P.15 Fock-Bargmann Representation

In the conventional representation the Hilbert space of vectors is formed by the space of complex valued, square intergrable, coordinate or momentum functions $\psi(q)$ and $\tilde{\psi}(p)$. No analyticity conditions are placed on these complex functions. However, there exists a representation in which any state vector is described by an entire analytic function (a function that is analytic in every open set of the complex plane $\mathbb{C}$ is called an entire analytic function). This is called the Fock-Bargmann representation.

Applications are.... Ashtekar variables quantization of simple model. Where explicitly one can get the inner production from the requirement that real observables should correspond to self-adjoint operators.

An arbitrary state $|\psi>$ of the Hilbert space can be expanded in the Harmonic oscillator basis states $\{|n>\}$,

$$|\psi> = \sum_{n=0}^{\infty} c_n |n>, \qquad <\psi|\psi> = \sum_{n=0}^{\infty} |c_n|^2 = 1 \tag{P.202}$$

For the coherent state

$$|\alpha> = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n> \tag{P.203}$$

The projection of the state $|\psi>$ onto the coherent state $|z>$ is

$$<\alpha|\psi> = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} c_n \frac{\overline{\alpha}^n}{\sqrt{n!}} \tag{P.204}$$

then

$$\begin{aligned} <\alpha|\psi> &= \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} c_n u_n(\overline{\alpha}) \\ &= \exp(-|\alpha|^2/2) \psi(\overline{\alpha}) \end{aligned} \tag{P.205}$$

so we have

$$\psi(z) = \sum_{n=0}^{\infty} c_n u_n(z), \qquad u_n(z) = \frac{z^n}{\sqrt{n!}}. \tag{P.206}$$

The series (P.206) converges uniformally in any compact domain of the complex plane $\mathbb{C}$ because of the condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$ (see next section on the Weierstrass M-test). Further as a consequence of this (see the section following the next section), $\psi(z)$ an entire analytic function.

normalized according to

$$\|\psi\|^2 = <\psi|\psi> = \int \exp(-|z|^2)|\psi(z)|^2 d\mu(z) \ < \infty \tag{P.207}$$

Tha scalar product of two entire functions, satifying (P.207), is defined by

$$<\psi_1|\psi_2> = \int \exp(-|z|^2)\overline{\psi}_1(z)\psi_2(z)d\mu(z) \tag{P.208}$$

P.15.1 Weierstrass M-test

Theorem P.15.1 *Let $\sum_{k=1}^{\infty} f_k(z)$ be a series of functions, with each function defined on a subset U of $\mathbb{C}$. Suppose $\sum_{k=1}^{\infty} M_k$ is a series of real numbers such that:*

(i) $0 \leq |f_k(z)| \leq M_k$;

(ii) the series $\sum_{k=1}^{\infty} M_k$ converges

then $\sum_{k=1}^{\infty} f_k(z)$ converges uniformally.

Proof. For a series to be uniformally convergent, given any $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, we have

$$|\sum_{k=1}^{\infty} f_k(z)| < \epsilon,$$

for all $z \in U$. Since $\sum_{k=1}^{\infty} M_k$ converges, we know that we can find an N so that for all $n \geq N$, we have

$$\sum_{k=1}^{\infty} M_k < \epsilon.$$

Since $0 \geq |f_k(z)| \leq M_k$, for all $z \in U$, we have

$$|\sum_{k=1}^{\infty} f_k(z)| \leq \sum_{k=1}^{\infty} |f_k(z)| \leq \sum_{k=1}^{\infty} M_k < \epsilon,$$

$\square$

Example 1.

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

$$\left|\frac{z^{n+1}}{z^n}\right| = |z|, \tag{P.209}$$

the Taylor series converges for $|z| < 1.$, but fails to converge for $|z| > 1$. $|z| = 1$ is said to be the *radius of convergence*

Consider the geometric series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k.$$

for $|z| < 1$.

we show that this series converges uniformally on any disc $|z| \leq a < 1$. Set

$$M_k = a^k.$$

For all z, we have $0 < |z|^n \leq a^n$. We know that the geometric series for $a < 1$, hence the geometric series converges uniformaly on any disc with radius less than 1 about the origin of the complex plane $\mathbb{C}$.

Example 2. Consider the series

$$\sum_{k=1}^{\infty} \frac{z^k}{k!}.$$

we show that this series converges uniformally on any disc $|z| \leq a$. Set

$$M_k = \frac{a^k}{k!}.$$

For all z , we have $0 < |z|^n/n! \leq a^n/n!$. Thus as the series $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} a^k/k!$ converges, we will have uniform convergence. Convergence comes from the ratio test

$$\lim_{k\to\infty}\frac{M_{k+1}}{M_k}=\lim_{k\to\infty}\frac{\frac{a^{k+1}}{(k+1)!}}{\frac{a^k}{k!}}=\lim_{k\to\infty}\frac{a}{k+1}=0.$$

We find the Taylor series for e^z converges uniformaly on any disc about the origin of the complex plane $\mathbb{C}$.

Example 3. Consider the function

$$\frac{1}{1-(1/z)}=\sum_{k=0}^{\infty}\frac{1}{z^k}.$$

for $|z|>1$.

Set

$$M_k=\frac{1}{a^k}.$$

For all z, we have $0<|1/z|^n\leq(1/a)^n$. Hence the function converges uniformaly outside any disc of radius more than 1 about the origin of the complex plane $\mathbb{C}$.

Coherent state expansion

The same proof slightly modified will give the required result.

Consider the series

$$f(z)=\sum_k\frac{c_n}{\sqrt{k!}}z^k \tag{P.210}$$

where $\sum_n|c_n|^2=1$.

Suppose, given ϵ, there exists an integer N such that for all $n\geq N$, we have

$$|\sum_{k=n}^{\infty}f_k(z)|^2<\epsilon^2, \tag{P.211}$$

for all $z\in U$, then for all $n\geq N$, we have

$$|\sum_{k=n}^{\infty}f_k(z)|<\epsilon,$$

for all $z \in U$, i.e., the series $\sum_{k=1}^{\infty} f_k(z)$ is uniformally convergent.

We will prove the series (P.210) is uniformally convergent by proving it satifies the statement regarding (P.211). As $\sum_k |c_k|^2 = 1$ we have

$$\sum_{k=n} \frac{|c_k|^2}{k!} < \infty$$

from which follows that we can find an N so that for all $n \geq N$, we have $\sum_{k=n} |c_k|^2/k! < \epsilon^2$. Therefore

$$|\sum_{k=n}^{\infty} f_k(z)|^2 \leq \sum_{k=n}^{\infty} |f_k(z)|^2 \leq \sum_{k=n}^{\infty} |c_k|^2/k! < \epsilon^2,$$

proving the inequality (P.211).

□

P.15.2 From the Space of Normalizable Functions to the Space of Entire Analytic Functions

Let U be an open set of the complex plane $\mathbb{C}$. A function $f(z)$ is analytic at z_0 if and only if in a neighbourhood of z_0, $f(z)$ is equal to a uniformaly convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{P.212}$$

Recall a squence of functions $f_k(z)$ converges uniformaly to a function $f(z)$ if eventually all the functions $f_k(z)$ fall within any $\epsilon-$tube about the limit function $f(z)$.

If case.

Here we take the sequence functions to be *partial sums*. Let $h_1(z), h_2(z), \ldots$ be a sequence of functions. The series of functions

$$f(z) = \sum_{k=1}^{\infty} h_k(z)$$

converges uniformaly to a function $f(z)$ if the sequence of partial sums $f_1(z) = h_1(z)$, $f_2(z) = h_1(z) + h_2(z)$, $f_3(z) = h_1(z) + h_2(z) + h_3(z) \ldots$ converges uniformaly to $f(z)$.

An example of particular interest is when $h_k(z) = a_k(z - z_0)^k$ and

$$f_k(z) = \sum_{n=0}^{k} a_n(z - z_0)^n. \tag{P.213}$$

This allows for a notion of uniform convergence for series. A series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformaly in an open set U of the complex plane $\mathbb{C}$ if the sequence of polynomials $\{\sum_{n=0}^{N} a_n(z - z_0)^n\}$ converges uniformaly in U.

Theorem P.15.2 *Let the sequence $\{f_n(z)\}$ of analytic functions converge uniformaly on an open set U to a function $f : U \to \mathbb{C}$. Then the function $f(z)$ is also analytic and the sequence of derivatives $(f_n'(z))$ converge pointwise to the derivative $f'(z)$ on the set U.*

By this theorem, since polynomials are analytic, we conclude that if

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{P.214}$$

is a uniformaly convergent series, then the function $f(z)$ is analytic.

Only if case.

Suppose we have a function f which is analytic about a point z_0. Take a closed contour C around z_0. By the cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw, \tag{P.215}$$

For any z inside C. Now we know for the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

for $|r| < 1$, it follows for all w and z with $|z - z_0| < |w - z_0|$.

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n. \end{aligned} \tag{P.216}$$

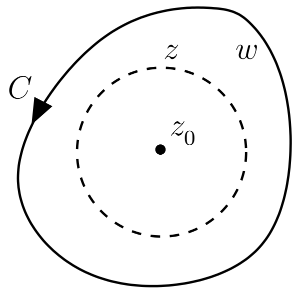

Figure P.2: contfornth. The contour used to evaluate the nth derivative, $|w-z_0| > |z-z_0|$.

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw \\
&= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad \text{(P.217)}
\end{aligned}
$$

swaping the integral and summation follows from the fact that the series $\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$ converges uniformaly. The expansion (P.217) is finite wherever the series does.

Where we used the Cauchy Integral Formula

$$
f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw.
$$

Prove the first of these statements. From the triangle inequality

$$
\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz| \le |\max f(z)|L \qquad \text{(P.218)}
$$

The series of partial sums

$$
s_k(z) = \sum_{n=0}^{k} f_n(z) \qquad \text{(P.219)}
$$

Uniform convergence implies that, given any $\epsilon > 0$, there exists some N such that $|f(z) - s_k(z)| < \epsilon$ everywhere on C. It follows

$$\left| \int_C f(z)dz - \int_C s_n(z)dz \right| \leq \epsilon L \tag{P.220}$$

or as $s_n(z)$ is a finite sum we can interchange integration and summation,

$$\left| \int_C f(z)dz - \sum_{k=0}^{n} \int_C f_k(z)dz \right| \leq \epsilon L \tag{P.221}$$

By choosing n large enough ϵL can be arbitary small, so

$$\lim_{n\to\infty} \sum_{k=0}^{n} \int_C f_k(z)dz = \int_C f(z)dz \tag{P.222}$$

or

$$\sum_{n=0}^{\infty} \int_C f_n(z)dz = \int_C \sum_{n=0}^{\infty} f_n(z)dz. \tag{P.223}$$

□

The proof

$$\begin{aligned} \frac{f(\xi) - f(z)}{\xi - z} &= \frac{1}{2\pi i} \int_C \left[\frac{f(w)}{(w-\xi)} - \frac{f(w)}{(w-z)} \right] \frac{dw}{w-z} \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-\xi)(w-z)} dw \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} \left(1 + \frac{\xi - z}{w - \xi} \right) dw. \end{aligned} \tag{P.224}$$

writing $\xi - z = \epsilon e^{i\theta}$

$$\left| \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} - \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} dw \right| \le \frac{1}{2\pi} \lim_{\epsilon \to 0} \epsilon \oint_C \frac{|f(w)||dw|}{|(w-z) - \epsilon e^{i\theta}||w-z|^2} \tag{P.225}$$

We replace $|w - z|$ by its maximum value, m, and $|f(w)|$ by its maximum value, M, we obtain

$$\left| \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} - \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} dw \right| \le \frac{1}{2\pi} \frac{ML}{m^2} \lim_{\epsilon \to 0} \frac{\epsilon}{m - \epsilon} = 0, \tag{P.226}$$

where L is the length of the contour. Repeating the process, we obtain for the nth derivative

$$f^{(n)} = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw. \tag{P.227}$$

Proof of theorem (P.15.2).

(a) We show term-by-term differentiation or integration of a power series yields a new power series with the same radius of curvature.

(b) The uniform-convergence property of a power series implies that term-by-term integration yields the integral of the sum function. We show that the integrated sum function is single-valued and analytic within the radius of convergence.

(c) We show that a power series converges to an analytic function within its circle of convergence.

□

(a) The convergence of a complex series is determined by the ratio test. The power series converreges for $|z| < R$, but fails to converge for $|z| > R$.

$$\begin{aligned} \frac{d}{dz} s(z) &= \frac{d}{dz} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \frac{d}{dz} \left(a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots \right) \\ &= \sum_{n=0}^{\infty} a_{n+1} (n+1)(z - z_0)^n \end{aligned} \tag{P.228}$$

The ratio test

$$\lim_{k\to\infty}\frac{M_{k+1}}{M_k}=\lim_{k\to\infty}\frac{(k+1)a_{k+1}}{ka_k}=\lim_{k\to\infty}\frac{a_{k+1}}{a_k} \tag{P.229}$$

Hence the differentiation of a power series yields a new power series with the same radius of curvature. Similarly for integration:

$$\begin{aligned}\int s(z)dz &= \int\sum_{n=0}^{\infty}\frac{a_n}{n}(z-z_0)^n dz=\int\big(a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots\big)dz\\ &= a_{-1}+\sum_{n=1}^{\infty}\frac{a_{n-1}}{n}(z-z_0)^n,\end{aligned} \tag{P.230}$$

the ratio test gives

$$\lim_{k\to\infty}\frac{M_{k+1}}{M_k}=\lim_{k\to\infty}\frac{ka_{k+1}}{(k+1)a_k}=\lim_{k\to\infty}\frac{a_{k+1}}{a_k} \tag{P.231}$$

(b) The integral sum fucntion is $F(z)=\sum_{n=0}^{\infty}\int_C f_n(z)dz$. $F(z)$ is also

$$F(z)=\int_{z_0}^{z}\sum_{n=0}^{\infty}f_n(w)dw=\int_a^z f(w)dw$$

a is a fixed point and z an arbitrary point of the region, $F(z)$ only depends on a and z.

$$\begin{aligned}\frac{F(z+\Delta z)-F(z)}{\Delta z} &= \frac{1}{\Delta z}\int_a^{z+\Delta z}f(z')dz'-\frac{1}{\Delta z}\int_a^z f(z')dz'\\ &= \frac{1}{\Delta z}\int_z^{z+\Delta z}f(z')dz'\\ &= \frac{f(z)}{\Delta z}\int_z^{z+\Delta z}dz'+\frac{1}{\Delta z}\int_z^{z+\Delta z}[f(z')-f(z)]dz'\end{aligned} \tag{P.232}$$

Take the inequality

$$\left|\frac{1}{\Delta z}\int_a^{z+\Delta z}[f(z')-f(z)]dz'\right|\le\max|[f(z')-f(z)]|. \tag{P.233}$$

The RHS tends to zero as $\Delta z\to 0$ because $f(z)$ is continuous and therefore from (P.232) we have

$$\frac{d}{dz}F(z) = f(z).$$

Hence the derivative exists and is single-valued, it is equal to $f(z)$. As its derivative is finite and unique, the integrated sum function $F(z)$ is analytic.

(c) If a fucntion $F(z)$ can be represented by

$$F(z) = \frac{1}{2\pi i}\int_C \frac{I(w)}{w-z}dw \tag{P.234}$$

and $I(z)$ is contiuous on C, then, $F(z)$ is analytic at any point z which doesn't lie on C.

In particular, a function $F(z)$ that is analytic in some region can be expressed in this region by Cauchy's integral formula - C will be an arbitrary closed contour encircling the point z and $I(z) = F(z)$.

Proof. Consider the expression

$$\Delta := \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{I(w)}{(w-z)^2}dw \right| \tag{P.235}$$

Using the integral formula (P.234) for $F(z)$ and $F(z+\Delta z)$, one obtains

$$\Delta = \left| \frac{\Delta z}{2\pi}\int_C \frac{I(w)}{(w-z-\Delta z)(w-z)^2}dw \right| \tag{P.236}$$

Since z and $z+\Delta z$ dont lie on C and $I(w)$ is continuous on C, the integrand is bounded, therefore $\Delta \to 0$ as $\Delta z \to 0$. This proves the differentiability of $F(z)$. An analogous result holds for the nth derivavtive of $F(z)$.

□

Thus, $F(z)$ also has a second derivative and hence $f(z)$ is differentiable throughout the region. And so the power series converges to an analytic function within its circle of convergence.

□