Irreducible representations of $SU(2)$.

We write a vector in $\mathcal{C}^2$ as a pair of complex numbers (z_1, z_2). We define tne element of a Hilbert space $\mathcal{H}_\ell$ is a polynomial in z_1 and z_2 that is a linear combination of polynomials

$$f(z_1, z_2) = z_1^p z_2^q \tag{U.145}$$

where the total degree is $p + q = 2\ell$.

$$P \in \mathbb{C}[z_1, z_2], \qquad g = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}, \qquad z = (z_1, z_2), \tag{U.146}$$

$ad + bc = 1$

$$g^{-1} = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \tag{U.147}$$

and

$$zg := g^{-1}z = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (az_1 + bz_2, -cz_1 + dz_2). \tag{U.148}$$

$$P_k(z_1, z_2) = z_1^k z_2^{2\ell - k}, \quad 0 \geq k \geq 2\ell, \tag{U.149}$$

$$\begin{aligned} f(g^{-1}z) &= f\Big[\begin{pmatrix} a & b \\ -c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\Big] \\ &= (az_1 + bz_2)^k (-cz_1 + dz_2)^{2\ell - k} \end{aligned} \tag{U.150}$$

Taking the basis of mononials of degree n in the order

$$P_\ell(z_1, z_2) = \frac{z_1^{\ell+j} z_2^{\ell-j}}{\sqrt{(\ell+j)!(\ell-j)!}}, \quad -\ell \geq j \geq \ell, \quad \ell = 0, \frac{1}{2}, 1, \ldots \tag{U.151}$$

where ℓ is an integer or half integer

$$\frac{z_1^{2\ell}}{\sqrt{(2\ell)!}}, \frac{z_1^{2\ell-1} z_2}{\sqrt{(2\ell-1)!(1)!}}, \frac{z_1^{2\ell-2} z_2^2}{\sqrt{(2\ell-2)!(2)!}}, \ldots, \frac{z_1 z_2^{2\ell-1}}{\sqrt{(1)!(2\ell-1)!}}, \frac{z_2^{2\ell}}{\sqrt{(2\ell)!}}. \tag{U.152}$$

These are the basis vector of a vector space we denote V_ℓ. Since the monomial set is closed under the linear transformation g, it will provide a $(2\ell + 1) \times (2\ell + 1)$ matrix representation.

For any $g \in SU(2)$, let $U_\ell(g)$ be the linear transformation of $\mathcal{H}_\ell$ given by

$$(U_\ell(g)f)(v) = f(g^{-1}v) \tag{U.153}$$

for all $f \in \mathcal{H}_\ell$ and $v \in \mathbb{C}^2$. This is a representation: $U_\ell(1)$ is the identity, and for any $g, h \in SU(2)$ we have

$$\begin{aligned} (U_\ell(g)U_\ell(h)f)(v) &= (U_\ell(g)f)(g^{-1}v) \\ &= f(h^{-1}g^{-1}v) \\ &= f((gh)^{-1}v) \\ &= (U_\ell(gh)f)(v) \end{aligned} \tag{U.154}$$

for all $f \in \mathcal{H}_\ell$, $v \in \mathbb{C}^2$.

Matrix elements

Setting

$$\phi^\ell_k(z_1, z_2) = \frac{z_1^{\ell+k} z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}} \tag{U.155}$$

The matrix elements are defined by

$$U_\ell(g)\phi^\ell_n = \sum_{k=\ell}^{-\ell} \phi^\ell_k(z_1, z_2)\ \pi_\ell(g)_{kn} \tag{U.156}$$

From (U.154)

$$\begin{aligned} U_\ell(g)U_\ell(h)\phi^\ell_n &= U_\ell(g) \sum_{j=-\ell}^{\ell} \phi^\ell_j\ \pi_\ell(h)_{jn} \\ &= \sum_{j=-\ell}^{\ell} \sum_{k=-\ell}^{\ell} \phi^\ell_k\ \pi_\ell(g)_{kj}\pi_\ell(h)_{jn} \\ &= U_\ell(gh)\phi^\ell_n \\ &= \sum_{k=-\ell}^{\ell} \phi^\ell_k\ \pi_\ell(gh)_{kn} \end{aligned} \tag{U.157}$$

that is

$$\sum_{k=-\ell}^{\ell} \frac{z_1^{\ell+k} z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}} \Big(\sum_{j=-\ell}^{\ell} \pi_\ell(g)_{kj} \pi_\ell(h)_{jn} - \pi_\ell(gh)_{kn} \Big) = 0. \tag{U.158}$$

Setting $z_2 = 1$ and multiplying both sides by $z_1^{-\ell-m-1}$, integrating z_1 around the unit circle about the origin of the z_1-complex plane one obtains

$$\sum_{j=-\ell}^{\ell} \pi_\ell(g)_{mj} \pi_\ell(h)_{jn} = \pi_\ell(gh)_{mn} \tag{U.159}$$

i.e. $\pi_\ell(g)_{kj}$ is a matrix representation of $SU(2)$.

We defined the representation $\pi_\ell(g)_{km}$ by

$$\begin{aligned} U_\ell(g)\phi_m^\ell(z_1, z_2) &= \frac{1}{\sqrt{(\ell+m)!(\ell-m)!}} U_\ell(g) z_1^{\ell+m} z_2^{\ell-m} \\ &= \frac{(\alpha z_1 - \overline{\beta} z_2)^{\ell+m} (\beta z_1 + \overline{\alpha} z_2)^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}} \\ &= \sum_{k=\ell}^{-\ell} \phi_k^\ell(z_1, z_2)\ \pi_\ell(g)_{km} \end{aligned} \tag{U.160}$$

The matrix elements can be expressed

$$\pi_\ell(g)_{mn} = \alpha^{m+n} (-\overline{\beta})^{\ell-m} \beta^{\ell-n} \sum_{s=0}^{\ell-n} \frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}{(m+n+s)!(\ell-m-s)!(\ell-n-s)!s!} \left(-\frac{|\alpha|}{|\beta|} \right)^s \tag{U.161}$$

To arrive at this we rearange the binomial expansion

$$\begin{aligned} & (\alpha z_1 - \overline{\beta} z_2)^{\ell+n} (\beta z_1 + \overline{\alpha} z_2)^{\ell-n} \\ = & \left(\sum_{t=0}^{\ell+n} (-1)^{\ell+n-t} \begin{pmatrix} \ell+n \\ t \end{pmatrix} \alpha^t \overline{\beta}^{\ell+n-t} z_1^t z_2^{\ell+n-t} \right) \left(\sum_{k=0}^{\ell-n} \begin{pmatrix} \ell-n \\ k \end{pmatrix} \beta^{\ell-n-k} \overline{\alpha}^k z_1^{\ell-n-k} z_2^k \right) \\ = & \sum_{m=-\ell}^{\ell} \alpha^{m+n} (-\beta^*)^{\ell-m} \beta^{\ell-n} \left(\sum_k \begin{pmatrix} \ell+n \\ m+n+k \end{pmatrix} \begin{pmatrix} \ell-n \\ k \end{pmatrix} \left(-\left| \frac{\alpha}{\beta} \right| \right)^k \right) z_1^{\ell+t-n-k} z_1^{\ell-t+n+k} \end{aligned} \tag{U.162}$$

Noting $1/(-N)! = 0$ for positive integer N, the summation over k in the last line above is obviously non-zero at from 0 to $\ell-m$ (some of these terms can also be zero). To derive this we will work backwards, we will start from the last line and get back out $(\alpha z_1 - \overline{\beta} z_2)^{\ell+n}(\beta z_1 + \overline{\alpha} z_2)^{\ell-n}$. Let us set $Q_{m,n,k} = \alpha^{m+n+k}(-\beta^*)^{\ell-m-k}(\alpha^*)^k \beta^{\ell-n-k} \cdot z_1^{\ell+m'} z_1^{\ell-m'}$, then we can write

$$\begin{aligned}
& \sum_{m=-\ell}^{\ell} \sum_{k=0}^{\ell-n} \begin{pmatrix} \ell+n \\ m+n+k \end{pmatrix} \begin{pmatrix} \ell-n \\ k \end{pmatrix} Q_{m,n,k} \\
= & \sum_{k=0}^{\ell-n} \sum_{m=-\ell}^{\ell} \begin{pmatrix} \ell+n \\ m+n+k \end{pmatrix} \begin{pmatrix} \ell-n \\ k \end{pmatrix} Q_{m,n,k} \\
= & \sum_{k=0}^{\ell-n} \sum_{t=n+k-\ell}^{\ell+n+k} \begin{pmatrix} \ell+n \\ t \end{pmatrix} \begin{pmatrix} \ell-n \\ k \end{pmatrix} Q_{t-n-k,n,k} \\
= & \sum_{t=n-\ell}^{\ell+n} \begin{pmatrix} \ell+n \\ t \end{pmatrix} \begin{pmatrix} \ell-n \\ 0 \end{pmatrix} Q_{t-n,n,0} + \sum_{t=n+1-\ell}^{\ell+n} \begin{pmatrix} \ell+n \\ t \end{pmatrix} \begin{pmatrix} \ell-n \\ 1 \end{pmatrix} Q_{t-n-1,n,1} + \ldots \\
\ldots & + \sum_{t=0}^{\ell+n} \begin{pmatrix} \ell+n \\ t \end{pmatrix} \begin{pmatrix} \ell-n \\ \ell-n \end{pmatrix} Q_{t-\ell,n,\ell-n} \\
= & \sum_{t=0}^{\ell+n} \sum_{k=0}^{\ell-n} \begin{pmatrix} \ell+n \\ t \end{pmatrix} \begin{pmatrix} \ell-n \\ k \end{pmatrix} Q_{t-n-k,n,k}
\end{aligned} \tag{U.163}$$

where we have used again $1/(-N)! = 0$ for positive integer N in going from the fourth line to the last one. Now $Q_{t-n-k,n,k} = \alpha^t \beta^{\ell-n-k} \overline{\alpha}^k \overline{\beta}^{\ell+n-t} \cdot z_1^{\ell+t-n-k} z_1^{\ell-t+n+k}$ and we have proven (U.162).

We divide the last line above by $[(\ell+n)!(\ell-n)!]^{1/2}$ and get $(\alpha z_1 - \overline{\beta} z_2)^{\ell+n}(\beta z_1 + \overline{\alpha} z_2)^{\ell-n}/[(\ell+n)!(\ell-n)!]^{1/2}$. We can then read off the matrix elements (U.161) from (U.160).

□

Examples

$\ell = 0, 1/2, 1, 3/2, \ldots$

For $2\ell =$ to an even integer, the representatioon is the $2\ell+1$ dimensional tensorial representation of $SO(3)$. For $2\ell =$ to an odd integer, π_ℓ is a spinor representation. For matter, 1/2 describe elementary particles of half-integer spin.

(1) $\ell = 1/2$

$$
\begin{aligned}
\pi_{1/2}(g)_{\frac{1}{2}\frac{1}{2}} &= \alpha \sum_{k=0}^{0} = \alpha \\
\pi_{1/2}(g)_{-\frac{1}{2}-\frac{1}{2}} &= \alpha^{-1}(-\overline{\beta})\beta \sum_{k=0}^{1} \frac{1}{(k-1)!(1-k)!(1-k)!k!} \left(-\left|\frac{\alpha}{\beta}\right|\right)^k = \overline{\alpha} \\
\pi_{1/2}(g)_{-\frac{1}{2}\frac{1}{2}} &= = \beta
\end{aligned}
\tag{U.164}
$$

$$
\pi_{1/2} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{U.165}
$$

(2) $\ell = 1$:

$$
\pi_1 \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\overline{\beta} & |\alpha|^2 - |\beta|^2 & \sqrt{2}\overline{\alpha}\beta \\ \overline{\beta}^2 & -\sqrt{2}\overline{\alpha}\overline{\beta} & \overline{\alpha}^2 \end{pmatrix} \tag{U.166}
$$

Unitarity of the representation

the transpose $g \to g^T$

$$
\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \to \pi_\ell(g^T)_{mn} = \pi_\ell(g)_{nm} = (\pi_\ell(g)^T)_{mn} \tag{U.167}
$$

the complex conjugate $g \to \overline{g}$

$$
\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \to \pi_\ell(\overline{g})_{mn} = (\overline{\pi_\ell(g)})_{mn} \tag{U.168}
$$

Combining both operations $g \to g^\dagger$, this induces $\pi_\ell(g^\dagger) = \pi_\ell(g)^\dagger$ and so we can write

$$\pi_\ell(g)^\dagger = \pi_\ell(g^\dagger) = \pi_\ell(g^{-1}) = \pi_\ell(g)^{-1}. \tag{U.169}$$

Therefore the representation is unitary.

$\square$

According to Euler's theorem, every rotation R in $\mathbb{R}^3$ can be written as $R = R_3(\phi)R_2(\theta)R_3(\psi)$, see fig (U.7).

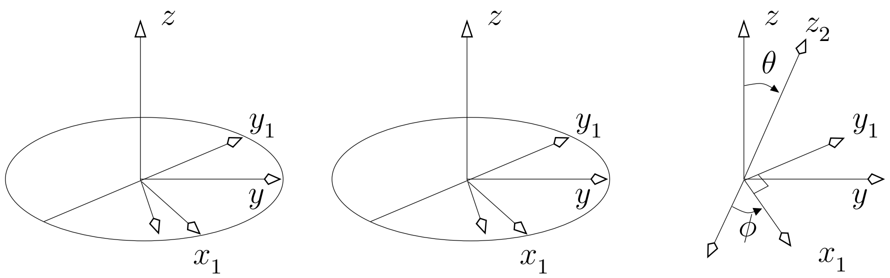

Figure U.7: EulerTherm.

$$\begin{aligned}
\pi_{\frac{1}{2}}(\phi,\theta,\psi) &= \exp\left(-i\frac{\phi}{2}\sigma_3\right)\exp\left(-i\frac{\theta}{2}\sigma_2\right)\exp\left(-i\frac{\psi}{2}\sigma_3\right) \\
&= \begin{pmatrix} \exp(-i\frac{\phi}{2}) & 0 \\ 0 & \exp(i\frac{\phi}{2}) \end{pmatrix}\begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}\begin{pmatrix} \exp(-i\frac{\psi}{2}) & 0 \\ 0 & \exp(i\frac{\psi}{2}) \end{pmatrix}
\end{aligned} \tag{U.170}$$

Together

$$\begin{pmatrix} \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & -\exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \\ \exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \end{pmatrix} \tag{U.171}$$

Irreducibility

According to Schur's lemma, the representation is irreducible if the matrix which commutes with all the elements of the representation is a constant matrix. This we will use to prove the representations (labelled by ℓ) are irreducible. Let M be a matrix that commutes with $\pi_\ell(\theta)$

$$M\pi_\ell(\theta) - \pi_\ell(\theta)M = 0. \tag{U.172}$$

By considering special cases we can show that any such matrix is a constant matrix. First consider the case in (U.161) $\alpha = e^{-im\theta/2}$ $\beta = 0$

$$\pi_\ell(0,0,R_3(\phi))_{mn} = \delta_{mn}e^{-im\phi} \tag{U.173}$$

where $m,n = \ell, \ell-1, \cdots -\ell+1, -\ell$,

$$(e^{-in\theta} - e^{-im\theta})M_{mn} = 0. \tag{U.174}$$

implying that M is diagonal. Write $M_{mn} = M_m\delta_{mn}$ (no summation), then (U.172) takes the form

$$(M_m - M_n)\pi_\ell(\theta)_{mn} = 0. \tag{U.175}$$

Now set $m = \ell$ in (U.161), the ℓth row is given by

$$\pi_\ell(\theta)_{\ell n} = [(2\ell)!/(\ell+n)!(\ell-n)!]^{1/2}\alpha^{\ell+n}\beta^{\ell-n} \tag{U.176}$$

From this and setting $m = \ell$ in (U.175) we get

$$(M_\ell - M_n)(\alpha/\beta)^n = 0, \quad \text{for all } n \tag{U.177}$$

which implies $M_n = M_\ell$ for all n. Thus any matrix M that commutes with the representation $\pi_\ell(\theta)$ is a constant matrix and therefore, the representation is irreducible.

□

Clebsch-Gordon coefficients.

Vector addition

$$\chi^{(j_1)}\chi^{(j_2)} = \sum_{m_2=-j_2}^{j_2} e^{-im_2\theta} \sum_{m_1=-j_1}^{j_1} e^{-im_1\theta} \tag{U.181}$$

Set $m = m_1 + m_2$, and assume $j_1 \geq j_2$ without loss of generality. Then

$$\begin{aligned} \chi^{(j_1)}\chi^{(j_2)} &= (e^{+j_2\theta} + \cdots + e^{-j_2\theta}) \left(\frac{e^{-i(j_1+1)\theta} - e^{ij_1\theta}}{e^{-i\theta} - 1} \right) \\ &= \frac{e^{-i(j_1+j_2+1)\theta} - e^{i(j_1+j_2)\theta}}{e^{-i\theta} - 1} + \cdots + \frac{e^{-i(j_2-j_1+1)\theta} - e^{i(j_2-j_1)\theta}}{e^{-i\theta} - 1} \\ &= \chi^{(j_1+j_2)} + \chi^{(j_1+j_2-1)} + \cdots + \chi^{(j_1-j_2)} \end{aligned} \tag{U.182}$$

Thus

$$\pi_{j_1} \otimes \pi_{j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \pi_j \tag{U.183}$$

For example

$$\pi_{1/2} \otimes \pi_{1/2} = \pi_0 \oplus \pi_1 \tag{U.184}$$

The composition of two spin haaalf particle is the direct sum of a scalar (singlet) and a spin one (doublet). Or two $j = 1/2$ edges of a spin network shares a tri-valent vertex with either a $j = 0$ or $j = 1$ edge.

$$\pi_1 \otimes \pi_1 = \pi_0 \oplus \pi_1 \oplus \pi_2 \tag{U.185}$$

$$\begin{aligned} \mathbf{J}^2\psi(j,m) &= j(j+1)\psi(j,m) \\ J_z\psi(j,m) &= m\psi(j,m) \end{aligned} \tag{U.186}$$

□