Intruction to the use of Spinors in General Relativity

0.1 Null Tetrads and Spinor Analysis

0.1.1 Null Tetrads

We start with four linerly independent vector fields e_i^a, where i serves to label the vectors.

$$\sum_i \alpha_i e_i^a = 0 \quad implies\ \alpha_i = 0.$$

At a particular point, we define a matrix of scalars g_{ij}, called the frame metric, by

$$g_{ij} = g_{ab} e_i^a e_j^b. \tag{1}$$

Since e_i^a are linearly independent and that g_{ab} is non-singular, it follows that the matrix g_{ij} is non-singular and hence invertible. To see this first consider the eigenvector equation

$$\sum_j g_{ij} v_j = \lambda v_i$$

as g_{ij} is real and symmetric it has eigenvales. To prove g_{ij} is non-singular we assume $\lambda = 0$. Then

$$\sum_j g_{ij} v_j = e_i^a [\sum_j g_{ab} e_j^b v_j] = 0.$$

As the e_i^a are linearly independent, this implies

$$\sum_j g_{ab} e_j^b v_j = 0.$$

Using that g_{ab} is invertible, this implies

$$\sum_j e_j^b v_j = 0$$

and again by linearly independence of e_j^b we have $v_j = 0$. Hence there is no eigenvector with eigenvalue zero and so g_{ij} is invertible.

We denote the inverse as g^{ij},

$$g_{ij}g^{jk} = \delta_i^k. \tag{2}$$

we then use the frame metric to raise and lower frame indices. Then we can write

$$\delta_j^i = g^{ik}g_{kj} = g^{ik}g_{ab}e_k^a e_j^b = e_a^i e_j^a. \tag{3}$$

Using this it is easy to verify the inverse relationship to (1) is

$$g_{ab} = g_{ij}e_a^i e_b^j \tag{4}$$

as

$$\begin{aligned} g_{ab}e_i^a e_j^b &= (g_{kl}e_a^k e_b^l)e_i^a e_j^b \\ &= g_{kl}(e_a^k e_i^a)(e_b^l e_j^b) \\ &= g_{ij} \end{aligned}$$

Suppose for a given spacetime we have defined an orthonormal tetrad as follows:

$$\begin{array}{ll} v^a & \text{timelikevector} \\ i^a, j^a, k^a & \text{spacelikevectors.} \end{array} \tag{5}$$

We can construct a null teterad via

$$e_{\mathbf{0}}^{\ a} = l^a := \frac{1}{\sqrt{2}}(v^a + i^a), \tag{6}$$

$$e_{\mathbf{1}}^{\ a} = n^a := \frac{1}{\sqrt{2}}(v^a - i^a) \tag{7}$$

in whcih case l^a and n^a are *null* vectors, that is

$$l^a l_a = n^a n_a = 0 \tag{8}$$

and satisfy the normalization condition

$$l^a n_a = 1. \tag{9}$$

next we introduce a *complex* null vector defined by

$$m^a := \frac{1}{\sqrt{2}}(j^a + ik^a) \tag{10}$$

together with its complex conjugate

$$m^a := \frac{1}{\sqrt{2}}(j^a - ik^a) \tag{11}$$

It easily follows that the vector are null,

$$m^a m_a = \overline{m}^a \overline{m}_a = 0, \tag{12}$$

and satisfy the normalization condition

$$m^a \overline{m}_a = -1. \tag{13}$$

Including

$$e_{\mathbf{2}}^{\ a} = m^a, \quad e_{\mathbf{3}}^{\ a} = \overline{m}^a, \tag{14}$$

then the defines the frame metric via (1) is then

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{15}$$

and the inverse frame metric is

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0. \end{pmatrix} \tag{16}$$

$$
\begin{aligned}
g^{ab} &= g_{ij} e^i_a e^j_b \\
&= (l_a, n_a, m_a \overline{m}_a) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} l^b \\ n^b \\ m^b \\ \overline{m}^b \end{pmatrix} \\
&= l_a n_b + n_a l_b - m_a \overline{m}_b - m_b \overline{m}_a.
\end{aligned}
$$

The metric g_{ab} is decomposed into products the null tetrads according to

$$
g_{ab} = l_a n_a + l_b n_a - m_a \overline{m}_b - m_b \overline{m}_a. \tag{17}
$$

$$
\begin{aligned}
g^{ab} &= g^{ij} e^a_i e^b_j \\
&= (l^a, n^a, m^a \overline{m}^a) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} l^b \\ n^b \\ m^b \\ \overline{m}^b \end{pmatrix} \\
&= l^a n^b + n^a l^b - m^a \overline{m}^b - m^b \overline{m}^a.
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&n \cdot l = -1 \quad n \cdot m = 0 \quad n \cdot \overline{m} = 0 \\
&l \cdot m = 0 \quad l \cdot \overline{m} = 0 \quad m \cdot \overline{m} = 1.
\end{aligned} \tag{19}
$$

Example

Conbsider the flat Minkowski metric, written in spherical polar coordinates:

$$
ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2
$$

$$
\begin{aligned}
l_\mu &= \frac{1}{\sqrt{2}}(1, 1, 0, 0), \qquad n_\mu = \frac{1}{\sqrt{2}}(1, -1, 0, 0) \\
m_\mu &= \frac{1}{\sqrt{2}}(0, 0, r, ir\sin\theta), \qquad \overline{m}_\mu = \frac{1}{\sqrt{2}}(0, 0, r, -ir\sin\theta)
\end{aligned} \tag{20}
$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2\theta \end{pmatrix} \tag{21}$$

$$\begin{aligned} l \cdot l &= g^{\mu\nu} l_\mu l_\nu \\ &= \frac{1}{2}(1, 1, 0, 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{22}$$

$$\begin{aligned} n \cdot n &= g^{\mu\nu} n_\mu n_\nu \\ &= \frac{1}{2}(1, -1, 0, 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{23}$$

$$\begin{aligned} l \cdot n &= g^{\mu\nu} l_\mu n_\nu \\ &= \frac{1}{2}(1, 1, 0, 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= 1. \end{aligned} \tag{24}$$

$$\begin{aligned} m \cdot m &= g^{\mu\nu} m_\mu m_\nu \\ &= \frac{1}{2}(0, 0, r, ir\sin\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r \\ ir\sin\theta \end{pmatrix} \\ &= \frac{1}{2}(0, 0, r, ir\sin\theta) \begin{pmatrix} 0 \\ 0 \\ -1/r \\ -i/r\sin\theta \end{pmatrix} \\ &= 0. \end{aligned} \tag{25}$$

Obviously $\overline{m} \cdot \overline{m} = 0$.

$$
\begin{aligned}
m \cdot \overline{m} &= g^{\mu\nu} m_\mu m_\nu \\
&= \frac{1}{2}(0, 0, r, ir\sin\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r \\ -ir\sin\theta \end{pmatrix} \\
&= \frac{1}{2}(0, 0, r, ir\sin\theta) \begin{pmatrix} 0 \\ 0 \\ -1/r \\ i/r\sin\theta \end{pmatrix} \\
&= -1.
\end{aligned} \tag{26}
$$

Obviously $l \cdot m = l \cdot \overline{m} = 0$ and $n \cdot m = n \cdot \overline{m} = 0$

0.1.2 Newman-Penrose Formulism

The Weyl tensor in terms of the curvature tensor is

$$
\begin{aligned}
C_{abcd} &= R_{abcd} + \frac{1}{2}\left(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}\right) \\
&\quad + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R.
\end{aligned} \tag{27}
$$

Define the scalars

$$
\begin{aligned}
\Psi_0 &= -C_{abdc} l^a m^b l^c m^d, \\
\Psi_1 &= iC_{abdc} l^a m^b l^c n^d, \\
\Psi_2 &= C_{abdc} l^a m^b \overline{m}^c n^d, \\
\Psi_3 &= -iC_{abdc} l^a n^b \overline{m}^c n^d, \\
\Psi_4 &= -C_{abdc} \overline{m}^a n^b \overline{m}^c n^d.
\end{aligned} \tag{28}
$$

The tetrad components of the Ricci tensor are given by:

$$
\begin{aligned}
\Phi_{00} &= \frac{1}{2}R_{ab}l^a l^b, & \Phi_{01} &= i\frac{1}{2}R_{ab}l^a m^b, \\
\Phi_{02} &= -\frac{1}{2}R_{ab}m^a m^b, & \Phi_{10} &= -i\frac{1}{2}R_{ab}l^a \overline{m}^b, \\
\Phi_{11} &= \frac{1}{4}R_{ab}(l^a n^b + m^a \overline{m}^b), & \Phi_{12} &= i\frac{1}{2}R_{ab}n^a m^b, \\
\Phi_{20} &= -\frac{1}{2}R_{ab}\overline{m}^a \overline{m}^b, & \Phi_{21} &= -i\frac{1}{2}R_{ab}n^a \overline{m}^b, \\
\Phi_{22} &= \frac{1}{2}R_{ab}n^a n^b.
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\kappa &= -m^a l^b \nabla_b l_a & \epsilon &= \frac{1}{2}(\overline{m}^a l^b \nabla_b m_a - n^a l^b \nabla_b l_a) & \pi &= \overline{m}^a l^b \nabla_b n_a \\
\sigma &= -m^a m^b \nabla_b l_a & \beta &= \frac{1}{2}(\overline{m}^a m^b \nabla_b m_a - n^a m^b \nabla_b l_a) & \mu &= \overline{m}^a m^b \nabla_b n_a \\
\rho &= -m^a \overline{m}^b \nabla_b l_a & \alpha &= \frac{1}{2}(\overline{m}^a \overline{m}^b \nabla_b m_a - n^a \overline{m}^b \nabla_b l_a) & \lambda &= \overline{m}^a \overline{m}^b \nabla_b n_a \\
\tau &= -m^a n^b \nabla_b l_a & \gamma &= \frac{1}{2}(\overline{m}^a n^b \nabla_b m_a - n^a n^b \nabla_b l_a) & \nu &= \overline{m}^a n^b \nabla_b n_a
\end{aligned}
\tag{30}
$$

0.1.3 Spinor Analysis in GR

Spinors and Vectors

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{31}
$$

We form the matrix

$$
u^{AA'} = u^0\sigma_0^{AA'} + u^1\sigma_1^{AA'} + u^2\sigma_2^{AA'} + u^3\sigma_3^{AA'} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{32}
$$

We demand that this matrix by Hermitian

$$
u^\dagger = u \tag{33}
$$

as then and only then are the coordinates (t, x, y, z) guareanteed to be real. One can represent the effect of the Lorenz transformation by matrix multiplication

$$
u' = LuL^\dagger \tag{34}
$$

Here u' is also Hermitian

$$\begin{aligned}
(u')^\dagger &= (LuL^\dagger)^\dagger \\
&= (L^\dagger)^\dagger u^\dagger L^\dagger \\
&= LuL^\dagger \\
&= u'.
\end{aligned} \tag{35}$$

and the new coordinates (t', x', y', z') are guareanteed to be real too. Explicitly

$$\begin{pmatrix} t'+z' & x'+iy' \\ x'-iy' & t'-z' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \tag{36}$$

A Lorentx transformation is defined as a linear operation that leaves the interval invariant:

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2 \tag{37}$$

Note that the determinant of u is

$$det\ u = t^2 - x^2 - y^2 - z^2 \tag{38}$$

so that the condition for the preservation of the interval is

$$det\ u' = det\ u \tag{39}$$

or

$$(det\ L)(det\ u')(det\ L^\dagger) = det\ u \tag{40}$$

This requirement is fulfilled by

$$det\ L = 1. \tag{41}$$

We take the transformation rule for a two component quantitiy ξ^A to be

$$\xi'^A = L^A{}_B \xi^B. \tag{42}$$

This is the definition of a two-component spinor. To recover the formula $u' = LuL^{\dagger}$, namely,

$$u'^{AB'} = L^{A}{}_{C}\overline{L}^{B'}{}_{D'}u^{CD'} \tag{43}$$

we introduce another spinor $\eta^{B'}$ which transforms according to the conjugate of the Lorentz transformation

$$\eta'^{A'} = \overline{L}^{A'}{}_{B'}\eta^{B'} \tag{44}$$

because then the transformation law for a second rank spinor $\xi^{A}\eta^{B'}$,

$$\xi'^{A}\eta'^{B'} = L^{A}{}_{C}\overline{L}^{B'}{}_{D'}\xi^{C}\eta^{D'}, \tag{45}$$

is the same transformation law for $u^{AA'}$.

We see in a sense a spinor is the "square-root of a vector".

Dual spinor

In components with respect to this basis we have

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{46}$$

We define ϵ^{AB} as

$$\epsilon^{AB} = -(\epsilon^{-1})^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{47}$$

We can use ϵ_{AB} lower indices of spinors,

$$\epsilon_{AB}\xi^{A}\eta^{B} = (\epsilon_{AB}\xi^{A})\eta^{B}. \tag{48}$$

The quantity in the brackets is the dual of ξ^{B}, and so we have

$$\xi_{B} = \epsilon_{AB}\xi^{A} = \xi^{A}\epsilon_{AB} = -\epsilon_{BA}\xi^{A} \tag{49}$$

Using the inverse of ϵ_{AB} we have

$$(\epsilon^{-1})^{BC}\xi_B = \xi^A \epsilon_{AB}(\epsilon^{-1})^{BC} = \xi^A \delta_A{}^C, \tag{50}$$

where $\delta_A{}^C$ is the spinor Kronecker delta. Note th eposition of the indices

$$-\epsilon^{BC}\xi_B = \xi^A \delta_A{}^C = \xi^C. \tag{51}$$

These can be used for the raising or lowering of spinor indices, in a way analogous to g^{ab} and g_{ab} for tensors, but here one has to be careful with the up-down rule.

$$\xi_B = \xi^A \epsilon_{AB}, \qquad \xi^C = \epsilon^{CB}\xi_B. \tag{52}$$

For multiple-component spinors, for example we have

$$\xi^A{}_D{}^C = \xi^{ABC}\epsilon_{BD} \tag{53}$$

Null vectors.

A space time vector corresponding to a spinor of the form

$$X^{AA'} = \alpha^A \beta^{A'} \tag{54}$$

is null as its determinate is zero,

$$\epsilon_{AB}\epsilon_{A'B'}(\alpha^A\beta^{A'})(\alpha^B\beta^{B'}) = (\alpha^A\alpha_A)(\beta^{A'}\beta_{A'}) = 0. \tag{55}$$

In fact any null vector has this spinorial form. A four-vector is null if and only if $\det(X^{AA'}) = 0$. This means rows/columbs must be linearly dependent.

A real null vector satisfies the condition $\beta = \overline{\alpha}$ whenever X is.

Converting spinors into vectors

$$u^{AA'} = x^\mu \sigma_\mu{}^{AA'} \tag{56}$$

As

$$u_{AA'} = u^{BB'}\epsilon_{BA}\epsilon_{B'A'}$$

we define the matrix ${\sigma^\mu}_{AA'}$ via

$$u_{AA'} = x_\mu {\sigma^\mu}_{AA'} \tag{57}$$

we have

$$\begin{aligned} u_{AA'} &= x^\mu {\sigma_\mu}^{BB'} \epsilon_{BA} \epsilon_{B'A'} \\ &= x_\mu (\eta^{\mu\nu} {\sigma_\nu}^{BB'} \epsilon_{BA} \epsilon_{B'A'}) \\ &= x_\mu {\sigma^\mu}_{AA'} \end{aligned} \tag{58}$$

implying

$${\sigma^\mu}_{AA'} = \eta^{\mu\nu} {\sigma_\nu}^{BB'} \epsilon_{BA} \epsilon_{B'A'}. \tag{59}$$

Let us calculate this from

$${\sigma_0}^{AA'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {\sigma_1}^{AA'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad {\sigma_2}^{AA'} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad {\sigma_3}^{AA'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{60}$$

Firstly

$$\begin{aligned} {\sigma^0}_{AA'} &= \eta^{0\mu} {\sigma_\mu}^{BB'} \epsilon_{BA} \epsilon_{B'A'} \\ &= -(\epsilon^T)_{AB} {\sigma_0}^{BB'} \epsilon_{B'A'} \\ &= -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{61}$$

then

$$\begin{aligned} {\sigma^1}_{AA'} &= \eta^{1\mu} {\sigma_\mu}^{BB'} \epsilon_{BA} \epsilon_{B'A'} \\ &= (\epsilon^T)_{AB} {\sigma_1}^{BB'} \epsilon_{B'A'} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{62}$$

then

$$\begin{aligned}
{\sigma^2}_{AA'} &= \eta^{2\mu}{\sigma_\mu}^{BB'}\epsilon_{BA}\epsilon_{B'A'} \\
&= (\epsilon^T)_{AB}{\sigma_2}^{BB'}\epsilon_{B'A'} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
{\sigma^3}_{AA'} &= \eta^{3\mu}{\sigma_\mu}^{BB'}\epsilon_{BA}\epsilon_{B'A'} \\
&= (\epsilon^T)_{AB}{\sigma_3}^{BB'}\epsilon_{B'A'} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{64}$$

Altogether we have

$$\begin{aligned}
{\sigma^0}_{AA'} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {\sigma^1}_{AA'} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad {\sigma^2}_{AA'} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
{\sigma^3}_{AA'} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{65}$$

We have the following orthogonality and normalisation relations

$${\sigma_\mu}^{AA'}{\sigma^\mu}_{BB'} = -2\delta^A_B\delta^{A'}_{B'} \tag{66}$$

and

$${\sigma_\mu}^{AA'}{\sigma^\nu}_{AA'} = -2\delta^\nu_\mu \tag{67}$$

Proof:

(i) Proof of (66)

We check by direct calculation. There arer 16 possibilities

$A = A'$ and $B \neq B'$

$$
\begin{aligned}
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{00'} \sigma^{\mu}{}_{01'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{00'} \sigma^{\mu}{}_{10'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{11'} \sigma^{\mu}{}_{01'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{11'} \sigma^{\mu}{}_{10'} &= 0 \qquad (68)
\end{aligned}
$$

$B = B'$ and $A \neq A'$

$$
\begin{aligned}
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{01'} \sigma^{\mu}{}_{00'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{10'} \sigma^{\mu}{}_{00'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{01'} \sigma^{\mu}{}_{11'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{10'} \sigma^{\mu}{}_{11'} &= 0 \qquad (69)
\end{aligned}
$$

$A = A'$ and $B = B'$

$$
\begin{aligned}
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{00'} \sigma^{\mu}{}_{00'} &= -2 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{00'} \sigma^{\mu}{}_{11'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{11'} \sigma^{\mu}{}_{00'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_{\mu}{}^{11'} \sigma^{\mu}{}_{11'} &= -2 \qquad (70)
\end{aligned}
$$

$A \neq A'$ and $B \neq B'$

$$\begin{aligned}
\sum_{\nu=0}^{3} \sigma_\mu{}^{01'} \sigma^\mu{}_{01'} &= -2 \\
\sum_{\nu=0}^{3} \sigma_\mu{}^{01'} \sigma^\mu{}_{10'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_\mu{}^{10'} \sigma^\mu{}_{01'} &= 0 \\
\sum_{\nu=0}^{3} \sigma_\mu{}^{10'} \sigma^\mu{}_{10'} &= -2
\end{aligned} \tag{71}$$

confirming (66).

(ii) Proof of (67).

Calculate $\sigma_\mu{}^{AA'} \sigma^\nu{}_{AA'}$. First say $\mu \neq \nu$

$$\sigma_\mu{}^{AA'} \sigma^\nu{}_{AA'} = Tr[-\sigma_\mu^T \sigma_\nu] = 0 \tag{72}$$

as what appears in the bracket is the positive or negative of one of the sigma matrices. Now say $\mu = \nu$, first $\mu = 0$

$$\sigma_0{}^{AA'} \sigma^0{}_{AA'} = Tr[-\sigma_0^T \sigma_0] = -2.$$

Now say $\mu = 1$

$$\sigma_1{}^{AA'} \sigma^1{}_{AA'} = Tr[-\sigma_1^T \sigma_1] = -2.$$

Similarly for σ_3. Now say $\mu = 2$

$$\sigma_2{}^{AA'} \sigma^2{}_{AA'} = Tr[\sigma_2^T \sigma_2] = -2.$$

Now say $\mu = 3$

$$\sigma_3{}^{AA'} \sigma^3{}_{AA'} = Tr[\sigma_3^T \sigma_3] = -2.$$

Altogether

$$\sigma_{\mu}{}^{AA'}\sigma^{\nu}{}_{AA'} = -2\delta^{\nu}_{\mu} \tag{73}$$

□

Now

$$\begin{aligned}
\sigma^{\nu}{}_{AA'}u^{AA'} &= \sigma^{\nu}{}_{AA'}(x^{\mu}\sigma_{\mu}{}^{AA'}) \\
&= x^{\mu}\sigma^{\nu}{}_{AA'}\ \sigma_{\mu}{}^{AA'} \\
&= -2x^{\mu}\delta^{\nu}_{\mu} \\
&= -2x^{\nu}
\end{aligned}$$

therefore the contravariant components of the vector are

$$x^{\mu} = -\frac{1}{2}\sigma^{\mu}{}_{AA'}u^{AA'} \tag{74}$$

Now

$$\begin{aligned}
\sigma_{\nu}{}^{AA'}u_{AA'} &= \sigma_{\nu}{}^{AA'}(x_{\mu}\sigma^{\mu}{}_{AA'}) \\
&= x_{\mu}\sigma^{\mu}{}_{AA'}\ \sigma_{\nu}{}^{AA'} \\
&= -2x_{\nu}
\end{aligned}$$

and the covariant components of the vector are

$$x_{\mu} = -\frac{1}{2}\sigma_{\mu}{}^{AA'}u_{AA'}. \tag{75}$$

A tensor T can be expressed in spinor language, for example for a mixed tensor

$$T_{AA'}{}^{BB'CC'} = \sigma^{\mu}{}_{AA'}\sigma_{\nu}{}^{BB'}\sigma_{\sigma}{}^{CC'}T_{\mu}{}^{\nu\sigma} \tag{76}$$

and the inverting this via

$$T_{\mu}{}^{\nu\sigma} = \left(-\frac{1}{2}\right)^{3}\sigma_{\mu}{}^{AA'}\sigma^{\nu}{}_{BB'}\sigma^{\sigma}{}_{CC'}T_{AA'}{}^{BB'CC'} \tag{77}$$

$$\epsilon_{AA'BB'CC'DD'} = \epsilon_{\mu\nu\sigma\rho}\sigma^{\mu}{}_{AA'}\sigma^{\nu}{}_{BB'}\sigma^{\sigma}{}_{CC'}\sigma^{\rho}{}_{DD'} \tag{78}$$

Spin basis

It is easy to find if two spinors μ_A and λ_A are linearly independent $\mu_A = Const.\lambda_A$; there scalar product $\lambda_A \mu^A = 0$. Therefore a nonvanishing scalar product

$$\lambda_A \mu^A \neq 0 \tag{79}$$

is a necessary and sufficient condition for the linear independence of two spinors.

A general spinor can be written as a linear combination of two basis spinors:

$$\xi = \xi^0 o + \xi^1 \imath. \tag{80}$$

$$o^A = (1,0), \qquad \imath^A = (0,1) \tag{81}$$

The conition that $(o, \imath)$ is a spin basis are

$$\epsilon_{AB} o^A o^B = \epsilon_{AB} \imath^A \imath^B = 0, \quad \epsilon_{AB} o^A \imath^B = 1. \tag{82}$$

$$\epsilon_{AB} = o_A \imath_B - \imath_A o_A \tag{83}$$

since both sides give the same result when applied to o^A or $\imath^A$:

$$\begin{aligned} \epsilon_{AB} o^A &= o_A o^A \imath_B - \imath_A o^A o_B = o_B \\ \epsilon_{AB} \imath^A &= o_A \imath^A \imath_B - \imath_A \imath^A o_B = \imath_B \end{aligned} \tag{84}$$

Jacobi identity

We have the Jacobi identity

$$\epsilon_{A[B} \epsilon_{CD]} = 0 = \epsilon_{AB} \epsilon_{CD} + \epsilon_{AC} \epsilon_{DB} + \epsilon_{AD} \epsilon_{BC} \tag{85}$$

Lemma 0.1.1 *Let $\tau_{\ldots CD \ldots}$ be a multivalent spinor. Then*

$$\tau_{\ldots AB \ldots} = \tau_{\ldots (AB) \ldots} + \frac{1}{2} \epsilon_{AB} \tau_{\ldots C}{}^{C}{}_{\ldots} \tag{86}$$

Proof:

It is sufficient to consider the case where τ has valence two. So multiply (85) with the CD indices raised with τ_{CD},

$$(\epsilon_{AB}\epsilon^{CD} + \epsilon_A{}^C\epsilon^D{}_B + \epsilon_A{}^D\epsilon_B{}^C)\tau_{CD} = \epsilon_{AB}\tau_C{}^C - \tau_{AB} + \tau_{BA} = 0, \tag{87}$$

or

$$\tau_{[AB]} = \frac{1}{2}\epsilon_{AB}\tau_C{}^C. \tag{88}$$

So that

$$\tau_{AB} = \tau_{(AB)} + \tau_{[AB]} = \tau_{(AB)} + \frac{1}{2}\epsilon_{AB}\tau_C{}^C. \tag{89}$$

□

The alternating tensor. The alternating, as defined before, by $\epsilon_{abcd} = \epsilon_{[abcd]}$, with $\epsilon_{0123} = 1$. We prove that

$$\epsilon_{abcd} = i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}. \tag{90}$$

First note this corresponds to a real tensor since complex conjugation interchanges the two terms with i replaced with $-i$. We check that it is anti-symmetric under interchanging a and b:

$$\epsilon_{abcd} = i\epsilon_{BC}\epsilon_{AD}\epsilon_{B'D'}\epsilon_{A'C'} - i\epsilon_{BD}\epsilon_{AC}\epsilon_{B'C'}\epsilon_{A'D'} = -\epsilon_{bacd}. \tag{91}$$

Similarly it can be shown to be anti-symmetric under the interchange of c and d. Finally we consider the interchange between b and c. We have using the Jacobi identity (85),

$$\begin{aligned}
\epsilon_{abcd} + \epsilon_{acbd} &= i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'} \\
&+ \; i\epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{C'B'} - i\epsilon_{AD}\epsilon_{CB}\epsilon_{A'B'}\epsilon_{C'D'} \\
&= i(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AB}\epsilon_{CD})\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}(\epsilon_{A'C'}\epsilon_{D'B'} + \epsilon_{A'B'}\epsilon_{C'D'}) \\
&= i(-\epsilon_{AD}\epsilon_{BC})\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}(-\epsilon_{A'D'}\epsilon_{B'C'}) \\
&= 0.
\end{aligned} \tag{92}$$

From anti-symmetry in the pairs ab, bc and cd, it follows that we have total anti-symmetry:

$$\epsilon_{abcd} = \epsilon_{[abcd]}. \tag{93}$$

An equivalent expression for ϵ_{abcd} is:

$$i\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}. \tag{94}$$

We use the Jacobi identity (85) to show they are equivalent:

$$\begin{aligned}
& i\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} \\
= \; & -i(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC})\epsilon_{A'C'}\epsilon_{B'D'} + i\epsilon_{AC}\epsilon_{BD}(\epsilon_{A'C'}\epsilon_{D'B'} + \epsilon_{A'D'}\epsilon_{B'C'}) \\
= \; & -i\epsilon_{AC}\epsilon_{DB}\epsilon_{A'C'}\epsilon_{B'D'} \underbrace{- i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}} \\
& i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{D'B'} \underbrace{+ i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'}} \\
= \; & i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}.
\end{aligned} \tag{95}$$

Null tetrads and spinors

We can contract the $\sigma^{\hat{a}}{}_{AA'}$ with a tetrad $e_{\hat{a}}{}^{a}$

$$u^a = u^{\hat{a}} e_{\hat{a}}{}^{a} = u^{AA'} \sigma^{\hat{a}}{}_{AA'} e_{\hat{a}}{}^{a}, \tag{96}$$

so that

$$u^{AA'} = \sigma_{\hat{a}}{}^{AA'} u^{\hat{a}}. \tag{97}$$

$$k^a = \pm \kappa^A \overline{\kappa}^{A'} \tag{98}$$

$$s^a = 2^{-1/2}(\kappa^A \overline{\mu}^{A'} + \mu^A \overline{\kappa}^{A'}), \tag{99}$$

is a unit spacetime vector orthogonal to k^a, and is unique up to an additive multiple of k^a.

Another real unit spacelike vector orthogonal to k^a is

$$t^a = 2^{-1/2}(\kappa^A \overline{\mu}^{A'} - \mu^A \overline{\kappa}^{A'}) \tag{100}$$

$$\tau^{AA'} = \xi o^A \overline{o}^{A'} + \eta \iota^A \overline{\iota}^{A'} + \zeta o^A \overline{\iota}^{A'} + \sigma \iota^A \overline{o}^{A'} \tag{101}$$

$$l^a = o^A \overline{o}^{A'}, \quad n^a = \iota^A \overline{\iota}^{A'}, \quad m^a = o^A \overline{\iota}^{A'}, \overline{m}^a = \iota^A \overline{o}^{A'}, \tag{102}$$
$$l_a = o_A \overline{o}_{A'}, \quad n_a = \iota_A \overline{\iota}_{A'}, \quad m_a = o_A \overline{\iota}_{A'}, \overline{m}_a = \iota_A \overline{o}_{A'}. \tag{103}$$

$$l^a l_a = o_A \overline{o}_{A'} o^A \overline{o}^{A'} = 0 \tag{104}$$

Theorem: Suppose $\tau_{AB\ldots C}$ is totally symmetric. Then there exists univalent spinors $\alpha_A, \beta_B, \ldots, \gamma_C$ such that

$$\tau_{AB\ldots C} = \alpha_{(A} \beta_B \ldots \gamma_{C)}. \tag{105}$$

The $\alpha, \beta, \ldots \gamma$ are called *principal spinors of* τ. The corresponding *null directions of* τ.

Proof:

First we let $\xi^A = (x, y)$ and define

$$\tau(\xi) = \tau_{AB\ldots C} \xi^A \xi^B \ldots \xi^C. \tag{106}$$

For simplicity, let us consider the simple case of a valent 2 spinor, τ_{AB}. Now

$$\tau_{AB} = \tau_{AB} \xi^A \xi^B$$

is obviously a polynomial of degree 2 in (complex) x and y:

$$\begin{aligned} \tau_{00} x^2 - \tau_{01} xy - \tau_{10} yx + \tau_{11} y^2 &= \tau_{00} x^2 - 2! \tau_{01} xy + \tau_{11} y^2 \\ &= y^2 \left[\tau_{00} \left(\frac{x}{y} \right)^2 - 2! \tau_{01} \left(\frac{x}{y} \right) + \tau_{11} \right] \\ &= y^2 p_2 \left(\frac{x}{y} \right). \end{aligned} \tag{107}$$

The polynomial in (x/y) can be factorised

$$\tau_{00} (x/y - a_1)(x/y - a_2) \tag{108}$$

where a_1 and a_2 are roots of the equation $p_2(x/y) = 0$. So

$$\begin{aligned}
\tau(\xi) &= \tau_{AB}\xi^A\xi^B \\
&= y^2 p_2\left(\frac{x}{y}\right) \\
&= y^2(\alpha_0\frac{x}{y} - \alpha_1)(\beta_0\frac{x}{y} - \beta_1) \\
&= (\alpha_0 x - \alpha_1 y)(\beta_0 x - \beta_1 y) \\
&= \alpha_A\beta_B\xi^A\xi^B
\end{aligned} \tag{109}$$

Therefore

$$\tau_{AB}\xi^A\xi^B = \alpha_A\beta_B\xi^A\xi^B \tag{110}$$

$$\begin{aligned}
\tau_{00}x^2 - 2!\tau_{01}xy + \tau_{11}y^2 &= \tau_{00}x^2 - 2!\tau_{10}xy + \tau_{11} \\
&= \alpha_0\beta_0 x^2 - \alpha_0\beta_1 xy - \alpha_1\beta_0 yx + \alpha_1\beta_1 y^2
\end{aligned} \tag{111}$$

Differentiating $\partial^2/\partial x^p \partial y^q$ for $p + q = 2$ we get

$$\begin{aligned}
\tau_{00} &= \alpha_{(0}\beta_{0)} \\
\tau_{01} &= \alpha_{(0}\beta_{1)} \\
\tau_{11} &= \alpha_{(1}\beta_{1)}
\end{aligned} \tag{112}$$

or

$$\tau_{AB} = \alpha_{(A}\beta_{B)} \tag{113}$$

For τ with valence n

$$\begin{aligned}
\tau(\xi) &= \tau_{AB\ldots C}\xi^A\xi^B\ldots\xi^C \\
&= y^n p_n\left(\frac{x}{y}\right) \\
&= y^2(\alpha_0\frac{x}{y} - \alpha_1)(\beta_0\frac{x}{y} - \beta_1)\ldots(\gamma_0 - \gamma_1\frac{x}{y}) \\
&= (\alpha_0 x - \alpha_1 y)(\beta_0 x - \beta_1 y)\ldots(\gamma_0 x - \gamma_1 y) \\
&= \alpha_A\beta_B\ldots\gamma_C\xi^A\xi^B\ldots\xi^C
\end{aligned} \tag{114}$$

Differentiating as $\partial^2/\partial x^p \partial y^q$ for $p, q = 0, 1, \ldots, n$, such that $p + q = n$ we obtain

$$\tau_{AB\ldots C} = \alpha_{(A}\beta_B \ldots \gamma_{C)}. \tag{115}$$

$\square$

The spinor equivalent of $T_{ab} = -T_{ba}$ is $T_{ABA'B'} = -T_{BAB'A'}$. Define

$$\phi_{AB} = \frac{1}{2} T_{ABC'}{}^{C'}$$

By

$$T_{ABC'}{}^{C'} = T_{ABA'B'}\epsilon^{A'B'} = -T_{BAB'A'}\epsilon^{A'B'} = T_{BAA'B'}\epsilon^{A'B'} = T_{BAC'}{}^{C'}$$

we have

$$\phi_{AB} = \phi_{BA}.$$

$$\begin{aligned} T_{ABA'B'} &= T_{AB(A'B')} + T_{AB[A'B']} \\ &= T_{AB(A'B')} + \phi_{AB}\epsilon_{A'B'} \end{aligned} \tag{116}$$

where $\phi_{AB} = \frac{1}{2} T_{ABC'}{}^{C'}$. Aplying this again

$$T_{ABA'B'} = T_{(AB)(A'B')} + \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\overline{\phi}_{A'B'} \tag{117}$$

Note that

$$\begin{aligned} T_{(AB)(A'B')} &= \frac{1}{2!}\frac{1}{2!}(T_{ABA'B'} + T_{BAB'A'} + T_{BAA'B'} + T_{ABB'A'}) \\ &= \frac{1}{2!}\frac{1}{2!}(T_{ABA'B'} - T_{ABA'B'} + T_{BAA'B'} - T_{BAA'B'}) \\ &= 0. \end{aligned} \tag{118}$$

So that we have

$$T_{ABA'B'} = \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\overline{\phi}_{A'B'}. \tag{119}$$

The dual of T is defined by $T^*_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}T_{cd}$. Let us calculate $T^*_{ABA'B'}$,

$$
\begin{aligned}
T^*_{ABA'B'} &= \frac{1}{2}i(\epsilon_{AB}\epsilon^{CD}\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'} - \epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'B'}\epsilon^{C'D'})T_{CDC'D'} \\
&= \frac{i}{2}(\epsilon_{AB}\epsilon^{CD}\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'} - \epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'B'}\epsilon^{C'D'})(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\overline{\phi}_{C'D'}) \\
&= \frac{i}{2}[\epsilon_{AB}(\epsilon^{CD}\phi_{CD})(\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'}\epsilon_{C'D'}) + \epsilon_{AB}(\epsilon^{CD}\epsilon_{CD})(\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'}\overline{\phi}_{C'D'}) \\
&\quad - (\epsilon_A{}^C\epsilon_B{}^D\phi_{CD})\epsilon_{A'B'}(\epsilon^{C'D'}\epsilon_{C'D'}) - (\epsilon_A{}^C\epsilon_B{}^D\epsilon_{CD})\epsilon_{A'B'}(\epsilon^{C'D'}\overline{\phi}_{C'D'})] \\
&= i(\epsilon_{AB}\overline{\phi}_{A'B'} - \phi_{AB}\epsilon_{A'B'}) \qquad (120)
\end{aligned}
$$

where we used $\epsilon^{CD}\phi_{CD} = \epsilon^{C'D'}\phi_{C'D'} = 0$ and $\epsilon^{CD}\epsilon_{CD} = \epsilon^{C'D'}\epsilon_{C'D'} = 2$. From this and (119) we have

$$T_{ab} + iT^*_{ab} = 2\phi_{AB}\epsilon_{A'B'} \qquad (121)$$

which proves that ϕ_{AB} and T_{ab} are fully equivalent as

$$T_{ab} = Re\ (2\phi_{AB}\epsilon_{A'B'}).$$

We can decompose ϕ_{AB} as it is symmetric,

$$\phi_{AB} = \alpha_{(A}\beta_{B)} \qquad (122)$$

If α and β are proportional then α is called a repeated principal spinor of ϕ, and ϕ is called algebraically special.

0.1.4 Curvature Spinors

$$R_{abcd} = R_{AA'BB'CC'DD'} \qquad (123)$$

From $R_{abcd} = -R_{bacd}$

$$R_{abcd} = \frac{1}{2}R_{AX'B}{}^{X'}{}_{CC'DD'}\epsilon_{A'B'} + \frac{1}{2}R_{XA'}{}^{X}{}_{B'CC'DD'}\epsilon_{AB} \qquad (124)$$

We have the symmetries

$$
\begin{aligned}
R_{AX'B}{}^{X'}{}_{CC'DD'} &= R_{(A|X'|B)}{}^{X'}{}_{CC'DD'} \\
R_{XA'}{}^{X}{}_{B'CC'DD'} &= R_{X(A'}{}^{X}{}_{B')CC'DD'}
\end{aligned} \tag{125}
$$

From $R_{abcd} = -R_{abdc}$ we then have

$$
\begin{aligned}
R_{abcd} &= X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\
&\quad + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}
\end{aligned} \tag{126}
$$

where

$$
X_{ABCD} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'}, \quad \Phi_{ABC'D'} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{YC'}{}^{Y}{}_{D'} \tag{127}
$$

The complex conucates appear to make R_{abcd} real. We have the symmetries

$$
\begin{aligned}
R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'} &= R_{(A|X'|B)}{}^{X'}{}_{CY'D}{}^{Y'} = R_{(A|X'|B)}{}^{X'}{}_{(C|Y'|D)}{}^{Y'} \\
R_{XA'}{}^{X}{}_{B'YC'}{}^{Y}{}_{D'} &= R_{X(A'}{}^{X}{}_{B')YC'}{}^{Y}{}_{D'} = R_{(A|X'|B)}{}^{X'}{}_{Y(C'}{}^{Y}{}_{D')}
\end{aligned} \tag{128}
$$

or

$$
\begin{aligned}
X_{ABCD} &= X_{(AB)CD} = X_{(AB)(CD)} = X_{AB(CD)} \\
\Phi_{ABC'D'} &= \Phi_{(AB)C'D'} = \Phi_{(AB)(C'D')} = \Phi_{AB(C'D')}
\end{aligned} \tag{129}
$$

The interchange symmetry

$$
\begin{aligned}
& X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
= \; & X_{CDAB}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Phi}_{C'D'AB}\epsilon_{A'B'}\epsilon_{CD} + \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} + \overline{X}_{C'D'A'B'}\epsilon_{AB}\epsilon_{CD}
\end{aligned} \tag{130}
$$

Contracting both sides with $\epsilon^{A'B'}\epsilon^{C'D'}$ gives

$$
X_{ABCD} = X_{CDAB} \tag{131}
$$

Contracting both sides with $\epsilon^{A'B'}\epsilon^{CD}$ gives

$$\Phi_{ABC'D'} = \overline{\Phi}_{C'D'AB} \tag{132}$$

Therefore the interchange symmetry is equivalent to

$$X_{ABCD} = X_{CDAB}, \qquad \overline{\Phi}_{ABC'D'} = \Phi_{ABC'D'} \tag{133}$$

The second of these equations implies that $\Phi_{ABA'B'}$ corresponds to a real tensor Φ_{ab} while $\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} = \Phi_{BAB'A'}$ implies $\Phi_{ab} = \Phi_{ba}$ and $\Phi_{C}{}^{C}{}_{C'}{}^{C'} = \Phi_{ABA'B'}\epsilon^{AB}\epsilon^{A'B'} = \Phi_{(AB)(A'B')}\epsilon^{AB}\epsilon^{A'B'} = 0$ implies $\Phi_{a}{}^{a} = 0$, altogether

$$\Phi_{ABA'B'} = \Phi_{ab} = \Phi_{ba} = \overline{\Phi}_{ab}, \quad \Phi_{a}{}^{a} = 0 \tag{134}$$

that is Φ_{ab} is real, symmetric and trace-free. Note also $X_{ABCD} = X_{(AB)(CD)}$ and $X_{ABCD} = X_{CDAB}$ implies

$$\begin{aligned}
X_{A(BC)D}\epsilon^{AD} &= \frac{1}{2}(X_{ABCD} + X_{ACBD})\epsilon^{AD} \\
&= \frac{1}{2}(X_{BADC} + X_{BDAC})\epsilon^{AD} \\
&= X_{B(AD)C}\epsilon^{AD} \\
&= 0
\end{aligned} \tag{135}$$

that is

$$X_{A(BC)}{}^{A} = 0. \tag{136}$$

The first type of dual of $R^*_{abcd} = \frac{1}{2}\epsilon_{cd}{}^{ef}R_{abef}$

$$\begin{aligned}
R^*_{ABCDA'B'C'D'} &= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'})R_{ABEFA'B'E'F'} \\
&= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'}) \\
&\quad (X_{ABEF}\epsilon_{A'B'}\epsilon_{E'F'} + \Phi_{ABE'F'}\epsilon_{A'B'}\epsilon_{EF} \\
&\quad + \overline{\Phi}_{A'B'EF}\epsilon_{AB}\epsilon_{E'F'} + \overline{X}_{A'B'E'F'}\epsilon_{AB}\epsilon_{EF}) \\
&= \frac{1}{2}i(2\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + 2\overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
&\quad - 2X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - 2\overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}) \\
&= iR_{ABCDA'B'D'C'}
\end{aligned} \tag{137}$$

Similarly we have

$$ {}^*R_{abcd} = {}^*R_{ABCDA'B'C'D'} = iR_{ABCDB'A'C'D'} \tag{138} $$

Also

$$ \begin{aligned} {}^*R^*{}_{abcd} &= \frac{1}{4}\epsilon_{ab}{}^{ef}\epsilon_{cd}{}^{gh}R_{efgh} \\ &= {}^*R^*_{ABCDA'B'C'D'} \\ &= i\,{}^*R_{ABCDA'B'D'C'} \\ &= -R_{ABCDB'A'D'C'} \end{aligned} \tag{139} $$

Altogether

$$ \begin{aligned} R^*{}_{abcd} &= iR_{ABCDA'B'D'C'} \\ {}^*R_{abcd} &= iR_{ABCDB'A'C'D'} \\ {}^*R^*{}_{abcd} &= -R_{ABCDB'A'D'C'} \end{aligned} \tag{140} $$

Clearly all three duals satisfy share the anti-symmetry of R_{abcd} ($R_{abcd} = R_{[ab][cd]}$). In addition as

$$ \epsilon_{ab}{}^{ef}\epsilon_{cd}{}^{gh}R_{efgh} = \epsilon_{ab}{}^{ef}\epsilon_{cd}{}^{gh}R_{ghef} = \epsilon_{cd}{}^{ef}\epsilon_{ab}{}^{gh}R_{efgh} $$

i.e.

$$ {}^*R^*{}_{abcd} = {}^*R^*{}_{cdab} \tag{141} $$

and

$$ {}^*R^*{}_{a[bcd]} = \frac{1}{4}\epsilon_{a[b}{}^{ef}\epsilon_{cd]}{}^{gh}R_{efgh} = \frac{1}{4}\epsilon_{a[b}{}^{ef}\epsilon_{c]d}{}^{gh}R_{efgh} = 0 $$

i.e.

$$ {}^*R^*{}_{a[bcd]} = 0. \tag{142} $$

$$
\begin{aligned}
R_{abcd} &= X + \Phi + \overline{\Phi} + \overline{X} \\
R^{*}{}_{abcd} &= -iX + i\Phi - i\overline{\Phi} + i\overline{X} \\
{}^{*}R_{abcd} &= -iX - i\Phi + i\overline{\Phi} + i\overline{X} \\
{}^{*}R^{*}{}_{abcd} &= -X + \Phi + \overline{\Phi} - \overline{X}
\end{aligned} \tag{143}
$$

Because of $X_{ABCD} = X_{CDAB}$

$$
X_{CB}{}^{C}{}_{D} = \epsilon^{AC} X_{ABCD} = \epsilon^{AC} X_{CDAB} = -\epsilon^{AC} X_{ADCB} = -X_{CD}{}^{C}{}_{B} \tag{144}
$$

and so

$$
X_{CB}{}^{C}{}_{D} = 3\Upsilon \epsilon_{BD}. \tag{145}
$$

We translate the symmetry $R_{[abc]d} = 0$ or equivalently $R_{a[bcd]} = 0$ into spinors. To simplify the calculation we establish an equivalence, first note

$$
R^{*}{}_{ab}{}^{bc} = \frac{1}{2}\epsilon^{bcef} R_{abef} = -\frac{1}{2}\epsilon^{cbef} R_{a[bef]} \tag{146}
$$

therefore $R_{a[bcd]} = 0$ implies $R^{*}{}_{ab}{}^{bc} = 0$. Next

$$
\begin{aligned}
R_{a[bcd]} &= \frac{1}{3!}\delta^{efg}_{bcd} R_{aefg} \\
&= -\frac{2}{3!}\epsilon_{hbcd}\frac{1}{2}\epsilon^{ehfg} R_{aefg} \\
&= -\frac{2}{3!}\epsilon_{hbcd} R^{*}{}_{ae}{}^{eh}
\end{aligned} \tag{147}
$$

therefore $R^{*}{}_{ab}{}^{bc} = 0$ implies $R_{a[bcd]} = 0$. Thus $R_{a[bcd]} = 0$ is equivalent to

$$
R^{*}{}_{ab}{}^{bc} = 0. \tag{148}
$$

To obtain the cyclic identity

$$
\begin{aligned}
R^*{}_{ab}{}^{b}{}_{c} &= -iX_{AB}{}^{B}{}_{C}\epsilon_{A'B'}\epsilon^{B'}{}_{C'} + i\Phi_{AB}{}^{B'}{}_{C'}\epsilon_{A'B'}\epsilon^{B}{}_{C} \\
&\quad - i\overline{\Phi}_{A'B'}{}^{B}{}_{C}\epsilon_{AB}\epsilon^{B'}{}_{C'} + i\overline{X}_{A'B'}{}^{B'}{}_{C'}\epsilon_{AB}\epsilon^{B}{}_{C} \\
&= -iX_{AB}{}^{B}{}_{C}\epsilon_{A'B'}\epsilon^{B'}{}_{C'} + i\overline{X}_{A'B'}{}^{B'}{}_{C'}\epsilon_{AB}\epsilon^{B}{}_{C} \\
&\quad + i(\Phi_{ACA'C'} - \Phi_{ACA'C'}) \\
&= 0 \qquad (149)
\end{aligned}
$$

implies

$$
X_{AB}{}^{B}{}_{C}\epsilon_{A'C'} = \overline{X}_{A'B'}{}^{B'}{}_{C'}\epsilon_{AC} \qquad (150)
$$

or

$$
\Upsilon\epsilon_{AC}\epsilon_{A'C'} = \overline{\Upsilon}\epsilon_{A'C'}\epsilon_{AC} \qquad (151)
$$

or on contracting with $\epsilon^{AC}\epsilon^{A'C'}$ gives

$$
\Upsilon = \overline{\Upsilon} \qquad (152)
$$

Let us define Ψ_{ABCD} by

$$
\Psi_{ABCD} = X_{ABCD} - \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) \qquad (153)
$$

where

$$
\Upsilon = \frac{1}{6}\epsilon^{AC}\epsilon^{BD}X_{ABCD}. \qquad (154)
$$

The symmetries $X_{ABCD} = X_{(AB)(CD)}$

$$
\begin{aligned}
\Psi_{(AB)(CD)} - \Psi_{ABCD} &= X_{(AB)(CD)} - X_{ABCD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) \\
&\quad - \frac{\Upsilon}{(2!)^2}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD} + \epsilon_{BC}\epsilon_{AD} + \epsilon_{AC}\epsilon_{BD} \\
&\quad + \epsilon_{AD}\epsilon_{BC} + \epsilon_{BD}\epsilon_{AC} + \epsilon_{BD}\epsilon_{AC} + \epsilon_{AD}\epsilon_{BC}) \\
&= 0. \qquad (155)
\end{aligned}
$$

or

$$\Psi_{ABCD} = \Psi_{(AB)(CD)} \tag{156}$$

and $X_{ABCD} = X_{CDAB}$ imply

$$\begin{aligned}
\Psi_{ABCD} - \Psi_{CDAB} &= X_{ABCD} - X_{CDAB} \\
&\quad - \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) + \Upsilon(\epsilon_{CA}\epsilon_{DB} + \epsilon_{DA}\epsilon_{CB}) \\
&= 0.
\end{aligned} \tag{157}$$

or

$$\Psi_{ABCD} = \Psi_{CDAB}. \tag{158}$$

By construction

$$\begin{aligned}
\epsilon^{AC}\Psi_{ABCD} &= \epsilon^{AC}X_{ABCD} - \epsilon^{AC}\Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}) \\
&= 3\Upsilon\epsilon_{BD} - \Upsilon(2\epsilon_{BD} + \epsilon_{BD}) \\
&= 0.
\end{aligned} \tag{159}$$

Therefore Ψ_{ABCD} is totally symmetric

$$\Psi_{ABCD} = \Psi_{(ABCD)}. \tag{160}$$

We have now found the spinor equivalents of all the symmetries of R_{abcd}. Next we compute the Ricci tensor $R_{ac} = R_{abc}{}^{b}$. From (126) we get

$$\begin{aligned}
R_{ac} &= R_{abc}{}^{b} \\
&= X_{ABC}{}^{B}\epsilon_{A'B'}\epsilon_{C'}{}^{B'} + \Phi_{ABC'}{}^{B'}\epsilon_{A'B'}\epsilon_{C}{}^{B} \\
&\quad + \overline{\Phi}_{A'B'C}{}^{B}\epsilon_{AB}\epsilon_{C'}{}^{B'} + \overline{X}_{A'B'C'}{}^{B'}\epsilon_{AB}\epsilon_{C}{}^{B} \\
&= 6\Upsilon\epsilon_{AC}\epsilon_{A'C'} - 2\Phi_{ACA'C'}
\end{aligned} \tag{161}$$

or

$$R_{ab} = 6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}. \tag{162}$$

which may written

$$R_{ab} = 6\Upsilon g_{ab} - 2\Phi_{ab}. \tag{163}$$

Hence, for the scalar curvature $R = R_a{}^a$ we find, using $\Phi_a{}^a = 0$

$$R = 24\Upsilon. \tag{164}$$

Thus

$$\Phi_{ab} = -\frac{1}{2}(R_{ab} - \frac{1}{4}Rg_{ab}) \tag{165}$$

$$\begin{aligned} R_{ABCDA'B'C'D'} &= (\Psi_{ABCD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}))\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\ &\quad + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + (\overline{\Psi}_{A'B'C'D'} + \Upsilon(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{B'C'}\epsilon_{A'D'}))\epsilon_{AB}\epsilon_{CD} \\ &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\epsilon_{A'B'}\epsilon_{C'D'} \\ &\quad + c.c. \end{aligned} \tag{166}$$

Weyl tensor

Define the real tensor

$$C_{ABCDA'B'C'D'} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \tag{167}$$

Rearanging (166) gives

$$\begin{aligned} C_{ABCDA'B'C'D'} &= R_{ABCDA'B'C'D'} \\ &\quad -\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} - \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\epsilon_{A'B'}\epsilon_{C'D'} \\ &\quad -\overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} - \Upsilon(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{B'C'}\epsilon_{A'D'})\epsilon_{AB}\epsilon_{CD} \end{aligned} \tag{168}$$

We use the Jacobi identity

$$\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{A'C'}\epsilon_{B'D'} = 0$$

on the terms proportional to Υ,

$$
\begin{aligned}
& -\Upsilon(\epsilon_{AC}\epsilon_{BD}+\epsilon_{BC}\epsilon_{AD})\epsilon_{A'B'}\epsilon_{C'D'}+c.c. \\
= & -\Upsilon(\epsilon_{AC}\epsilon_{BD}+\epsilon_{BC}\epsilon_{AD})(\epsilon_{A'C'}\epsilon_{B'D'}-\epsilon_{A'D'}\epsilon_{B'C'})+c.c. \\
= & -\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'}-\epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'} \\
& -\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'}+\epsilon_{BC}\epsilon_{AD}\epsilon_{A'C'}\epsilon_{B'D'})+c.c \\
= & -2\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'}-\epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'}) \qquad (169)
\end{aligned}
$$

Substituting this into (168)

$$
\begin{aligned}
C_{ABCDA'B'C'D'} = & R_{ABCDA'B'C'D'}-\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD}-\Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} \\
& -2\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'}-\epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'}) \qquad (170)
\end{aligned}
$$

Comparing

$$
-\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD}-\Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} \qquad (171)
$$

with

$$
\Phi_{ACA'C'}\epsilon_{B'D'}\epsilon_{BD}-\Phi_{ADA'D'}\epsilon_{B'C'}\epsilon_{BC}-\Phi_{BCB'C'}\epsilon_{A'D'}\epsilon_{AD}+\Phi_{BDB'D'}\epsilon_{A'C'}\epsilon_{AC} \qquad (172)
$$

for the distinct indices

$A=0, B=0, C=0, D=0;$
$A=1, B=1, C=1, D=1;$
$A=1, B=1, C=1, D=0;$
$A=0, B=1, C=1, D=1;$
$A=0, B=1, C=0, D=1;$
$A=0, B=0, C=0, D=1;$
$A=0, B=1, C=0, D=0;$

proves they are equivalent. Substituting (172) into (170) gives

$$
\begin{aligned}
C_{ABCDA'B'C'D'} = & R_{ABCDA'B'C'D'}+\Phi_{ACA'C'}\epsilon_{B'D'}\epsilon_{BD}-\Phi_{ADA'D'}\epsilon_{B'C'}\epsilon_{BC} \\
& -\Phi_{BCB'C'}\epsilon_{A'D'}\epsilon_{AD}+\Phi_{BDB'D'}\epsilon_{A'C'}\epsilon_{AC} \\
& -2\Upsilon(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'}-\epsilon_{BC}\epsilon_{B'C'}\epsilon_{AD}\epsilon_{A'D'}) \qquad (173)
\end{aligned}
$$

Converting into tensors gives

$$\begin{aligned} C_{ABCDA'B'C'D'} &= R_{abcd} + (\Phi_{ac}g_{db} - \Phi_{ad}g_{bc} - \Phi_{bc}g_{ad} + \Phi_{bd}g_{ac}) - 2\Upsilon(g_{ac}g_{bd} - g_{bc}g_{ad}) \\ &= R_{abcd} - \frac{1}{2}(R_{ac}g_{db} - R_{ad}g_{bc} - R_{bc}g_{ad} + R_{bd}g_{ac}) \\ &\quad + \frac{1}{8}R(g_{ac}g_{db} - g_{ad}g_{bc} - g_{bc}g_{ad} + g_{bd}g_{ac}) - \frac{R}{12}(g_{ac}g_{bd} - g_{bc}g_{ad}) \\ &= R_{abcd} - \frac{1}{2}(R_{ac}g_{db} - R_{ad}g_{bc} - R_{bc}g_{ad} + R_{bd}g_{ac}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R \end{aligned} \tag{174}$$

This agrees with the definition of the Weyl tensor

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R \tag{175}$$

so we can identify

$$C_{ABCDA'B'C'D'} = C_{abcd} \tag{176}$$

Worked exercise:

Use that C_{abcd} only differes from R_{abcd} by terms involving the Ricci tensor and the Ricci scalar, that C_{abcd} have the same symmetries as R_{abcd}:

$$R_{abcd} = -R_{abdc} = -R_{bacd} \tag{177}$$

and that

$$C_{acb}{}^{c} = 0$$

to determine C_{abcd}.

Solution:

The most general expression for C_{abcd} involving R_{abcd}, R_{ab}, R and g_{ab} is

$$\begin{aligned} C_{abcd} &= R_{abcd} + [C_1R_{ab}g_{cd} + C_2R_{ac}g_{bd} + C_3R_{ad}g_{bc}] + [C_4R_{bc}g_{ad} + C_5R_{bd}g_{ac}] + C_6R_{cd}g_{ab} \\ &\quad + R(C_7g_{ab}g_{cd} + C_8g_{ac}g_{bd} + C_9g_{ad}g_{bc}). \end{aligned} \tag{178}$$

Using

$$\begin{aligned} C_{abcd} &= -C_{bacd} \\ &= R_{abcd} - [C_1 R_{ab} g_{cd} + C_2 R_{bc} g_{ad} + C_3 R_{bd} g_{ac}] - [C_4 R_{ac} g_{bd} + C_5 R_{ad} g_{bc}] - C_6 R_{cd} g_{ab} \\ &\quad - R(C_7 g_{ab} g_{cd} + C_8 g_{bc} g_{ad} + C_9 g_{bd} g_{ac}) \end{aligned} \tag{179}$$

which implies $C_1 = 0$ and $C_2 = -C_4$ and $C_3 = -C_5$ and $C_6 = 0$ and $C_7 = 0$ and $C_8 = -C_9$. So that

$$\begin{aligned} C_{abcd} &= R_{abcd} + C_2(R_{ac} g_{bd} - R_{bc} g_{ad}) + C_3(R_{ad} g_{bc} - R_{bd} g_{ac}) \\ &\quad + C_8 R(g_{ac} g_{bd} - g_{ad} g_{bc}) \end{aligned} \tag{180}$$

Now using

$$\begin{aligned} C_{abcd} &= -C_{abdc} \\ &= R_{abcd} - C_2(R_{ad} g_{bc} - R_{bd} g_{ac}) - C_3(R_{ac} g_{bd} - R_{bc} g_{ad}) \\ &\quad - C_8 R(g_{ad} g_{bc} - g_{ac} g_{bd}) \end{aligned} \tag{181}$$

So that $C_2 = -C_3$ and so

$$\begin{aligned} C_{abcd} &= R_{abcd} + C_2(R_{ac} g_{bd} - R_{bc} g_{ad} - R_{ad} g_{bc} + R_{bd} g_{ac}) \\ &\quad + C_8 R(g_{ac} g_{bd} - g_{ad} g_{bc}). \end{aligned} \tag{182}$$

Now using $C_{adc}{}^{d} = 0$ we obtain

$$0 = R_{ac} + C_2(4R_{ac} - R_{ac} - R_{ac} + R g_{ac}) + C_8 R(4g_{ac} - g_{ac})$$

or

$$0 = (1 + 2C_2) R_{ac} + (C_2 + 3C_8) R g_{ac}.$$

So that

$$C_{abcd} \;=\; R_{abcd} - \frac{1}{2}(R_{ac}g_{bd} - R_{bc}g_{ad} - R_{ad}g_{bc} + R_{bd}g_{ac}) + \frac{1}{6}R(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{183}$$

$\square$

The dual of the Weyl tensor is

$$C^*{}_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef}C_{abef}$$

In spinor notation

$$\begin{aligned} C^*_{ABCDA'B'C'D'} &= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'})C_{ABEFA'B'E'F'} \\ &= \frac{1}{2}i(\epsilon_{CD}\epsilon^{EF}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'} - \epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'D'}\epsilon^{E'F'}) \\ &\quad \Psi_{ABEF}\epsilon_{A'B'}\epsilon_{E'F'} + \overline{\Psi}_{A'BE'F'}\epsilon_{AB}\epsilon_{EF} \\ &= i(\overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} - \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}) \end{aligned} \tag{184}$$

Therfore

$$C_{\ abcd} + iC^*{}_{abcd} = 2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} \tag{185}$$

so that $C_{\ abcd}$ and Ψ_{ABCD} are equivalent,

$$C_{\ abcd} = Re\ (2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}). \tag{186}$$

As Ψ_{ABCD} is totally symetric we can decompose it as

Petrov classification

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}. \tag{187}$$

There are six distict cases which constitute the so-called Petrov classification

Type I or $\{1,1,1,1\}$. None of the four pricipal null directions coincide.

Type II or $\{2,1,1\}$. Two directions coincide.

Type D or $\{2,2\}$. Two are two different pairs of repeated null directions.

Type III or $\{3,1\}$. Three pricipal null directions coincide.

Type N or $\{4\}$. All four pricipal null directions coincide.

Type O The Weyl tensor vanishes and spacetime is conformally flat.

Spinor Covariant Derivative

$$(\nabla_{AA'}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_t + \partial_z & \partial_x - i\partial_y \\ \partial_x i + \partial_y & \partial_t - \partial_z \end{pmatrix} \tag{188}$$

where the ∂_a are derivatives with respect to inertial coordinates.

0.1.5 Curvature in spinors

$$u^a R_{abcd} = 2\nabla_{[c}\nabla_{d]} u_b \tag{189}$$

It is easily seen that $\nabla_{[c}\nabla_{d]}$ annihiltes all scalar fields:

$$\begin{aligned} \nabla_{[c}\nabla_{d]}\phi &= \nabla_c \partial_d \phi - \nabla_d \partial_c \phi \\ &= \partial_c \partial_d \phi - \partial_d \partial_c \phi + (\Gamma^e_{cd} - \Gamma^e_{dc})\partial_e \phi = 0. \end{aligned} \tag{190}$$

$$\begin{aligned} u^a R_{abcd} &= 2\nabla_{[c}\nabla_{d]}(u_{\hat{b}} e^{\hat{b}}{}_b) \\ &= 2u_{\hat{b}} \nabla_{[c}\nabla_{d]} e^{\hat{b}}{}_b \quad (\text{since } \nabla_{[c}\nabla_{d]} u_{\hat{b}} = 0) \\ &= 2(u^a e_{\hat{b}a}) \nabla_{[c}\nabla_{d]} e^{\hat{b}}{}_b \end{aligned} \tag{191}$$

It follows that

$$R_{abcd} = 2e_{\hat{b}a} \nabla_{[c}\nabla_{d]} e^{\hat{b}}{}_b. \tag{192}$$

$$2\nabla_{[c}\nabla_{d]} = \nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'} \tag{193}$$

The spinor equivalent is

$$
\begin{aligned}
R_{ABCDA'B'C'D'} &= 2\epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}\nabla_{[c}\nabla_{d]}\left(\epsilon^{\hat{B}}{}_{B}\epsilon^{\hat{B}'}{}_{B'}\right) \\
&= \epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}(\nabla_c\nabla_d - \nabla_d\nabla_c)\left(\epsilon^{\hat{B}}{}_{B}\epsilon^{\hat{B}'}{}_{B'}\right) \\
&= \epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}[\nabla_c(\epsilon^{\hat{B}'}{}_{B'}\nabla_d\epsilon^{\hat{B}}{}_{B} + \epsilon^{\hat{B}}{}_{B}\nabla_d\epsilon^{\hat{B}'}{}_{B'}) \\
&\quad - \nabla_d(\epsilon^{\hat{B}'}{}_{B'}\nabla_c\epsilon^{\hat{B}}{}_{B} + \epsilon^{\hat{B}}{}_{B}\nabla_c\epsilon^{\hat{B}'}{}_{B'}) \\
&= \epsilon_{\hat{B}A}\epsilon_{\hat{B}'A'}\epsilon^{\hat{B}'}{}_{B'}(\nabla_c\nabla_d - \nabla_d\nabla_c)\epsilon^{\hat{B}}{}_{B} + c.c. \\
&= 2\epsilon_{\hat{B}A}\epsilon_{B'A'}\nabla_{[c}\nabla_{d]}\epsilon^{\hat{B}}{}_{B} + c.c. \qquad (194)
\end{aligned}
$$

We define

$$
\begin{aligned}
\square_{CD} &= \epsilon^{C'D'}\nabla_{[CC'}\nabla_{DD']} \\
&= \frac{1}{2}\epsilon^{C'D'}(\nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'}) \\
&= \frac{1}{2}(\nabla_{CC'}\nabla_{D}{}^{C'} + \nabla_{DD'}\nabla_{C}{}^{D'}) \\
&= \nabla_{C'(C}\nabla_{D)}{}^{C'}. \qquad (195)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{CC'}\nabla_{DD'} &= \nabla_{C'(C}\nabla_{D)D'} + \frac{1}{2}\epsilon_{CD}\nabla_{C'E}\nabla^{E}{}_{D'} \\
&= \nabla_{(C'(C}\nabla_{D)D')} + \frac{1}{2}\epsilon_{C'D'}\nabla_{E'(C}\nabla_{D)}{}^{E'} \\
&\quad + \frac{1}{2}\epsilon_{CD}\nabla_{E(C'}\nabla_{D')}{}^{E} + \frac{1}{4}\epsilon_{CD}\epsilon_{C'D'}\nabla_{EE'}\nabla^{EE'} \qquad (196)
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'} \\
&= \nabla_{(C'(C}\nabla_{D)D')} + \frac{1}{2}\epsilon_{C'D'}\nabla_{E'(C}\nabla_{D)}{}^{E'} \\
&\quad + \frac{1}{2}\epsilon_{CD}\nabla_{E(C'}\nabla_{D')}{}^{E} - \frac{1}{4}\epsilon_{CD}\epsilon_{C'D'}\nabla_{EE'}\nabla^{EE'} \\
&\quad - \nabla_{(D'(D}\nabla_{C)C')} - \frac{1}{2}\epsilon_{D'C'}\nabla_{E'(D}\nabla_{C)}{}^{E'} \\
&\quad - \frac{1}{2}\epsilon_{DC}\nabla_{E(D'}\nabla_{C')}{}^{E} - \frac{1}{4}\epsilon_{DC}\epsilon_{D'C'}\nabla_{EE'}\nabla^{EE'} \qquad (197)
\end{aligned}
$$

Or

$$
\begin{aligned}
\nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'} &= \epsilon_{C'D'}\nabla_{E'(C}\nabla_{D)}{}^{E'} + \epsilon_{CD}\nabla_{E(D'}\nabla_{C')}{}^{E} \\
&= \epsilon_{C'D'}\square_{CD} + \epsilon_{CD}\square_{C'D'}.
\end{aligned} \tag{198}
$$

Consider the term

$$
\epsilon_{\hat{B}A}\square_{CD}\epsilon^{\hat{B}}{}_{B} \tag{199}
$$

This is obviously symmetric in CD and from

$$
\begin{aligned}
0 &= \square_{CD}(\epsilon_{\hat{B}A}\epsilon^{\hat{B}}{}_{B}) \\
&= \epsilon_{\hat{B}A}\square_{CD}\epsilon^{\hat{B}}{}_{B} + \epsilon^{\hat{B}}{}_{B}\square_{CD}\epsilon_{\hat{B}A} \\
&= \epsilon_{\hat{B}A}\square_{CD}\epsilon^{\hat{B}}{}_{B} - \epsilon_{\hat{B}B}\square_{CD}\epsilon^{\hat{B}}{}_{A}
\end{aligned} \tag{200}
$$

we see it is symmeric in AB. We decompose it into symmetric spinors

$$
\begin{aligned}
\epsilon_{\hat{B}A}\square_{CD}\epsilon^{\hat{B}}{}_{B} &= \frac{1}{3}\epsilon_{\hat{B}A}(\square_{CD}\epsilon^{\hat{B}}{}_{B} + \square_{DB}\epsilon^{\hat{B}}{}_{C} + \square_{BC}\epsilon^{\hat{B}}{}_{D}) \\
&\quad + \frac{1}{3}\epsilon_{\hat{B}A}(\square_{CD}\epsilon^{\hat{B}}{}_{B} - \square_{BD}\epsilon^{\hat{B}}{}_{C}) \\
&\quad + \frac{1}{3}\epsilon_{\hat{B}A}(\square_{CD}\epsilon^{\hat{B}}{}_{B} - \square_{CB}\epsilon^{\hat{B}}{}_{D}) \\
&= \epsilon_{\hat{B}A}\square_{(CD}\epsilon^{\hat{B}}{}_{B)} - \frac{1}{3}\epsilon_{\hat{B}A}\epsilon_{CB}\square_{D}{}^{F}\epsilon^{\hat{B}}{}_{F} - \frac{1}{3}\epsilon_{\hat{B}A}\epsilon_{DB}\square_{C}{}^{F}\epsilon^{\hat{B}}{}_{F} \\
&= \epsilon_{\hat{B}A}\square_{(CD}\epsilon^{\hat{B}}{}_{B)} - \frac{1}{3}\epsilon_{CB}\epsilon_{\hat{B}(A}\square_{D)}{}^{F}\epsilon^{\hat{B}}{}_{F} - \frac{1}{6}\epsilon_{AD}\epsilon_{CB}\epsilon_{\hat{B}E}\square^{EF}\epsilon^{\hat{B}}{}_{F} \\
&\quad - \frac{1}{3}\epsilon_{DB}\epsilon_{\hat{B}(A}\square_{C)}{}^{F}\epsilon^{\hat{B}}{}_{F} - \frac{1}{6}\epsilon_{AC}\epsilon_{DB}\epsilon_{\hat{B}E}\square^{EF}\epsilon^{\hat{B}}{}_{F} \\
&= \epsilon_{\hat{B}A}\square_{(CD}\epsilon^{\hat{B}}{}_{B)} - (\frac{1}{6}\epsilon_{\hat{B}E}\square^{EF}\epsilon^{\hat{B}}{}_{F})(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{CB}) \\
&\quad - \frac{1}{3}[\epsilon_{CB}\epsilon_{\hat{B}(A}\square_{D)}{}^{F}\epsilon^{\hat{B}}{}_{F} + \epsilon_{DB}\epsilon_{\hat{B}(A}\square_{C)}{}^{F}\epsilon^{\hat{B}}{}_{F}]
\end{aligned} \tag{201}
$$

Using the symmetry in AB and CD,

$$
\begin{aligned}
\epsilon_{\hat{B}A}\square_{CD}\epsilon^{\hat{B}}{}_{B} &= \epsilon_{\hat{B}(A}\square_{|(CD)|}\epsilon^{\hat{B}}{}_{B)} \\
&= \epsilon_{\hat{B}A}\square_{(CD}\epsilon^{\hat{B}}{}_{B)} - (\frac{1}{6}\epsilon_{\hat{B}E}\square^{EF}\epsilon^{\hat{B}}{}_{F})2\epsilon_{(A(C}\epsilon_{D)B)} \\
&= \Psi_{ABCD} - 2\Upsilon\epsilon_{(A(C}\epsilon_{D)B)}
\end{aligned} \tag{202}
$$

where

$$\Psi_{ABCD} = \epsilon_{\hat{B}A}\square_{(CD}\epsilon^{\hat{B}}{}_{B)} = \Psi_{(ABCD)} \tag{203}$$

and

$$\Upsilon = \frac{1}{6}\epsilon_{\hat{B}E}\square^{EF}\epsilon^{\hat{B}}{}_{F}. \tag{204}$$

Also we write

$$\epsilon_{\hat{B}A}\square_{C'D'}\epsilon^{\hat{B}}{}_{B} = \Phi_{ABC'D'} \tag{205}$$

which is symmetric in AB and $C'D'$. Combining all of this into (194)

$$R_{ABCDA'B'C'D'} = \epsilon_{A'B'}\epsilon_{C'D'}\left[\Psi_{ABCD} - 2\Upsilon\epsilon_{(A(C}\epsilon_{D)B)}\right] + \epsilon_{A'B'}\epsilon_{CD}\Phi_{ABC'D'} + c.c. \tag{206}$$

0.1.6 Spinor Form of the Ricci Identies

Recall the covariant derivative satisfies the product rule

$$\nabla_a(T_A S_B) = T_A\nabla_a S_B + S_B\nabla_a T_A. \tag{207}$$

Then

$$\begin{aligned}(\nabla_a\nabla_b - \nabla_b\nabla_a)(T_A S_B) &= \nabla_a(T_A\nabla_b S_B + S_B\nabla_b T_A) \\ &\quad -\nabla_b(T_A\nabla_a S_B + S_B\nabla_a T_A) \\ &= T_A\nabla_a\nabla_b S_B + S_B\nabla_a\nabla_b T_A \\ &\quad -T_A\nabla_b\nabla_a S_B + S_B\nabla_b\nabla_a T_A \\ &= 2T_A\nabla_{[a}\nabla_{b]}S_B + 2S_B\nabla_{[a}\nabla_{b]}T_A \end{aligned} \tag{208}$$

$$2\nabla_{[a}\nabla_{b]} = \epsilon_{A'B'}\square_{AB} + \epsilon_{AB}\square_{A'B'}$$

$$\square_{AB}(\phi_{\mathcal{C}}\chi_{\mathcal{D}}) = \phi_{\mathcal{C}}\square_{AB}\chi_{\mathcal{D}} + \chi_{\mathcal{D}}\square_{AB}\phi_{\mathcal{C}} \tag{209}$$

We consider the self-dual null bivector

$$T^{ab} = \xi^A \xi^B \xi^{A'B'} \tag{210}$$

The Ricci identity says

$$2\nabla_{[a}\nabla_{b]} T^{cd} = R_{abe}{}^{c} T^{ed} + R_{abe}{}^{d} T^{ce} \tag{211}$$

Using (208)

$$2\xi^C \epsilon^{C'D'} \nabla_{[a}\nabla_{b]} \xi^D + 2\xi^D \epsilon^{C'D'} \nabla_{[a}\nabla_{b]} \xi^C = R_{abEE'}{}^{CC'} \xi^E \xi^D \epsilon^{E'D'} + R_{abEE'}{}^{DD'} \xi^C \xi^E \epsilon^{C'E'} \tag{212}$$

or

$$4\epsilon^{C'D'} \xi^{(C} \nabla_{[a}\nabla_{b]} \xi^{D)} = R_{abEE'}{}^{CC'} \xi^E \xi^D \epsilon^{E'D'} + R_{abEE'}{}^{DD'} \xi^C \xi^E \epsilon^{C'E'} \tag{213}$$

Consider

$$\begin{aligned} \epsilon_{C'D'} R_{AA'BB'EE'}{}^{CC'} \epsilon^{E'D'} &= \delta_{C'}^{E'} (X_{ABE}{}^{C} \epsilon_{A'B'} \epsilon_{E'}{}^{C'} + \Phi_{ABE'}{}^{C'} \epsilon_{A'B'} \epsilon_{E}{}^{C} \\ &\quad + \Phi_{A'B'E}{}^{C} \epsilon_{AB} \epsilon_{E'}{}^{C'} + \overline{X}_{A'B'E'}{}^{C'} \epsilon_{AB} \epsilon_{E}{}^{C}) \\ &= 2X_{ABE}{}^{C} \epsilon_{A'B'} + \Phi_{ABC'}{}^{C'} \epsilon_{A'B'} \epsilon_{E}{}^{C} \\ &\quad + 2\Phi_{A'B'E}{}^{C} \epsilon_{AB} + \overline{X}_{A'B'C'}{}^{C'} \epsilon_{AB} \epsilon_{E}{}^{C} \\ &= 2X_{ABE}{}^{C} \epsilon_{A'B'} + 2\Phi_{A'B'E}{}^{C} \epsilon_{AB} \end{aligned} \tag{214}$$

where we used $\Phi_{ABC'}{}^{C'} = \epsilon^{C'D'} \Phi_{ABC'D'} = 0$, and $\overline{X}_{A'B'C'}{}^{C'} = \epsilon^{C'D'} \overline{X}_{A'B'C'D'} = 0$. We also have

$$\begin{aligned} \epsilon_{C'D'} R_{AA'BB'EE'}{}^{DD'} \epsilon^{C'E'} &= \epsilon_{D'C'} R_{AA'BB'EE'}{}^{DD'} \epsilon^{E'C'} \\ &= \epsilon_{C'D'} R_{AA'BB'EE'}{}^{DC'} \epsilon^{E'D'} \\ &= 2X_{ABE}{}^{D} \epsilon_{A'B'} + 2\Phi_{A'B'E}{}^{D} \epsilon_{AB} \end{aligned} \tag{215}$$

where in the first step we made the replacements $\epsilon_{C'D'} = -\epsilon_{D'C'}$ $\epsilon^{C'E'} = -\epsilon^{E'C'}$, and in the second we swapped the dummy variables C' and D', then comparison with the LHS of (214) gives the last line. Substitution of these results into the contraction of (213) with $\epsilon_{C'D'}$ gives

$$2\xi^{(D}\nabla_{[a}\nabla_{b]}\xi^{C)} = \xi^{(D}(\epsilon_{A'B'}X_{ABE}{}^{C)} + \epsilon_{AB}\Phi_{A'B'E}{}^{C)})\xi^E \tag{216}$$

If we have

$$\phi^{(A}\chi^{B)} = 0$$

then either $\phi^A = 0$ or $\chi^B = 0$

We obtain

$$2\nabla_{[a}\nabla_{b]}\xi^C = (\epsilon_{A'B'}X_{ABD}{}^C + \epsilon_{AB}\Phi_{A'B'D}{}^C)\xi^D \tag{217}$$

or

$$\epsilon_{A'B'}\Box_{AB}\xi^C + \epsilon_{AB}\Box_{A'B'}\xi^C = (\epsilon_{A'B'}X_{ABE}{}^C + \epsilon_{AB}\Phi_{A'B'E}{}^C)\xi^E \tag{218}$$

contracting with $\epsilon^{A'B'}$ gives

$$\Box_{AB}\xi_C = X_{ABCD}\xi^D \tag{219}$$

contracting with ϵ^{AB} gives

$$\Box_{A'B'}\xi_C = \Phi_{A'B'CD}\xi^D \tag{220}$$

If we substitute $X_{ABCD} = \Psi_{ABCD} + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})$ in (219) we obtain

$$\begin{aligned}\Box_{AB}\xi_C &= \Psi_{ABCD}\xi^D + \Upsilon(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\xi^D \\ &= \Psi_{ABCD}\xi^D - \Upsilon(\xi_B\epsilon_{AC} + \xi_A\epsilon_{BC})\end{aligned} \tag{221}$$

Symmetrising over ABC gives

$$\Box_{(AB}\xi_{C)} = \Psi_{ABCD}\xi^D \tag{222}$$

Contracting (219) with ϵ^{BC} and using $\epsilon^{BC}X_{ABCD} = 3\Upsilon\epsilon_{AD}$ we obtain

$$\Box_{AB}\xi^B = -3\Upsilon\xi_A \tag{223}$$

We collect these formula together:

$$\begin{aligned}
&\Box_{AB}\xi_C = \Psi_{ABCD}\xi^D - 2\Upsilon\xi_{(A}\epsilon_{B)C}\\
&\Box_{(AB}\xi_{C)} = \Psi_{ABCD}\xi^D\\
&\Box_{AB}\xi^B = -3\Upsilon\xi_A\\
&\Box_{A'B'}\xi_C = \xi^D\Phi_{CDA'B'}.
\end{aligned} \tag{224}$$

These are the spinor forms for the Ricci identities.

0.1.7 Einstein's equations

The vacuum field equations are

$$R_{ab} = 0. \tag{225}$$

or in terms of spinors

$$6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'} = 0. \tag{226}$$

Symmetrising (226) over AB implies

$$\Phi_{ab} = \Phi_{ABA'B'} = 0 \quad \text{then} \quad \Upsilon = 0. \tag{227}$$

Obviously (231) implies (226), thus they are equivalent.

If a cosmological constant is included in the field equations are

$$R_{ab} - \frac{1}{2}g_{ab}R - \Lambda g_{ab} = 0. \tag{228}$$

Contracting gives $R = -4\Lambda$ which upon substitution back into the field equations gives

$$R_{ab} = -\Lambda g_{ab} \tag{229}$$

which in spinor form becomes

$$6\Upsilon\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'} = -\Lambda\epsilon_{AB}\epsilon_{A'B'} \tag{230}$$

and is equivalent to

$$\Phi_{ab} = \Phi_{ABA'B'} = 0, \qquad \Upsilon = -\frac{1}{6}\Lambda. \tag{231}$$

In the general case, where sources are present, the fielf equations with cosmological term are

$$G_{ab} - \Lambda g_{ab} = 8\pi G T_{ab} \tag{232}$$

using $R = 24\Upsilon$, this can be written

$$\Phi_{ab} + (3\Upsilon + \frac{1}{2}\Lambda)g_{ab} = -4\pi G T_{ab} \tag{233}$$

which we rewrite as

$$\Phi_{ab} + (3\Upsilon + \frac{1}{2}\Lambda)g_{ab} = -4\pi G[(T_{ab} - \frac{1}{4}T^c_c g_{ab}) + \frac{1}{4}T^c_c g_{ab}] \tag{234}$$

As Φ_{ab} represents the trace-free part of the RHS we have

$$\Phi_{ab} = -4\pi G(T_{ab} - \frac{1}{4}T^c_c g_{ab}) \tag{235}$$

and

$$\Upsilon = -\frac{1}{3}\pi G T^c_c - \frac{1}{6}\Lambda \tag{236}$$

Recall that

$$X_{AB}{}^{B}{}_{C} = 3\Upsilon\epsilon_{AC}$$

thus the vanishing of Υ which occurs for vacuum field equations implies that X_{ABCD} is symmetric in BC and since it is symetric in AB and CD, it is symmetric in all its indices. The curvature tensor $R_{ABCDA'B'C'D'}$ is given by the Weyl tensor (in accordance with) $C_{ABCDA'B'C'D'}$. In a vacuum the curvature can be fully characterised by a totally symmetric four-index spinor.

0.1.8 Spinor form of the Bianchi identity

Recall the Bianchi identity

$$\nabla_{[a}R_{bc]de} = 0. \tag{237}$$

Consider

$$\begin{aligned} \nabla^a {}^*R_{abcd} &= \frac{1}{2}\epsilon_{ab}{}^{ef}\nabla^a R_{efcd} \\ &= \frac{1}{2}\epsilon^a{}_b{}^{ef}\nabla_{[a}R_{ef]cd} \end{aligned} \tag{238}$$

This proves $\nabla_{[a}R_{bc]de} = 0$ implies $\nabla^a {}^*R_{abcd} = 0$. Now consider

$$\begin{aligned} \nabla_{[a}R_{bc]de} &= \frac{1}{3!}\delta^{efg}_{abc}\nabla_f R_{ghde} \\ &= \frac{1}{3!}\epsilon_{iabc}\epsilon^{ifgh}\nabla_f R_{ghde} \\ &= -\frac{2}{3!}\epsilon^i{}_{abc}\nabla^f\left(\frac{1}{2}\epsilon_{fi}{}^{gh}R_{ghde}\right) \\ &= -\frac{2}{3!}\epsilon^i{}_{abc}\nabla^f {}^*R_{fide} \end{aligned} \tag{239}$$

This proves $\nabla^a {}^*R_{abcd} = 0$ implies $\nabla_{[a}R_{bc]de} = 0$, therefore the Bianchi identity is equivalent to

$$\nabla^a {}^*R_{abcd} = 0 \tag{240}$$

From (**??**) this in spinor form is

$$\begin{aligned} \nabla^a {}^*R_{abcd} &= -i\nabla^{AA'}[X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\ &\qquad - \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} - \overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}] \\ &= 0. \end{aligned} \tag{241}$$

Or

$$\epsilon_{C'D'}\nabla^A{}_{B'}X_{ABCD} + \epsilon_{CD}\nabla^A{}_{B'}\Phi_{ABC'D} - \epsilon_{C'D'}\nabla_B{}^{A'}\Phi_{CDA'B'} - \epsilon_{CD}\nabla_B{}^{A'}\overline{X}_{A'B'C'D'} = 0 \tag{242}$$

Contracting with $\epsilon^{C'D'}$ gives

$$\nabla^A{}_{B'} X_{ABCD} = \nabla_B{}^{A'} \Phi_{CDA'B'}. \tag{243}$$

Contracting with ϵ^{CD} gives its complex conjugate. Thus (243) is the spinor form of the Bianchi identity.

0.1.9 Newman-Penrose Formalism in Spinor Form

Newman-Penrose scalars in terms of spinors

Now

$$\begin{aligned} m^a \nabla l_a &= o^A \bar{\imath}^{A'} \nabla(o_A \bar{o}_{A'}) \\ &= o^A \bar{\imath}^{A'} (o_A \nabla \bar{o}_{A'} + \bar{o}_{A'} \nabla o_A) \\ &= o^A \nabla o_A. \end{aligned} \tag{244}$$

Note $\nabla(o^A \imath_A) = 0$ implies

$$o^A \nabla \imath_A = \imath^A \nabla o_A \tag{245}$$

Now consider

$$\begin{aligned} \frac{1}{2}(n^a \nabla l_a - \overline{m}^a \nabla m_a) &= \frac{1}{2}(\imath^A \bar{\imath}^{A'} \nabla(o_A \bar{o}_{A'}) - \imath^A \bar{o}^{A'} \nabla(o_A \bar{\imath}_{A'})) \\ &= \frac{1}{2}[\imath^A \bar{\imath}^{A'} (o_A \nabla \bar{o}_{A'} + \bar{o}_{A'} \nabla o_A) - \imath^A \bar{o}^{A'} (o_A \nabla \bar{\imath}_{A'} + \bar{\imath}_{A'} \nabla o_A)] \\ &= \frac{1}{2}(\bar{\imath}^{A'} \nabla \bar{o}_{A'} + \imath^A \nabla o_A - \bar{o}^{A'} \nabla \bar{\imath}_{A'} + \imath^A \nabla o_A) \\ &= \imath^A \nabla o_A = o^A \nabla \imath_A. \end{aligned} \tag{246}$$

$$\begin{aligned} -\overline{m}^a \nabla n_a &= -\imath^A \bar{o}^{A'} \nabla(\imath_A \bar{\imath}_{A'}) \\ &= -\imath^A \bar{o}^{A'} (\imath_A \nabla \bar{\imath}_{A'} + \bar{\imath}_{A'} \nabla \imath_A) \\ &= \imath^A \nabla \imath_A. \end{aligned} \tag{247}$$

Altogether we have

$$
\begin{aligned}
m^a \nabla l_a &= o^A \nabla o_A \\
\frac{1}{2}(n^a \nabla l_a - \overline{m}^a \nabla m_a) &= o^A \nabla \imath_A = \imath^A \nabla o_A \\
-\overline{m}^a \nabla n_a &= \imath^A \nabla \imath_A
\end{aligned} \tag{248}
$$

Recall $D = l^a \nabla_a$, $\Delta = n^a \nabla_a$, $\delta = m^a \nabla_a$, and $\overline{\delta} = \overline{m}^a \nabla_a$. The Newman-Penrose scalars can then be written:

$$
\begin{array}{lllllllll}
\kappa &=& o^A D o_A & \epsilon &=& o^A D \imath_A & \pi &=& \imath^A D \imath_A \\
\sigma &=& o^A \delta o_A & \beta &=& o^A \delta \imath_A & \mu &=& \imath^A \delta \imath_A \\
\rho &=& o^A \overline{\delta} o_A & \alpha &=& o^A \overline{\delta} \imath_A & \lambda &=& \imath^A \overline{\delta} \imath_A \\
\tau &=& o^A \Delta o_A & \gamma &=& o^A \Delta \imath_A & \nu &=& \imath^A \Delta \imath_A
\end{array} \tag{249}
$$

Consider $D o_A$, we can write

$$
D o_A = a o_A + b\, \imath_A \tag{250}
$$

we can determine a and b by contracting with $\imath^A$ and o^A respectively,

$$
a = \imath^A D o_A, \qquad b = -o^A D o_A, \tag{251}
$$

so that, using (249),

$$
D o_A = \epsilon \nabla o_A - \kappa \imath_A. \tag{252}
$$

We can derive the following in a similar manner:

$$
\begin{aligned}
D o_A &= \epsilon o_A - \kappa \imath_A, \quad D \imath_A = \pi o_A - \epsilon \imath_A, \\
\Delta o_A &= \gamma o_A - \tau \imath_A, \quad \Delta \imath_A = \nu o_A - \gamma \imath_A, \\
\delta o_A &= \beta o_A - \sigma \imath_A, \quad \delta \imath_A = \mu o_A - \beta \imath_A, \\
\overline{\delta} o_A &= \alpha o_A - \rho \imath_A, \quad \overline{\delta} \imath_A = \lambda o_A - \alpha \imath_A.
\end{aligned} \tag{253}
$$

Weyl tensor written in terms of spinors

We move to the Newman-Penrose components of the Weyl tensor. Consider $\Psi_0 = C_{abcd}l^a m^b l^c m^d$

$$\begin{aligned}
C_{abcd}l^a m^b l^c m^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})o^A\overline{o}^{A'}o^B\overline{\imath}^{B'}o^C\overline{o}^{C'}o^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}o^A o^B o^C o^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) + \ldots \epsilon_{AB}o^A o^B \ldots \\
&= \Psi_{ABCD}o^A o^B o^C o^D. \qquad (254)
\end{aligned}$$

Consider $\Psi_1 = C_{abcd}l^a m^b l^c n^d$

$$\begin{aligned}
C_{abcd}l^a m^b l^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})o^A\overline{o}^{A'}o^B\overline{\imath}^{B'}o^C\overline{o}^{C'}\imath^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}o^A o^B o^C \imath^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) \\
&= \Psi_{ABCD}o^A o^B o^C \imath^D. \qquad (255)
\end{aligned}$$

Consider $\Psi_2 = C_{abcd}l^a m^b \overline{m}^c n^d$

$$\begin{aligned}
C_{abcd}l^a m^b \overline{m}^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})o^A\overline{o}^{A'}o^B\overline{\imath}^{B'}\imath^C\overline{o}^{C'}\imath^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}o^A o^B \imath^C \imath^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) \\
&= \Psi_{ABCD}o^A o^B \imath^C \imath^D. \qquad (256)
\end{aligned}$$

Consider $\Psi_3 = C_{abcd}l^a n^b \overline{m}^c n^d$

$$\begin{aligned}
C_{abcd}l^a n^b \overline{m}^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})o^A\overline{o}^{A'}\imath^B\overline{\imath}^{B'}\imath^C\overline{o}^{C'}\imath^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}o^A \imath^B \imath^C \imath^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) + \ldots \epsilon_{CD}\imath^C\imath^D \\
&= \Psi_{ABCD}o^A \imath^B \imath^C \imath^D. \qquad (257)
\end{aligned}$$

Lastly, consider $\Psi_4 = C_{abcd}\overline{m}^a n^b \overline{m}^c n^d$

$$\begin{aligned}
C_{abcd}\overline{m}^a n^b \overline{m}^c n^d &= (\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D}\epsilon_{AB}\epsilon_{CD})\imath^A\overline{o}^{A'}\imath^B\overline{\imath}^{B'}\imath^C\overline{o}^{C'}\imath^D\overline{\imath}^{D'} \\
&= \Psi_{ABCD}\imath^A \imath^B \imath^C \imath^D(\epsilon_{A'B'}\overline{o}^{A'}\overline{\imath}^{B'})(\epsilon_{C'D'}\overline{o}^{C'}\overline{\imath}^{D'}) \\
&= \Psi_{ABCD}\imath^A \imath^B \imath^C \imath^D. \qquad (258)
\end{aligned}$$

The Newman-Penrose components of the Weyl tensor written in terms of spinors are given by:

$$
\begin{aligned}
\Psi_0 &= \Psi_{ABCD} o^A o^B o^C o^D, \\
\Psi_1 &= \Psi_{ABCD} o^A o^B o^C \imath^D, \\
\Psi_2 &= \Psi_{ABCD} o^A o^B \imath^C \imath^D, \\
\Psi_3 &= \Psi_{ABCD} o^A \imath^B \imath^C \imath^D, \\
\Psi_4 &= \Psi_{ABCD} \imath^A \imath^B \imath^C \imath^D. \qquad (259)
\end{aligned}
$$

Ricci tensor written in terms of spinors

We move to the tetrad components of the Ricci tensor written in terms of spinors. Consider $\Phi_{00} = -\frac{1}{2} R_{ab} l^a l^b$

$$
\begin{aligned}
-\frac{1}{2} R_{ab} l^a l^b &= -\frac{1}{2}(6\Upsilon \epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}) o^A \overline{o}^{A'} o^B \overline{o}^{B'} \\
&= \Phi_{ABA'B'} o^A o^B \overline{o}^{A'} \overline{o}^{B'}. \qquad (260)
\end{aligned}
$$

Consider $\Phi_{01} = -\frac{1}{2} R_{ab} l^a m^b$

$$
\begin{aligned}
-\frac{1}{2} R_{ab} l^a m^b &= -\frac{1}{2}(6\Upsilon \epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}) o^A \overline{o}^{A'} o^B \overline{\imath}^{B'} \\
&= \Phi_{ABA'B'} o^A o^B \overline{o}^{A'} \overline{\imath}^{B'}. \qquad (261)
\end{aligned}
$$

Consider $\Phi_{02} = -\frac{1}{2} R_{ab} m^a m^b$

$$
\begin{aligned}
-\frac{1}{2} R_{ab} m^a m^b &= -\frac{1}{2}(6\Upsilon \epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}) o^A \overline{\imath}^{A'} o^B \overline{\imath}^{B'} \\
&= \Phi_{ABA'B'} o^A o^B \overline{\imath}^{A'} \overline{\imath}^{B'}. \qquad (262)
\end{aligned}
$$

Consider $\Phi_{10} = -\frac{1}{2} R_{ab} l^a \overline{m}^b = \overline{\Phi_{01}}$, hence

$$
\Phi_{10} = \Phi_{ABA'B'} o^A \imath^B \overline{o}^{A'} \overline{o}^{B'}. \qquad (263)
$$

Consider $\Phi_{20} = -\frac{1}{2} R_{ab} \overline{m}^a \overline{m}^b = \overline{\Phi_{02}}$, hence

$$\Phi_{20} = \Phi_{ABA'B'} i^A \imath^B \overline{o}^{A'} \overline{o}^{B'}. \tag{264}$$

Consider $\Phi_{11} = -\frac{1}{4} R_{ab}(l^a n^b + m^a \overline{m}^b)$

$$\begin{aligned}
-\frac{1}{4} R_{ab}(l^a n^b + m^a \overline{m}^b) &= -\frac{1}{4}(6\Upsilon \epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'})(o^A \overline{o}^{A'} \imath^B \overline{\imath}^{B'} + o^A \overline{\imath}^{A'} \imath^B \overline{o}^{B'}) \\
&= \frac{1}{2}\Phi_{ABA'B'}(o^A \imath^B \overline{o}^{A'} \overline{\imath}^{B'} + o^A \imath^B \overline{\imath}^{A'} \overline{o}^{B'}) \\
&\quad - \frac{3\Upsilon}{2}\epsilon_{AB}\epsilon_{A'B'}(o^A \imath^B \overline{o}^{A'} \overline{\imath}^{B'} + o^A \imath^B \overline{\imath}^{A'} \overline{o}^{B'}) \\
&= \Phi_{ABA'B'} o^A \imath^B \overline{o}^{A'} \overline{\imath}^{B'}.
\end{aligned} \tag{265}$$

where we used the symmetry in $A'B'$ of $\Phi_{ABA'B'}$. Consider $\Phi_{12} = -\frac{1}{2} R_{ab} n^a m^b$

$$\begin{aligned}
-\frac{1}{2} R_{ab} n^a m^b &= -\frac{1}{2}(6\Upsilon \epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}) \imath^A \overline{\imath}^{A'} o^B \overline{\imath}^{B'} \\
&= \Phi_{ABA'B'} \imath^A \overline{\imath}^{A'} o^B \overline{\imath}^{B'} \\
&= \Phi_{ABA'B'} o^A \imath^B \overline{\imath}^{A'} \overline{\imath}^{B'}.
\end{aligned} \tag{266}$$

where we used the symmetry in AB of $\Phi_{ABA'B'}$. Consider $\Phi_{21} = -\frac{1}{2} R_{ab} n^a \overline{m}^b = \overline{\Phi_{12}}$, hence

$$\Phi_{21} = \Phi_{ABA'B'} i^A \imath^B \overline{o}^{A'} \overline{\imath}^{B'}. \tag{267}$$

Consider $\Phi_{22} = -\frac{1}{2} R_{ab} n^a n^b$

$$\begin{aligned}
-\frac{1}{2} R_{ab} n^a n^b &= -\frac{1}{2}(6\Upsilon \epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}) \imath^A \overline{\imath}^{A'} \imath^B \overline{\imath}^{B'} \\
&= \Phi_{ABA'B'} \imath^A \overline{\imath}^{A'} \imath^B \overline{\imath}^{B'} \\
&= \Phi_{ABA'B'} \imath^A \imath^B \overline{\imath}^{A'} \overline{\imath}^{B'}.
\end{aligned} \tag{268}$$

Collecting these results together the tetrad components of the Ricci tensor written in terms of spinors are given by:

$$
\begin{aligned}
\Phi_{00} &= \Phi_{ABA'B'} o^A o^B \overline{o}^{A'} \overline{o}^{B'}, \\
\Phi_{01} &= \Phi_{ABA'B'} o^A o^B \overline{o}^{A'} \overline{\imath}^{B'}, \\
\Phi_{02} &= \Phi_{ABA'B'} o^A o^B \overline{\imath}^{A'} \overline{\imath}^{B'}, \\
\Phi_{10} &= \Phi_{ABA'B'} o^A \imath^B \overline{o}^{A'} \overline{o}^{B'}, \\
\Phi_{11} &= \Phi_{ABA'B'} o^A \imath^B \overline{o}^{A'} \overline{\imath}^{B'}, \\
\Phi_{12} &= \Phi_{ABA'B'} o^A \imath^B \overline{\imath}^{A'} \overline{\imath}^{B'}, \\
\Phi_{20} &= \Phi_{ABA'B'} \imath^A \imath^B \overline{o}^{A'} \overline{o}^{B'}, \\
\Phi_{21} &= \Phi_{ABA'B'} \imath^A \imath^B \overline{o}^{A'} \overline{\imath}^{B'}, \\
\Phi_{22} &= \Phi_{ABA'B'} \imath^A \imath^B \overline{\imath}^{A'} \overline{\imath}^{B'}.
\end{aligned}
\tag{269}
$$

Lorentz transformations

Class I transformation:

They correspond to

$$
(\hat{o}, \hat{\imath}) = (o, \imath + ao) \tag{270}
$$

we have

$$
\begin{aligned}
\hat{l} &= \hat{o}^A \overline{\hat{o}}^{A'} = o^A \overline{o}^{A'} \\
\hat{m} &= \hat{o}^A \overline{\hat{\imath}}^{A'} = o^A \overline{\imath}^{A'} + \overline{a} o^A \overline{o}^{A'} \\
\hat{n} &= \hat{\imath}^A \overline{\hat{\imath}}^{A'} = \imath^A \overline{\imath}^{A'} + a o^A \overline{\imath}^{A'} + \overline{a} \imath^A \overline{o}^{A'} + a\overline{a} o^A \overline{o}^{A'}
\end{aligned}
\tag{271}
$$

or

$$
\hat{l} = l, \quad \hat{m} = m + \overline{a} l, \quad \hat{n} = n + am + \overline{am} + a\overline{a} l. \tag{272}
$$

Class II transformation:

They correspond to

$$
(\hat{o}, \hat{\imath}) = (o + b\imath, \imath) \tag{273}
$$

we have

$$
\begin{aligned}
\hat{n} &= \hat{\imath}^A \bar{\hat{\imath}}^{A'} = \imath^A \bar{\imath}^{A'} \\
\hat{m} &= \hat{o}^A \bar{\hat{\imath}}^{A'} = o^A \bar{\imath}^{A'} + b\imath^A \bar{\imath}^{A'} \\
\hat{l} &= \hat{o}^A \bar{\hat{o}}^{A'} = o^A \bar{o}^{A'} + \bar{b} o^A \bar{\imath}^{A'} + b\imath^A \bar{o}^{A'} + b\bar{b}\imath^A \bar{\imath}^{A'}
\end{aligned} \tag{274}
$$

or

$$
\hat{n} = n, \quad \hat{m} = m + bn, \quad \hat{l} = l + \bar{b}m + b\overline{m} + b\bar{b}l. \tag{275}
$$

Class III transformation:

They correspond to

$$
(\hat{o}, \hat{\imath}) = (\lambda o, \lambda^{-1}\imath), \qquad \lambda = c\exp(i\theta), \tag{276}
$$

we have

$$
\hat{l} = c^2 l, \quad \hat{n} = c^{-2} n, \quad , \quad \hat{m} = e^{2i\theta} m. \tag{277}
$$

Transformation of Weyl scalars

The Weyl scalars transform under class I transformations as

$$
\begin{aligned}
\hat{\Psi}_0 &= \Psi_0, \\
\hat{\Psi}_1 &= \Psi_1 + a\Psi_0, \\
\hat{\Psi}_2 &= \Psi_2 + 2a\Psi_1 + a^2\Psi_0, \\
\hat{\Psi}_3 &= \Psi_3 + 3a\Psi_2 + 3a^2\Psi_1 + a^3\Psi_0, \\
\hat{\Psi}_4 &= \Psi_4 + 4a\Psi_3 + 6a^2\Psi_2 + 4a^3\Psi_1 + a^4\Psi_0.
\end{aligned} \tag{278}
$$

The Weyl scalars transform under class II transformations as

$$
\begin{aligned}
\hat{\Psi}_0 &= \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4, \\
\hat{\Psi}_1 &= \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4, \\
\hat{\Psi}_2 &= \Psi_2 + 2b\Psi_3 + b^2\Psi_4, \\
\hat{\Psi}_3 &= \Psi_3 + b\Psi_4, \\
\hat{\Psi}_4 &= \Psi_4.
\end{aligned} \tag{279}
$$

The Weyl scalars transform under class III transformations as

$$
\begin{aligned}
\tilde{\Psi}_0 &= c^4 e^{4i\theta}\Psi_0, \\
\tilde{\Psi}_1 &= c^2 e^{2i\theta}\Psi_1, \\
\tilde{\Psi}_2 &= \Psi_2, \\
\tilde{\Psi}_3 &= c^{-2} e^{-2i\theta}\Psi_3, \\
\tilde{\Psi}_4 &= c^{-4} e^{-4i\theta}\Psi_4.
\end{aligned} \tag{280}
$$

0.1.10 Petrov Classification

Recall

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}. \tag{281}$$

There are six distict cases which constitute the so-called Petrov classification

Type I or $\{1,1,1,1\}$. None of the four pricipal null directions coincide.

Type II or $\{2,1,1\}$. Two directions coincide.

Type D or $\{2,2\}$. Two are two different pairs of repeated null directions.

Type III or $\{3,1\}$. Three pricipal null directions coincide.

Type N or $\{4\}$. All four pricipal null directions coincide.

Type O The Weyl tensor vanishes and spacetime is conformally flat.

Petrov classification via scalars

The condition for $e_{\hat{0}}{}^a$ to be a principal null vector is

Type I is when $\Psi_0 = \Psi_4 = 0$ and $\Psi_1, \Psi_2, \Psi_3 \neq 0$

Condition for double:

Type II is when $\Psi_0 = \Psi_1 = \Psi_4 = 0$ and $\Psi_2, \Psi_3 \neq 0$

Condition for a pair of doubles:

Type D is when $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and $\Psi_2 \neq 0$

Condition for triple:

Type III is when $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$ and $\Psi_3 \neq 0$

Condition for quadruple:

Type N is when $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ and $\Psi_4 \neq 0$

Type O is when $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$.

Proof:

Assume spacetime is not conformally flat and not all Weyl scalars vanish. Let $\Psi_4 \neq 0$. If it happens to be zero in the chosen frame, we can make it non-zero by a rotation of class I. By a class II transformation, $\hat{\Psi}_0$ can be made to vanish if b is a root of the equation

$$b^4\Psi_4 + 4b^3\Psi_3 + 6b^2\Psi_2 + 4b\Psi_1 + \Psi_0 = 0 \tag{282}$$

This always has four roots and the correspondingnew directions of l,

$$l + \bar{b}m + b\overline{m} + b\bar{b}n, \tag{283}$$

are the principal null directions of the Weyl tensor.

We can easily derive from (279) that

$$\begin{aligned}
\frac{1}{4}\frac{d}{db}\hat{\Psi}_0(b) &= b^3\Psi_4 + 3b^2\Psi_3 + 3b\Psi_2 + \Psi_1 = \hat{\Psi}_1(b) \\
\frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b) &= b^2\Psi_4 + 2b\Psi_3 + \Psi_2 = \hat{\Psi}_2(b) \\
\frac{1}{2}\frac{d}{db}\hat{\Psi}_2(b) &= b\Psi_4 + \Psi_3 = \hat{\Psi}_3(b) \\
\frac{d}{db}\hat{\Psi}_3(b) &= \Psi_4 = \hat{\Psi}_4(b),
\end{aligned} \tag{284}$$

and that

$$\hat{\Psi}_0(b) = \Psi_4(b - b_1)(b - b_2)(b - b_3)(b - b_4). \tag{285}$$

We have

$$
\begin{aligned}
\hat{\Psi}_1(b) &= \frac{1}{4}\frac{d}{db}\hat{\Psi}_0(b) \\
&= \frac{\Psi_4}{4}\frac{d}{db}(b-b_1)(b-b_2)(b-b_3)(b-b_4) \\
&= \frac{\Psi_4}{4}[(b-b_2)(b-b_3)(b-b_4)+(b-b_1)(b-b_3)(b-b_4) \\
&\quad +(b-b_1)(b-b_2)(b-b_4)+(b-b_2)(b-b_3)(b-b_4)]
\end{aligned}
\tag{286}
$$

and

$$
\begin{aligned}
\hat{\Psi}_2(b) &= \frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b) \\
&= \frac{\Psi_4}{6}[(b-b_1)(b-b_2)+(b-b_1)(b-b_3)+(b-b_1)(b-b_4) \\
&\quad +(b-b_2)(b-b_3)+(b-b_2)(b-b_4)+(b-b_3)(b-b_4)
\end{aligned}
\tag{287}
$$

and

$$
\begin{aligned}
\hat{\Psi}_3(b) &= \frac{1}{2}\frac{d}{db}\hat{\Psi}_2(b) \\
&= \frac{\Psi_4}{4}(4b-b_1-b_2-b_3-b_4)
\end{aligned}
\tag{288}
$$

and finally

$$
\hat{\Psi}_4(b) = \frac{d}{db}\hat{\Psi}_3(b) = \Psi_4. \tag{289}
$$

a) **Petrov type I**. All four roots are distinct. Then a rotation of class II with parameter $b = b_1$ (say) we can make

$$
\hat{\Psi}_0 = b_1^4\Psi_4 + 4b_1^3\Psi_3 + 6b_1^2\Psi_2 + 4b_1\Psi_1 + \Psi_0 = 0.
$$

Taking $\hat{\Psi}_0$ as a fucntion of b, we have

For (286) the parameter $b = b_1$

$$
\hat{\Psi}_1 = \hat{\Psi}_1(b=b_1) = \frac{\Psi_4}{2}(b_1-b_2)(b_1-b_3)(b_1-b_4) \neq 0. \tag{290}
$$

For (287)

$$\hat{\Psi}_2(b) \;=\; \frac{\Psi_4}{6}[(b-b_1)(3b-b_2-b_3-b_4)+(b-b_2)(2b-b_3-b_4)+(b-b_3)(b-b_4)] \tag{291}$$

$$\hat{\Psi}_2=\hat{\Psi}_2(b=b_1)=\frac{\Psi_4}{6}[(b_1-b_2)(b_1-b_3)+(b_1-b_2)(b_1-b_4)+(b_1-b_3)(b_1-b_4)]. \tag{292}$$

As we have already seen

$$\hat{\Psi}_3(b)=\frac{\Psi_4}{4}(4b-b_1-b_2-b_3-b_4) \tag{293}$$

so that

$$\hat{\Psi}_3=\hat{\Psi}_3(b=b_1)=\frac{\Psi_4}{4}[(b_1-b_2)+(b_1-b_3)+(b_1-b_4)]. \tag{294}$$

We have that $\hat{\Psi}_1$ is guaranteed to be non-zero. If we have $Re(b_1)>Re(b_2),Re(b_3),Re(b_4)$ or $Im(b_1)>Im(b_2),Im(b_3),Im(b_4)$ then $\hat{\Psi}_2,\hat{\Psi}_3>0$.

A rotation of class I (which does not effect Ψ_0) we can make Ψ_4 vanish with appropriate value of parameter a

$$\hat{\hat{\Psi}}_4=\Psi_4+4a\hat{\Psi}_3+6a^2\hat{\Psi}_2+4a^3\hat{\Psi}_1=0. \tag{295}$$

We also have that $\hat{\Psi}_1$ is invariant under this rotation and so

$$\hat{\hat{\Psi}}_1=\hat{\Psi}_1\neq 0. \tag{296}$$

Only if the above polynomial (295) has three distinct roots can $\hat{\hat{\Psi}}_2$ and $\hat{\hat{\Psi}}_3$ be non-zero.

b) **Petrov type II**. Two roots coincide $b_1=b_2$ the other two different and distinct $b_1\neq b_3\neq b_4$ and $b_3\neq b_4$. We have

$$\hat{\Psi}_0(b)=\Psi_4(b-b_1)^2(b-b_3)(b-b_4) \tag{297}$$

and under a transformation of class II we have (using (286) with $b_1 = b_2$)

$$\hat{\Psi}_1(b) = \Psi_4(b-b_1)[2(b-b_3)(b-b_4) + (b-b_1)(b-b_3) + (b-b_1)(b-b_4)] \tag{298}$$

Setting $b = b_1$ we make Ψ_0, and Ψ_1 vanish simultaneously.

$$\hat{\Psi}_2 = \frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b)\Big|_{b=b_1} = \frac{2}{3}\Psi_4(b_1-b_3)(b_1-b_4) \neq 0. \tag{299}$$

and from (288)

$$\hat{\Psi}_3 = \hat{\Psi}_3(b=b_1) = \frac{\Psi_4}{4}(2b_1 - b_3 - b_4) \tag{300}$$

Under a transformation of class I with parameter a we uneffect the vanishing of Ψ_0 and Ψ_1 while make Ψ_4 vanish if we chose a to be a root of

$$\hat{\hat{\Psi}}_4 = \Psi_4 + 4a\hat{\Psi}_3 + 6a^2\hat{\Psi}_2 = 0. \tag{301}$$

Under such a transformation we have

$$\hat{\hat{\Psi}}_2 = \hat{\Psi}_2 \neq 0 \tag{302}$$

and

$$\hat{\hat{\Psi}}_3 = \hat{\Psi}_3 + 3a\hat{\Psi}_2$$

We have

$$\frac{1}{4}\frac{d}{da}\hat{\hat{\Psi}}_4(a) = \hat{\Psi}_3 + 3a\hat{\Psi}_2 = \hat{\hat{\Psi}}_3$$

So $\hat{\hat{\Psi}}_3$ will vanish if (301) has repeated roots. The condition for the qudratic equation (301) to have repeated roots reduces to

$$2\hat{\Psi}_3^2 - 3\hat{\Psi}_2\Psi_4 = 0.$$

From (300) we have

$$
\begin{aligned}
2\hat{\Psi}_3^2 &= 2\frac{\Psi_4^2}{16}[(b_1-b_3)+(b_1-b_4)]^2 \\
&= \frac{\Psi_4^2}{8}[(b_1-b_3)^2+2(b_1-b_3)(b_1-b_4)+(b_1-b_4)^2],
\end{aligned}
$$

and from (299) we have

$$3\hat{\Psi}_2\Psi_4 = 2\Psi_4^2(b_1-b_3)(b_1-b_4).$$

We write

$$2\hat{\Psi}_3^2 - 3\hat{\Psi}_2\Psi_4 = \frac{1}{8}[(b_1-b_3)^2 - 14(b_1-b_3)(b_1-b_4)+(b_1-b_4)^2]$$

Put $\rho = b_1 - b_4$ then look for roots for the quadratic equation in γ,

$$\gamma^2 - 14\gamma\rho + \rho^2 = 0.$$

Say α is a root, if it happens that $b_3 = b_1 - \alpha$ then $2\hat{\Psi}_3^2 - 3\hat{\Psi}_2\Psi_4 = 0$ and (301) has repeated roots.

c) **Petrov type D**. We have two distinct double roots b_1 and b_2. And so putting $b_3 = b_1$ and $b_4 = b_2$ we have

$$\hat{\Psi}_0(b) = \Psi_4(b-b_1)^2(b-b_2)^2 \tag{303}$$

$$\hat{\Psi}_1(b) = \frac{1}{2}\Psi_4(b-b_1)(b-b_2)(2b-b_1-b_2). \tag{304}$$

$$\hat{\Psi}_2(b) = \frac{1}{3}\Psi_4[(b-b_2)(b-b_2)+\frac{1}{2}(2b-b_1-b_2)^2]. \tag{305}$$

$$\hat{\Psi}_3(b) = \frac{1}{2}\Psi_4(2b-b_1-b_2). \tag{306}$$

$$\hat{\Psi}_4 = \Psi_4. \tag{307}$$

With the choice, $b = b_1$

$$\hat{\Psi}_0 = \hat{\Psi}_1 = 0 \qquad \hat{\Psi}_2 = \frac{1}{6}\Psi_4(b_1 - b_2)^2$$
$$\hat{\Psi}_3 = \frac{1}{2}\Psi_4(b_1 - b_2), \quad \text{and} \quad \hat{\Psi}_4 = \Psi_4 \tag{308}$$

We now subject the frame to a class I transformation with parameter a. First we also have

$$\hat{\hat{\Psi}}_0 = \hat{\hat{\Psi}}_1 = 0 \tag{309}$$

which follows from

$$\begin{aligned} \hat{\hat{\Psi}}_0 &= \hat{\Psi}_0 = 0 \\ \hat{\hat{\Psi}}_1 &= \hat{\Psi}_1 + a\hat{\Psi}_0 = 0. \end{aligned} \tag{310}$$

Then

$$\begin{aligned} \hat{\hat{\Psi}}_2 &= \hat{\Psi}_2 + 2a\hat{\Psi}_1 + \hat{\Psi}_0 \\ &= \hat{\Psi}_2 \\ &= \frac{1}{6}\Psi_4(b_1 - b_2)^2 \neq 0 \end{aligned} \tag{311}$$

Consider

$$\begin{aligned} \hat{\hat{\Psi}}_3 &= \hat{\Psi}_3 + 3a\hat{\Psi}_2 \\ &= \frac{1}{2}\Psi_4(b_1 - b_2) + 3a\frac{1}{6}\Psi_4(b_1 - b_2)^2 \\ &= \frac{1}{2}\Psi_4(b_1 - b_2)[1 + a(b_1 - b_2)] \end{aligned} \tag{312}$$

and

$$\begin{aligned} \hat{\hat{\Psi}}_4 &= \Psi_4 + 4a\hat{\Psi}_3 + 6a^2\hat{\Psi}_2 \\ &= \Psi_4 + 4a\frac{1}{2}\Psi_4(b_1 - b_2) + 6a^2\frac{1}{6}\Psi_4(b_1 - b_2)^2 \\ &= \Psi_4[1 + a(b_1 - b_2)]^2. \end{aligned} \tag{313}$$

With the choice

$$a = -(b_1 - b_2)^{-1} \tag{314}$$

we have

$$\hat{\hat{\Psi}}_3 = \hat{\hat{\Psi}}_4 = 0. \tag{315}$$

Thus Ψ_0, Ψ_1, Ψ_3, and Ψ_4 have all been reduced to zero with Ψ_2 the only nonvanishing scalar.

d) **Petrov type III**. Three roots coincide $b_1 = b_2 = b_3 \neq b_4$. With a class II transformation with parameter b_1 then we can make Ψ_0, Ψ_1, and Ψ_2 vanish. We have

$$\begin{aligned} \hat{\Psi}_3(b) &= \frac{1}{4}\Psi_4(4b - b_4 - 3b_1) \\ \hat{\Psi}_4(b) &= \Psi_4 \end{aligned} \tag{316}$$

Putting $b = b_1$

$$\begin{aligned} \hat{\Psi}_3 &= \frac{1}{4}(b_1 - b_4) \neq 0. \\ \hat{\Psi}_4 &= \Psi_4 \end{aligned} \tag{317}$$

Then by a subsequent transformation of class I with parameter a we have

$$\begin{aligned} \hat{\hat{\Psi}}_3 &= \hat{\Psi}_3 \neq 0 \\ \hat{\hat{\Psi}}_4 &= \hat{\Psi}_4 + 4a\hat{\Psi}_3 \\ &= \Psi_4 + \Psi_4 a(b_1 - b_4) \\ &= \Psi_4(1 + a(b_1 - b_4)) \end{aligned} \tag{318}$$

With the choice $a = -(b_1 - b_4)^{-1}$ we can make Ψ_4 vanish. And Ψ_3 is the only non-zero scalar.

e) **Petrov type N**. All four roots coincide with on distinct root b_1. Then a transformation of class II with parameter b_1 we can make Ψ_0, Ψ_1, Ψ_2, and Ψ_3 vanish simultaneously and Ψ_4 will be the only non-vanishing scalar, as is easily seen from:

$$
\begin{aligned}
\hat{\Psi}_1(b) &= \frac{1}{4}\frac{d}{db}\hat{\Psi}_0(b) = \Psi_4\frac{1}{4}\frac{d}{db}(b-b_1)^4 = \Psi_4(b-b_1)^3 \\
\hat{\Psi}_2(b) &= \frac{1}{3}\frac{d}{db}\hat{\Psi}_1(b) = \Psi_4(b-b_1)^2 \\
\hat{\Psi}_3(b) &= \frac{1}{2}\frac{d}{db}\hat{\Psi}_2(b) = \Psi_4(b-b_1) \\
\hat{\Psi}_4(b) &= \frac{d}{db}\hat{\Psi}_3(b) = \Psi_4. \qquad (319)
\end{aligned}
$$

□

0.1.11 Equivalence of Petrov Classification Schemes

We are interested in the roots of $\hat{\Psi}_4(b) = 0$ which is quartic in b and so can be written, $\hat{\Psi}_0(b) = \Psi_4(b-b_1)(b-b_2)(b-b_3)(b-b_4)$.

Case (a) First we consider the case where the four roots b_1, b_2, b_3, b_4 are distinct. Write

$$
\begin{aligned}
\rho_1^A &:= o^A + b_1 i^A \\
\rho_2^A &:= o^A + b_2 i^A \\
\rho_3^A &:= a^A + b_3 i^A \\
\rho_4^A &:= a^A + b_4 i^A \qquad (320)
\end{aligned}
$$

Then $\hat{\Psi}_0 = 0$ implies the 4 equations for $\alpha, \beta, \gamma, \delta$:

$$
\begin{aligned}
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C\rho_1^D &= 0 \\
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_2^A\rho_2^B\rho_2^C\rho_2^D &= 0 \\
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_3^A\rho_3^B\rho_3^C\rho_3^D &= 0 \\
\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_4^A\rho_4^B\rho_4^C\rho_4^D &= 0 \qquad (321)
\end{aligned}
$$

which reduce to the 4 equations:

$$(\alpha_A\rho_1^A)(\beta_B\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0 \qquad (322)$$
$$(\alpha_A\rho_2^A)(\beta_B\rho_2^B)(\gamma_C\rho_2^C)(\delta_D\rho_2^D) = 0 \qquad (323)$$
$$(\alpha_A\rho_3^A)(\beta_B\rho_3^B)(\gamma_C\rho_3^C)(\delta_D\rho_3^D) = 0 \qquad (324)$$
$$(\alpha_A\rho_4^A)(\beta_B\rho_4^B)(\gamma_C\rho_4^C)(\delta_D\rho_4^D) = 0 \qquad (325)$$

Now we use that for spinors, $\alpha_A \rho^A = 0$ if and only if α is proportional to ρ (we write $\alpha_A = \lambda \rho_A$).

We are considering the case where all the roots b_1, b_2, b_3, b_4 are all different and as such that the spinors $\rho_1, \rho_2, \rho_3, \rho_4$ are not proportional to each other. Then (322) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1 \rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (323) cant then vanish because ρ_1 is not proportional to ρ_2, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_2 \rho_{2A} = \lambda_2(o_A + b_2 i_A)$. The first two brackets of (324) cant vanish, so at least one of the other two vanish, say it is the 3rd bracket then $\gamma_A = \lambda_3 \rho_{3A} = \lambda_3(o_A + b_3 i_A)$. The first 3 brackets of (325) can't vanish and so it must be the last bracket that vanishes, and so $\delta_A = \lambda_4 \rho_{4A} = \lambda_4(o_A + b_4 i_A)$. And So $\Psi_{ABCD} = \alpha_{(A} \beta_B \gamma_C \delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ are all distinct and each representing a principal null direction.

Case (b) We consider the case where just two roots coincide, say $b_1 = b_2$. As $\rho_1 = \rho_2$ we have three independent equations from $\hat{\Psi}_0 = 0$:

$$(\alpha_A \rho_1^A)(\beta_B \rho_1^B)(\gamma_C \rho_1^C)(\delta_D \rho_1^D) = 0 \tag{326}$$
$$(\alpha_A \rho_3^A)(\beta_B \rho_3^B)(\gamma_C \rho_3^C)(\delta_D \rho_3^D) = 0 \tag{327}$$
$$(\alpha_A \rho_4^A)(\beta_B \rho_4^B)(\gamma_C \rho_4^C)(\delta_D \rho_4^D) = 0 \tag{328}$$

Then (326) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1 \rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (327) cant then vanish because ρ_1 is not proportional to ρ_3, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_3 \rho_{3A} = \lambda_3(o_A + b_3 i_A)$. The first two brackets of (328) cant vanish, so at least one of the other two vanish, say it is the 3rd bracket then $\gamma_A = \lambda_4 \rho_{4A} = \lambda_4(o_A + b_4 i_A)$.

It can easily be shown that with parameter $b = b_1 (= b_2)$ $\hat{\Psi}_0$ and $\hat{\Psi}_1$ will vanish. So we also have the equation

$$\alpha_{(A} \beta_B \gamma_C \delta_{D)} \; \rho_1^A \rho_1^B \rho_1^C i^D = 0$$

or

$$\rho_{1(A} \rho_{3B} \rho_{4C} \delta_{D)} \; \rho_1^A \rho_1^B \rho_1^C i^D = 0$$

which reduce to

$$(\rho_{1A} i^A)(\rho_{3B} \rho_1^B)(\rho_{4C} \rho_1^C)(\delta_D \rho_1^D) = 0$$

implying $\delta_A = \rho_{1A} = \lambda_1(o_A + b_1 i_a)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with two directions coinciding.

Case (c) Two distinct double roots b_1 and b_2. As $\rho_1 = \rho_3$ and $\rho_2 = \rho_4$ we have two independent equations from $\hat{\Psi}_0 = 0$:

$$(\alpha_A\rho_1^A)(\beta_B\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0 \tag{329}$$
$$(\alpha_A\rho_2^A)(\beta_B\rho_2^B)(\gamma_C\rho_2^C)(\delta_D\rho_2^D) = 0 \tag{330}$$

Then (329) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (330) cant then vanish because ρ_1 is not proportional to ρ_4, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_2\rho_{2A} = \lambda_2(o_A + b_2 i_A)$.

It is easily shown that with parameter $b = b_1$ we have $\hat{\Psi}_1 = 0$

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0$$

or

$$\rho_{1(A}\rho_{2B}\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0$$

which reduces to

$$(\rho_{1A}i^A)(\rho_{2B}\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0$$

So that at least one of the last two brackets vanish. Say the third bracket vanishes, then $\gamma_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$.

It is easily shown that with parameter $b = b_2$ we have $\Psi_1 = 0$

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_2^A\rho_2^B\rho_2^C i^D = 0$$

or

$$\rho_{1(A}\rho_{2B}\rho_{1C}\delta_{D)}\ \rho_2^A\rho_2^B\rho_2^C i^D = 0$$

which reduces to

$$(\rho_{1A}\rho_2^A)(\rho_{2B}i^B)(\rho_{1C}\rho_2^C)(\delta_D\rho_2^D) = 0$$

So that at least one of the last two brackets vanish. Say the third bracket vanishes, then $\delta_A = \lambda_2\rho_{2A} = \lambda_2(o_A + b_2 i_A)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with two different pairs repeated.

Case (d) Three roots coincide and $b = b_1 (= b_2 = b_3)$. As $\rho_1 = \rho_2 = \rho_3$ we have two independent equations from $\hat{\Psi}_0 = 0$:

$$(\alpha_A\rho_1^A)(\beta_B\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0 \tag{331}$$
$$(\alpha_A\rho_4^A)(\beta_B\rho_4^B)(\gamma_C\rho_4^C)(\delta_D\rho_4^D) = 0 \tag{332}$$

Then (331) is zero if and only if one of at least one of the brackets vanish. Say the first bracket is one that vanishes, so we can say $\alpha_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$. The first bracket in (332) cant then vanish because ρ_1 is not proportional to ρ_4, and so one of the other brackets must vanish. Say the second bracket is one that vanishes, and so $\beta_A = \lambda_4\rho_{4A} = \lambda_4(o_A + b_4 i_A)$.

It is easily shown that with parameter $b = b_1 (= b_2 = b_3)$ $\hat{\Psi}_0$, $\hat{\Psi}_1$ and $\hat{\Psi}_2$ will vanish simultaneously. So we also have the equations

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0 \tag{333}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B i^C i^D = 0 \tag{334}$$

(333) is

$$\rho_{1(A}\rho_{4B}\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0$$

which reduces to

$$(\rho_{1A}i^A)(\rho_{4B}\rho_1^B)(\gamma_C\rho_1^C)(\delta_D\rho_1^D) = 0$$

This implies that $\gamma_A = \lambda_1\rho_{1A} = \lambda_1(o_A + b_1 i_A)$. (334) then reads

$$\rho_{1(A}\rho_{4B}\rho_{1C}\delta_{D)}\ \rho_1^A\rho_1^B i^C i^D = 0$$

which reduces to

$$(\rho_{1A} i^A)(\rho_{4B} \rho_1^B)(\rho_{1C} i^C)(\delta_D \rho_1^D) = 0$$

which means that $\delta_D = \rho_{1D} = \lambda_1(o_a + b_1 i_A)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with three directions coinciding.

Case (e) All roots coincide and we have for $b = b_1$ that $\hat{\Psi}_0 = \hat{\Psi}_1 = \hat{\Psi}_2 = \hat{\Psi}_3 = 0$. We have the equations:

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C\rho_1^D = 0 \tag{335}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B\rho_1^C i^D = 0 \tag{336}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A\rho_1^B i^C i^D = 0 \tag{337}$$
$$\alpha_{(A}\beta_B\gamma_C\delta_{D)}\ \rho_1^A i^B i^C i^D = 0 \tag{338}$$

(335) reduces to

$$(\alpha_A \rho_1^A)(\beta_B \rho_1^B)(\gamma_C \rho_1^C)(\delta_D \rho_1^D) = 0$$

At least one of the brackets vanish, say the first. So that $\alpha_A = \lambda_1 \rho_{1A} = \lambda_1(o_A + b_1 i_A)$. (336) reduces to

$$(\rho_{1A} i^A)(\beta_B \rho_1^B)(\gamma_C \rho_1^C)(\delta_D \rho_1^D) = 0.$$

At least one the last three brackets vanish, say the second bracket vanishes. Then $\beta_A = \lambda_1 \rho_{1A} = \lambda_1(o_a + b_1 i_A)$. (337) reduces to

$$(\rho_{1A} i^A)(\rho_{1B} i^B)(\gamma_C \rho_1^C)(\delta_D \rho_1^D) = 0.$$

At least one the last two brackets vanish, say the third bracket vanishes. Then $\gamma_A = \lambda_1 \rho_{1A} = \lambda_1(o_a + b_1 i_A)$. (338) reduces to

$$(\rho_{1A} i^A)(\rho_{1B} i^B)(\rho_{1C} i^C)(\delta_D \rho_1^D) = 0.$$

The last bracket must vanish, therefore $\delta_A = \lambda_1 \rho_{1A} = \lambda_1(o_a + b_1 i_A)$. So now we have that $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ where the spinors $\alpha_A, \beta_A, \gamma_A, \delta_A$ each represent a principal null direction with all four directions coinciding.

Proof of the converse:

We now prove the converse.

Case (a) Assume

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$$

where

$$\begin{aligned}
\alpha &= \alpha_0 o_A + \alpha_1 i_A, \\
\beta &= \beta_0 o_A + \beta_1 i_A, \\
\gamma &= \gamma_0 o_A + \gamma_1 i_A, \\
\delta &= \delta_0 o_A + \delta_1 i_A
\end{aligned} \tag{339}$$

are disinct (not proportional to each other). We then wish to show that all roots of

$$\Psi_0 \to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^2\Psi_3 + b^4\Psi_4 = 0$$

are distinct.

$$\begin{aligned}
\Psi_0 &\to \Psi_{ABCD}(o^A + bi^A)(o^A + bi^A)(o^A + bi^A)(o^A + bi^A) \\
&= \alpha_{(A}\beta_B\gamma_C\delta_{D)}(o^A + bi^A)(o^A + bi^A)(o^A + bi^A)(o^A + bi^A) \\
&= \alpha_A\beta_B\gamma_C\delta_D(o^A + bi^A)(o^A + bi^A)(o^A + bi^A)(o^A + bi^A) \\
&= (\alpha_A o^A + b\,\alpha_A i^A)(\beta_B o^B + b\,\beta_B i^B)(\gamma_C o^C + b\,\gamma_C i^C)(\delta_D o^D + b\delta_D i^D) \\
&= (-\alpha_1 + b\alpha_0)(-\beta_0 + b\beta_0)(-\gamma_0 + b\gamma_1)(-\delta_0 + b\delta_1) \\
&= 0.
\end{aligned} \tag{340}$$

We see that it has 4 distinct roots.

Case (b) Assume

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\gamma_C\delta_{D)}$$

where α, β, γ are distinct, then

$$
\begin{aligned}
\Psi_0 &\rightarrow \Psi_{ABCD}(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)(o^A+bi^A) \\
&= \alpha_A\alpha_B\gamma_C\delta_D(o^A+bi^A)(o^A+bi^A)(o^A+bi^A)(o^A+bi^A) \\
&= (\alpha_A o^A + b\ \alpha_A i^A)(\alpha_B o^B + b\ \alpha_B i^B)(\gamma_C o^C + b\ \gamma_C i^C)(\delta_D o^D + b\delta_D i^D) \\
&= (-\alpha_1 + b\alpha_0)^2(-\gamma_0 + b\gamma_1)(-\delta_0 + b\delta_1) \\
&= 0. \qquad (341)
\end{aligned}
$$

We see that two roots coincide.

The other cases work through the same way.

0.1.12 Petrov classification via Eigenbivectors of the Weyl Tensor

Eigenbivectors of the Weyl Spinor

Given any spin-frame (o^A, i^A) we can construct a corresponding orthonormal basis $\overset{1}{\delta}_{AB}$, $\overset{2}{\delta}_{AB}$, and $\overset{3}{\delta}_{AB}$ ($\overset{\alpha}{\delta}_{AB}\overset{\beta}{\delta}{}^{AB} = \delta_{\alpha\beta}$) for $\mathcal{C}_{(AB)}$

$$
\overset{1}{\delta}_{AB} = -\frac{i}{\sqrt{2}}(o_A o_B - i_A i_B), \quad \overset{2}{\delta}_{AB} = \frac{1}{\sqrt{2}}(o_A o_B + i_A i_B), \quad \overset{3}{\delta}_{AB} = i\sqrt{2}o_{(A}i_{B)}. \qquad (342)
$$

We have

$$
\begin{aligned}
\overset{\alpha}{\delta}_{AB}\overset{\alpha}{\delta}{}^{CD} &= -\frac{i}{\sqrt{2}}(o_A o_B - i_A i_B)\cdot -\frac{i}{\sqrt{2}}(o^C o^D - i^C i^D) \\
&\quad +\frac{1}{\sqrt{2}}(o_A o_B + i_A i_B)\cdot \frac{1}{\sqrt{2}}(o^C o^D + i^C i^D) \\
&\quad +\frac{i}{\sqrt{2}}(o_A i_B + o_B i_A)\cdot \frac{i}{\sqrt{2}}(o^C i^D + o^D i^C) \\
&= \frac{1}{2}\big(- o_A o_B o^C o^D + o_A o_B i^C i^D + i_A i_B o^C o^D - i_A i_B i^C i^D\big) \\
&\quad +\frac{1}{2}\big(o_A o_B o^C o^D + o_A o_B i^C i^D + i_A i_B o^C o^D + i_A i_B i^C i^D\big) \\
&\quad +\frac{1}{2}\big(- o_A i_B o^C i^D - o_A i_B o^D i^C - o_B i_A o^C i^D - o_B i_A o^D i^C\big) \\
&= \frac{1}{2}\big(o_A o_B i^C i^D - o_A i_B i^C o^D - i_A o_B o^C i^D + i_A i_B o^C o^D \\
&\quad + o_B o_A i^C i^D - o_B i_A i^C o^D - i_B o_A o^C i^D + i_B i_A o^C o^D\big) \\
&= \frac{1}{2}(o_A i^C - i_A o^C)(o_B i^D - i_B o^D) + \frac{1}{2}(o_B i^C - i_B o^C)(o_A i^D - i_A o^D) \\
&= \frac{1}{2}(\delta_A{}^C \delta_B{}^D + \delta_B{}^C \delta_A{}^D) \qquad (343)
\end{aligned}
$$

The components of ϕ_{AB} with respect to the basis (342) are

$$
\overset{1}{\phi} = \frac{-i}{\sqrt{2}}(\phi_{00} - \phi_{11}), \quad \overset{2}{\phi} = \frac{1}{\sqrt{2}}(\phi_{00} + \phi_{11}), \quad \overset{3}{\phi} = i\sqrt{2}\phi_{01}. \qquad (344)
$$

The components of $\Psi_{AB}{}^{CD}$ with respect to the basis (342) are

$$
\mathbf{\Psi} = \begin{pmatrix} \frac{1}{2}(-\Psi_0 + 2\Psi_1 - \Psi_4) & \frac{-i}{2}(\Psi_0 - \Psi_4) & (\Psi_1 - \Psi_3) \\ \frac{-i}{2}(\Psi_0 - \Psi_4) & \frac{1}{2}(\Psi_0 + 2\Psi_1 + \Psi_4) & i(\Psi_1 + \Psi_3) \\ (\Psi_1 - \Psi_3) & i(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix} \qquad (345)
$$

From (343)

$$
\begin{aligned}
\overset{\alpha\alpha}{\Psi} &= \Psi_{AB}{}^{CD}\,\overset{\alpha}{\delta}{}^{AB}\overset{\alpha}{\delta}_{CD} \\
&= \Psi_{AB}{}^{CD}\frac{1}{2}(\delta^A_C\delta^B_D + \delta^A_D\delta^B_C) \\
&= \Psi_{AB}{}^{AB} \\
&= \Psi_{ABCD}\epsilon^{CA}\epsilon^{DB} \\
&= 0. \qquad (346)
\end{aligned}
$$

From (343) we have that the eigen-equation

$$\overset{\alpha\beta}{\Psi}\,\phi_\beta = \lambda\phi_\alpha \tag{347}$$

can be written:

$$\begin{aligned}
\overset{\alpha\beta}{\Psi}\,\phi_\beta &= \Psi_{AB}{}^{CD}\,\overset{\alpha\,AB}{\delta}\,\overset{\beta}{\delta}_{CD}\,\phi_{EF}\,\overset{\beta}{\delta}{}^{EF} \\
&= \Psi_{AB}{}^{CD}\,\overset{\alpha\,AB}{\delta}\,\frac{1}{2}(\delta^E_C\delta^F_D + \delta^E_D\delta^F_C)\phi_{EF} \\
&= \Psi_{AB}{}^{CD}\phi_{CD}\,\overset{\alpha\,AB}{\delta} \\
&= \lambda\phi_{AB}\,\overset{\alpha AB}{\delta}\quad.
\end{aligned} \tag{348}$$

Therefore the expressing the eigen-equation

$$\Psi_{AB}{}^{CD}\phi_{CD} = \lambda\phi_{AB} \tag{349}$$

in components according to the basis (342), we see that λ is also an eigenvalue in the normal sense of the matrix $\boldsymbol{\Psi}$. If $\lambda_1, \lambda_2, \lambda_3$ are the three eigenvalues of $\boldsymbol{\Psi}$ we have

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= \Psi_{AB}{}^{AB} = 0 \\
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= \Psi_{AB}{}^{CD}\Psi_{CD}{}^{AB} =: I \\
\lambda_1^3 + \lambda_2^3 + \lambda_3^3 &= \Psi_{AB}{}^{CD}\Psi_{CD}{}^{EF}\Psi_{EF}{}^{AB} =: J.
\end{aligned} \tag{350}$$

From

$$\begin{aligned}
0 &= (\lambda_1 + \lambda_2 + \lambda_3)^3 - 3(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \\
&= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + 3(\lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_2^2\lambda_3 + \lambda_2^2\lambda_1 + \lambda_3^2\lambda_1 + \lambda_3^2\lambda_2) + 6\lambda_1\lambda_2\lambda_3 \\
&\quad - 3(\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + (\lambda_2 + \lambda_3)\lambda_1^2 + (\lambda_1 + \lambda_3)\lambda_2^2 + (\lambda_1 + \lambda_2)\lambda_3^2) \\
&= -2(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + 6\lambda_1\lambda_2\lambda_3
\end{aligned}$$

we have

$$J = 3\lambda_1\lambda_2\lambda_3. \tag{351}$$

Note that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are roots of

$$6\lambda^3 - 3I\lambda - 2J = 0$$

as is easily seen from

$$6(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 6\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\lambda - 6\lambda_1\lambda_2\lambda_3$$

Using

$$\begin{aligned}\Psi_{ABCD} &= \Psi_0 i_A i_B i_C i_D - 4\Psi_1 o_{(A} i_B i_C i_{D)} + 6\Psi_2 o_{(A} o_B i_C i_{D)} \\ &\quad - 4\Psi_1 o_{(A} o_B o_C i_{D)} + \Psi_4 o_A o_B o_C o_D \end{aligned} \tag{352}$$

which follows as it gives (259) upon contraction with appropriate combination of o's and i's. For example

$$\begin{aligned}\Psi_{ABCD} o^A o^B i^C i^D &= (\Psi_0 i_A i_B i_C i_D - 4\Psi_1 o_{(A} i_B i_C i_{D)} + 6\Psi_2 o_{(A} o_B i_C i_{D)} \\ &\quad - 4\Psi_1 o_{(A} o_B o_C i_{D)} + \Psi_4 o_A o_B o_C o_D) o^A o^B i^C i^D \\ &= 6\Psi_2 o_{(A} o_B i_C i_{D)} o^A o^B i^C i^D \\ &= 6\Psi_2 \frac{1}{4!}(\cdots + o_C o_D i_A i_B + o_D o_C i_A i_B + o_C o_D i_B i_A \\ &\quad + o_D o_C i_B i_A + \cdots) o^A o^B i^C i^D \\ &= \Psi_2. \end{aligned} \tag{353}$$

where we used $o_A o^A = i_A i^A = 0$ and $o_A i^A = 1 = -i_A o^A$.

We obtain

$$I = 2\Psi_0\Psi_4 - 8\Psi_1\Psi_3 + 6\Psi_2^2 \tag{354}$$

$$J = 6 \det \begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix}. \tag{355}$$

From (351) we have that the determinant of $\mathbf{\Psi}$ is $\frac{1}{3}J$

We have

$$I^3 - 6J^2 = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2 \quad \text{subject to } \lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{356}$$

which can be shown by verifying:

$$\begin{aligned}&(\lambda_1^2 + \lambda_2^2 + (-\lambda_1 - \lambda_2)^2)^3 - 6(\lambda_1^3 + \lambda_2^3 + (-\lambda_1 - \lambda_2)^3)^2 \\ &\qquad = (\lambda_1 - \lambda_2)^2(\lambda_2 - (-\lambda_1 - \lambda_2))^2((-\lambda_1 - \lambda_2) - \lambda_1)^2.\end{aligned} \tag{357}$$

Equation (356) establishes that two or more of the λ's are equal is equivalent to $I^3 = 6J^2$.

Eigenbivectors of the Weyl Tensor

$$C_{ab}{}^{cd}X_{cd} = \mu X_{ab}, \tag{358}$$

where $X_{ab} = -X_{ab}$. In terms of spinors:

$$X_{ab} = \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\xi_{A'B'} \tag{359}$$

where ϕ_{AB} and $\xi_{A'B'}$ are both symmetric. Defing X^*_{ab}:

$$X^*_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}X_{cd}$$

and using

$$\epsilon_{abcd} = i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'})$$

we obtain

$$\begin{aligned}X^*_{ABA'B'} &= \frac{1}{2}i(\epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'}{}^{D'}\epsilon_{B'}{}^{C'} - \epsilon_A{}^D\epsilon_B{}^C\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'})X_{CDC'D'} \\ &= \frac{1}{2}i(\epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'}{}^{D'}\epsilon_{B'}{}^{C'} - \epsilon_A{}^D\epsilon_B{}^C\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'})(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\xi_{C'D'}) \\ &= \frac{1}{2}i(\phi_{AB}\epsilon_{B'A'} + \epsilon_{AB}\xi_{B'A'} - \phi_{BA}\epsilon_{A'B'} - \epsilon_{BA}\xi_{A'B'}) \\ &= i(\epsilon_{AB}\xi_{A'B'} - \phi_{AB}\epsilon_{A'B'}).\end{aligned} \tag{360}$$

As such we have

$$X_{ab} + iX^*_{ab} = 2\phi_{AB}\epsilon_{A'B'}, \qquad X_{ab} - iX^*_{ab} = 2\epsilon_{AB}\xi_{A'B'}. \tag{361}$$

The Weyl tensor in terms of spinors is

$$C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'} \tag{362}$$

Defining

$$C^*_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef}C_{efcd}X^{cd}$$

We obtain

$$\begin{aligned}
C^*_{ABCDA'B'C'D'} &= \frac{1}{2}i(\epsilon_A{}^E\epsilon_B{}^F\epsilon_{A'}{}^{F'}\epsilon_{B'}{}^{E'} - \epsilon_A{}^F\epsilon_B{}^E\epsilon_{A'}{}^{E'}\epsilon_{B'}{}^{F'}) \\
&\quad (\Psi_{EFCD}\epsilon_{E'F'}\epsilon_{C'D'} + \epsilon_{EF}\epsilon_{CD}\overline{\Psi}_{E'F'C'D'}) \\
&= \frac{1}{2}i(\Psi_{ABCD}\epsilon_{B'A'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{B'A'C'D'} \\
&\quad -\Psi_{BACD}\epsilon_{A'B'}\epsilon_{C'D'} - \epsilon_{BA}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'}) \\
&= i(\overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} - \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'})
\end{aligned} \tag{363}$$

Then

$$C_{abcd} + iC^*_{abcd} = 2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}, \qquad C_{abcd} - iC^*_{abcd} = 2\overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \tag{364}$$

From (358) we obtain

$$(C_{ab}{}^{cd} + i\frac{1}{2}\epsilon_{ab}{}^{ef}C_{ef}{}^{cd})X_{cd} = \mu(X_{ab} + i\frac{1}{2}\epsilon_{ab}{}^{ef}X_{ef}) \tag{365}$$

which in terms of spinors is

$$2\Psi_{AB}{}^{CD}\epsilon_{A'B'}\epsilon^{C'D'}(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\xi_{C'D'}) = \mu 2\phi_{AB}\epsilon_{A'B'}$$

or

$$\Psi_{AB}{}^{CD}\phi_{CD} = \frac{1}{2}\mu\phi_{AB} \tag{366}$$

where we have used $\epsilon^{C'D'}\epsilon_{C'D'} = 2$ and $\epsilon^{C'D'}\xi_{C'D'} = 0$.

From (358) we obtain

$$(C_{ab}{}^{cd} - i\frac{1}{2}\epsilon_{ab}{}^{ef}C_{ef}{}^{cd})X_{cd} = \mu(X_{ab} - i\frac{1}{2}\epsilon_{ab}{}^{ef}X_{ef}) \tag{367}$$

which in terms of spinors is

$$2\overline{\Psi}_{A'B'}{}^{C'D'}\epsilon_{AB}\epsilon^{CD}(\phi_{CD}\epsilon_{C'D'} + \epsilon_{CD}\xi_{C'D'}) = \mu 2\epsilon_{AB}\xi_{A'B'}$$

or

$$\overline{\Psi}_{A'B'}{}^{C'D'}\xi_{C'D'} = \frac{1}{2}\mu\xi_{A'B'} \tag{368}$$

where we have used $\epsilon^{CD}\epsilon_{CD} = 2$ and $\epsilon^{CD}\phi_{CD} = 0$.

0.1.13 Focussing and Shearing of Null Curves

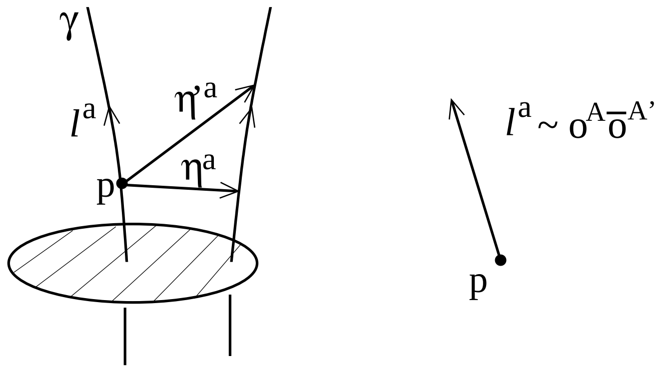

Figure 1: visualflownull.

$$[l, \eta]^a = l^b\nabla_b\eta^a - \eta^b\nabla_b l^a = 0. \tag{369}$$

$$Do^A = D\iota^A = 0. \tag{370}$$

This means that o and ι are parallelly propergated along γ and so remain a spin basis at each point of γ.

recalling that $D = l^a \nabla_a$

$$D(l^a \eta_a) = l^a D\eta_a = l^a l^b \nabla_b \eta_a = l^a \eta^b \nabla_b l^a = \frac{1}{2}\eta^b \nabla_b (l^2) = 0 \tag{371}$$

We now construct the NP tetrad $(l, n, m, \overline{m})$

$$\begin{aligned} \eta^a &= ul^a + \overline{z}m^a + z\overline{m}^a \\ &= uo^A\overline{o}^{A'} + \overline{z}o^A\overline{\imath}^{A'} + z\overline{o}^{A'}\imath^A. \end{aligned} \tag{372}$$

Now, since $l^b \nabla_b \eta^a = \eta^b \nabla_b l^a$ we have

$$D\eta^a = l^b \nabla_b \eta^a = \eta^b \nabla_b l^a, \tag{373}$$

so that,

$$D\eta^a = uDl^a + \overline{z}\delta l^a + z\overline{\delta} l^a. \tag{374}$$

In terms of spinors this reads

$$o^A\overline{o}^{A'}Du + o^A\overline{\imath}^{A'} + \imath^A\overline{o}^{A'} = \overline{z}o^A\overline{\delta}\overline{o}^{A'} + \overline{z}\overline{o}^{A'}\delta o^A + zo^A\overline{\delta}\overline{o}^{A'} + z\overline{o}^{A'}\overline{\delta}o^A. \tag{375}$$

Multipling by $o_A \overline{\imath}_{A'}$ gives

$$-Dz = \overline{z}o_A\delta o^A - zo_A\overline{\delta}o^A, \tag{376}$$

i.e.,

$$Dz = -\rho z - \sigma\overline{z}. \tag{377}$$

The interpretation of z is as follows. Consider the projection of η^a onto the spacelike surface 2-dimensional subspace $T_\perp$ as introduced in subsection?? Recall that this surface was spanned by $m^a, \overline{m}^a$, or equivalently $e_{\hat{1}}{}^a, e_{\hat{2}}{}^a$. Write the projection as

$$\begin{aligned} \sqrt{2}(xe_{\hat{1}} - ye_{\hat{2}}) &= x(m + \overline{m})/i + y(m - \overline{m})/i \\ &= \overline{z}m + z\overline{m}, \end{aligned} \tag{378}$$

where $z = x + iy$, which is consistent with (372).

Suppose first that $\sigma = 0$ while ρ is real, i.e. $Dz = -\rho z$, or

$$Dx = -\rho x, \quad Dy = -\rho y. \tag{379}$$

This is isotrpic magnificatoin at a rate of $-\rho$. Next suppose that $\sigma = 0$ while $\rho = -i\omega$, so that $Dz = i\omega z$, or

$$Dx = -\omega y, \quad Dy = \omega x. \tag{380}$$

This corresponds to a rotation with angular velocity ω. Next consider the case where $\rho = 0$ and σ is real. Then

$$Dx = -\sigma x, \quad Dy = \sigma y, \tag{381}$$

which represents a volume-preserving shear at a rate with principle axes along the x and y axes.

0.1.14 Goldberg Sachs Theorem

Equivalence relations for geodesic shearfree null congruences

Lemma 0.1.2 *The following three conditions are equivalent*

a) $l^a = o^A \bar{o}^{A'}$ *corresponds to geodesic shearfree null congruences;*

b) $\kappa = \sigma = 0$*;*

c) $o^A o^B \nabla_{AA'} o_B = 0$.

Proof:

b) implies c) and **c) implies b):**

Recall that $\kappa = o^A o^B \bar{o}^{B'} \nabla_{BB'} o_A$ and $\sigma = o^A o^B \bar{\iota}^{B'} \nabla_{BB'} o_A$. We write

$$o^A o^B \nabla_{BB'} o_A = c\bar{o}_{B'} + d\bar{\iota}_{B'} \tag{382}$$

then

$$\kappa = o^A o^B \overline{o}^{B'} \nabla_{BB'} o_A = c\overline{o}^{B'}\overline{o}_{B'} + d\overline{o}^{B'}\overline{\imath}_{B'} = -d. \tag{383}$$

and

$$\sigma = o^A o^B \overline{\imath}^{B'} \nabla_{BB'} o_A = c\overline{\imath}^{B'}\overline{o}_{B'} + d\overline{\imath}^{B'}\overline{\imath}_{B'} = c. \tag{384}$$

so that

$$o^A o^B \nabla_{BB'} o_A = \sigma \overline{o}_{B'} - \kappa \overline{\imath}_{B'} \tag{385}$$

This establishes the equivalence of b) and c).

b) implies a)

$$\kappa = o^A D o_A = o^A o^B \overline{o}^{B'} \nabla_{BB'} o_A = 0.$$

and

$$\sigma = o^A \delta o_A = o^A o^B \overline{\imath}^{B'} \nabla_{BB'} o_A = 0.$$

We write

$$\begin{aligned} l^b \nabla_b l_a &= D(o_A \overline{o}_{A'}) \\ &= o_A D\overline{o}_{A'} + \overline{o}_{A'} D o_A \\ &= a o_A \overline{o}_{A'} + b o_A \overline{\imath}_{A'} + \overline{b}\imath_A \overline{o}_{A'} + c\imath_A \overline{\imath}_{A'} \end{aligned} \tag{386}$$

Contracting the second and third line on the RHS with $o^A\overline{o}^{A'}$ implies $c = 0$. Next, contracting the second and third line on the RHS with $\imath^A\overline{o}^{A'}$ implies

$$o_A \imath^A (\overline{o}^{A'} D\overline{o}_{A'}) = \kappa = 0 = -b = -\overline{b}.$$

Next, contracting the second and third line on the RHS with $\imath^A\overline{\imath}^{A'}$ implies

$$\overline{\imath}^{A'} D\overline{o}_{A'} + \imath^A D o_A = \epsilon + \overline{\epsilon} = a$$

.

We find

$$l^b\nabla_b l_a = a l_a. \tag{387}$$

which the non-affinely parameterised geodesic equation.

a) implies b):

$l^b\nabla_b l_a = a l_a$ says

$$D(o_A\overline{o}_{A'}) = o_A D\overline{o}_{A'} + \overline{o}_{A'} D o_A = a o_A \overline{o}_{A'} \tag{388}$$

Contracting with o^A implies

$$\overline{o}_{A'}(o^A D o_A) = 0,$$

or

$$o^A D o_A = \kappa = 0. \tag{389}$$

□

Integrability conditions

$$A^a\nabla_a\phi = f, \qquad B^a\nabla_a\phi = g, \tag{390}$$

where A^a, B^a are locally tranverse vector fields which are surface forming,

$$A^a\nabla_a B^b - B^a\nabla_a A^b = \alpha A^b + \beta B^b. \tag{391}$$

Theorem 0.1.3 *A necessary and sufficient condition for solutions of the system (390) to exist is*

$$A^a\nabla_a g - B^a\nabla_a f = \alpha f + \beta g. \tag{392}$$

Proof:

Suppose ϕ is a solution of (390) then (391) implies

$$
\begin{aligned}
& A^a\nabla_a(B^b\nabla_b\phi) - B^a\nabla_a(A^b\nabla_b\phi) \\
= \;& (A^a\nabla_a B^b - B^a\nabla_a A^b)\nabla_b\phi + (A^a B^b - B^a A^b)\nabla_a\nabla_b\phi \\
= \;& (\alpha A^a + \beta B^a)\nabla_a\phi \\
= \;& \alpha f + \beta g
\end{aligned} \tag{393}
$$

No suppose conversely that (399) holds. We choose a special coordinate system such that

$$
A^a\nabla_a = \frac{\partial}{\partial y^1}
$$

Then we may solve $A^a\nabla_a\phi = f$ via

$$
\phi(y^1, y^\alpha) = \int_{\tilde{y}}^{y^1} f(s, y^\alpha)ds \tag{394}
$$

where $\alpha = 2, \ldots, n$. Let $B^a\nabla_a\phi - g = h$. We may choose $\tilde{y} = \tilde{y}(y^\alpha)$ to set $h = 0$ on an intial surface $y^1 = const$. Now

$$
\begin{aligned}
A^a\nabla_a h &= A^a\nabla_a(B^b\nabla_b\phi) - A^a\nabla_a g \\
&= B^a\nabla_a(A^b\nabla_b\phi) + (A^a\nabla_a B^b - B^a\nabla_a A^b)\nabla_b\phi - A^a\nabla_a g \\
&= B^a\nabla_a f + (\alpha A^a + \beta B^a)\nabla_a\phi - A^a\nabla_a g \\
&= (\alpha A^a + \beta B^a)\nabla_a\phi - (A^a\nabla_a g - B^a\nabla_a f) \\
&= \alpha f + \beta B^a\nabla_a\phi - \alpha f - \beta g \\
&= \beta h.
\end{aligned} \tag{395}
$$

Since $h = 0$ intially we see that $h = 0$ and so $B^a\nabla_a\phi = g$, i.e. ϕ solves (390).

□

Theorem 0.1.4 *(Sommers 1976) The integrable condition for the equation*

$$
\xi^A\nabla_{AA'}x = \alpha_{A'} \tag{396}
$$

in complex x for ξ^A analytic shearfree null geodesics is

$$
\xi^A\xi^B\nabla_A{}^{A'}\alpha_{A'} = \alpha_{A'}\xi^A\nabla_A{}^{A'}\xi^B. \tag{397}
$$

Proof: We write the equation to be solved in component form relative to some basis $(o^A, \imath^A)$,

$$\xi^A \overline{o}^{A'} \nabla_{AA'} x = \overline{o}^{A'} \alpha_{A'}, \qquad \xi^A \imath^{A'} \nabla_{AA'} x = \imath^{A'} \alpha_{A'} \tag{398}$$

putting it into the form (390) where we identify $\phi = x$, $A^a = \xi^A \overline{o}^{A'}$, $B^a = \xi^A \imath^{A'}$, $f = \overline{o}^{A'} \alpha_{A'}$, and $g = \imath^{A'} \alpha_{A'}$. The integrability condition is then

$$\xi^A \overline{o}^{A'} \nabla_{AA'} (\overline{\imath}^{B'} \alpha_{B'}) - \xi^A \overline{\imath}^{A'} \nabla_{AA'} (\overline{o}^{B'} \alpha_{B'}) = \alpha \overline{o}^{A'} \alpha_{A'} + \beta \overline{\imath}^{A'} \alpha_{A'}. \tag{399}$$

where α and β are given by

$$\xi^A \overline{o}^{A'} \nabla_{AA'} (\xi^B \overline{\imath}^{B'}) - \xi^A \overline{\imath}^{A'} \nabla_{AA'} (\xi^B \overline{o}^{B'}) = \alpha \xi^B \overline{o}^{B'} + \beta \xi^B \overline{\imath}^{B'}. \tag{400}$$

Transvecting (400) with $\alpha_{B'}$ and equating to (399), we find

$$\xi^A \xi^B \{ \overline{o}^{A'} \nabla_{AA'} (\overline{\imath}^{B'} \alpha_{B'}) - \overline{\imath}^{A'} \nabla_{AA'} (\overline{o}^{B'} \alpha_{B'}) \} = \alpha_{B'} \xi^A \{ \overline{o}^{A'} \nabla_{AA'} (\xi^B \overline{\imath}^{B'}) - \imath^{A'} \nabla_{AA'} (\xi^B \overline{o}^{B'}) \} \tag{401}$$

or

$$\xi^A \xi^B \{ (\overline{o}^{A'} \overline{\imath}^{B'} - \overline{\imath}^{A'} \overline{o}^{B'}) \nabla_{AA'} \alpha_{B'} \} = \alpha_{B'} \xi^A \{ (\overline{o}^{A'} \overline{\imath}^{B'} - \imath^{A'} \overline{o}^{B'}) \nabla_{AA'} \xi^B) \} \tag{402}$$

Using $\overline{o}^{A'} \overline{\imath}^{B'} - \overline{\imath}^{A'} \overline{o}^{B'} = \epsilon^{A'B'}$, (402) becomes

$$\xi^A \xi^B \nabla_A{}^{A'} \alpha_{A'} = \alpha_{A'} \xi^A \nabla_A{}^{A'} \xi^B \tag{403}$$

□

Some relations

If ξ is geodesic shearfree it satifies

$$\xi^A \xi^B \nabla_{AA'} \xi_B = 0 \tag{404}$$

We define $\overline{\eta}_{A'}$ as the proportinality factor in the equation

$$\xi^A \nabla_{AA'} \xi_B = \xi_B \overline{\eta}_{A'} \tag{405}$$

Consider

$$\begin{aligned}
\xi_B \xi^A \nabla_{AA'} \overline{\eta}^{A'} &= \xi_B \xi^A \nabla_{AA'} \overline{\eta}^{A'} + \xi^A \overline{\eta}^{A'} \nabla_{AA'} \xi_B \\
&= -\xi^A \nabla_A{}^{A'} (\xi_B \overline{\eta}_{A'}) \\
&= \xi^A \nabla_{AA'} (\xi^C \nabla_C{}^{A'} \xi_B) \\
&= (\nabla_C{}^{A'} \xi_B) \xi^A \nabla_{AA'} \xi^C + \xi^A \xi^C \nabla_{AA'} \nabla_C{}^{A'} \xi_B \\
&= (\nabla_C{}^{A'} \xi_B) \xi^C \overline{\eta}_{A'} + \xi^A \xi^C \nabla_{AA'} \nabla_C{}^{A'} \xi_B \\
&= 0 + \xi^A \xi^C \square_{AC} \xi_B \\
&= \Psi_{ABCD} \xi^A \xi^C \xi^D.
\end{aligned} \tag{406}$$

where we used (221). Therefore we have the first relation

$$\xi_B \xi^A \nabla_{AA'} \overline{\eta}^{A'} = \Psi_{ABCD} \xi^A \xi^C \xi^D. \tag{407}$$

Now taking the derivative of $\xi^A \xi^D \nabla_A{}^{A'} \xi_D = 0$ gives

$$\begin{aligned}
0 &= \nabla_{BA'} (\xi^A \xi^D \nabla_A{}^{A'} \xi_D) \\
&= \xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D + (\nabla_{BA'} \xi^A) \xi^D \nabla_A{}^{A'} \xi_D + (\nabla_{BA'} \xi^D) \xi^A \nabla_A{}^{A'} \xi_D \\
&= \xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D + 2 (\nabla_{BA'} \xi^D) \xi^A \nabla_{(A}{}^{A'} \xi_{D)}
\end{aligned} \tag{408}$$

The first term of the last line

$$\begin{aligned}
\xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D &= \frac{1}{2} \xi^A \xi^D (\nabla_{BA'} \nabla_A{}^{A'} \xi_D + \nabla_{DA'} \nabla_A{}^{A'} \xi_B) + \frac{1}{2} \xi^A \xi^D \epsilon_{BD} \nabla_{CA'} \nabla_A{}^{A'} \xi^C \\
&= \frac{1}{2} \xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D + \frac{1}{2} [\xi^A \xi^D \nabla_{DA'} \nabla_A{}^{A'} \xi_B - \xi^A \xi_B \nabla_{DA'} \nabla_A{}^{A'} \xi^D]
\end{aligned}$$

which implies

$$\xi^A \xi^D \nabla_{BA'} \nabla_A{}^{A'} \xi_D = \xi^A \xi^D \nabla_{DA'} \nabla_A{}^{A'} \xi_B - \xi^A \xi_B \nabla_{DA'} \nabla_A{}^{A'} \xi^D \tag{409}$$

Using (221) and (223)

$$
\begin{aligned}
\xi^A\xi^D\nabla_{BA'}\nabla_A{}^{A'}\xi_D &= \xi^A\xi^D\nabla_{A'(D}\nabla_{A)}{}^{A'}\xi_B - \xi^A\xi_B(2\nabla_{A'(D}\nabla_{A)}{}^{A'}\xi^D - \nabla_{AA'}\nabla_D{}^{A'}\xi^D) \\
&= \xi^A\xi^D\Box_{AD}\xi_B - 2\xi^A\xi_B\Box_{AD}\xi^D + \xi^A\xi_B\nabla_{AA'}\nabla_D{}^{A'}\xi^D \\
&= \Psi_{ABCD}\xi^A\xi^C\xi^D + \xi^A\xi_B\nabla_{AA'}\nabla_D{}^{A'}\xi^D
\end{aligned}
\tag{410}
$$

We introduce, analogously to (405), the proportinality factor $\overline{\zeta}_{A'}$

$$
\begin{aligned}
\xi_A\overline{\zeta}_{A'} &:= \xi^B\nabla_{AA'}\xi_B \\
&= \frac{1}{2}\xi^B(\nabla_{AA'}\xi_B + \nabla_{BA'}\xi_A + \epsilon_{DA'}\nabla_{DA'}\xi^D) \\
&= \frac{1}{2}\xi_A(\overline{\zeta}_{A'} + \overline{\eta}_{A'} - \nabla_{DA'}\xi^D)
\end{aligned}
\tag{411}
$$

Hence

$$
\overline{\zeta}_{A'} = \overline{\eta}_{A'} - \nabla_{DA'}\xi^D. \tag{412}
$$

The second term in (408) becomes

$$
\begin{aligned}
2(\nabla_{BA'}\xi^D)\xi^A\nabla_{(A}{}^{A'}\xi_{D)} &= (\nabla_{BA'}\xi^D)\xi_D(\overline{\eta}^{A'} + \overline{\zeta}^{A'}) \\
&= -\xi_B\overline{\zeta}_{A'}(\overline{\eta}^{A'} + \overline{\zeta}^{A'}) \\
&= \xi_B\overline{\eta}_{A'}\overline{\zeta}^{A'} \\
&= \xi_B\overline{\eta}_{A'}(\overline{\eta}^{A'} - \nabla_D{}^{A'}\xi^D) \\
&= -\xi_B\overline{\eta}_{A'}\nabla_D{}^{A'}\xi^D
\end{aligned}
\tag{413}
$$

The sum of the RHS of (410) and (413) is equal to zero by (408) hence

$$
\xi_B\xi^A\nabla_{AA'}\nabla_D{}^{A'}\xi^D - \xi_B\overline{\eta}_{A'}\nabla_D{}^{A'}\xi^D = -\Psi_{ABCD}\xi^A\xi^C\xi^D. \tag{414}
$$

Goldberg Sachs Theorem

Theorem 0.1.5 *In a spacetime which satifies the vacuum field equations $R_{\mu\nu} = 0$ any two of the following conditions imply the third:*

a) The Weyl tensor Ψ_{ABCD} is algebraically special with $n-$fold repeated principal spinor o, $(n = 2, 3, 4)$;

b) either spacetime is flat or o generates a geodesic shearfree congruence;

c) $\nabla^{AA'}\Psi_{ABCD}$ contracted with $(5 - n)$ o's vanishes.

Proof:

a) and b) imply c): $n = 2$.

Assume a), then $\Psi_{ABCD} = \xi_{(A}\xi_B\alpha_C\beta_{D)}$. Obviously

$$\xi^A\xi^B\xi^C\Psi_{ABCD} = 0,$$

and so

$$\xi^A\xi^B\xi^C\nabla^{DD'}\Psi_{ABCD} + 3\Psi_{ABCD}\xi^A\xi^B\nabla^{DD'}\xi^C = 0. \tag{415}$$

Now we use b). If spacetime is flat then condition c) obviously holds. If instead ξ is geodesic and shearfree

a) and c) imply b): $n = 2$.

Condition a) implies (415). If c) holds then (415) imjplies

$$\Psi_{ABCD}\xi^A\xi^B\nabla^{DD'}\xi^C = 0,$$

which on substitution of $\Psi_{ABCD} = \xi_{(A}\xi_B\alpha_C\beta_{D)}$ implies

$$\begin{aligned}
0 &= \xi_{(A}\xi_B\alpha_C\beta_{D)}\xi^A\xi^B\nabla^{DD'}\xi^C \\
&= \frac{1}{24}[\cdots + \alpha_A\beta_B(\xi_C\xi_D + \xi_D\xi_C) + \alpha_B\beta_A(\xi_C\xi_D + \xi_D\xi_C) + \ldots]\xi^A\xi^B\nabla^{DD'}\xi^C \\
&= \frac{1}{6}(\alpha_{(A}\beta_{B)}\xi^A\xi^B)\xi_C\xi_D\nabla^{DD'}\xi^C
\end{aligned} \tag{416}$$

and so

$$\xi_C\xi_D\nabla^{DD'}\xi^C = 0$$

which proves b).

b) and c) imply a): $n=2$.

Consider the relation

$$\xi^A\xi^B\xi^C\Psi_{ABCD} = x\xi_D.$$

We wish to show that $x=0$. Taking derivatives we have

$$\xi^A\xi^B\xi^C\nabla^{DD'}\Psi_{ABCD} + 3\Psi_{ABCD}\xi^A\xi^B\nabla^{DD'}\xi^C = \xi_D\nabla^{DD'}x + x\nabla^{DD'}\xi_D. \qquad (417)$$

Assuming c ($n=2$), the first term on the LHS vanishes. Assume $\Psi_{ABCD} = \xi_{(A}\alpha_B\beta_C\gamma_{D)}$, then we have for the second term on the LHS

$$\begin{aligned} 3\xi_{(A}\alpha_B\beta_C\gamma_{D)}\xi^A\xi^B\nabla^{DD'}\xi^C &= \frac{3}{4}[\xi_A\alpha_{(B}\beta_C\gamma_{D)} + \xi_B\alpha_{(A}\beta_C\gamma_{D)} \\ &\quad + \xi_C\alpha_{(A}\beta_B\gamma_{D)} + \xi_D\alpha_{(A}\beta_B\gamma_{C)}]\xi^A\xi^B\nabla^{DD'}\xi^C \\ &= \frac{3}{4}[\xi_C\alpha_{(A}\beta_B\gamma_{D)} + \xi_D\alpha_{(A}\beta_B\gamma_{C)}]\xi^A\xi^B\nabla^{DD'}\xi^C \end{aligned}$$

which on using (405) and (411) becomes

$$\begin{aligned} &\frac{3}{4}\xi^A\xi^B[\alpha_{(A}\beta_B\gamma_{D)}\xi_C\nabla^{DD'}\xi^C + \alpha_{(A}\beta_B\gamma_{C)}\xi_D\nabla^{DD'}\xi^C] \\ = \; & -\frac{3}{4}[\xi^A\xi^B\xi^D\overline{\zeta}^{D'}\alpha_{(A}\beta_B\gamma_{D)} + \xi^A\xi^B\xi^C\overline{\eta}^{D'}\alpha_{(A}\beta_B\gamma_{C)}] \\ = \; & -3x(\overline{\zeta}^{D'} + \overline{\eta}^{D'}) \end{aligned} \qquad (418)$$

where we used

$$4x = \xi^A\xi^B\xi^C\alpha_{(A}\beta_B\gamma_{C)} \qquad (419)$$

which follows from

$$x\xi_D = \xi^A\xi^B\xi^C\Psi_{ABCD} = \xi^A\xi^B\xi^C\xi_{(A}\alpha_B\beta_C\gamma_{D)} = \frac{1}{4}\xi^A\xi^B\xi^C\xi_D\alpha_{(A}\beta_B\gamma_{C)}.$$

(417) now reads

$$
\begin{aligned}
-3x(\overline{\zeta}^{D'} + \overline{\eta}^{D'}) &= \xi_D \nabla^{DD'} x + x\nabla^{DD'} \xi_D \\
&= \xi_D \nabla^{DD'} x + x(\overline{\zeta}^{D'} - \overline{\eta}^{D'})
\end{aligned} \tag{420}
$$

This becomes

$$
\xi^A \nabla_{AA'}(lnx) = 2\overline{\eta}_{A'} + 4\overline{\zeta}_{A'} = 6\overline{\eta}_{A'} - 4\nabla_{DA'}\xi^D. \tag{421}
$$

To check if a solution lnx of this equation exists we substitute its RHS for $\alpha_{A'} = 6\overline{\eta}_{A'} - 4\nabla_{DA'}\xi^D$ into the integrability theorem (0.1.4). This yields

$$
\begin{aligned}
-\xi^A \xi_B \nabla_{AA'}(6\overline{\eta}^{A'} - 4\nabla_D{}^{A'}\xi^D) &= \alpha_{A'}\xi^A \nabla_A{}^{A'}\xi_B \\
&= \xi_B \alpha_{A'} \overline{\eta}^{A'} \\
&= -\xi_B \overline{\eta}_{A'}(6\overline{\eta}^{A'} - 4\nabla_D{}^{A'}\xi^D) \\
&= 4\xi_B \overline{\eta}_{A'} \nabla_D{}^{A'}\xi^D
\end{aligned} \tag{422}
$$

Rearanging this gives

$$
-6\xi_B \xi^A \nabla_{AA'}\overline{\eta}^{A'} = -4(\xi_B \xi^A \nabla_{AA'}\nabla_D{}^{A'}\xi^D - \xi_B \overline{\eta}_{A'}\nabla_D{}^{A'}\xi^D) \tag{423}
$$

Substitution of the identities (407) and (414) gives

$$
-6\Psi_{ABCD}\xi^A\xi^B\xi^D = -4(-\Psi_{ABCD}\xi^A\xi^B\xi^D)
$$

or

$$
\Psi_{ABCD}\xi^A\xi^B\xi^D = 0
$$

which says $x = 0$, contary to our assumption.

□

0.1.15 Tetrad Formulism and the Cartan Structure Equations

$$T^{\hat{b}\ldots}_{\hat{a}\ldots} = e_{\hat{a}}{}^{a} e^{\hat{b}}{}_{b} \cdots T^{b\ldots}_{a\ldots} \tag{424}$$

$$T^{b\ldots}_{a\ldots} = e^{\hat{a}}{}_{a} e_{\hat{b}}{}^{b} \cdots T^{\hat{b}\ldots}_{\hat{a}\ldots} \tag{425}$$

The directional derivative along a tetrad vector is denoted by a comma or by ∂_a

$$T^{\hat{b}\ldots}_{\hat{a}\ldots,\hat{c}} = \partial_{\hat{c}} T^{\hat{b}\ldots}_{\hat{a}\ldots} = e_{\hat{c}}{}^{c} \frac{\partial}{\partial x^c} T^{\hat{b}\ldots}_{\hat{a}\ldots} \tag{426}$$

The tetrad components of the covariant derivative are denoted by a semicolon

$$T^{\hat{b}\ldots}_{\hat{a}\ldots;\hat{c}} = e^{\hat{a}}{}_{a} e_{\hat{b}}{}^{b} \cdots e_{\hat{c}}{}^{c} T^{b\ldots}_{a\ldots;c} \tag{427}$$

They are given by

$$T^{\hat{b}\ldots}_{\hat{a}\ldots;c} = T^{\hat{b}\ldots}_{\hat{a}\ldots,\hat{c}} - \Gamma^{\hat{d}}{}_{\hat{a}\hat{c}} T^{\hat{b}\ldots}_{\hat{d}\ldots} - \cdots + \Gamma^{\hat{b}}{}_{\hat{d}\hat{c}} T^{\hat{d}\ldots}_{\hat{a}\ldots} + \cdots \tag{428}$$

where the $\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$ are the Ricci rotation coefficients

$$\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} = -e^{\hat{a}}{}_{a;b} e_{\hat{b}}{}^{a} e_{\hat{c}}{}^{b} \tag{429}$$

and take the place of the Christoffel symbols in the tetradad formulism. The rotation coefficients also appear in the commutator of two directional derivatives along tetrad vectors

$$\begin{aligned}
2T^{\cdots}_{\ldots,[\hat{a}\hat{b}]} &= T^{\cdots}_{\ldots,\hat{a}\hat{b}} - T^{\cdots}_{\ldots,\hat{b}\hat{a}} \\
&= e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a} T^{\cdots}_{\ldots,a})_{,b} - e_{\hat{a}}{}^{a}(e_{\hat{b}}{}^{b} T^{\cdots}_{\ldots,b})_{,a} \\
&= e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a})_{,b} T^{\cdots}_{\ldots,a} - e_{\hat{a}}{}^{a}(e_{\hat{b}}{}^{b})_{,a} T^{\cdots}_{\ldots,b} + e_{\hat{a}}{}^{a} e_{\hat{b}}{}^{b}(T^{\cdots}_{\ldots,ab} - T^{\cdots}_{\ldots,ba}) \\
&= T^{\cdots}_{\ldots,\hat{c}}[e^{\hat{c}}{}_{a}(e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a})_{,b} - e_{\hat{a}}{}^{b}(e_{\hat{b}}{}^{a})_{,b})] \\
&= T^{\cdots}_{\ldots,\hat{c}}[e^{\hat{c}}{}_{a}(e_{\hat{b}}{}^{b}[(e_{\hat{a}}{}^{a})_{,b} + \Gamma^{a}_{bd} e_{\hat{a}}{}^{d}] - e_{\hat{a}}{}^{b}[(e_{\hat{b}}{}^{a})_{,b} + \Gamma^{a}_{bd} e_{\hat{a}}{}^{d}])] \\
&= T^{\cdots}_{\ldots,\hat{c}}[e^{\hat{c}}{}_{a}(e_{\hat{b}}{}^{b}(e_{\hat{a}}{}^{a})_{;b} - e_{\hat{a}}{}^{b}(e_{\hat{b}}{}^{a})_{;b})] \\
&= 2T^{\cdots}_{\ldots,\hat{c}} \Gamma^{\hat{c}}{}_{[\hat{a}\hat{b}]}
\end{aligned} \tag{430}$$

The tetrad vector determine linear differential forms

$$e^{\hat{a}}{}_{a} dx^a = e^{\hat{a}}, \tag{431}$$

in terms of which the metric form is given by

$$ds^2 = e_{\hat{a}} e^{\hat{a}} = e_{\hat{a}a} e^{\hat{a}}{}_{b} dx^a dx^b = g_{ab} dx^a dx^b \tag{432}$$

The exterior product of two linear differential forms $A = A_a dx^a$ and $B = B_a dx^a$ is the anti-symmetric multiplication

$$\begin{aligned} A \wedge B &= -B \wedge A \\ &= A_a B_b dx^a \wedge dx^b \\ &= A_{[a} B_{b]} dx^a \wedge dx^b \end{aligned} \tag{433}$$

The exterior derivative of a linear differential form is

$$dA = A_{b,a} dx^a \wedge dx^b = A_{[b,a]} dx^a \wedge dx^b \tag{434}$$

Cartan Structure Equations

We define

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^c \tag{435}$$

$$\mathcal{R}^{\hat{a}}{}_{\hat{b}} = R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}}. \tag{436}$$

The Cartan structure equations are

$$de^{\hat{a}} = e^{\hat{b}} \wedge \Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}} \tag{437}$$

$$\frac{1}{2}\mathcal{R}^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{f}} \wedge \Gamma^{\hat{f}}{}_{\hat{b}} \tag{438}$$

Proof:

First equation

$$
\begin{aligned}
de^{\hat{a}} &= d(e^{\hat{a}}{}_a dx^a) \\
&= \partial_b e^{\hat{a}}{}_b dx^b \wedge dx^a \\
&= -(\partial_b e^{\hat{a}}{}_a - \Gamma^d_{ba} e^{\hat{a}}{}_d) dx^a \wedge dx^b \\
&= -\nabla_b e^{\hat{a}}{}_a dx^a \wedge dx^b \\
&= -(\nabla_b e^{\hat{a}}{}_a) \delta^a_c \delta^b_d \, dx^c \wedge dx^d \\
&= -(\nabla_b e^{\hat{a}}{}_a) e_{\hat{b}}{}^a e_{\hat{c}}{}^b e^{\hat{b}}{}_c e^{\hat{c}}{}_d dx^c \wedge dx^d \\
&= -(\nabla_b e^{\hat{a}}{}_a) e_{\hat{b}}{}^a e_{\hat{c}}{}^b e^{\hat{b}} \wedge e^{\hat{c}} \\
&= \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}}. \qquad (439)
\end{aligned}
$$

Second equation:

$$
\begin{aligned}
R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}} &= e^{\hat{a}}{}_a e_{\hat{b}}{}^b R^a{}_{b\hat{c}\hat{d}} e^{\hat{c}}{}_c e^{\hat{d}}{}_d dx^c \wedge dx^d \\
&= e^{\hat{a}}{}_a e_{\hat{b}}{}^b R^a{}_{bcd} dx^c \wedge dx^d \\
&= 2 e^{\hat{a}}{}_a e_{\hat{b}}{}^b (\partial_c \Gamma^a_{bd} + \Gamma^e_{bd} \Gamma^a_{ce}) dx^c \wedge dx^d \qquad (440)
\end{aligned}
$$

Consider $d\Gamma^{\hat{a}}{}_{\hat{b}}$

$$
\begin{aligned}
d\Gamma^{\hat{a}}{}_{\hat{b}} &= d(\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{c}}{}_b dx^b) \\
&= d[(e^{\hat{a}}{}_c e_{\hat{c}}{}^d \nabla_d e_{\hat{b}}{}^c) e^{\hat{c}}{}_b dx^b] \\
&= d(e^{\hat{a}}{}_c \nabla_b e_{\hat{b}}{}^c dx^b) \\
&= d([e^{\hat{a}}{}_c \partial_b e_{\hat{b}}{}^c + e^{\hat{a}}{}_c \Gamma^c_{bd} e_{\hat{b}}{}^d] dx^b) \\
&= [(\partial_a e^{\hat{a}}{}_c)(\partial_b e_{\hat{b}}{}^c) + e^{\hat{a}}{}_c \partial_a \partial_b e_{\hat{b}}{}^c + (\partial_a e^{\hat{a}}{}_c) \Gamma^c_{bd} e_{\hat{b}}{}^d \\
&\quad + e^{\hat{a}}{}_c \Gamma^c_{bd} (\partial_a e_{\hat{b}}{}^d) + e^{\hat{a}}{}_c (\partial_a \Gamma^c_{bd}) e_{\hat{b}}{}^d] dx^a \wedge dx^b. \qquad (441)
\end{aligned}
$$

The term $e^{\hat{a}}{}_c \partial_a \partial_b e_{\hat{b}}{}^c dx^a \wedge dx^b$ vanishes. Now

$$
\begin{aligned}
\Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} &= (e^{\hat{a}}{}_c \nabla_a e_{\hat{c}}{}^c dx^a) \wedge (e^{\hat{c}}{}_d \nabla_b e_{\hat{b}}{}^d dx^b) \\
&= [e^{\hat{a}}{}_c \partial_a e_{\hat{c}}{}^c + e^{\hat{a}}{}_c \Gamma^c_{ae} e_{\hat{c}}{}^e] e^{\hat{c}}{}_d [\partial_b e_{\hat{b}}{}^d + \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= [-e_{\hat{c}}{}^c (\partial_a e^{\hat{a}}{}_c) + e^{\hat{a}}{}_c \Gamma^c_{ae} e_{\hat{c}}{}^e] e^{\hat{c}}{}_d [\partial_b e_{\hat{b}}{}^d + \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= [-(\partial_a e^{\hat{a}}{}_d) + e^{\hat{a}}{}_c \Gamma^c_{ad}][\partial_b e_{\hat{b}}{}^d + \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= -(\partial_a e^{\hat{a}}{}_c)(\partial_b e_{\hat{b}}{}^c) - (\partial_a e^{\hat{a}}{}_c) \Gamma^c_{bd} e_{\hat{b}}{}^d + e^{\hat{a}}{}_c \Gamma^c_{ad} (\partial_b e_{\hat{b}}{}^d) \\
&\quad + e^{\hat{a}}{}_c \Gamma^c_{ad} \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \qquad (442)
\end{aligned}
$$

Combining (441) and (442)

$$
\begin{aligned}
d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} &= [e^{\hat{a}}{}_c \Gamma^c_{bd} (\partial_a e_{\hat{b}}{}^d) + e^{\hat{a}}{}_c \Gamma^c_{ad} (\partial_b e_{\hat{b}}{}^d) \\
&\quad e^{\hat{a}}{}_c (\partial_a \Gamma^c_{bd}) e_{\hat{b}}{}^d + e^{\hat{a}}{}_c \Gamma^c_{ad} \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= [e^{\hat{a}}{}_c (\partial_a \Gamma^c_{bd}) e_{\hat{b}}{}^d + e^{\hat{a}}{}_c \Gamma^c_{ad} \Gamma^d_{bf} e_{\hat{b}}{}^f] dx^a \wedge dx^b \\
&= e^{\hat{a}}{}_a e_{\hat{b}}{}^b (\partial_c \Gamma^a_{db} + \Gamma^a_{ce} \Gamma^e_{db}) dx^c \wedge dx^d \\
&= \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}}. \qquad (443)
\end{aligned}
$$

□

Equations (437) determine the anti-symmetric part of $\Gamma^{\hat{a}}{}_{[\hat{b}\hat{c}]}$ of the rotation coefficients. Define

$$
\Gamma_{\hat{a}\hat{b}\hat{c}} = g_{\hat{a}\hat{d}} \Gamma^{\hat{d}}{}_{\hat{b}\hat{c}}. \qquad (444)
$$

The vanishing of the covariant derivatives of the metric tensor

$$
\begin{aligned}
0 &= \nabla_{\hat{c}} g_{\hat{a}\hat{b}} = g_{\hat{a}\hat{b},\hat{c}} - \Gamma^{\hat{d}}_{\hat{a}\hat{c}} g_{\hat{d}\hat{b}} - \Gamma^{\hat{d}}_{\hat{b}\hat{c}} g_{\hat{a}\hat{d}} \\
&= g_{\hat{a}\hat{b},\hat{c}} - \Gamma_{\hat{b}\hat{a}\hat{c}} - \Gamma_{\hat{a}\hat{b}\hat{c}}
\end{aligned}
$$

implies the symmetric part $\Gamma_{(\hat{a}\hat{b})\hat{c}}$ of the rotation coefficients

$$
\Gamma_{(\hat{a}\hat{b})\hat{c}} = \frac{1}{2} \partial_{\hat{c}} g_{\hat{a}\hat{b}}. \qquad (445)
$$

The expressions for $\Gamma^{\hat{a}}{}_{[\hat{b}\hat{c}]}$ and $\Gamma_{(\hat{a}\hat{b})\hat{c}}$ determine all rotation coefficients, and (438) then determine all the components of the curvature tensor.

Specialisation to rigid tetrads

When we limit ourselves to the case of rigid tetrads where the $g_{\hat{a}\hat{b}}$ are constants. The rotation coefficients are the anti-symmetric in the first two indicies

$$
\Gamma_{\hat{a}\hat{b}\hat{c}} = -\Gamma_{\hat{b}\hat{a}\hat{c}},
$$

and are determined by (437)

$$
\begin{aligned}
\Gamma_{\hat{a}\hat{b}\hat{c}} &= \frac{1}{2}(\Gamma_{\hat{a}\hat{b}\hat{c}} + \Gamma_{\hat{c}\hat{a}\hat{b}}) + \frac{1}{2}(\Gamma_{\hat{b}\hat{c}\hat{a}} + \Gamma_{\hat{a}\hat{b}\hat{c}}) - \frac{1}{2}(\Gamma_{\hat{c}\hat{a}\hat{b}} + \Gamma_{\hat{b}\hat{c}\hat{a}}) \\
&= \frac{1}{2}(\Gamma_{\hat{a}\hat{b}\hat{c}} - \Gamma_{\hat{a}\hat{c}\hat{b}}) + \frac{1}{2}(\Gamma_{\hat{b}\hat{c}\hat{a}} - \Gamma_{\hat{b}\hat{a}\hat{c}}) - \frac{1}{2}(\Gamma_{\hat{c}\hat{a}\hat{b}} - \Gamma_{\hat{c}\hat{b}\hat{a}}) \\
&= \Gamma_{\hat{a}[\hat{b}\hat{c}]} + \Gamma_{\hat{b}[\hat{c}\hat{a}]} - \Gamma_{\hat{c}[\hat{a}\hat{b}]} \qquad (446)
\end{aligned}
$$

The curvature one forms $\Gamma_{\hat{a}\hat{b}}$ satisfy

$$
\Gamma_{\hat{a}\hat{b}} = -\Gamma_{\hat{b}\hat{a}} \qquad (447)
$$

and therefore has six independent components. They are obtained from the first Cartan structure equation (437),

$$
de^{\hat{a}} = \Gamma^{\hat{a}}{}_{\hat{b}} \wedge e^{\hat{b}}.
$$

Once we have calculated the curvature one forms, we obtain the Ricci rotation coefficients using

$$
\Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{c}}. \qquad (448)
$$

We calculate the curvature two forms $\mathcal{R}^{\hat{a}}{}_{\hat{b}}$ using the second Cartan structure equation (438),

$$
\frac{1}{2}\mathcal{R}^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{f}} \wedge \Gamma^{\hat{f}}{}_{\hat{b}}
$$

which are related to the Riemann tensor as

$$
\mathcal{R}^{\hat{a}}{}_{\hat{b}} = R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}} \qquad (449)
$$

After calculating the Riemann tensor we obtain the Ricci tensor by

$$
R_{\hat{a}\hat{b}} = R_{\hat{b}\hat{a}} = R^{\hat{c}}{}_{\hat{a}\hat{b}\hat{c}} = g^{\hat{c}\hat{d}} R_{\hat{c}\hat{a}\hat{b}\hat{d}}. \qquad (450)
$$

There are 10 independent components. We then calculate the Rici scalar by

$$
R = g^{\hat{a}\hat{b}} R_{\hat{a}\hat{b}}. \qquad (451)
$$

0.1.16 Specialisation to Null Tetrads

$$e_{\hat{0}} = l, \quad e_{\hat{1}} = n, \quad e_{\hat{2}} = m, \quad e_{\hat{3}} = \overline{m}, \tag{452}$$

$$g_{\hat{a}\hat{b}} = g^{\hat{a}\hat{b}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{453}$$

$$e^{\hat{a}} = g^{\hat{a}\hat{b}} e_{\hat{b}} \tag{454}$$

$$e^{\hat{0}} = e_{\hat{1}} = n, \quad e^{\hat{1}} = e_{\hat{0}} = l, \quad e^{\hat{2}} = -e_{\hat{3}} = -\overline{m}, \quad e^{\hat{3}} = -e_{\hat{2}} = -m, \tag{455}$$

Independent curvature one forms

There are six independent curvature one forms $\Gamma_{\hat{a}\hat{b}}$ as it is anti-symmetric. The non-zero curvature one forms are

$$\Gamma_{\hat{0}\hat{1}}, \Gamma_{\hat{0}\hat{2}}, \Gamma_{\hat{0}\hat{3}}, \Gamma_{\hat{1}\hat{2}}, \Gamma_{\hat{1}\hat{3}}, \Gamma_{\hat{2}\hat{3}} \tag{456}$$

However

$$\Gamma_{\hat{0}\hat{1}} = \overline{\Gamma_{\hat{0}\hat{1}}}, \quad \Gamma_{\hat{0}\hat{2}} = \overline{\Gamma_{\hat{0}\hat{3}}}, \quad \Gamma_{\hat{1}\hat{3}} = \overline{\Gamma_{\hat{1}\hat{2}}}, \quad \Gamma_{\hat{2}\hat{3}} = -\overline{\Gamma_{\hat{2}\hat{3}}}.$$

The collection (456) is equivalent to the collection

$$\Gamma_{\hat{1}\hat{0}} + \Gamma_{\hat{2}\hat{3}}, \Gamma_{\hat{0}\hat{2}}, \Gamma_{\hat{0}\hat{3}}, \Gamma_{\hat{1}\hat{2}}, \Gamma_{\hat{1}\hat{3}}, \Gamma_{\hat{1}\hat{0}} - \Gamma_{\hat{2}\hat{3}} \tag{457}$$

This collection is made up of the one forms

$$\Gamma_{\hat{0}\hat{3}}, \quad \Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}, \quad \Gamma_{\hat{1}\hat{2}} \tag{458}$$

and their complex cojugates. The colection (458) are taken as our six independent curvature one forms.

Independent rotation coefficients

Recall the Ricci rotation coefficients are given by

$$\Gamma_{\hat{a}\hat{b}\hat{c}} = -e_{\hat{a}a;b}e_{\hat{b}}{}^{a}e_{\hat{c}}{}^{b}$$

$$\begin{aligned}
\kappa &= \Gamma_{\hat{2}\hat{0}\hat{0}} & \epsilon &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{0}} + \Gamma_{\hat{2}\hat{3}\hat{0}}) & \pi &= \Gamma_{\hat{1}\hat{3}\hat{0}} \\
\sigma &= \Gamma_{\hat{2}\hat{0}\hat{2}} & \beta &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{2}} + \Gamma_{\hat{2}\hat{3}\hat{2}}) & \mu &= \Gamma_{\hat{1}\hat{3}\hat{2}} \\
\rho &= \Gamma_{\hat{2}\hat{0}\hat{3}} & \alpha &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{3}} + \Gamma_{\hat{2}\hat{3}\hat{3}}) & \lambda &= \Gamma_{\hat{1}\hat{3}\hat{3}} \\
\tau &= \Gamma_{\hat{2}\hat{0}\hat{1}} & \gamma &= \frac{1}{2}(\Gamma_{\hat{1}\hat{0}\hat{1}} + \Gamma_{\hat{2}\hat{3}\hat{1}}) & \nu &= \Gamma_{\hat{1}\hat{3}\hat{3}}
\end{aligned} \tag{459}$$

Independent curvature two forms

Before we write down the first Cartan structure equations for these independent one forms we need the following:

$$\begin{aligned}
\Gamma_{\hat{0}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{3}} &= g^{\hat{c}\hat{d}}\Gamma_{\hat{0}\hat{c}} \wedge \Gamma_{\hat{d}\hat{3}} \\
&= g^{\hat{0}\hat{1}}\Gamma_{\hat{0}\hat{0}} \wedge \Gamma_{\hat{1}\hat{3}} + g^{\hat{1}\hat{0}}\Gamma_{\hat{0}\hat{1}} \wedge \Gamma_{\hat{0}\hat{3}} + g^{\hat{2}\hat{3}}\Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{3}\hat{3}} + g^{\hat{3}\hat{2}}\Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{2}\hat{3}} \\
&= \Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{1}\hat{0}} - \Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{2}\hat{3}}
\end{aligned} \tag{460}$$

$$\begin{aligned}
\Gamma_{\hat{2}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{3}} &= g^{\hat{c}\hat{d}}\Gamma_{\hat{2}\hat{c}} \wedge \Gamma_{\hat{d}\hat{3}} \\
&= g^{\hat{0}\hat{1}}\Gamma_{\hat{2}\hat{0}} \wedge \Gamma_{\hat{1}\hat{3}} + g^{\hat{1}\hat{d}}\Gamma_{\hat{2}\hat{1}} \wedge \Gamma_{\hat{0}\hat{3}} + g^{\hat{2}\hat{3}}\Gamma_{\hat{2}\hat{2}} \wedge \Gamma_{\hat{3}\hat{3}} + g^{\hat{3}\hat{2}}\Gamma_{\hat{2}\hat{3}} \wedge \Gamma_{\hat{2}\hat{3}} \\
&= \Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{1}\hat{2}} - \Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{1}\hat{3}}
\end{aligned} \tag{461}$$

$$\begin{aligned}
\Gamma_{\hat{1}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{0}} &= g^{\hat{c}\hat{d}}\Gamma_{\hat{1}\hat{c}} \wedge \Gamma_{\hat{d}\hat{0}} \\
&= g^{\hat{0}\hat{1}}\Gamma_{\hat{1}\hat{0}} \wedge \Gamma_{\hat{1}\hat{0}} + g^{\hat{1}\hat{0}}\Gamma_{\hat{1}\hat{1}} \wedge \Gamma_{\hat{0}\hat{0}} + g^{\hat{2}\hat{3}}\Gamma_{\hat{1}\hat{2}} \wedge \Gamma_{\hat{3}\hat{0}} + g^{\hat{3}\hat{2}}\Gamma_{\hat{1}\hat{3}} \wedge \Gamma_{\hat{2}\hat{0}} \\
&= -\Gamma_{\hat{0}\hat{3}} \wedge \Gamma_{\hat{1}\hat{2}} - \Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{1}\hat{3}}
\end{aligned} \tag{462}$$

$$
\begin{aligned}
\Gamma_{\hat{1}\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{2}} &= g^{\hat{c}\hat{d}}\Gamma_{\hat{1}\hat{c}} \wedge \Gamma_{\hat{d}\hat{2}} \\
&= g^{\hat{0}\hat{1}}\Gamma_{\hat{1}\hat{0}} \wedge \Gamma_{\hat{1}\hat{2}} + g^{\hat{1}\hat{0}}\Gamma_{\hat{1}\hat{1}} \wedge \Gamma_{\hat{0}\hat{2}} + g^{\hat{2}\hat{3}}\Gamma_{\hat{1}\hat{2}} \wedge \Gamma_{\hat{3}\hat{2}} + g^{\hat{3}\hat{2}}\Gamma_{\hat{1}\hat{3}} \wedge \Gamma_{\hat{2}\hat{2}} \\
&= \Gamma_{\hat{1}\hat{0}} \wedge \Gamma_{\hat{1}\hat{2}} - \Gamma_{\hat{2}\hat{3}} \wedge \Gamma_{\hat{1}\hat{2}} \qquad (463)
\end{aligned}
$$

The first Cartan structure equations are then

$$
\begin{aligned}
d\Gamma_{\hat{0}\hat{3}} + \Gamma_{\hat{0}\hat{3}} \wedge (-\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) &= \frac{1}{2}R_{\hat{0}\hat{3}\hat{a}\hat{b}}e^{\hat{a}} \wedge e^{\hat{b}} \qquad (464) \\
d(\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) - 2\Gamma_{\hat{0}\hat{2}} \wedge \Gamma_{\hat{1}\hat{3}} &= \frac{1}{2}(R_{\hat{2}\hat{3}\hat{a}\hat{b}} + R_{\hat{1}\hat{0}\hat{a}\hat{b}})e^{\hat{a}} \wedge e^{\hat{b}} \qquad (465) \\
d\Gamma_{\hat{1}\hat{2}} + (-\Gamma_{\hat{2}\hat{3}} + \Gamma_{\hat{1}\hat{0}}) \wedge \Gamma_{\hat{1}\hat{2}} &= \frac{1}{2}R_{\hat{1}\hat{2}\hat{a}\hat{b}}e^{\hat{a}} \wedge e^{\hat{b}} \qquad (466)
\end{aligned}
$$

Independent components of the Ricci tensor

$$
R_{\hat{a}\hat{b}} = R_{\hat{b}\hat{a}} = R^{\hat{c}}{}_{\hat{a}\hat{b}\hat{c}} = g^{\hat{c}\hat{d}}R_{\hat{c}\hat{a}\hat{b}\hat{d}} \qquad (467)
$$

$$
\begin{aligned}
R_{\hat{3}\hat{3}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{3}\hat{3}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{3}\hat{3}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{3}\hat{3}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{3}\hat{3}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{3}\hat{3}\hat{2}} \\
&= 2R_{\hat{0}\hat{3}\hat{3}\hat{1}}. \qquad (468)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{3}\hat{0}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{3}\hat{0}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{3}\hat{0}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{3}\hat{0}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{3}\hat{0}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{3}\hat{0}\hat{2}} \\
&= -R_{\hat{0}\hat{3}\hat{2}\hat{3}} - R_{\hat{0}\hat{3}\hat{1}\hat{0}}. \qquad (469)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{0}\hat{0}} &= g^{\hat{c}\hat{d}}R_{\hat{c}\hat{0}\hat{0}\hat{d}} \\
&= g^{\hat{0}\hat{1}}R_{\hat{0}\hat{0}\hat{0}\hat{1}} + g^{\hat{1}\hat{0}}R_{\hat{1}\hat{0}\hat{0}\hat{0}} + g^{\hat{2}\hat{3}}R_{\hat{2}\hat{0}\hat{0}\hat{3}} + g^{\hat{3}\hat{2}}R_{\hat{3}\hat{0}\hat{0}\hat{2}} \\
&= -2R_{\hat{0}\hat{3}\hat{2}\hat{0}}. \qquad (470)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{2}\hat{3}} &= g^{\hat{c}\hat{d}} R_{\hat{c}\hat{2}\hat{3}\hat{d}} \\
&= g^{\hat{0}\hat{1}} R_{\hat{0}\hat{2}\hat{3}\hat{1}} + g^{\hat{1}\hat{0}} R_{\hat{1}\hat{2}\hat{3}\hat{0}} + g^{\hat{2}\hat{3}} R_{\hat{2}\hat{2}\hat{3}\hat{3}} + g^{\hat{3}\hat{2}} R_{\hat{3}\hat{2}\hat{3}\hat{2}} \\
&= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{2}\hat{3}\hat{0}} + R_{\hat{0}\hat{2}\hat{3}\hat{1}} \\
&= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{2}\hat{3}\hat{0}} - (R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{0}\hat{1}\hat{2}\hat{3}}) \\
&= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}}. \qquad (471)
\end{aligned}
$$

where we used the cyclic identity $R_{\hat{0}\hat{2}\hat{3}\hat{1}} + R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{0}\hat{1}\hat{2}\hat{3}} = 0$.

$$
\begin{aligned}
R_{\hat{1}\hat{0}} &= g^{\hat{c}\hat{d}} R_{\hat{c}\hat{1}\hat{0}\hat{d}} \\
&= g^{\hat{0}\hat{1}} R_{\hat{0}\hat{1}\hat{0}\hat{1}} + g^{\hat{1}\hat{0}} R_{\hat{1}\hat{1}\hat{0}\hat{0}} + g^{\hat{2}\hat{3}} R_{\hat{2}\hat{1}\hat{0}\hat{3}} + g^{\hat{3}\hat{2}} R_{\hat{3}\hat{1}\hat{0}\hat{2}} \\
&= +R_{\hat{1}\hat{0}\hat{1}\hat{0}} - R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{0}\hat{2}\hat{3}\hat{1}} \\
&= +R_{\hat{1}\hat{0}\hat{1}\hat{0}} - R_{\hat{0}\hat{3}\hat{1}\hat{2}} - (R_{\hat{0}\hat{3}\hat{1}\hat{2}} + R_{\hat{2}\hat{3}\hat{0}\hat{1}}) \\
&= R_{\hat{2}\hat{3}\hat{0}\hat{1}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}} \qquad (472)
\end{aligned}
$$

where we used the same cyclic identity again.

$$
\begin{aligned}
R_{\hat{1}\hat{1}} &= g^{\hat{c}\hat{d}} R_{\hat{c}\hat{1}\hat{1}\hat{d}} \\
&= g^{\hat{0}\hat{1}} R_{\hat{0}\hat{1}\hat{1}\hat{1}} + g^{\hat{1}\hat{0}} R_{\hat{1}\hat{1}\hat{1}\hat{0}} + g^{\hat{2}\hat{3}} R_{\hat{2}\hat{1}\hat{1}\hat{3}} + g^{\hat{3}\hat{2}} R_{\hat{3}\hat{1}\hat{1}\hat{2}} \\
&= 2R_{\hat{1}\hat{2}\hat{1}\hat{3}}. \qquad (473)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{3}\hat{1}} &= g^{\hat{c}\hat{d}} R_{\hat{c}\hat{3}\hat{1}\hat{d}} \\
&= g^{\hat{0}\hat{1}} R_{\hat{0}\hat{3}\hat{1}\hat{1}} + g^{\hat{1}\hat{0}} R_{\hat{1}\hat{3}\hat{1}\hat{0}} + g^{\hat{2}\hat{3}} R_{\hat{2}\hat{3}\hat{1}\hat{3}} + g^{\hat{3}\hat{2}} R_{\hat{3}\hat{3}\hat{1}\hat{2}} \\
&= -R_{\hat{2}\hat{3}\hat{1}\hat{3}} + R_{\hat{1}\hat{0}\hat{1}\hat{3}}. \qquad (474)
\end{aligned}
$$

Altogether

$$
\begin{aligned}
R_{\hat{2}\hat{2}} &= 2R_{\hat{0}\hat{2}\hat{2}\hat{1}}, \qquad (475) \\
R_{\hat{3}\hat{0}} &= -R_{\hat{0}\hat{3}\hat{2}\hat{3}} - R_{\hat{0}\hat{3}\hat{1}\hat{0}}, \qquad (476) \\
R_{\hat{0}\hat{0}} &= -2R_{\hat{0}\hat{3}\hat{2}\hat{0}}, \qquad (477) \\
& \qquad (478) \\
R_{\hat{2}\hat{3}} &= R_{\hat{2}\hat{3}\hat{3}\hat{2}} + R_{\hat{1}\hat{0}\hat{2}\hat{3}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}}, \qquad (479) \\
R_{\hat{1}\hat{0}} &= R_{\hat{2}\hat{3}\hat{0}\hat{1}} + R_{\hat{1}\hat{0}\hat{1}\hat{0}} - 2R_{\hat{0}\hat{3}\hat{1}\hat{2}}, \qquad (480) \\
R_{\hat{1}\hat{1}} &= 2R_{\hat{1}\hat{2}\hat{1}\hat{3}}, \qquad (481) \\
R_{\hat{3}\hat{1}} &= -R_{\hat{2}\hat{3}\hat{1}\hat{3}} + R_{\hat{1}\hat{0}\hat{1}\hat{3}} \qquad (482)
\end{aligned}
$$

where $R_{\hat{0}\hat{0}}$, $R_{\hat{2}\hat{3}}$, $R_{\hat{1}\hat{0}}$, $R_{\hat{1}\hat{1}}$ are real and $R_{\hat{2}\hat{2}}$, $R_{\hat{2}\hat{0}}$, $R_{\hat{2}\hat{1}}$ are complex conjugates of $R_{\hat{3}\hat{3}}$, $R_{\hat{3}\hat{0}}$, $R_{\hat{3}\hat{1}}$, respectively:

$$\begin{aligned} R_{\hat{3}\hat{3}} &= \overline{R_{\hat{2}\hat{2}}} \\ R_{\hat{2}\hat{0}} &= \overline{R_{\hat{3}\hat{0}}} \\ R_{\hat{2}\hat{1}} &= \overline{R_{\hat{3}\hat{1}}} \end{aligned} \tag{483}$$

We have 10 independent terms altogether.

The Ricci scalar is given by

$$\begin{aligned} R &= g^{\hat{a}\hat{b}} R_{\hat{a}\hat{b}} \\ &= g^{\hat{0}\hat{1}} R_{\hat{0}\hat{1}} + g^{\hat{1}\hat{0}} R_{\hat{1}\hat{0}} + g^{\hat{2}\hat{3}} R_{\hat{2}\hat{3}} + g^{\hat{3}\hat{2}} R_{\hat{3}\hat{2}} \\ &= 2(R_{\hat{1}\hat{0}} - R_{\hat{2}\hat{3}}). \end{aligned} \tag{484}$$

The Weyl tensor and Petrov classification

$$\begin{aligned} \Psi_0 &= C_{abcd} l^a m^b l^c m^d \\ &= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} gcb)] l^a m^b l^c m^d \\ &= R_{abcd} e_{\hat{0}}{}^a e_{\hat{2}}{}^b e_{\hat{0}}{}^c e_{\hat{2}}{}^d \\ &= R_{\hat{0}\hat{2}\hat{0}\hat{2}}. \end{aligned} \tag{485}$$

$$\begin{aligned} \Psi_1 &= C_{abcd} l^a m^b l^c n^d \\ &= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} g_{cb})] l^a m^b l^c n^d \\ &= R_{abcd} l^a m^b l^c n^d - \frac{1}{2}(-R_{bc} m^b l^c) \\ &= R_{abcd} e_{\hat{0}}{}^a e_{\hat{2}}{}^b e_{\hat{0}}{}^c e_{\hat{1}}{}^d + \frac{1}{2} R_{bc} m^b l^c \\ &= R_{\hat{0}\hat{2}\hat{0}\hat{1}} + \frac{1}{2} R_{\hat{2}\hat{0}}. \end{aligned} \tag{486}$$

$$
\begin{aligned}
\Psi_2 &= C_{abcd} l^a m^b \overline{m}^c n^d \\
&= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} g_{cb})] l^a m^b l^c n^d \\
&= R_{abcd} l^a m^b l^c n^d - \frac{1}{2}(-R_{bc} m^b l^c) \\
&= R_{abcd} e_{\hat{0}}{}^a e_{\hat{2}}{}^b e_{\hat{3}}{}^c e_{\hat{1}}{}^d + \frac{1}{2} R_{bc} e_{\hat{2}}{}^b e_{\hat{3}}{}^c \\
&= R_{\hat{0}\hat{2}\hat{3}\hat{1}} + \frac{1}{2} R_{\hat{2}\hat{3}}.
\end{aligned} \tag{487}
$$

$$
\begin{aligned}
\Psi_3 &= C_{abcd} l^a n^b \overline{m}^c n^d \\
&= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} g_{cb})] l^a n^b \overline{m}^c n^d \\
&= R_{abcd} l^a n^b \overline{m}^c n^d - \frac{1}{2}(-R_{bc} n^b \overline{m}^c) \\
&= R_{abcd} e_{\hat{0}}{}^a e_{\hat{1}}{}^b e_{\hat{3}}{}^c e_{\hat{1}}{}^d + \frac{1}{2} R_{bc} e_{\hat{1}}{}^b e_{\hat{3}}{}^c \\
&= R_{\hat{0}\hat{2}\hat{0}\hat{1}} + \frac{1}{2} R_{\hat{1}\hat{3}}.
\end{aligned} \tag{488}
$$

$$
\begin{aligned}
\Psi_4 &= C_{abcd} \overline{m}^a n^b \overline{m}^c n^d \\
&= [R_{abcd} - \frac{1}{2}(R_{ac} g_{db} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + \frac{R}{6}(g_{ac} g_{db} - g_{ad} gcb)] \overline{m}^a n^b \overline{m}^c n^d \\
&= R_{abcd} \overline{m}^a n^b \overline{m}^c n^d \\
&= R_{abcd} e_{\hat{3}}{}^a e_{\hat{1}}{}^b e_{\hat{3}}{}^c e_{\hat{1}}{}^d \\
&= R_{\hat{3}\hat{1}\hat{3}\hat{1}}.
\end{aligned} \tag{489}
$$