Derivation of Vacuum Field Equations of General Relativity

Chapter 1

Derivation of Vacuum Field Equations

1.1 Newtonian Theory of Gravity

We give a quick reminder of Newtoian theory of gravity in differential for. A distribution of matter of mass density $\rho(x, y, z)$ gives rise to a gravitational potential ϕ which satisfies Poisson's equation

$$\vec{\nabla}^2 \phi = 4\pi G \rho \tag{1.1}$$

where the Laplacian operator $\vec{\nabla}^2$ is given in Cartesian coordinates by

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In regions where there is no matter, Poisson's equation reduces to Laplace's equation

$$\vec{\nabla}^2 \phi = 0 \tag{1.2}$$

1.2 The Newtonian Equation of Deviation

We examine two neigbouring test particles in free fall in a gravitational field in Newtonian theory in Euclidian space. The first particle's curve is denoted by

$$x^\alpha(t)$$

and its equation of motion is given in terms of the gravitational potential $\phi(x)$

$$\ddot{x}^\alpha(t) = -\partial^\alpha \phi(x^a(t)) \tag{1.3}$$

The second particle is located at $y^a(t)$ with

$$y^a(t) = x^a(t) + \eta^\alpha(t) \tag{1.4}$$

defining a small connecting vector $\eta^\alpha(t)$ between the curves of the two particles. The second particle's equation of motion is

$$\begin{aligned} \ddot{x}^\alpha + \ddot{\eta}^\alpha &= -\partial^\alpha \phi(x^a(t) + \eta^\alpha(t)) \\ &= -\partial^\alpha \phi(x^a(t)) - \eta^\beta \partial_\beta \partial^\alpha \phi(x^a(t)) \end{aligned} \tag{1.5}$$

This implies the equation for the connecting vector

$$\ddot{\eta}^\alpha = -\eta^\beta \partial_\beta \partial^\alpha \phi. \tag{1.6}$$

This is called the Newtonian equation of deviation. Let

$$K^\alpha{}_\beta = K_\beta{}^\alpha = \partial^\alpha \partial_\beta \phi, \tag{1.7}$$

then the Newtonian equation of deviation can be written

$$\ddot{\eta}^\alpha + K^\alpha{}_\beta \eta^\beta = 0. \tag{1.8}$$

Laplace's equation, (1.2), can be expressed as

$$K^\alpha{}_\alpha = 0. \tag{1.9}$$

In the next sections we investigate the analogies between the Newtonian expression for the variation of the freely falling particles and the General Relativistic expression for the geodesic deviation. We then look at the vanishing of the trace of the analogy of $K^\alpha{}_\beta$ to find the vacuum field equations of General Relativity.

1.3 Equation of geodesic deviation

Write

$$x^a = x^a(\tau, \nu) \tag{1.10}$$

where τ is the proper time along the geodesic C_1 and ν paramaterises a curve connecdting the geodesic C_2. We define

$$v^a = \frac{dx^a}{d\tau} \tag{1.11}$$

and

$$\xi^a = \frac{dx^a}{d\nu}. \tag{1.12}$$

Then v^a is the tangent vector to the timelike geodesic at each point along C_1 and ξ^a is a connecting vector connecting the neighbouring curves.

Now

$$\begin{aligned} [v, \xi]^a &:= v^b \partial_b \xi^a - \xi^b \partial_b v^a \\ &= \frac{dx^a}{d\tau}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{d\nu}\right) - \frac{dx^a}{d\nu}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{d\tau}\right) \\ &= \frac{d^2x^a}{d\tau d\nu} - \frac{d^2x^a}{d\nu d\tau} \\ &= 0. \end{aligned} \tag{1.13}$$

We can replace partial derivatives with covariant ones:

$$\begin{aligned} 0 &= v^b \partial_b \xi^a - \xi^b \partial_b v^a \\ &= v^b \partial_b (\xi^a + \Gamma^a_{bc}\xi^c) - \xi^b(\partial_b v^a + \Gamma^a_{bc} v^c) \\ &= v^b \nabla_b \xi^a - \xi^b \nabla_b v^c \end{aligned} \tag{1.14}$$

where we have used $\Gamma^a_{bc} = \Gamma^a_{cb}$. Let use the directional derivative notation: $\nabla_X \equiv X^b \nabla_b$. So we have from (1.14)

$$\nabla_v \xi^a = \nabla_\xi v^a \tag{1.15}$$

Applying the directional derivative ∇_v to both sides gives

$$\nabla_v \nabla_v \xi^a = \nabla_v \nabla_\xi v^a. \tag{1.16}$$

Consider the identity

$$\nabla_X(\nabla_Y Z^a) - \nabla_Y(\nabla_X Z^a) - \nabla_{[X,Y]} Z^a = R^a{}_{bcd} Z^b X^c Y^d \tag{1.17}$$

If we set $X^a = Z^a = v^a$ and $Y^a = \xi^a$, then then the second is $\nabla_\xi(\nabla_v v^a)$ vanishes because v^a is tangent to a geodesic,

$$\nabla_v v^a = v^b \nabla_v v^a = 0. \tag{1.18}$$

The third term, $[v, \xi]^b \nabla_b v^a$, vanishes by (1.13). Thus (1.17) becomes

$$\nabla_v \nabla_\xi v^a - R^a{}_{bcd} v^b v^c \xi^d = 0. \tag{1.19}$$

Substituting (1.16) into this we obtain

$$\nabla_v \nabla_v \xi^a - R^a{}_{bcd} v^b v^c \xi^d = 0. \tag{1.20}$$

By definition

$$\frac{D^2 \xi^a}{D\tau^2} \equiv \nabla_v \nabla_v \xi^a$$

and so, the geodesic deviation equation is

$$\frac{D^2 \xi^a}{D\tau^2} = R^a{}_{bcd} v^b v^c \xi^d. \tag{1.21}$$

We are only interested in the spatial part of ξ^a and its geodesic deviation equation. To this end we introduce the projection operator which acts on tensors and gives the spatial information orthogonal to the timelike vector v^a.

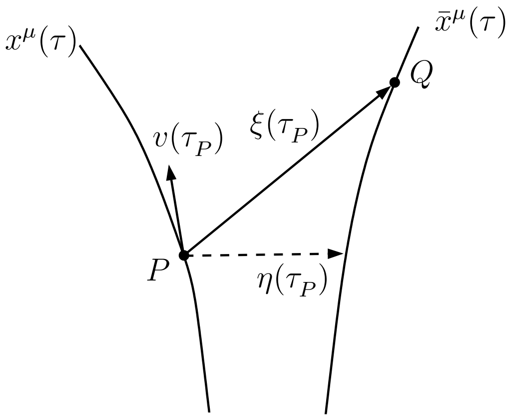

Figure 1.1: η is the orthogonal connecting vector.

1.3.1 Projection operator

First note that since $d\tau^2 = g_{ab}dx^a dx^b$,

$$v^a v_a = g_{ab}v^a v^b = g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = 1, \tag{1.22}$$

i.e. v^a is a unit tangent vector. The projection operator defined by

$$h^a{}_b := \delta^a{}_b - v^a v_b \tag{1.23}$$

it projects tensors into the three-space orthogonal to v^a

Obviously

$$h^a{}_b v^b = v^a - v^a v_b v^b = 0$$

If $w^a u_a = 0$ then

$$h^a{}_b w^b = w^a - v^a v_b w^b = w^a$$

and if $h^a{}_b w^b = w^a$ then

$$-v^a v_b w^b = w^a$$

and so

$$-v_b w^b = w^a v_a \quad \Rightarrow \quad w^a v_a = 0$$

$$\begin{aligned} h^a{}_b h^b{}_c &= (\delta^a{}_b - v^a v_b)(\delta^b{}_c - v^b v_c) \\ &= (\delta^a{}_c - 2v^a v_c + v^a v_b v^b v_c \\ &= \delta^a{}_c - v^a v_c = h^a{}_c \end{aligned} \tag{1.24}$$

$$h^a{}_a := \delta^a{}_a - v^a v_a = 4 - 1 = 3$$

Obviously

$$h_{ab} = h_{ba}$$

1.3.2 Orthogonal connecting vector and its equation of geodesic deviation

We thus define the orthogonal connecting vector η^a by

$$\eta^a := h^a{}_b \xi^b. \tag{1.25}$$

In the following we will use that

$$v_a \xi^b \nabla_b v^a = 0$$

which comes from

$$\begin{aligned} 0 &= \xi^b \nabla_b (1) \\ &= \xi^b \nabla_b (v^a v_a) \\ &= v_a \xi^b \nabla_b v^a + v^a \xi^b \nabla_b v_a \\ &= 2v_a \xi^b \nabla_b v^a \end{aligned} \tag{1.26}$$

since the covariant derivative of 1 is zero. This is because we can always go to a frame in free fall and use cartesian coordinates in which the connection Γ^a_{bc} vanishes. As

$$\xi^a = \eta^a + v^a v_b \xi^b$$

we have

$$\begin{aligned}
\frac{D\xi^a}{D\tau} &= v^c \nabla_c \xi^a \\
&= v^c \nabla_c (\eta^a + v^a v_b \xi^b) \\
&= v^c \nabla_c \eta^a + (v^c \nabla_c v^a) v_b \xi^b + v^a (v^c \nabla_c v_b) \xi^b + v^a v_b (v^c \nabla_c \xi^b) \\
&= v^c \nabla_c \eta^a + v^a v_b (\xi^c \nabla_c v^b) \\
&= \frac{D\eta^a}{D\tau},
\end{aligned} \tag{1.27}$$

where we used the geodesic equation $v^b \nabla_b v^a = 0$ and (1.26). We also have

$$R^a{}_{bcd} v^b v^c \xi^d = R^a{}_{bcd} v^b v^c (\eta^d + v^d v_e \xi^e) = R^a{}_{bcd} v^b v^c \eta^d \tag{1.28}$$

since $R^a{}_{bcd}$ is anti-symmetric in c and d.

So we have

$$\frac{D^2\eta^a}{D\tau^2} - R^a{}_{bcd} v^b v^c \eta^d = 0 \tag{1.29}$$

which is the same as (1.21) but with ξ^a replaced with η^a. However, this is still a four-vector equation whereas the Newtonian deviation equation is a three-vector equation.

1.4 The Newtonian Correspondence

At any point on the curve C_1, we introduce an orthogonal frame of three unit spacelike vectors

$$e_\alpha{}^a = (e_{\mathbf{1}}{}^a, e_{\mathbf{2}}{}^a, e_{\mathbf{3}}{}^a)$$

which are all orthogonal to v^a and where α is a bold label running from 1 to 3. We define

$$e_\alpha{}^a = v^a,$$

Combine these four vectors

$$e_{\mathbf{i}}{}^{a} \quad (\mathbf{i} = 0, 1, 2, 3).$$

They satisfy the orthonormaility relations

$$e_{\mathbf{i}}{}^{a} e_{\mathbf{j}a} = \eta_{\mathbf{ij}} \tag{1.30}$$

where $\eta_{\mathbf{ij}}$ is the Minkowski metric, that is,

$$\eta_{\mathbf{ij}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The four vectors are said to for a tetrad. We have introduced the frame notation for convience.

1.4.1 Frame field - and the precise analogue with η^α

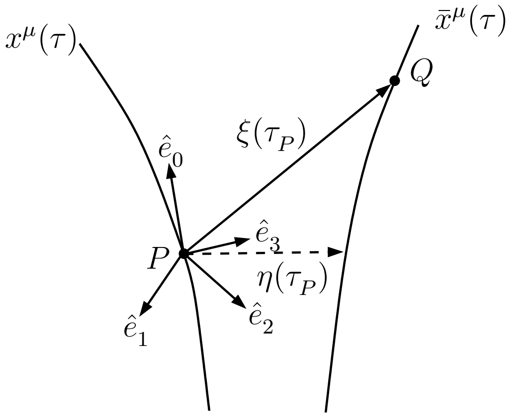

Figure 1.2: We find the spatial frame components η^α of the orthogonal connecting vector by projecting onto a spatial frame field. This is the precise analogue of the Newtonian connecting vector.

Treating $e_{\mathbf{i}}{}^{a}$ as a 4×4 matrix, we can define its inverse $e^{\mathbf{j}}{}_{a}$ by

$$e_{\mathbf{i}}{}^{a} e^{\mathbf{j}}{}_{a} = \delta_{\mathbf{i}}^{\mathbf{j}} \tag{1.31}$$

Multiply this by $e^{\mathbf{i}}{}_b$, we have

$$(e^{\mathbf{i}}{}_b e_{\mathbf{i}}{}^a) e^{\mathbf{j}}{}_a = e^{\mathbf{j}}{}_b$$

this implies

$$e^{\mathbf{i}}{}_b e_{\mathbf{i}}{}^a = \delta^a_b \tag{1.32}$$

We propagate the frame along C_1 by parallel propagation.

$$\frac{D}{D\tau}(e_{\mathbf{i}}{}^a) = 0 \tag{1.33}$$

We define spatial frame components of the orthogonal connecting vector η^α

$$\eta^\alpha = e^\alpha{}_a \eta^a \tag{1.34}$$

This is the precise analogue with η^α from the Newtonian equation of deviation. Note that

$$\eta^{\mathbf{0}} = e^{\mathbf{0}}{}_a \eta^a = e^{\mathbf{0}}{}_a h^a{}_b \xi^b = (v_a h^a{}_b) \xi^b = 0 \tag{1.35}$$

To find the spatial part of (1.29) we contract with $e^\alpha{}_a$, and then using parallel progation of the frame, we find

$$\frac{D^2 \eta^\alpha}{D\tau^2} - R^a{}_{bcd} e^\alpha{}_a v^b v^c \eta^d = 0 \tag{1.36}$$

Now

$$\begin{aligned} \eta^d &= \delta^d_c \eta^c \\ &= e_{\mathbf{i}}{}^d e^{\mathbf{i}}{}_c \eta^c \\ &= e_{\mathbf{0}}{}^d e^{\mathbf{0}}{}_c \eta^c + e_\beta{}^d e^\beta{}_c \eta^c = e_\beta{}^d \eta^\beta \end{aligned} \tag{1.37}$$

Using this the spatial part of the equation of geodesic deviation becomes

$$\frac{D^2 \eta^\alpha}{D\tau^2} + K_\beta{}^\alpha \eta^\beta = 0, \tag{1.38}$$

where

$$K^{\alpha}{}_{\beta} = -R^{a}{}_{bcd} e^{\alpha}{}_{a} v^{b} v^{c} e_{\beta}{}^{d} \tag{1.39}$$

We now have the analogue of the Newtonian deviation equation (1.8).

1.5 The Vacuum Field Equations

From the analogies between the Newtonian expression for the variation of the freely falling particles and the General Relativistic expression for the geodesic deviation, we investigate the vanishing of the trace (1.39), namely,

$$R^{a}{}_{bcd} e^{\alpha}{}_{a} v^{b} v^{c} e_{\alpha}{}^{d} = 0 \tag{1.40}$$

Let us introduce a special coordinate system in which

$$e_{i}{}^{a} = \delta_{i}{}^{a} \tag{1.41}$$

Then (1.40)

$$R^{\alpha}{}_{00\alpha} = 0.$$

Since the Riemann tensor is anti-symmetric in the last pair of indices

$$R^{0}{}_{000} = -R^{0}{}_{000} = 0.$$

Therefore

$$R^{a}{}_{00a} = 0.$$

Then

$$\begin{aligned} 0 &= R^{a}{}_{00a} \\ &= R^{a}{}_{bca} \delta^{b}_{0} \delta^{c}_{0} \\ &= R^{a}{}_{bca} v^{b} v^{c} \\ &= -R^{a}{}_{bac} v^{b} v^{c} \\ &= -R_{bc} v^{b} v^{c} \end{aligned} \tag{1.42}$$

As $R_{bc}v^b v^c$ is a scalar if it vanishes in one coordinate system it must vanish in all coordinate systems. Moreover, since it vanishes for all observers (world lines passing through P), it vanises for all v^a at P. We prove this in the following. Now

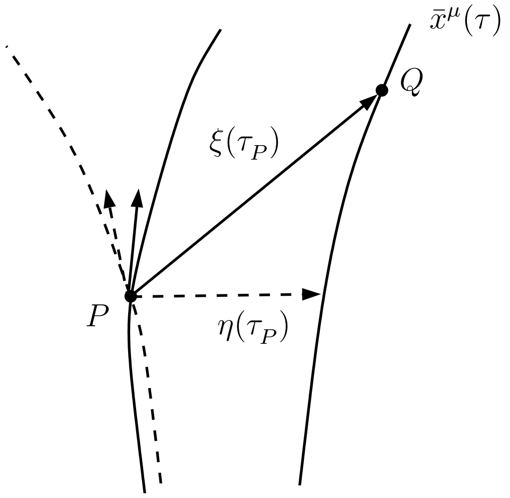

Figure 1.3: Different world line passing through P corresponds to different observer with different v^a.

$$R_{bc}v^b v^c = 0$$

for arbitrary timelike vector v^a (note that v^a need not be normalised). Let

$$v^a = u^a + \sum_{\alpha=1}^{3} \lambda^\alpha w^a_\alpha,$$

where $u^a u_a = 1$, $w^a_\alpha w_{\alpha a} = -1$, $u_a w^a_\alpha = 0$

$$v^a v_a = (u^a + \sum_{\alpha=1}^{3} \lambda^\alpha w^a_\alpha)(u_a + \sum_{\beta=1}^{3} \lambda^\beta w_{\beta a}) = u^a u_a - \sum_{\alpha=1}^{3} (\lambda^\alpha)^2 w^a_\alpha w_{\alpha a} = 1 - \sum_{\alpha=1}^{3} (\lambda^\alpha)^2$$

we must have $0 \leq \sum_{\alpha=1}^{3} (\lambda^\alpha)^2 < 1$, λ^α arbitrary otherwise.

$$R_{ab}(u^a + \sum_{\alpha=1}^{3} \lambda^\alpha w^a_\alpha)(u^b + \sum_{\beta=1}^{3} \lambda^\beta w^b_\alpha) = 0$$

or

$$R_{ab}u^a u^b + 2\sum_{\alpha=1}^{3} \lambda^\alpha R_{ab} u^a w^b_\alpha + \sum_{\alpha,\beta=1}^{3} \lambda^\alpha \lambda^\beta R_{ab} w^a_\alpha w^b_\beta = 0 \tag{1.43}$$

Consider the special coordinate system in which $u^a = \delta^a_0$ and $w^a_\alpha = \delta^a_\alpha$

Choose $\lambda^\alpha = 0$ then

$$R_{ab}u^a u^b = R_{ab}\delta^a_0 \delta^b_0 = R_{00} = 0.$$

Differentiate with respect to λ^α and put $\lambda^\alpha = 0$

$$R_{ab}u^a w^b_\alpha = R_{ab}\delta^a_0 \delta^b_\alpha = R_{0\alpha} = R_{\alpha 0} = 0 \tag{1.44}$$

Differentiate with respect to λ^α and λ^β then

$$R_{ab}w^a_\alpha w^c_\beta = R_{ab}\delta^b_\alpha \delta^b_\beta = R_{\alpha\beta} = 0$$

Since altogether $R_{ab} = 0$ holds in our special coordinate system it holds in all coordinate systems. So we have

$$[R_{ab}]_P = 0.$$

And finally since P is arbitrary, we find the vacuum field equation

$$R_{ab} = 0. \tag{1.45}$$