Spin-Networks

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Chapter 1

Spin Networks

Mostly follows "A Spin Network Primer" [?]

1.1 Diagrammatic Mathematics

Diagrammatic algebra designed too handle the combinatorics of irreducible representations, all the familiar results of representation theory have diagrammatical form.

1.1.1 Line, Bend ad Loop

Consider the tensor

$$(\delta^A_B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which can be represented diagramatically as in fig 1.1

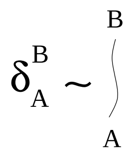

Figure 1.1: Diagramatically representation of δ^A_B

Consider the two antisymmetric tensors

$$
(\epsilon_{AB}) = (\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.1}
$$

We associate a curve with a matrix with two upper (lower) indices. The first trial for ϵ_{AB} we look at is in fig (1.2) and for ϵ^{AB} fig (1.3)

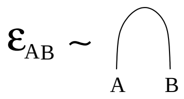

Figure 1.2: A trial diagrammatic representation of ϵ_{AB}.

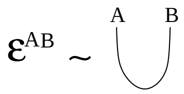

Figure 1.3: A trial diagrammatic representation of ϵ_{AB}.

This fits well with the diagramatitics of $\delta_A^C \epsilon_{CB} = \epsilon_{AB}$. We soon find trouble with this choose however. Firstly:

$$
\delta_A^C \epsilon_{CD} \epsilon^{DE} \delta_E^B = -\delta_A^B
$$

and straightening a line yields a minus sign:

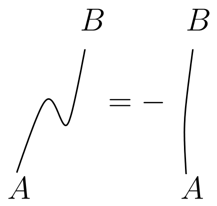

Figure 1.4: First problem with diagrammatic representation of ϵ_{AB} and ϵ^{AB}.

Secondaly, as a conseqence of

$$
\epsilon_{AD} \epsilon_{BC} \epsilon^{CD} = -\epsilon_{AB},
$$

However, these "topological" difficulties can be fixed by modifying the definition

$$
\epsilon_{AB} \to \tilde{\epsilon}_{AB} = i\epsilon_{AB}.
$$

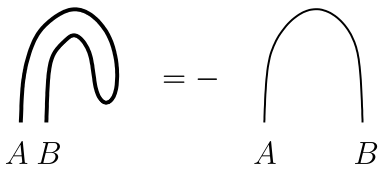

Figure 1.5: Second problem with diagrammatic representation of ϵ_{AB} and ϵ^{AB}.

The diagrams in fig (1.4) and fig (1.5) will henceforth be associated with $\tilde{\epsilon}_{AB}$ and $\tilde{\epsilon}^{AB}$ respectively.

As the indices take two values, we have the identity

$$\epsilon_{[EB}\epsilon_{C]F} = 0 \tag{1.2}$$

which reduces to

$$\epsilon_{EB}\epsilon_{CF} + \epsilon_{BC}\epsilon_{EF} + \epsilon_{CE}\epsilon_{BF} = 0 \tag{1.3}$$

Contracting this with ϵ^{EA} and ϵ^{FD}, then using $\epsilon_{EB}\epsilon^{EA} = \delta^A_B$, $\epsilon_{CF}\epsilon^{FD} = -\delta^D_C$ and $\epsilon_{EF}\epsilon^{EA}\epsilon^{FD} = \epsilon^{AD}$ etc we obtain the so-called binor identity:

$$\epsilon_{AC}\epsilon^{BD} = \delta^B_A\delta^D_C - \delta^D_A\delta^B_C \tag{1.4}$$

Using the definitions of the $\tilde{\epsilon}$ matrices, the binor identity becomes

$$\tilde{\epsilon}_{AC}\tilde{\epsilon}^{BD} - \delta^D_A\delta^B_C + \delta^B_A\delta^D_C = 0. \tag{1.5}$$

Then introducing the rule that we assign a minus sign to each crossing, equation (1.5) can be diagrammatically, represented as in fig 1.6.

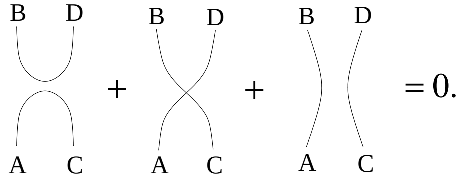

Figure 1.6: The diagrammatic representation of the binor indentity $\tilde{\epsilon}_{AC}\tilde{\epsilon}^{BD} - \delta^D_A\delta^B_C + \delta^B_A\delta^D_C = 0$

For more than

$\delta_A^C \epsilon_{CD} = \epsilon_{AB}$.

$$\delta_A^C \epsilon_{CD} \epsilon^{DE} \delta_E^B = \epsilon_{AD} \epsilon^{DB} = -\delta_A^B, \tag{1.6}$$

$$\epsilon_{AD} \epsilon_{BC} \epsilon^{CD} = -\epsilon_{AB} \tag{1.7}$$

$$\epsilon_{AB} \to \tilde{\epsilon}_{AB} = i\epsilon_{AB} \tag{1.8}$$

$$\delta_A^D \tag{1.9}$$

Using these rules, we can show that these strands behave as would thin strings in the plane; one can arbitrary deform a graphical expression without changing its meaning.

In translating a diagram into tensor notation, we use

1. assign a minus sign to each
2. assign a minus sign to each crossing

1.1.2 Symmetrizing Products of Delta Functions

Define the $D^{A'B'}_{(A\,B)}$ as the symmetric product of two delta functions:

$$D^{A'B'}_{(A\,B)} := \frac{1}{2!} \left(\delta_A^{A'} \delta_B^{B'} + \delta_B^{A'} \delta_A^{B'} \right) \tag{1.10}$$

$D^{A'B'}_{(A\,B)}$ are projectors i.e.

$$D^{A'B'}_{(C\,D)} D^{C\,D}_{(A\,B)} = \frac{1}{2!} \left(\delta_C^{A'} \delta_D^{B'} + \delta_D^{A'} \delta_C^{B'} \right) D^{C\,D}_{(A\,B)} = D^{(A'B')}_{(A\,B)} = D^{A'B'}_{(A\,B)} \tag{1.11}$$

More generally $D^{A'B'...D'}_{(A\,B\,...D)}$, the symmetric product of n delta functions, is a projector:

$$\begin{aligned} D^{A'B'...D'}_{(E\,F\,...H)} D^{EF...H}_{(A\,B\,...D)} &= \frac{1}{n!} \left(\delta_E^{A'} \delta_F^{B'} \dots \delta_H^{H'} + \dots \right) D^{EF...H}_{(A\,B\,...D)} \\ &= D^{(EF...H)}_{(A\,B\,...D)} = D^{EF...H}_{(A\,B\,...D)}. \end{aligned} \tag{1.12}$$

This general result can be represented diagramatically as in fig 1.1.2.

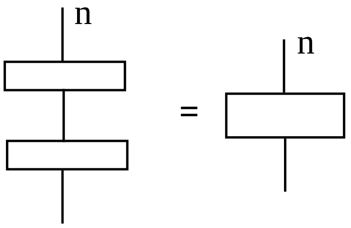

Figure 1.7: The symmetric product of n delta functions, is a projector.

Also note we have the result that if $D^{A'B'...D'_1...D'_k}_{(A\;B\;...D_1\;...D_k)}$, the symmetric product of n delta functions, and if $D^{D'_1...D'_k}_{(D_1...D_k)}$, the symmetric product of k delta functions ($k < n$):

$$\begin{aligned} D^{A'B'...D'_1...D'_k}_{(A\;B\;...H_1...H_k)} D^{H_1...H_k}_{(D_1\;...D_k)} &= D^{A'B'...D'_1...D'_k}_{(A\;B\;...H_1...H_k)} \frac{1}{k!}\left(\delta^{H_1}_{D_1}\ldots\delta^{H_k}_{D_k} + \ldots\right) \\ &= D^{A'B'...(D'_1...D'_k)}_{(A\;B\;...H_1...H_k)} = D^{A'B'...D'_1...D'_k}_{(A\;B\;...H_1...H_k)}. \end{aligned} \tag{1.13}$$

This result can be represented diagramatically as in fig (1.1.2).

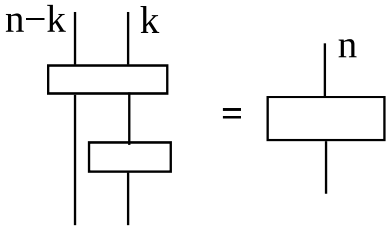

Figure 1.8: Applying the symmetric product of k delta functions to the symmetric product of n delta functions results in the symmetric product of n delta functions.

1.1.3 Jones-Wenzl Projectors

Starting from the binor identity

$$-\tilde{\epsilon}^{A'B'}\tilde{\epsilon}_{AB} = \delta^{A'}_A\delta^{B'}_B - \delta^{A'}_B\delta^{B'}_A, \tag{1.14}$$

a simple rearrangement gives

$$\frac{1}{2}(\delta^{A'}_A\delta^{B'}_B + \delta^{B'}_A\delta^{A'}_B) = \delta^{A'}_A\delta^{B'}_B + \frac{1}{2}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'B'} \tag{1.15}$$

Written in tha stanard form (see in a momemt)

$$\delta^{A'}_{(A}\delta^{B'}_{B)} = \delta^{A'}_{A}\delta^{B'}_{B} - \mu_1\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'B'} \tag{1.16}$$

where $\mu_1 = -1/2$. Which is diagramatically represented in fig 1.1.3.

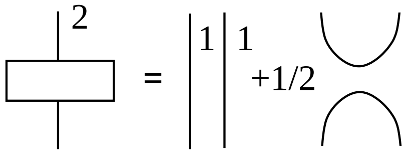

Figure 1.9: Diagrammatical representation of equation (1.16) with $\mu_1 = -1/2$.

Jones-Wenzl Projectors for $n = 3$

We can rearange the symmetric product of the three deltas as follows

$$\begin{aligned} 3\delta^{(A'}_{A}\delta^{B'}_{B}\delta^{C')}_{C} &= \delta^{A'}_{A}\delta^{(B'}_{B}\delta^{C')}_{C} + \delta^{A'}_{B}\delta^{(B'}_{A}\delta^{C')}_{C} + \delta^{A'}_{C}\delta^{B'}_{(B}\delta^{C'}_{A)} \\ &= 3\delta^{A'}_{A}\delta^{(B'}_{B}\delta^{C')}_{C} - \left(\delta^{A'}_{A}\delta^{(B'}_{B}\delta^{C')}_{C} - \delta^{A'}_{B}\delta^{(B'}_{A}\delta^{C')}_{C}\right) - \left(\delta^{A'}_{A}\delta^{(B'}_{B}\delta^{C')}_{C} - \delta^{A'}_{C}\delta^{(B'}_{B}\delta^{C')}_{A}\right) \end{aligned} \tag{1.17}$$

This rearangment, (1.17), can be represented diagramatically as in fig (1.1.3)

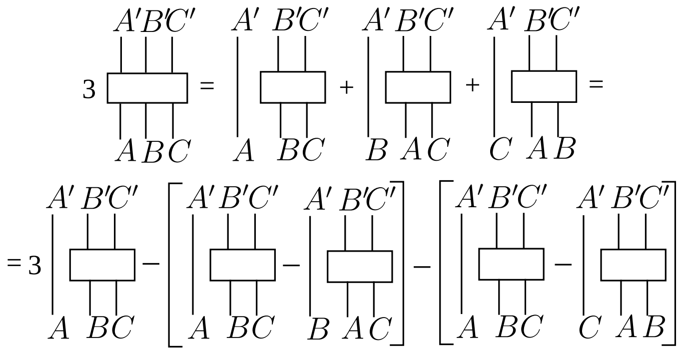

Figure 1.10: graphmath7. Diagramatical representation of (1.17).

Multiply (1.14) by $\delta^{C'}_{C}$ and symmetrize over the upper indices B' and C' to get

$$\delta^{A'}_{A}\delta^{(B'}_{B}\delta^{C')}_{C} - \delta^{A'}_{B}\delta^{(B'}_{A}\delta^{C')}_{C} = -\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta^{C')}_{C}, \tag{1.18}$$

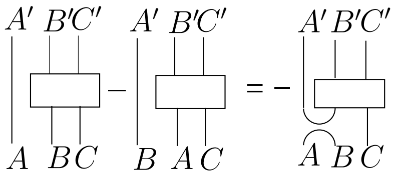

Figure 1.11: Diagramatical representation of (1.18)

which is expressed diagramatically as fig.(1.1.3)

Using this in fig.(1.1.3) we obtain the equation displayed in fig (1.1.3)

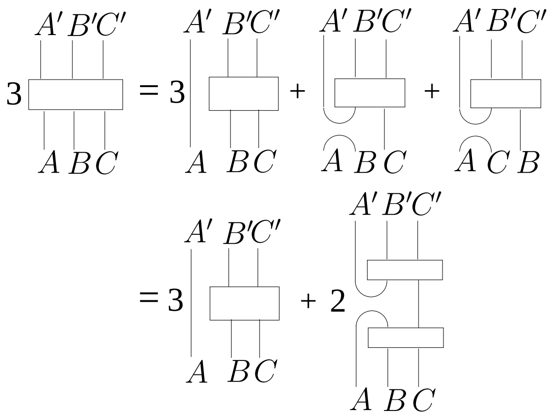

Figure 1.12: graphmath8.

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \frac{1}{3}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C')} + \frac{1}{3}\tilde{\epsilon}_{AC}\tilde{\epsilon}^{A'(C'}\delta_B^{B')} \tag{1.19}$$

we obtain

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \frac{2}{3}\tilde{\epsilon}_{A(B}\delta_{C)}^{(C'}\tilde{\epsilon}^{B')A'} \tag{1.20}$$

Or

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \mu_1\tilde{\epsilon}_{A(B}\delta_{C)}^{(C'}\tilde{\epsilon}^{B')A'} \tag{1.21}$$

where

$$\mu_1 = -2/3. \tag{1.22}$$

This is represented in fig (1.1.3)

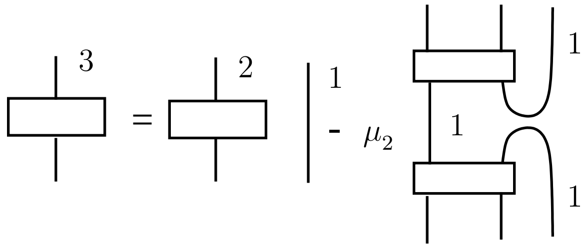

Figure 1.13: Diagrelfor3. Compact diagrammatical representation of (1.21)

Jones-Wenzl Projectors for Arbitrary n

We now consider the symmetric product of n δ's. We have:

$$
\begin{aligned}
n\delta_A^{(A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_F^{F')} &= \delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \dots + \delta_F^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_A^{F')} \\
&= n\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')}\right) - \dots \\
&\quad \dots - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_F^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_A^{F')}\right)
\end{aligned}
\tag{1.23}
$$

This is represented by diagram (graphmath11)

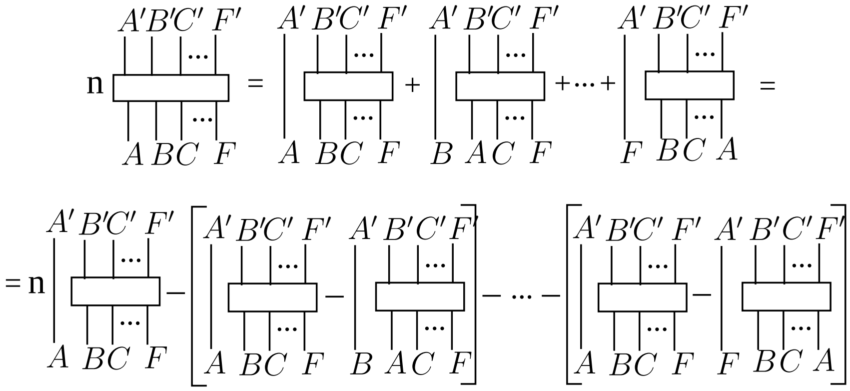

Figure 1.14: graphmath11. Representing equation (1.23).

Multiply (1.14) by $\delta_C^{C'}\cdots\delta_F^{F'}$ and symmetrize over the upper indicies $B', C', \dots, F'$ to get

$$
\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')} = -\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} \tag{1.24}
$$

This is represented by diagram graphmath12

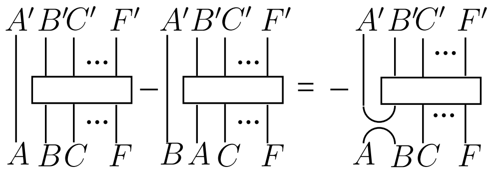

Figure 1.15: graphmath12. Diagrammatical representation of (1.24)

Substituting (1.24) into (1.23) we obtain

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_F^{F')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \frac{1}{n}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} + \dots + \frac{1}{n}\tilde{\epsilon}_{AF}\tilde{\epsilon}^{A'(F'}\delta_C^{C'}\dots\delta_B^{B')} \tag{1.25}$$

We obviously have

$$\underbrace{\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} + \tilde{\epsilon}_{AC}\tilde{\epsilon}^{A'(C'}\delta_B^{B'}\dots\delta_F^{F')}\dots + \tilde{\epsilon}_{AF}\tilde{\epsilon}^{A'(F'}\delta_C^{C'}\dots\delta_B^{B')}}_{n-1 \text{ terms}}$$
$$= +(n-1)\tilde{\epsilon}_{A(B}\delta_C^{C'}\dots\delta_{F)}^{(F'}\tilde{\epsilon}^{B')A'}. \tag{1.26}$$

we obtain

$$\delta_{(A}^{A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_{F)}^{F'} = \delta_A^{A'}\delta_{(B}^{B'}\delta_C^{C'}\dots\delta_{F)}^{F'} + (n-1)\epsilon_{A(B}\delta_C^{C'}\dots\delta_{F)}^{(F'}\epsilon^{B')A'} \tag{1.27}$$

This is represented in fig (1.1.3)

In compact form we have fig (1.1.3):

1.1.4 Contractions of Symmetrised Lines

We perform a contraction the symmetrised lines as given by fig (1.1.4), and we denote the resulting value Δ_n. For example $\Delta_1 = -2$:

$$\Delta_1 = \delta_A^{A'}\left(\tilde{\epsilon}_{A'C}\delta_D^C\tilde{\epsilon}^{DA}\right) = -\delta_A^{A'}\delta_{A'}^A = -2.$$

As an example, we explicitly work out the value of Δ_2 using the graphical method as shown in fig (graphmath2A). We find that the result is $\Delta_2 = 3$.

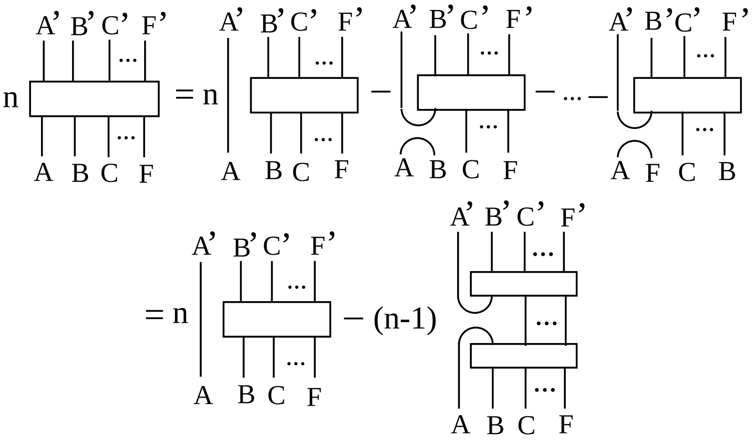

Figure 1.16: graphmath13.

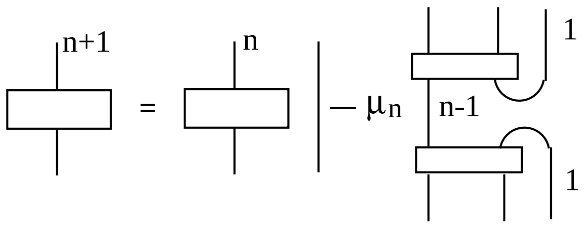

Figure 1.17: .

In order to find the value of Δ_n for $n > 2$ we derive recursive relations.

$$\Delta_{n+1} = -(2 + \mu_n)\Delta_n$$

Rearranged

$$\mu_n = -\frac{\Delta_{n+1}}{\Delta_n} - 2$$

We know from fig (1.1.3) that $\mu_n = \frac{n-1}{n}$, therefore

$$\begin{aligned}
\Delta_n &= -2\Delta_{n-1} + \frac{n-1}{n}\Delta_{n-1} \\
&= \Delta_{n-1}[\frac{n-1}{n} - 2] \\
&= -\Delta_{n-1}[\frac{n+1}{n}]
\end{aligned} \tag{1.28}$$

Employing this recursive relation we obtain,

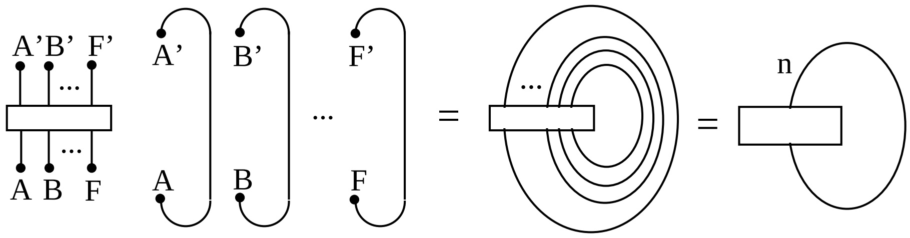

Figure 1.18: graphmath2. The contraction of the symmetric product of deltas, the resulting values denoted Δ_n.

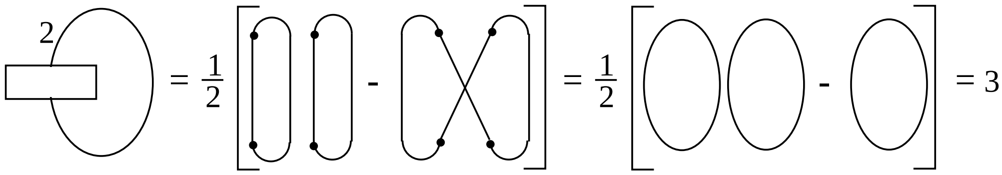

Figure 1.19: graphmath2A. Calculation of Δ_2.

$$
\begin{aligned}
\Delta_n &= -\Delta_{n-1}[\frac{n+1}{n}] \\
&= (-1)^2\Delta_{n-2}[\frac{n}{n-1}][\frac{n+1}{n}] \\
&= (-1)^2\Delta_{n-2}[\frac{n+1}{n-1}] \\
&= (-1)^3\Delta_{n-3}[\frac{n+1}{n-2}] \\
&\vdots \\
&= (-1)^{n-1}\frac{(n+1)}{2}\Delta_1 \\
&= (-1)^n(n+1) \qquad (1.29)
\end{aligned}
$$

where we have used $\Delta_1 = -2$.

So that

$$\Delta_n = (-1)^n(n+1). \qquad (1.30)$$

We derive a recursive relation between Δ_{n+2}, Δ_{n+1} and Δ_n.

The one contraction of an $(n+1)-$symmetrised prodct is proportional to an $n-$symmetrised product, as shown in fig graphmath3

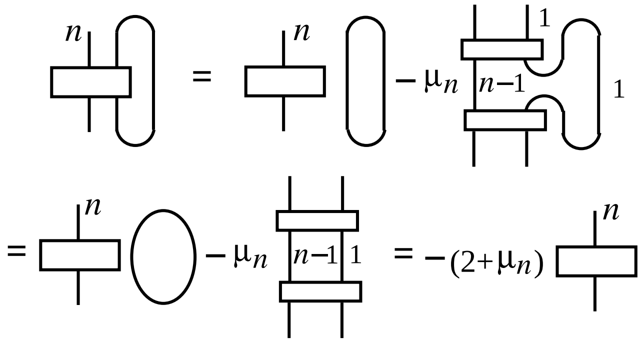

Figure 1.20: graphmath3A .

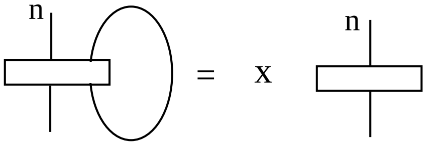

Figure 1.21: graphmath3 .

By definition of Δ_n, we see that x is given by Δ_{n+1}/Δ_n. (see fig graphmath1)

Now, if

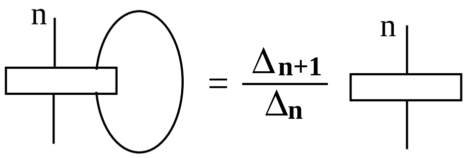

Figure 1.22: graphmath1.

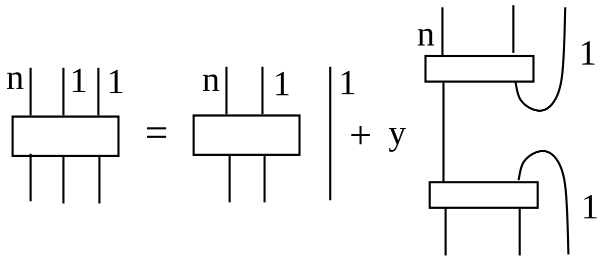

Figure 1.23: P math4.

it follows that

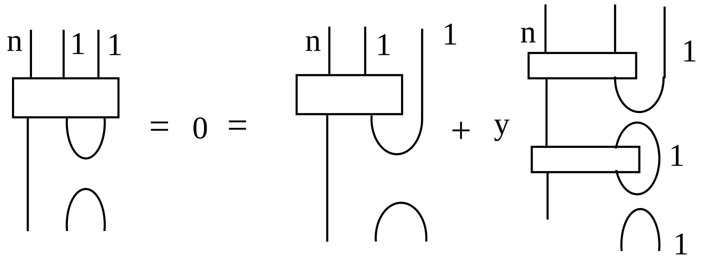

Figure 1.24: graphmath5.

Hence

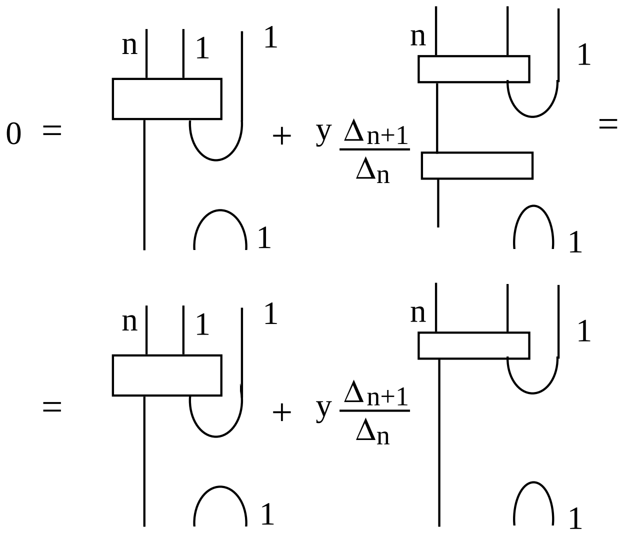

Figure 1.25: graphmath6.

Therefore $y = \Delta_n/\Delta_{n+1}$ and the recursion takes the form

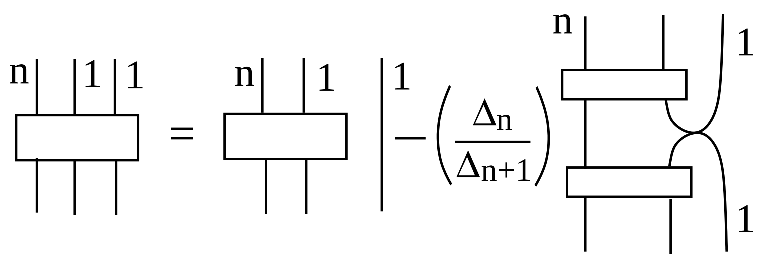

Figure 1.26: P math4.

It follows that

$$\Delta_{n+2} = -2\Delta_{n+1} - \Delta_n \tag{1.31}$$

with $\Delta_1 = -2$ and $\Delta_2 = 3$. This obviously has a unique solution which is

$$\Delta_n = (-1)^n(n+1), \tag{1.32}$$

as is easily checked:

$$
\begin{aligned}
-2\Delta_{n+1} - \Delta_n &= -2(-1)^{n+1}(n+2) - (-1)^n(n+1) \\
&= (-1)^{n+2}[2(n+2) - (n+1)] \\
&= (-1)^{n+2}(n+3) \\
&= \Delta_{n+2},
\end{aligned}
\tag{1.33}
$$

with

$$\Delta_1 = (-1)^1(1+1)$$

$$\Delta_2 = (-1)^2(2+1).$$

in agreement with calculations we have already performed.

Alternatively, first note that we have the recursion relation between Δ_n and Δ_{n+1}

$$
\begin{aligned}
\Delta_{n+1} + \Delta_n &= (-1)(\Delta_n + \Delta_{n-1}) \\
&= (-1)^2(\Delta_{n-1} + \Delta_{n-2}) \\
&\quad \ldots \\
&= (-1)^{n-1}(\Delta_2 + \Delta_1) \\
&= (-1)^{n-1}(3-2) = (-1)^{n-1}
\end{aligned}
\tag{1.34}
$$

From which we obatin

$$
\begin{aligned}
\Delta_n &= -\Delta_{n-1} + (-1)^n \\
&= \Delta_{n-2} + 2(-1)^n \\
&= -\Delta_{n-3} + 3(-1)^n \\
&\quad \ldots \\
&= (-1)^{n-1}(\Delta_1 + (n-1)(-1)^n \\
&= (-1)^n(n+1).
\end{aligned}
\tag{1.35}
$$

each containing a turn back.

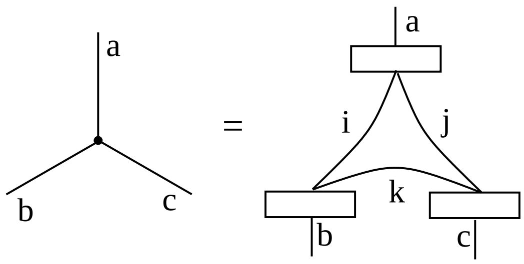

Figure 1.27: Definition of 3-vertex.

1.1.5 3-Vertices

We define a 3-vertex as in fig...

The "internal" labels i, j, k are positive integers determined by the external labels a, b, c via

$$\begin{aligned} i &= \frac{a+b-c}{2} \\ j &= \frac{a+c-b}{2} \\ k &= \frac{b+c-a}{2}. \end{aligned} \tag{1.36}$$

We consider the "bubble" diagram.

Lemma 1.1.1 *The network is zero if $a \neq b$. If $a = b$, then*

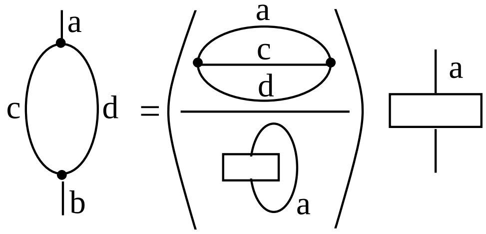

Figure 1.28: P.

Proof:

Assume that $a > b$.

where

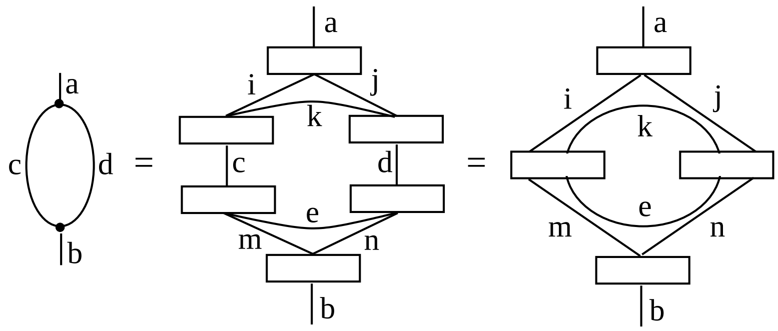

Figure 1.29: graphmath15.

$$
\begin{aligned}
i &= \frac{a+c-d}{2} & l &= \frac{c+d-b}{2} \\
j &= \frac{a+d-c}{2} & m &= \frac{c+b-d}{2} \\
k &= \frac{c+d-a}{2} & n &= \frac{b+d-c}{2}.
\end{aligned}
\tag{1.37}
$$

Rewritting, we find $e = (a-b)/2$

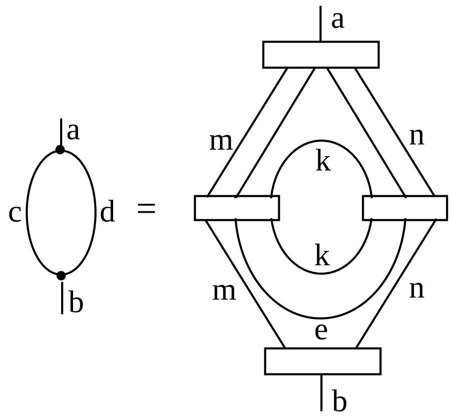

Figure 1.30: graphmath16.

Consider expanding each of the two middle projectors into their sum of products of δ's. It followes that each term will contain a turn-back with respect to the $a-$projector above and give zero.

Now assume that $a = b$. Consider

Hence

and

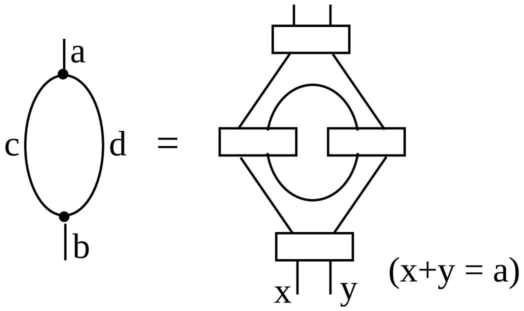

Figure 1.31: graphmath17.

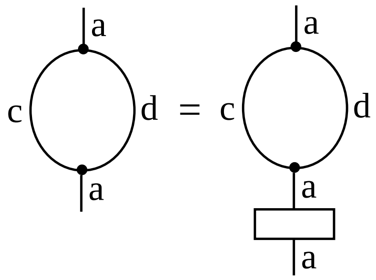

Figure 1.32: graphmath18.

Consider expanding each of the two middle projectors into their sum of products of δ's. Only straight-ahead terms survive the extra projector at the bottom. Thus

Hence

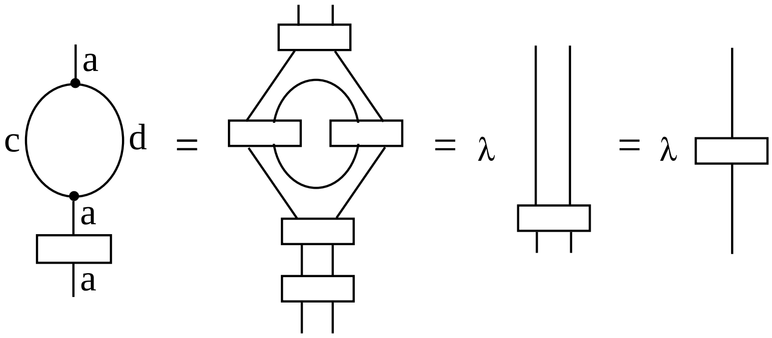

Figure 1.33: graphmath18A.

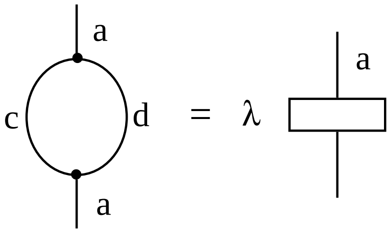

Figure 1.34: graphmath19.

□

$$N_{abc}\begin{pmatrix} a & b & c \\ m_a & m_b & m_c \end{pmatrix} \tag{1.38}$$

$$N_{abc} = \left[\frac{(a+b-c)!(b+c-a)!(c+a-b)!}{2^2(a+b+c+2)!}\right]^{1/2} \tag{1.39}$$

where

$$\begin{aligned} m &= \frac{a+b-c}{2}, \\ n &= \frac{b+c-a}{2}, \\ p &= \frac{a+c-b}{2}. \end{aligned} \tag{1.40}$$

□

Definition Definition $Net(m, n, p)$ is defined as the diagram on the far right of fig (1.1.2)

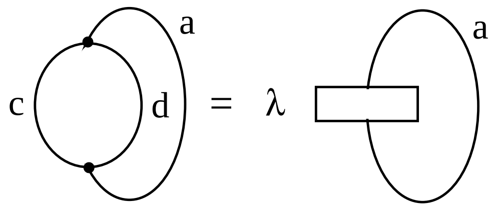

Figure 1.35: graphmath20.

Lemma 1.1.2

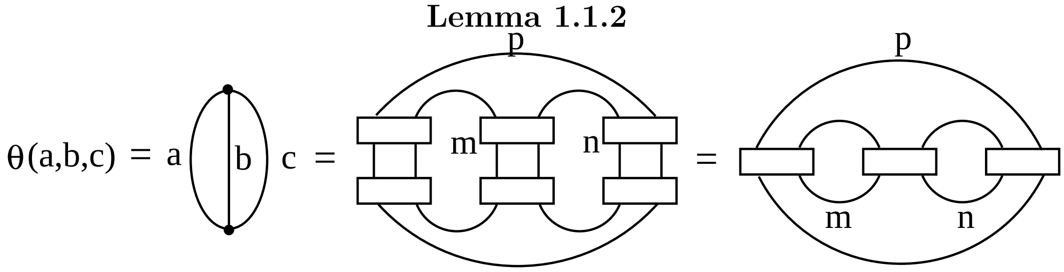

Figure 1.36: graphmath21.

□

We will now be working to evaluate $Net(m, n, p)$.

In the case $p = 0$, we get

$$Net(m, n, 0) \tag{1.41}$$

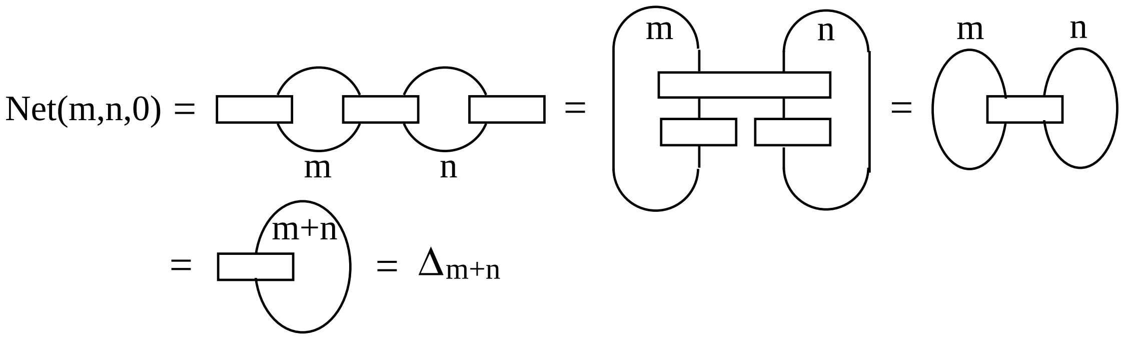

Figure 1.37: graphmath22.

$Net(n-1, 1, 1)$ we will need to get the eigenvalue of the area operator. $Net(m, n, 1) =$ is easy to deal with.

$$Net(m, n, 1) = -(2 + \mu_m + \mu_n)\Delta_{m+n}. \tag{1.42}$$

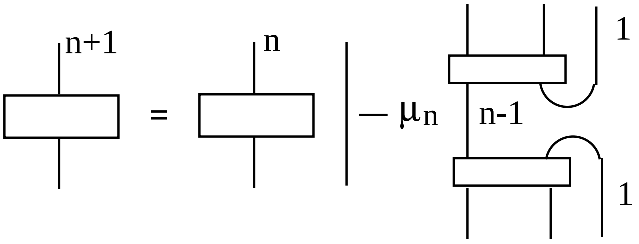

Figure 1.38: This one.

Applying this to $Net(m, n, 1)$ as shown below.

We see that the last term is equivalent to (1.1.5) and so is zero.

The first network is $-2Net(m, n, 0)$, where $Net(m, n, 0)$ has already been calculated in the previous Lemma as Δ_{m+n}. The second and third nets are each equivalent to $Net(m, n, 0)$. The forth network vanishes. Thus

$$Net(m, n, 1) = -(2\Delta_{m+n} + \mu_m \Delta_{m+n} + \mu_n \Delta_{m+n}).$$

Definition Let $Net(m, n, p_e, p_i)$, for $p_e + p_i = p - 1 \geq 1$

Lemma 1.1.3 *Similarly,*

$$\begin{aligned} Net(m, n, p) \;&=\; (-2 - \mu_{m+p-1} - \mu_{n+p-1})Net(m, n, p-1) \\ &\quad + \mu_{m+p-1}\mu_{n+p-1}Net(m, n, 1, p-2) \end{aligned} \tag{1.43}$$

□

Proof:

The last network is equivalent to $Net(m, n, 1, p-2)$ as demonstrated in fig (1.1.5)

Lemma 1.1.4 *We have*

(a) $Net(m, n, p-1, 0) = (-2 - \mu_m - \mu_n)Net(m, n, p-1)$

(b) $Net(m, n, p_e, p_i) = (-2 - \mu_{m+p_i} - \mu_{n+p_i})Net(m, n, p-1) + \mu_{m+p_i}\mu_{n+p_i}Net(m, n, p_e + 1, p_i - 1)$

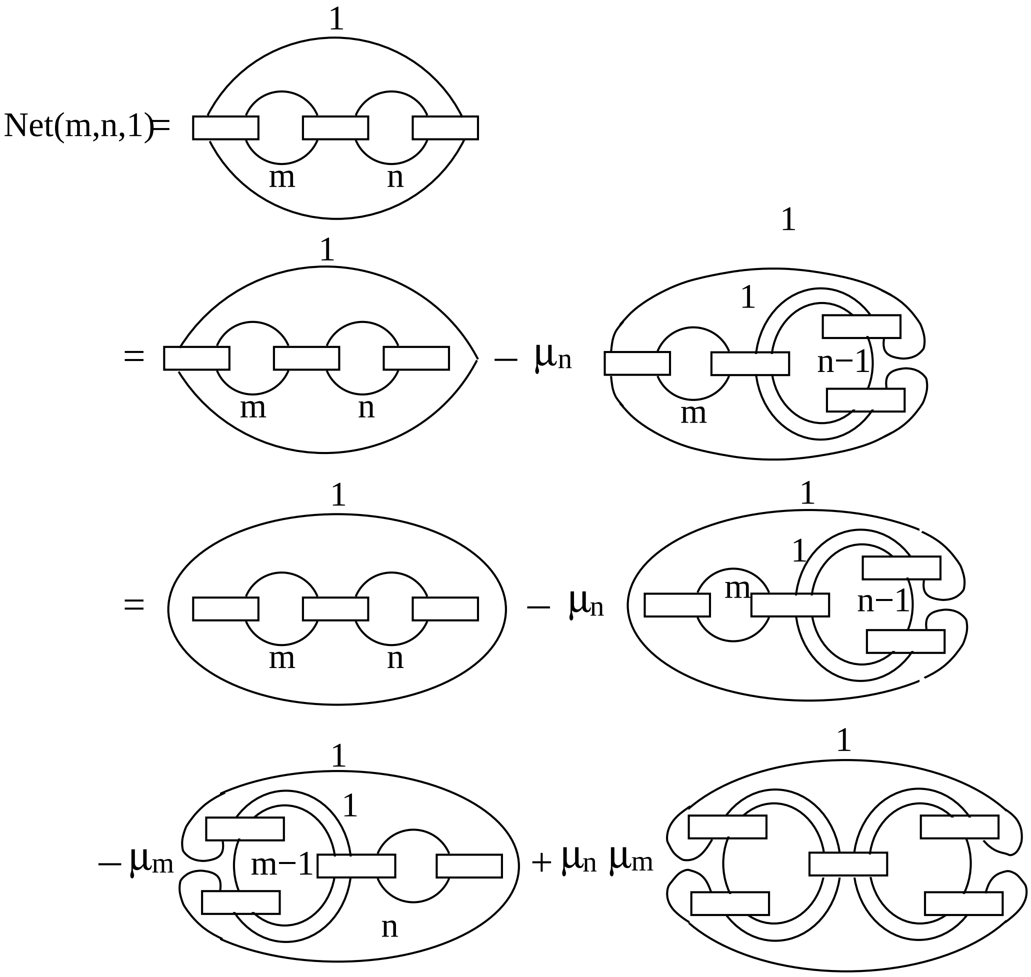

Figure 1.39: graphmath23.

Proof:

The last network is equivalent to $Net(m, n, p_e+, p_i - 1)$ as demonstrated in fig (1.1.5).

□

Recursion relation for Net

(a) $Net(m, n, p-1, 0) = (-2 - \mu_m - \mu_n)Net(m, n, p-1)$

(b) $Net(m, n, p_e, p_i) = (-2 - \mu_{m+p_i} - \mu_{n+p_i})Net(m, n, p-1) + \mu_{m+p_i}\mu_{n+p_i}Net(m, n, p_e + 1, p_i - 1)$

Starting with (1.43) and using (b) over again $p-2$ times, and then finally using (a) we can obtain a realtion:

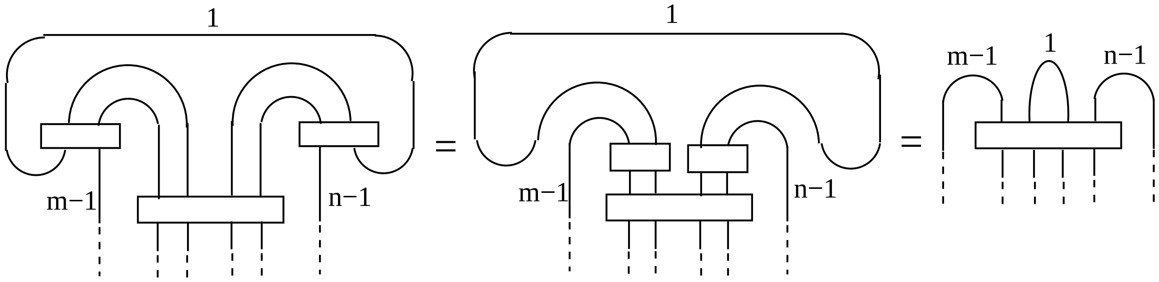

Figure 1.40: graphmath25.

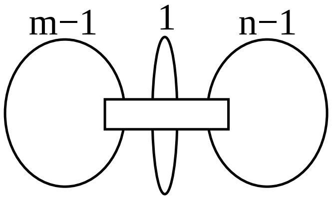

Figure 1.41: graphmath24.

$$Net(m,n,p) = \rho(m,n,p)Net(m,n,p-1). \tag{1.44}$$

To simplify the analysis, we introduce the following. Since $\mu_{m+j} = \Delta_{m-1+j}/\Delta_{m+j}$,

$$-2 - \mu_{m+j} - \mu_{n+j} = \frac{-2\Delta_{m+j}\Delta_{n+j} - \Delta_{m-1+j}\Delta_{n+j} - \Delta_{m+j}\Delta_{n-1+j}}{\Delta_{m+j}\Delta_{n+j}}$$

Write

$$\alpha_j = -2\Delta_{m+j}\Delta_{n+j} - \Delta_{m-1+j}\Delta_{n+j} - \Delta_{m+j}\Delta_{n-1+j}$$

and

$$\beta_j = \Delta_{m+j}\Delta_{n+j}$$

First (1.43) becomes

$$Net(m,n,p) = \frac{\alpha_{p-1}}{\beta_{p-1}}Net(m,n,p-1) + \frac{\beta_{p-2}}{\beta_{p-1}}Net(m,n,1,p-2) \tag{1.45}$$

then we would use (b) with $p_e = 1$ and $p_i = p-2$,

$\mathrm{Net}(m,n,p_e,p_i) =$

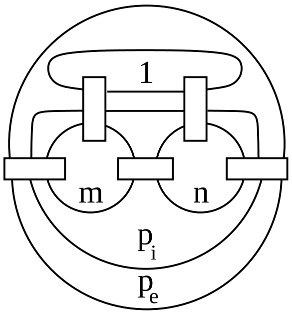

Figure 1.42: graphmath26.

$$Net(m, n, 1, p-2) = \frac{\alpha_{p-2}}{\beta_{p-2}} Net(m, n, p-1) + \frac{\beta_{p-3}}{\beta_{p-2}} Net(m, n, 2, p-3) \tag{1.46}$$

and next we would use (b) with $p_e = 2$ and $p_i = p-3$,

$$Net(m, n, 2, p-3) = \frac{\alpha_{p-3}}{\beta_{p-3}} Net(m, n, p-1) + \frac{\beta_{p-4}}{\beta_{p-3}} Net(m, n, 3, p-4) \tag{1.47}$$

and so on until

$$Net(m, n, p-3, 2) = \frac{\alpha_2}{\beta_2} Net(m, n, p-1) + \frac{\beta_1}{\beta_2} Net(m, n, p-2, 1) \tag{1.48}$$

$$\begin{aligned} Net(m, n, p-2, 1) &= \frac{\alpha_1}{\beta_1} Net(m, n, p-1) + \frac{\beta_0}{\beta_1} Net(m, n, p-1, 0) \\ &= \frac{\alpha_1}{\beta_1} Net(m, n, p-1) + \frac{\beta_0}{\beta_1} \frac{\alpha_0}{\beta_0} Net(m, n, p-1) \end{aligned} \tag{1.49}$$

where in the last line we used (a). Putting it together,

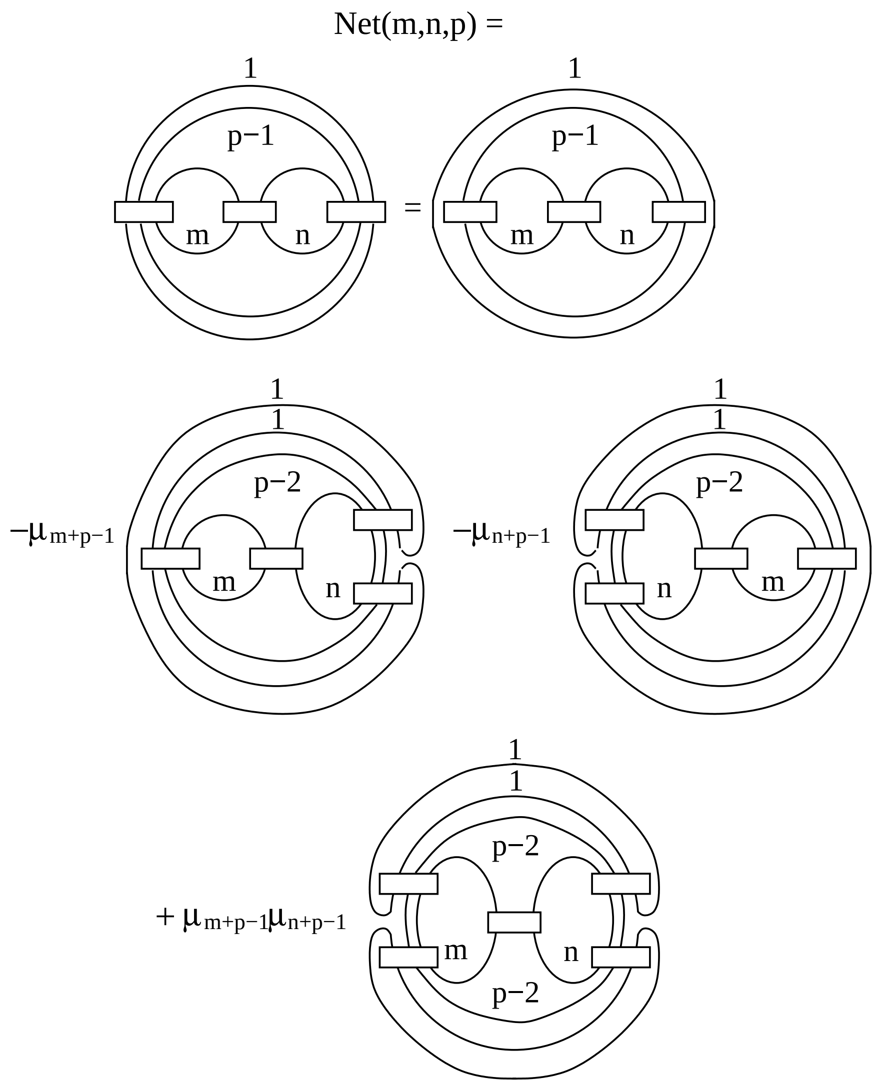

Figure 1.43: Netmnp.

$$
\begin{aligned}
Net(m,n,p) &= \frac{\alpha_{p-1}}{\beta_{p-1}} Net(m,n,p-1) + \frac{\beta_{p-2}}{\beta_{p-1}} Net(m,n,1,p-2) \\
&= (\frac{\alpha_{p-1}+\alpha_{p-2}}{\beta_{p-1}}) Net(m,n,p-1) + \frac{\beta_{p-3}}{\beta_{p-1}} Net(m,n,2,p-3) \\
&= (\frac{\alpha_{p-1}+\alpha_{p-2}+\alpha_{p-3}}{\beta_{p-1}}) Net(m,n,p-1) + \frac{\beta_{p-4}}{\beta_{p-1}} Net(m,n,2,p-4) \\
&= \cdots \\
&= \frac{1}{\beta_{p-1}} \left(\sum_{j=0}^{p-1} \alpha_j \right) Net(m,n,p-1)
\end{aligned} \tag{1.50}
$$

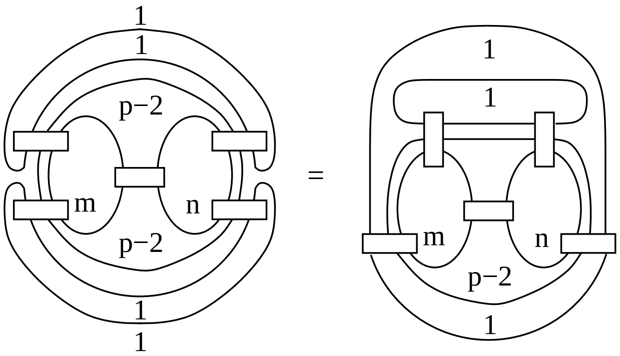

Figure 1.44: Netmn1p-2. Equivalence of last network in fig (1.1.5) with $Net(m, n, 1, p-2)$.

Therefore

$$\rho(m,n,p) = \frac{1}{\beta_{p-1}} \sum_{j=0}^{p-1} \alpha_j. \tag{1.51}$$

Or, upon using $\Delta_{k+2} = -2\Delta_{k+1} - \Delta_k$,

$$\begin{aligned}
\rho(m,n,p) &= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \big(-2\Delta_{m+j}\Delta_{n+j} - \Delta_{m+j-1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \Big((-2\Delta_{m+j} - \Delta_{m+j-1})\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\Big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \Big(\Delta_{m+j+1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\Big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \Big(\Delta_{m+1}\Delta_n - \Delta_m\Delta_{n-1} \\
&\qquad \Delta_{m+2}\Delta_{n+1} - \Delta_{m+1}\Delta_n \\
&\qquad \Delta_{m+3}\Delta_{n+2} - \Delta_{m+2}\Delta_{n+1} \\
&\qquad \cdots \\
&\qquad \Delta_{m+j+1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1} \\
&\qquad \cdots \\
&\qquad \Delta_{m+p-1}\Delta_{n+p-2} - \Delta_{m+p-2}\Delta_{n+p-3} \\
&\qquad \Delta_{m+p}\Delta_{n+p-1} - \Delta_{m+p-1}\Delta_{n+p-2}\Big) \\
&= \frac{\Delta_{m+p}\Delta_{n+p-1} - \Delta_m\Delta_{n-1}}{\Delta_{m+p-1}\Delta_{n+p-1}}
\end{aligned} \tag{1.52}$$

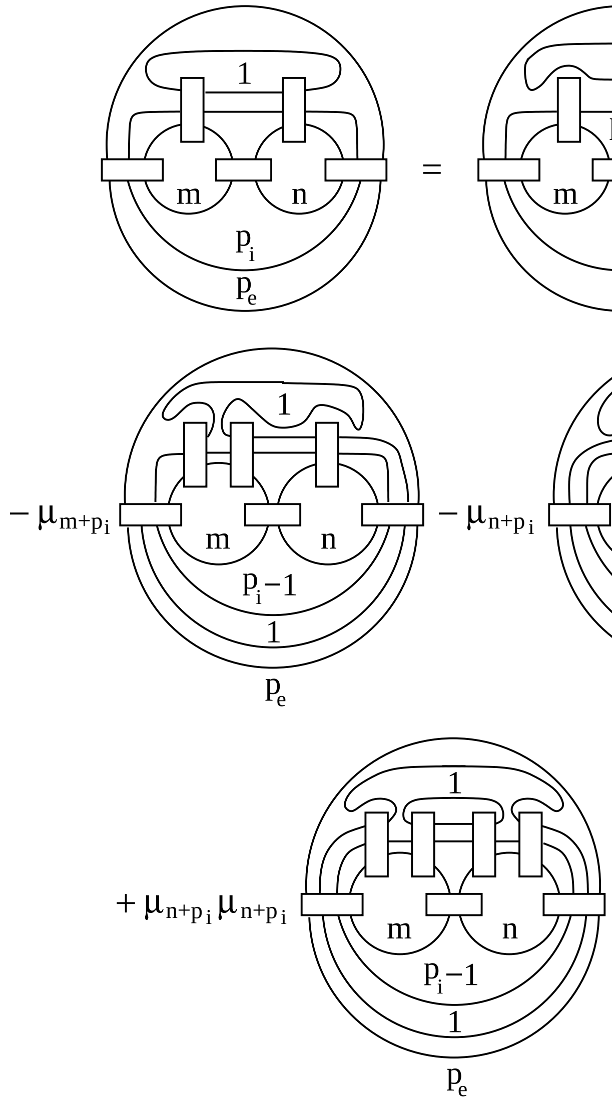

Figure 1.45: Netmnpepi.

We can simplify futher,

$$\begin{aligned}
\Delta_{m+p}\Delta_{n+p-1} - \Delta_m\Delta_{n-1} &= (-1)^{m+p}(m+p+1)(-1)^{n+p-1}(n+p) - (-1)^m(m+1)(-1)^{n-1}n \\
&= (-1)^{m+n+2p-1}[(m+p+1)(n+p) - (m+1)n] \\
&= (-1)^{m+n+2p-1}[np + (m+p+1)p] \\
&= (-1)^{m+n+p}(m+n+p+1)(-1)^{p-1}p \\
&= \Delta_{m+n+p}\Delta_{p-1} \qquad (1.53)
\end{aligned}$$

Therefore

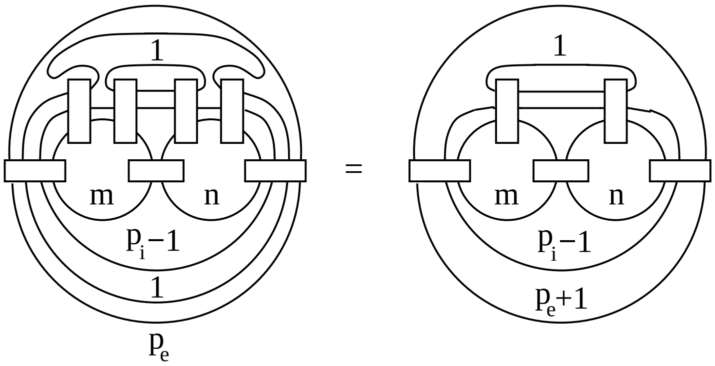

Figure 1.46: Netmnpe+1pi-1.

$$\rho(m,n,p) = \frac{\Delta_{m+n+p}\Delta_{p-1}}{\Delta_{m+p-1}\Delta_{n+p-1}} \tag{1.54}$$

Denote

$$\Delta_n! := \Delta_n\Delta_{n-1}\Delta_{n-2}\cdots\Delta_1.$$

For $\theta(a,b,c)$,

$$\begin{aligned}
\theta(a,b,c) &= \rho(m,n,p)Net(m,n,p-1) \\
&= \left(\prod_{j=1}^{p}\rho(m,n,j)\right)Net(m,n,0) \\
&= \left(\prod_{j=1}^{p}\rho(m,n,j)\right)\Delta_{m+n}
\end{aligned} \tag{1.55}$$

Hence, by (1.54)

$$\begin{aligned}
\theta(a,b,c) &= \prod_{j=1}^{p}\left[\frac{\Delta_{m+n+j}\Delta_{j-1}}{\Delta_{m+j-1}\Delta_{n+j-1}}\right]\Delta_{m+n} \\
&= \frac{\left(\Delta_{m+n+p}\Delta_{m+n+p-1}\cdots\Delta_{m+n}\right)\Delta_{p-1}!}{\left(\Delta_{m+p-1}\Delta_{m+p-2}\cdots\Delta_{m}\right)\left(\Delta_{n+p-1}\Delta_{n+p-2}\cdots\Delta_{n}\right)} \\
&= \frac{\Delta_{m+n+p}!\Delta_{n-1}!\Delta_{m-1}!\Delta_{p-1}!}{\Delta_{m+p-1}!\Delta_{n+p-1}!\Delta_{m+n-1}!}
\end{aligned} \tag{1.56}$$

The minus signs in the factorial simplifies as follows

$$\begin{aligned}\Delta_{m+n+p}! &= (-1)^{m+n+p}(m+n+p+1)(-1)^{m+n+p-1}(m+n+p)\cdots(-1)^1 2! \\ &= (-1)^{(m+n+p)+(m+n+p-1)+\cdots+1}(m+n+p+1)! \\ &= (-1)^{(m+n+p)(m+n+p+1)/2}(m+n+p+1)! \end{aligned} \tag{1.57}$$

So that we get

$$\begin{aligned}\Delta_{m+n+p}! &= (-1)^{(m+n+p)(m+n+p+1)/2}(m+n+p+1)! \\ \Delta_{m+n-1}! &= (-1)^{(m+n-1)(m+n)/2}(m+n)! \\ \Delta_{m-1}! &= (-1)^{(m-1)m/2}m! \end{aligned} \tag{1.58}$$

Collecting the exponents of (-1) in (1.56) is

$$\begin{aligned}&\frac{1}{2}\big[(m+n+p)(m+n+p+1)+(n-1)n+(m-1)m+(p-1)p+ \\ &+(m+p-1)(m+p)+(n+p-1)(n+p)+(m+n-1)(m+n)\big] \\ &=\frac{1}{2}\big[(m+n+p)^2+n^2+m^2+p^2 \\ &+(m+p)^2-(m+p)+(n+p)^2-(n+p)+(m+n)^2-(m+n)\big] \\ &=\frac{1}{2}\big[(m^2+n^2+p^2+2mn+2mp+2np)+n^2+m^2+p^2+ \\ &+2m^2+2n^2+2p^2+2mp+2np+2mn-2(m+n+p)\big] \\ &=2m^2+2n^2+2p^2+2mn+2np+2pm-m-n-p\equiv m+n+p+2k. \end{aligned} \tag{1.59}$$

where k is an integer.

Therefore,

$$\theta(a,b,c)=\frac{(-1)^{m+n+p}(m+n+p+1)!m!n!p!}{(m+n)!(n+p)!(m+p)!} \tag{1.60}$$

where

$$
\begin{aligned}
m &= \frac{a+b-c}{2}, \\
n &= \frac{b+c-a}{2}, \\
p &= \frac{a+c-b}{2}. \qquad (1.61)
\end{aligned}
$$

$$
\begin{aligned}
m+p &= 2a \\
m+n &= 2b \\
n+p &= 2c \\
m+n+p &= 2a+2b+2c. \qquad (1.62)
\end{aligned}
$$

TET

Recoupling formula

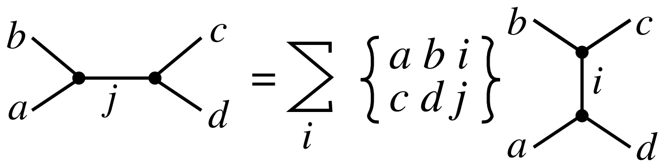

Figure 1.47: recouplefig. The recoupling equation

The tetrahedron network.

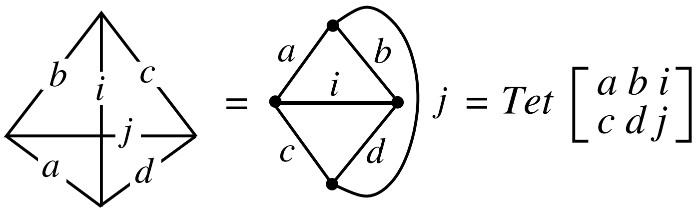

Figure 1.48: TetDef.

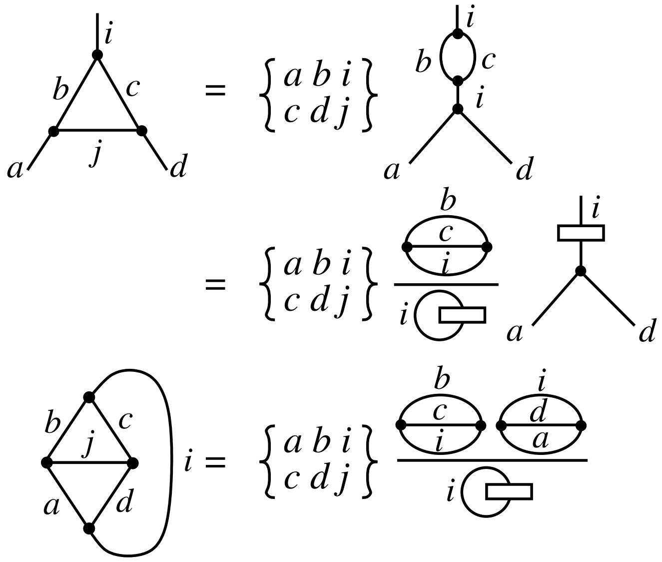

Figure 1.49: 6jandTET.

The tetrahedron formula for recoupling theoery.

The evaluation of the tetrahedron network.

$$\left\{\begin{array}{ccc} a & b & i \\ c & d & j \end{array}\right\} = \frac{Tet\begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix}\Delta_i}{\theta(a,d,i)\theta(b,c,j)} \tag{1.63}$$

$$Tet\begin{bmatrix} A & B & E \\ C & D & F \end{bmatrix} = \frac{\mathcal{I}}{\mathcal{E}} \sum_{m \le S \le M} \frac{(-1)^S (S+1)!}{\prod_i (S-a_i)! \prod_j (b_j - S)!} \tag{1.64}$$

where

$$
\begin{aligned}
a_1 &= \frac{A+D+E}{2}, & b_1 &= \frac{B+D+E+F}{2} \\
a_2 &= \frac{B+C+E}{2}, & b_2 &= \frac{A+C+E+F}{2} \\
a_3 &= \frac{A+B+F}{2}, & b_3 &= \frac{A+B+C+D}{2} \\
a_4 &= \frac{C+D+F}{2}, & & \\
m &= \max\{a_i\}, & M &= \min\{b_j\} \\
\mathcal{E} &= A!B!C!D!E!F!, & \mathcal{I} &= \prod_{ij}(b_j - a_i)!.
\end{aligned} \tag{1.65}
$$

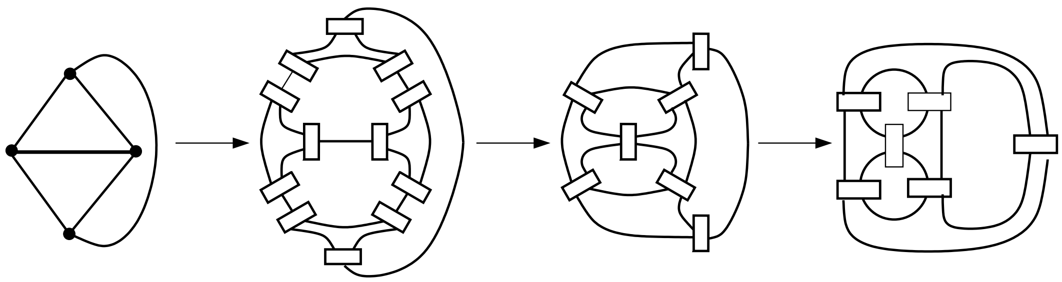

Figure 1.50: Tetfig2.

The $6j-$sybols have a number of properties including the orthogonal identity

$$
\sum_l \left\{ \begin{array}{ccc} a & b & l \\ c & d & j \end{array} \right\} \left\{ \begin{array}{ccc} d & a & i \\ b & c & l \end{array} \right\} = \delta_i^j \tag{1.66}
$$

and the Biedenharn-Elliot or Pentagon identity

$$
\sum_l \left\{ \begin{array}{ccc} d & i & l \\ e & m & c \end{array} \right\} \left\{ \begin{array}{ccc} a & b & f \\ e & l & i \end{array} \right\} \left\{ \begin{array}{ccc} a & f & k \\ d & d & l \end{array} \right\} = \left\{ \begin{array}{ccc} a & b & k \\ c & d & i \end{array} \right\} \left\{ \begin{array}{ccc} k & b & f \\ e & m & c \end{array} \right\} \tag{1.67}
$$

The reduction formula

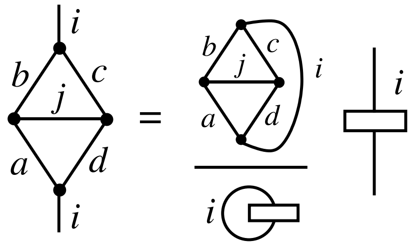

Figure 1.51: reductfigs.

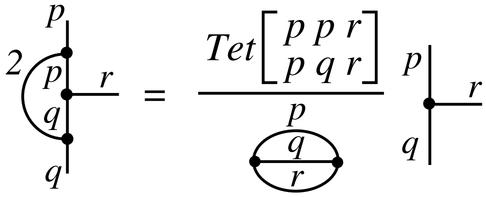

Figure 1.52: reductfigs2.

Change of basis for 4-valent spin networks.

(1)

Answers:

Rotate the network on the RHS by clockwise and apply the recoupling identity again.

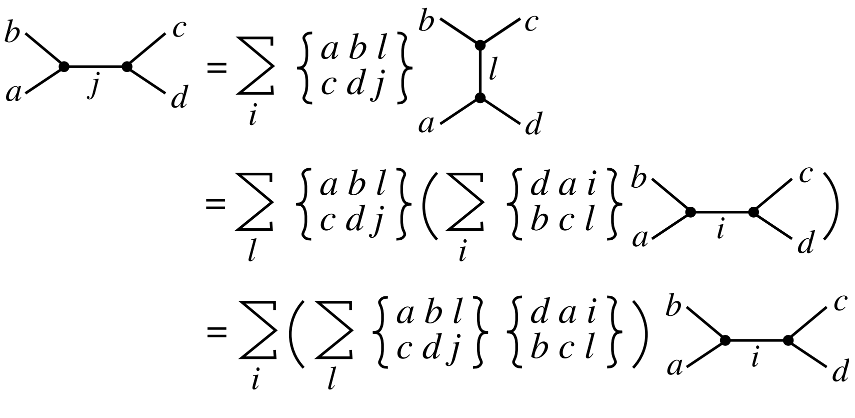

Figure 1.53: recoupfig2. Proof of the orthogonality identity.