (2j_1+1)(2j_2+1). \qquad (3.33)
\end{aligned}
$$

**Summarising the properties of the Clebsch-Gordon coefficients:**

- The coefficient $C^{(j_1j_2j)}{}_{m_1m_2}{}^{m}$ is trivially zero unless the conditions

$$
|j_2 - j_1| \leq j \leq j_1 + j_2 \qquad \text{and} \qquad j_1 + j_2 + j = \text{ integer} \qquad (3.34)
$$

are met. These conditions are called the Clebsch-Gordon conditions, or the triangular conditions, for the spins $j_1, j_2, j$.

- Moreover, $C^{(j_1j_2j)}{}_{m_1m_2}{}^{m} = 0$ whenever $m \neq m_1 + m_2$.

- The orthogonality relations of the Clebsch-Gordan coeeficients reads

$$
\sum_{jm} C^{(j_1j_2j)}{}_{m_1m_2}{}^{m} C^{(j_1j_2j)m_1'm_2'}{}_{m} = \delta^{m_1'}_{m1}\,\delta^{m_2'}_{m2}. \qquad (3.35)
$$

and

$$
\sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_{m} C^{(j_1j_2j')}{}_{m_1m_2}{}^{m'} = \delta_{jj'}\delta^{m}_{m'}. \qquad (3.36)
$$

## 3.3 Definition of Clebsch-Gordan Coefficients Phase Factors

We draw a figure in which for fixed $j_1 = 2$ and $j_2 = 4$ all 45 states are represented by dots in the $(m_1, m_2)-$plane and in the $(jm)-$plane for the uncoupled and coupled set respectively. This diagram is used to help see how the states of $\mathcal{H}_{j_1 j_2}$ are mapped from the uncoupled set $|j_1 m_1 j_2 m_2\rangle$ onto the coupled set $|(j_1 j_2) jm\rangle$ and vice-versa.

In figure 4.1 we have indicated which points belong to the same subspace $\mathcal{H}_{j_1 j_2 m}$. Any one given state (e.g. one of the circles o in figure 4.1 (b)) is a linear combination of the corresponding subset (e.g. all circles in figure 4.1 (a)).

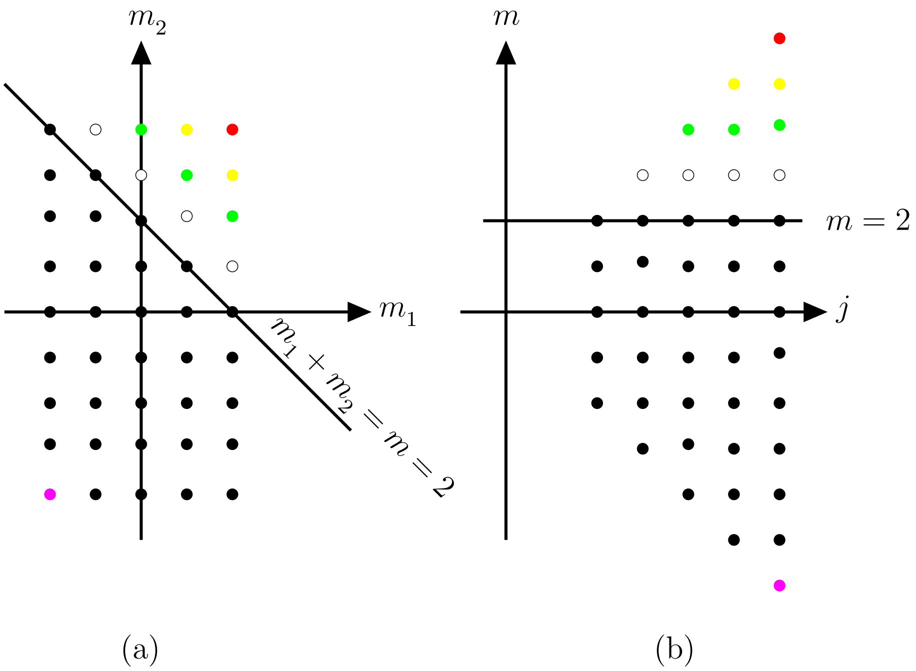


Figure 3.1: Fixed $j_1 = 2$ and $j_2 = 4$. (a) $|j_1 m_1 j_2 m_2\rangle$. (b) $|(j_1 j_2) jm\rangle$.

We immediately see that there are two subspaces $\mathcal{H}_{j_1 j_2 m}$ , namely with $m = j_1 + j_2$ (indicated by the red circle) and $m = -j_1 - j_2$ (indicated by the purple circle) which consist of one single state each. Here the unitary transformation consists at most of multiplication by a complex number of modulus 1.

This phase is arbitrary. The usual convention is to fix it to be +1 for the red-circle-states. Hence the Clebsch-Gordan coefficient is +1:

$$\begin{aligned} |(j_1 j_2), j_1 + j_2, j_1 + j_2\rangle &= |j_1 j_1 j_2 j_2\rangle \\ \langle j_1 j_1 j_2 j_2 | (j_1 j_2), j_1 + j_2, j_1 + j_2\rangle &= 1. \end{aligned} \tag{3.37}$$

This convention fixes uniquely all those Clebsch-Gordan coefficients belonging to the states in the column the red circle in 4.1 (b), for which $j = j_1 + j_2$. The calculation of these Clebsch-Gordan coefficients is achieved by the lowering operators

$$\mathbf{J}_- = \mathbf{J}_-^{(1)} + \mathbf{J}_-^{(2)}. \tag{3.38}$$

Applying $J_-$ to $|(j_1j_2)j_1+j_2, j_1+j_2\rangle$ repeatedly, we descend from this state step by step down the column below the red circle in figure 4.1 (b) until we arrive at the purple circle, namely $|(j_1j_2)j_1+j_2, -j_1-j_2\rangle$.

Applying $J_-^{(1)} + J_-^{(2)}$ to $|j_1j_1j_2j_2\rangle$ repeatedly, we generate those linear combinations of the states of figure 4.1 (a) which make up the states $|(j_1j_2)j_1+j_2, m\rangle$.

Thus, applying $J_-$ in figure 4.1 (b) three times to the red circle will give $o$; applying $J_-^{(1)} + J_-^{(2)}$ in figure 4.1 (a) three times to the red circle will give a linear combination which is equal to $|(j_1j_2)j, j-3\rangle$.

According to (2.21)

$$\begin{aligned}
J_-|(j_1j_2),j,m\rangle &= \sqrt{j(j+1)-m(m-1)}|(j_1j_2),j,m-1\rangle \\
J_-^{(1)}|j_1m_1j_2m_2\rangle &= \sqrt{j_1(j_1+1)-m_1(m_1-1)}|j_1,m_1-1,j_2m_2\rangle \\
J_-^{(2)}|j_1m_1j_2m_2\rangle &= \sqrt{j_2(j_2+1)-m_2(m_2-1)}|j_1m_1j_2,m_2-1\rangle
\end{aligned} \tag{3.39}$$

Thus applying $J_-^{(1)} + J_-^{(2)}$ to any state $|j_1m_1j_2m_2\rangle$ generates a superposition of two states:

$$\begin{aligned}
\left(J_-^{(1)} + J_-^{(2)}\right)|j_1m_1j_2m_2\rangle &= \sqrt{j_1(j_1+1)-m_1(m_1-1)}|j_1,m_1-1,j_2m_2\rangle \\
&\quad + \sqrt{j_2(j_2+1)-m_2(m_2-1)}|j_1m_1j_2,m_2-1\rangle
\end{aligned} \tag{3.40}$$

As this superposition involves real positive coefficients (the square roots in (3.40)) it is clear that by repeated application of $J_-$ and and of $J_-^{(1)} + J_-^{(2)}$ respectively only sums of products of these real, positive roots can occur. Thus, $n-$fold application of $J_-$ to (3.37) leads to

$$\left(J_-\right)^n|(j_1j_2)j_1+j_2,j_1+j_2\rangle = \left(J_-^{(1)} + J_-^{(2)}\right)^n|j_1j_1j_2j_2\rangle \tag{3.41}$$

with real positive coeffients on both sides. On the LHS $n$ square roots multiply; writing $\sqrt{j(j+1)-m(m-1)} = \sqrt{(j+m)(j-m+1)}$, we obtain

$$
\begin{aligned}
\left(J_-\right)^n |(j_1 j_2) jj\rangle &= \left(J_-\right)^{n-1} \sqrt{(2j)(+1)} |(j_1 j_2) j, j-1\rangle \\
&= \left(J_-\right)^{n-2} \sqrt{(2j)(2j-1)(1)(2)} |(j_1 j_2) j, j-2\rangle \\
&\vdots \\
&= \left(J_-\right)^{n-3} \sqrt{(2j)(2j-1)\cdots(2j-n+1)(1)(2)\cdots(n)} |(j_1 j_2) j, j-n\rangle \\
&= \sqrt{\frac{(2j)!n!}{(2j-n)!}} |(j_1 j_2) j, j-n\rangle. \qquad (3.42)
\end{aligned}
$$

Putting $j - n = m$ yields

$$
|(j_1 j_2) jm\rangle = \sqrt{\frac{(j+m)!}{(j-m)!(2j)!}} \left(J_-\right)^{j-m} |(j_1 j_2) jj\rangle. \qquad (3.43)
$$

For our particular case (3.41) we must put $j = j_1 + j_2$ and obtain

$$
|(j_1 j_2) jm\rangle = \sqrt{\frac{(j_1 + j_2 + m)!}{(j_1 + j_2 - m)!(2j_1 + 2j_2)!}} \left(J_-^{(1)} + J_-^{(2)}\right)^{j_1 + j_2 - m} |j_1 j_1 j_2 j_2\rangle. \qquad (3.44)
$$

We now return to figure 4.1. We have just described how we would explicitly caculate the Clebsch-Gordan coefficients belonging to the states $|(j_1 j_2) j_1 + j_2, m\rangle$ which make up the column furthest to the right in figure 4.1 (b). We now do the same for the next column, the one to the left of it. The state on top of it is $|(j_1 j_2) j_1 + j_2 - 1, j_1 + j_2 - 1\rangle$, indicated by the yellow circle. There is another state, also indicated by a yellow circle, which belongs to the $j = j_1 + j_2$ column which has already been determined. Both these yellow-circle-states in figure 4.1 (b) are linear combinations of the two yellow-circle-states in figure 4.1 (a). Since there are (up to an aritrary phase) only two orthogonal linear combinations possible (of which one is already used), there remains only one linear combination for the top state yellow-circle-state of the second column (this description we are giving here, we have already gone through in section 3.2). The phase , however, is arbitrary and must again be fixed by convention. Once this is done, all other states of the secon column are found by repeated application of $J_-$ to the highest state, the yellow-circle-state.

Obviously, when we move to the top state of the next column in figure 4.1 (b), two linear combinations of the three possible ones are already used up and the only remaining one is fixed (up to an arbitrary phase). Its phase will be chosen by convention and then by repeated application of $J_-$ all states of that column can be calculated.

This situation is found each time we wish to determine the linear combination of a state at the top of a column in figure 4.1 (b), i.e. a state of the form $|j_1 j_2 jj\rangle$. All possible mutual orthogonal linear combinations, except one, have been used up, and for this last one we must only fix the phase, then the rest is uniquely determined.

We fix the phases for the top states ($m = j$) of figure 4.1 (b) by the following phase convention: In the linear combination

$$|(j_1 j_2) jj\rangle = \sum_{m_1+m_2=j} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2|jj\rangle$$

the CGC with the higest $m_1$ is real and $\geq 0$:

$$\langle j_1 j_1 j_2, j - j_1 | jj\rangle \text{ real and } \geq 0.$$

That is, in the linear combinations making up the top states of figure 4.1 (b), the states of the rightmost column ($m_1 = j_1$) of figure 4.1 (a) always enter with a positive coefficient.

It is this phase convention that which causes the two states $|(j_1 j_2) jm\rangle$ and $|(j_2 j_1) jm\rangle$ to differ by a phase factor, as we will see in chapter 4.

## 3.4 Derivation of the Clebsch-Gordan Coefficients

The problem of determining the Clebsch-Gordan coefficients reduces to finding the linear combinations of

$$f_{m_1}{}^{j_1}(\eta_1, \xi_1) \times f_{m_2}{}^{j_2}(\eta_2, \xi_2)$$

with fixed $j_1$, $j_2$ which are eiegenstates of

$$\begin{aligned} J_z &= J_z^{(1)} + J_z^{(2)} \\ &= -\frac{1}{2}\left(\xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} - \eta_1 \frac{\partial}{\partial \eta_1} - \eta_2 \frac{\partial}{\partial \eta_2}\right) \end{aligned} \tag{3.45}$$

with eigenvalues $m$ and of $\mathbf{J}^2 = (\mathbf{J}_1 + \mathbf{J}_2)^2$ with eigenvalues $j(j+1)$:

$$\psi_m{}^j = \sum_{m_1 m_2} f_{m_1}{}^{j_1} f_{m_2}{}^{j_2} C^{(j_1 j_2 j) m_1 m_2}{}_m \tag{3.46}$$

We determine $\psi_j{}^j$ directly and then generate the other $\psi_m{}^j$ by repeated application of the $\mathbf{J}_-$. The Clebsch-Gordan coefficients will then be obtained by simply expanding $\psi_m{}^j$ in $f_{m_1}{}^{j_1} f_{m_2}{}^{j_2}$.

The $\psi_j{}^j$ satisfies

$$J_+\psi_j{}^j = 0 \quad \text{and} \quad J_z\psi_j{}^j = j\psi_j{}^j.$$

The general solution of $J_+\psi_j{}^j = 0$ or

$$\left(\eta_1\frac{\partial}{\partial\xi_1} + \eta_2\frac{\partial}{\partial\xi_2}\right)\psi_j{}^j = 0$$

is

$$\psi_j{}^j = \varphi(\eta_1\xi_2 - \eta_2\xi_1, \eta_1, \eta_2).$$

Since it is a homogeneous function of degree $2j_1$ in $\eta_1, \xi_1$ and of degree $2j_2$ in $\eta_2, \xi_2$

$$\psi_j{}^j = \sum_x (\xi_1\eta_2 - \xi_2\eta_1)^x \eta_1^{2j_1-x}\eta_2^{2j_2-x} C_x. \tag{3.47}$$

Let us just check that this expression has the correct scaling properties:

$$\begin{aligned}
\sum_x ((\rho\xi_1)\eta_2 - \xi_2(\rho\eta_1))^x (\rho\eta_1)^{2j_1-x}\eta_2^{2j_2-x} C_x &= \rho^{2j_1}\sum_x (\xi_1\eta_2 - \xi_2\eta_1)^x \eta_1^{2j_1-x}\eta_2^{2j_2-x} C_x \\
\sum_x (\xi_1(\rho\eta_2) - (\rho\xi_2)\eta_1)^x \eta_1^{2j_1-x}(\rho\eta_2)^{2j_2-x} C_x &= \rho^{2j_2}\sum_x (\xi_1\eta_2 - \xi_2\eta_1)^x \eta_1^{2j_1-x}\eta_2^{2j_2-x} C_x.
\end{aligned}$$

From $\mathbf{J}_z\psi_j{}^j = j\psi_j{}^j$

$$\begin{aligned}
& \mathbf{J}_z\psi_j{}^j = \\
= \;& -\frac{1}{2}\left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2} - \eta_1\frac{\partial}{\partial\eta_1} - \eta_2\frac{\partial}{\partial\eta_2}\right)\psi_j{}^j \\
= \;& -\frac{1}{2}\left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2} - \eta_1\frac{\partial}{\partial\eta_1} - \eta_2\frac{\partial}{\partial\eta_2}\right)\sum_x (\xi_1\eta_2 - \xi_2\eta_1)^x \eta_1^{2j_1-x}\eta_2^{2j_2-x} C_x \\
= \;& -\frac{1}{2}\sum_x \eta_1^{2j_1-x}\eta_2^{2j_2-x} C_x \left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2} - \eta_1\frac{\partial}{\partial\eta_1} - \eta_2\frac{\partial}{\partial\eta_2}\right)(\xi_1\eta_2 - \xi_2\eta_1)^x + \\
& -\frac{1}{2}\sum_x (\xi_1\eta_2 - \xi_2\eta_1)^x \left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2} - \eta_1\frac{\partial}{\partial\eta_1} - \eta_2\frac{\partial}{\partial\eta_2}\right)\eta_1^{2j_1-x}\eta_2^{2j_2-x} C_x \\
= \;& 0 + \sum_x (\xi_1\eta_2 - \xi_2\eta_1)^x \eta_1^{2j_1-x}\eta_2^{2j_2-x} C_x (j_1 + j_2 - x) \\
= \;& j\psi_j{}^j
\end{aligned}$$

we get

$$\sum_x (\xi_1\eta_2 - \xi_2\eta_1)^x \eta_1^{2j_1-x}\eta_2^{2j_2-x}(j_1+j_2-j-x)C_x = 0 \tag{3.48}$$

(we see that $\mathbf{J}_z$ commutes with $\xi_1\eta_2 - \xi_2\eta_1$). It follows that $C_x = 0$ unless $x = j_1 + j_2 - j$, and therefore

$$\psi_j{}^j = C(\xi_1\eta_2 - \xi_2\eta_1)^{j_1+j_2-j}\eta_1^{j+j_1-j_2}\eta_2^{j-j_1+j_2}. \tag{3.49}$$

We now determine the constant $C$. Expanding the right-hand side of (3.49)

$$\begin{aligned}
\psi_j{}^j &= C\sum_{\ell=0}^{j_1+j_2-j}\binom{j_1+j_2-j}{\ell}(-1)^\ell(\xi_2\eta_1)^\ell(\xi_1\eta_2)^{j_1+j_2-j-\ell}\eta_1^{j+j_1-j_2}\eta_2^{j-j_1+j_2} \\
&= C\sum_{\ell=0}^{j_1+j_2-j}(-1)^\ell(j_1+j_2-j)!\frac{\eta_1^{\ell+j+j_1-j_2}\xi_1^{j_1+j_2-j-\ell}\eta_2^{2j_2-\ell}\xi_2^\ell}{(\ell)!(j_1+j_2-j-\ell)!} \\
&= C\sum_{m_1=j-j_2}^{j_1}(-1)^{j_2-j+m_1}(j_1+j_2-j)!\frac{\eta_1^{j_1+m_1}\xi_1^{j_1-m_1}\eta_2^{j_2+j-m_1}\xi_2^{j_2-j+m_1}}{(j_2-j+m_1)!(j_1-m_1)!} \\
&= C\sum_{m_1=j-j_2}^{j_1}(-1)^{j_1-m_1}(j_1+j_2-j)!\left[\frac{(j_1+m_1)!(j_2+j-m_1)!}{(j_1-m_1)!(j_2-j+m_1)!}\right]^{\frac{1}{2}}\times \\
&\quad \times(-1)^{j_1-m_1}\frac{\eta_1^{j_1+m_1}\xi_1^{j_1-m_1}}{\sqrt{(j_1+m_1)!(j_1-m_1)!}}(-1)^{j_2-j+m_1}\frac{\eta_2^{j_2+j-m_1}\xi_2^{j_2-j+m_1}}{\sqrt{(j_2+j-m_1)!(j_2-j+m_1)!}} \\
&= C\sum_{m_1=j-j_2}^{j_1}(-1)^{j_1-m_1}(j_1+j_2-j)!\left[\frac{(j_1+m_1)!(j_2+j-m_1)!}{(j_1-m_1)!(j_2-j+m_1)!}\right]^{\frac{1}{2}} f_{m_1}{}^{j_1} f_{j-m_1}{}^{j_2}
\end{aligned} \tag{3.50}$$

We must make the transformation from the $f_{m_1}{}^{j_1} f_{m_2}{}^{j_2}$ to the $\psi_m{}^j$ unitary. By (3.36)

$$\sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_m C^{(j_1j_2j)}{}_{m_1m_2}{}^m = 1. \tag{3.51}$$

Note that this condition is independent of the value of $m$, and so we may take $m = j$.

$$\sum_{m_1m_2} C^{(j_1j_2j)m_1m_2}{}_j C^{(j_1j_2j)}{}_{m_1m_2}{}^j = 1. \tag{3.52}$$

From (3.50) we have the condition

$$C^2[(j_1+j_2-j)!]^2\sum_m \frac{(j_1+m)!(j_2+j-m)!}{(j_1-m)!(j_2-j+m)!}=1. \tag{3.53}$$

We use the result (proved in appendix B) that:

$$\begin{aligned}\sum_{m=j-j_1}^{j_1}\frac{(j_1+m_1)!(j_2+j-m)!}{(j_1-m_1)!(j_2-j+m)!} &= (j_2-j_1+j)!(j_1-j_2+j)!\binom{j_1+j_2+j+1}{j_1+j_2-j}\\ &= \frac{(j+j_2-j_1)!(j+j_1-j_2)!(j_1+j_2+j+1)!}{(j_1+j_2-j)!(2j+1)!}.\end{aligned} \tag{3.54}$$

Substituting this into (3.53) we obtain

$$C=\left[\frac{(2j+1)!}{(j+j_1-j_2)!(j+j_2-j_1)!(j_1+j_2-j)!(j_1+j_2+j+1)!}\right]^{\frac{1}{2}} \tag{3.55}$$

The phase is arbitrary and is chosen to agree with convention.

By applying $\mathbf{J}_-$ to (3.49) $j-m$ times we find an explicit expression for $\psi_m^{\ j}$. Recalling that

$$\psi_m^{\ j}=\frac{1}{[(j+m+1)(j-m)]^{\frac{1}{2}}}\mathbf{J}_-\psi_{m+1}^{\ \ j}$$

we prove by induction that

$$\psi_m^{\ j}=\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}}(\mathbf{J}_-)^{j-m}\psi_j^{\ j} \tag{3.56}$$

$$\begin{aligned}\psi_m^{\ j} &= \frac{1}{[(j+m+1)(j-m)]^{\frac{1}{2}}}\mathbf{J}_-\psi_{m+1}^{\ \ j}\\ &= \frac{1}{[(j+m+1)(j-m)]^{\frac{1}{2}}}\mathbf{J}_-\left(\left[\frac{(j+m+1)!}{(j-m-1)!(2j)!}\right]^{\frac{1}{2}}(\mathbf{J}_-)^{j-m-1}\psi_j^{\ j}\right)\\ &= \left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}}(\mathbf{J}_-)^{j-m}\psi_j^{\ j}\end{aligned} \tag{3.57}$$

We use that $\mathbf{J}_- = -[\xi_1(\partial/\partial\eta_1) + \xi_2(\partial/\partial\eta_2)]$ commutes with $\xi_1\eta_2 - \xi_2\eta_1$ when applying $\mathbf{J}_-$ to (3.49)

$$\begin{aligned}
&\psi_m{}^j = \\
&= \left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}} (\mathbf{J}_-)^{j-m}\psi_j{}^j \\
&= C\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}} (-1)^{j-m}(\xi_1\eta_2 - \xi_2\eta_1)^{j_1+j_2-j}\left(\xi_1\frac{\partial}{\partial\eta_1} + \xi_2\frac{\partial}{\partial\eta_2}\right)^{j-m} \eta_1^{j+j_1-j_2}\eta_2^{j-j_1+j_2}
\end{aligned} \tag{3.58}$$

## 3.5 Example Calculation

### 3.5.1 First column

We first consider a specific example, the example given in section 3.3 where we consider $j_1 = 2$ and $j_2 = 4$. First consider the top states in further most column in figure 4.1 (b). Then $j = j_1 + j_2 = 6$ and using (3.49) we have

$$\begin{aligned}
\psi_6{}^6 &= C[4!8!]^{\frac{1}{2}}\frac{\eta_1^{2j_1}}{[4!]^{\frac{1}{2}}}\frac{\eta_2^{2j_2}}{[8!]^{\frac{1}{2}}} \\
&= \frac{\eta_1^{2j_1}}{[4!]^{\frac{1}{2}}}\frac{\eta_2^{2j_2}}{[8!]^{\frac{1}{2}}}
\end{aligned} \tag{3.59}$$

We see that for this top state the Clebsch-Gordan coefficient is $+1$.

Using the binomial theorem and Leibnitz rule:

$$\begin{aligned}
&\psi_m{}^j = \\
&= \left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}} (\mathbf{J}_-)^{j-m}\psi_j{}^j \\
&= C\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}} (-1)^{j-m}\left(\xi_1\frac{\partial}{\partial\eta_1} + \xi_2\frac{\partial}{\partial\eta_2}\right)^{j-m} \eta_1^{2j_1}\eta_2^{2j_2} \\
&\left\{\sum_{x=0}^{j-m}\binom{j-m}{x}\left(\xi_2^x\frac{\partial^x}{\partial\eta_2^x}\eta_2^{j-j_1+j_2}\right)\left(\xi_1^{j-m-x}\frac{\partial^{j-m-x}}{\partial\eta_1^{j-m-x}}\eta_1^{j+j_1-j_2}\right)\right\}
\end{aligned} \tag{3.60}$$

First:

$$
\begin{aligned}
\psi_{j-1}{}^{j} &= C\left[\frac{(2j+1)!}{(2j)!}\right]^{\frac{1}{2}}(-1)\left(\xi_1\frac{\partial}{\partial\eta_1}+\xi_2\frac{\partial}{\partial\eta_2}\right)\eta_1^{2j_1}\eta_2^{2j_2} \\
&= C\left[\frac{(2j+1)!}{(2j)!}\right]^{\frac{1}{2}}(-1)\left(2j_1\eta_1^{2j_1-1}\xi_1\eta_2^{2j_2}+2j_2\eta_1^{2j_1}\eta_2^{2j_2-1}\xi_2\right) \qquad (3.61)
\end{aligned}
$$

Bottom state:

$$
\begin{aligned}
\psi_{-6}{}^{6} &= C\left[\frac{1}{(2j)!}\right]\left(\xi_1\frac{\partial}{\partial\eta_1}+\xi_2\frac{\partial}{\partial\eta_2}\right)^{2j_1+2j_2}\eta_1^{2j_1}\eta_2^{2j_2} \\
&= C\left[\frac{1}{(2j)!}\right]\sum_{x=0}^{2j}\binom{2j}{x}\left(\xi_2^x\frac{\partial^x}{\partial\eta_2^x}\eta_2^{2j_2}\right)\left(\xi_1^{2j_1+2j_2-x}\frac{\partial^{2j_1+2j_2-x}}{\partial\eta_1^{2j_1+2j_2-x}}\eta_1^{2j_1}\right) \\
&= C\left[\frac{1}{(2j)!}\right]\frac{(2j)!}{(2j_1)!(2j_2)!}(2j_1)!\xi_1^{2j_1}(2j_2)!\xi_2^{2j_2} \\
&= \frac{\xi_1^{2j_1}}{[(2j_1)!]^{\frac{1}{2}}}\frac{\xi_2^{2j_2}}{[(2j_2)!]^{\frac{1}{2}}} \qquad (3.62)
\end{aligned}
$$

### 3.5.2 Second column

$j = j_1 + j_2 - 1$. Using (3.49) we have

$$
\begin{aligned}
\psi_j{}^j &= C(\xi_1\eta_2-\xi_2\eta_1)\eta_1^{2j_1-1}\eta_2^{2j_2-1} \\
&= C(\eta_1^{2j_1-1}\xi_1\eta_2^{2j_2}-\eta_1^{2j_1}\eta_2^{2j_2-1}\xi_2) \qquad (3.63)
\end{aligned}
$$

The state with the highest value of $m_2$ is real and $\geq 0$.

### 3.5.3 Third column

$j = j_1 + j_2 - 2$. Using (3.49) we have

$$
\begin{aligned}
\psi_j{}^j &= C(\xi_1\eta_2-\xi_2\eta_1)^2\eta_1^{2j_1-2}\eta_2^{2j_2-2} \\
&= C(\eta_1^{2j_1-2}\xi_1^2\eta_2^{2j_2}-) \qquad (3.64)
\end{aligned}
$$

The state with the highest value of $m_2$ is real and $\geq 0$.

$$\psi_j{}^j = C(\xi_1\eta_2 - \xi_2\eta_1)^{j_1+j_2-j}\eta_1^{j+j_1-j_2}\eta_2^{j-j_1+j_2}. \tag{3.65}$$

## 3.6 General Calculation

Using the binomial theorem and Leibnitz rule:

$$\begin{aligned}
\psi_m{}^j &= C\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}}(-1)^{j-m}(\mathbf{J}_-)^{j-m}\psi_j{}^j \\
&= C\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}}(-1)^{j-m}\left\{\sum_{\ell=0}^{j_1+j_2-j}\binom{j_1+j_2-j}{\ell}(-1)^\ell(\xi_2\eta_1)^\ell(\xi_1\eta_2)^{j_1+j_2-j-\ell}\right\} \\
&\quad \left\{\sum_{x=0}^{j-m}\binom{j-m}{x}\left(\xi_2^x\frac{\partial^x}{\partial\eta_2^x}\eta_2^{j-j_1+j_2}\right)\left(\xi_1^{j-m-x}\frac{\partial^{j-m-x}}{\partial\eta_1^{j-m-x}}\eta_1^{j+j_1-j_2}\right)\right\} \\
&= C\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}}(-1)^{j-m}\left\{\sum_{\ell=0}^{j_1+j_2-j}\binom{j_1+j_2-j}{\ell}(-1)^\ell\eta_1^\ell\xi_1^{j_1+j_2-j-\ell}\eta_2^{j_1+j_2-j-\ell}\xi_2^\ell\right\} \\
&\quad \left\{\sum_{x=0}^{j-m}\binom{j-m}{x}\xi_1^{j-m-x}\xi_2^x\left(\frac{(j+j_1-j_2)!}{(j_1-j_2+m+x)!}\eta_1^{m+j_1-j_2+x}\right)\times\right. \\
&\quad \left.\times\left(\frac{(j-j_1+j_2)!}{(j-j_1+j_2-x)!}\eta_2^{j-j_1+j_2-x}\right)\right\} \\
&= C\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}}(-1)^{j-m}\left\{\sum_{x=0}^{j-m}\sum_{\ell=0}^{j_1+j_2-j}\eta_1^{\ell+m+j_1-j_2+x}\xi_1^{j_1+j_2-\ell-m-x}\eta_2^{2j_2-\ell-x}\xi_2^{\ell+x}\right. \\
&\quad \left.(-1)^\ell\binom{j_1+j_2-j}{\ell}\binom{j-m}{x}\frac{(j+j_1-j_2)!}{(j_1-j_2+m+x)!}\frac{(j-j_1+j_2)!}{(j-j_1+j_2-x)!}\right\}
\end{aligned} \tag{3.66}$$

If the substitution $\ell = j_2 - m + m_1 - x$ is made for the dummy summation variable

$$\begin{aligned}
\psi_m{}^j &= C\left[\frac{(j+m)!}{(j-m)!(2j)!}\right]^{\frac{1}{2}}(-1)^{j-m}\left\{\sum_{x=0}^{j-m}\sum_{m_1=m-j_2+x}^{j_1-j+m+x}\eta_1^{j_1+m_1}\xi_1^{j_1-m_1}\eta_2^{j_2+m-m_1}\xi_2^{j_2+m_1-m}\right. \\
&\quad \left.(-1)^{j_2-m+m_1-x}\binom{j_1+j_2-j}{j_2-m+m_1-x}\binom{j-m}{x}\frac{(j+j_1-j_2)!}{(j_1-j_2+m+x)!}\frac{(j-j_1+j_2)!}{(j-j_1+j_2-x)!}\right\}
\end{aligned} \tag{3.67}$$

or

$$
\begin{aligned}
&{\psi_m}^j \\
= \;& C\left\{\sum_{x=0}^{j-m}\sum_{m_1=m-j_2}^{j_1-j+m+x}\left[\frac{(j+m)!}{(j-m)!(2j)!}(j_1+m_1)!(j_1-m_1)!(j_2+m-m_1)!(j_2+m_1-m)!\right]^{\frac{1}{2}}(-1)^{j-m}\right. \\
& (-1)^{-j_1-j_2-m}(-1)^{j_1-m_1}\frac{\eta_1^{j_1+m_1}\xi_1^{j_1-m_1}}{[(j_1+m_1)!(j_1-m_1)!]^{\frac{1}{2}}}(-1)^{j_2-m+m_1}\frac{\eta_2^{j_2+m-m_1}\xi_2^{j_2+m_1-m}}{[(j_2+m-m_1)!(j_2+m_1-m)!]^{\frac{1}{2}}} \\
& \left.(-1)^{j_2-m+m_1-x}\binom{j_1+j_2-j}{j_2-m+m_1-x}\binom{j-m}{x}\frac{(j+j_1-j_2)!}{(j_1-j_2+m+x)!}\frac{(j-j_1+j_2)!}{(j-j_1+j_2-x)!}\right\}
\end{aligned}
\tag{3.68}
$$

or

$$
\begin{aligned}
&{\psi_m}^j \\
= \;& C\left\{\sum_{x=0}^{j-m}\sum_{m_1=m-j_2}^{j_1-j+m+x}\left[\frac{(j+m)!}{(j-m)!(2j)!}(j_1+m_1)!(j_1-m_1)!(j_2+m-m_1)!(j_2+m_1-m)!\right]^{\frac{1}{2}}\right. \\
& (-1)^{j-m}(-1)^{-j_1-j_2-m}{f_{m_1}}^{j_1}{f_{m-m_1}}^{j_1} \\
& \left.(-1)^{j_2-m+m_1-x}\binom{j_1+j_2-j}{j_2-m+m_1-x}\binom{j-m}{x}\frac{(j+j_1-j_2)!}{(j_1-j_2+m+x)!}\frac{(j-j_1+j_2)!}{(j-j_1+j_2-x)!}\right\}
\end{aligned}
\tag{3.69}
$$

We can now pick out the coefficient of ${f_{m_1}}^{j_1}{f_{m_2}}^{j_2}$; it is the Clebsch-Gordan coefficient we wish to determine:

$$
\begin{aligned}
&{C^{(j_1j_2j)m_1m_2}}_m = (-1)^{j+j_-m+m_1}\Bigg[(2j-1)(j+m)!(j-m)!\times \\
&\times\frac{(j_1+m_1)!(j_2+m_2)!(j_1-m_1)!(j_2-m_2)!(j_1+j_2-j)!(j+j_1-j_2)!(j-j_1+j_2)!}{(j+j_1+j_2+1)!}\Bigg]^{\frac{1}{2}}\delta_{m_1+m_2,m} \\
&\times\sum_x(-1)^x\frac{1}{x!(j-m-x)!(j_2-m_2x)!(j_1-j+m_2+x)!(j_1-j_2+m+x)!(j-j_1+j_2-x)!}
\end{aligned}
\tag{3.70}
$$

If the substitution $x=j-j_1-m_2+x'$ is made for the dummy summation variable, this agrees with the formula as given by Racah.

# Chapter 4

# Symmetry Properties of the Clebsch-Gordan coefficients and the $3j$-Symbol

## 4.1 Sketch of Symmetry Properties

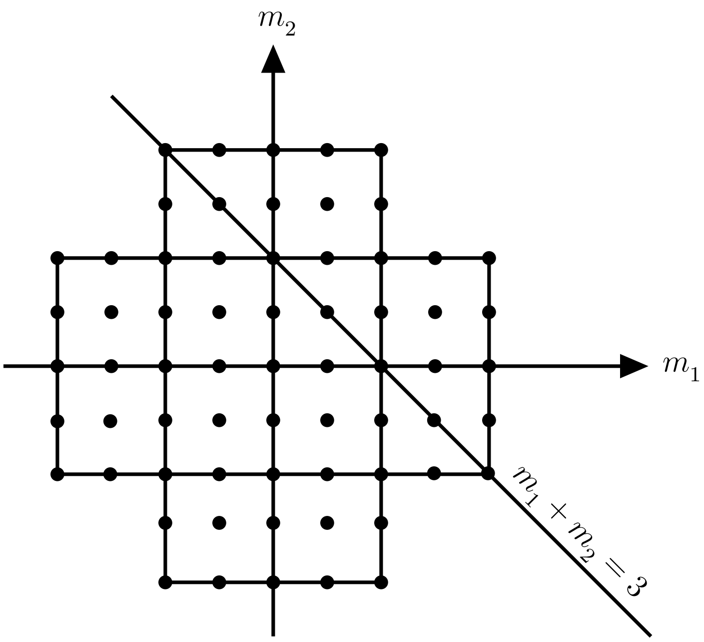


Figure 4.1:

## 4.2 Symmetry Properties of the Clebsch-Gordan coefficients

$$C^{(j_2 j_1 j_3) m_2 m_1}{}_{m_3} = (-1)^{-j_1 - j_2 + j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{m_3} \tag{4.1}$$

$$C^{(j_1 j_2 j_3) -m_1 -m_2}{}_{-m_3} = (-1)^{-j_1 - j_2 + j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{m_3} \tag{4.2}$$

$$C^{(j_1 j_3 j_2) m_1 -m_3}{}_{-m_2} = (-1)^{-j_1 + m_1} \left( \frac{2j_2 + 1}{2j_3 + 1} \right)^{\frac{1}{2}} C^{(j_1 j_2 j_3) m_1 m_2}{}_{m_3} \tag{4.3}$$

These three symmetry relations are easily derived from (). First consider the square root expression. Inspection of the shows that it is invariant under the interchanges carried out in (4.1) and (4.2). There are phase changes that come from the sum over $x$.

It should be noticed that chaqnging the signs of the three magnetic quantum numbers in this sum, giving

$$\sum_x \frac{(-1)^x}{x!} [(j_1 + j_2 - j_3 - x)!(j_1 + m_1 - x)!(j_3 - j_2 - m_1 + x)!(j_3 - j_1 + m_2 + x)! \tag{4.4}$$

is equivalent to interchanging $j_1$ with $j_2$ and $m_1$ with $m_2$. This verifies that the phases in (4.1) and (4.2) must be the same. If a new dummy sumation variable $x'$ is introduced by

$$x = j_1 + j_2 - j_3 - x$$

This sum over $x$ in () becomes

$$\begin{aligned} &(-1)^{j_1 + j_2 - j_3} \sum_{x'} \frac{(-1)^{x'}}{x'!} [(j_1 + j_2 - j_3 - x')!(j_3 - j_2 + m_1 + x')! \\ &\times (j_3 - j_1 - m_2 + x')!(j_1 - m_1 - x')!(j_2 + m_2 - x')!]^{-1} \end{aligned} \tag{4.5}$$

## 4.3 The $3j$-Symbol

### 4.3.1 Definition of the $3j$-symbol

The Wigner $3j$-symbol is defined by contracting the Clebsch-Gordan coefficient with the $\epsilon$ tensor, and multiplying with a numerical factor (which is inserted in order to optimise the symmetry properties of the resulting object):

$$\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \frac{1}{\sqrt{2j_3+1}}(-1)^{j_1-j_2+j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{n} \epsilon^{(j_3)}_{n m_3} \\
&= \frac{1}{\sqrt{2j_3+1}}(-1)^{j_1-j_2+j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{n} (-1)^{-2j_3} \epsilon^{(j_3)}_{m_3 n} \\
&= \frac{1}{\sqrt{2j_3+1}}(-1)^{j_1-j_2-j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{n} (-1)^{j_3-m_3} \delta_{m_3,-n} \\
&= \frac{1}{\sqrt{2j_3+1}}(-1)^{j_1-j_2-m_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{-m_3} \qquad (4.6)
\end{aligned}$$

### 4.3.2 Symmetry properties of the $3j$-symbol

Interchanging any two columns in the symbol produces a factor of $(-1)^{j_1+j_2+j_3}$. We first prove this when interchanging the first and second column. Note that the relation (4.1) is independent of the value of $m_3$ and so we may write

$$C^{(j_2 j_1 j_3) m_2 m_1}{}_{-m_3} = (-1)^{-j_1-j_2+j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{-m_3}$$

Using this we obtain

$$
\begin{aligned}
&\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \\
&= \frac{1}{\sqrt{2j_3+1}}(-1)^{-j_1+j_2-m_3} C^{(j_2 j_1 j_3) m_2 m_1}{}_{-m_3} \\
&= \frac{1}{\sqrt{2j_3+1}}(-1)^{-j_1+j_2-m_3} \left[(-1)^{-j_1-j_2+j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{-m_3}\right] \\
&= (-1)^{-j_1+j_2}(-1)^{-j_1-j_2+j_3}(-1)^{-j_1+j_2} \left[\frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} C^{(j_1 j_2 j_3) m_2 m_1}{}_{-m_3}\right] \\
&= (-1)^{j_1+j_2+j_3-4j_1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\
&= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
\end{aligned} \tag{4.7}
$$

Next

$$
\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \frac{1}{\sqrt{2j_3+1}}(-1)^{j_1-j_2+j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{n} (-1)^{j_3-m_3} \delta_{n,m_3} \\
&= \frac{1}{\sqrt{2j_3+1}}(-1)^{j_1-j_2+j_3} C^{(j_1 j_2 j_3) m_1 m_2}{}_{-m_3}
\end{aligned} \tag{4.8}
$$

This symmetry implies in particular that the symbol under cyclic permutations of its columns.

Comparing (4.1) and (4.2) we see that the same phase factor results from reversing the signs of all the magnetic numbers:

$$
\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{4.9}
$$

# Appendix A

# Euler's Theorem

Suppose that a function $f(z_1, \ldots, z_n)$ is homogeneous of degree $D$ in the variables $z_1, \ldots, z_n$. By definition this means that, on multiplying each $z_i$ by an arbitrary factor $\rho$, the value of $f$ is multiplied by $\rho^D$:

$$f(\rho z_1, \ldots, \rho z_n) = \rho^D f(z_1, \ldots, z_n). \tag{A.1}$$

Let us differentiate both sides with respect to $\rho$ and multiply by $\rho$. We find that

$$\rho \left[ \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} \right] f(\rho z_1, \ldots, \rho z_n) = D \rho^D f(z_1, \ldots, z_n). \tag{A.2}$$

Putting $\rho = 1$ we obtain

$$\left[ \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} \right] f(z_1, \ldots, z_n) = D f(z_1, \ldots, z_n). \tag{A.3}$$

## A.1 Homogeneous function of one variable

By Euler's equation (A.3) we have

$$z \frac{df(z)}{dz} = D f(\rho z) \tag{A.4}$$

or

$$\frac{d\ln[f(z)]}{d\ln z} = D \tag{A.5}$$

which is easily solved

$$\ln[f(z)] - \ln[f(1)] = D\ln z \tag{A.6}$$

so we have

$$f(z) = cz^D \tag{A.7}$$

where $c = f(1)$.

# Appendix B

# A Summation

## B.1 Binomial identities

### B.1.1 Negative binomial identity

We first define a negative binomial. This is given by

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k \tag{B.1}$$

We obviously have

$$\begin{aligned}
\binom{-n}{0} &= 1 \\
\binom{-n}{1} &= (-n) \\
\binom{-n}{2} &= (-n)(-n-1)/2! \\
\binom{-n}{3} &= (-n)(-n-1)(-n-2)/3! \\
&\vdots \\
\binom{-n}{k} &= (-n)(-n-1)(-n-k+1)/k!.
\end{aligned} \tag{B.2}$$

We now prove the negative binomial identity,

$$\binom{-n}{k} = \binom{n+k-1}{k}(-1)^k. \tag{B.3}$$

There is a combinatorial proof: $\binom{n+k-1}{k}$ is the number of ways of placing k indistinguishable balls, o, in n boxes. Consider $k$ balls

$$\underbrace{\text{oo}\cdots\text{o}}_{k \text{ balls}}$$

and insert $n-1$ bars to form $n$ boxes

$$\underbrace{\text{oo}}_{k_1 \text{ balls}} \mid \underbrace{\text{ooo}}_{k_2 \text{ balls}} \mid \cdots \mid \underbrace{\text{oooo}}_{k_n \text{ balls}}$$

which is the same as counting the number of ways we can have: $\sum_{i=1}^{n} k_i = k$. This is the same as counting the number of times you get the term $x^k$ in the expansion

$$(1 + x + \cdots + x^{k_1} + \cdots)(1 + x + \cdots + x^{k_2} + \cdots)\cdots(1 + x + \cdots + x^{k_n} + \cdots).$$

And so

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k. \tag{B.4}$$

Comparing this to (B.1) gives

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k x^k \tag{B.5}$$

which gives

$$\binom{n+k-1}{k} = \binom{-n}{k}(-1)^k \tag{B.6}$$

### B.1.2 Vandermonde identity

The Vandermonde identity

$$\binom{m+n}{r} = \sum_{k=0}^{m+n} \binom{n}{k}\binom{m}{r-k}. \tag{B.7}$$

is easily proved. We have

$$\begin{aligned}
\sum_{r=0}^{m+n} \binom{m+n}{r} x^r &= (1+x)^n(1+x)^m \\
&= \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right) \\
&= \sum_{r=0}^{m+n} \left(\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}\right) x^r
\end{aligned} \tag{B.8}$$

where we have relied on the convention that the binomial coefficients are zero for all $i > m$ and $j > n$. Equating coefficients gives (B.7).

### B.1.3 Chu-Vandermonde identity

First we have

$$(1+x)^{-m-n} = \sum_{r=0}^{\infty} \binom{-m-n}{r} x^r \tag{B.9}$$

We also have

$$\begin{aligned}
(1+x)^{-m-n} &= (1+x)^{-m}(1+x)^{-n} \\
&= \left(\sum_{i=0}^{\infty} \binom{-m}{i} x^i\right)\left(\sum_{j=0}^{\infty} \binom{-n}{i} x^j\right) \\
&= \sum_{r=0}^{\infty} \left(\sum_{k=0}^{r} \binom{-m}{k}\binom{-n}{r-k}\right) x^r.
\end{aligned} \tag{B.10}$$

Equating coefficients we have

$$\binom{-m-n}{r} = \sum_{k=0}^{r} \binom{-m}{i} \binom{-n}{r-k} \tag{B.11}$$

or upon using $s = -m$ and $t = -n$,

$$\binom{s+t}{r} = \sum_{k=0}^{r} \binom{s}{k} \binom{t}{r-k}. \tag{B.12}$$

### B.1.4 Main identity

We wish to prove

$$\sum_{m=j-j_1}^{j_1} \frac{(j_1+m_1)!(j_2+j-m)!}{(j_1-m_1)!(j_2-j+m)!} = (j_2-j_1+j)!(j_1-j_2+j)! \binom{j_1+j_2+j+1}{j_1+j_2-j}. \tag{B.13}$$

This can be written

$$\sum_{m=j-j_1}^{j_1} \binom{j_1+m}{j_2-j+m} \binom{j_2+j-m}{j_1-m} = \binom{j_1+j_2+j+1}{j_1+j_2-j}. \tag{B.14}$$

We obtain

$$\begin{aligned}
&\sum_{m=j-j_1}^{j_1} \binom{j_1+m}{j_2-j+m} \binom{j_2+j-m}{j_1-m} = \\
&= \sum_{m=j-j_1}^{j_1} \binom{-j_1+j_2-j-1}{j_2-j+m} (-1)^{-j_2-j+m} \binom{-j_2-j+j_1-1}{j_1-m} (-1)^{j_1-m} \\
&= (-1)^{j_1+j_2-j} \sum_{m=0}^{j_1+j_2-j} \binom{-j_1+j_2-j-1}{m} \binom{-j_2-j+j_1-1}{j_1+j_2-j-m} \\
&= (-1)^{j_1+j_2-j} \binom{-2j-2}{j_1+j_2-j} \\
&= \binom{j_1+j_2+j+1}{j_1+j_2-j}.
\end{aligned} \tag{B.15}$$

where in the second line we have used the negative binomial identity (B.3), in the third line we have adjusted $m$ so that we sum from 0, in the fourth line we have used the Chu-Vandermonde identity (B.12), and in the last line we have used the negative binomial identity (B.6).