Determinant of the Jacobian for a Coordinate Transformation

Contents

List of Figures

Chapter 1

Two Dimensions

We consider the case $n = 2$ first. Consider a two-dimensional integral I_2 for the integration of an arbitrary function $f(x_1, x_2)$:

$$I_2 = \int_{-\infty}^{\infty} f(x_1, x_2) dx_1 dx_2 \tag{1.1}$$

The Cartesian coordinates x_1 and x_2 are transformed into curvilinear coordinates q_1 and q_2 with the transformation relations

$$q_1 = q_1(x_1, x_2), \quad q_2 = q_2(x_1, x_2). \tag{1.2}$$

which are assumed to be invertible and non-singular. Thus, x_1 and x_2 can be expressed in terms of q_1 and q_2:

$$x_1 = x_1(q_1, q_2), \quad x_2 = x_2(q_1, q_2). \tag{1.3}$$

We will make repeated use of the formula

$$\delta(g(x)) = \frac{\delta(x - x_0)}{|g'(x_0)|} \tag{1.4}$$

where x_0 is the root of $g(x) = 0$ (we will only need the case when there is a single root).

$$
\begin{aligned}
I_2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,\ldots,x_n)dx_1\ldots dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int dq_1\delta[q_1-q_1(x_1,x_2)]\right)f(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int\int_{-\infty}^{\infty}\delta[q_1-q_1(x_1,x_2)]f(x_1,x_2)dx_1dq_1dx_2 \\
&= \int_{-\infty}^{\infty}\int\int_{-\infty}^{\infty}\frac{\delta[x_1-x_1(q_1,x_2)]}{\left|\frac{\partial q_1}{\partial x_1}\right|_{x_1=x_1(q_1,x_2)}}f(x_1,x_2)dx_1dq_1dx_2 \qquad (1.5)
\end{aligned}
$$

where we have used (1.4) with $g(x_1)=q_1-q_1(x_1,x_2)$. The $x_1(q_1,x_2)$ denotes the root of the equation $q_1-q_1(x_1,x_2)=0$. so that

$$
I_2=\int_{-\infty}^{\infty}\int\frac{f\left(x_1(q_1,x_2),x_2\right)}{\mathcal{G}_1(q_1,x_2)}dq_1dx_2 \qquad (1.6)
$$

We will use the abbreviation $f(q_1,x_2)\equiv f\left(x_1(q_1,x_2),x_2\right)$ and $\mathcal{G}_1(q_1,x_2)\equiv\left|\frac{\partial q_1}{\partial x_1}\right|_{x_1=x_1(q_1,x_2)}$.

The delta function for q_2 will be $\delta[q_2-q_2\left(x_1(q_1,x_2),x_2\right)]\equiv\delta[q_i-q_i(q_1,x_2)]$. So I_n can bewritten

$$
\begin{aligned}
I_2 &= \int\int_{-\infty}^{\infty}\left(\int\delta[q_2-q_2(q_1,x_2)]dq_2\right)\frac{f(q_1,x_2)}{\mathcal{G}_1(q_1,x_2)}dq_1dx_2 \\
&= \int\int\int_{-\infty}^{\infty}\delta[q_2-q_2(q_1,x_2)]\frac{f(q_1,x_2)}{\mathcal{G}_1(q_1,x_2)}dx_2dq_1dq_2 \\
&= \int\int\int_{-\infty}^{\infty}\frac{\delta[x_2-x_2(q_1,q_2)]}{\left|\left(\frac{\partial q_2}{\partial x_2}\right)_{(1)}\right|_{x_2=x_2(q_1,q_2)}}\frac{f(q_1,x_2)}{\mathcal{G}_1(q_1,x_2)}dx_2dq_1dq_2 \qquad (1.7)
\end{aligned}
$$

where we have used (1.4) with $g(x_1)=q_2-q_2(q_1,x_2)$. The $x_2(q_1,q_2)$ denotes the root of the equation $q_2-q_2(q_1,x_2)=0$, and

$$
\left(\frac{\partial q_2}{\partial x_2}\right)_{(1)}:=\frac{\partial}{\partial x_2}q_2(x_1(q_1,x_2),x_2). \qquad (1.8)
$$

So that

$$
I_2=\int\int\frac{f\left(q_1,x_2(q_1,q_2)),x_2(q_1,q_2)\right)}{\tilde{\mathcal{G}}_2(q_1,q_2)\mathcal{G}_1(q_1,q_2)}dq_1dq_2. \qquad (1.9)
$$

where $\mathcal{G}_1(q_1, x_2(q_1, q_2)) = \mathcal{G}_1(q_1, q_2)$ and $\tilde{\mathcal{G}}_2(q_1, x_2(q_1, q_2)) = \mathcal{G}_2(q_1, q_2)$.

The total differentials of q_1 and x_1 can be expressed as

$$\begin{aligned} dq_1 &= \frac{\partial q_1}{\partial x_1} dx_1 + \frac{\partial q_1}{\partial x_2} dx_2 && (1.10) \\ dx_1 &= \left(\frac{\partial q_1}{\partial x_1}\right)_{(1)} dq_1 + \left(\frac{\partial q_1}{\partial x_2}\right)_{(1)} dx_2 && (1.11) \end{aligned}$$

where $q_1 = q_1(x_1, x_2)$ and $x_1 = x_1(q_1, x_2)$. Replacing dx_1 in (1.10), we obtain

$$dq_1 = \frac{\partial q_1}{\partial x_1} \left(\frac{\partial q_1}{\partial x_1}\right)_{(1)} dq_1 + \left[\frac{\partial q_1}{\partial x_1} \left(\frac{\partial x_1}{\partial x_2}\right)_{(1)} + \frac{\partial q_1}{\partial x_2}\right] dx_2. \tag{1.12}$$

Comparing the coefficients of the differentials of both sides, we find that

$$\begin{aligned} \frac{\partial q_1}{\partial x_1} \left(\frac{\partial q_1}{\partial x_1}\right)_{(1)} &= 1, && (1.13) \\ \frac{\partial q_1}{\partial x_1} \left(\frac{\partial x_1}{\partial x_2}\right)_{(1)} + \frac{\partial q_1}{\partial x_2} &= 0 && (1.14) \end{aligned}$$

We have

$$\begin{aligned} \left(\frac{\partial q_2}{\partial x_2}\right)_{(1)} &= \frac{\partial}{\partial x_2} q_2(x_1(q_1, x_2), x_2) \\ &= \frac{\partial q_2}{\partial x_1} \left(\frac{\partial x_1}{\partial x_2}\right)_{(1)} + \frac{\partial q_2}{\partial x_2} \\ &= \frac{\dfrac{\partial q_1}{\partial x_1} \dfrac{\partial q_2}{\partial x_2} + \dfrac{\partial q_2}{\partial x_1} \dfrac{\partial q_1}{\partial x_1} \left(\dfrac{\partial x_1}{\partial x_2}\right)_{(1)}}{\dfrac{\partial q_1}{\partial x_1}} \\ &= \frac{\det \begin{pmatrix} \dfrac{\partial q_1}{\partial x_1} & \dfrac{\partial q_2}{\partial x_1} \\ \dfrac{\partial q_1}{\partial x_2} & \dfrac{\partial q_2}{\partial x_2} \end{pmatrix}}{\det \left(\dfrac{\partial q_1}{\partial x_1}\right)}. && (1.15) \end{aligned}$$

So

$$\tilde{\mathcal{G}}_2 =: \frac{\mathcal{G}_2}{\mathcal{G}_1} = \frac{\left|\det\begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} \end{pmatrix}\right|}{\left|\det\left(\frac{\partial q_1}{\partial x_1}\right)\right|} \tag{1.16}$$

Substituting this into (1.9) we have

$$\begin{aligned} I_2 &= \int\int \frac{F(q_1,q_2)}{\frac{\mathcal{G}_2(q_1,q_2)}{\mathcal{G}_1(q_1,q_2)}\mathcal{G}_1(q_1,q_2)} dq_1 dq_2 \\ &= \int\int F(q_1,q_2) \frac{1}{\left|\det\begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} \end{pmatrix}\right|} dq_1 dq_2. \end{aligned} \tag{1.17}$$

where we have swapped the abbreviation $f(q_1, q_2)$ with $F(q_1, q_2) = f(x_1(q_1, q_2), x_2(q_1, q_2))$. As $x_i = x_i(q_1(x_1, x_2), q_2(x_1, x_2))$ we have

$$\delta_i^j = \frac{\partial x_i}{\partial x_j} = \sum_k \frac{\partial x_i}{\partial q_k}\frac{\partial q_k}{\partial x_j} \tag{1.18}$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} \end{pmatrix}. \tag{1.19}$$

We use this to rewrite (1.17) as

$$I_2 = \int\int F(q_1,q_2) \left|\det\begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} \end{pmatrix}\right| dq_1 dq_2. \tag{1.20}$$

$\square$

Chapter 2

Arbitrary Dimensions

Define an $n-$dimensional integral I_n:

$$I_n = \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) dx_1 \ldots dx_n \tag{2.1}$$

Cartesian coordinates x_i to the curvilinear coordinates q_i with transformation functions

$$q_i = q_i(x_1, \ldots, x_n), \quad \text{for } 1 \leq i \leq n. \tag{2.2}$$

We insert $\int \cdots \int \prod_{i=1}^{n} \delta[q_i - q_i(x_1, \ldots, x_n)] dq_1 \ldots dq_n = 1$ into the integral (2.1):

$$\begin{aligned} I_n &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\int \cdots \int \prod_{i=1}^{n} \delta[q_i - q_i(x_1, \ldots, x_n)] dq_1 \ldots dq_n \right) f(x_1, \ldots, x_n) dx_1 \ldots dx_n \\ &= \int \cdots \int \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \delta[q_i - q_i(x_1, \ldots, x_n)] f(x_1, \ldots, x_n) dx_1 \ldots dx_n dq_1 \ldots dq_n \end{aligned} \tag{2.3}$$

We now focus on integrating out the variable x_1:

$$
\begin{aligned}
I_n &= \cdots \int_{-\infty}^{\infty} \delta[q_1 - q_1(x_1, \ldots, x_n)] \cdots f(x_1, \ldots, x_n) dx_1 dx_2 \ldots dx_n dq_1 \ldots dq_n \\
&= \cdots \int_{-\infty}^{\infty} \delta[q_1 - q_1(x_1, \ldots, x_n)] \cdots f(x_1, \ldots, x_n) dx_1 dx_2 \ldots dx_n dq_1 \ldots dq_n \\
&= \cdots \int_{-\infty}^{\infty} \frac{\delta[x_1 - x_1(q_1, x_2 \ldots, x_n)]}{\left|\frac{\partial q_1}{\partial x_1}\right|_{x_1 = x_1(q_1, x_2, \ldots, x_n)}} \cdots f(x_1, \ldots, x_n) dx_1 dx_2 \ldots dx_n dq_1 \ldots dq_n \\
&= \cdots \frac{1}{\mathcal{G}_1(q_1, x_2, \ldots, x_n)} \cdots f(x_1(q_1, x_2 \ldots, x_n), \ldots, x_n) dx_2 \ldots dx_n dq_1 \ldots dq_n
\end{aligned}
\tag{2.4}
$$

where we used (1.4) with $g(x_1) = q_1 - q_1(x_1, \ldots, x_n)$. The $x_1(q_1, x_2 \ldots, x_n)$ denotes the root of the equation $q_1 - q_1(x_1, \ldots, x_n) = 0$. And

$$
\mathcal{G}_1(q_1, x_2, \ldots, x_n) = \left|\frac{\partial q_1}{\partial x_1}\right|_{x_1 = x_1(q_1, x_2, \ldots, x_n)}. \tag{2.5}
$$

We will abbreviate $f(x_1(q_1, x_2 \ldots, x_n), \ldots, x_n)$ as $f(q_1, x_2, \ldots, x_n)$.

Next we integrate out the variable x_2:

$$
\begin{aligned}
I_n &= \cdots \int_{-\infty}^{\infty} \delta[q_2 - q_2(q_1, x_2, \ldots, x_n)] \cdots f(q_1, x_2, \ldots, x_n) dx_2 \ldots dx_n dq_1 \ldots dq_n \\
&= \cdots \int_{-\infty}^{\infty} \delta[q_1 - q_1(q_1, x_2, \ldots, x_n)] \cdots f(q_1, x_2, \ldots, x_n) dx_2 \ldots dx_n dq_1 \ldots dq_n \\
&= \cdots \int_{-\infty}^{\infty} \frac{\delta[x_2 - x_2(q_1, q_2 \ldots, x_n)]}{\left|\left(\frac{\partial q_2}{\partial x_2}\right)_{(1)}\right|_{x_2 = x_2(q_1, q_2, \ldots, x_n)}} \cdots f(q_1, x_2, \ldots, x_n) dx_2 \ldots dx_n dq_1 \ldots dq_n \\
&= \cdots \frac{1}{\mathcal{G}_1(q_1, q_2, \ldots, x_n)} \frac{\mathcal{G}_1(q_1, q_2, \ldots, x_n)}{\mathcal{G}_2(q_1, q_2, \ldots, x_n)} \cdots f(q_1, q_2, \ldots, x_n) dx_2 \ldots dx_n dq_1 \ldots dq_n
\end{aligned}
\tag{2.6}
$$

where we used (1.4) with $g(x_2) = q_2 - q_2(q_1, x_2, \ldots, x_n)$. The $x_1(q_1, x_2 \ldots, x_n)$ denotes the root of the equation $q_1 - q_1(x_1, \ldots, x_n) = 0$.

$$
\frac{\mathcal{G}_2(q_1, q_2, \ldots, x_n)}{\mathcal{G}_1(q_1, q_2, \ldots, x_n)} = \left|\frac{\partial q_1}{\partial x_1}\right|_{x_1 = x_1(q_1, q_2, x_3, \ldots, x_n)}. \tag{2.7}
$$

Contnuing this way we obtain

$$\begin{aligned}
I_n &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int\cdots\int \frac{1}{\mathcal{G}_1}\times\frac{\mathcal{G}_1}{\mathcal{G}_2}\times\cdots\times\frac{\mathcal{G}_{i-1}}{\mathcal{G}_i}\prod_{j=i+1}^{n}\delta[q_j - q_j(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)] \\
&\quad \times f(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)dx_{i+1}\ldots dx_n dq_1\ldots dq_n \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int\cdots\int \frac{1}{\mathcal{G}_i(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)}\prod_{j=i+1}^{n}\delta[q_j - q_j(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)] \\
&\quad \times f(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)dx_{i+1}\ldots dx_n dq_1\ldots dq_n. \qquad (2.8)
\end{aligned}$$

Which we will now prove by induction. The base case has been proven, so we turn to the inductive part. We assume (2.8) and prove the corresponding result for i replaced by $i+1$. The integration over x_{i+1} is carried out as per usual:

$$\begin{aligned}
I_n &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int\cdots\int \frac{1}{\mathcal{G}_i(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)}\frac{\delta[x_{i+1} - x_{i+1}(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)]}{\left|\left(\frac{\partial q_{i+1}}{\partial x_{i+1}}\right)_{(i)}\right|_{x_{i+1}=x_{i+1}(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)}} \\
&\quad \prod_{j=i+2}^{n}\delta[q_j - q_j(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)]f(q_1,\ldots,q_i,x_{i+1},\ldots,x_n)dx_{i+1}\ldots dx_n dq_1\ldots dq_n \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int\cdots\int \frac{1}{\mathcal{G}_i(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)}\frac{1}{\left|\left(\frac{\partial q_{i+1}}{\partial x_{i+1}}\right)_{(i)}\right|_{x_{i+1}=x_{i+1}(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)}} \\
&\quad \prod_{j=i+2}^{n}\delta[q_j - q_j(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)]f(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)dx_{i+2}\ldots dx_n dq_1\ldots dq_n. \\
&\qquad (2.9)
\end{aligned}$$

We now only need to establish that

$$\left|\left(\frac{\partial q_{i+1}}{\partial x_{i+1}}\right)_{(i)}\right|_{x_{i+1}=x_{i+1}(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)} = \frac{\mathcal{G}_i(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)}{\mathcal{G}_{i+1}(q_1,\ldots,q_{i+1},x_{i+2},\ldots,x_n)} \qquad (2.10)$$

To that end, the total differentials of $q_1,\ldots,q_i,x_1,\ldots,x_i$ can be expessed by

$$\begin{aligned}
dq_1 &= \sum_{a=1}^{n} \frac{\partial q_1}{\partial x_a} dx_a, \\
&\vdots \\
dq_i &= \sum_{a=1}^{n} \frac{\partial q_i}{\partial x_a} dx_a, \\
dx_1 &= \sum_{a=1}^{i} \left[\left(\frac{\partial x_1}{\partial q_a} \right)_{(i)} dq_a \right] + \sum_{a=i+1}^{n} \left[\left(\frac{\partial x_1}{\partial q_a} \right)_{(i)} dq_a \right], \\
&\vdots \\
dx_i &= \sum_{a=1}^{i} \left[\left(\frac{\partial x_1}{\partial q_a} \right)_{(i)} dq_a \right] + \sum_{a=i+1}^{n} \left[\left(\frac{\partial x_i}{\partial q_a} \right)_{(i)} dq_a \right].
\end{aligned} \tag{2.11}$$

We need the partial derivative of $q_{i+1} = q_{i+1}(x_1(q_1, \ldots, x_{i+1}), \ldots, x_i(q_1, \ldots, x_{i+1}), x_{i+1}, x_{i+2}, \cdots x_n)$ with respect to x_{i+1}:

$$\left(\frac{\partial q_{i+1}}{\partial x_{i+1}} \right)_{(i)} = \sum_{a=1}^{i} \left[\frac{\partial q_{i+1}}{\partial x_a} \left(\frac{\partial x_a}{\partial x_{i+1}} \right)_{(i)} \right] + \frac{\partial q_{i+1}}{\partial x_{i+1}}. \tag{2.12}$$

Substituting $dx_1, \ldots, dx_i$ into dq_j, we obtain

$$dq_j = \sum_{a=1}^{i} \frac{\partial q_j}{\partial x_a} \left[\sum_{b=1}^{i} \left(\frac{\partial x_a}{\partial q_b} \right)_{(i)} dq_b + \sum_{b=i+1}^{n} \left(\frac{\partial x_a}{\partial x_b} \right)_{(i)} dx_b \right] + \sum_{a=i+1}^{n} \left(\frac{\partial q_j}{\partial x_a} \right)_{(i)} dx_a, \tag{2.13}$$

where $1 \leq j \leq i$. Because the coefficient of dx_{i+1} in (2.13) should be zero, we obtain the following equation

$$\begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \cdots & \frac{\partial q_1}{\partial x_i} \\ \vdots & \ddots & \vdots \\ \frac{\partial q_i}{\partial x_1} & \cdots & \frac{\partial q_i}{\partial x_i} \end{pmatrix} \begin{pmatrix} \left(\frac{\partial x_1}{\partial x_{i+1}} \right)_{(i)} \\ \vdots \\ \left(\frac{\partial x_i}{\partial x_{i+1}} \right)_{(i)} \end{pmatrix} = - \begin{pmatrix} \frac{\partial q_1}{\partial x_{i+1}} \\ \vdots \\ \frac{\partial q_i}{\partial x_{i+1}}. \end{pmatrix} \tag{2.14}$$

By making use of Cramer's rule

$$\left(\frac{\partial x_j}{\partial x_{i+1}} \right)_{(i)} = - \frac{\det \left[(\mathcal{J}^{-1}_{[i \times i]})^{(j)} (\partial \mathfrak{q}_i) \right]}{\det \left[\mathcal{J}^{-1}_{[i \times i]} \right]} \tag{2.15}$$

where

$$
\begin{aligned}
\mathcal{J}^{-1}_{[i\times i]} &= \begin{pmatrix} \dfrac{\partial q_1}{\partial x_1} & \cdots & \dfrac{\partial q_1}{\partial x_i} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial q_i}{\partial x_1} & \cdots & \dfrac{\partial q_i}{\partial x_i} \end{pmatrix}, \\
\partial \mathbb{q}_i &= \begin{pmatrix} \dfrac{\partial q_1}{\partial x_{i+1}} \\ \vdots \\ \dfrac{\partial q_i}{\partial x_{i+1}}. \end{pmatrix}
\end{aligned}
\tag{2.16}
$$

and $(\mathcal{J}^{-1}_{[i\times i]})^{(j)}(\partial \mathbb{q}_i)$ is identical to $\mathcal{J}^{-1}_{[i\times i]}$ except that the $j-$th column is replaced with $\partial \mathbb{q}_i$. Substituting (2.15) into (2.12) we obtain

$$
\left(\frac{\partial q_{i+1}}{\partial x_{i+1}}\right)_{(i)} = \frac{\partial q_{i+1}}{\partial x_{i+1}} - \sum_{j=1}^{i} \frac{\det\left[(\mathcal{J}^{-1}_{[i\times i]})^{(j)}(\partial \mathbb{q}_i)\right]}{\det\left[\mathcal{J}^{-1}_{[i\times i]}\right]} \frac{\partial q_{i+1}}{\partial x_j} \tag{2.17}
$$

For example when $i = 3$ this would read explicitly and simlify as follows

$$\left(\frac{\partial q_4}{\partial x_4}\right)_{(3)} = -\det\begin{pmatrix} \frac{\partial q_1}{\partial x_4} & \frac{\partial q_1}{\partial x_2} & \frac{\partial q_1}{\partial x_3} \\ \frac{\partial q_2}{\partial x_4} & \frac{\partial q_2}{\partial x_2} & \frac{\partial q_2}{\partial x_3} \\ \frac{\partial q_3}{\partial x_4} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_3} \end{pmatrix} \frac{\partial q_4}{\partial x_1} - \det\begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_4} & \frac{\partial q_1}{\partial x_3} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_4} & \frac{\partial q_2}{\partial x_3} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_4} & \frac{\partial q_3}{\partial x_3} \end{pmatrix} \frac{\partial q_4}{\partial x_2}$$

$$- \det\begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \frac{\partial q_1}{\partial x_4} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \frac{\partial q_2}{\partial x_4} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_4} \end{pmatrix} \frac{\partial q_4}{\partial x_3} + \det\begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \frac{\partial q_1}{\partial x_3} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \frac{\partial q_2}{\partial x_3} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_3} \end{pmatrix} \frac{\partial q_4}{\partial x_4}$$

$$= \det\begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \frac{\partial q_1}{\partial x_3} & \frac{\partial q_1}{\partial x_4} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \frac{\partial q_2}{\partial x_3} & \frac{\partial q_2}{\partial x_4} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_3} & \frac{\partial q_3}{\partial x_4} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_3} & \frac{\partial q_3}{\partial x_4} \end{pmatrix} \tag{2.18}$$

Similarly we have for (2.17)

$$\begin{aligned} \left(\frac{\partial q_{i+1}}{\partial x_{i+1}}\right)_{(i)} &= \frac{1}{\det\left[\mathcal{J}^{-1}_{[i\times i]}\right]} \left(-\sum_{j=1}^{i} \det\left[(\mathcal{J}^{-1}_{[i\times i]})^{(j)}(\partial \mathbb{q}_i)\right] \frac{\partial q_{i+1}}{\partial x_j} + \det\left[\mathcal{J}^{-1}_{[i\times i]}\right] \frac{\partial q_{i+1}}{\partial x_{i+1}} \right) \\ &= \frac{1}{\det\left[\mathcal{J}^{-1}_{[i\times i]}\right]} \sum_{j=1}^{i+1} (-1)^{i+1-j} \mathcal{M}_{i+1\,j}(\mathcal{J}^{-1}_{[(i+1)\times(i+1)]}) \\ &= \frac{\det\left[\mathcal{J}^{-1}_{[(i+1)\times(i+1)]}\right]}{\det\left[\mathcal{J}^{-1}_{[i\times i]}\right]} \end{aligned} \tag{2.19}$$

where $\mathcal{M}_{i\,j}(\mathbb{A})$ is the ij minor of the square matrix $\mathbb{A}$, i.e. the determinant of the matrix obtained by removing the $i-$th row and $j-$th column from $\mathbb{A}$. Hence, (2.19) yields

$$\left| \left(\frac{\partial q_{i+1}}{\partial x_{i+1}}\right)_{(i)} \right| = \frac{\mathcal{G}_{i+1}}{\mathcal{G}_i}. \tag{2.20}$$

Completing the inductive step of the argument.

As $x_i = x_i(q_1(x_1, \dots x_n), \dots, q_n(x_1, \dots x_n))$ we have

$$\delta_i^j = \frac{\partial x_i}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} \frac{\partial q_k}{\partial x_j} \tag{2.21}$$

and so

$$\mathbb{1}_{[n\times n]} = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \cdots & \frac{\partial q_1}{\partial x_i} \\ \vdots & \ddots & \vdots \\ \frac{\partial q_i}{\partial x_1} & \cdots & \frac{\partial q_i}{\partial x_i} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_i} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_i}{\partial q_1} & \cdots & \frac{\partial x_i}{\partial q_i} \end{pmatrix} \tag{2.22}$$

where $\mathbb{1}_{[n\times n]}$ is the $n \times n$ unit matrix. So that finally,

$$I_n = \int \cdots \int F(q_1, \dots q_n) \left| \det \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_n} \end{pmatrix} \right| dq_1 \dots dq_n \tag{2.23}$$

where we have swapped the abbreviation $f(q_1, \dots, q_n)$ with

$$F(q_1, \dots, q_n) = f(x_1(q_1, \dots, q_n), \dots, x_n(q_1, \dots, q_n)).$$