Infinite summation technique

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Chapter 1

Summation technique

1.1 Summation - $\zeta(2k)$

We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n^{2k}} &= \frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\int_0^\infty e^{-y}y^{2k-1}dy \\
&= \frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\int_0^\infty e^{-nx}x^{2k-1}dx \\
&= \frac{1}{(2k-1)!}\int_0^\infty \frac{x^{2k-1}}{e^x-1}dx \\
&= \frac{1}{(2k)!}\int_0^\infty \frac{x^{2k}e^x}{(e^x-1)^2}dx \\
&= \frac{1}{2\cdot(2k)!}\int_{-\infty}^\infty \frac{x^{2k}e^x}{(e^x-1)^2}dx.
\end{aligned}$$

This integral is not immediately amenable to complex contour integration. However, we have

$$\begin{aligned}
\int_{-\infty}^\infty \frac{x^{2k}e^x}{(e^x-1)^2}dx - \int_{-\infty}^\infty \frac{x^{2k}e^x}{(e^x+1)^2}dx &= \int_{-\infty}^\infty \frac{4x^{2k}e^{2x}}{(e^{2x}-1)^2}dx \\
&= 2^{1-2k}\int_{-\infty}^\infty \frac{x^{2k}e^x}{(e^x-1)^2}dx \qquad (1.1)
\end{aligned}$$

Which rearranged gives

$$\int_{-\infty}^{\infty} \frac{x^{2k}e^x}{(e^x-1)^2}dx = \frac{1}{1-2^{1-2k}} \int_{-\infty}^{\infty} \frac{x^{2k}e^x}{(e^x+1)^2}dx \tag{1.2}$$

So that

$$\sum_{n=0}^{\infty} \frac{1}{n^{2k}} = \frac{1}{2\cdot(2k)!}\frac{1}{1-2^{1-2k}} \int_{-\infty}^{\infty} \frac{x^{2k}e^x}{(e^x+1)^2}dx$$

We next perform the integral on the RHS.

Performing the integral:

Consider the integral:

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}e^x}{(e^x+1)^2}dx$$

where $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. Then

$$\int_{-\infty}^{\infty} \frac{x^{2k}e^x}{(e^x+1)^2}dx = \frac{\partial^{2k}}{\partial\alpha^{2k}} \int_{-\infty}^{\infty} \frac{e^{\alpha x}e^x}{(e^x+1)^2}dx\bigg|_{\alpha=0}$$

We now evaluate this integral using complex analysis. Consider the rectangular contour, C, in the figure

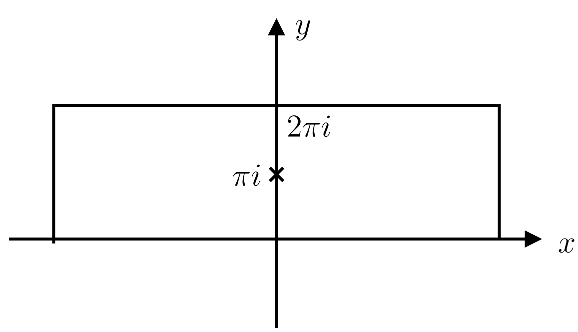

Figure 1.1: .

and the integral

$$\oint_C \frac{e^{\alpha z} e^z}{(e^z+1)^2} dz$$

whose integrand has a pole at πi. The integral along the vertical edges vanishes as:

$$f(z) = \frac{e^{\alpha(x+iy)} e^{(x+iy)}}{(e^{x+iy}+1)^2} = \begin{cases} e^{-(1-\alpha)(x+iy)} & x \to \infty \\ e^{(1+\alpha)(x+iy)} & x \to -\infty \end{cases}$$

So that

$$\begin{aligned} \oint_C \frac{e^{\alpha z} e^z}{(e^z+1)^2} dz &= \int_{-\infty}^{\infty} \frac{e^{\alpha x} e^x}{(e^x-1)^2} dx - e^{i2\alpha\pi} \int_{-\infty+i2\pi}^{\infty+i2\pi} \frac{e^{\alpha x} e^x}{(e^x-1)^2} dx \\ &= (1-e^{i2\alpha\pi}) \int_{-\infty}^{\infty} \frac{e^{\alpha x} e^x}{(e^x+1)^2} dx. \end{aligned}$$

Which rearranged is

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{\alpha x} e^x}{(e^x+1)^2} dx &= \frac{2\pi i}{1-e^{i2\alpha\pi}} \frac{1}{2\pi i} \oint_C \frac{e^{\alpha z} e^z}{(e^z+1)^2} dz \\ &= \frac{2\pi i}{1-e^{i2\alpha\pi}} Res_{z=\pi i}[f(z)]. \end{aligned}$$

We calculate the residue. Expand about pole, $z_0 = i\pi$:

$$\begin{aligned} \frac{1}{(e^z+1)^2} &= \frac{1}{(e^{z_0+(z-z_0)} - e^{z_0})^2} \\ &= e^{-2z_0} \frac{1}{[(z-z_0) + \frac{1}{2!}(z-z_0)^2 + \cdots]^2} \\ &= e^{-2z_0} \frac{1}{(z-z_0)^2[(1 + \frac{1}{2!}(z-z_0) + \cdots]^2} \\ &= e^{-2z_0} \frac{1}{(z-z_0)^2[1 + (z-z_0) + \cdots]} \\ &= e^{-2z_0} \frac{1}{(z-z_0)^2} [(1 - (z-z_0) + \cdots] \\ &= e^{-2z_0} \frac{1}{(z-z_0)^2} - e^{-2z_0} \frac{1}{z-z_0} + \cdots \end{aligned}$$

Using this we can find the residue:

$$\begin{aligned}
\frac{e^{\alpha z}e^{z}}{(e^{z}+1)^2} &= e^{-2z_0}\frac{e^{(\alpha+1)z_0+(\alpha+1)(z-z_0)}}{(z-z_0)^2} - e^{-2z_0}\frac{e^{(\alpha+1)z_0}}{z-z_0} + \cdots \\
&= e^{-2z_0}\frac{e^{(\alpha+1)z_0}[1+(\alpha+1)(z-z_0)+\cdots]}{(z-z_0)^2} - e^{-2z_0}\frac{e^{(\alpha+1)z_0}}{z-z_0} + \cdots \\
&= e^{-2z_0}\frac{e^{(\alpha+1)z_0}}{(z-z_0)^2} + e^{-2z_0}\frac{\alpha e^{(\alpha+1)z_0}}{z-z_0} + \cdots
\end{aligned}$$

So that

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{\alpha x}e^{x}}{(e^{x}+1)^2}dx &= \frac{2\pi i}{1-e^{i2\alpha\pi}}\cdot \alpha e^{i(\alpha+1)\pi} \\
&= \frac{\alpha\pi}{\sin(\alpha\pi)}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n^{2k}} &= \frac{1}{1-2^{1-2k}}\frac{1}{2\cdot(2k)!}\int_{-\infty}^{\infty}\frac{x^{2k}e^{x}}{(e^{x}+1)^2}dx \\
&= \frac{1}{1-2^{1-2k}}\frac{1}{2\cdot(2k)!}\ \frac{\partial^{2k}}{\partial\alpha^{2k}}\frac{\alpha\pi}{\sin(\alpha\pi)}\bigg|_{\alpha=0}
\end{aligned}$$

We have the series expansion of $x\pi\csc x\pi$ in terms of Bernoulli numbers:

$$\frac{x\pi}{\sin(x\pi)} = \sum_{n=0}^{\infty}\frac{2(2^{2n-1}-1)(-1)^{n+1}\pi^{2n}B_{2n}}{(2n)!}x^{2n}.$$

Employing this we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n^{2k}} &= \tfrac{1}{1-2^{1-2k}}\tfrac{1}{2\cdot(2k)!}\ \frac{\partial^{2k}}{\partial\alpha^{2k}}\tfrac{\alpha\pi}{\sin(\alpha\pi)}\bigg|_{\alpha=0} \\
&= \tfrac{2^{2k-1}}{2^{2k-1}-1}\tfrac{1}{2\cdot(2k)!}\cdot 2(2^{2k-1}-1)(-1)^{k+1}\pi^{2k}B_{2k} \\
&= (-1)^{k+1}(2\pi)^{2k}\tfrac{B_{2k}}{2(2k)!}. \qquad (1.3)
\end{aligned}$$

1.2 Summation over odd integers

General sum, its integral representations, and evaluation of integral

The general sum, $\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^{2k}}$, has the following essentially equivalent integral representations:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = \frac{1}{4 \cdot (2k)!} \int_{-\infty}^{\infty} \frac{x^{2k} e^x}{(e^x - 1)^2} dx + \frac{1}{4 \cdot (2k)!} \int_{-\infty}^{\infty} \frac{x^{2k} e^x}{(e^x + 1)^2} dx$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = (1 - 2^{-2k}) \frac{1}{2 \cdot (2k)!} \int_{-\infty}^{\infty} \frac{x^{2k} e^x}{(e^x - 1)^2} dx$$

which I derive below. Comparing these you find

$$\int_{-\infty}^{\infty} \frac{x^{2k} e^x}{(e^x - 1)^2} dx = \frac{2^{2k-1}}{2^{2k-1} - 1} \int_{-\infty}^{\infty} \frac{x^{2k} e^x}{(e^x + 1)^2} dx$$

This relation was proven directly - see (1.2). Substituting this into the second integral representation you get another integral representation

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = \frac{2^{2k} - 1}{2^{2k} - 2} \frac{1}{2 \cdot (2k)!} \int_{-\infty}^{\infty} \frac{x^{2k} e^x}{(e^x + 1)^2} dx$$

Below we will evaluate this integral and obtain the expected result:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = (1 - 2^{-2k}) \times (-1)^{k+1} (2\pi)^{2k} \frac{B_{2k}}{2(2k)!}.$$

where B_{2k} are the Bernoulli numbers.

Deriving integral representations of sum

I now derive these integral representations. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1-\cos \pi n}{n^{2k}} \\
&= \frac{1}{2\cdot(2k-1)!} \sum_{n=1}^{\infty} \frac{1-\cos \pi n}{n^{2k}} \int_0^{\infty} e^{-y} y^{2k-1} dy \\
&= \frac{1}{2\cdot(2k-1)!} \sum_{n=1}^{\infty} [1-\cos \pi n] \int_0^{\infty} e^{-nx} x^{2k-1} dx \\
&= \frac{1}{2\cdot(2k-1)!} \sum_{n=1}^{\infty} \int_0^{\infty} [1-\cos \pi n] e^{-nx} x^{2k-1} dx \\
&= \frac{1}{4\cdot(2k-1)!} \sum_{n=1}^{\infty} \int_0^{\infty} (2e^{-nx} - e^{-nx+i\pi n} - e^{-nx+-i\pi n}) x^{2k-1} dx \\
&= \frac{1}{4\cdot(2k-1)!} \int_0^{\infty} \left(\frac{2}{e^x-1} - \frac{1}{e^{x-i\pi}-1} - \frac{1}{e^{x+i\pi}-1} \right) x^{2k-1} dx \\
&= \frac{1}{2\cdot(2k-1)!} \int_0^{\infty} \frac{x^{2k-1}}{e^x-1} dx + \frac{1}{2\cdot(2k-1)!} \int_0^{\infty} \frac{x^{2k-1}}{e^x+1} dx \\
&= \frac{1}{2\cdot(2k)!} \int_0^{\infty} \frac{x^{2k}e^x}{(e^x-1)^2} dx + \frac{1}{2\cdot(2k)!} \int_0^{\infty} \frac{x^{2k}e^x}{(e^x+1)^2} dx \\
&= \frac{1}{4\cdot(2k)!} \int_{-\infty}^{\infty} \frac{x^{2k}e^x}{(e^x-1)^2} dx + \frac{1}{4\cdot(2k)!} \int_{-\infty}^{\infty} \frac{x^{2k}e^x}{(e^x+1)^2} dx
\end{aligned}$$

where we have integrated by parts and then extended the range of integration. This is first integral representation the sum quoted at the beginning.

Recall the identity

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = (1-2^{-2k}) \sum_{n=0}^{\infty} \frac{1}{n^{2k}}.$$

We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n^{2k}} &= \frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\int_0^{\infty}e^{-y}y^{2k-1}dy \\
&= \frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}e^{-nx}x^{2k-1}dx \\
&= \frac{1}{(2k-1)!}\int_0^{\infty}\frac{x^{2k-1}}{e^x-1}dx \\
&= \frac{1}{(2k)!}\int_0^{\infty}\frac{x^{2k}e^x}{(e^x-1)^2}dx \\
&= \frac{1}{2\cdot(2k)!}\int_{-\infty}^{\infty}\frac{x^{2k}e^x}{(e^x-1)^2}dx.
\end{aligned}$$

Substituting this into the identity just mentioned we get the second integral representation of the sum quoted above.

As noted at the beginning, another representation follows from the first two:

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k}} = \frac{2^{2k}-1}{2^{2k}-2}\frac{1}{2\cdot(2k)!}\int_{-\infty}^{\infty}\frac{x^{2k}e^x}{(e^x+1)^2}dx.$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k}} &= \frac{2^{2k}-1}{2^{2k}-2}\frac{1}{2\cdot(2k)!}\int_{-\infty}^{\infty}\frac{x^{2k}e^x}{(e^x+1)^2}dx \\
&= \frac{2^{2k}-1}{2^{2k}-2}\frac{1}{2\cdot(2k)!}\left.\frac{\partial^{2k}}{\partial\alpha^{2k}}\frac{\alpha\pi}{\sin(\alpha\pi)}\right|_{\alpha=0}.
\end{aligned}$$

We have the series expansion of $x\pi\csc x\pi$ in terms of Bernoulli numbers:

$$\frac{x\pi}{\sin(x\pi)} = \sum_{n=0}^{\infty}\frac{2(2^{2n-1}-1)(-1)^{n+1}\pi^{2n}B_{2n}}{(2n)!}x^{2n}.$$

Employing this we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} &= \frac{2^{2k}-1}{2^{2k}-2} \frac{1}{2\cdot(2k)!} \left.\frac{\partial^{2k}}{\partial\alpha^{2k}} \frac{\alpha\pi}{\sin(\alpha\pi)}\right|_{\alpha=0} \\
&= \frac{2^{2k}-1}{2^{2k}-2} \frac{1}{2\cdot(2k)!} \cdot 2(2^{2k-1}-1)(-1)^{k+1}\pi^{2k}B_{2k} \\
&= (1-2^{-2k})(-1)^{k+1}(2\pi)^{2k}\frac{B_{2k}}{2(2k)!}.
\end{aligned}$$

1.3 Alternating sums

The general sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}}$$

for $k = 0, 1, 2, \ldots$ can be easily performed using the above method.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} &= \sum_{n=1}^{\infty} \frac{\sin\dfrac{\pi n}{2}}{n^{2k+1}} \\
&= \frac{1}{(2k)!} \sum_{n=1}^{\infty} \frac{\sin\dfrac{\pi n}{2}}{n^{2k+1}} \int_0^{\infty} e^{-y} y^{2k} dy \\
&= \frac{1}{(2k)!} \sum_{n=1}^{\infty} \sin\frac{\pi n}{2} \int_0^{\infty} e^{-nx} x^{2k} dx \\
&= \frac{1}{(2k)!2i} \sum_{n=1}^{\infty} \int_0^{\infty} (e^{-nx+\frac{i\pi n}{2}} - e^{-nx+\frac{-i\pi n}{2}}) x^2 dx \\
&= \frac{1}{(2k)!2i} \int_0^{\infty} \left(\frac{1}{1-e^{-x+\frac{i\pi}{2}}} - \frac{1}{1-e^{-x+\frac{-i\pi}{2}}} \right) x^{2k} dx \\
&= \frac{1}{(2k)!} \int_0^{\infty} \frac{x^{2k}}{e^x+e^{-x}} dx \\
&= \frac{1}{(2k)!2} \int_{-\infty}^{\infty} \frac{x^{2k}}{e^x+e^{-x}} dx
\end{aligned}$$

We will evaluate

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x + e^{-x}} dx$$

for $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$ by considering the rectangular contour integral (see figure) of

$$\oint_C \frac{e^{\alpha z}}{e^z + e^{-z}} dz$$

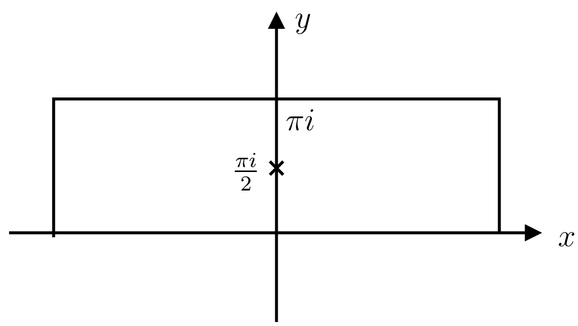

Figure 1.2: .

The integral along the vertical edges vanishes as:

$$f(z) = \frac{e^{\alpha(x+iy)}}{e^{x+iy} + e^{-x-iy}} = \begin{cases} e^{(\alpha-1)(x+iy)} & x \to \infty \\ e^{(\alpha+1)(x+iy)} & x \to -\infty \end{cases}$$

So that

$$\begin{aligned}\oint_C \frac{e^{\alpha z}}{e^z + e^{-z}} dz &= \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x + e^{-x}} dx + e^{\alpha \pi i} \int_{-\infty + i\pi}^{\infty + i\pi} \frac{e^{\alpha x}}{e^x + e^{-x}} dx \\ &= (1 + e^{\alpha \pi i}) \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x + e^{-x}} dx\end{aligned}$$

and so

$$
\begin{aligned}
\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{\alpha x}}{e^x+e^{-x}}dx &= \frac{i\pi}{2(1+e^{\alpha\pi i})}\frac{1}{2\pi i}\oint_C \frac{e^{\alpha z}}{e^z+e^{-z}}dz \\
&= \frac{i\pi}{2(1+e^{\alpha\pi i})}\lim_{z\to\frac{i\pi}{2}}(z-\frac{i\pi}{2})\frac{e^{\alpha z}}{e^z+e^{-z}} \\
&= \frac{i\pi}{2(1+e^{\alpha\pi i})}\lim_{z\to\frac{i\pi}{2}}\frac{e^{\alpha z}}{e^z-e^{-z}} \\
&= \frac{\pi}{4(1+e^{\alpha\pi i})}e^{\alpha\frac{i\pi}{2}} \\
&= \frac{\pi}{8}\frac{1}{\cos(\frac{\alpha\pi}{2})}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
\frac{1}{4}\int_{-\infty}^{\infty}\frac{x^2}{e^x+e^{-x}}dx &= \frac{\pi}{8}\frac{\partial^2}{\partial\alpha^2}\left.\frac{1}{\cos(\frac{\alpha\pi}{2})}\right|_{\alpha=0} \\
&= \frac{\pi^2}{16}\frac{\partial}{\partial\alpha}\left.\frac{\sin(\frac{\alpha\pi}{2})}{\cos^2(\frac{\alpha\pi}{2})}\right|_{\alpha=0} \\
&= \frac{\pi^3}{32}\left.\frac{\cos^3(\frac{\alpha\pi}{2})+2\sin^2(\frac{\alpha\pi}{2})\cos(\frac{\alpha\pi}{2})}{\cos^4(\frac{\alpha\pi}{2})}\right|_{\alpha=0} \\
&= \frac{\pi^3}{32}.
\end{aligned}
$$

So finally,

$$
\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.
$$

In post 5 I proved

$$
\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{\alpha x}}{e^x+e^{-x}}dx = \frac{\pi}{4}\frac{1}{\cos(\frac{\alpha\pi}{2})}.
$$

Using this, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}} &= \frac{1}{(2k)!2}\int_{-\infty}^{\infty}\frac{x^{2k}}{e^x+e^{-x}}dx \\
&= \frac{\pi}{(2k)!4}\frac{\partial^{2k}}{\partial\alpha^{2k}}\left.\frac{1}{\cos(\frac{\alpha\pi}{2})}\right|_{\alpha=0} \qquad (*)
\end{aligned}
$$

Euler numbers are defined by

$$
\frac{2}{e^t+e^{-t}} = \sum_{n=0}^{\infty}\frac{E_n}{n!}t^n.
$$

So that

$$
\sec t = \sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}t^{2n}
$$

and

$$
\sec\frac{\pi t}{2} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}E_{2n}}{4^k(2n)!}t^{2n}
$$

Using this in $(*)$, results in:

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}} &= \frac{\pi}{(2k)!4}\frac{\partial^{2k}}{\partial\alpha^{2k}}\left.\frac{1}{\cos(\frac{\alpha\pi}{2})}\right|_{\alpha=0} \\
&= \pi^{2k+1}\frac{(-1)^k E_{2k}}{(2k)!4^{k+1}}
\end{aligned}
$$

The first few Euler numbers are $E_0 = 1$, $E_2 = -1$, and $E_4 = 5$.

Interchanging summation and integration

Note in the above, I made an interchange of summation and integration. This needs to be justified.

Case $k > 0$:

As

$$\begin{aligned}
\frac{1}{(2k)!}\sum_{n=1}^{\infty}\int_0^{\infty}\left|\sin\frac{\pi n}{2}x^{2k}e^{-nx}\right|dx &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k+1}} \\
&< \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2} \\
&= (1-2^{-2})\sum_{n=1}^{\infty}\frac{1}{n^2} \\
&< (1-2^{-2})(1+\lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)}) \\
&< (1-2^{-2})(1+\lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right)) \\
&= (1-2^{-2})(1+1-\lim_{N\to\infty}\frac{1}{N}) = (1-2^{-2})2 < \infty
\end{aligned}$$

you can use Fubini to justify interchanging summation and integration:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\sin\frac{\pi n}{2}x^{2k}e^{-nx}dx = \int_0^{\infty}\sum_{n=1}^{\infty}\sin\frac{\pi n}{2}x^{2k}e^{-nx}dx.$$

Case $k = 0$ (Leibniz formula for π):

As

$$\sum_{n=1}^{\infty}\int_0^{\infty}\left|\sin\frac{\pi n}{2}e^{-nx}\right|dx = \sum_{n=0}^{\infty}\frac{1}{2n+1} = \infty$$

you can't use Fubini to justify this interchange. An equivalent check:

$$
\begin{aligned}
\int_0^\infty \sum_{n=1}^\infty \left|\sin\frac{\pi n}{2} e^{-nx}\right| dx &= \int_0^\infty (e^{-x}+e^{-3}+e^{-5x}+\cdots)dx \\
&= \int_0^\infty \frac{1}{e^x-e^{-x}} dx \\
&= \left[\frac{1}{2}\ln\left(\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}\right)\right]_0^\infty = \infty
\end{aligned}
$$

You can, however, use the dominated convergence theorem to prove the interchange is legitimate. The proof for $k \geq 0$ is not much more difficult that the proof for $k = 0$, so I give the proof for $k \geq 0$.

Using the dominated convergence theorem $k \geq 0$:

I now justify interchanging summation and integration using the dominated convergence theorem. Define $f_N(x) = \sum_{n=1}^N \sin\frac{\pi n}{2} x^{2k} e^{-nx}$. Then, for appropriate integers ℓ, ℓ' taken from $1, 2, \ldots$, we have

$$
\begin{aligned}
f_N(x) &= \begin{cases} (e^{-x}-e^{-3x}+e^{-5x}-e^{-7x}+\cdots+e^{-(4\ell-3)x})x^{2k} & \text{odd number of terms} \\ (e^{-x}-e^{-3x}+e^{-5x}-e^{-7x}+\cdots-e^{-(4\ell'-1)x})x^{2k} & \text{even number of terms} \end{cases} \\
&= \begin{cases} e^{-x}\dfrac{1+e^{-(4\ell-1)x}}{1+e^{-2x}}x^{2k} & \text{odd number of terms} \\ e^{-x}\dfrac{1-e^{-(4\ell'+1)x}}{1+e^{-2x}}x^{2k} & \text{even number of terms} \end{cases}
\end{aligned}
$$

So the limit function for $x > 0$ is

$$
\lim_{N\to\infty} f_N(x) = f(x) = \frac{x^{2k}}{e^x+e^{-x}}
$$

So $f_N(x)$ converges to the function $f(x)$ except at $x = 0$:

$$
\lim_{N\to\infty} |f(x)-f_N(x)| = \begin{cases} \lim_{\ell\to\infty} e^{-x}\dfrac{e^{-(4\ell-1)x}}{1+e^{-2x}}x^{2k} = 0 & \text{odd number of terms} \\ \lim_{\ell'\to\infty} e^{-x}\dfrac{e^{-(4\ell'+1)x}}{1+e^{-2x}}x^{2k} = 0 & \text{even number of terms} \end{cases}
$$

It is not a problem that you don't have convergence at the single point $x = 0$?

Also,

$$0 \leq f_N(x) \leq \frac{1+e^{-2x}}{1+e^{-2x}} e^{-x} x^{2k} = x^{2k} e^{-x}$$

So

$$|f_N(x)| \leq x^{2k} e^{-x} = g(x)$$

The function $g(x)$ is integrable. By the dominated convergence theorem

$$\lim_{N\to\infty} \int_0^\infty \sum_{n=1}^{N} \sin\frac{\pi n}{2} e^{-nx} dx = \int_0^\infty \lim_{N\to\infty} \sum_{n=1}^{N} \sin\frac{\pi n}{2} e^{-nx} dx$$

Chapter 2

Inverting Fourier series

2.1 List of sums

I aim to evaluate the following sums through an integral representation of them:

(i)

For fixed h such that $0 < h < 2\pi$,

$$\frac{h}{2\pi} + \sum_{n=1}^{\infty} \frac{\sin nh}{\pi n} \cos(nx) + \sum_{n=1}^{\infty} \frac{1 - \cos nh}{\pi n} \sin(nx)$$

(ii)

$$\sum_{n=0}^{\infty} \frac{\cos(2\pi(2n-1)x)}{(2n-1)^2}$$

(iii)

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx)$$

(iv)

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos(2nx)$$

(v)

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{2k}}$$

(vi)

$$\frac{1}{2\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(an)}{a^2n^2}\cos(nx)$$

(vii)

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\sin(2\pi nx)$$

(viii)

$$\frac{\pi^2}{2\pi} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

(ix)

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)x)$$

(x)

$$\sum_{n=0}^{\infty}\frac{\cos(2\pi(2n+1)x)}{(2n+1)^{2k}}$$

(xi)

$$\sum_{n=0}^{\infty} \frac{\cos(\pi(2n+1)x)}{(2n+1)^{2k}}$$

(xii)

$$\sum_{n=0}^{\infty} \frac{\sin(\pi(2n+1)x)}{(2n+1)^{2k+1}}$$

(xiii)

$$\frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-in} e^{inx}$$
$$= \frac{\sinh a\pi}{a\pi} + \frac{2a \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \cos(nx) - \frac{2 \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{a^2+n^2} \sin(nx)$$

2.2 Sum (i)

Deriving the step function from its Fourier series

For $0 < x < 2\pi$ and fixed h such that $0 < h < 2\pi$,

$$\frac{h}{2\pi} + \sum_{n=1}^{\infty} \frac{\sin nh}{\pi n} \cos(nx) + \sum_{n=1}^{\infty} \frac{1-\cos nh}{\pi n} \sin(nx)$$
$$= \frac{h}{2\pi} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} [\sin(n(x+h)) - \sin(n(x-h))] + \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin(nx)$$
$$- \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} [\sin(n(x+h)) + \sin(n(x-h))] \sin(nx)$$
$$= \frac{h}{2\pi} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n(x-h)) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$$

We have:

$$
\begin{aligned}
&[-\sum_{n=1}^{\infty}\frac{1}{n}\sin(n(x-h))+\sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)] \\
&=\sum_{n=1}^{\infty}[-\frac{1}{n}\sin(n(x-h))+\frac{1}{n}\sin(nx)]\int_0^{\infty}e^{-u}du \\
&=\sum_{n=1}^{\infty}[-\sin(n(x-h))+\sin(nx)]\int_0^{\infty}e^{-nu}du \\
&=\sum_{n=1}^{\infty}\int_0^{\infty}[-\sin(n(x-h))+\sin(nx)]e^{-nu}du \\
&=\sum_{n=1}^{\infty}[-\sin(n(x-h))+\sin(nx)]e^{-nu}du \\
&=\frac{1}{2i}\int_0^{\infty}\sum_{n=1}^{\infty}[-(e^{-nu+in(x-h)}-e^{-nu-in(x-h)})+e^{-nu+inx}-e^{-nu-inx}]du \\
&=\frac{1}{2i}\int_0^{\infty}\left(-\frac{1}{1-e^{-u+i(x-h)}}+\frac{1}{1-e^{-u-i(x-h)}}+\frac{1}{1-e^{-u+ix}}-\frac{1}{1-e^{-u-ix}}\right)du
\end{aligned}
$$

To justify interchanging summation and integration in the above, we can't use Fubini:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\int_0^{\infty}|\sin(nx)e^{-nu}|du &\leq \sum_{n=1}^{\infty}\int_0^{\infty}e^{-nu}du \\
&=\sum_{n=1}^{\infty}\frac{1}{n}=\infty
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{\sin(nx)}{n} &= \sum_{n=1}^{\infty}\frac{\sin(nx)}{n}\int_0^{\infty}e^{-u}du \\
&= \sum_{n=1}^{\infty}\sin(nx)\int_0^{\infty}e^{-nu}du \\
&= \frac{1}{2i}\sum_{n=1}^{\infty}\int_0^{\infty}(e^{-nu+inx}-e^{-nu+-inx})du \\
&= \frac{1}{2i}\int_0^{\infty}\left(\frac{1}{1-e^{-u+ix}}-\frac{1}{1-e^{-u-ix}}\right)du \qquad (2.1)
\end{aligned}
$$

Continuing

$$\frac{1}{2i}\int_0^\infty \left(\frac{1}{1-e^{-u+ix}} - \frac{1}{1-e^{-u-ix}}\right)du = \frac{1}{2i}\int_0^\infty \left(\frac{e^{ix}e^{-u}u}{(1-e^{-u+ix})^2} - \frac{e^{-ix}e^{-u}u}{(1-e^{-u-ix})^2}\right)du$$
$$= \frac{1}{2i}\int_0^\infty \left(\frac{e^{ix}e^{u}u}{(e^u-e^{ix})^2} - \frac{e^{-ix}e^{u}u}{(e^u-e^{-ix})^2}\right)du$$
$$= \frac{1}{4i}\int_{-\infty}^\infty \left(\frac{e^{ix}e^{u}u}{(e^u-e^{ix})^2} - \frac{e^{-ix}e^{u}u}{(e^u-e^{-ix})^2}\right)du$$
$$= \frac{1}{2}Im\int_{-\infty}^\infty \frac{e^{ix}e^{u}u}{(e^u-e^{ix})^2}du$$

Consider the contour integral

$$\oint_C \frac{e^{ix}e^z z^2}{(e^z-e^{ix})^2}dz = \int_{-\infty}^\infty \frac{e^{ix}e^u u^2}{(e^u-e^{ix})^2} - \int_{-\infty+2\pi i}^{\infty+2\pi i} \frac{e^{ix}e^u(u+2\pi i)^2}{(e^u-e^{ix})^2}$$
$$= 4\pi i\int_{-\infty}^\infty \frac{e^{ix}e^u u}{(e^u-e^{ix})^2} + 4\pi^2\int_{-\infty}^\infty \frac{e^{ix}e^u}{(e^u-e^{ix})^2}$$

We easily have $\int_{-\infty}^\infty \dfrac{e^{ix}e^u}{(e^u-e^{ix})^2} = \int_{-\infty}^\infty \frac{d}{du}\left(\dfrac{e^{ix}}{e^u-e^{ix}}\right) = 1$. The above rearranged gives:

$$\int_{-\infty}^\infty \frac{e^{ix}e^u u}{(e^u-e^{ix})^2} = -\pi i + \frac{1}{2}Res_{z=xi}\left[\frac{e^{ix}e^z z^2}{(e^z-e^{ix})^2}\right]$$

$Res_{z=xi}[\cdots] = 2(\pi x i)$. So finally,

$$\sum_{n=0}^\infty \frac{\sin(nx)}{n} = -\frac{1}{2}\int_{-\infty}^\infty \frac{e^{ix}e^u u}{(e^u-e^{ix})^2}$$
$$= \frac{\pi - x}{2}$$

It is obvious fromm the contour integral that $\sum_{n=0}^\infty \dfrac{\sin(nx)}{n}$ is discontinuous at $x = 2\pi$; the poles jumps from $z = 2\pi i$ to $z = 0$ as x is increased past 2π.

The integral (2.1) could also be written as

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{\sin(nx)}{n} &= \sum_{n=1}^{\infty}\frac{\sin(nx)}{n}\int_0^\infty e^{-u}du \\
&= \frac{1}{2i}\int_0^\infty\left(\frac{1}{1-e^{-u}(\cos(x)+i\sin(x))}-\frac{1}{1-e^{-u}(\cos(x)-i\sin(x))}\right)du \\
&= \int_0^\infty \frac{e^{-u}\sin(x)}{(1-e^{-u}\cos(x))^2+e^{-2u}\sin^2(x)}du \\
&= \sin(x)\int_0^\infty \frac{e^{-u}}{1-2e^{-u}\cos(x)+e^{-2u}}du \\
&= \frac{\sin(x)}{2}\int_{-\infty}^\infty \frac{e^{u}}{1-2e^{u}\cos(x)+e^{2u}}du \\
&= \frac{\sin(x)}{2}\int_0^\infty \frac{dw}{1-2w\cos(x)+w^2}
\end{aligned}
$$

However, it is less obvious that there is a discontiuity as x increase past 2π. This integral is easily done,

$$
\begin{aligned}
\frac{\sin(x)}{2}\int_0^\infty \frac{dw}{(w-\cos(x))^2+\sin^2(x)} &= \frac{1}{2}\left[\tan^{-1}\left(\frac{w-\cos(x)}{\sin(x)}\right)\right]_0^\infty \\
&= \frac{1}{2}\left(\frac{\pi}{2}+\tan^{-1}\left(\frac{\cos(x)}{\sin(x)}\right)\right) \\
&= \frac{1}{2}\left(\frac{\pi}{2}+\tan^{-1}\left(-\frac{\sin(x-\frac{\pi}{2})}{\cos(x-\frac{\pi}{2})}\right)\right) \\
&= \frac{\pi-x}{2}.
\end{aligned}
$$

So

$$
\sum_{n=0}^{\infty}\frac{\sin(nx)}{n} = \frac{\pi-x}{2} \qquad \text{for} \qquad 0<x<2\pi.
$$

and

$$
\sum_{n=0}^{\infty}\frac{\sin(n(x-h))}{n} = \begin{cases} \frac{\pi-(x-h+2\pi)}{2} & 0<x<h \\ \frac{\pi-(x-h)}{2} & h<x<2\pi \end{cases}
$$

So that

$$
\begin{aligned}
&\frac{h}{2\pi} + \sum_{n=1}^{\infty} \frac{\sin nh}{\pi n} \cos(nx) + \sum_{n=1}^{\infty} \frac{1 - \cos nh}{\pi n} \sin(nx) \\
&= \frac{h}{2\pi} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n(x-h)) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx) \\
&= \begin{cases} \frac{h}{2\pi} + \frac{1}{\pi}\frac{\pi - x}{2} - \frac{1}{\pi}\frac{\pi-(x-h+2\pi)}{2} & 0 < x < h \\ \frac{h}{2\pi} + \frac{1}{\pi}\frac{\pi - x}{2} - \frac{1}{\pi}\frac{\pi-(x-h)}{2} & h < x < 2\pi \end{cases} \\
&= \begin{cases} 1 & 0 < x < h \\ 0 & h < x < 2\pi \end{cases}
\end{aligned}
$$

We have found that the Fourier series is that of the step function.

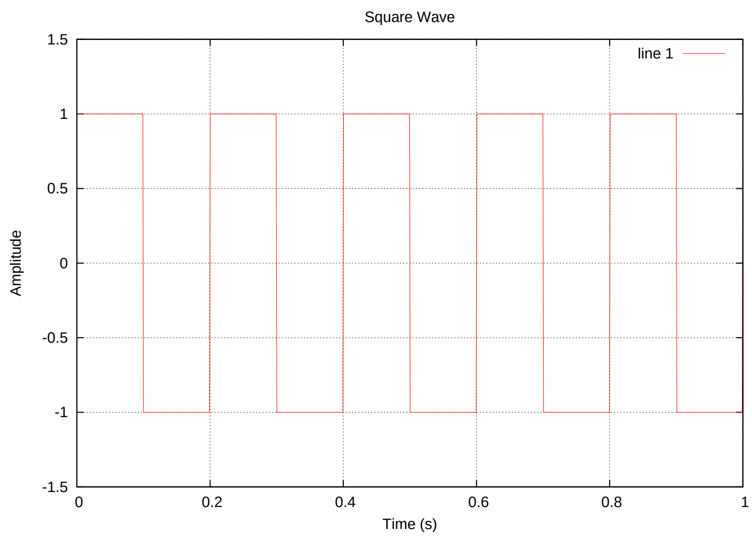

Figure 2.1: Plot of square wave.

2.3 Sum (ii)

Deriving the $|x|$ function from its Fourier series

We have:

$$
\sum_{n=0}^{\infty} \frac{\cos(2\pi(2n-1)x)}{(2n-1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} [1 - \cos n\pi] \frac{\cos(2\pi n x)}{n^2}
$$

So we need to only calculate

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{n=1}^{\infty}\cos n\pi \frac{\cos(2\pi n x)}{n^2}\int_0^\infty ue^{-u}du\\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\cos n\pi \cos(2\pi n x)\int_0^\infty ue^{-nu}du\\
&= -\frac{1}{4}\sum_{n=1}^{\infty}[\cos(2\pi n x + n\pi) + \cos(2\pi n x - n\pi)]\int_0^\infty ue^{-nu}du\\
&= -\frac{1}{8}\sum_{n=1}^{\infty}\int_0^\infty [(e^{-nu+in2\pi(x+\frac{1}{2})} + e^{-nu-in2\pi(x+\frac{1}{2})}) + (e^{-nu+in2\pi(x-\frac{1}{2})} + e^{-nu-in2\pi(x-\frac{1}{2})})]udu\\
&= -\frac{1}{8}\int_0^\infty \sum_{n=1}^{\infty}[(e^{-nu+in2\pi(x+\frac{1}{2})} + e^{-nu-in2\pi(x+\frac{1}{2})}) + (e^{-nu+in2\pi(x-\frac{1}{2})} + e^{-nu-in2\pi(x-\frac{1}{2})})]udu\\
&= -\frac{1}{8}\int_0^\infty \left(\frac{1}{e^{u-2\pi(x+\frac{1}{2})i}-1} + \frac{1}{e^{u+2\pi(x+\frac{1}{2})i}-1} + \frac{1}{e^{u-2\pi(x-\frac{1}{2})i}-1} + \frac{1}{e^{u+2\pi(x-\frac{1}{2})i}-1}\right)udu
\end{aligned}
$$

To justify interchanging summation and integration in the above, we can use Fubini:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\int_0^\infty |\cos n\pi \cos(2\pi n x) ue^{-nu}|du &\leq \sum_{n=1}^{\infty}\int_0^\infty ue^{-nu}du\\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}\\
&< 1 + \lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)}\\
&= 1 + \lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n}\right)\\
&= 1 + 1 - \lim_{N\to\infty}\frac{1}{N} = 2 < \infty
\end{aligned}
$$

From the above

$$
\begin{aligned}
&-\frac{1}{2}\sum_{n=1}^{\infty}\cos n\pi\frac{\cos(2\pi nx)}{n^2}\int_0^{\infty} ue^{-u}du \\
&= -\frac{1}{8}\int_0^{\infty}\left(\frac{1}{e^{u-2\pi(x+\frac{1}{2})i}-1}+\frac{1}{e^{u+2\pi(x+\frac{1}{2})i}-1}+\frac{1}{e^{u-2\pi(x-\frac{1}{2})i}-1}+\frac{1}{e^{u+2\pi(x-\frac{1}{2})i}-1}\right)udu \\
&= -\frac{1}{8}\int_0^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u}{e^u-e^{2\pi(x+\frac{1}{2})i}}+\frac{e^{-2\pi(x+\frac{1}{2})i}u}{e^u-e^{-2\pi(x+\frac{1}{2})i}}+\frac{e^{2\pi(x-\frac{1}{2})i}u}{e^u-e^{2\pi(x-\frac{1}{2})i}}+\frac{e^{-2\pi(x-\frac{1}{2})i}u}{e^u-e^{-2\pi(x-\frac{1}{2})i}}\right)du \\
&= -\frac{1}{16}\int_0^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u^2}{(e^u-e^{2\pi(x+\frac{1}{2})i})^2}+\frac{e^{-2\pi(x+\frac{1}{2})i}u^2}{(e^u-e^{-2\pi(x+\frac{1}{2})i})^2}+\frac{e^{2\pi(x-\frac{1}{2})i}u^2}{(e^u-e^{2\pi(x-\frac{1}{2})i})^2}+\frac{e^{-2\pi(x-\frac{1}{2})i}u^2}{(e^u-e^{-2\pi(x-\frac{1}{2})i})^2}\right)du \\
&= -\frac{1}{32}\int_{-\infty}^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u^2}{(e^u-e^{2\pi(x+\frac{1}{2})i})^2}+\frac{e^{-2\pi(x+\frac{1}{2})i}u^2}{(e^u-e^{-2\pi(x+\frac{1}{2})i})^2}+\frac{e^{2\pi(x-\frac{1}{2})i}u^2}{(e^u-e^{2\pi(x-\frac{1}{2})i})^2}+\frac{e^{-2\pi(x-\frac{1}{2})i}u^2}{(e^u-e^{-2\pi(x-\frac{1}{2})i})^2}\right)du \\
&= -\frac{1}{16}Re\int_{-\infty}^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u^2}{(e^u-e^{2\pi(x+\frac{1}{2})i})^2}+\frac{e^{2\pi(x-\frac{1}{2})i}u^2}{(e^u-e^{2\pi(x-\frac{1}{2})i})^2}\right)du
\end{aligned}
$$

Define

$$
I_{\pm}(\alpha)=\int_{-\infty}^{\infty}\frac{e^{2\pi(x\pm\frac{1}{2})i}e^{\alpha u}e^u}{(e^u-e^{2\pi(x\pm\frac{1}{2})i})^2}du
$$

for $-\frac{1}{2}\leq\alpha\leq\frac{1}{2}$. Then

$$
\sum_{n=0}^{\infty}\frac{\cos(2\pi(2n-1)x)}{(2n-1)^2} = \frac{1}{8}\frac{\partial^2}{\partial\alpha^2}[ReI(\alpha)-\frac{1}{2}ReI_+(\alpha)-\frac{1}{2}ReI_-(\alpha)]\bigg|_{\alpha=0} \qquad (*).
$$

In order to evaluate $I_{\pm}(\alpha)$, consider the contour in the figure.

and the integral

$$
\oint_C\frac{e^{2\pi(x+\pm\frac{1}{2})i}e^{\alpha z}e^z}{(e^z-e^{2\pi(x+\pm\frac{1}{2})i})^2}
$$

whose integrand has pole at $2\pi(x+\frac{1}{2})i$ when $0<x<\frac{1}{2}$ and pole at $2\pi(x-\frac{1}{2})i$ when $\frac{1}{2}<x<1$. The integral along the vertical edges vanishes as:

$$f_\pm(z) = \frac{e^{2\pi(x\pm\frac{1}{2})i}e^{(\alpha+1)(u+iv)}}{(e^{u+iv}+e^{2\pi(x\pm\frac{1}{2})i})^2} = \begin{cases} e^{2\pi(x\pm\frac{1}{2})i}e^{(\alpha-1)(u+iv)} & u\to\infty \\ \dfrac{e^{(\alpha+1)(u+iv)}}{e^{2\pi(x\pm\frac{1}{2})i}} & u\to-\infty \end{cases}$$

Combining everything

$$I_+(\alpha) = \begin{cases} -e^{\alpha 2\pi(x+\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & 0<x<\frac{1}{2} \\ -e^{\alpha 2\pi(x+\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & \frac{1}{2}<x<1 \end{cases}$$
$$I_-(\alpha) = \begin{cases} -e^{\alpha 2\pi(x-\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & 0<x<\frac{1}{2} \\ -e^{\alpha 2\pi(x-\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & \frac{1}{2}<x<1 \end{cases}$$

Taylor expanding $I(\alpha)$ to get $\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2}$

We Taylor expand $-e^{\alpha 2\pi(x+\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi}$:

$$\begin{aligned}&-e^{\alpha 2\pi(x+\frac{1}{2})i-\alpha\pi i}\frac{\alpha\pi}{\sin\alpha\pi} \\ &= -\left[1+\alpha\pi i(2x)-\frac{1}{2}\alpha^2\pi^2(2x)^2+\frac{1}{6}\alpha^2\pi^2+\cdots\right]\end{aligned}$$

and we Taylor expand $-e^{\alpha 2\pi(x-\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi}$:

$$-e^{\alpha 2\pi(x-\frac{1}{2})i-\alpha\pi i}\frac{\alpha\pi}{\sin\alpha\pi} = -\left[1+\alpha\pi i(2x-2)-\frac{1}{2}\alpha^2\pi^2(2x-2)^2+\frac{1}{6}\alpha^2\pi^2+\cdots\right]$$

Then,

$$\begin{aligned}&\frac{1}{8}\frac{\partial^2}{\partial\alpha^2}[I(\alpha)-\frac{1}{2}I_+(\alpha)-\frac{1}{2}I_-(\alpha)]\Big|_{\alpha=0} \\ &= \begin{cases} \frac{1}{2}\pi^2\left(x-\frac{1}{2}\right)^2-\dfrac{\pi^2}{24}-\frac{1}{2}\left(\pi^2x^2-\dfrac{\pi^2}{12}\right) & 0<x<\frac{1}{2} \\ \frac{1}{2}\pi^2\left(x-\frac{1}{2}\right)^2-\dfrac{\pi^2}{24}-\frac{1}{2}\left(\pi^2\left(x-1\right)^2-\dfrac{\pi^2}{12}\right) & \frac{1}{2}<x<1 \end{cases} \\ &= \begin{cases} \frac{\pi^2}{2}\left(-x\right) & 0<x<\frac{1}{2} \\ \frac{\pi^2}{2}\left(x-\frac{1}{2}\right) & \frac{1}{2}<x<1 \end{cases}\end{aligned}$$

So that

$$\sum_{n=0}^{\infty} \frac{\cos(2\pi(2n-1)x)}{(2n-1)^2} = \begin{cases} \frac{\pi^2}{8} - \frac{\pi}{4} 2\pi x & 0 < x < \frac{1}{2} \\ \frac{\pi^2}{8} - \frac{\pi}{4}(2\pi - 2\pi x) & \frac{1}{2} < x < 1 \end{cases}$$

or

$$|x| = \frac{\pi^2}{4} - 2\sum_{n=0}^{\infty} \frac{\cos(2\pi(2n-1)x)}{(2n-1)^2}$$

on the interval $-\frac{1}{2} < x < \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{\cos((2n-1)y)}{(2n-1)^2} = \begin{cases} \frac{\pi^2}{8} - \frac{\pi}{4} y & 0 < y < \pi \\ \frac{\pi^2}{8} - \frac{\pi}{4}(2\pi - y) & \pi < y < 2\pi \end{cases}$$

—-

————-

2.4 Sum (iii)

Deriving the $|\sin x|$ function from its Fourier series

For $-\pi \leq x < \pi$,

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos(2nx) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} [\frac{1}{2n-1} \cos(2nx) - \frac{1}{2n+1} \cos(2nx)] \tag{2.2}$$

We have for the first sum on the RHS:

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{2n-1} \cos(2nx) \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \cos(2nx) \int_0^{\infty} e^{-u} du \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} \cos(2nx) e^{-(2n-1)u} du \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} \cos(2nx) e^{-2nu+u} du \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (e^{i2nx} + e^{-i2nx}) e^{-2nu+u} du \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (e^{-2n+i2nx} + e^{-2n-i2nx}) e^{u} du \\
&= \frac{1}{2} \int_0^{\infty} \left(\frac{1}{e^{2u-i2x}-1} + \frac{1}{e^{2u+i2x}-1} \right) e^{u} du
\end{aligned}
$$

There is a similar expression for the second sum on the RHS of ()

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{2n-1} \cos(nx) - \sum_{n=1}^{\infty} \frac{1}{2n+1} \cos(2nx) \\
&= \frac{1}{2} \int_0^{\infty} \left(\frac{1}{e^{2u-i2x}-1} + \frac{1}{e^{2u+i2x}-1} \right) e^{u} du - \frac{1}{2} \int_0^{\infty} \left(\frac{1}{e^{2u-i2x}-1} + \frac{1}{e^{2u+i2x}-1} \right) e^{-u} du \\
&= \frac{1}{2} \int_0^{\infty} \left(\frac{e^{i2x}}{e^{2u}-e^{i2x}} + \frac{e^{-i2x}}{e^{2u}-e^{-ix}} \right) e^{u} du - \frac{1}{2} \int_0^{\infty} \left(\frac{e^{i2x}}{e^{2u}-e^{i2x}} + \frac{e^{-i2x}}{e^{2u}-e^{-i2x}} \right) e^{-u} du \\
&= \frac{1}{2} \int_1^{\infty} \left(\frac{e^{i2x}}{w^2-e^{i2x}} + \frac{e^{-i2x}}{w^2-e^{-i2x}} \right) dw - \frac{1}{2} \int_1^{\infty} \left(\frac{e^{i2x}}{w^2(w^2-e^{i2x})} + \frac{e^{-i2x}}{w^2(w^2-e^{-i2x})} \right) dw \\
&= \frac{1}{2} \int_1^{\infty} \left(\frac{e^{i2x}}{w^2-e^{i2x}} + \frac{e^{-i2x}}{w^2-e^{-i2x}} \right) dw - \frac{1}{2} \int_1^{\infty} \left(\frac{1}{w^2-e^{i2x}} - \frac{1}{w^2} + \frac{1}{w^2-e^{-i2x}} - \frac{1}{w^2} \right) dw \\
&= \frac{1}{2} \int_1^{\infty} \left(\frac{e^{i2x}}{w^2-e^{i2x}} + \frac{e^{-i2x}}{w^2-e^{-i2x}} \right) dw + \frac{1}{2} \int_0^{1} \left(\frac{e^{-i2x}}{w^2-e^{-i2x}} + \frac{e^{i2x}}{w^2-e^{i2x}} \right) dw + \int_1^{\infty} \frac{1}{w^2} dw \\
&= \frac{1}{2} \int_0^{\infty} \left(\frac{e^{i2x}}{w^2-e^{i2x}} + \frac{e^{-i2x}}{w^2-e^{-i2x}} \right) dw + \left[-\frac{1}{w} \right]_1^{\infty} \\
&= Re \int_0^{\infty} \frac{e^{i2x}}{w^2-e^{i2x}} dw + 1
\end{aligned}
$$

We consider the integral

$$\oint_C \frac{e^{i2x}}{z^2 - e^{i2x}} \ln z dz$$

Estimates

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)dz\right| &\leq \int_0^{2\pi} \left|\frac{e^{2\pi xi}(\ln R + i\theta)}{R^2 e^{i2\theta} - e^{2xi}} Rie^{i\theta}\right| d\theta \\ &\int_0^{2\pi} \frac{\ln R + 2\pi}{R^2 - 1} R d\theta \\ &= \mathcal{O}(R^{-1}\ln R)\end{aligned}$$

$$\begin{aligned}\left|\int_{\Gamma_\epsilon} f(z)dz\right| &\leq \int_0^{2\pi} \left|\frac{e^{2\pi xi}(\ln \epsilon + i\theta)}{\epsilon^2 e^{i2\theta} - e^{2xi}} \epsilon e^{i\theta}\right| d\theta \\ &\int_0^{2\pi} \frac{|\ln \epsilon| + 2\pi}{1 - \epsilon^2} \epsilon d\theta \\ &= \mathcal{O}(\epsilon \ln \epsilon)\end{aligned}$$

In the limits $R \to \infty$ and $\epsilon \to 0$ we have

$$\begin{aligned}\oint_C \frac{e^{i2x}}{z^2 - e^{i2x}} \ln z dz &= \int_0^\infty \frac{e^{i2x} \ln w}{w^2 - e^{i2x}} dw - \int_0^\infty \frac{e^{i2x}(\ln w + 2\pi i)}{w^2 - e^{i2x}} dw \\ &= -2\pi i \int_0^\infty \frac{e^{i2x}}{w^2 - e^{i2x}} dw\end{aligned}$$

$$\frac{e^{i2x}}{z^2 - e^{i2x}} \ln z = \frac{e^{i2x}}{(z - e^{ix})(z - e^{ix + i\pi})} \ln z$$

Case $0 < x < \pi$:

$$\begin{aligned}Res_{z=e^{ix}} \frac{e^{i2x}}{z^2 - e^{i2x}} \ln z &= \frac{e^{i2x}}{2e^{ix}} ix = \frac{e^{ix}}{2} ix \\ Res_{z=e^{ix+i\pi}} \frac{e^{i2x}}{z^2 - e^{i2x}} \ln z &= \frac{e^{i2x}}{2e^{ix+i\pi}} (ix + i\pi) = -\frac{e^{ix}}{2}(ix + i\pi)\end{aligned}$$

Case $\pi < x < 2\pi$:

$$Res_{z=e^{ix}} \frac{e^{i2x}}{z^2 - e^{i2x}} \ln z = \frac{e^{i2x}}{2e^{ix}} ix = \frac{e^{ix}}{2} ix$$
$$Res_{z=e^{ix-i\pi}} \frac{e^{i2x}}{z^2 - e^{i2x}} \ln z = \frac{e^{i2x}}{2e^{ix+i\pi}} (ix - i\pi) = -\frac{e^{ix}}{2} (ix - i\pi)$$

$$\begin{aligned} \int_0^\infty \frac{e^{i2x}}{w^2 - e^{i2x}} dw &= -Res \frac{e^{i2x}}{z^2 - e^{i2x}} \ln z \\ &= \begin{cases} \frac{e^{ix}}{2} i\pi & 0 < x < \pi \\ -\frac{e^{ix}}{2} i\pi & \pi < x < 2\pi \end{cases} \end{aligned}$$

Combining everything

$$\begin{aligned} \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx) &= \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} [\frac{1}{2n-1} \cos(2nx) - \frac{1}{2n+1} \cos(2nx)] \\ &= Re \int_0^\infty \frac{e^{i2x}}{w^2 - e^{i2x}} dw + 1 \\ &= \begin{cases} \frac{2}{\pi} - \frac{2}{\pi} [-\frac{\sin x}{2} \pi + 1] & 0 < x < \pi \\ \frac{2}{\pi} - \frac{2}{\pi} [\frac{\sin x}{2} \pi + 1] & \pi < x < 2\pi \end{cases} \\ &= \begin{cases} = -\sin x & 0 < x < \pi \\ -\sin x & \pi < x < 2\pi \end{cases} \\ &= |\sin x| \end{aligned}$$

2.5 Sum (iv)

Deriving the $|\cos x|$ function from its Fourier series

For $-\pi \leq x < \pi$,

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \cos(2nx)$$

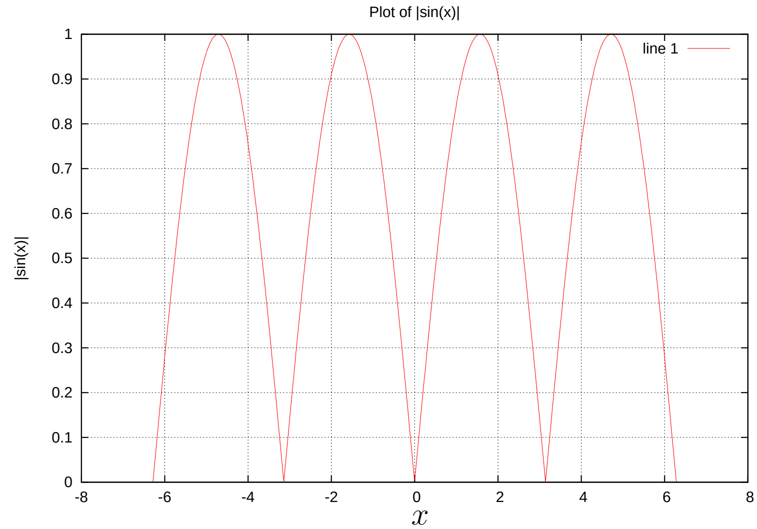

Figure 2.2: Plot of $|\sin x|$.

We have just proved:

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos(2nx) = |\sin x|$$

From $\cos x = \sin(x + \frac{\pi}{2})$ and $\cos(2n(x + \frac{\pi}{2})) = (-1)^n \cos(2nx)$, we have:

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}\cos(2nx) = |\cos x|$$

—-

2.6 Sum (v)

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}(x).$$

or

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}(\frac{x}{2\pi}).$$

We will evaluate the general Fourier series

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{2k}}$$

for $k = 1, 2, \ldots$. We have:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\cos(2\pi nx)}{n^{2k}} \\
&= \frac{1}{(2k-1)!} \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{2k}} \int_0^{\infty} u^{2k-1} e^{-u} du \\
&= \frac{1}{(2k-1)!} \sum_{n=1}^{\infty} \cos(2\pi nx) \int_0^{\infty} u^{2k-1} e^{-nu} du \\
&= \frac{1}{(2k-1)!} \frac{1}{2} \sum_{n=1}^{\infty} \int_0^{\infty} (e^{-nu+in2\pi x} + e^{-nu-in2\pi x}) u^{2k-1} du \\
&= \frac{1}{2(2k-1)!} \int_0^{\infty} \sum_{n=1}^{\infty} (e^{-nu+in2\pi x} + e^{-nu-in2\pi x}) u^{2k-1} du \\
&= \frac{1}{2(2k-1)!} \int_0^{\infty} \left(\frac{1}{e^{u-2\pi xi} - 1} + \frac{1}{e^{u+2\pi xi} - 1} \right) u^{2k-1} du
\end{aligned}$$

To justify interchanging summation and integration in the above, we can use Fubini:

$$
\begin{aligned}
\frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}|\cos(2\pi nx)u^{2k-1}e^{-nu}|du &\leq \frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}u^{2k-1}e^{-nu}du \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \\
&< \sum_{n=1}^{\infty}\frac{1}{n^2} \\
&< 1+\lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)} \\
&= 1+\lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right) \\
&= 1+1-\lim_{N\to\infty}\frac{1}{N} = 2 < \infty
\end{aligned}
$$

From the above

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{\cos(2\pi nx)}{n^2} &= \frac{1}{2(2k-1)!}\int_0^{\infty}\left(\frac{1}{e^{u-2\pi xi}-1}+\frac{1}{e^{u+2\pi xi}-1}\right)u^{2k-1}du \\
&= \frac{1}{2(2k-1)!}\int_0^{\infty}\left(\frac{e^{2\pi xi}u^{2k-1}}{e^u-e^{2\pi xi}}+\frac{e^{-2\pi xi}u^{2k-1}}{e^u-e^{-2\pi xi}}\right)du \\
&= \frac{1}{2(2k)!}\int_0^{\infty}\left(\frac{e^{2\pi xi}e^u u^{2k}}{(e^u-e^{2\pi xi})^2}+\frac{e^{-2\pi xi}e^u u^{2k}}{(e^u-e^{-2\pi xi})^2}\right)du \\
&= \frac{1}{4(2k)!}\int_{-\infty}^{\infty}\left(\frac{e^{2\pi xi}e^u u^{2k}}{(e^u-e^{2\pi xi})^2}+\frac{e^{-2\pi xi}e^u u^{2k}}{(e^u-e^{-2\pi xi})^2}\right)du \\
&= \frac{1}{2(2k)!}Re\int_{-\infty}^{\infty}\frac{e^{2\pi xi}e^u u^{2k}}{(e^u-e^{2\pi xi})^2}du
\end{aligned}
$$

where I have done an integration by parts. Extending the range of integration is valid if you use both terms.

Define

$$
I(\alpha) = \int_{-\infty}^{\infty}\frac{e^{2\pi xi}e^{\alpha u}e^u}{(e^u-e^{2\pi xi})^2}du
$$

for $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. Then

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2k}} = \frac{1}{2(2k)!} \frac{\partial^{2k}}{\partial \alpha^{2k}} ReI(\alpha)\Bigg|_{\alpha=0} \qquad (*).$$

In order to evaluate $I(\alpha)$, consider the contour in the figure.

rectangle contour.jpg

and the integral

$$\oint_C \frac{e^{2\pi x i} e^{\alpha z} e^z}{(e^z - e^{2\pi x i})^2}$$

whose integrand has pole at $2\pi x i$. The integral along the vertical edges vanishes as:

$$f(z) = \frac{e^{2\pi x i} e^{(\alpha+1)(u+iv)}}{(e^{u+iv} + e^{2\pi x i})^2} = \begin{cases} e^{2\pi x i} e^{(\alpha-1)(u+iv)} & u \to \infty \\ \dfrac{e^{(\alpha+1)(u+iv)}}{e^{2\pi x i}} & u \to -\infty \end{cases}$$

So that

$$\begin{aligned} \oint_C \frac{e^{2\pi x i} e^{\alpha z} e^z}{(e^z - e^{2\pi x i})^2} &= \int_{-\infty}^{\infty} \frac{e^{2\pi x i} e^{\alpha u}}{(e^u - e^{2\pi x i})^2} du - e^{\alpha 2\pi i} \int_{-\infty+2\pi i}^{\infty+2\pi i} \frac{e^{2\pi x i} e^{\alpha u}}{(e^u - e^{2\pi x i})^2} du \\ &= (1 - e^{\alpha 2\pi i}) \int_{-\infty}^{\infty} \frac{e^{2\pi x i} e^{\alpha u}}{(e^u - e^{2\pi x i})^2} du \end{aligned}$$

Which rearranged is

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{2\pi x i} e^{\alpha u} e^u}{(e^u - e^{2\pi x i})^2} du &= \frac{2\pi i}{1 - e^{\alpha 2\pi i}} \frac{1}{2\pi i} \oint_C \frac{e^{2\pi x i} e^{\alpha z} e^z}{(e^z - e^{2\pi x i})^2} dz \\ &= \frac{2\pi i}{1 - e^{\alpha 2\pi i}} Res[f(z)] \end{aligned}$$

We wish to expand the integrand in powers of $z - z_0$ about pole $z_0 = 2\pi x i$. First note:

$$
\begin{aligned}
\frac{1}{(e^z - e^{z_0})^2} &= \frac{e^{-2z_0}}{(e^{z-z_0} - 1)^2} \\
&= \frac{e^{-2z_0}}{(z - z_0)^2} - \frac{e^{-2z_0}}{z - z_0} + \cdots
\end{aligned}
$$

Then note:

$$
\begin{aligned}
\frac{e^{z_0} e^{\alpha z} e^z}{(e^z - e^{z_0})^2} &= e^{-z_0} \frac{e^{(\alpha+1)z_0 + (\alpha+1)(z-z_0)}}{(z - z_0)^2} - e^{-z_0} \frac{e^{(\alpha+1)z_0}}{z - z_0} + \cdots \\
&= e^{-z_0} \frac{e^{(\alpha+1)z_0}[1 + (\alpha+1)(z - z_0) + \cdots]}{(z - z_0)^2} - e^{-z_0} \frac{e^{(\alpha+1)z_0}}{z - z_0} + \cdots \\
&= \frac{e^{\alpha z_0}}{(z - z_0)^2} + \frac{\alpha e^{\alpha z_0}}{z - z_0} + \cdots
\end{aligned}
$$

Combining everything

$$
\begin{aligned}
I(\alpha) &= \int_{-\infty}^{\infty} \frac{e^{2\pi x i} e^{\alpha u}}{(e^u - e^{2\pi x i})^2} du \\
&= 2\pi i \frac{1}{1 - e^{\alpha 2\pi i}} \cdot \alpha e^{\alpha 2\pi x i} \\
&= -e^{\alpha 2\pi x i - \alpha \pi i} \frac{\alpha \pi}{\sin \alpha \pi}
\end{aligned}
$$

This expression is related to the generating function for Bernoulli polynomials, as we establish in a moment. First we take the case $k = 1$ and Taylor expand in α to obtain the coefficient of α^2.

Taylor expanding $I(\alpha)$ to get $\sum_{n=1}^{\infty} \frac{\cos 2\pi x}{n^2}$

We Taylor expand $I(\alpha)$:

$$
\begin{aligned}
I(\alpha) &= -e^{\alpha 2\pi x i - \alpha \pi i} \frac{\alpha \pi}{\sin \alpha \pi} \\
&= -\left[1 + \alpha \pi i(2x - 1) - \frac{1}{2}\alpha^2 \pi^2 (2x - 1)^2 + \cdots\right] \frac{1}{1 - \frac{1}{3!}\alpha^2 \pi^2 + \cdots} \\
&= -\left[1 + \alpha \pi i(2x - 1) - \frac{1}{2}\alpha^2 \pi^2 (2x - 1)^2 + \cdots\right] \left[1 + \frac{1}{6}\alpha^2 \pi^2 + \cdots\right] \\
&= -\left[1 + \alpha \pi i(2x - 1) - \frac{1}{2}\alpha^2 \pi^2 (2x - 1)^2 + \frac{1}{6}\alpha^2 \pi^2 + \cdots\right]
\end{aligned}
$$

Then, for $k = 1$:

$$\frac{1}{4}\frac{\partial^2}{\partial\alpha^2}I(\alpha)\bigg|_{\alpha=0} = \pi^2\left(x - \frac{1}{2}\right)^2 - \frac{\pi^2}{12}$$
$$= \pi^2\left(x^2 - x + \frac{1}{6}\right)$$

and so

$$\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^2} = \pi^2\left(x^2 - x + \frac{1}{6}\right).$$

Bernoulli polynomials

We return to the evaluation of the general sum, $\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{2k}}$. The generating function for the Bernoulli polynomials is

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}.$$

We have

$$\frac{te^{xt}}{\sinh\frac{t}{2}} = \frac{2te^{t(x+\frac{1}{2})}}{e^t - 1} = 2\sum_{k=0}^{\infty} B_k(x + \frac{1}{2})\frac{t^k}{k!}$$

We perform $t \mapsto i2\pi t$ and $x \mapsto x - \frac{1}{2}$,

$$\frac{\pi t}{\sin\pi t}e^{i2\pi tx - i\pi t} = \sum_{k=0}^{\infty} B_k(x)\frac{(i2\pi t)^k}{k!}$$

We recognise the LHS as our expression for $-I(\alpha)$ with α replaced with t. Taking the real part:

$$Re\frac{\pi t}{\sin\pi t}e^{i2\pi tx - i\pi t} = \sum_{k=0}^{\infty} B_{2k}(x)\frac{(-1)^k(2\pi)^{2k}}{(2k)!}t^{2k}$$

Using this in $(*)$ we obtain the general formula:

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2k}} = -\frac{1}{2(2k)!} \frac{\partial^{2k}}{\partial \alpha^{2k}} Re\, e^{\alpha 2\pi x i - \alpha \pi i} \frac{\alpha \pi}{\sin \alpha \pi} \bigg|_{\alpha=0}$$
$$= \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}(x)$$

For $k = 1$,

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^2} = \pi^2 B_2(x).$$

We need to compute $B_2(x)$. We next derive an explicit formula for Bernoulli polynomials.

Explicit formula for Bernoulli polynomials

The generating function for Bernoulli numbers is

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

From the generating function for Bernoulli polynomials and Cauchy product of series:

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{tx}$$
$$= \left(\sum_{k=0}^{\infty} \frac{B_k t^k}{k!}\right)\left(\sum_{l=0}^{\infty} \frac{x^l t^l}{l!}\right)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} B_{n-k} x^k \frac{t^n}{n!}$$

We read off

$$B_n(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} B_{n-k} x^k$$

The first few Bernoulli polynomials are

$$
\begin{aligned}
B_0(x) &= 1, & B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\
B_1(x) &= x - \frac{1}{2}, & B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\
B_2(x) &= x^2 - x + \frac{1}{6}, & B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, & &\vdots
\end{aligned}
$$

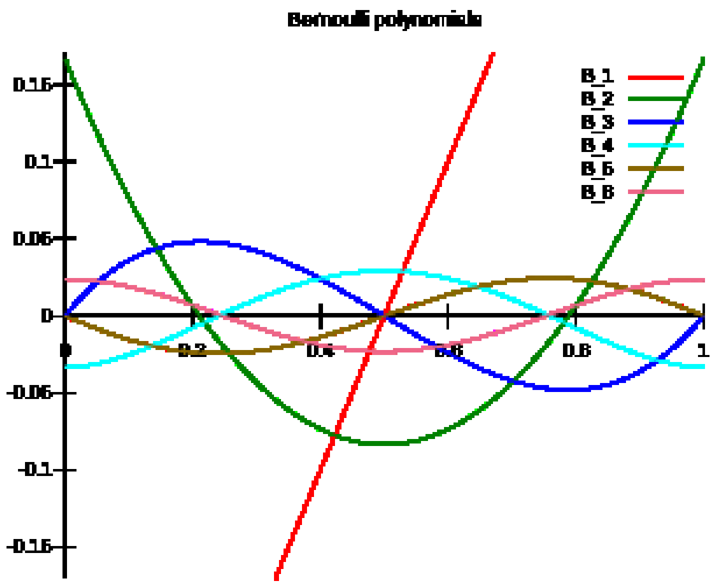

Figure 2.3: Plot of Bernoulli polynomials.

It is easy to expand the generating function for Bernoulli numbers to find $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_2 = \frac{1}{6}$. Substituting them into the expression for $B_2(x)$,

$$
\begin{aligned}
B_2(x) &= B_0 + 2B_1 x + B_2 x^2 \\
&= x^2 - x + \frac{1}{6}.
\end{aligned}
$$

So finally,

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^2} = \pi^2 \left(x^2 - x - \frac{1}{6} \right).$$

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}(x).$$

or

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}(\frac{x}{2\pi}).$$

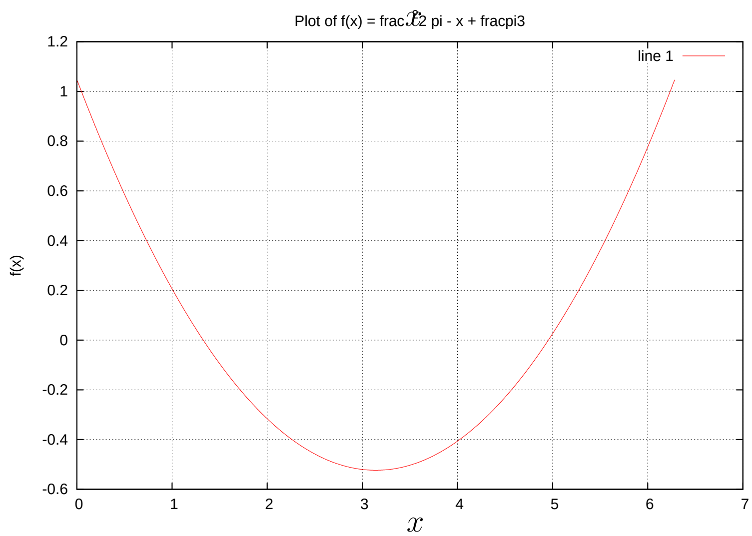

Figure 2.4: Plot of $B_{2k}(\frac{x}{2\pi}) = \frac{x^2}{2\pi} - x + \frac{\pi}{3}$ between 0 and 2π.

2.7 (vi)

We wish to evaluate:

$$\frac{1}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos(an)}{a^2 n^2} \cos(nx)$$

It can be written as

$$\frac{1}{2\pi} + \frac{2}{\pi a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) - \frac{1}{2} \frac{2}{\pi a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n(x-a)) - \frac{1}{2} \frac{2}{\pi a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n(x+a)).$$

As $B_2(x) = x^2 - x + \frac{1}{6}$ we have from the previous result, we have in particular:

$$\begin{aligned} \frac{2}{\pi a^2} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} &= \frac{2}{\pi a^2} \frac{(2\pi)^2}{2(2)!} B_2(\frac{x}{2\pi}) \\ &= a^{-2} \left(\frac{x^2}{2\pi} - x + \frac{\pi}{3} \right) \end{aligned} \tag{2.3}$$

for $x \in [0, 2\pi)$. So that

$$\begin{aligned} \frac{2}{\pi a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n(x-a)) &= \begin{cases} a^{-2} \left(\frac{(x-a+2\pi)^2}{2\pi} - (x-a+2\pi) + \frac{\pi}{3} \right) & \text{if } 0 \le x < a \\ a^{-2} \left(\frac{(x-a)^2}{2\pi} - (x-a) + \frac{\pi}{3} \right) & \text{if } a \le x < 2\pi \end{cases} \\ &= \begin{cases} a^{-2} \left(\frac{(x-a)^2}{2\pi} + (x-a) + \frac{\pi}{3} \right) & \text{if } 0 \le x < a \\ a^{-2} \left(\frac{(x-a)^2}{2\pi} - (x-a) + \frac{\pi}{3} \right) & \text{if } a \le x < 2\pi \end{cases} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \frac{2}{\pi a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n(x+a)) &= \begin{cases} a^{-2} \left(\frac{(x+a)^2}{2\pi} - (x+a) + \frac{\pi}{3} \right) & \text{if } 0 \le x < 2\pi - a \\ a^{-2} \left(\frac{(x+a-2\pi)^2}{2\pi} - (x+a-2\pi) + \frac{\pi}{3} \right) & \text{if } 2\pi - a \le x < 2\pi \end{cases} \\ &= \begin{cases} a^{-2} \left(\frac{(x+a)^2}{2\pi} - (x+a) + \frac{\pi}{3} \right) & \text{if } 0 \le x < 2\pi - a \\ a^{-2} \left(\frac{(x+a)^2}{2\pi} - 3(x+a) + 4\pi + \frac{\pi}{3} \right) & \text{if } 2\pi - a \le x < 2\pi \end{cases} \end{aligned} \tag{2.5}$$

Combing (2.3), (2.4), and (2.5) we obtain:

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nx)-\frac{1}{2}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n(x-a))-\frac{1}{2}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n(x+a))]$$

$$= \begin{cases} \frac{x^2}{2\pi}-x-\frac{1}{2}[\frac{(x-a)^2}{2\pi}+(x-a)]-\frac{1}{2}[\frac{(x+a)^2}{2\pi}-(x+a)] & \text{if } 0\le x<a \\ \frac{x^2}{2\pi}-x-\frac{1}{2}[\frac{(x-a)^2}{2\pi}-(x-a)]-\frac{1}{2}[\frac{(x+a)^2}{2\pi}-(x+a)] & \text{if } a\le x<2\pi-a \\ \frac{x^2}{2\pi}-x-\frac{1}{2}[\frac{(x-a)^2}{2\pi}-(x-a)]-\frac{1}{2}[\frac{(x+a)^2}{2\pi}-3(x+a)+4\pi] & \text{if } 2\pi-a\le x<2\pi \end{cases}$$

$$= \begin{cases} -\frac{a^2}{2\pi}+a-x & \text{if } 0\le x<a \\ -\frac{a^2}{2\pi} & \text{if } a\le x<2\pi-a \\ -\frac{a^2}{2\pi}+x-2\pi+a & \text{if } 2\pi-a\le x<2\pi \end{cases} \tag{2.6}$$

$$\frac{1}{2\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(an)}{a^2n^2}\cos(nx)=\begin{cases} a^{-2}(a-x) & \text{if } 0\le x<a \\ 0 & \text{if } a\le x<2\pi-a \\ a^{-2}[x-(2\pi-a)][& \text{if } 2\pi-a\le x<2\pi \end{cases}$$

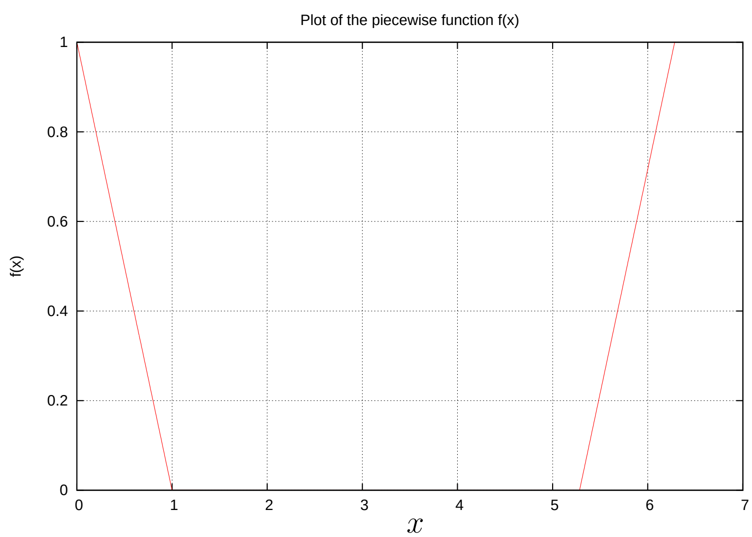

Figure 2.5: Plot of triangle wave for $a=1$ from 0 to 2π.

2.8 (vii)

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\sin(2\pi nx)$$

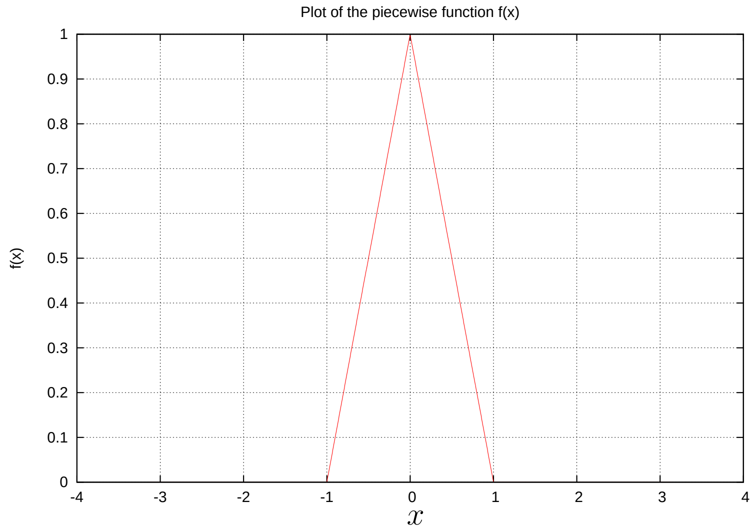

Figure 2.6: Plot of triangle wave for $a = 1$ from $-\pi$ to π.

for $k = 1, 2, \ldots$. We have:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\sin(2\pi nx)}{n^{2k+1}} \\
&= \frac{1}{(2k)!} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{2k+1}} \int_0^{\infty} u^{2k} e^{-u} du \\
&= \frac{1}{(2k)!} \sum_{n=1}^{\infty} \sin(2\pi nx) \int_0^{\infty} u^{2k} e^{-nu} du \\
&= \frac{1}{(2k)!} \frac{1}{2i} \sum_{n=1}^{\infty} \int_0^{\infty} (e^{-nu+in2\pi x} - e^{-nu-in2\pi x}) u^{2k} du \\
&= \frac{1}{2i(2k)!} \int_0^{\infty} \sum_{n=1}^{\infty} (e^{-nu+in2\pi x} - e^{-nu-in2\pi x}) u^{2k} du \\
&= \frac{1}{2i(2k)!} \int_0^{\infty} \left(\frac{1}{e^{u-2\pi xi} - 1} - \frac{1}{e^{u+2\pi xi} - 1} \right) u^{2k} du
\end{aligned}
$$

To justify interchanging summation and integration in the above, we can use Fubini:

$$
\begin{aligned}
\frac{1}{(2k)!}\sum_{n=1}^{\infty}\int_0^{\infty}|\sin(2\pi nx)u^{2k}e^{-nu}|du &\leq \frac{1}{(2k)!}\sum_{n=1}^{\infty}\int_0^{\infty}u^{2k}e^{-nu}du \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{2k+1}} \\
&< \sum_{n=1}^{\infty}\frac{1}{n^2} \\
&< 1+\lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)} \\
&= 1+\lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right) \\
&= 1+1-\lim_{N\to\infty}\frac{1}{N}=2<\infty
\end{aligned}
$$

From the above

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{\sin(2\pi nx)}{n^2} &= \frac{1}{2i(2k)!}\int_0^{\infty}\left(\frac{1}{e^{u-2\pi xi}-1}-\frac{1}{e^{u+2\pi xi}-1}\right)u^{2k}du \\
&= \frac{1}{2i(2k)!}\int_0^{\infty}\left(\frac{e^{2\pi xi}u^{2k}}{e^u-e^{2\pi xi}}-\frac{e^{-2\pi xi}u^{2k}}{e^u-e^{-2\pi xi}}\right)du \\
&= \frac{1}{2i(2k+1)!}\int_0^{\infty}\left(\frac{e^{2\pi xi}e^u u^{2k+1}}{(e^u-e^{2\pi xi})^2}-\frac{e^{-2\pi xi}e^u u^{2k+1}}{(e^u-e^{-2\pi xi})^2}\right)du \\
&= \frac{1}{4i(2k+1)!}\int_{-\infty}^{\infty}\left(\frac{e^{2\pi xi}e^u u^{2k+1}}{(e^u-e^{2\pi xi})^2}-\frac{e^{-2\pi xi}e^u u^{2k+1}}{(e^u-e^{-2\pi xi})^2}\right)du \\
&= \frac{1}{2(2k+1)!}Im\int_{-\infty}^{\infty}\frac{e^{2\pi xi}e^u u^{2k+1}}{(e^u-e^{2\pi xi})^2}du
\end{aligned}
$$

where I have done an integration by parts. Extending the range of integration is valid if you use both terms.

Define

$$
I(\alpha)=\int_{-\infty}^{\infty}\frac{e^{2\pi xi}e^{\alpha u}e^u}{(e^u-e^{2\pi xi})^2}du
$$

for $-\frac{1}{2}\leq\alpha\leq\frac{1}{2}$ as before. Then

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^{2k}} = \frac{1}{2(2k+1)!} \frac{\partial^{2k+1}}{\partial \alpha^{2k+1}} Im I(\alpha)\Big|_{\alpha=0} \qquad (*).$$

We have, derived in a previous section,

$$I(\alpha) = -e^{\alpha 2\pi x i - \alpha \pi i} \frac{\alpha \pi}{\sin \alpha \pi}$$

This expression is related to the generating function for Bernoulli polynomials, as we establish in a previous section:

$$\frac{\pi t}{\sin \pi t} e^{i2\pi t x - i\pi t} = \sum_{k=0}^{\infty} B_k(x) \frac{(i2\pi t)^k}{k!}$$

We recognise the LHS as our expression for $-I(\alpha)$ with α replaced with t. Taking the imaginary part:

$$Im \frac{\pi t}{\sin \pi t} e^{i2\pi t x - i\pi t} = \sum_{k=0}^{\infty} B_{2k+1}(x) \frac{(-1)^k (2\pi)^{2k+1}}{(2k+1)!} t^{2k+1}$$

Using this in $(*)$ we obtain the general formula:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^{2k+1}} &= -\frac{1}{2(2k+1)!} \frac{\partial^{2k+1}}{\partial \alpha^{2k+1}} Im\ e^{\alpha 2\pi x i - \alpha \pi i} \frac{\alpha \pi}{\sin \alpha \pi}\Big|_{\alpha=0} \\ &= \frac{(-1)^{k+1} (2\pi)^{2k+1}}{2(2k+1)!} B_{2k+1}(x) \end{aligned}$$

2.9 Sum (viii)

$$\frac{\pi^2}{2\pi} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

We have from () that for $x \in [0, 1)$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2\pi n x) = \pi^2 \left(x - \frac{1}{2} \right)^2 - \frac{\pi^2}{12}$$

Performing the shift $x \mapsto x + \frac{1}{2}$ results in

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(2\pi n x) = \pi^2 x^2 - \frac{\pi^2}{12}$$

for $x \in [-\frac{1}{2}, \frac{1}{2})$ or

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) = \frac{x^2}{4} - \frac{\pi^2}{12}$$

So that for $x \in [-\pi, \pi)$ we have

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) = x^2$$

2.10 Sum (ix)

$$\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin((2n-1)x)$$

2.11 Sum (x)

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi(2n-1)x)}{(2n-1)^{2k}}$$

Deriving the functions from its Fourier series

We have:

$$\sum_{n=0}^{\infty} \frac{\cos(2\pi(2n-1)x)}{(2n-1)^{2k}} = \frac{1}{2}\sum_{n=1}^{\infty}[1-\cos n\pi]\frac{\cos(2\pi nx)}{n^{2k}}$$

We already know that

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{4(2k)!}B_{2k}(x).$$

So we need to only calculate

$$\begin{aligned}
&= -\frac{1}{2(2k-1)!}\sum_{n=1}^{\infty}\cos n\pi \frac{\cos(2\pi nx)}{n^2}\int_0^{\infty} u^{2k-1}e^{-u}du \\
&= -\frac{1}{2(2k-1)!}\sum_{n=1}^{\infty}\cos n\pi \cos(2\pi nx)\int_0^{\infty} u^{2k-1}e^{-nu}du \\
&= -\frac{1}{4(2k-1)!}\sum_{n=1}^{\infty}[\cos(2\pi nx+n\pi)+\cos(2\pi nx-n\pi)]\int_0^{\infty} u^{2k-1}e^{-nu}du \\
&= -\frac{1}{8(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}[(e^{-nu+in2\pi(x+\frac{1}{2})}+e^{-nu-in2\pi(x+\frac{1}{2})})+(e^{-nu+in2\pi(x-\frac{1}{2})}+e^{-nu-in2\pi(x-\frac{1}{2})})]u^{2k-1}du \\
&= -\frac{1}{8(2k-1)!}\int_0^{\infty}\sum_{n=1}^{\infty}[(e^{-nu+in2\pi(x+\frac{1}{2})}+e^{-nu-in2\pi(x+\frac{1}{2})})+(e^{-nu+in2\pi(x-\frac{1}{2})}+e^{-nu-in2\pi(x-\frac{1}{2})})]u^{2k-1}du \\
&= -\frac{1}{8(2k-1)!}\int_0^{\infty}\left(\frac{1}{e^{u-2\pi(x+\frac{1}{2})i}-1}+\frac{1}{e^{u+2\pi(x+\frac{1}{2})i}-1}+\frac{1}{e^{u-2\pi(x-\frac{1}{2})i}-1}+\frac{1}{e^{u+2\pi(x-\frac{1}{2})i}-1}\right)u^{2k-1}du
\end{aligned}$$

To justify interchanging summation and integration in the above, we can use Fubini:

$$\frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}|\cos n\pi\cos(2\pi nx)ue^{-nu}|du \leq \sum_{n=1}^{\infty}\int_0^{\infty}u^{2k-1}e^{-nu}du$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$< 1+\lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)}$$
$$= 1+\lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$
$$= 1+1-\lim_{N\to\infty}\frac{1}{N} = 2 < \infty$$

From the above

$$\begin{aligned}
&-\frac{1}{2(2k-1)!}\sum_{n=1}^{\infty}\cos n\pi\frac{\cos(2\pi nx)}{n^2}\int_0^{\infty}ue^{-u}du\\
&= -\frac{1}{8(2k-1)!}\int_0^{\infty}\left(\frac{1}{e^{u-2\pi(x+\frac{1}{2})i}-1}+\frac{1}{e^{u+2\pi(x+\frac{1}{2})i}-1}+\frac{1}{e^{u-2\pi(x-\frac{1}{2})i}-1}+\frac{1}{e^{u+2\pi(x-\frac{1}{2})i}-1}\right)u^{2k-1}du\\
&= -\frac{1}{8(2k-1)!}\int_0^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u^{2k-1}}{e^u-e^{2\pi(x+\frac{1}{2})i}}+\frac{e^{-2\pi(x+\frac{1}{2})i}u^{2k-1}}{e^u-e^{-2\pi(x+\frac{1}{2})i}}+\frac{e^{2\pi(x-\frac{1}{2})i}u^{2k-1}}{e^u-e^{2\pi(x-\frac{1}{2})i}}+\frac{e^{-2\pi(x-\frac{1}{2})i}u}{e^u-e^{-2\pi(x-\frac{1}{2})i}}\right)du\\
&= -\frac{1}{8(2k)!}\int_0^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u^{2k}}{(e^u-e^{2\pi(x+\frac{1}{2})i})^2}+\frac{e^{-2\pi(x+\frac{1}{2})i}u^{2k}}{(e^u-e^{-2\pi(x+\frac{1}{2})i})^2}+\frac{e^{2\pi(x-\frac{1}{2})i}u^{2k}}{(e^u-e^{2\pi(x-\frac{1}{2})i})^2}+\frac{e^{-2\pi(x-\frac{1}{2})i}u^{2k}}{(e^u-e^{-2\pi(x-\frac{1}{2})i})^2}\right)du\\
&= -\frac{1}{16(2k)!}\int_{-\infty}^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u^{2k}}{(e^u-e^{2\pi(x+\frac{1}{2})i})^2}+\frac{e^{-2\pi(x+\frac{1}{2})i}u^{2k}}{(e^u-e^{-2\pi(x+\frac{1}{2})i})^2}+\frac{e^{2\pi(x-\frac{1}{2})i}u^{2k}}{(e^u-e^{2\pi(x-\frac{1}{2})i})^2}+\frac{e^{-2\pi(x-\frac{1}{2})i}u^{2k}}{(e^u-e^{-2\pi(x-\frac{1}{2})i})^2}\right)du\\
&= -\frac{1}{8(2k)!}Re\int_{-\infty}^{\infty}\left(\frac{e^{2\pi(x+\frac{1}{2})i}u^{2k}}{(e^u-e^{2\pi(x+\frac{1}{2})i})^2}+\frac{e^{2\pi(x-\frac{1}{2})i}u^{2k}}{(e^u-e^{2\pi(x-\frac{1}{2})i})^2}\right)du
\end{aligned}$$

Define

$$I_{\pm}(\alpha) = \int_{-\infty}^{\infty}\frac{e^{2\pi(x\pm\frac{1}{2})i}e^{\alpha u}e^u}{(e^u-e^{2\pi(x\pm\frac{1}{2})i})^2}du$$

for $-\frac{1}{2}\leq\alpha\leq\frac{1}{2}$. Then

$$\sum_{n=0}^{\infty}\frac{\cos(2\pi(2n-1)x)}{(2n-1)^2}=\frac{1}{8(2k)!}\frac{\partial^{2k}}{\partial\alpha^{2k}}[ReI(\alpha)-\frac{1}{2}ReI_+(\alpha)-\frac{1}{2}ReI_-(\alpha)]\Big|_{\alpha=0}.$$

Recall

$$I_+(\alpha)=\begin{cases}-e^{\alpha 2\pi(x+\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & 0<x<\frac{1}{2}\\ -e^{\alpha 2\pi(x+\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & \frac{1}{2}<x<1\end{cases}$$
$$I_-(\alpha)=\begin{cases}-e^{\alpha 2\pi(x-\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & 0<x<\frac{1}{2}\\ -e^{\alpha 2\pi(x-\frac{1}{2})i-\alpha\pi i}\dfrac{\alpha\pi}{\sin\alpha\pi} & \frac{1}{2}<x<1\end{cases}$$

and

$$ReI(\alpha)=\sum_{k=0}^{\infty}B_{2k}\frac{(-1)^{k+1}(2\pi)^{2k}}{(2k)!}\alpha^{2k}.$$

So that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\cos(2\pi(2n-1)x)}{(2n-1)^2}&=\frac{1}{8(2k)!}\frac{\partial^{2k}}{\partial\alpha^{2k}}[ReI(\alpha)-\frac{1}{2}ReI_+(\alpha)-\frac{1}{2}ReI_-(\alpha)]\Big|_{\alpha=0}\\&=\frac{(-1)^{k+1}(2\pi)^{2k}}{8(2k)!}[B_{2k}(x)-\frac{1}{2}B_{2k}(x+\frac{1}{2})-\frac{1}{2}B_{2k}(x-\frac{1}{2})]\end{aligned}$$

2.12 Sum (xi)

$$\sum_{n=0}^{\infty}\frac{\cos(\pi(2n+1)x)}{(2n+1)^{2k}}$$

$0\leq x<1$

We will evaluate the general Fourier series

$$\sum_{n=1}^{\infty} \frac{\cos(\pi(2n-1)x)}{(2n-1)^{2k}}$$

for $k = 1, 2, \ldots$. We have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\cos(\pi(2n-1)x)}{(2n-1)^{2k}} &= \frac{1}{2}\sum_{n=1}^{\infty}[1-\cos(\pi n)]\frac{\cos(\pi n x)}{n^{2k}} \\
&= \frac{1}{4}\sum_{n=1}^{\infty}\frac{[2\cos(\pi n x) - \cos(\pi n(x+1)) - \cos(\pi n(x-1))]}{n^{2k}}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{\infty}[1-\cos(\pi n)]\frac{\cos(\pi n x)}{n^{2k}} \\
&= \frac{1}{4(2k-1)!}\sum_{n=1}^{\infty}[2\cos(\pi n x) - \cos(\pi n(x+1)) - \cos(\pi n(x-1))]\frac{1}{n^{2k}}\int_0^{\infty} u^{2k-1}e^{-u}du \\
&= \frac{1}{4(2k-1)!}\sum_{n=1}^{\infty}[2\cos(\pi n x) - \cos(\pi n(x+1)) - \cos(\pi n(x-1))]\int_0^{\infty} u^{2k-1}e^{-nu}du \\
&= \frac{1}{8(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}(2e^{-nu+in\pi x} + 2e^{-nu-in\pi x} - e^{-nu+in\pi(x+1)} - e^{-nu-in\pi(x+1)} \\
&- e^{-nu+in\pi(x-1)} - e^{-nu-in\pi(x-1)})u^{2k-1}du \\
&= \frac{1}{8(2k-1)!}\int_0^{\infty}\sum_{n=1}^{\infty}(2e^{-nu+in\pi x} + 2e^{-nu-in\pi x} - e^{-nu+in\pi(x+1)} - e^{-nu-in\pi(x+1)} \\
&- e^{-nu+in\pi(x-1)} - e^{-nu-in\pi(x-1)})u^{2k-1}du \\
&= \frac{1}{8(2k-1)!}\int_0^{\infty}\left(\frac{2}{e^{u-\pi xi}-1} + \frac{2}{e^{u+\pi xi}-1} - \frac{1}{e^{u-\pi(x+1)i}-1} - \frac{1}{e^{u+\pi(x+1)i}-1}\right. \\
&\left. - \frac{1}{e^{u-\pi(x-1)i}-1} - \frac{1}{e^{u+\pi(x-1)i}-1}\right)u^{2k-1}du \\
&= \frac{1}{4(2k-1)!}\int_0^{\infty}\left(\frac{1}{e^{u-\pi xi}-1} + \frac{1}{e^{u+\pi xi}-1} + \frac{1}{e^{u-\pi xi}+1} + \frac{1}{e^{u+\pi xi}+1}\right)u^{2k-1}du \\
&= \frac{1}{4(2k-1)!}\int_0^{\infty}\left(\frac{1}{e^{u-\pi xi}-1} + \frac{1}{e^{u+\pi xi}-1} + \frac{1}{e^{u-\pi xi}+1} + \frac{1}{e^{u+\pi xi}+1}\right)u^{2k-1}du \\
&= \frac{1}{2(2k)!}Re\int_0^{\infty}\left(\frac{e^{\pi xi}e^{u}u^{2k}}{(e^u - e^{\pi xi})^2} + \frac{e^{\pi xi}e^{u}u^{2k}}{(e^u + e^{\pi xi})^2}\right)du \\
&= \frac{1}{4(2k)!}Re\int_{-\infty}^{\infty}\left(\frac{e^{\pi xi}e^{u}u^{2k}}{(e^u - e^{\pi xi})^2} + \frac{e^{\pi xi}e^{u}u^{2k}}{(e^u + e^{\pi xi})^2}\right)du
\end{aligned}$$

To justify interchanging summation and integration in the above, we can use Fubini:

$$\begin{aligned}
\frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}|\cos(2\pi nx)u^{2k-1}e^{-nu}|du &\leq \frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}u^{2k-1}e^{-nu}du \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \\
&< \sum_{n=1}^{\infty}\frac{1}{n^2} \\
&< 1+\lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)} \\
&= 1+\lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right) \\
&= 1+1-\lim_{N\to\infty}\frac{1}{N} = 2 < \infty
\end{aligned}$$

Define

$$I(\alpha) = \int_{-\infty}^{\infty}\frac{e^{\pi xi}e^{\alpha u}e^{u}}{(e^u-e^{\pi xi})^2}du + \int_{-\infty}^{\infty}\frac{e^{\pi xi}e^{\alpha u}e^{u}}{(e^u+e^{\pi xi})^2}du$$

for $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$ as before. Then

$$\sum_{n=1}^{\infty}\frac{\cos(\pi(2n-1)x)}{(2n-1)^{2k}} = \frac{1}{4(2k)!}\frac{\partial^{2k}}{\partial\alpha^{2k}}ReI(\alpha)\Big|_{\alpha=0} \qquad (*).$$

In order to evaluate $I(\alpha)$, consider the contour in the figure.

rectangle contour.jpg

and the integral

$$\oint_C\left(\frac{e^{\pi xi}e^{\alpha z}e^{z}}{(e^z-e^{\pi xi})^2}+\frac{e^{\pi xi}e^{\alpha z}e^{z}}{(e^z+e^{\pi xi})^2}\right)dz$$

whose first integrand has pole at πxi and second integrand has pole at $\pi xi+\pi i$ when $0 \leq x < \frac{1}{2}$ and pole at $\pi xi - \pi i$ when $\frac{1}{2} \leq x < 1$. The integral along the vertical edges vanishes as:

$$f(z) = \frac{e^{\pi xi} e^{(\alpha+1)(u+iv)}}{(e^{u+iv} - e^{\pi xi})^2} + \frac{e^{\pi xi} e^{(\alpha+1)(u+iv)}}{(e^{u+iv} + e^{\pi xi})^2} = \begin{cases} 2e^{\pi xi} e^{(\alpha-1)(u+iv)} & u \to \infty \\ 2\dfrac{e^{(\alpha+1)(u+iv)}}{e^{\pi xi}} & u \to -\infty \end{cases}$$

So that

$$\begin{aligned}
&\oint_C \left(\frac{e^{\pi xi} e^{\alpha z} e^z}{(e^z - e^{\pi xi})^2} + \frac{e^{\pi xi} e^{\alpha z} e^z}{(e^z + e^{\pi xi})^2} \right) dz \\
&= \int_{-\infty}^{\infty} \left(\frac{e^{\pi xi} e^{\alpha u} e^u}{(e^u - e^{\pi xi})^2} + \frac{e^{\pi xi} e^{\alpha u} e^u}{(e^u + e^{\pi xi})^2} \right) du + e^{\alpha \pi i} \int_{-\infty+\pi i}^{\infty+\pi i} \left(\frac{e^{\pi xi} e^{\alpha u} e^u}{(e^u + e^{\pi xi})^2} + \frac{e^{\pi xi} e^{\alpha u} e^u}{(e^u - e^{\pi xi})^2} \right) du \\
&= (1 + e^{\alpha \pi i}) \int_{-\infty}^{\infty} \left(\frac{e^{\pi xi} e^{\alpha u} e^u}{(e^u - e^{\pi xi})^2} + \frac{e^{\pi xi} e^{\alpha u} e^u}{(e^u + e^{\pi xi})^2} \right) du
\end{aligned}$$

Which rearranged is

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{e^{\pi xi} e^{\alpha u}}{(e^u - e^{\pi xi})^2} + \frac{e^{\pi xi} e^{\alpha u}}{(e^u + e^{\pi xi})^2} \right) du &= \frac{2\pi i}{1 + e^{\alpha \pi i}} \frac{1}{2\pi i} \oint_C \left(\frac{e^{\pi xi} e^{\alpha z} e^z}{(e^z - e^{\pi xi})^2} + \frac{e^{\pi xi} e^{\alpha z} e^z}{(e^z + e^{\pi xi})^2} \right) dz \\
&= \frac{2\pi i}{1 + e^{\alpha \pi i}} Res[f(z)]
\end{aligned}$$

We wish to expand the integrand in powers of $z - z_0$ about pole z_0. First note:

$$\begin{aligned}
\frac{1}{(e^z - e^{z_0})^2} &= \frac{e^{-2z_0}}{(e^{z-z_0} - 1)^2} \\
&= \frac{e^{-2z_0}}{(z - z_0)^2} - \frac{e^{-2z_0}}{z - z_0} + \cdots
\end{aligned}$$

Then note:

$$\begin{aligned}
\frac{e^{z_0} e^{\alpha z} e^z}{(e^z - e^{z_0})^2} &= e^{-z_0} \frac{e^{(\alpha+1)z_0 + (\alpha+1)(z-z_0)}}{(z - z_0)^2} - e^{-z_0} \frac{e^{(\alpha+1)z_0}}{z - z_0} + \cdots \\
&= e^{-z_0} \frac{e^{(\alpha+1)z_0}[1 + (\alpha+1)(z - z_0) + \cdots]}{(z - z_0)^2} - e^{-z_0} \frac{e^{(\alpha+1)z_0}}{z - z_0} + \cdots \\
&= \frac{e^{\alpha z_0}}{(z - z_0)^2} + \frac{\alpha e^{\alpha z_0}}{z - z_0} + \cdots
\end{aligned}$$

$$\frac{e^{\pi xi}e^{\alpha z}e^{z}}{(e^{z}-e^{\pi xi})^2} = \frac{e^{\alpha\pi xi}}{(z-z_0)^2} + \frac{\alpha e^{\alpha z_0}}{z-z_0} + \cdots$$

Combining everything

$$\begin{aligned}
I(\alpha) &= \int_{-\infty}^{\infty}\left(\frac{e^{\pi xi}e^{\alpha u}e^{u}}{(e^{u}-e^{\pi xi})^2} + \frac{e^{\pi xi}e^{\alpha u}e^{u}}{(e^{u}+e^{\pi xi})^2}\right)du \\
&= 2\pi i\frac{1}{1+e^{\alpha\pi i}}\cdot\alpha e^{\alpha\pi xi} \\
&= ie^{\alpha\pi(x-1/2)i}\frac{\alpha\pi}{\cos\frac{\alpha\pi}{2}}
\end{aligned}$$

$$\frac{e^{t\pi(x-\frac{1}{2})i}}{\cos\frac{t\pi}{2}} = \sum_{n=0}^{\infty}E_n(x)\frac{(i\pi t)^n}{n!}$$

So that

$$I(\alpha) = \frac{e^{t\pi(x-\frac{1}{2})i}}{\cos\frac{t\pi}{2}} = \sum_{n=0}^{\infty}E_n(x)\frac{(i\pi\alpha)^{n+1}}{n!}$$

Using this in () we have:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\cos(\pi(2n-1)x)}{(2n-1)^{2k}} &= \frac{1}{4(2k)!}\,\frac{\partial^{2k}}{\partial\alpha^{2k}}ReI(\alpha)\bigg|_{\alpha=0} \\
&= \frac{(-1)^k\pi^{2k}}{4(2k-1)!}E_{2k-1}(x)
\end{aligned}$$

The first few Euler polynomials are

$$\begin{aligned}
E_0(x) &= 1, & E_4(x) &= x^4-2x^3+x, \\
E_1(x) &= x-\frac{1}{2}, & E_5(x) &= x^5-\frac{5}{2}x^4+\frac{5}{2}x^2-\frac{1}{2}, \\
E_2(x) &= x^2-x, & E_6(x) &= x^6-3x^5+5x^3-3x, \\
E_3(x) &= x^3-\frac{3}{2}x^2+\frac{1}{4}, & &\vdots
\end{aligned}$$

2.13 Sum (xii)

$$\sum_{n=0}^{\infty} \frac{\sin(\pi(2n+1)x)}{(2n+1)^{2k+1}}$$

$0 \leq x < 1$

We will evaluate the general Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin(\pi(2n-1)x)}{(2n-1)^{2k}}$$

for $k = 1, 2, \ldots$. We have:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin(\pi(2n-1)x)}{(2n-1)^{2k}} &= \frac{1}{2}\sum_{n=1}^{\infty}[1-\cos(\pi n)]\frac{\sin(\pi n x)}{n^{2k}} \\ &= \frac{1}{4}\sum_{n=1}^{\infty}\frac{[2\cos(\pi n x) - \sin(\pi n(x+1)) + \sin(\pi n(x-1))]}{n^{2k}}\end{aligned}$$

$$\frac{1}{2}\sum_{n=1}^{\infty}[1-\cos(\pi n)]\frac{\sin(\pi nx)}{n^{2k+1}}$$
$$= \frac{1}{4(2k)!}\sum_{n=1}^{\infty}[2\sin(\pi nx)-\sin(\pi n(x+1))-\sin(\pi n(x-1))]\frac{1}{n^{2k+1}}\int_0^{\infty}u^{2k}e^{-u}du$$
$$= \frac{1}{4(2k)!}\sum_{n=1}^{\infty}[2\sin(\pi nx)-\sin(\pi n(x+1))-\sin(\pi n(x-1))]\int_0^{\infty}u^{2k}e^{-nu}du$$
$$= \frac{1}{8i(2k)!}\sum_{n=1}^{\infty}\int_0^{\infty}(2e^{-nu+in\pi x}-2e^{-nu-in\pi x}-e^{-nu+in\pi(x+1)}+e^{-nu-in\pi(x+1)}$$
$$-e^{-nu+in\pi(x-1)}+e^{-nu-in\pi(x-1)})u^{2k}du$$
$$= \frac{1}{8i(2k)!}\int_0^{\infty}\sum_{n=1}^{\infty}(2e^{-nu+in\pi x}-2e^{-nu-in\pi x}-e^{-nu+in\pi(x+1)}+e^{-nu-in\pi(x+1)}$$
$$-e^{-nu+in\pi(x-1)}+e^{-nu-in\pi(x-1)})u^{2k}du$$
$$= \frac{1}{8i(2k)!}\int_0^{\infty}\left(\frac{2}{e^{u-\pi xi}-1}-\frac{2}{e^{u+\pi xi}-1}-\frac{1}{e^{u-\pi(x+1)i}-1}+\frac{1}{e^{u+\pi(x+1)i}-1}\right.$$
$$\left.-\frac{1}{e^{u-\pi(x-1)i}-1}+\frac{1}{e^{u+\pi(x-1)i}-1}\right)u^{2k}du$$
$$= \frac{1}{4i(2k)!}\int_0^{\infty}\left(\frac{1}{e^{u-\pi xi}-1}-\frac{1}{e^{u+\pi xi}-1}+\frac{1}{e^{u-\pi xi}+1}-\frac{1}{e^{u+\pi xi}+1}\right)u^{2k}du$$
$$= \frac{1}{4i(2k)!}\int_0^{\infty}\left(\frac{1}{e^{u-\pi xi}-1}-\frac{1}{e^{u+\pi xi}-1}+\frac{1}{e^{u-\pi xi}+1}-\frac{1}{e^{u+\pi xi}+1}\right)u^{2k}du$$
$$= \frac{1}{(2k+1)!}Im\int_0^{\infty}\left(\frac{e^{\pi xi}e^{u}u^{2k+1}}{(e^u-e^{\pi xi})^2}+\frac{e^{\pi xi}e^{u}u^{2k+1}}{(e^u+e^{\pi xi})^2}\right)du$$
$$= \frac{1}{4(2k+1)!}Im\int_{-\infty}^{\infty}\left(\frac{e^{\pi xi}e^{u}u^{2k+1}}{(e^u-e^{\pi xi})^2}+\frac{e^{\pi xi}e^{u}u^{2k+1}}{(e^u+e^{\pi xi})^2}\right)du$$

To justify interchanging summation and integration in the above, we can use Fubini:

$$
\begin{aligned}
\frac{1}{(2k)!}\sum_{n=1}^{\infty}\int_0^{\infty}|\sin(2\pi nx)u^{2k}e^{-nu}|du &\le \frac{1}{(2k)!}\sum_{n=1}^{\infty}\int_0^{\infty}u^{2k}e^{-nu}du \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{2k+1}} \\
&< \sum_{n=1}^{\infty}\frac{1}{n^2} \\
&< 1+\lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)} \\
&= 1+\lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right) \\
&= 1+1-\lim_{N\to\infty}\frac{1}{N} = 2<\infty
\end{aligned}
$$

Defingn $I(\alpha)$ as in (**??**):

$$
I(\alpha)=\frac{e^{\alpha\pi(x-\frac{1}{2})i}}{\cos\dfrac{\alpha\pi}{2}}=\sum_{n=0}^{\infty}E_n(x)\frac{(i\pi\alpha)^{n+1}}{n!}
$$

We have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sin(\pi(2n-1)x)}{(2n-1)^{2k+1}} &= \frac{1}{4(2k+1)!}\,\frac{\partial^{2k+1}}{\partial\alpha^{2k+1}}ImI(\alpha)\bigg|_{\alpha=0} \\
&= \frac{(-1)^k\pi^{2k+1}}{4(2k)!}E_{2k}(x).
\end{aligned}
$$

2.14 Sum (xiii)

I aim to evaluate the following sum through an integral representation of it:

$$
\begin{aligned}
&\frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a-in}e^{inx} \\
&=\frac{\sinh a\pi}{a\pi}+\frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos(nx)-\frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2+n^2}\sin(nx)
\end{aligned}
$$

Through detailed a calculation (presented below), I have found that this sum can be expressed as:

$$2\frac{\sinh a\pi}{\pi}Im\int_0^\infty \frac{e^{ix}e^{-iau}}{e^u+e^{ix}}du+\frac{\sinh a\pi}{a\pi}$$

(although, I haven't justified a step in the calculation where you interchange a summation and integration).

Calculational details: Expressing the sum as an integral

We write

$$\begin{aligned}
&\frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a-in}e^{inx}\\
&=\frac{\sinh a\pi}{\pi}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{ia+n}ie^{inx}-\sum_{n=1}^{\infty}\frac{(-1)^n}{-ia+n}ie^{-inx}+\frac{1}{a}\right)\\
&=i\frac{\sinh a\pi}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{ia+n}e^{inx+in\pi}-\sum_{n=1}^{\infty}\frac{1}{-ia+n}e^{-inx-in\pi}\right)+\frac{\sinh a\pi}{a\pi}\\
&=-2\frac{\sinh a\pi}{\pi}Im\sum_{n=1}^{\infty}\frac{1}{ia+n}e^{in(x+\pi)}\int_0^\infty e^{-u}du+\frac{\sinh a\pi}{a\pi}\\
&=-2\frac{\sinh a\pi}{\pi}Im\sum_{n=1}^{\infty}\int_0^\infty e^{in(x+\pi)}e^{-(n+ia)u}du+\frac{\sinh a\pi}{a\pi}\\
&=-2\frac{\sinh a\pi}{\pi}Im\int_0^\infty\sum_{n=1}^{\infty}e^{-n(u-ix-i\pi)}e^{-iau}du+\frac{\sinh a\pi}{a\pi}\\
&=-2\frac{\sinh a\pi}{\pi}Im\int_0^\infty\frac{e^{-iau}}{e^{u-ix-i\pi}-1}du+\frac{\sinh a\pi}{a\pi}\\
&=2\frac{\sinh a\pi}{\pi}Im\int_0^\infty\frac{e^{-iau}}{e^{u-ix}+1}du+\frac{\sinh a\pi}{a\pi}\\
&=2\frac{\sinh a\pi}{\pi}Im\int_0^\infty\frac{e^{ix}e^{-iau}}{e^u+e^{ix}}du+\frac{\sinh a\pi}{a\pi}
\end{aligned}$$

You would need to justify interchanging the summation and integration, which I haven't done. You can write:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\int_0^{\infty}|Im(-1)^n e^{inx}e^{-(n+ia)u}|du &= \sum_{n=1}^{\infty}\int_0^{\infty}|(-1)^n\sin(nx+a)e^{-nu}|du \\
&\leq \sum_{n=1}^{\infty}\int_0^{\infty}e^{-nu}du \\
&= \sum_{n=1}^{\infty}\frac{1}{n} = \infty
\end{aligned}
$$

Therefore, the series of integrals may diverge, but it could also be finite. Without further analysis, we cannot definitively conclude whether the series of integrals converges or diverges. Thus, we cannot apply Fubini's theorem to justify the interchange of summation and integration based solely on this inequality.

It is unclear whether or how the dominated convergence theorem can be applied to justify the interchange of summation and integration.

Define $I(b) = \int_0^{\infty}\frac{e^{ix}e^{-iau}}{e^{u/b}+e^{ix}}du$. Then

$$
\begin{aligned}
\frac{\partial}{\partial b}\int_0^{\infty}\frac{e^{ix}e^{-iau}}{e^{u/b}+e^{ix}}du &= \int_0^{\infty}\frac{\partial}{\partial b}\frac{e^{ix}e^{-iau}}{e^{u/b}+e^{ix}}du \\
&= \int_0^{\infty}\frac{e^{ix}e^{-iau}(-u/b^2)e^{u/b}}{(e^{u/b}+e^{ix})^2}du \\
&= -\int_0^{\infty}\frac{e^{ix}e^{-iabu}ue^{u}}{(e^{u}+e^{ix})^2}du \qquad (*)
\end{aligned}
$$

where in the last line I have made the substitution $u \mapsto bu$. Note that:

$$
\int_0^1\frac{\partial}{\partial b}\int_0^{\infty}\frac{e^{ix}e^{-iau}}{e^{u/b}+e^{ix}}dudb = \left[\int_0^{\infty}\frac{e^{ix}e^{-iau}}{e^{u/b}+e^{ix}}du\right]_0^1 = \int_0^{\infty}\frac{e^{ix}e^{-iau}}{e^{u}+e^{ix}}du
$$

Integrating both sides of $(*)$ gives

$$
\begin{aligned}
\int_0^{\infty}\frac{e^{ix}e^{-iau}}{e^{u}+e^{ix}}du &= -\int_0^1\int_0^{\infty}\frac{e^{ix}e^{-iabu}u}{(e^{u}+e^{ix})^2}dudb \\
&= -\int_0^{\infty}\int_0^1 e^{-iabu}db\frac{e^{ix}u}{(e^{u}+e^{ix})^2}du \\
&= -\frac{i}{a}\int_0^{\infty}\frac{e^{ix}(e^{-iau}-1)e^{u}}{(e^{u}+e^{ix})^2}du
\end{aligned}
$$

Taking the complex conjugate of the above result, substituting $u \to -u$, and rearranging, we get:

$$\begin{aligned}\int_0^\infty \frac{e^{-ix}e^{iau}}{e^u+e^{-ix}}du &= \frac{i}{a}\int_0^\infty \frac{e^{-ix}(e^{iau}-1)e^u}{(e^u+e^{-ix})^2}du \\ &= \frac{i}{a}\int_{-\infty}^0 \frac{e^{-ix}(e^{-iau}-1)e^{-u}}{(e^{-u}+e^{-ix})^2}du \\ &= \frac{i}{a}\int_{-\infty}^0 \frac{e^{ix}(e^{-iau}-1)e^{u}}{(e^{u}+e^{ix})^2}du\end{aligned}$$

Note:

$$\int_{-\infty}^\infty \frac{e^{ix}e^u}{(e^u+e^{ix})^2}du = \int_{-\infty}^\infty \left(-\frac{d}{du}\frac{e^{ix}}{e^u+e^{ix}}\right)du = \left[-\frac{e^{ix}}{e^u+e^{ix}}\right]_{-\infty}^\infty = 1.$$

Using the above results, we have:

$$\begin{aligned}Im\int_0^\infty \frac{e^{ix}e^{-iau}}{e^u+e^{ix}}du &= -\frac{i}{2}\left(\int_0^\infty \frac{e^{ix}e^{-iau}}{e^u+e^{ix}}du - \int_0^\infty \frac{e^{-ix}e^{iau}}{e^u+e^{-ix}}du\right) \\ &= \frac{1}{2a}\int_{-\infty}^\infty \frac{e^{ix}(e^{-iau}-1)e^u}{(e^u+e^{ix})^2}du \\ &= \frac{1}{2a}\int_{-\infty}^\infty \frac{e^{ix}e^{-iau}e^u}{(e^u+e^{ix})^2}du - \frac{1}{2a}\int_{-\infty}^\infty \frac{e^{ix}e^u}{(e^u+e^{ix})^2}du \\ &= \frac{1}{2a}\int_{-\infty}^\infty \frac{e^{ix}e^{-iau}e^u}{(e^u+e^{ix})^2}du - \frac{1}{2a} \qquad (**).\end{aligned}$$

In order to evaluate the integral in the last line of $(**)$, consider the rectangular contour C in the figure

and the contour integral

$$\oint_C \frac{e^{ix}e^{-iaz}e^z}{(e^z+e^{ix})^2}dz$$

whose integrand has pole at $ix+i\pi$ for $-\pi < x < \pi$. The integral along the vertical edges vanishes as:

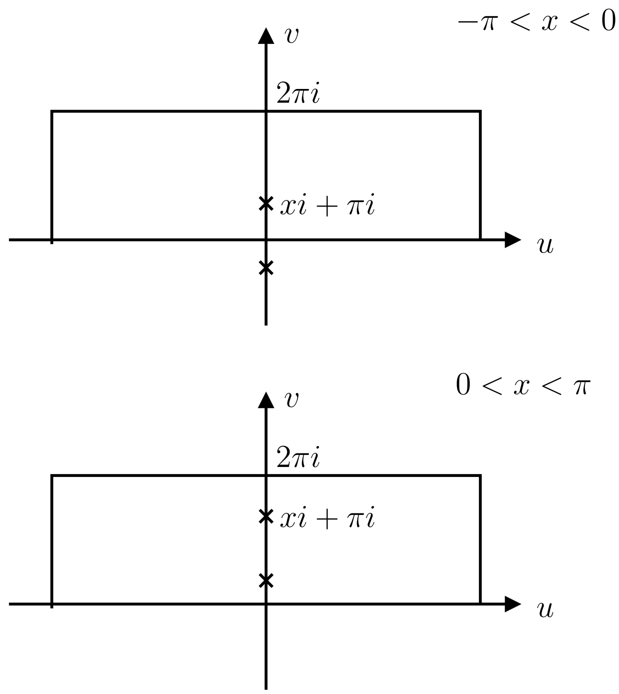

Figure 2.7: .

$$f(z) = \frac{e^{ix} e^{-ia(u+iv)} e^{(u+iv)}}{(e^{u+iv} + e^{ix})^2} = \begin{cases} e^{ix} e^{-ia(u+iv)} e^{-(u+iv)} & u \to \infty \\ e^{-ix} e^{-ia(u+iv)} e^{(u+iv)} & u \to -\infty \end{cases}$$

Thus

$$\oint_C \frac{e^{ix} e^{-iaz} e^z}{(e^z + e^{ix})^2} dz = (1 - e^{a2\pi}) \int_{-\infty}^{\infty} \frac{e^{ix} e^{-iau} e^u}{(e^u + e^{ix})^2} du$$

which rearranged

$$\begin{aligned} \frac{1}{2a} \int_{-\infty}^{\infty} \frac{e^{ix} e^{-iau} e^u}{(e^u + e^{ix})^2} du &= \frac{\pi i}{a(1 - e^{a2\pi})} \frac{1}{2\pi i} \oint_C \frac{e^{ix} e^{-iaz} e^z}{(e^z + e^{ix})^2} dz \\ &= \frac{\pi i}{a(1 - e^{a2\pi})} Res_{z=ix+i\pi} \left[\frac{e^{ix} e^{-iaz} e^z}{(e^z + e^{ix})^2} \right] \end{aligned}$$

For $-\pi \leq x < \pi$ there is a pole in rectangular contour at $z_0 = ix + i\pi$. We wish to expand the integrand in powers of $z - z_0$ about pole z_0. First note:

$$\begin{aligned}\frac{1}{(e^z-e^{z_0})^2} &= \frac{e^{-2z_0}}{(e^{z-z_0}-1)^2}\\ &= \frac{e^{-2z_0}}{(z-z_0)^2} - \frac{e^{-2z_0}}{z-z_0} + \cdots\end{aligned}$$

Then note:

$$\begin{aligned}-\frac{e^{z_0}e^{-iaz}e^z}{(e^z-e^{z_0})^2} &= -e^{-z_0}\frac{e^{(-ia+1)z_0+(-ia+1)(z-z_0)}}{(z-z_0)^2} + e^{-z_0}\frac{e^{(-ia+1)z_0}}{z-z_0} + \cdots\\ &= -e^{-z_0}\frac{e^{(-ia+1)z_0}[1+(-ia+1)(z-z_0)+\cdots]}{(z-z_0)^2} + e^{-z_0}\frac{e^{(-ia+1)z_0}}{z-z_0} + \cdots\\ &= -\frac{e^{-iaz_0}}{(z-z_0)^2} - \frac{-iae^{-iaz_0}}{z-z_0} + \cdots\end{aligned}$$

Combining everything

$$\begin{aligned}\frac{1}{2a}\int_{-\infty}^{\infty}\frac{e^{ix}e^{-iau}e^u}{(e^u+e^{ix})^2}du &= \frac{\pi i}{a(1-e^{a2\pi})}Res_{z=ix+i\pi}\left[\frac{e^{ix}e^{-iaz}e^z}{(e^z+e^{ix})^2}\right]\\ &= \frac{\pi i}{a(1-e^{a2\pi})}\cdot iae^{a(x+\pi)}\\ &= \frac{\pi e^{ax}}{e^{a\pi}-e^{-a\pi}}.\end{aligned}$$

Using this in $(**)$ we finally have:

$$Im\int_0^{\infty}\frac{e^{ix}e^{-iau}}{e^u+e^{ix}}du = \frac{\pi e^{ax}}{2\sinh a\pi} - \frac{1}{2a}.$$

$$\frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a-in}e^{inx} = e^{ax}$$

$$\frac{1}{a}\int_0^{\pi}\frac{e^{ix}e^{-ia\epsilon e^{i\theta}}e^{\epsilon e^{i\theta}}}{(e^{\epsilon e^{i\theta}}+e^{ix})^2}i\epsilon e^{-i\theta}d\theta = \frac{1}{a}\int_0^{\pi}\frac{e^{ix}(1+\epsilon(-ia+1)e^{i\theta}+\cdots)}{(1+\epsilon e^{i\theta}+e^{ix}+\cdots)^2}i\epsilon e^{-i\theta}d\theta$$

—-

$$\left|\int_0^{2\pi} \frac{e^{ix}(\epsilon e^{i\theta})^{-ia}}{(\epsilon + e^{ix})^2} i\epsilon e^{i\theta} d\theta\right| \leq \int_0^{2\pi} \frac{[\epsilon|\cos(a\ln\epsilon)| + \epsilon|\sin(a\ln\epsilon)|]e^{a\theta}}{(1-\epsilon)^2} d\theta$$

This tends to zero as $\epsilon \to 0$.

$$\begin{aligned}\left|\int_0^{2\pi} \frac{e^{ix}(Re^{i\theta})^{-ia}}{(R + e^{ix})^2} iRe^{i\theta} d\theta\right| &= \int_0^{2\pi} \frac{R[|\cos(a\ln R)| + |\sin(a\ln R)|]e^{a\theta}}{(R-1)^2} d\theta \\ &= \mathcal{O}(R^{-1})\end{aligned}$$

So that

$$\begin{aligned}2\pi i Res_{z=-e^{ix}}\left[\frac{e^{ix}z^{-ia}}{(z+e^{ix})^2}\right] &= (1 - e^{a2\pi})\frac{1}{2a}\int_0^\infty \frac{e^{ix}w^{-ia}}{(w+e^{ix})^2} dw \\ &= \end{aligned}$$

—-

2.15 Summary

(i)

For $0 < x < 2\pi$ and fixed h such that $0 < h < 2\pi$,

$$\frac{h}{2\pi} + \sum_{n=1}^\infty \frac{\sin nh}{\pi n}\cos(nx) + \sum_{n=1}^\infty \frac{1-\cos nh}{\pi n}\sin(nx) = \begin{cases} 1 & 0 < x < h \\ 0 & h < x < 2\pi \end{cases}$$

This is a pulse wave. A special case would be the square wave corresponding to $h = \pi$:

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{\pi n}\sin((2n-1)x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

If we ommit the constant term and double the amplitude you get the square wave that alternates in value between +1 and −1:

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{\pi n}\sin((2n-1)x) = \begin{cases} 1 & 0 < x < \pi \\ = 1 & \pi < x < 2\pi \end{cases}$$

There is the time-shifted square wave

$$\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{\pi n}\sin((2n-1)x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

(ii)

$$\sum_{n=0}^{\infty}\frac{\cos(2\pi(2n-1)x)}{(2n-1)^2} = |x|$$

(iii)

$$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx) = |\sin x|$$

(iv)

For $-\pi \leq x < \pi$,

$$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos(2nx) = |\cos x|$$

(v)

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{2k}}$$

(vi)

$$\frac{1}{2\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos(an)}{a^2n^2}\cos(nx)$$

(vii)

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}}\sin(2\pi nx)$$

(viii)

$$\frac{\pi^2}{2\pi} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

(ix)

$$\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin((2n-1)x)$$

(x)

$$\sum_{n=0}^{\infty} \frac{\cos(2\pi nx)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!}B_{2k}(x)$$

$$\sum_{n=0}^{\infty} \frac{\cos(2\pi(2n+1)x)}{(2n+1)^{2k}}$$

(xi)

$$\sum_{n=1}^{\infty} \frac{\cos(\pi(2n-1)x)}{(2n-1)^{2k}} = \frac{(-1)^k\pi^{2k}}{4(2k-1)!}E_{2k-1}(x)$$

(xii)

$$\sum_{n=1}^{\infty} \frac{\sin(\pi(2n-1)x)}{(2n-1)^{2k+1}} = \frac{(-1)^k \pi^{2k+1}}{4(2k)!} E_{2k}(x)$$

(xiii) For $a \in \mathbb{R}\ \{0\}$ e^{ax} for $[-\pi, \pi)$:

$$\begin{aligned}
&\frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-in} e^{inx} \\
&= \frac{\sinh a\pi}{a\pi} + \frac{2a \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \cos(nx) - \frac{2 \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{a^2+n^2} \sin(nx) \\
&= e^{ax}
\end{aligned}$$

Chapter 3

More Methods

We have this:

$$\ln(1 - e^{ix}) = -\sum_{n=1}^{\infty} \frac{e^{inx}}{n}$$

You can split $\ln(1 - e^{ix})$ into its real and imaginary parts:

$$\begin{aligned}\ln(1 - e^{ix}) &= \ln|1 - e^{inx}| + i\theta \\ &= \ln\left|2\sin\frac{x}{2}\right| + i\theta\end{aligned}$$

where

$$\tan\theta = \frac{\sin x}{\cos x - 1} = -\frac{\cos(x/2)}{\sin(x/2)} = -\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

So

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n} = -\ln\left|2\sin\frac{x}{2}\right| + i\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

We take the imaginary part:

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi}{2} - \frac{x}{2} \qquad (*)$$

Integrating this gives

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^x \left(\frac{\pi}{2} - \frac{x}{2}\right) dx$$
$$= -\frac{\pi}{2}x + \frac{x^2}{4}$$

Rearranging, and integrating again

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = -\frac{\pi}{4}x^2 + \frac{x^3}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}x$$

Putting $x = \frac{\pi}{2}$ we get:

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n^3} = -\pi^3 \frac{5}{3 \cdot 32} + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad (**)$$

We can find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by integrating $(*)$ from 0 to π. Integrating the LHS of $(*)$ results in:

$$-\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$
$$= 2(\sum_{n=0}^{\infty} \frac{1}{(2n-1)^2})$$
$$= 2(\sum_{n=0}^{\infty} \frac{1}{n^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2})$$
$$= \frac{3}{2} \sum_{n=0}^{\infty} \frac{1}{n^2}$$

Integrating the RHS of $(*)$ results in

$$\int_0^\pi (\frac{\pi}{2} - \frac{x}{2})dx = \frac{\pi^2}{4}.$$

Equating the last two results, we obtain:

$$\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Substituting this into $(**)$, finally we get:

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n^3} = -\pi^3 \frac{5}{3 \cdot 32} + \frac{\pi^3}{12} = \frac{\pi^3}{32}.$$

Checking convergenge of $\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$

You can prove $\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges everywhere on the unit circle except at $z = 1$. Fix z in the unit circle, i.e. $|z| = 1$. We apply Dirichlet's test:

If $\{a_n\}$ are real numbers and $\{b_n\}$ complex numbers such that: (i) $a_1 \geq a_2 \geq \cdots$ (ii) $\lim_{n\to\infty} a_n = 0$ (iii) There exists M such that $\left|\sum_{n=1}^{N} b_n\right| \leq M$ for all $N \in \mathbb{N}$; then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Choose $a_n = 1/n$ and $b_n = z^n$, then the first two conditions obviously hold. For the third condition:

$$\left|\sum_{n=1}^{N} z^n\right| = \left|\frac{z - z^{N+1}}{1-z}\right| \leq \frac{2}{|1-z|}$$

for all $N \in \mathbb{N}$. This shows the series converges for $|z| = 1$ where $z \neq 1$.

3.1 Generalising

Integrating

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^3}=-\frac{\pi}{4}x^2+\frac{x^3}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}x$$

we get

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^4}=\frac{\pi}{12}x^3-\frac{x^4}{48}+\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x}{2}+\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Summarising:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}&=\frac{\pi}{2}-\frac{x}{2}\\
\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}&=-\frac{\pi}{2}x+\frac{x^2}{2\cdot 2}+\sum_{n=1}^{\infty}\frac{1}{n^2}\\
\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^3}&=-\frac{\pi}{2}\frac{x^2}{2!}+\frac{x^3}{2\cdot 3!}+\sum_{n=1}^{\infty}\frac{1}{n^2}x\\
\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^4}&=\frac{\pi}{2}\frac{x^3}{3!}-\frac{x^4}{2\cdot 4!}-\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x^2}{2}+\sum_{n=1}^{\infty}\frac{1}{n^4}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^5}&=\frac{\pi}{2}\frac{x^4}{4!}-\frac{x^5}{2\cdot 5!}-\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x^3}{3!}+\sum_{n=1}^{\infty}\frac{1}{n^4}x\\
\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^6}&=-\frac{\pi}{2}\frac{x^5}{5!}+\frac{x^6}{2\cdot 6!}+\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x^4}{4!}-\sum_{n=1}^{\infty}\frac{1}{n^4}\frac{x^2}{2!}+\sum_{n=1}^{\infty}\frac{1}{n^6}\\
\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^7}&=-\frac{\pi}{2}\frac{x^6}{6!}+\frac{x^7}{2\cdot 7!}+\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x^4}{4!}-\sum_{n=1}^{\infty}\frac{1}{n^4}\frac{x^3}{3!}+\sum_{n=1}^{\infty}\frac{1}{n^6}x\\
\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^8}&=\frac{\pi}{2}\frac{x^7}{7!}-\frac{x^8}{2\cdot 8!}-\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x^5}{5!}+\sum_{n=1}^{\infty}\frac{1}{n^4}\frac{x^4}{4!}-\sum_{n=1}^{\infty}\frac{1}{n^6}\frac{x^2}{2}+\sum_{n=1}^{\infty}\frac{1}{n^8}
\end{aligned}$$

We have the general formula

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^{2k+1}} = (-1)^k\left(\frac{\pi}{2}\frac{x^{2k}}{(2k)!} - \frac{x^{2k+1}}{2\cdot(2k+1)!} + \textstyle\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x^{2k-1}}{(2k-1)!} - \cdots + (-1)^k\textstyle\sum_{n=1}^{\infty}\frac{1}{n^{2k}}x\right) \tag{3.1}$$

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^{2k}} = (-1)^k\left(\frac{\pi}{2}\frac{x^{2k-1}}{(2k-1)!} - \frac{x^{2k}}{2\cdot(2k)!} - \textstyle\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{x^{2k-2}}{(2k-2)!} - \cdots + (-1)^k\textstyle\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right) \tag{3.2}$$

General forumla

$$\begin{aligned}
-\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{2k}} + \sum_{n=1}^{\infty}\frac{1}{n^{2k}} &= -\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k}} \\
&= 2\sum_{n=0}^{\infty}\frac{1}{(2n-1)^{2k}} \\
&= 2\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}} - \sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}}\right) \\
&= 2(1-2^{-2k})\sum_{n=1}^{\infty}\frac{1}{n^{2k}}
\end{aligned}$$

Using this in (3.2)

$$\begin{aligned}
&-\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^{2k}} + \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \\
&= (-1)^{k-1}\left(\frac{\pi^{2k}}{2(2k-1)!} - \frac{\pi^{2k}}{2\cdot(2k)!} - \sum_{n=1}^{\infty}\frac{1}{n^2}\frac{\pi^{2k-2}}{(2k-2)!} - \cdots + (-1)^{k-1}\sum_{n=1}^{\infty}\frac{1}{n^{2k-2}}\frac{\pi^2}{2!}\right)
\end{aligned}$$

gives

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^{2k}} \\
&= \frac{(-1)^{k-1}}{2(1-2^{-2k})}\left(\frac{\pi^{2k}}{2(2k-1)!} - \frac{\pi^{2k}}{2\cdot(2k)!} - \sum_{n=1}^{\infty}\frac{1}{n^2}\frac{\pi^{2k-2}}{(2k-2)!} - \cdots + (-1)^{k-1}\sum_{n=1}^{\infty}\frac{1}{n^{2k-2}}\frac{\pi^2}{2!}\right)
\end{aligned}$$

Chapter 4

Partial fraction expansions of trigonometric functions

(a)

$$\pi \cot \pi x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} \tag{4.1}$$

(b)

$$\pi \csc \pi x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} \tag{4.2}$$

(c)

$$\pi^2 \csc^2 \pi x = \sum_{n=-\infty}^{\infty} \frac{1}{(x+n)^2} \tag{4.3}$$

(d)

$$\pi \sec \pi x = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(n+\frac{1}{2})^2 - x^2} \tag{4.4}$$

(e)

$$\pi \sec \pi x = 2x \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2 - x^2} \tag{4.5}$$

Proof of (a):

We will evaluate:

$$\sum_{n=-\infty}^{\infty} \frac{1}{x+n} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2-n^2}$$

We take WLG we can take $-1 < x < 1$. This is because if we had $k < x < k+1$ we could write $\sum_{n=-\infty}^{\infty} \frac{1}{x+k+n} = \sum_{n=-\infty}^{\infty} \frac{1}{x+n}$. We need to choose $-1 < x < 1$ so that our integrals converge. We have

$$\begin{aligned}
\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2-n^2} &= \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} - \frac{1}{-x+n} \right) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \int_0^{\infty} e^{-(n+x)u} du - \sum_{n=1}^{\infty} \int_0^{\infty} e^{-(n-x)u} du \\
&= \frac{1}{x} + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-(n+x)u} du - \int_0^{\infty} \sum_{n=1}^{\infty} e^{-(n-x)u} du \\
&= \frac{1}{x} + \int_0^{\infty} \frac{e^{-xu}}{e^u-1} du - \int_0^{\infty} \frac{e^{xu}}{e^u-1} du \\
&= \frac{1}{x} + \int_0^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!} \frac{u^k}{e^u-1} du - \int_0^{\infty} \sum_{k=1}^{\infty} \frac{x^k}{k!} \frac{u^k}{e^u-1} du \\
&= \frac{1}{x} - 2 \int_0^{\infty} \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!} \frac{u^{2k-1}}{e^u-1} du \\
&= \frac{1}{x} - 2 \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!} \int_0^{\infty} \frac{u^{2k-1}}{e^u-1} du \qquad (4.6)
\end{aligned}$$

But

$$\begin{aligned}\int_0^\infty \frac{u^k}{e^u-1}du &= \int_0^\infty \frac{e^{-u}u^k}{1-e^{-u}}du \\ &= \int_0^\infty \sum_{n=1}^\infty u^k e^{-nu}du \\ &= \sum_{n=1}^\infty \int_0^\infty u^k e^{-nu}du \\ &= \sum_{n=1}^\infty \frac{1}{n^{k+1}} \int_0^\infty u^k e^{-u}du \\ &= k!\sum_{n=1}^\infty \frac{1}{n^{k+1}} \\ &= \Gamma(k+1)\zeta(k+1)\end{aligned}$$

Substituting this into (4.6) and using the formula for $\zeta(2k)$ (see (1.3)) gives:

$$\begin{aligned}\frac{1}{x}+2x\sum_{n=1}^\infty \frac{1}{x^2-n^2} &= \frac{1}{x}-2\sum_{k=1}^\infty \zeta(2k)x^{2k-1} \\ &= \frac{1}{x}+2\sum_{k=1}^\infty (-1)^k 4^k \pi^{2k}\frac{B_{2k}}{2(2k)!}x^{2k-1}\end{aligned} \tag{4.7}$$

The generating function for the Bernoulli numbers is $\frac{x}{e^x-1}=\sum_{n=0}^\infty \frac{B_n}{n!}x^n$. We have

$$\begin{aligned}\frac{x}{2}\coth\frac{x}{2} &= = \frac{x}{2}\frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}} \\ &= \frac{x}{2}\frac{e^x+1}{e^x-1} = \frac{x}{e^x-1}+\frac{x}{2} = \frac{x}{2}+\sum_{n=0}^\infty \frac{B_n}{n!}x^n\end{aligned} \tag{4.8}$$

Using $B_0=1$, $B_1=-\frac{1}{2}$, and that $B_{2k+1}=0$ for $k\geq 1$, we have

$$\frac{x}{2}\coth\frac{x}{2} = \sum_{n=0}^\infty \frac{B_{2n}}{(2n)!}x^{2n} \tag{4.9}$$

That $x\cot(x)=(ix)\coth(ix)$ implies:

$$x \cot x = \sum_{n=0}^{\infty} \frac{4^n (-1)^n B_{2n}}{(2n)!} x^{2n} \tag{4.10}$$

So that

$$\pi z \cot \pi z = \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n} B_{2n}}{(2n)!} z^{2n} \tag{4.11}$$

Comparing this to (4.7) we finally obtain

$$\pi \cot \pi x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}.$$

Proof of (b):

We will evaluate:

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x+n} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2}$$

We take WLG we can take $-1 < x < 1$. This is because if we had $k < x < k+1$ we could write $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x+n} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x+k+n}$. We need to choose $-1 < x < 1$ so that our integrals converge. We have

$$
\begin{aligned}
&\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-n^2}\\
&=\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{x+n}-\frac{(-1)^n}{-x+n}\right)\\
&=\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{\cos\pi n}{x+n}-\frac{\cos\pi n}{-x+n}\right)\\
&=\frac{1}{x}+\sum_{n=1}^{\infty}\cos\pi n\int_0^{\infty}e^{-(n+x)u}du-\sum_{n=1}^{\infty}\cos\pi n\int_0^{\infty}e^{-(n-x)u}du\\
&=\frac{1}{x}+\frac{1}{2}\int_0^{\infty}\sum_{n=1}^{\infty}(e^{i\pi n}+e^{-i\pi n})e^{-(n+x)u}du-\int_0^{\infty}\sum_{n=1}^{\infty}(e^{i\pi n}+e^{-i\pi n})e^{-(n-x)u}du\\
&=\frac{1}{x}+\frac{1}{2}\int_0^{\infty}\left(\frac{e^{-xu}}{e^{u-i\pi}-1}+\frac{e^{-xu}}{e^{u+i\pi}-1}\right)du-\frac{1}{2}\int_0^{\infty}\left(\frac{e^{xu}}{e^{u-i\pi}-1}+\frac{e^{xu}}{e^{u+i\pi}-1}\right)du\\
&=\frac{1}{x}-\int_0^{\infty}\frac{e^{-xu}}{e^u+1}du+\int_0^{\infty}\frac{e^{xu}}{e^u+1}du
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-n^2}\\
&=\frac{1}{x}-\int_0^{\infty}\frac{e^{-xu}}{e^u+1}du+\int_0^{\infty}\frac{e^{xu}}{e^u+1}du\\
&=\frac{1}{x}+\left[\frac{e^{-xu}}{x(e^u+1)}\right]_0^{\infty}+\frac{1}{x}\int_0^{\infty}\frac{e^{-xu}e^u}{(e^u+1)^2}du+\left[\frac{e^{xu}}{x(e^u+1)}\right]_0^{\infty}+\frac{1}{x}\int_0^{\infty}\frac{e^{xu}e^u}{(e^u+1)^2}du\\
&=\frac{1}{x}\int_0^{\infty}\frac{e^{-xu}e^u}{(e^u+1)^2}du+\frac{1}{x}\int_0^{\infty}\frac{e^{xu}e^u}{(e^u+1)^2}du\\
&=\frac{1}{x}\int_{-\infty}^{\infty}\frac{e^{xu}e^u}{(e^u+1)^2}du
\end{aligned}
$$

Consider the contour integral in fig 4.1

and contour integral:

$$\frac{1}{x}\oint\frac{e^{xz}e^z}{(e^z+1)^2}dz$$

We have

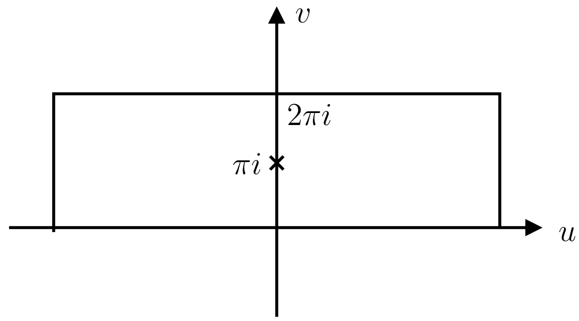

Figure 4.1: .

$$\frac{1}{x}\oint \frac{e^{xz}e^{z}}{(e^z+1)^2}dz = \frac{1}{x}\int_{-\infty}^{\infty} \frac{e^{xu}e^{u}}{(e^u+1)^2}du - \frac{e^{i2\pi x}}{x}\int_{-\infty}^{\infty} \frac{e^{xu}e^{u}}{(e^u+1)^2}du$$

Which rearranged results in

$$\begin{aligned}\frac{1}{x}\int_{-\infty}^{\infty} \frac{e^{xu}e^{u}}{(e^u+1)^2}du &= \frac{2\pi i}{1-e^{i2\pi x}}\frac{1}{x}\frac{1}{2\pi i}\oint \frac{e^{xz}e^{z}}{(e^z+1)^2}dz \\ &= \frac{2\pi i}{1-e^{i2\pi x}}\frac{1}{x}Res_{z=i\pi}\left[\frac{e^{xz}e^{z}}{(e^z+1)^2}\right]\end{aligned}$$

We have

$$\begin{aligned}\frac{e^{xz}e^{z}}{(e^z+1)^2} &= \frac{e^{(x+1)z}}{(z-z_0)^2} - \frac{e^{xz}e^{z}}{z-z_0} + \cdots \\ &= \frac{e^{(x+1)z_0}(1+(x+1)(z-z_0)+\cdots)}{(z-z_0)^2} - \frac{e^{xz_0}e^{z_0}}{z-z_0} + \cdots \\ &= \frac{e^{(x+1)z_0}}{(z-z_0)^2} + \frac{e^{xz_0}e^{z_0}x}{z-z_0} + \cdots\end{aligned}$$

$$\begin{aligned}\frac{1}{x}\int_{-\infty}^{\infty} \frac{e^{xu}e^{u}}{(e^u+1)^2}du &= \frac{2\pi i}{1-e^{i2\pi x}}\frac{1}{x}\cdot(-xe^{i\pi x}) \\ &= \frac{\pi}{\sin \pi x}\end{aligned}$$

So finally,

$$\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} = \frac{\pi}{\sin \pi x}$$

Alternatively:

$$\begin{aligned}
&\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} \\
&= \frac{1}{x} - \int_0^{\infty} \frac{e^{-xu}}{e^u + 1} du + \int_0^{\infty} \frac{e^{xu}}{e^u + 1} du \\
&= \frac{1}{x} - \int_0^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!} \frac{u^k}{e^u + 1} du + \int_0^{\infty} \sum_{k=1}^{\infty} \frac{x^k}{k!} \frac{u^k}{e^u + 1} du \\
&= \frac{1}{x} + 2 \int_0^{\infty} \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!} \frac{u^{2k-1}}{e^u + 1} du \\
&= \frac{1}{x} + 2 \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!} \int_0^{\infty} \frac{u^{2k-1}}{e^u + 1} du
\end{aligned}$$

But

$$\begin{aligned}
\int_0^{\infty} \frac{u^k}{e^u + 1} du &= \int_0^{\infty} \frac{e^{-u} u^k}{1 + e^{-u}} du \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} u^k (-1)^n e^{-nu} du \\
&= \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} u^k e^{-nu} du \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{k+1}} \int_0^{\infty} u^k e^{-u} du \\
&= k! \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{k+1}} \\
&= (1 - 2^{-k}) \Gamma(k+1) \zeta(k+1)
\end{aligned}$$

Substituting this into $(*)$ and using the formula for $\zeta(2k)$ gives:

$$
\begin{aligned}
\frac{\pi}{\sin \pi x} &= \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} \\
&= \frac{1}{x} + 2 \sum_{k=1}^{\infty} \zeta(2k) x^{2k-1} \\
&= \frac{1}{x} - 2 \sum_{k=1}^{\infty} (-1)^k (1 - 2^{-2k+1}) 4^k \pi^{2k} \frac{B_{2k}}{2(2k)!} x^{2k-1}
\end{aligned}
$$

Proof of (c):

From (a) we have

$$
\pi \frac{d}{dx} \cot \pi x = \frac{d}{dx} \sum_{n=-\infty}^{\infty} \frac{1}{x+n} \tag{4.12}
$$

or

$$
\pi^2 \frac{-\sin^2 \pi x - \cos^2 \pi x}{\sin^{\pi} x} = - \sum_{n=-\infty}^{\infty} \frac{1}{(x+n)^2} \tag{4.13}
$$

or

$$
\pi^2 \frac{1}{\sin^{\pi} x} = \sum_{n=-\infty}^{\infty} \frac{1}{(x+n)^2}. \tag{4.14}
$$

Proof of (d):

We will evaluate:

$$
\sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(n+\frac{1}{2})^2 - x^2}
$$

We take WLG we can take $-1 < 2x < 1$. This is because if we had $k < 2x < k+1$ we could write $\sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(n+\frac{1}{2})^2 - x^2} = 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+2x} + \frac{1}{n-2x} \right) = 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+k+2x} + \frac{1}{n+k-2x} \right)$. We need to choose $-1 < 2x < 1$ so that our integrals converge. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{(n+\frac{1}{2})^2-x^2}\\
&=4\sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{(2n+1)^2-4x^2}\\
&=4\sum_{n=0}^{\infty}\sin\frac{\pi n}{2}\frac{n}{n^2-4x^2}\\
&=2\sum_{n=0}^{\infty}\sin\frac{\pi n}{2}\left(\frac{1}{n+2x}+\frac{1}{n-2x}\right)\\
&=\frac{1}{i}\sum_{n=1}^{\infty}\int_0^{\infty}(e^{i\frac{\pi n}{2}}-e^{-i\frac{\pi n}{2}})e^{-(n+2x)u}du+\frac{1}{i}\sum_{n=1}^{\infty}\int_0^{\infty}(e^{i\frac{\pi n}{2}}-e^{-i\frac{\pi n}{2}})e^{-(n-2x)u}du\\
&=\frac{1}{i}\int_0^{\infty}\frac{e^{-2xu}}{e^{u-i\frac{\pi}{2}}-1}du-\frac{1}{i}\int_0^{\infty}\frac{e^{-2xu}}{e^{u+i\frac{\pi}{2}}-1}du+\frac{1}{i}\int_0^{\infty}\frac{e^{2xu}}{e^{u-i\frac{\pi}{2}}-1}du-\frac{1}{i}\int_0^{\infty}\frac{e^{2xu}}{e^{u+i\frac{\pi}{2}}-1}du\\
&=\int_0^{\infty}\frac{e^{-2xu}}{e^u-i}du+\int_0^{\infty}\frac{e^{-2xu}}{e^u+i}du+\int_0^{\infty}\frac{e^{2xu}}{e^u-i}du+\frac{1}{i}\int_0^{\infty}\frac{e^{2xu}}{e^u+i}du\\
&=2\int_0^{\infty}\frac{e^{-2xu}e^u}{e^{2u}+1}du+2\int_0^{\infty}\frac{e^{2xu}e^u}{e^{2u}+1}du\\
&=\int_0^{\infty}\frac{e^{-xu}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}du+\int_0^{\infty}\frac{e^{xu}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}du\\
&=\int_{-\infty}^{\infty}\frac{e^{xu}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}du
\end{aligned}$$

Consider the contour integral in fig 4.1 and contour integral:

$$\int_C\frac{e^{xz}}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}dz.$$

But

$$\begin{aligned}
\int_C\frac{e^{xz}}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}dz&=\int_0^{\infty}\frac{e^{xu}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}du-e^{xi2\pi}\int_0^{\infty}\frac{e^{xu}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}du\\
&=(1+e^{xi2\pi})\int_0^{\infty}\frac{e^{xu}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}du
\end{aligned}$$

So

$$\int_0^\infty \frac{e^{xu}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}du = \frac{2\pi i}{1+e^{xi2\pi}}\frac{1}{2\pi i}\int_C \frac{e^{xz}}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}dz$$
$$= \frac{2\pi i}{1+e^{i2\pi x}} Res_{z=i\pi}[\frac{e^{xz}e^{\frac{z}{2}}}{e^z+1}]$$
$$= \frac{2\pi i}{1+e^{i2\pi x}} \cdot -e^{i\pi x}e^{i\frac{\pi}{2}}$$
$$= \frac{\pi}{\cos \pi x}$$

Proof of (e):

We will evaluate:

$$2x\sum_{n=0}^\infty \frac{1}{(n+\frac{1}{2})^2-x^2}$$

We take WLG we can take $-1<2x<1$. This is because if we had $k<2x<k+1$ we could write $2x\sum_{n=0}^\infty \frac{1}{(n+\frac{1}{2})^2-x^2} = \sum_{n=0}^\infty \left(\frac{1}{n-2x}+\frac{1}{n+2x}\right) = \sum_{n=0}^\infty \left(\frac{1}{n+k-2x}-\frac{1}{n+k+2x}\right)$. We need to choose $-1<2x<1$ so that our integrals converge. We have

$$2x\sum_{n=0}^\infty \frac{1}{(n+\frac{1}{2})^2-x^2}$$
$$= 8x\sum_{n=0}^\infty \frac{1}{(2n+1)^2-4x^2}$$
$$= 4x\sum_{n=0}^\infty [1-\cos\pi n]\frac{1}{n^2-4x^2}$$
$$= \sum_{n=0}^\infty [1-\cos\pi n]\left(\frac{1}{n-2x}-\frac{1}{n+2x}\right)$$
$$= \frac{1}{2}\sum_{n=0}^\infty \int_0^\infty (2-e^{i\pi n}-e^{-i\pi n})e^{-(n-2x)u}du - \frac{1}{2}\sum_{n=0}^\infty \int_0^\infty (2-e^{i\pi n}-e^{-i\pi n})e^{-(n+2x)u}du$$
$$= \frac{1}{2}\int_0^\infty \frac{2e^{2xu}}{1-e^{-u}}du - \frac{1}{2}\int_0^\infty \frac{2e^{2xu}}{1-e^{-u-i\pi}}du + \frac{1}{2}\int_0^\infty \frac{2e^{-2xu}}{1-e^{-u}}du - \frac{1}{2}\int_0^\infty \frac{2e^{-2xu}}{1-e^{-u-i\pi}}du$$
$$= \int_0^\infty \frac{e^{2xu}}{1-e^{-u}}du - \int_0^\infty \frac{e^{2xu}}{e^{-u}+1}du - \int_0^\infty \frac{e^{-2xu}}{1-e^{-u}}du + \int_0^\infty \frac{e^{-2xu}}{e^{-u}+1}du$$
$$= 2\int_0^\infty \frac{e^{2xu}e^u}{e^{2u}-1}du - 2\int_0^\infty \frac{e^{-2xu}e^u}{e^{2u}-1}du$$
$$= 2\int_{-\infty}^\infty \frac{e^{2xu}e^u}{e^{2u}-1}du$$

Consider the contour in fig 4.2

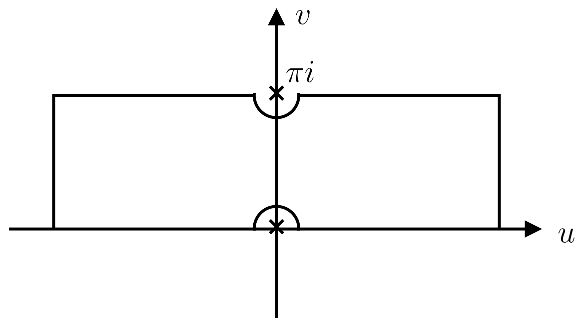

Figure 4.2: .

and the contour integral:

$$2\oint \frac{e^{2xz}e^{z}}{e^{2z}-1}dz$$

As no poles are contained within the contour, the contour integral will be equal to zero.

We evaluate the integral around a small semi-circle centered at $z=\epsilon e^{i\theta}$ (note it is clockwise):

$$\begin{aligned} -2\int_{\pi}^{0}\frac{e^{2x\epsilon e^{i\theta}}e^{\epsilon e^{i\theta}}}{e^{2\epsilon e^{i\theta}}-1}i\epsilon e^{i\theta}d\theta &= -2\int_{\pi}^{0}\frac{1}{2\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta+\mathcal{O}(\epsilon) \\ &= \pi i+\mathcal{O}(\epsilon) \end{aligned}$$

We evaluate the integral around a small semi-circle centered at $z=i\pi+\epsilon e^{i\theta}$ (note it is clockwise):

$$\begin{aligned} -2\int_{0}^{-\pi}\frac{e^{2x(i\pi+\epsilon e^{i\theta})}e^{i\pi+\epsilon e^{i\theta}}}{e^{2(i\pi+\epsilon e^{i\theta})}-1}i\epsilon e^{i\theta}d\theta &= 2e^{2xi\pi}\int_{0}^{-\pi}\frac{e^{2x\epsilon e^{i\theta}}e^{\epsilon e^{i\theta}}}{e^{2\epsilon e^{i\theta}}-1}i\epsilon e^{i\theta}d\theta \\ &= 2e^{2xi\pi}\int_{0}^{-\pi}\frac{1}{2\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta+\mathcal{O}(\epsilon) \\ &= -\pi i e^{2xi\pi}+\mathcal{O}(\epsilon) \end{aligned}$$

Using these results, we have

$$
\begin{aligned}
0 &= 2\oint \frac{e^{2xz}e^{z}}{e^{2z}-1}du \\
&= 2\int_{-\infty}^{-\epsilon} \frac{e^{2xu}e^{u}}{e^{2u}-1}du + \pi i + 2\int_{\epsilon}^{\infty} \frac{e^{2xu}e^{u}}{e^{2u}-1}du \\
&+ 2e^{i2\pi x}\int_{-\infty}^{-\epsilon} \frac{e^{2xu}e^{u}}{e^{2u}-1}du - \pi i e^{2xi\pi} + 2e^{i2\pi x}\int_{\epsilon}^{\infty} \frac{e^{2xu}e^{u}}{e^{2u}-1}du + \mathcal{O}(\epsilon) \\
&= 2(1+e^{i2\pi x})\int_{-\infty}^{-\epsilon} \frac{e^{2xu}e^{u}}{e^{2u}-1}du + 2(1+e^{i2\pi x})\int_{\epsilon}^{\infty} \frac{e^{2xu}e^{u}}{e^{2u}-1}du + \pi i(1-e^{2xi\pi}) + \mathcal{O}(\epsilon)
\end{aligned}
$$

Taking the limit $\epsilon \to 0$ we oabtain:

$$
\pi i(-1+e^{-2xi\pi}) = 2(1+e^{i2\pi x})\int_{-\infty}^{\infty} \frac{e^{2xu}e^{u}}{e^{2u}-1}du
$$

Which rearranged gives

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{2xu}e^{u}}{e^{2u}-1}du &= -\pi i \frac{e^{ix\pi}-e^{-ix\pi}}{e^{i\pi x}+e^{-i\pi x}} \\
&= \pi \tan \pi x.
\end{aligned}
$$

So finally,

$$
2x\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2 - x^2} = \pi \tan \pi x
$$

Alternatively

$$
\begin{aligned}
2x\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2 - x^2} &= 2\int_0^{\infty} \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}\frac{u^n}{e^u - e^{-u}}du - 2\int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^n}{n!}\frac{u^n}{e^u - e^{-u}}du \\
&= 4\int_0^{\infty} \sum_{n=1}^{\infty} 2^{2n-1}\frac{x^{2n-1}}{(2n-1)!}\frac{u^{2n-1}e^u}{e^{2u}-1}du \\
&= 4\sum_{n=1}^{\infty} 2^{2n-1}\frac{x^{2n-1}}{(2n-1)!}\int_0^{\infty} \frac{u^{2n-1}e^u}{e^{2u}-1}du
\end{aligned}
$$

But

$$\begin{aligned}
\int_0^\infty \frac{u^{2n-1}e^u}{e^{2u}-1}du &= \int_0^\infty u^{2n-1}\sum_{k=0}^\infty e^{-(2k+1)u}du \\
&= \sum_{k=0}^\infty \frac{1}{(2k+1)^{2n}}\int_0^\infty u^{2n-1}e^{-u}du \\
&= \left(\sum_{k=1}^\infty \frac{1}{k^{2n+1}} - \sum_{k=1}^\infty \frac{1}{(2k)^{2n}}\right)\Gamma(2n) \\
&= (1-2^{-2n})\zeta(2n)\Gamma(2n)
\end{aligned}$$

Then

$$\begin{aligned}
2x\sum_{n=1}^\infty \frac{2n+1}{(n+\frac{1}{2})^2-x^2} &= 2\sum_{n=0}^\infty (2^{2n}-1)\zeta(2n)\Gamma(2n)\frac{x^{2n-1}}{(2n-1)!} \\
&= \sum_{n=0}^\infty 2(2^{2n}-1)\zeta(2n)x^{2n-1} \\
&= \sum_{n=0}^\infty \frac{(-1)^{n-1}(2^{2n}-1)(2\pi)^{2n}B_{2n}}{(2n)!}x^{2n-1} \\
&= \sum_{n=0}^\infty \frac{(-1)^n(2\pi)^{2n}B_{2n}}{(2n)!}x^{2n-1} - 2\sum_{n=0}^\infty \frac{(-1)^n(2\pi)^{2n}B_{2n}}{(2n)!}(2x)^{2n-1} \\
&= \pi\cot\pi x - 2\pi\cot 2\pi x \\
&= \pi\left(\cot\pi x - 2\frac{\cos^2\pi x - \sin^2\pi x}{2\sin\pi x\cos\pi x}\right) \\
&= \pi\tan\pi x
\end{aligned}$$

Appendix A

Fubini's Theorem and the Dominated Convegence Theorem

Appendix B

Bernoulli numbers and Euler numbers

Appendix C

Bernoulli polynomials and Euler polynomials

C.1 Bernoulli polynomials

The generating function for the Bernoulli polynomials is

$$\frac{te^{tx}}{e^t - 1} = \sum_n^{\infty} B_n(x) \frac{t^n}{n!} \tag{C.1}$$

Explicit expressions for low degrees:

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x - \frac{1}{2} \\
B_2(x) &= x^2 - x - \frac{1}{6} \\
B_3(x) &= x^3 - \frac{3}{2}x^2 - \frac{1}{2}x \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}.
\end{aligned} \tag{C.2}$$

Derivatives

We have

$$\sum_{n=0}^{\infty} B_n'(x) \frac{t^n}{n!} = \frac{t^2 e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} n B_{n-1}(x) \frac{t^n}{n!} \tag{C.3}$$

which implies

$$B_n'(x) = nB_{n-1}(x). \tag{C.4}$$

Symmetries:

$$\begin{aligned} B_n(1-x) &= (-1)^n B_n(x), \quad n \geq 0 & \text{(C.5)} \\ (-1)^n B_n(-x) &= B_n(x) - nx^{n-1} & \text{(C.6)} \end{aligned}$$

proof:

We have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(1-x)\frac{t^n}{n!} &= \frac{te^{t(1-x)}}{e^t - 1} \\ &= \frac{te^t e^{t(-x)}}{e^t - 1} \\ &= \frac{(-t)e^{(-t)x}}{e^{-t} - 1} \\ &= \sum_{n=0}^{\infty}(-1)^n B_n(x)\frac{t^n}{n!}. & \text{(C.7)} \end{aligned}$$

Equating coefficient of eams power of t gives (C.5).

We have

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n B_n(-x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} &= \frac{-te^{tx}}{e^{-t} - 1} - \frac{te^{tx}}{e^t - 1} \\ &= te^{tx}\left[\frac{e^t}{e^t - 1} - \frac{1}{e^t - 1}\right] \\ &= te^{tx} \\ &= \sum_{n=0}^{\infty} x^n \frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty} nx^n \frac{t^n}{n!}. & \text{(C.8)} \end{aligned}$$

Equating coefficient of eams power of t gives (C.6).

$\square$

Representation by a differential operator

The Bernoulli polynomials are also given by

$$B_n(x) = \frac{D}{e^D - 1} x^n \tag{C.9}$$

where $D = d/dx$ is differentiation with respect to x and the fraction is expanded as a formal power series. This follows from:

$$\sum_{n=0}^{\infty} \frac{D}{e^D - 1} x^n \frac{t^n}{n!} = \frac{D}{e^D - 1} e^{tx} = \frac{te^{tx}}{e^t - 1}. \tag{C.10}$$

Representation by an integral operator

The Bernoulli polynomials are also the unique polynomials determined by

$$\int_x^{x+1} B_n(u) du = x^n. \tag{C.11}$$

The integral transform

$$(Tf)(x) = \int_x^{x+1} f(u) du, \tag{C.12}$$

on polynomials f. If $F(x) = \int^x f(u)du$, then

$$\begin{aligned} (Tf)(x) &= \int_x^{x+1} \frac{d}{du} F(u) du \\ &= F(x+1) - F(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dx^n} F(x) - F(x) \\ &= \sum_{n=0}^{\infty} \frac{D^n}{(n+1)!} f(x) \\ &= \frac{e^D - 1}{D} f(x) \end{aligned} \tag{C.13}$$

Inversion

$$
\begin{aligned}
x^n &= \frac{e^D - 1}{D} B_n(x) \\
&= \sum_{m=0}^{\infty} \frac{D^m}{(m+1)!} B_n(x) \\
&= \sum_{m=0}^{n} \frac{n!}{(m+1)!(n-m)!} B_{n-m}(x) \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \frac{(n+1)!}{(n+1-k)!k!} B_k(x) \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(x). \qquad \text{(C.14)}
\end{aligned}
$$

C.2 Euler polynomials

The generating function for the Euler polynomials is

$$
\frac{2e^{tx}}{e^t + 1} = \sum_{n}^{\infty} E_n(x) \frac{t^n}{n!} \qquad \text{(C.15)}
$$

Explicit expressions for low degrees:

$$
\begin{aligned}
E_0(x) &= 1 \\
E_1(x) &= x - \frac{1}{2} \\
E_2(x) &= x^2 - x \\
E_3(x) &= x^3 - \frac{3}{2}x^3 + \frac{1}{4} \\
E_4(x) &= x^4 - 2x^3 + x. \qquad \text{(C.16)}
\end{aligned}
$$

Symmetries:

$$
\begin{aligned}
E_n(1-x) &= (-1)^n E_n(x), \quad n \geq 0 \qquad \text{(C.17)} \\
(-1)^n E_n(-x) &= -E_n(x) + 2x^n. \qquad \text{(C.18)}
\end{aligned}
$$

proof:

We have

$$
\begin{aligned}
\sum_{n=0}^{\infty} E_n(1-x)\frac{t^n}{n!} &= \frac{2e^{t(1-x)}}{e^t+1} \\
&= \frac{2e^t e^{t(-x)}}{e^t+1} \\
&= \frac{2e^{(-t)x}}{e^{-t}+1} \\
&= \sum_{n=0}^{\infty}(-1)^n E_n(x)\frac{t^n}{n!}. \qquad \text{(C.19)}
\end{aligned}
$$

Equating coefficients of the same power of t gives (C.17).

We have

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n E_n(-x)\frac{t^n}{n!} + \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} &= \frac{2e^{tx}}{e^{-t}+1} + \frac{2e^{tx}}{e^t+1} \\
&= 2e^{tx}\left[\frac{e^t}{e^t+1} + \frac{1}{e^t+1}\right] \\
&= 2e^{tx} \\
&= \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}. \qquad \text{(C.20)}
\end{aligned}
$$

Equating coefficient of the same power of t gives (C.18).

□