## C.12 Active Diffeomorphisms and the Lie Derivative

Up until now we have only considered coordinate transformation, that is, passive diffeomorphisms. We now move onto active diffeomorphisms. As their formulas look very alike the two are easily mixed up. But as we have seen in chapter 1 they are quite different, active diffeomorphisms relate *distinct* spacetime geometries, whereas a coordinate transformation merely represents the same spacetime geometry in a different coordinate system.

In chapter 1 we defined an active diffeomorphism as simultaneously dragging the metric and matter fields over the spacetime manifold while keeping the coordinate lines 'attached' (fig C.12). This is called a **pushforward**.

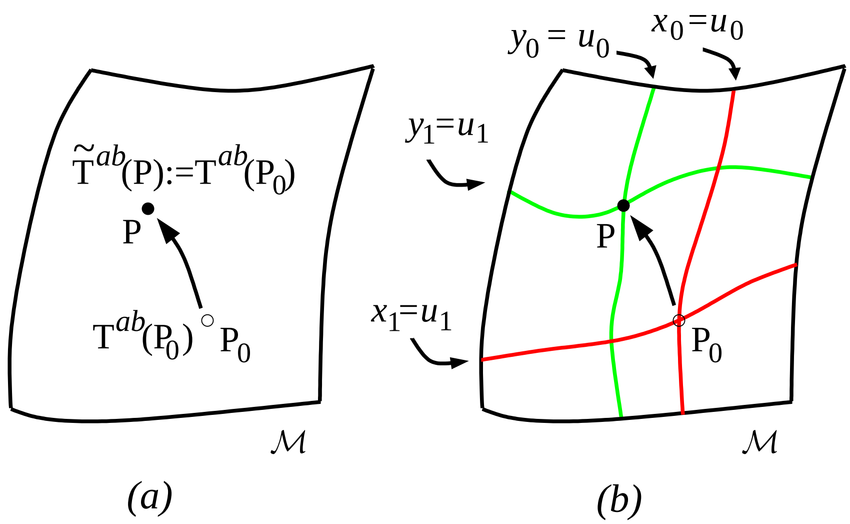


Figure C.20: activeDiffGeom. A pushforward of the tensor $T_{ab}(x)$, i.e. $T_{ab}(x) \to \tilde{T}_{ab}(y)$.

Let us slightly modify the definition of an active diffeomorphism by requiring that after we have dragged the fields across the manifold we perform a coordinate transformation back to the original coordinates. An active diffeomorphism defined this way then relates different space-time geometries and matter field configurations in the same coordinate system.

They relate $g_{ab}(x)$ to $\tilde{g}_{ab}(h(x))$ by the Jacobian matrix of the coordinate transformation $x \mapsto h(x)$,

$$\tilde{g}_{ab}(h(x)) = \Lambda_c^a \Lambda_d^b g_{ab}(x) \tag{C.490}$$

Two metrics related by an active diffeomorphism, viewed in the same coordinate system, compared at the same point also have 'transformation matrices', however, these have a different geometric interpretation!

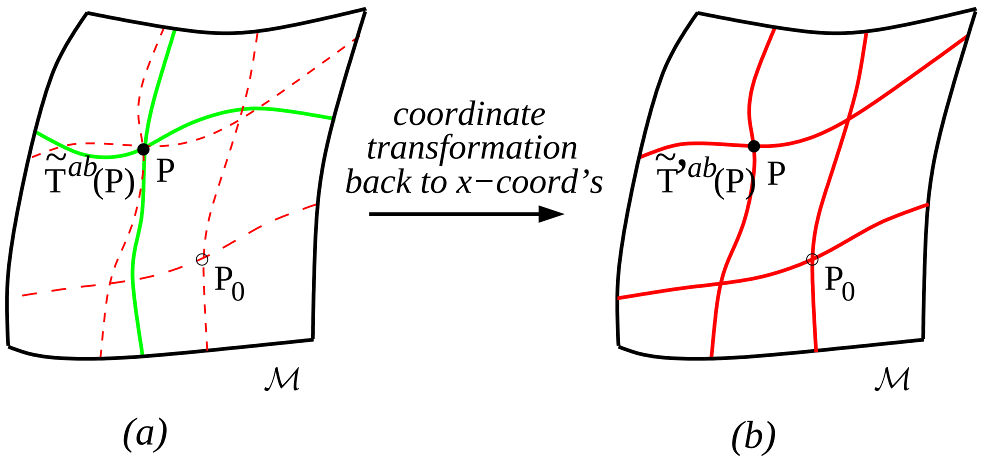


Figure C.21: activeDiffGeom1. The red dashed lines in (a) are the $x-$coordinate lines of the point $P$. We perform a coordinate transformation back to the original coordinate system. The pushed-forward tensor $\tilde{T}_{ab}(y)$ transforms to $\tilde{T}'_{ab}(x)$, i.e. $\tilde{T}_{ab}(y) \to \tilde{T}'_{ab}(x)$.

$$\tilde{g}_{ab}(x) = \frac{\partial h^c(x)}{\partial x^a} \frac{\partial h^d(x)}{\partial x^b} g_{cd}(h(x)) \tag{C.491}$$

The fact that the coordinate values do not change, while the tensor fields do, distinguishes the active diffeomorphism from a simple coordinate transformation.

Passive diffeomorphism invariance refers to invariance under change of coordinates, i.e. the same object represented in different coordinate systems. Choose a (local) coordinate system for $S$ in which the metirc $g_{ab}(x)$. (If the map $h$ sends each point to the same point of the manifold $\mathcal{M}$, then in the second system $S'$ the metric given by $\tilde{g}_{ab}(h(x))$, $f(x)$ being the coordinates on $\mathcal{M}$ of the second system.)

Any theory can be made invariant under passive diffeomorphisms because a dynamical system doesn't care which coordinate system you use to describe it. However, general relativity is the only theory invariant under active diffeomorphisms and this invariance is a property of the dynamical theory itself.

### Maths Tools for Manifold Without a Metric

In the previous section we reviewed metrics on manifolds, these are important in classical general relativity and are what a physicists is most likely to be familiar with. As we have learned, in reality it is only **geometry up to active diffeomorphisms** that has physical meaning. As we have empathized, in formulating the quantum theory we prefer not to employ metrics with its direct relation to the notion of distance.

There is a rich geometric structure of the manifold without a metric defined on it. Important tools of the Lie derivative and differential forms which have nothing to do with metrics. These will be important in the quantum theory where we will avoid introducing a background metric whenever possible.

### C.12.1 Mapping a Manifold to Itself Along Integral Curves

We start by considering a congruence of curves defined such that only one curve goes through each point in the manifold. Then, given any one curve of the congruence,

$$x^\mu = x^\mu(u), \tag{C.492}$$

we can use it to define the tangent vector field $dx^\mu/du$ along the curve. If we do this for every curve in the congruence, then we end up with a vector field $X^\mu$ (given by $dx^\mu/du$ at every point) defined over the whole manifold, then this can be used to define a congruence of curves in the manifold called the orbits or trajectories of $X^\mu$.

a smooth, non-intersecting family of curves on a manifold then the tangent vectors at each point can be taken together to form a vector field on the manifold.

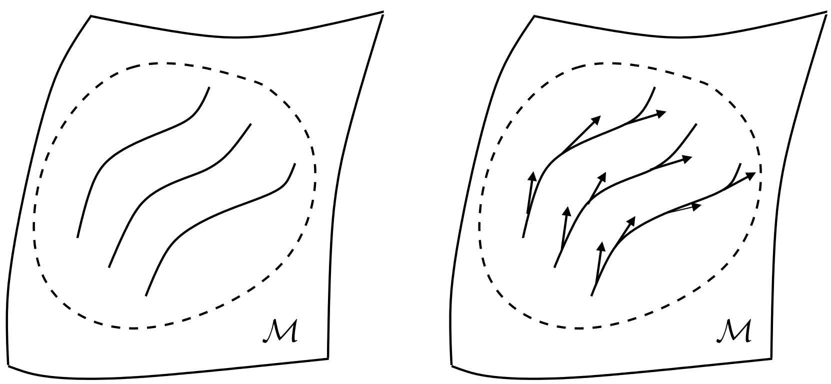


Figure C.22: The tangent vector field resulting from a congruence of curves.

These curves are obtained by solving differential equations

$$\frac{dx^\mu}{du} = X^\mu(x(u)) \tag{C.493}$$

Let $x^i$ be a local coordinate system and let $x_p^i$ be the coordinates of $p$. The equation of the integral curve is

$$\frac{d}{dt}x^i(t) = X^i(x^m(t)),$$

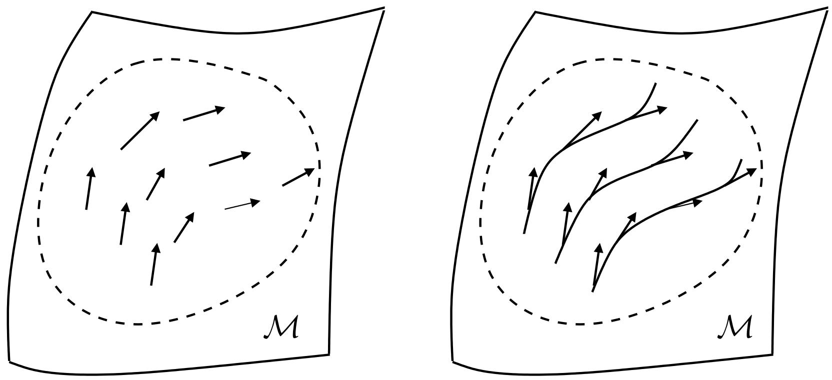


Figure C.23: The local congruence of curves resulting from vector field.

with initial conditions $x^i(0) = x^i_p$. Provided $X$ is smooth the theory of ordinary differential equations guarantees the existence and uniqueness, (at least locally, i.e., for small $t$), of a solution. Uniqueness implies that no two curves in the congruence intersect (at least locally).

**Definition** A congruence of curves is a family of curves such that precisely one curve of the family passes through each point. It is a geodesic congruence if the curves are geodesics.

**Active Diffeormorphisms**

## C.12.2 The Lie Derivative

This is called an *active transformation*. The *passive transformation* is a coordinate transformation.

A contravariant vector flow determines a local congruence of curves,

$$x^a = x^a(u),$$

where the tangent vector field to the congruence is

$$\frac{dx^a}{du} = X^a.$$

at least locally, a vector field generates a unique integral flow about any given point $p$. We use this flow to take a tensor to a nearby point and hence form a derivative. This derivative is called the Lie derivative.

$$\sigma^b(\epsilon, p) = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.494}$$

The Lie derivative of a scalar field $f \in \mathcal{C}^\infty(\mathcal{M})$. Let $X$ be a vector field on $\mathcal{M}$ we define the Lie derivative of $f$ along $X$ to be

$$\mathcal{L}_X f(p) = \lim_{\epsilon \to 0} \frac{f(\sigma(\epsilon, p)) - f(p)}{\epsilon} \tag{C.495}$$

which is the usual directional derivative along $X$.

point transform

$$x'^b = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.496}$$

We generate a new vector (with vector components in the $x'$ coordinates). By definition its components are related to $T^a(x)$ by a pushforward

$$\tilde{T}'^a(x') := T^a\left(x^c + \epsilon X^c(x)\right) = T^a(x) + \epsilon X^c(x)\partial_c T^{ab}(x) + \mathcal{O}(\epsilon^2). \tag{C.497}$$

We now wish to transform this tensor to the $x-$coordinates so we can compare it with the original tensor $T^{ab}(x)$. Using (C.496) we have

$$\frac{\partial x^a}{\partial x'^c} = \delta^a_c - \epsilon \partial_c X^a + \mathcal{O}(\epsilon^2) \tag{C.498}$$

The parameter distance derivative of an object along the vector field is the *Lie derivative.*

$$\begin{aligned} \tilde{T}^a(x) &= \frac{\partial x^a}{\partial x'^c} \tilde{T}'^c(x') \\ &= (\delta^a_c - \epsilon \partial_c X^a)(T^c(x) + \epsilon X^e \partial_e T^c) + \mathcal{O}(\epsilon^2) \\ &= T^a(x) + [X^e \partial_e T^a - \partial_c X^a T^c(x)]\epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{C.499}$$

$$\mathcal{L}_X T^a = \lim_{\epsilon \to 0} \frac{\tilde{T}^a(x) - T^a(x)}{\epsilon} \tag{C.500}$$

$$\mathcal{L}_X T_a(x) = X^c \partial_c T_a + T_b \partial_c X^a \tag{C.501}$$

What is the Lie derivative for a tensor $T^{ab}(x)$? We generate a new tensor (with tensor components in the $x'$ coordinates). By definition its components are related to $T^{ab}(x)$ by a pushforward

$$\tilde{T}'^{ab}(x') := T^{ab}\left(x^c + \epsilon X^c(x)\right) = T^{ab}(x) + \epsilon X^c(x)\partial_c T^{ab}(x) + \mathcal{O}(\epsilon^2). \tag{C.502}$$

We now wish to transform this tensor to the $x-$coordinates so we can compare it with the original tensor $T^{ab}(x)$. Using (C.498) again. The parameter distance derivative of an object along the vector field is the *Lie derivative*.

$$\begin{aligned}\tilde{T}^{ab}(x) &= \frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}\tilde{T}'^{cd}(x') \\ &= (\delta^a_c - \epsilon\partial_c X^a)(\delta^b_d - \epsilon\partial_d X^b)(T^{cd}(x) + \epsilon X^e\partial_e T^{cd}) + \mathcal{O}(\epsilon^2) \\ &= T^{ab}(x) + [X^e\partial_e T^{ab} - \partial_c X^a T^{cb}(x) - \partial_d X^b T^{ad}(x)]\epsilon + \mathcal{O}(\epsilon^2)\end{aligned} \tag{C.503}$$

$$\mathcal{L}_X T^{ab} = \lim_{\epsilon\to 0}\frac{\tilde{T}^{ab}(x) - T^{ab}(x)}{\epsilon} \tag{C.504}$$

$$\mathcal{L}_X T^{ab} = X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a. \tag{C.505}$$

$$\mathcal{L}_X T_a(x) = X^c\partial_c T_a + T_b\partial_c X^a, \qquad \mathcal{L}_X T_{ab}(x) = X^c\partial_c T_{ab} + T_{cb}\partial_a X^c + T_{ac}\partial_b X^c$$

The first term of the Lie derivative, $X^c\partial_c$, corresponds to the pushforward, shifting the tensor to another point in the manifold. The remaining terms arise from the coordinate transformation back to the original coordinates. Is it coordinate invariant? Does it have the same form in all coordinate systems? In fact (C.501) is equivalent to:

$$X^c\nabla_c T^a - T^c\nabla_c X^a \tag{C.506}$$

since

$$\begin{aligned}\mathcal{L}_X T^a(x) &= X^c\partial_c T^a - T^c\partial_c X^a \\ &= X^c(\partial_c T^a + \Gamma^a_{dc}T^d) - T^c(\partial_c X^a + \Gamma^a_{dc}X^d) \\ &= X^c\nabla_c T^a - T^c\nabla_c X^a\end{aligned} \tag{C.507}$$

where we have used that the connection is symmetric in its lower indices. Similarly, (C.505) is equivalent to:

$$
\begin{aligned}
& X^c\nabla_c T^{ab} - T^{ac}\nabla_c X^b - T^{cb}\nabla_c X^a \\
= \; & X^c(\partial_c T^{ab} + \Gamma^a_{dc}T^{db} + \Gamma^b_{dc}T^{ad}) - T^{ac}(\partial_c X^b + \Gamma^b_{dc}X^d) - T^{cb}(\partial_c X^a + \Gamma^a_{dc}X^d) \\
= \; & X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a + T^{db}X^c(\Gamma^a_{dc} - \Gamma^a_{cd}) + T^{ad}X^c(\Gamma^b_{dc} - \Gamma^b_{cd}) \\
= \; & X^c\partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a
\end{aligned} \tag{C.508}
$$

In general, the partial derivatives appearing in Lie derivatives can be replaced by covariant derivatives. Hence, the combination of pushback and coordinate transformation make the Lie derivative a tensor in the tangent space at $x^a$.

$$
\tilde{T}'(q) = T(h_\epsilon(p)) \qquad \tilde{T}(p) = h_{\epsilon *}[T(h_\epsilon(p))] \tag{C.509}
$$

We can write down the coordinate free equation

$$
(\mathcal{L}_X T)(p) = \frac{h_{\epsilon *}[T(h_\epsilon(p))] - T(p)}{\epsilon} \tag{C.510}
$$

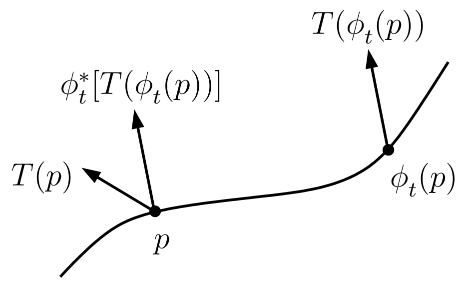


Figure C.24: .

the original tensor components at a different point. distinguishes the Lie derivative from the directional derivative.

the curve passing through P is given by $x^1$ varying, with $x^2, x^3, x^4$ all constant along the curve, and such that

$$
X^\alpha \stackrel{*}{=} \delta^\alpha_1 = (1, 0, 0, 0) \tag{C.511}
$$

along this curve. The notation used in means that the equation holds only in a particular coordinate system. Then it follows that

$$
X = X^\alpha \partial_\alpha = \partial_1, \tag{C.512}
$$

and equation reduces to

$$L_X T_{\alpha\beta} \stackrel{*}{=} \partial_1 T_{\alpha\beta} \tag{C.513}$$

Thus, in this special coordinate system, Lie differentiation reduces to ordinary differentiation.

If we have a map $\phi$ from a manifold $\mathcal{M}$ to another manifold $\mathcal{N}$, and we choose a point $x \in \mathcal{M}$, we can *push forward* a vector from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$, by a head-to-head and tail-to-tail map. If the vector has components $X^\mu$ and the map takes the point with coordinates $x^\mu$ to one with coordinates $\xi(x)$, the vector $\phi_* X$ has components

$$(\phi_* X)^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} X^\nu. \tag{C.514}$$

This looks like the transformation formula for contravariant vector components under a coordinate transformation, but we are doing an active transformation, changing a vector into a different one.

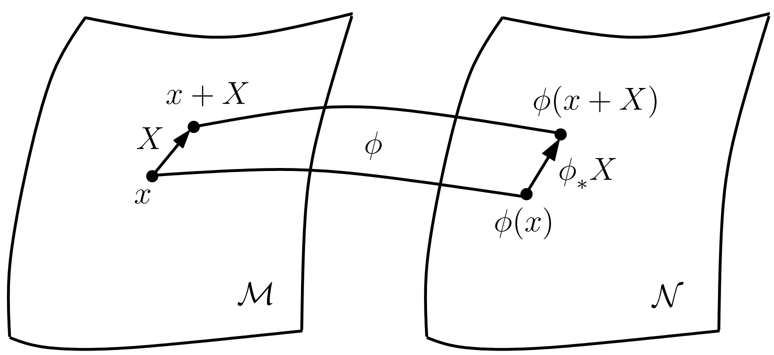


Figure C.25: pullbackDef0. Pushing forward a vector $X$ from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$.

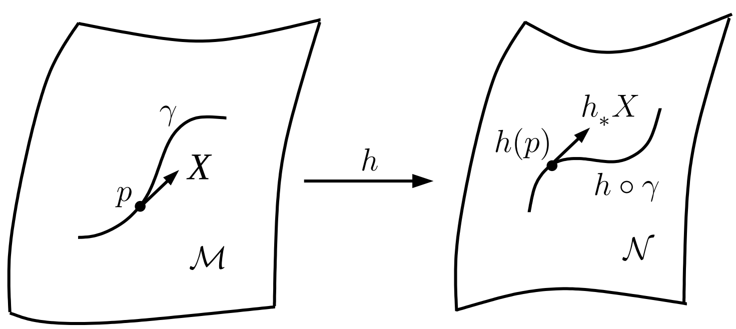


Figure C.26: The **push-forward map** $h_*$ that maps the tangent spaces of $\mathcal{M}$ *linearly* into the tangent spaces of $\mathcal{N}$.

pushforward $\phi_*$ (**??**)

Recall that a one-form maps a vector to a number. Given a one-form $\omega$ on $\mathcal{N}$, we define $\phi^*\omega$ as a one-form on $\mathcal{M}$ by specifying what we get when we plug the vector $X$ at $x \in \mathcal{M}$ into it. This we do by pushing the $X$ forward to $T\mathcal{N}_{\phi(x)}$, plugging it into $\omega$, and declaring the result to be the evaluation of $\phi^*\omega$ on the $X$. Symbolically

$$[\phi^*\omega](X) = \omega(\phi_* X). \tag{C.515}$$

or in components

$$[\phi^*\omega]_a X^a = \omega_a [\phi_* X]^a. \tag{C.516}$$

We work a coordinate system $(x^1, \ldots, x^m)$, such that $x^1$ is the parameter along the integral curves and the other coordinates are choosen any way. In this coordinate system and the components of the tensor pulled back from $\phi_t(p)$ to $p$ are simply

$$\phi_{t\star}[T^{a_1 \ldots a_k}_{b_1 \ldots b_l}(\phi_t(p))] \stackrel{*}{=} T^{a_1 \ldots a_k}_{b_1 \ldots b_l}(x^1 + t, x^2, \ldots, x^n). \tag{C.517}$$

In this coordinate system the Lie derivative becomes

$$\mathcal{L}_V T^{a_1 \ldots a_k}_{b_1 \ldots b_l} \stackrel{*}{=} \frac{\partial}{\partial x^1} T^{a_1 \ldots a_k}_{b_1 \ldots b_l}, \tag{C.518}$$

**Coordinate-Free Description**

We will prove

$$\mathcal{L}_X Y = [X, Y]. \tag{C.519}$$

$$\sigma^b(\epsilon, p) = x^b(p) + \epsilon X^b(p) + \mathcal{O}(\epsilon^2) \tag{C.520}$$

for any $f$

$$
\begin{aligned}
Y_{\sigma(\epsilon)} f &= \sum_a Y^b(\sigma(\epsilon)) \frac{\partial}{\partial x^a}\Big|_{\sigma(\epsilon)} f \\
&= \sum_a (Y^b + \epsilon X^c \partial_c Y^a)\Big|_{\sigma(\epsilon)} f + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b + \epsilon X^c \partial_c Y^a)(\partial f + \epsilon X^c \partial_a \partial_c f) + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b \partial_a + \epsilon X^c \partial_c Y^a \partial_a + \epsilon Y^a X^c \partial_c \partial_a) f + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{C.521}
$$

Therefore

$$
\begin{aligned}
\sigma(-\epsilon)_* Y_{\sigma(\epsilon)} f &= Y_{\sigma(\epsilon)}(f \circ \sigma(-\epsilon)_*) \\
&= \sum_a Y^a\ (h \circ \sigma(\epsilon)) \frac{\partial}{\partial x^a}\Big|_p (h \circ \sigma(-\epsilon)) + \mathcal{O}(\epsilon^2) \\
&= \sum_a (Y^b \partial_a + \epsilon X^c \partial_c Y^a \partial_a + \epsilon Y^a X^c \partial_c \partial_a)(f - \epsilon \partial_b f X^b) + \mathcal{O}(\epsilon^2) \\
&= \sum_a Y^b \partial_a f + \epsilon(\partial_c Y^a X^c - Y^c \partial_c X^a) \partial_a f + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{C.522}
$$

From which

$$
\begin{aligned}
\frac{\sigma(-\epsilon)_* Y_{\sigma(\epsilon)}(f) - Y(f)}{\epsilon} &= (\partial_c Y^a X^c - Y^c \partial_c X^a) \partial_a f + \mathcal{O}(\epsilon^2) \\
&= [X, Y]^a \partial_a f + \mathcal{O}(\epsilon^2) \\
&= [X, Y]^a (f) + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{C.523}
$$

**The Lie derivative of a covariant tensors**

$$
L_X Y^\alpha = X^\beta \partial_\beta Y^\alpha - Y^\beta \partial_\beta X^\alpha \tag{C.524}
$$

The Lie derivative of a covariant vector field $Y_\alpha$ is given by

$$
L_X Y_\alpha = X^\beta \partial_\beta Y_\alpha + Y^\beta \partial_\alpha X^\beta \tag{C.525}
$$

## The Lie Derivative

there is a coordinate system in which

$$\mathcal{L}_{\vec{N}}\vec{M} = N^a\partial_a M_b - M^a\partial_a N_b \tag{C.526}$$

It satisfies the Leibniz rule

$$L_X(Y^a Z_{bc}) = Y^a(L_X Z_{bc}) + (L_X Y^a)Z_{bc}. \tag{C.527}$$

It is type-preserving; that is, the Lie derivative of s tensor of type(p,q) is again a tensor of type (p,q).

The Lie derivative of a scalar field $\phi$ is simply an ordinary derivative in the direction of $X$

$$L_X\phi = X\phi = X^a\partial_a\phi \tag{C.528}$$

Now, given the Lie derivative of a vector and a scalar, we can apply the Leibniz rule to deduce the Lie derivative of a covariant vector field $Y_a$: consider the Lie derivative of the scalar formed by the contraction of an arbitrary vector $Z^a$ with an arbitrary covector $Y_a$.

$$L_X(Y_c Z^c) = X^b\partial_b(Y_c Z^c) = Z^c X^b\partial_b Y_c + Y_c X^b\partial_b Z^c \tag{C.529}$$

whereas the Leibniz rule gives

$$L_X(Y_c Z^c) = Y_c L_X Z^c + (L_X Y_c)Z^c. \tag{C.530}$$

$$Z^c L_X Y_c = Z^c X^b\partial_b Y_c + Z^c Y_b\partial_a X^c. \tag{C.531}$$

but as $Z^c$ is arbitrary this means

$$L_X Y_a == X^b\partial_b Y_a + Y_b\partial_a X^b. \tag{C.532}$$

$$\bar{T} := \phi_{\Delta\lambda}T(p_0) \quad \overline{x}^a(P) = x^a(P_0) \tag{C.533}$$

$$\mathcal{L}_\xi T := \lim_{\Delta\lambda}\frac{\phi_{\Delta\lambda}T(x) - T(x)}{\Delta\lambda} \tag{C.534}$$

$$\mathcal{L}_X T^{a\dots}_{b\dots} = X^c\partial_c T^{a\dots}_{b\dots} - T^{c\dots}_{b\dots}\partial_c X^a - \dots + T^{a\dots}_{c\dots}\partial_b X^c + \cdots \tag{C.535}$$