Raychaundhuri Equations

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Chapter 1

Introduction

1.1 Timelike Congruences

We are considering a smooth congruence of timelike geodesics. The geodesics are parameterized by proper tiem τ...

....so that the vector fields V^a of tangents is normalised to unit length: $V^a V_a = -1$.

We define the transverse metric or "spatial metric" by

$$h_{ab} = g_{ab} + V_a V_b. \tag{1.1}$$

It is easily seen that the four tensor

$$h^a{}_b = g^{ac} h_{cb} = \delta^a{}_b + V^a V_b \tag{1.2}$$

is the projection operator onto the subspace of the tangent space perpendicular to V_a as

$$h^a{}_b V^b = (\delta^a{}_b + V^a V_b) V^b = 0 \quad \text{and if} \quad W^a V_a = 0 \quad \text{then} \quad h^a{}_b W^b = W^a. \tag{1.3}$$

Also

$$h^a_c h^c_b = h^a_c (\delta^c_b + V_b V^c) = h^a_b \tag{1.4}$$

which is the property of a projector.

Also we have

$$h^{ab}h_b^{\ c} = h_b^{\ c}(g^{ab} + V^aV^b) = h^{ac}. \tag{1.5}$$

Similarly

$$h_a^{\ b}h_{bc} = h_a^{\ b}(g_{bc} + V_bV_c) = h_{ac}. \tag{1.6}$$

And

$$\begin{aligned} h^{ab}h_{bc} = h^{ab}(g_{bc} + V_bV_c) &= h^a_{\ c} + V_c(g^{ab} + V^aV^b)V_b \\ &= h^a_{\ c} \end{aligned} \tag{1.7}$$

And

$$\begin{aligned} h_{ab}h^{ab} &= h_{ab}(g^{ab} + V^aV^b) \\ &= h_{ab}g^{ab} \\ &= (g_{ab} + V_aV_b)g^{ab} \\ &= 3. \end{aligned} \tag{1.8}$$

1.2 The expansion, twist and shear of the field

The expansion, shear and twist of the field are defined by

$$\theta_{ab} = h_a^{\ c}h_b^{\ d}\nabla_{(d}V_{c)} \tag{1.9}$$

$$\begin{aligned} \theta &= h^{ab}h_a^{\ c}h_b^{\ d}\nabla_{(d}V_{c)} \\ &= h^{cd}\nabla_{(d}\eta_{c)} \end{aligned} \tag{1.10}$$

The shear tensor is

$$\sigma_{ab} = h_a^{\ c}h_b^{\ d}\nabla_{(d}V_{c)} - \frac{1}{3}\theta h_{ab}. \tag{1.11}$$

It is symmetric, i.e. $\sigma_{ab} = \sigma_{ba}$, and trace-free:

$$
\begin{aligned}
h^{ab}\sigma_{ab} &= h^{ab}(h_a{}^c h_b{}^d \nabla_{(d}V_{c)} - \frac{1}{3}\theta h_{ab}) \\
&= h^{cd}\nabla_{(d}V_{c)} - \frac{1}{3}\theta h^{ab}h_{ab} \\
&= \theta - \frac{1}{3}3\theta = 0.
\end{aligned} \tag{1.12}
$$

The vorticity tensor is

$$
\omega_{ab} = h_a{}^c h_b{}^d \nabla_{[d}V_{c]}. \tag{1.13}
$$

the expansion is

$$
\begin{aligned}
\theta &= h^{ab}h_a{}^c h_b{}^d \nabla_{(d}V_{c)} \\
&= h^{cd}\nabla_d V_c.
\end{aligned} \tag{1.14}
$$

Note that we can raise the indices of θ_{ab}, σ_{ab}, and ω_{ab} using either h^{ab} or g^{ab}. Take the shear tensor for example,

$$
\begin{aligned}
h^{ae}\sigma_{eb} &= h^{ae}(h_e{}^c h_b{}^d \nabla_{(d}V_{c)} - \frac{1}{3}\theta h_{eb}) \\
&= h^{ac}h_b{}^d \nabla_{(d}V_{c)} - \frac{1}{3}\theta h^a{}_b \\
&= g^{ae}\sigma_{eb} \\
&= \sigma^a{}_b.
\end{aligned} \tag{1.15}
$$

where we have used (1.5) and (1.7).

1.3 As we have geodesics: The expansion, twist and shear of the field

The expansion, shear and twist of the field can then be simplified

$$\begin{aligned}
\theta_{ab} &= h_a{}^c h_b{}^d \nabla_{(d} V_{c)} \\
&= h_a{}^c (g_b{}^d + V_b V^d) \nabla_{(d} V_{c)} \\
&= h_a{}^c \frac{1}{2} [g_b{}^d \nabla_d V_c + \nabla_c (g_b{}^d V_d)] \\
&= h_a{}^c \nabla_{(b} V_{c)} \\
&= \nabla_{(b} V_{c)} \qquad (1.16)
\end{aligned}$$

where we have used the geodesic equation $V^c \nabla_c V_a = 0$ and the identity $V^c \nabla_a V_c = 0$.

$$\begin{aligned}
\theta &= h^{ab} h_a{}^c h_b{}^d \nabla_{(d} V_{c)} \\
&= h^{cd} \nabla_{(d} \eta_{c)} \qquad (1.17)
\end{aligned}$$

The shear tensor simplifies,

$$\begin{aligned}
\sigma_{ab} &= h_a{}^c h_b{}^d \nabla_{(d} V_{c)} - \frac{1}{3} \theta h_{ab} \\
&= \nabla_{(b} V_{a)} - \frac{1}{3} \theta h_{ab} \qquad (1.18)
\end{aligned}$$

where we used (1.16).

The vorticity tensor simplifies,

$$\begin{aligned}
\omega_{ab} &= h_a{}^c h_b{}^d \nabla_{[d} V_{c]} \\
&= \nabla_{[b} V_{a]}. \qquad (1.19)
\end{aligned}$$

where again we have used the geodesic equation $V^c \nabla_c V_a = 0$ and the identity $V^c \nabla_a V_c = 0$.

The expansion is

$$\begin{aligned}
\theta &= h^{ab} h_a{}^c h_b{}^d \nabla_{(d} V_{c)} \\
&= h^{cd} \nabla_d V_c. \qquad (1.20)
\end{aligned}$$

Chapter 2

The Raychaudhuri equations

Note that $\nabla_b V_a$ can be decomposed as

$$\nabla_b V_a = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} \tag{2.1}$$

$$\begin{aligned} V^c \nabla_c (\nabla_b V_a) &= V^c \nabla_b \nabla_c V_d + V^c (\nabla_c \nabla_b - \nabla_b \nabla_c) V_d \\ &= V^c \nabla_b \nabla_c V_d + R_{cba}{}^d V^c V_d \\ &= \nabla_b (V^c \nabla_c V_d) - (\nabla_b V^c)(\nabla_c V_a) + R_{cba}{}^d V^c V_d \\ &= -(\nabla^c V_b)(\nabla_a V_c) + R_{cba}{}^d V^c V_d \end{aligned} \tag{2.2}$$

where we used $V^c \nabla_c V_d = 0$. In the following it should be kept in mind that θ_{ab} and σ_{ab} are symmetric while ω_{ab} is anitsymmetric. Taking the trace over a and b we obtain,

2.1 Raychaundhuri equation for scalar expansion

$$\begin{aligned} V^c \nabla_c \theta &= -\left(\frac{1}{3}\theta h^{ca} + \sigma^{ca} + \omega^{ca}\right)\left(\frac{1}{3}\theta h_{ac} + \sigma_{ac} + \omega_{ac}\right) - R_{cd} V^c V^d \\ &= -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd} V^c V^d \end{aligned} \tag{2.3}$$

and so

$$\frac{d\theta}{ds} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab} - R_{cd} V^c V^d \tag{2.4}$$

where $\omega_{ab}\omega^{ab} \geq 0$ and $\sigma_{ab}\sigma^{ab} \geq 0$. This is the Raychaudhuri equation and is of great importance for the singularity theorem. Note that vorticity induces expansion which is in analogy with centrifugal force while shear induces cotraction.

2.2 The trace-free, symmetric part

The trace-free part of a tensor X_{ab} is

$$T_{ab} = X_{ab} - \frac{1}{3}h_{ab}h^{cd}X_{cd}$$

The symmetric part of a tensor T_{ab} is

$$\frac{1}{2}(T_{ab} + T_{ba}) \equiv T_{(ab)} = X_{(ab)} - \frac{1}{3}h_{ab}h^{cd}X_{cd}.$$

Or

$$T_{(ab)} = \left(\delta^c_{(a}\delta^d_{b)} - \frac{1}{3}h_{ab}h^{cd}\right)X_{cd}.$$

2.2.1 The trace-free, symmetric part of $V^c\nabla_c(\nabla_b V_a)$

We wish to obtain the trace-free, symmetric part of the equation $V^c\nabla_c(\nabla_b V_a) = -(\nabla_b V^c)(\nabla_c V_a) + R_{cbad}V^cV^d$. First consider the LHS, the symmetric, trace-free part is

$$V^c\nabla_c(\nabla_{(b}V_{a)}) - \frac{1}{3}h_{ab}h^{ef}V^c\nabla_c(\nabla_f V_e). \tag{2.5}$$

We have

$$\begin{aligned} & V^c\nabla_c(h_{ab}h^{ef}\nabla_f V_e) = \\ = \; & (V^c\nabla_c h_{ab})h^{ef}\nabla_f V_e + h_{ab}(V^c\nabla_c h^{ef})\nabla_f V_e + h_{ab}h^{ef}V^c\nabla_c(\nabla_f V_e), \end{aligned}$$

but as $V^c\nabla_c h_{ab} = V^c\nabla_c(g_{ab} + V_a V_b) = 0$, where we used the geodesic equation, we have $V^c\nabla_c(h_{ab}h^{ef}\nabla_f V_e) = h_{ab}h^{ef}V^c\nabla_c(\nabla_f V_e)$. Using this in (2.5) gives

$$V^c\nabla_c(\nabla_{(b}V_{a)}) - \frac{1}{3}h_{ab}h^{ef}V^c\nabla_c(\nabla_f V_e) = V^c\nabla_c\big(\nabla_{(b}V_{a)} - \frac{1}{3}h_{ab}h^{ef}\nabla_f V_e\big) = V^c\nabla_c\sigma_{ab}. \tag{2.6}$$

We now consider the symmetric, trace-free part of the RHS:

$$\begin{aligned}
&V^c\nabla_c\sigma_{ab} = \\
&= -\left[(\nabla_c V_{(a})(\nabla_{b)}V^c) - \frac{1}{3}h_{ab}h^{ef}(\nabla_c V_e)(\nabla_f V^c)\right] + R_{c(ba)d}V^cV^d \\
&-\frac{1}{3}h_{ab}h^{ef}R_{cfed}V^cV^d \\
&= -\frac{1}{2}\left[\left(\frac{1}{3}\theta h_{ac} + \sigma_{ac} + \omega_{ac}\right)\left(\frac{1}{3}\theta h^c{}_b + \sigma^c{}_b + \omega^c{}_b\right) + a \leftrightarrow b\right] \\
&+\frac{1}{3}h_{ab}h^{ef}\left(\frac{1}{3}\theta h_{ec} + \sigma_{ec} + \omega_{ec}\right)\left(\frac{1}{3}\theta h^c{}_f + \sigma^c{}_f + \omega^c{}_f\right) \\
&+R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}h^{ef}R_{cfed}V^cV^d \\
&= -\left(\frac{1}{9}\theta^2 h_{ab} + \frac{2}{3}\theta\sigma_{ab} + \sigma_{ac}\sigma^c{}_b + \omega_{ac}\omega^c{}_b\right) \\
&+\frac{1}{3}h_{ab}h^{ef}\left(\frac{1}{3}\theta^2 + \frac{2}{3}\theta h^{ef}\sigma_{ef} + \frac{2}{3}\theta h^{ef}\omega_{ef} + \sigma_{ec}\sigma^{ce} + 2\sigma_{ec}\omega^{ce} + \omega_{ec}\omega^{ce}\right) \\
&+R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}h^{ef}R_{cfed}V^cV^d \\
&= -\left(\frac{1}{9}\theta^2 h_{ab} + \frac{2}{3}\theta\sigma_{ab} + \sigma_{ac}\sigma^c{}_b + \omega_{ac}\omega^c{}_b\right) + \frac{1}{3}h_{ab}\left(\frac{1}{3}\theta^2 + \sigma_{cd}\sigma^{cd} - \omega_{cd}\omega^{cd}\right) \\
&+R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}h^{ef}R_{cfed}V^cV^d \\
&= -\frac{2}{3}\theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + \frac{1}{3}h_{ab}\left(\sigma_{cd}\sigma^{cd} - \omega_{cd}\omega^{cd}\right) \\
&+R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}h^{ef}R_{cfed}V^cV^d
\end{aligned} \tag{2.7}$$

where we have used the anti-symetry of ω_{ab}.

2.2.2 Identities and an expression for the shear

The trace-free part of $R_{cbad}V^cV^d$ can be rewritten,

$$\begin{aligned}
R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}h^{ef}R_{cfed}V^cV^d &= R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}(g^{ef} + V^eV^f)R_{cfed}V^cV^d \\
&= R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}g^{ef}R_{cfed}V^cV^d \\
&= R_{c(ba)d}V^cV^d + \frac{1}{3}h_{ab}g^{ef}R_{fced}V^cV^d \\
&= R_{c(ba)d}V^cV^d + \frac{1}{3}h_{ab}R_{cd}V^cV^d. \qquad (2.8)
\end{aligned}$$

where we have used $R_{cfed}V^cV^dV^eV^f \equiv 0$ by the identities of the Riemann tensor, for example $R_{cfed} = -R_{cfde}$.

$$\begin{aligned}
-\frac{1}{3}h_{ab}V^c\nabla_c\theta &= -\frac{1}{3}h_{ab}V^c\nabla_c(h^{ef}\nabla_f V_e) \\
&= -\frac{1}{3}h_{ab}h^{ef}V^c\nabla_c(\nabla_f V_e) \qquad (2.9)
\end{aligned}$$

Which together with (2.2) gives

$$\begin{aligned}
& V^c\nabla_c(\nabla_{(b}V_{a)}) - \frac{1}{3}h_{ab}h^{ef}V^c\nabla_c(\nabla_f V_e) \\
= \; & -(\nabla_c V_{(a})(\nabla_{b)}V^c) + R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}V^c\nabla_c\theta \qquad (2.10)
\end{aligned}$$

This equation will establish the equivlence between taking the symmetric, trace-free part of (2.2) and $V^c\nabla_c\sigma_{ab}$, which will be considered in the next section.

2.2.3 Derivative of the shear

Here we directly take the derivative, $V^c\nabla_c$, of the shear $\sigma_{ab} = \nabla_{(b}V_{a)} - \frac{1}{3}\theta h_{ab}$,

$$
\begin{aligned}
V^c\nabla_c\sigma_{ab} &= V^c\nabla_c\big(\nabla_{(b}V_{a)} - \frac{1}{3}\theta h_{ab}\big) \\
&= V^c\nabla_c(\nabla_{(b}V_{a)}) - \frac{1}{3}h_{ab}V^c\nabla_c\theta \\
&= -(\nabla_c V_{(a})(\nabla_{b)}V^c) + R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}V^c\nabla_c\theta \\
&= -\frac{1}{2}\left[\left(\frac{1}{3}\theta h_{ac} + \sigma_{ac} + \omega_{ac}\right)\left(\frac{1}{3}\theta h^c{}_b + \sigma^c{}_b + \omega^c{}_b\right) + a \leftrightarrow b\right] \\
&\quad + R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}V^c\nabla_c\theta \\
&= -\frac{1}{2}\left[\left(\frac{1}{9}\theta^2 h_{ab} + \frac{2}{3}\theta\sigma_{ab} + \frac{2}{3}\theta\omega_{ab} + \sigma_{ac}\sigma^c{}_b + \sigma_a{}^c\omega_{cb} - \sigma^c{}_b\omega_{ca} + \omega_{ac}\omega^c{}_b\right) + a \leftrightarrow b\right] \\
&\quad + R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}V^c\nabla_c\theta \\
&= -\left(\frac{1}{9}\theta^2 h_{ab} + \frac{2}{3}\theta\sigma_{ab} + \sigma_{ac}\sigma^c{}_b + \omega_{ac}\omega^c{}_b\right) \\
&\quad + R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}\left(-\frac{1}{3}\theta^2 - \sigma_{cd}\sigma^{cd} + \omega_{cd}\omega^{cd} - R_{cd}V^cV^d\right) \\
&= -\frac{2}{3}\theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + \frac{1}{3}h_{ab}(\sigma_{cd}\sigma^{cd} - \omega^{cd}\omega_{cd}) + R_{c(ba)d}V^cV^d + \frac{1}{3}h_{ab}R_{cd}V^cV^d \\
&= -\frac{2}{3}\theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + \frac{1}{3}h_{ab}(\sigma_{cd}\sigma^{cd} - \omega_{cd}\omega^{cd}) \\
&\quad + R_{c(ba)d}V^cV^d - \frac{1}{3}h_{ab}h^{ef}R_{cfed}V^cV^d
\end{aligned}
\tag{2.11}
$$

where we used (2.7).

2.2.4 Symmetric trace-free part of $R_{cbad}V^cV^d$

The symmetric trace-free of $R_{cbad}V^cV^d$ is

$$
R_{cbad}V^cV^d - \frac{1}{3}h_{ab}h^{ef}R_{cefd}V^cV^d = \left(\delta^e_{(a}\delta^f_{b)} - \frac{1}{3}h_{ab}h^{ef}\right)R_{cefd}V^cV^d. \tag{2.12}
$$

Now because of symmetry properties of the Riemann tensor, the quantity $R_{cbad}V^cV^d$ is symmetric in a and b:

$$
\begin{aligned}
R_{cabd}V^cV^d &= R_{dabc}V^cV^d \\
&= R_{bcda}V^cV^d \\
&= R_{cbad}V^cV^d
\end{aligned}
\tag{2.13}
$$

where in the first line I have used that c and d are dummy indices, in the second line used the symmetry $R_{abcd} = R_{cdab}$, in the third line used the symmetry $R_{abcd} = -R_{bacd} = -R_{abdc}$. As such (2.12) slightly simplifies to

$$\left(\delta^e_b \delta^f_a - \frac{1}{3} h_{ab} h^{ef}\right) R_{cefd} V^c V^d. \tag{2.14}$$

Now the Riemann tensor can be expressed in terms of the Weyl tensor:

$$R_{cbad} = C_{cbad} - g_{c[d} R_{a]b} - g_{b[a} R_{d]c} - \frac{1}{3} R g_{c[a} g_{d]b}. \tag{2.15}$$

Notice it is anti-symmetric in a and d. Using that C_{cbad} has the same symmetries of R_{cbad} (in particular $R_{cbad} = R_{adcb}$), means that the difference $R_{cbad} - C_{cbad}$ in the above equation is also anti-symmetric in b and c.

We take (2.14) and decompose it as follows

$$\left(\delta^e_b \delta^f_a - \frac{1}{3} h_{ab} h^{ef}\right) (C_{cefd} + [R_{cefd} - C_{cefd}]) V^c V^d. \tag{2.16}$$

First note

$$\begin{aligned} \left(\delta^e_b \delta^f_a - \frac{1}{3} h_{ab} h^{ef}\right) C_{cefd} V^c V^d &= \left(\delta^e_b \delta^f_a - \frac{1}{3} h_{ab} (g^{ef} + V^e V^f)\right) C_{cefd} V^c V^d \\ &= C_{cbad} V^c V^d \end{aligned} \tag{2.17}$$

where we have used $C_{cefd} V^c V^d V^e V^f \equiv 0$ and that the Weyl tensor is trace-free in any two indices.

Using this in (2.16) we have

$$\left(\delta^e_b \delta^f_a - \frac{1}{3} h_{ab} h^{ef}\right) R_{cefd} V^c V^d = C_{cefd} V^c V^d + \left(\delta^e_b \delta^f_a - \frac{1}{3} h_{ab} h^{ef}\right) \left(R_{cefd} - C_{cefd}\right) V^c V^d. \tag{2.18}$$

Using $\delta^b_a = h_a{}^b - V_a V^b$ and the anti-symmetry in d and f, and the anti-symmetry in c and e of $R_{cefd} - C_{cefd}$ in the above equation we get,

$$\begin{aligned}\left(\delta_a^f\delta_b^e - \frac{1}{3}h_{ab}h^{ef}\right)R_{cefd}V^cV^d &= C_{cbad}V^cV^d + \left((h_a{}^f - V_aV^f)(h_b{}^e - V_bV^e) - \frac{1}{3}h_{ab}h^{ef}\right) \\ &\quad \left(R_{cefd} - C_{cefd}\right)V^cV^d \\ &= C_{cbad}V^cV^d + \left(h_a{}^f h_b{}^e - \frac{1}{3}h_{ab}h^{ef}\right)\left(R_{cefd} - C_{cefd}\right)V^cV^d\end{aligned} \tag{2.19}$$

We now just need $\left(R_{cefd} - C_{cefd}\right)V^cV^d$. Contracting both sides of (2.15) with V^cV^d we obtain

$$R_{cbad}V^cV^d = C_{cbad}V^cV^d - g_{c[d}R_{a]b}V^cV^d - g_{b[a}R_{d]c}V^cV^d - \frac{1}{3}Rg_{c[a}g_{d]b}V^cV^d.$$

The individual terms give

$$\begin{aligned}-g_{c[d}R_{a]b}V^cV^d &= -\frac{1}{2}[g_{cd}R_{ab} - g_{ca}R_{db}]V^cV^d \\ &= \frac{1}{2}[R_{ab} + V_aR_{bd}V^d] \\ -g_{b[a}R_{d]c}V^cV^d &= -\frac{1}{2}[g_{ab}R_{cd} - g_{bd}R_{ac}]V^cV^d \\ &= \frac{1}{2}[-g_{ab}R_{cd}V^cV^d + V_bR_{ac}V^c] \\ -\frac{1}{3}Rg_{c[a}g_{d]b}V^cV^d &= -\frac{1}{2}\cdot\frac{1}{3}R[g_{ca}g_{db} - g_{cd}g_{ab}]V^cV^d \\ &= -\frac{1}{2}\cdot\frac{1}{3}R[V_aV_b + g_{ab}]\end{aligned}$$

Bringing these results together we have

$$\begin{aligned}(R_{cbad} - C_{cbad})V^cV^d &= -g_{c[d}R_{a]b}V^cV^d - g_{b[a}R_{d]c}V^cV^d - \frac{1}{3}Rg_{c[a}g_{d]b}V^cV^d \\ &= \frac{1}{2}R_{ab} - \frac{1}{2}g_{ab}R_{cd}V^cV^d + V_{(a}R_{b)d}V^d - \frac{1}{2}\cdot\frac{1}{3}Rh_{ab}\end{aligned} \tag{2.20}$$

Using the above in (2.19) gives

$$
\begin{aligned}
\left(\delta_a^f\delta_b^e-\frac{1}{3}h_{ab}h^{ef}\right)R_{cefd}V^cV^d &= C_{cbad}V^cV^d+\left(h_a{}^fh_b{}^e-\frac{1}{3}h_{ab}h^{ef}\right)\\
&\quad \left(\frac{1}{2}R_{fe}-\frac{1}{2}g_{fe}R_{cd}V^cV^d+V_{(f}R_{e)d}V^d-\frac{1}{2}\cdot\frac{1}{3}Rh_{fe}\right)\\
&= C_{cbad}V^cV^d+\frac{1}{2}h_a{}^eh_b{}^fR_{ef}-\frac{1}{2}\cdot\frac{1}{3}h_{ab}h^{ef}R_{ef}+\\
&- \frac{1}{2}\left(h_a{}^fh_b{}^eg_{fe}-\frac{1}{3}h_{ab}h^{ef}g_{fe}\right)R_{cd}V^cV^d\\
&+ \underbrace{h_a{}^fh_b{}^eV_{(f}R_{e)d}V^d}_{=0}-\frac{1}{3}\underbrace{h_{ab}h^{ef}V_{(f}R_{e)d}V^d}_{=0}\\
&- \frac{1}{2}\cdot\frac{1}{3}h_a{}^fh_b{}^eRh_{fe}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}h_{ab}h^{ef}Rh_{fe}.
\end{aligned}
\tag{2.21}
$$

We have

$$
h_a{}^fh_{bf}=h_{ab} \qquad h_a{}^fh_b{}^eh_{fe}=h_{ab} \qquad h^{ef}h_{fe}=3 \qquad h^{ef}g_{fe}=h^{ef}(h_{fe}-V_fV_e)=3.
$$

Substituting this into (2.21) gives

$$
\left(\delta^e_{(a}\delta^f_{b)}-\frac{1}{3}h_{ab}h^{ef}\right)R_{cefd}V^cV^d=C_{cbad}V^cV^d+\frac{1}{2}\left(h_a{}^ch_b{}^dR_{cf}-\frac{1}{3}h_{ab}h^{cd}R_{cd}\right). \tag{2.22}
$$

And so (2.7) (or (2.11)) becomes

$$
\begin{aligned}
\frac{d\sigma_{ab}}{ds} &= -\frac{2}{3}\theta\sigma_{ab}-\sigma_{ac}\sigma^c{}_b-\omega_{ac}\omega^c{}_b+\frac{1}{3}h_{ab}(\sigma_{cd}\sigma^{cd}-\omega^{cd}\omega_{cd})\\
&\quad +C_{cbad}V^cV^d+\frac{1}{2}\left(h_{ac}h_{bd}R^{cd}-\frac{1}{3}h_{ab}h_{cd}R^{cd}\right).
\end{aligned}
\tag{2.23}
$$

Since the Wely tensor is trace-free is does not appear in the equation in the expansion equation(2.4). however as the terms $-2\sigma^2$ $(=-2\sigma^{ab}\sigma_{ab})$ occurs on the right-hand side of the expansion equation, the Weyl tensor induces convergence indirectly by inducing shear.

The Weyl tensor is the part of the Reiemann tensor not depending on the Ricci tensor. From the Einstein equations,

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}.$$

we see that it is the Rici tensor, R_{ab}, is determined locally by the matter distributionis. Thus the Weyl tensor is the part of the curvature which is not determined locally by the matter distribution.

2.3 Derivative of vorticity

And from the anti-symmetric part of (2.2) yields

$$\begin{aligned}
V^c \nabla_c \omega_{ab} &= V^c \nabla_c \big(\nabla_{[b} V_{a]} \big) \\
&= -(\nabla_c V_{[a})(\nabla_{b]} V^c) + R_{c[ba]d} V^c V^d \\
&= -(\nabla_c V_{[a})(\nabla_{b]} V^c) \\
&= -\frac{1}{2} [(\nabla_c V_a)(\nabla_b V^c) - a \leftrightarrow b] \\
&= -\frac{1}{2} \left[\left(\frac{1}{9} \theta^2 h_{ab} + \frac{2}{3} \theta \sigma_{ab} + \frac{2}{3} \theta \omega_{ab} + \sigma_{ac} \sigma^c{}_b + 2\sigma^c{}_b \omega_{ac} + \omega_{ac} \omega^c{}_b \right) - a \leftrightarrow b \right] \\
&= -\frac{2}{3} \theta \omega_{ab} - 2\sigma^c{}_{[b} \omega_{a]c} \qquad (2.24)
\end{aligned}$$

and so

$$\frac{d\omega_{ab}}{ds} = -\frac{2}{3} \theta \omega_{ab} - 2\sigma^c{}_{[b} \omega_{a]c}. \qquad (2.25)$$

2.4 Conjugate points

A pair of points $p, q \in \gamma$ if there is a jacobi field η^a which is not identically zero but vanishes at both p and q. Or p and q are conjugate if an an infinitesimally nearby geodesic intersects γ at both p and q.

Chapter 3

Raychaudhuri equations for null geodesics

3.1 Null congruences and transverse metric or "spatial metric"

The transverse metric is constructed by introducing an auxiliary null vector n^a with $k_a n^a = -1$. Then

$$h_{ab} = g_{ab} + k_a n_b + n_a k_b \tag{3.1}$$

It satifies

$$h_{ab} k^b = h_{ab} n^b = 0 \tag{3.2}$$

making it purely transverse (orthogonal to both k^a and n^a) and entirely two dimensional.

The tensor

$$h^a{}_b = \delta^a_b + k^a n_b + n^a k_b \tag{3.3}$$

is the projection operator onto the subspace of the space orthogonal to k^a and n^a as

$$h^a{}_b k^b = (\delta^a_b + k^a n_b + n^a k_b) k^b = 0, \quad h^a{}_b n^b = 0, \tag{3.4}$$

and if $x^a k_a = x^a n_a = 0$ then

$$h^a{}_b x^b = (\delta^a_b + k^a n_b + n^a k_b) x^b = x^a. \tag{3.5}$$

Also

$$h^a{}_c h^c{}_b = h^a{}_c (\delta^c_b + k^c n_b + n^c k_b) = h^a{}_b \tag{3.6}$$

which is the property of a projector.

We also have

$$\begin{aligned} h_{ab} h^{ab} &= h_{ab}(g^{ab} + k^a n^b + n^a k^b) \\ &= h_{ab} g^{ab} \\ &= (g_{ab} + k_a n_b + n_a k_b) g^{ab} \\ &= 2. \end{aligned} \tag{3.7}$$

Also

$$h^{ab} h^c{}_b = h^c{}_b (g^{ab} + k^a n^b + n^a k^b) = h^{ac} \tag{3.8}$$

and

$$h_a{}^b h_{bc} = h_a{}^b (g_{bc} + k_b n_c + n_b k_c) = h_{ac} \tag{3.9}$$

3.2 The expansion, shear and twist

$$\hat{B}_{ab} = h^c{}_a h^d{}_b B_{cd} \tag{3.10}$$

$$\begin{aligned} \hat{B}_{ab} &= (\delta^c_a + k_a n^c + n_a k^c)(\delta^d_b + k_b n^d + n_b k^d) B_{cd} \\ &= (\delta^c_a + k_a n^c + n_a k^c)(B_{cb} + k_b B_{cd} n^d) \\ &= (\delta^c_a + k_a n^c)(B_{cb} + k_b B_{cd} n^d) \\ &= B_{ab} + k_a n^c B_{cb} + k_b B_{ac} n^c + k_a k_b B_{cd} n^c n^d. \end{aligned} \tag{3.11}$$

Write

$$\hat{B}_{ab} = \hat{B}_{(ab)} + \hat{B}_{[ab]} \tag{3.12}$$

and make the decomposition

$$\hat{B}_{ab} = \frac{1}{2}\hat{\theta}h_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab} \tag{3.13}$$

where

$$\begin{aligned} \hat{\theta} &= h^{ab}\hat{B}_{(ab)} = g^{ab}\hat{B}_{ab} = \hat{B}^{a}{}_{a} \\ \hat{\sigma}_{ab} &= \hat{B}_{(ab)} - \frac{1}{2}\hat{\theta}h_{ab} \\ \hat{\omega}_{ab} &= \hat{B}_{[ab]} \end{aligned} \tag{3.14}$$

The expansion can be written

$$\begin{aligned} \hat{\theta} &= g^{ab}\hat{B}_{ab} \\ &= g^{ab}B_{ab} \end{aligned} \tag{3.15}$$

where we have used (3.11) and that B_{ab} is orthogonal to k^a. Therefore

$$\hat{\theta} = \nabla_a k^a. \tag{3.16}$$

3.2.1 Explicit formula

$$\begin{aligned} \hat{B}_{(ab)} &= \frac{1}{2}(h_a{}^c h_b{}^d + h_b{}^c h_a{}^d)\nabla_d k_c \\ &= h_a{}^c h_b{}^d \nabla_{(d}k_{c)} \end{aligned} \tag{3.17}$$

Similarly

$$\hat{\omega}_{ab} = \hat{B}_{[ab]} = h_a{}^c h_b{}^d \nabla_{[d}k_{c]} \tag{3.18}$$

and

$$
\begin{aligned}
\hat{\theta} &= h^{ab}\hat{B}_{(ab)} \\
&= h^{ab}h_a{}^c h_b{}^d \nabla_{(d}k_{c)} \\
&= h^{ab}\nabla_{(b}k_{a)}
\end{aligned} \tag{3.19}
$$

3.3 Raychaudhuri's equation for scalar expansion

As before

$$
\begin{aligned}
k^c\nabla_c(\nabla_b k_a) &= k^c\nabla_b\nabla_c k_d + k^c(\nabla_c\nabla_b - \nabla_b\nabla_c)k_d \\
&= k^c\nabla_b\nabla_c k_d + R_{cba}{}^d k^c k_d \\
&= \nabla_b(k^c\nabla_c k_d) - (\nabla_b k^c)(\nabla_c k_a) + R_{cba}{}^d k^c k_d \\
&= -(\nabla^c k_b)(\nabla_a k_c) + R_{cba}{}^d k^c k_d
\end{aligned} \tag{3.20}
$$

So that

$$
k^c\nabla_c(\nabla_b k_a) = -(\nabla^c k_b)(\nabla_a k_c) - R_{bcad}k^c k^d. \tag{3.21}
$$

Contracting gives

$$
k^c\nabla_c\hat{\theta} = -B^{ab}B_{ba} - R_{ab}k^a k^b. \tag{3.22}
$$

Now using (3.11), we have $B^{ab}B_{ba} = \hat{B}^{ab}B_{ba} = \hat{B}^{ab}\hat{B}_{ba}$ as $k^a B_{ba} = k^b B_{ba} = 0$ and $k_a\hat{B}^{ba} = k_b\hat{B}^{ba} = 0$.

We have

$$
\begin{aligned}
B^{ab}B_{ba} &= \hat{B}^{ab}\hat{B}_{ba} \\
&= \left[\frac{1}{2}\hat{\theta}h^{ab} + \hat{\sigma}^{ab} + \hat{\omega}^{ab}\right]\left[\frac{1}{2}\hat{\theta}h_{ba} + \hat{\sigma}_{ba} + \hat{\omega}_{ba}\right] \\
&= \frac{1}{2}\hat{\theta}^2 + \hat{\sigma}^{ab}\hat{\sigma}_{ab} - \hat{\omega}^{ab}\hat{\omega}_{ab}
\end{aligned} \tag{3.23}
$$

where we have used $h^{ab}h_{ab} = 2$, $h^{ab}\hat{\sigma}_{ab} = h^{ab}\hat{\omega}_{ab} = \hat{\sigma}^{ab}\hat{\omega}_{ab} = 0$.

This gives

$$\frac{\partial\hat{\theta}}{\partial\lambda} = -\frac{1}{2}\hat{\theta}^2 - \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - \widehat{R_{ab}k^a k^b}. \tag{3.24}$$

3.4 Expansion tensor

$$k^c\nabla_c(\nabla_b k_a) = -(\nabla^c k_b)(\nabla_a k_c) - R_{bcad}k^c k^d. \tag{3.25}$$

Write

$$\begin{aligned} k^e\nabla_e(h_a{}^c h_b{}^d\nabla_d k_c) &= k^c\nabla_c[(\delta_a{}^c + k_a n^c + n_a k^c)(\delta_b{}^d + k_b n^d + n_b k^d)\nabla_d k_c] \\ &= k^e\nabla_e[(\delta_a{}^c + k_a n^c)(\delta_b{}^d + k_b n^d)\nabla_d k_c] \\ &= k^e\nabla_e[\nabla_b k_a + k_b n^d\nabla_d k_a + k_a n^c\nabla_b k_c + k_a k_b n^c n^d\nabla_d k_c] \end{aligned} \tag{3.26}$$

3.5 The trace-free, symmetric part

The trace-free part of a tensor X_{ab} is

$$S_{ab} = X_{ab} - \frac{1}{2}\hat{h}_{ab}\hat{h}^{cd}X_{cd}$$

The symmetric part of the tensor S_{ab} is

$$\frac{1}{2}(S_{ab} + S_{ba}) = S_{(ab)} = X_{(ab)} - \frac{1}{2}\hat{h}_{ab}\hat{h}^{cd}X_{cd}$$

Or

$$S_{ab} = \left(\delta^c_{(a}\delta^d_{b)} - \frac{1}{2}\hat{h}_{ab}\hat{h}^{cd}\right)X_{cd}$$

3.6 Derivative of shear

3.7 Derivative of vorticity

3.8 The expansion, shear and twist - 2×2 notation

3.8.1 Defintions

We introduce vectors $\hat{e}^a_A$ $(A = 2, 3)$ that point in the two directions orthogonal to both k^a and N^a, and we choose them to be orthonormal, so that they satify

$$g_{ab}\hat{e}^a_A\hat{e}^b_B = \delta_{AB}.$$

We introduce the 2×2 matrix

$$B_{AB} = B_{ab}\hat{e}^a_A\hat{e}^b_B \tag{3.27}$$

$$\begin{aligned}
\hat{B}^{ab} &= B^{cd}h_c{}^a h_d{}^b \\
&= B_{cd}h^{ca}h^{db} \\
&= B_{cd}[\delta^{CA}\hat{e}^c_C\hat{e}^a_A][\delta^{DB}\hat{e}^d_D\hat{e}^b_B] \\
&= (B_{cd}\hat{e}^c_C\hat{e}^d_D)\delta^{CA}\delta^{DB}\hat{e}^a_A\hat{e}^b_B \\
&= B_{CD}\delta^{CA}\delta^{DB}\hat{e}^a_A\hat{e}^b_B \\
&= B^{AB}\hat{e}^a_A\hat{e}^b_B.
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\theta &= h_{ab}\hat{B}^{ab} \\
&= h_{ab}B^{AB}\hat{e}^a_A\hat{e}^b_B \\
&= B^{AB}g_{ab}\hat{e}^a_A\hat{e}^b_B \\
&= B^{AB}\delta_{AB}.
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
\sigma^{AB}\hat{e}_A^a\hat{e}_B^b &= \frac{1}{2}(B^{AB}+B^{BA}-\theta\delta^{AB})\hat{e}_A^a\hat{e}_B^b \\
&= \frac{1}{2}(\hat{B}^{ab}+\hat{B}^{ba}-\theta h^{ab}) \\
&= \hat{B}^{(ab)}-\frac{1}{2}\theta h^{ab} \\
&= \sigma^{ab}. \qquad (3.30)
\end{aligned}$$

$$\begin{aligned}
\omega^{AB}\hat{e}_A^a\hat{e}_B^b &= \frac{1}{2}(B^{AB}-B^{BA})\hat{e}_A^a\hat{e}_B^b \\
&= \frac{1}{2}(\hat{B}^{ab}-\hat{B}^{ba}) \\
&= \hat{\omega}^{ab} \qquad (3.31)
\end{aligned}$$

3.8.2 Dependence on choice of auxiliary null vector N^a

First

$$\begin{aligned}
k^aN'_a &= k^a(N_a+ck_a+c^A\hat{e}_{Aa}) \\
&= k^aN_a \\
&= -1. \qquad (3.32)
\end{aligned}$$

Next

$$\begin{aligned}
N'^aN'_a &= (N^a+ck^a+c^A\hat{e}_A^a)(N_a+ck_a+c^B\hat{e}_{Ba}) \\
&= 2cN^ak_a+c^Ac^B\hat{e}_A^a\hat{e}_{Ba} \\
&= -2c+c^Ac^B\delta_{AB} \qquad (3.33)
\end{aligned}$$

for this to vanish we must have $c=\frac{1}{2}c^Ac_A$.

Change in h^{ab}

$$\begin{aligned}
h'^{ab} &= g^{ab}+k^aN'^b+N'^ak^b \\
&= g^{ab}+k^a(N^b+ck^b+c^B\hat{e}_B^b)+(N^a+ck^a+c^A\hat{e}_A^a)k^b \\
&= h^{ab}+2ck^ak^b+c^Bk^a\hat{e}_B^b+c^A\hat{e}_A^ak^b \qquad (3.34)
\end{aligned}$$

Change in $\tilde{B}^{ab}$

$$
\begin{aligned}
\tilde{B}'^{ab} &= h'^{ca}h'^{db}B_{cd} \\
&= (h^{ac} + 2ck^a k^c + c^C k^a \hat{e}^c_C + c^A \hat{e}^a_A k^c)(h^{bd} + 2ck^b k^d + c^D k^b \hat{e}^d_D + c^B \hat{e}^b_B k^d)B_{cd} \\
&= (h^{ac} + c^C k^a \hat{e}^c_C)(h^{bd} + c^D k^b \hat{e}^d_D)B_{cd} \\
&= \tilde{B}^{ab} + (c^C h^{bd} B_{cd} \hat{e}^c_C)k^a + (c^D h^{ac} B_{cd} \hat{e}^d_D)k^b + (c^C c^D \hat{e}^c_C \hat{e}^d_D B_{cd})k^a k^b \\
&= (c^A c^B B_{AB})k^a k^b + (c^C \delta^{BD} \hat{e}^b_B \hat{e}^d_D B_{cd} \hat{e}^c_C)k^a + (c^D \delta^{AC} \hat{e}^a_A \hat{e}^c_C B_{cd} \hat{e}^d_D)k^b + B^{AB} \hat{e}^a_A \hat{e}^b_B \\
&= (c^A c^B B_{AB})k^a k^b + (c^C \delta^{BD} B_{CD})k^a \hat{e}^b_B + (c^D \delta^{AC} B_{CD})\hat{e}^a_A k^b + B^{AB} \hat{e}^a_A \hat{e}^b_B \\
&= (c^A c^B B_{AB})k^a k^b + (c^A B_A{}^B)k^a \hat{e}^b_B + (c^B B^A{}_B)\hat{e}^a_A k^b + B^{AB} \hat{e}^a_A \hat{e}^b_B \qquad (3.35)
\end{aligned}
$$

That is

$$
\tilde{B}'^{ab} = (c^A c^B B_{AB})k^a k^b + (c^A B_A{}^B)k^a \hat{e}^b_B + (c^B B^A{}_B)\hat{e}^a_A k^b + B^{AB} \hat{e}^a_A \hat{e}^b_B \qquad (3.36)
$$

Invariance of θ

From (3.36)

$$
\begin{aligned}
\theta' &= g_{ab}\tilde{B}'^{ab} \\
&= g_{ab} B^{AB} \hat{e}^a_A \hat{e}^b_B \\
&= g_{ab}\tilde{B}^{ab} \\
&= \theta. \qquad (3.37)
\end{aligned}
$$

Change in σ^{ab}

$$\begin{aligned}
\sigma'^{ab} &= \tilde{B}'^{(ab)} - \frac{1}{2}\theta' h'^{ab} \\
&= (c^A c^B B_{(AB)}) k^a k^b + \frac{1}{2} c^A B_A{}^B [k^a \hat{e}^b_B + k^b \hat{e}^a_B] + \frac{1}{2} c^B B^A{}_B [\hat{e}^a_A k^b + \hat{e}^b_A k^a] \\
&\quad + B^{(AB)} \hat{e}^a_A \hat{e}^b_B - \frac{1}{2}\theta[2ck^a k^b + c^B k^a \hat{e}^b_B + c^A \hat{e}^a_A k^b + h^{ab}] \\
&= (c^A c^B B_{(AB)}) k^a k^b + \frac{1}{2}[c^A B_A{}^B + c^A B^B{}_A] k^a \hat{e}^b_B + \frac{1}{2}[c^B B^A{}_B + c^B B_B{}^A] \hat{e}^a_A k^b \\
&\quad + B^{(AB)} \hat{e}^a_A \hat{e}^b_B - \frac{1}{2}\theta[c_A c^A k^a k^b + c^B k^a \hat{e}^b_B + c^A \hat{e}^a_A k^b + h^{ab}] \\
&= c^A c^B (B_{(AB)} - \frac{1}{2}\theta\delta_{AB}) k^a k^b + c_A (B^{(AB)} - \frac{1}{2}\theta\delta^{AB}) k^a \hat{e}^b_B \\
&\quad + c_B (B^{(AB)} - \frac{1}{2}\theta\delta^{AB}) \hat{e}^a_A k^b + (B^{(AB)} - \frac{1}{2}\theta\delta^{AB}) \hat{e}^a_A \hat{e}^b_B \\
&= c^A c^B \sigma_{AB} k^a k^b + c_A \sigma^{AB} k^a \hat{e}^b_B + c_B \sigma^{AB} \hat{e}^a_A k^b + \sigma^{AB} \hat{e}^a_A \hat{e}^b_B. \qquad (3.38)
\end{aligned}$$

That is

$$\sigma'^{ab} = \left(c^A c^B \sigma_{AB}\right) k^a k^b + \left(c^A \sigma_A{}^B\right) k^a \hat{e}^b_B + \left(c^B \sigma_B{}^A\right) \hat{e}^a_A k^b + \sigma^{AB} \hat{e}^a_A \hat{e}^b_B \qquad (3.39)$$

This shows that if $\sigma_{ab} = 0$ for one choice of N^a, then $\sigma_{ab} = 0$ for any other choice.

Invariance of $\sigma^{ab}\sigma_{ab}$

$$\begin{aligned}
\sigma'^{ab}\sigma'_{ab} &= \sigma'^{ab}\sigma'^{cd} g_{ac} g_{bd} \\
&= [(c^A c^B \sigma_{AB}) k^a k^b + (c^A \sigma_A{}^B) k^a \hat{e}^b_B + (c^B \sigma_B{}^A) \hat{e}^a_A k^b + \sigma^{AB} \hat{e}^a_A \hat{e}^b_B] \\
&\quad [(c^C c^D \sigma_{CD}) k^c k^d + (c^C \sigma_C{}^D) k^c \hat{e}^d_D + (c^D \sigma_D{}^C) \hat{e}^c_C k^d + \sigma^{CD} \hat{e}^c_C \hat{e}^d_D] g_{ac} g_{bd} \\
&= \sigma^{AB} \hat{e}^a_A \hat{e}^b_B \sigma^{CD} \hat{e}^c_C \hat{e}^d_D g_{ac} g_{bd} \\
&= \sigma^{AB} \sigma^{CD} \delta_{AC} \delta_{BD} \\
&= \sigma^{ab}\sigma_{ab} \qquad (3.40)
\end{aligned}$$

as

$$
\begin{aligned}
\sigma^{ab}\sigma_{ab} &= \sigma^{ab}\sigma^{cd}g_{ac}g_{bd} \\
&= \sigma^{AB}\hat{e}^a_A\hat{e}^b_B\sigma^{CD}\hat{e}^c_C\hat{e}^d_D g_{ac}g_{bd} \\
&= \sigma^{AB}\sigma^{CD}(\hat{e}^a_A\hat{e}^c_C g_{ac})(\hat{e}^b_B\hat{e}^d_D g_{bd}) \\
&= \sigma^{AB}\sigma^{CD}\delta_{AC}\delta_{BD} \\
&= \sigma^{AB}\sigma_{AB} \\
&= \sigma^{AB}\sigma_{BA} \\
&= Tr(\sigma^{AB}\sigma_{BC})
\end{aligned} \tag{3.41}
$$

Change in ω^{ab}

$$
\begin{aligned}
\omega'^{ab} &= \tilde{B}'^{[ab]} \\
&= \frac{1}{2}(c^A B_A{}^B)[k^a\hat{e}^b_B - k^b\hat{e}^a_B] + \frac{1}{2}(c^B B^A{}_B)[\hat{e}^a_A k^b - \hat{e}^b_A k^a] + B^{[AB]}\hat{e}^a_A\hat{e}^b_B \\
&= \frac{1}{2}(c^A B_A{}^B - c^A B^B{}_A)k^a\hat{e}^b_B + \frac{1}{2}(c^B B^A{}_B - c^B B_B{}^A)\hat{e}^a_A k^b + \omega^{AB}\hat{e}^a_A\hat{e}^b_B \\
&= (c_A\omega^{AB})k^a\hat{e}^b_B + (c_B\omega^{AB})\hat{e}^a_A k^b + \omega^{AB}\hat{e}^a_A\hat{e}^b_B
\end{aligned} \tag{3.42}
$$

That is

$$
\omega'^{ab} = \left(c^A\omega_A{}^B\right)k^a\hat{e}^b_B - \left(c^B\omega_B{}^A\right)\hat{e}^a_A k^b + \omega^{AB}\hat{e}^a_A\hat{e}^b_B \tag{3.43}
$$

This shows that if $\omega_{ab} = 0$ for one choice of N^a, then $\omega_{ab} = 0$ for any other choice. This proves hypersurface orthogonality for any choice of N^a.

Invariance of $\omega^{ab}\omega_{ab}$

$$
\begin{aligned}
\omega'^{ab}\omega'_{ab} &= \omega'^{ab}\omega'^{cd}g_{ac}g_{bd} \\
&= [(c^A\omega_A{}^B)k^a\hat{e}^b_B - (c^B\omega_B{}^A)\hat{e}^a_A k^b + \omega^{AB}\hat{e}^a_A\hat{e}^b_B] \\
&\quad [(c^C\omega_C{}^D)k^c\hat{e}^d_D - (c^D\omega_D{}^C)\hat{e}^c_C k^d + \omega^{CD}\hat{e}^c_C\hat{e}^d_D]g_{ac}g_{bd} \\
&= \omega^{AB}\hat{e}^a_A\hat{e}^b_B\omega^{CD}\hat{e}^c_C\hat{e}^d_D g_{ac}g_{bd} \\
&= \omega^{AB}\omega^{CD}\delta_{AC}\delta_{BD} \\
&= \omega^{ab}\omega_{ab}
\end{aligned} \tag{3.44}
$$

$$
\begin{aligned}
\omega^{ab}\omega_{ab} &= \omega^{AB}\omega_{AB} \\
&= -\omega^{AB}\omega_{BA} \\
&= -Tr(\omega^{AB}\omega_{BC})
\end{aligned} \tag{3.45}
$$

Invariance of θ, $\sigma^{ab}\sigma_{ab}$ and $\omega^{ab}\omega_{ab}$ implies that the Raychaudhuri equation is invariant under a change of N^a.

3.8.3 Evolution equations for shear and rotation

Introduce

$$R_{AB} = R_{acbd}\hat{e}^a_A k^c \hat{e}^b_B k^d \tag{3.46}$$

and

$$\Gamma_{AB} = \hat{e}_{Bc} k^d \nabla_d \hat{e}^c_A. \tag{3.47}$$

Note

$$\begin{aligned} 0 &= k^d \nabla_d (\hat{e}_{Bc}\hat{e}^c_A) \\ &= \hat{e}_{Bc} k^d \nabla_d \hat{e}^c_A + \hat{e}_{Ac} k^d \nabla_d \hat{e}^c_B \\ &= \Gamma_{AB} + \Gamma_{BA}. \end{aligned} \tag{3.48}$$

And so Γ_{AB} is antisymmetric. It is possible to set $\Gamma_{AB} = 0$ by choosing $\hat{e}^a_A$ to be parallel transported along the congruence.

In the following we need an identity. As k^a and $\hat{e}^a_A$ are orthogonal we have

$$\begin{aligned} 0 &= k^d \nabla_d (k_c \hat{e}^c_A) \\ &= \hat{e}^c_A (k^d \nabla_d k_c) + k_c k^d \nabla_d \hat{e}^c_A \\ &= k_c k^d \nabla_d \hat{e}^c_A. \end{aligned} \tag{3.49}$$

where we used the geodesic equation $k^d \nabla_d k_c = 0$.

Consider

$$
\begin{aligned}
\Gamma_A{}^C B_{CB} &= \delta^{CD}\Gamma_{AC}B_{DB} \\
&= \delta^{CD}(\hat{e}_{Cc}k^d\nabla_d\hat{e}^c_A)(B_{ab}\hat{e}^a_D\hat{e}^b_B) \\
&= (\delta^{CD}\hat{e}_{Cc}\hat{e}^a_D)B_{ab}\hat{e}^b_B k^d\nabla_d\hat{e}^c_A \\
&= h_c{}^a B_{ab}\hat{e}^b_B k^d\nabla_d\hat{e}^c_A \\
&= (\delta_c{}^a + k_c N^a + N_c k^a)B_{ab}\hat{e}^b_B k^d\nabla_d\hat{e}^c_A \\
&= \delta_c{}^a B_{ab}\hat{e}^b_B k^d\nabla_d\hat{e}^c_A \\
&= B_{ab}\hat{e}^b_B k^d\nabla_d\hat{e}^a_A
\end{aligned} \tag{3.50}
$$

where we have used (3.49) and $k^a B_{ab} = k^a\nabla_b k_a = \frac{1}{2}\nabla_b(k^a k_a) = 0$.

Consider

$$
\begin{aligned}
\Gamma_B{}^C B_{AC} &= \delta^{CD}\Gamma_{BC}B_{AD} \\
&= \delta^{CD}(\hat{e}_{Cc}k^d\nabla_d\hat{e}^c_B)(B_{ab}\hat{e}^a_A\hat{e}^b_D) \\
&= (\delta^{CD}\hat{e}_{Cc}\hat{e}^b_D)B_{ab}\hat{e}^a_A k^d\nabla_d\hat{e}^c_B \\
&= h_c{}^b B_{ab}\hat{e}^a_A k^d\nabla_d\hat{e}^c_B \\
&= B_{ab}\hat{e}^a_A k^d\nabla_d\hat{e}^b_B.
\end{aligned} \tag{3.51}
$$

where we have used (3.49) and $k^b B_{ab} = k^b\nabla_b k_a = 0$.

Combining them gives

$$
\begin{aligned}
\Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC} &= B_{ab}\hat{e}^b_B k^d\nabla_d\hat{e}^a_A + B_{ab}\hat{e}^a_A k^d\nabla_d\hat{e}^b_B \\
&= B_{ab}k^c\nabla_c(\hat{e}^a_A\hat{e}^b_B) \\
&= \nabla_b k_a k^c\nabla_c(\hat{e}^a_A\hat{e}^b_B).
\end{aligned} \tag{3.52}
$$

Also consider

$$
\begin{aligned}
B_{AC}B^C{}_B &= B_{AC}\delta^{CD}B_{DB} \\
&= \delta^{CD}(B_{ac}\hat{e}^a_A\hat{e}^c_C)(B_{db}\hat{e}^d_D\hat{e}^b_B) \\
&= (\delta^{CD}\hat{e}^c_C\hat{e}^d_D)B_{ac}B_{db}\hat{e}^a_A\hat{e}^b_B \\
&= h^{cd}B_{ac}B_{db}\hat{e}^a_A\hat{e}^b_B \\
&= (h^{cd} + k^c N^d + N^c k^d)B_{ac}B_{db}\hat{e}^a_A\hat{e}^b_B \\
&= g^{cd}B_{ac}B_{db}\hat{e}^a_A\hat{e}^b_B \\
&= B_{ac}B^c{}_b\hat{e}^a_A\hat{e}^b_B
\end{aligned} \tag{3.53}
$$

where we have used $k^c B_{ac} = k^c \nabla_c k_a = 0$ and $k^d B_{db} = k^d \nabla_b k_d = 0$.

Working out the main evolution equation proceeds as follows

$$
\begin{aligned}
\frac{dB_{AB}}{d\lambda} &= k^c \nabla_c B_{AB} \\
&= k^c \nabla_c (B_{ab} \hat{e}^a_A \hat{e}^b_B) \\
&= k^c \nabla_c (\nabla_b k_a \hat{e}^a_A \hat{e}^b_B) \\
&= \hat{e}^a_A \hat{e}^b_B (k^c \nabla_c \nabla_b k_a) + \nabla_b k_a [k^c \nabla_c (\hat{e}^a_A \hat{e}^b_B)] \\
&= [-(\nabla_b k^c)(\nabla_c k_a) - R_{acbd} k^c k^d] \hat{e}^a_A \hat{e}^b_B + \Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC} \\
&= -(\nabla_c k_a)(\nabla_b k^c) \hat{e}^a_A \hat{e}^b_B - R_{AB} + \Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC} \\
&= -B_{ac} B^c{}_b \hat{e}^a_A \hat{e}^b_B - R_{AB} + \Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC} \\
&= -B_{AC} B^C_{\ B} - R_{AB} + \Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC}
\end{aligned}
\tag{3.54}
$$

where we have substituted in (3.52) and (3.53). The final equation derived is

$$
\frac{dB_{AB}}{d\lambda} = -B_{AC} B^C_{\ B} - R_{AB} + \Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC}. \tag{3.55}
$$

3.8.4 Decomposition of matrices

$$
B_{AB} = \frac{1}{2} \theta \delta_{AB} + \sigma_{AB} + \omega_{AB}, \qquad R_{AB} = \frac{1}{2} \mathscr{R} \delta_{AB} + C_{AB} \tag{3.56}
$$

Need to prove:

$$
\mathscr{R} = R_{ab} k^a k^b \tag{3.57}
$$

We have by (2.15):

$$
\begin{aligned}
R_{AB} &= R_{acbd} \hat{e}^a_A k^c \hat{e}^b_B k^d \\
&= [C_{acbd} - g_{a[d} R_{b]c} - g_{c[b} R_{d]a} - \frac{1}{3} R g_{a[b} g_{d]c}] \hat{e}^a_A k^c \hat{e}^b_B k^d \\
&= C_{acbd} \hat{e}^a_A k^c \hat{e}^b_B k^d + \frac{1}{2} g_{ab} \hat{e}^a_A \hat{e}^b_B R_{cd} k^c k^d \\
&= C_{AB} + \frac{1}{2} \mathscr{R} \delta_{AB}.
\end{aligned}
\tag{3.58}
$$

We have that C_{AB} is symmetric:

$$\begin{aligned} C_{AB} &= C_{acbd}\hat{e}^a_A k^c \hat{e}^b_B k^d \\ &= C_{bdac}\hat{e}^a_A k^c \hat{e}^b_B k^d \\ &= C_{acbd}\hat{e}^b_A k^d \hat{e}^a_B k^c \\ &= C_{BA} \end{aligned} \tag{3.59}$$

where we first used $C_{acbd} = C_{bdac}$. It is trace-free:

$$\begin{aligned} \delta^{AB} C_{AB} &= C_{acbd}\delta^{AB}\hat{e}^a_A \hat{e}^b_B k^c k^d \\ &= C_{acbd} h^{ab} k^c k^d \\ &= C_{acbd}(h^{ab} + k^a N^b + N^a k^b) k^c k^d \\ &= (C_{acbd} g^{ab}) k^c k^d \\ &= 0 \end{aligned} \tag{3.60}$$

We introduce the parameterisation

$$\sigma_{AB} = \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix}, \qquad C_{AB} = \begin{pmatrix} C_+ & C_\times \\ C_\times & -C_+ \end{pmatrix}, \tag{3.61}$$

$$\omega_{AB} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{3.62}$$

and

$$\Gamma_{AB} = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}. \tag{3.63}$$

3.8.5 Explicit evolution equations

Expansion:

$$
\begin{aligned}
\frac{d\theta}{d\lambda} &= \frac{d(B_{AB}\delta^{AB})}{d\lambda} \\
&= -\delta^{AB}B_{AC}B^{C}{}_{B} - \delta^{AB}R_{AB} \\
&= -(\frac{1}{2}\theta\delta_{AC} + \sigma_{AC} + \omega_{AC})(\frac{1}{2}\theta\delta^{CA} + \sigma^{CA} + \omega^{CA}) - \mathscr{R} \\
&= -\frac{1}{2}\theta^2 - \sigma_{AC}\sigma^{CA} - \omega_{AC}\omega^{CA} - \mathscr{R} \\
&= -\frac{1}{2}\theta^2 - 2(\sigma_+^2 + \sigma_\times^2) + 2\omega^2 - \mathscr{R} \qquad (3.64)
\end{aligned}
$$

The term involving Γ_{AB} does not contribute here:

$$
\begin{aligned}
\delta^{AB}(\Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC}) &= \delta^{AB}\delta^{CD}(\Gamma_{AC}B_{DB} + \Gamma_{BD}B_{AC}) \\
&= \delta^{11}\delta^{11}(\Gamma_{11}B_{11} + \Gamma_{11}B_{11}) \\
&\quad +\delta^{22}\delta^{11}(\Gamma_{21}B_{12} + \Gamma_{21}B_{21}) \\
&\quad +\delta^{11}\delta^{22}(\Gamma_{12}B_{21} + \Gamma_{12}B_{12}) \\
&\quad +\delta^{22}\delta^{22}(\Gamma_{22}B_{22} + \Gamma_{22}B_{22}) \\
&= 0. \qquad (3.65)
\end{aligned}
$$

Shear:

$$
\begin{aligned}
\frac{d\sigma_{AB}}{d\lambda} &= \frac{d}{d\lambda}\left(B_{(AB)} - \frac{1}{2}\theta\delta_{AB}\right) \\
&= -\frac{1}{2}(B_{AC}B^{C}{}_{B} + B_{BC}B^{C}{}_{A}) - R_{AB} \\
&\quad +\frac{1}{2}(\Gamma_A{}^C B_{CB} + \Gamma_B{}^C B_{AC} + \Gamma_B{}^C B_{CA} + \Gamma_A{}^C B_{BC}) - \frac{1}{2}\frac{d\theta}{d\lambda}\delta_{AB} \\
&= -\frac{1}{2}\delta^{CD}(B_{AC}B_{DB} + B_{BC}B_{DA}) - R_{AB} \\
&\quad +\frac{1}{2}\delta^{CD}(\Gamma_{AC}B_{DB} + \Gamma_{BC}B_{AD} + \Gamma_{BC}B_{DA} + \Gamma_{AC}B_{BD}) - \frac{1}{2}\frac{d\theta}{d\lambda}\delta_{AB} \qquad (3.66)
\end{aligned}
$$

Using $A = 1$ and $B = 1$

$$
\begin{aligned}
\frac{1}{2}\delta^{CD}(\Gamma_{1C}B_{D1}+\Gamma_{1C}B_{1D}+\Gamma_{1C}B_{D1}+\Gamma_{1C}B_{1D}) &= \frac{1}{2}(\Gamma_{11}B_{11}+\Gamma_{11}B_{11} \\
&\quad +\Gamma_{11}B_{11}+\Gamma_{11}B_{11} \\
&\quad +\Gamma_{12}B_{21}+\Gamma_{12}B_{12} \\
&\quad +\Gamma_{12}B_{21}+\Gamma_{12}B_{12}) \\
&= \Gamma(B_{12}+B_{21}) \\
&= 2\Gamma\sigma_\times
\end{aligned}
\tag{3.67}
$$

and so

$$
\begin{aligned}
\frac{d\sigma_+}{d\lambda} &= -\frac{1}{2}\delta^{CD}(B_{1C}B_{D1}+B_{1C}B_{D1})-R_{11}+2\Gamma\sigma_\times-\frac{1}{2}\frac{d\theta}{d\lambda}\delta_{11} \\
&= -(B_{11}B_{11}+B_{12}B_{21})-\frac{1}{2}\mathscr{R}-C_+ +2\Gamma\sigma_\times-\frac{1}{2}\frac{d\theta}{d\lambda} \\
&= -[(\frac{1}{2}\theta+\sigma_+)^2+(\sigma_\times+\omega)(\sigma_\times-\omega)]-\frac{1}{2}\mathscr{R}-C_+ +2\Gamma\sigma_\times \\
&\quad -\frac{1}{2}[-\frac{1}{2}\theta^2-2(\sigma_+^2+\sigma_\times^2)+2\omega^2-\mathscr{R}] \\
&= -[\frac{1}{4}\theta^2-\theta\sigma_+ +(\sigma_+^2+\sigma_\times^2)-\omega^2]-\frac{1}{2}\mathscr{R}-C_+ +2\Gamma\sigma_\times \\
&\quad +[\frac{1}{4}\theta^2+(\sigma_+^2+\sigma_\times^2)-\omega^2+\frac{1}{2}\mathscr{R}] \\
&= -\theta\sigma_+ - C_+ + 2\Gamma\sigma_\times
\end{aligned}
\tag{3.68}
$$

Using $A=1$ and $B=2$. The term involving Γ_{AB}

$$
\frac{1}{2}\delta^{CD}(\Gamma_{AC}B_{DB}+\Gamma_{BD}B_{AC}-\Gamma_{BC}B_{DA}-\Gamma_{AD}B_{BC})
$$

gives the contributution

$$
\begin{aligned}
\frac{1}{2}\delta^{CD}(\Gamma_{1C}B_{D2}+\Gamma_{2D}B_{1C}+\Gamma_{1C}B_{D2}+\Gamma_{2D}B_{1C}) &= \frac{1}{2}(\Gamma_{11}B_{12}+\Gamma_{21}B_{11} \\
&\quad +\Gamma_{11}B_{12}+\Gamma_{21}B_{11} \\
&\quad \Gamma_{12}B_{22}+\Gamma_{22}B_{12} \\
&\quad +\Gamma_{12}B_{22}+\Gamma_{22}B_{12}) \\
&= -\Gamma(B_{11}-B_{22}) \\
&= -2\Gamma\sigma_+
\end{aligned}
\tag{3.69}
$$

and so

$$
\begin{aligned}
\frac{d\sigma_\times}{d\lambda} &= -\frac{1}{2}\delta^{CD}(B_{1C}B_{D2} + B_{2C}B_{D1}) - R_{12} - 2\Gamma\sigma_+ - \frac{1}{2}\frac{d\theta}{d\lambda}\delta_{12} \\
&= -\frac{1}{2}[(B_{11} + B_{22})(B_{12} + B_{21})] - C_{12} - 2\Gamma\sigma_+ \\
&= -\theta\sigma_\times - C_\times - 2\Gamma\sigma_+ \qquad (3.70)
\end{aligned}
$$

where we have used

$$B_{11} + B_{22} = \theta$$

$$B_{12} + B_{21} = 2\sigma_\times$$

Twist:

$$
\begin{aligned}
\frac{d\omega_{AB}}{d\lambda} &= \frac{dB_{[AB]}}{d\lambda} \\
&= -\frac{1}{2}(B_{AC}B^C{}_B - B_{BC}B^C{}_A) \\
&= -\frac{1}{2}\delta^{CD}(B_{AC}B_{DB} - B_{BC}B_{DA}) \qquad (3.71)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\omega}{d\lambda} &= -\frac{1}{2}\delta^{CD}(B_{1C}B_{D2} - B_{2C}B_{D1}) \\
&= -\frac{1}{2}(B_{11}B_{12} - B_{21}B_{11} + B_{12}B_{22} - B_{22}B_{21}) \\
&= -\frac{1}{2}[(B_{11} + B_{22})(B_{12} - B_{21}) \\
&= -\theta\omega \qquad (3.72)
\end{aligned}
$$

where we have used

$$B_{12} - B_{21} = 2\omega_{12} = 2\omega$$

The term involving Γ_{AB}

$$\frac{1}{2}\delta^{CD}(\Gamma_{AC}B_{DB} + \Gamma_{BD}B_{AC} - \Gamma_{BC}B_{DA} - \Gamma_{AD}B_{BC})$$

does not contribute here:

$$\begin{aligned}
\delta^{CD}(\Gamma_{1C}B_{D2} + \Gamma_{2D}B_{1C} - \Gamma_{1C}B_{D2} - \Gamma_{2D}B_{1C}) &= \Gamma_{11}B_{12} + \Gamma_{21}B_{11} \\
&\quad -\Gamma_{11}B_{12} - \Gamma_{21}B_{11} \\
&\quad \Gamma_{12}B_{22} + \Gamma_{22}B_{12} \\
&\quad -\Gamma_{12}B_{22} - \Gamma_{22}B_{12} \\
&= 0. \qquad (3.73)
\end{aligned}$$

Summary

$$\begin{aligned}
\frac{d\theta}{d\lambda} &= -\frac{1}{2}\theta^2 - 2(\sigma_+^2 + \sigma_\times^2) + 2\omega^2 - \mathscr{R} \\
\frac{d\sigma_+}{d\lambda} &= -\theta\sigma_+ - C_+ + 2\Gamma\sigma_\times \\
\frac{d\sigma_\times}{d\lambda} &= -\theta\sigma_\times - C_\times - 2\Gamma\sigma_+ \\
\frac{d\omega}{d\lambda} &= -\theta\omega \qquad (3.74)
\end{aligned}$$

$$\begin{aligned}
\sigma^{ab}\sigma_{ab} &= \sigma^{AB}\sigma_{AB} \\
&= \sigma^{AB}\sigma_{BA} \\
&= Tr(\sigma^{AB}\sigma_{BC}) \\
&= 2(\sigma_+^2 + \sigma_\times^2) \qquad (3.75)
\end{aligned}$$

Now

$$\begin{aligned}
\omega^{ab}\omega_{ab} &= \omega^{AB}\omega_{AB} \\
&= -Tr(\omega^{AB}\omega_{BC}) \\
&= 2\omega^2 \qquad (3.76)
\end{aligned}$$

We have

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} - R_{ab}k^a k^b \tag{3.77}$$

The other equations can be written

$$\begin{aligned} \frac{d\sigma_{AB}}{d\lambda} &= -\theta\sigma_{AB} - C_{AB} \\ \frac{d\omega_{AB}}{d\lambda} &= -\theta\omega_{AB} \end{aligned} \tag{3.78}$$

and as we assume that $\hat{e}^a_A$ are parallel propagated, (3.78) implies

$$\begin{aligned} \frac{d\sigma_{ab}}{d\lambda} &= -\theta\sigma_{ab} + \widehat{C_{cbad}k^c}k^d \\ \frac{d\omega_{ab}}{d\lambda} &= -\theta\omega_{ab} \end{aligned} \tag{3.79}$$

3.9 Optical scalars

Geodesic equation

$$l^b \nabla_b l^a = 0 \tag{3.80}$$

Here we define

$$\bar{\theta} = \frac{1}{2}\nabla_a l^a = \frac{1}{2} g^{ab}\nabla_b l^a = \frac{1}{2}\theta \tag{3.81}$$

$$\bar{\omega} = \left(\frac{1}{2}\nabla_{[b}l_{a]}\nabla^b l^a\right)^{\frac{1}{2}} = \left(\frac{1}{2}\omega^{ab}\omega_{ab}\right)^{\frac{1}{2}} \tag{3.82}$$

$$|\bar{\sigma}| = \left(\frac{1}{2}\nabla_{(b}l_{a)}\nabla^b l^a - \bar{\theta}^2\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\left(\nabla_{(b}l_{a)}\nabla^b l^a - \frac{1}{2}\theta^2\right)^{\frac{1}{2}} \tag{3.83}$$

Calculating $\sigma^{ab}\sigma_{ab}$, we find

$$
\begin{aligned}
\sigma^{ab}\sigma_{ab} &= (\nabla^{(b}l^{a)} - \frac{1}{2}h^{ab}\theta)(\nabla_{(b}l_{a)} - \frac{1}{2}h_{ab}\theta) \\
&= \nabla^{(b}l^{a)}\nabla_{(b}l_{a)} - \nabla_{(b}l_{a)}h^{ab}\theta + \frac{1}{4}h^{ab}h_{ab}\theta^2 \\
&= \nabla^{(b}l^{a)}\nabla_{(b}l_{a)} - \frac{1}{2}\theta^2 \qquad (3.84)
\end{aligned}
$$

and so

$$
|\bar{\sigma}| = \frac{1}{\sqrt{2}}\left(\sigma^{ab}\sigma_{ab}\right)^{\frac{1}{2}} \qquad (3.85)
$$

Define

$$
z = -\bar{\theta} + i\bar{\omega} \qquad (3.86)
$$

Then the two evolution equations for the optical scalars

$$
\frac{d\bar{\theta}}{d\lambda} = -\bar{\theta}^2 + \bar{\omega}^2 - |\bar{\sigma}|^2 - \frac{1}{2}R_{ab}l^a l^b \qquad (3.87)
$$

and

$$
\frac{d\bar{\omega}}{d\lambda} = -2\bar{\theta}\bar{\omega} \qquad (3.88)
$$

can be written as a single equation

$$
\frac{dz}{d\lambda} = z^2 + |\bar{\sigma}|^2 + \frac{1}{2}R_{ab}l^a l^b. \qquad (3.89)
$$

Rewriting (3.87)

$$
\frac{1}{2}\frac{d\theta}{d\lambda} = -\frac{1}{4}\theta^2 + \frac{1}{2}\omega^{ab}\omega_{ab} - \frac{1}{2}\sigma^{ab}\sigma_{ab} - \frac{1}{2}R_{ab}l^a l^b
$$

we see it identical to (3.77). Calculating

$$
\begin{aligned}
\frac{d\bar{\omega}}{d\lambda} &= \frac{d}{d\lambda}\left(\frac{1}{2}\omega^{ab}\omega_{ab}\right)^{\frac{1}{2}} \\
&= \frac{1}{4\bar{\omega}}\frac{d(\omega^{ab}\omega_{ab})}{d\lambda} \\
&= \frac{1}{2\bar{\omega}}\omega^{ab}\frac{d\omega_{ab}}{d\lambda} \\
&= \frac{1}{2\bar{\omega}}\omega^{ab}\cdot -\theta\omega_{ab} \\
&= -2\bar{\theta}\bar{\omega}. \qquad (3.90)
\end{aligned}
$$

we obtain (3.88).